# Existence of homoclinic constant sign solutions for a difference equation on the integers

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**Abstract**

We consider a difference equation involving the discrete $p$-Laplacian operator, depending on a positive real parameter $\lambda$. We prove, under convenient assumptions, that for $\lambda$ big enough the equations admits at least one homoclinic constant sign solution in $\mathbb{Z}$. Our method consists in two parts: first, we prove the existence of two Dirichlet-type solutions for the equation in the discrete interval $[-n, n]$, for all $n \in \mathbb{N}$ big enough; then, we show that such solutions converge to a homoclinic solution in $\mathbb{Z}$, as $n \to \infty$.

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## 1 Introduction

In the present paper we will deal with the following difference equation on $\mathbb{Z}$, with homoclinic asymptotic conditions, depending on a real parameter $\lambda > 0$:

$$(P^\lambda) \qquad \begin{cases} -\Delta_p x(k-1) = \lambda f(k, x(k)) & \text{for all } k \in \mathbb{Z} \\ x(-\infty) = x(+\infty) = 0 \end{cases}.$$

Here $p > 1$ is a real number, the discrete $p$-Laplacian operator is defined by

$$\Delta_p x(k-1) = \Delta\phi_p(\Delta x(k-1)) \text{ for all } k \in \mathbb{Z},$$

where $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, and $\Delta$ denotes the forward difference operator. Moreover, $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and we set

$$x(\pm\infty) = \lim_{k \to \pm\infty} x(k).$$

Solutions of problem $(P^\lambda)$ are mappings $x : \mathbb{Z} \to \mathbb{R}$ with a homoclinic behavior at $\pm\infty$, i.e. $x(k) \to 0$ as $|k| \to \infty$.

Boundary value problems (BVP's for short) for difference equations can be studied in several ways: usually, numerical analysis is employed together with fixed point methods or other tools from nonlinear operator theory (see the classical monographs of Agarwal [1], Kelley & Peterson [11] and Lakshmikantham & Trigiante [12]). In the last decade, the use of *variational methods* in such problems has

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gained increasing interest: this was made possible by Agarwal, Perera & O'Regan [2], who established a convenient variational framework for discrete BVP's, analogous to that used in differential equations.

Many authors have applied different results of critical point theory to prove existence and multiplicity results for the solutions of discrete BVP's: let us mention the works of Cabada, Iannizzotto & Tersian [4], Cai, Guo & Yu [6], Candito & Giovannelli [7], Faraci & Iannizzotto [8], Jiang & Zhou [10], Mihăilescu, Rădulescu & Tersian [13], and Ricceri [14]. All these papers deal with problems on *bounded* discrete intervals of the type

$$[m,n]=\{k\in\mathbb{Z}\ :\ m\leq k\leq n\},$$

which allows one to search for solutions in a *finite-dimensional* Banach space.

The issue of finding solutions for discrete BVP's on *unbounded* intervals is more delicate: in the works of Cabada & Cid [3] and Cabada & Tersian [5], such problems have been studied via an approximation method. We also mention the paper of Iannizzotto & Tersian [9], where a problem involving a coercive weight function is studied directly by variational methods on $\ell^p$-type spaces.

In the present paper we will apply a similar method to problem $(P^\lambda)$. First, under appropriate hypotheses on the reaction term $f$ (mainly sign conditions), we will consider a discrete Dirichlet-type BVP $(P_n^\lambda)$ on a bounded discrete interval $[-n,n]$ and prove that, for $\lambda>0$ big enough, $(P_n^\lambda)$ has at least two nontrivial solutions satisfying estimates independent of $n$. Moreover, we will prove that such solutions have constant sign and show monotonicity and convexity properties near $\pm n$ (Section 2). Then, we will pass to the limit as $n\to\infty$ and prove the existence of at least one nontrivial solution with constant sign for $(P^\lambda)$ (Section 3). Along the paper some examples are introduced to illustrate the obtained results.

Our work represents an advance in the relatively new field of variational methods applied to difference equations, in the spirit of Agarwal, Perera & O'Regan [2], combined with approximation techniques to deal with unbounded discrete domains similar to those in Cabada & Cid [3]. With respect to previous results in this field, we give a more precise description of the kind of solutions on $[-n,n]$ (constant sign, uniform estimates, monotonicity and convexity), which goes beyond a mere existence-multiplicity result. These properties are essentially used in the proof of the convergence of Dirichlet-type solution to a homoclinic solution on $\mathbb{Z}$, which inherits them.

## 2 Two Dirichlet-type solutions

This section is devoted to the study of the following discrete BVP with Dirichlet boundary conditions on the interval $[-n,n]$ ($n\in\mathbb{N}$), depending on a real parameter $\lambda>0$:

$$(P_n^\lambda)\qquad \begin{cases} -\Delta_p x(k-1)=\lambda f(k,x(k)) & \text{for all } k\in[-n,n] \\ x(-n-1)=x(n+1)=0 \end{cases}.$$

We will make the following hypotheses on the function $f$:

**H** $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is continuous. We set

$$F(k,t)=\int_0^t f(k,\tau)d\tau \text{ for all } k\in\mathbb{Z},\ t\in\mathbb{R},$$

and assume:

$(i)$ for all $k\in\mathbb{Z}$ there exist $\bar{a}_k\leq a_k\leq 0\leq b_k\leq\bar{b}_k$ s.t.

$$f(k,t)\begin{cases} >0 & \text{for all } t\in(-\infty,\bar{a}_k)\cup(a_k,0) \\ <0 & \text{for all } t\in(0,b_k)\cup(\bar{b}_k,+\infty) \end{cases};$$

$(ii)$ there exists $k_1 \in \mathbb{Z}$ s.t.

$$\max\left\{F(k_1, \bar{a}_{k_1}), F(k_1, \bar{b}_{k_1})\right\} > 0;$$

$(iii)$ there exists $k_2 \in \mathbb{N}$ s.t. $\bar{a}_k = a_k$ and $b_k = \bar{b}_k$ for all $k \in \mathbb{Z}$, $|k| \geq k_2$.

**Remark 1** From $\mathbf{H}(i)$ we have $f(k, 0) = 0$ for all $k \in \mathbb{Z}$, so problem $(P_n^\lambda)$ admits the trivial solution for all $\lambda > 0$, $n \in \mathbb{N}$. From $\mathbf{H}(ii), (iii)$ we have $k_2 > |k_1|$.

Throughout this section, we will always assume $n \geq k_2$.
We endow problem $(P_n^\lambda)$ with a convenient variational setting. Define

$$X_n = \{x : [-n-1, n+1] \to \mathbb{R} \ : \ x(-n-1) = x(n+1) = 0\},$$

$$\|x\|_n = \left[\sum_{k=-n}^{n+1} |\Delta x(k-1)|^p\right]^{\frac{1}{p}} \text{ for all } x \in X_n.$$

So, $(X_n, \|\cdot\|_n)$ is a $(2n+1)$-dimensional real Banach space and the norm $\|\cdot\|_n$ is equivalent to the maximum norm $\|\cdot\|_\infty$, according to the following inequalities:

(1) $\quad c_n'\|x\|_\infty \leq \|x\|_n \leq c_n''\|x\|_\infty$ for all $x \in X_n$

(for a precise computation of the constants $c_n', c_n'' > 0$, see Cabada, Iannizzotto & Tersian [4, Lemma 4]). Moreover, for all $\lambda > 0$ set

$$E_n^\lambda(x) = \frac{\|x\|_n^p}{p} - \lambda \sum_{k=-n}^{n} F(k, x(k)) \text{ for all } x \in X_n.$$

Then, $E_n^\lambda \in C^1(X_n)$ and $x \in X_n$ is a solution of problem $(P_n^\lambda)$ if and only if $x$ is a critical point of the functional $E_n^\lambda$ (see [4, Lemma 5]). Set

$$\gamma = \max\left\{\frac{2(-\bar{a}_{k_1})^p}{pF(k_1, \bar{a}_{k_1})}, \frac{2(\bar{b}_{k_1})^p}{pF(k_1, \bar{b}_{k_1})}\right\} > 0.$$

The following lemma gives some information about the critical points of $E_n^\lambda$:

**Lemma 2** *If $\mathbf{H}$ holds, then:*

$(i)$ *for all $\lambda > 0$, $0$ is a strict local minimizer of $E_n^\lambda$;*

$(ii)$ *for all $\lambda > \gamma$, there exists a global minimizer $\check{x}_n$ of $E_n^\lambda$ s.t. $E_n^\lambda(\check{x}_n) < 0$;*

$(iii)$ *for all $\lambda > \gamma$, there exists a critical point $\hat{x}_n$ of $E_n^\lambda$ s.t. $E_n^\lambda(\hat{x}_n) > 0$.*

**Proof.** First we prove $(i)$. For all $\lambda > 0$ we have $E_n^\lambda(0) = 0$. For all $k \in [-n, n]$, define $d_k > 0$ by

$$d_k = \begin{cases} \min\{-a_k, b_k\} & \text{if } a_k < 0 < b_k \\ b_k & \text{if } a_k = 0 < b_k \\ -a_k & \text{if } a_k < 0 = b_k \\ 1 & \text{if } a_k = 0 = b_k \end{cases}.$$

For all $x \in X_n$ satisfying

(2) $\quad 0 < \|x\|_n < c_n' \min_{k \in [-k_2, k_2]} \{d_k\},$

by (1) we have that

$$a_k < x(k) < b_k \text{ for all } k \in [-k_2, k_2], \text{ such that } a_k < 0 \text{ or } b_k > 0.$$

Now, $\mathbf{H}(i)$ implies that $F(k, x(k)) < 0$ for all $x \in X_n$ that fulfills (2). As a consequence $E_n^\lambda(x) > 0$ for all $\lambda > 0$ and $0$ is a strict local minimizer of $E_n^\lambda$.

Now we prove $(ii)$. Fix $\lambda > \gamma$ and assume $F(k_1, \bar{a}_{k_1}) > 0$. By $\mathbf{H}(i)$, we can easily see that $t \mapsto F(k,t)$ is bounded from above for all $k \in [-n, n]$. So, $E_n^\lambda$ is coercive. The functional $E_n^\lambda$, being continuous, has a global minimizer $\check{x}_n \in X_n$. Besides, if we put

$$\tilde{x}(k) = \begin{cases} \bar{a}_{k_1} & \text{if } k = k_1 \\ 0 & \text{if } k \neq k_1 \end{cases},$$

then we get

$$E_n^\lambda(\tilde{x}) = \frac{2(-\bar{a}_{k_1})^p}{p} - \lambda F(k_1, \bar{a}_{k_1}) < 0.$$

In particular, $E_n^\lambda(\check{x}_n) \leq E_n^\lambda(\tilde{x}) < 0$. (The case $F(k_1, \bar{b}_{k_1}) > 0$ is analogous.)

Finally, we prove $(iii)$. Again, we choose $\lambda > \gamma$. We already know that $E_n^\lambda \in C^1(X_n)$ is coercive, and that it has a strict local minimizer $0$ and a global minimizer $\check{x}_n$. Set

$$\Gamma = \{u \in C([0,1], X_n) \ : \ u(0) = 0, \ u(1) = \check{x}_n\},$$

and

$$\alpha = \inf_{u \in \Gamma} \max_{\tau \in [0,1]} E_n^\lambda(u(\tau)).$$

By the finite-dimensional version of the mountain pass theorem (see, for instance, Struwe [15], p. 74), there exists a critical point $\hat{x}_n \in X_n$ of $E_n^\lambda$ s.t. $E_n^\lambda(\hat{x}_n) = \alpha$. Since $0$ is a strict local minimizer, we have that for all $u \in \Gamma$ there is $\tau \in ]0,1[$ s.t. $E_n^\lambda(u(\tau)) > 0$, so $\alpha > 0$ and we are done. □

Lemma 2 above assures the existence of at least two non-zero solutions of $(P_n^\lambda)$, for all $\lambda > \gamma$. In the remaining part of the section, we will discuss some properties of non-zero solutions of $(P_n^\lambda)$. With this aim in mind, we will distinguish the case when $F(k_1, \bar{a}_{k_1}) > 0$ and the case when $F(k_1, \bar{b}_{k_1}) > 0$. In both cases, we will prove the existence, for $\lambda > 0$ big enough, of two constant sign solutions of $(P_n^\lambda)$ satisfying estimates independent of $n$.

First, if $F(k_1, \bar{a}_{k_1}) > 0$ we denote

$$\bar{a} = \min_{k \in [-k_2, k_2]} \bar{a}_k \text{ and } a = \max\{a_k \ : \ k \in [-k_2, k_2] \text{ s.t. } a_k < 0\},$$

while if $F(k_1, \bar{b}_{k_1}) > 0$ we denote

$$b = \min\{b_k \ : \ k \in [-k_2, k_2] \text{ s.t. } b_k > 0\} \text{ and } \bar{b} = \max_{k \in [-k_2, k_2]} \bar{b}_k.$$

Clearly $\bar{a} < a < 0$ and $0 < b < \bar{b}$. Moreover, we say that a mapping $x : [h,k] \to \mathbb{R}$ is *convex* if $\Delta x$ is increasing in $[h,k]$ and that $x$ is *concave* if $\Delta x$ is decreasing.

**Theorem 3** *If $F(k_1, \bar{a}_{k_1}) > 0$, then $(P_n^\lambda)$ admits at least two negative solutions for all $\lambda > \gamma$. Moreover, for all negative solution $x$ of $(P_n^\lambda)$, the following holds:*

$(i)$ $\bar{a} < \min_{k \in [-n,n]} x(k) < a$;

$(ii)$ *$x$ is decreasing and concave in $[-n-1,-k_2]$ and increasing and concave in $[k_2, n+1]$.*

**Proof.** We will prove the existence of two negative solutions of $(P_n^\lambda)$ via a truncation argument. Set

$$\tilde{f}(k,t) = \begin{cases} f(k,t) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases} \quad \text{for all } (k,t) \in [-n,n] \times \mathbb{R}.$$

Clearly, $\tilde{f}$ satisfies $\mathbf{H}$. Since

$$\gamma \geq \frac{2(-\bar{a}_{k_1})^p}{pF(k_1, \bar{a}_{k_1})},$$

by Lemma 2 we know that the truncated problem

$$(\tilde{P}_n^\lambda) \qquad \begin{cases} -\Delta_p x(k-1) = \lambda \tilde{f}(k, x(k)) & \text{for all } k \in [-n,n] \\ x(-n-1) = x(n+1) = 0 \end{cases}$$

admits at least two nonzero solutions for $\lambda > \gamma$.

**Claim.** If $x \in X_n$ is a nonzero solution of $(\tilde{P}_n^\lambda)$, then $x(k) < 0$ for all $k \in [-n,n]$.

We argue by contradiction: let $k \in [-n,n]$ be s.t. $x(k) \geq 0$, and we assume that $k$ is the smallest index with such property (so $\Delta x(k-1) \geq 0$). Then,

$$-\Delta_p x(k-1) = \lambda \tilde{f}(k, x(k)) = 0,$$

which implies

$$\Delta x(k) = \Delta x(k-1) \text{ (recall that } \phi_p \text{ is injective).}$$

Hence, if $\Delta x(k) > 0$, then $x(k+1) > 0$ and so again

$$-\Delta_p x(k) = \lambda \tilde{f}(k+1, x(k+1)) = 0.$$

Hence, we have that $x : [k, n+1] \to \mathbb{R}^+$ is increasing and $x(n+1) > 0$, a contradiction. If $\Delta x(k) = 0$, then $k = -n$ and

$$x(-n-1) = x(-n) = x(-n+1) = 0 \text{ (recall that } k \text{ is the smallest s.t. } x(k) \geq 0).$$

It is easily seen, then, that $x(k) = 0$ for all $k \in [-n,n]$, a contradiction. Thus, our Claim is proved.

Now, let $x$ be a nonzero (hence, negative) solution of $(\tilde{P}_n^\lambda)$. By definition of $\tilde{f}$, it is clear that $x$ also solves $(P_n^\lambda)$. Thus, $(P_n^\lambda)$ has at least two negative solutions for $\lambda > \gamma$.

Now we prove $(i)$. Let $x$ be a negative solution of $(P_n^\lambda)$. We argue by contradiction, assuming first that there exists $k \in [-n,n]$ s.t. $x(k) \leq \bar{a}$ (let $k$ be the smallest index with such property). Hence

$$-\Delta_p x(k-1) = \lambda f(k, x(k)) \geq 0 \text{ (because } x(k) \leq \bar{a}_k \text{ if } |k| < k_2, \text{ and } f(k,t) \geq 0 \text{ in } \mathbb{R}^- \text{ if } |k| \geq k_2).$$

So, we get

$$\Delta x(k) \leq \Delta x(k-1) < 0 \text{ (recall that } k \text{ is the smallest index s.t. } x(k) \leq \bar{a}),$$

which implies that

$$x(k+1) < x(k) \leq \bar{a}.$$

So, we again have

$$-\Delta_p x(k) = \lambda f(k+1, x(k+1)) \geq 0,$$

hence $x(k+2) < x(k+1)$ and so on. The mapping $x : [k, n+1] \to \mathbb{R}^-$ is decreasing, so $x(n+1) < 0$, a contradiction.

Now, arguing again by contradiction, assume that $a \leq x(k) < 0$ for all $k \in [-n, n]$. As above, we have

$$-\Delta_p x(-n-1) = \lambda f(-n, x(-n)) \geq 0,$$

hence

$$\Delta x(-n) \leq \Delta x(-n-1) < 0 \text{ (recall that } x \text{ is negative),}$$

which implies $x(-n+1) < x(-n)$. As above, we obtain that $x : [-n-1, n+1] \to \mathbb{R}^-$ is decreasing, so $x(n+1) < 0$, a contradiction.

Finally, we prove $(ii)$. Let $x$ be a negative solution of $(P_n^\lambda)$. From $\mathbf{H}(iii)$, for all $k \in [-n, -k_2]$ we have

$$-\Delta_p x(k-1) = \lambda f(k, x(k)) \geq 0,$$

hence

$$\Delta x(k) \leq \Delta x(k-1) < 0.$$

Thus, $x : [-n-1, -k_2] \to \mathbb{R}^-$ is decreasing and concave. In a similar way, we prove that $x : [k_2, n+1] \to \mathbb{R}^-$ is increasing and concave. □

In a similar way, we study the positive solutions of $(P_n^\lambda)$ when $F(k_1, \bar{b}_{k_1}) > 0$:

**Theorem 4** *If $F(k_1, \bar{b}_{k_1}) > 0$, then $(P_n^\lambda)$ admits at least two positive solutions for all $\lambda > \gamma$. Moreover, for all positive solution $x$ of $(P_n^\lambda)$, the following holds:*

*(i) $b < \max_{k \in [-n,n]} x(k) < \bar{b}$;*

*(ii) $x$ is increasing and convex in $[-n-1, -k_2]$ and decreasing and convex in $[k_2, n+1]$.*

We conclude this section by presenting an example:

**Example 5** For all $t \in \mathbb{R}$, denote $[t]$ the biggest integer less than or equal to $t$. Set

$$f(k,t) = \begin{cases} -t(t+1)^2 & \text{if } k = 0 \\ -t\left(t - [1/k]\right)\left(t - [1/k + 2/k^3]\right) & \text{if } k \neq 0 \end{cases} \quad \text{for all } (k,t) \in \mathbb{Z} \times \mathbb{R}.$$

This function satisfies $\mathbf{H}$ with $k_1 = \pm 1$ and $k_2 = 2$. Indeed, we have $\bar{a}_0 = a_0 = -1$, $b_0 = \bar{b}_0 = 0$, $\bar{a}_1 = a_1 = 0$, $b_1 = 1$, $\bar{b}_1 = 3$, $\bar{a}_k = a_k = 0 = b_k = \bar{b}_k$ for all $k \geq 2$, $\bar{a}_{-1} = -3$, $a_{-1} = -1$ and $b_{-1} = \bar{b}_{-1} = 0$, $\bar{a}_k = a_k = -1$ and $b_k = \bar{b}_k = 0$ for all $k \leq -2$. We set

$$\gamma = \frac{3^{p-2} 8}{p},$$

and by the results above we know that, for all $n \geq 2$ and $\lambda > \gamma$, problem $(P_n^\lambda)$ admits at least *four* non-zero constant sign solutions: a pair of negative solutions (with all the properties of Theorem 3 with $\bar{a} = -3$ and $a = -1$) and a pair of positive solutions (with all the properties of Theorem 4 with $b = 1$ and $\bar{b} = 3$).

## 3 Constructing the homoclinic constant sign solution

Now we will deduce from the results of the previous section the existence of a solution of problem $(P^\lambda)$, for $\lambda > 0$ big enough. We will closely follow the ideas of Cabada & Tersian [5]. To this end, we introduce the following hypotheses:

$\mathbf{H}'$ $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $\mathbf{H}(i) - (iii)$ plus the following conditions:

$(iv)$ if $F(k_1, \bar{a}_{k_1}) > 0$, then for all compact $A \subset [\bar{a}, 0[$ there exist $k_3 \in \mathbb{N}$ and a function $\psi_A : \mathbb{Z} \setminus [-k_3 + 1, k_3 - 1] \to \mathbb{R}$ s.t. $f(k, t) \geq \psi_A(k)$ for all $|k| \geq k_3$, $t \in A$ and

$$\sum_{k=k_3}^{+\infty} \psi_A(k) = \sum_{k=-\infty}^{-k_3} \psi_A(k) = +\infty;$$

$(v)$ if $F(k_1, \bar{b}_{k_1}) > 0$, then for all compact $B \subset ]0, \bar{b}]$ there exist $k_4 \in \mathbb{N}$ and a function $\psi_B : \mathbb{Z} \setminus [-k_4 + 1, k_4 - 1] \to \mathbb{R}$ s.t. $f(k, t) \leq \psi_B(k)$ for all $|k| \geq k_4$, $t \in B$ and

$$\sum_{k=k_4}^{+\infty} \psi_B(k) = \sum_{k=-\infty}^{-k_4} \psi_B(k) = -\infty.$$

**Remark 6** Note that constants $\bar{a} \leq a < 0 < b \leq \bar{b}$ and $\gamma$ do not depend on $n$.

Now we can state and prove our existence result for problem $(P^\lambda)$:

**Theorem 7** *If $\mathbf{H}'$ holds, then problem $(P^\lambda)$ admits at least one non-zero, constant sign solution for all $\lambda > \gamma$. In particular:*

$(i)$ *if $F(k_1, \bar{a}_{k_1}) > 0$, then there exists a solution $x : \mathbb{Z} \to \mathbb{R}$ of $(P^\lambda)$ s.t. $\bar{a} \leq x(k) < 0$ for all $k \in \mathbb{Z}$ and $\min_{k \in \mathbb{Z}} x(k) < a$;*

$(ii)$ *if $F(k_1, \bar{b}_{k_1}) > 0$, then there exists a solution $x : \mathbb{Z} \to \mathbb{R}$ of $(P^\lambda)$ s.t. $0 < x(k) \leq \bar{b}$ for all $k \in \mathbb{Z}$ and $\max_{k \in \mathbb{Z}} x(k) > b$.*

**Proof.** We prove $(i)$, arguing as in [3, 5]. Fix $\lambda > \gamma$. From Theorem 3 we know that, for all $n \in \mathbb{N}$, $n \geq k_2$ problem $(P_n^\lambda)$ admits at least two negative solutions, satisfying uniform estimates with respect to $n$. Set

$$r_n = \min \left\{ x(-n) \ : \ x \in X_n \text{ solution of } (P_n^\lambda),\ \bar{a} < x(k) < 0 \text{ for all } k \in [-n, n] \right\}.$$

We point out that the existence of $r_n$ follows from the fact that it is the minimum of a non empty, bounded from below and closed set of real numbers.

We denote $x_n$ the solution of $(P_n^\lambda)$ s.t. $x(-n) = r_n$. From the uniqueness of the solution of the initial value problem (recall that all solutions of $(P_n^\lambda)$ satisfy $x(-n-1) = 0$) we have that $x_n$ is unique. Moreover, from Theorem 3 we know that

$$\min_{k \in [-n, n]} x(k) < a$$

and $x$ is concave and decreasing in $[-n-1, -k_2]$ and concave and increasing in $[k_2, n+1]$.

We consider the bounded sequence $(x_n(0))_n$: passing if necessary to a subsequence, we have $x_n(0) \to x(0) \in [\bar{a}, 0]$ as $n \to \infty$. Then, we consider $(x_n(1))_n$, bounded as well: passing if necessary to a subsequence, we have $x_n(1) \to x(1) \in [\bar{a}, 0]$ as $n \to \infty$. In a similar way, we have $x_n(-1) \to x(-1) \in [\bar{a}, 0]$ as $n \to \infty$. By the continuity of functions $\phi_p$ and $f$, we pass to the limit in $(P_n^\lambda)$ as $n \to \infty$, and get

$$-\Delta_p x(-1) = \lambda f(0, x(0)).$$

Reasoning in the same way, we construct a mapping $x : \mathbb{Z} \to [\bar{a}, 0]$ which satisfies

(3) $\quad -\Delta_p x(k-1) = \lambda f(k, x(k))$ for all $k \in \mathbb{Z}$

and $x_n \to x$ uniformly on finite sets as $n \to \infty$.

Actually, we have $x(k) < 0$ for all $k \in \mathbb{Z}$. This is proved by an argument similar to the one used in Theorem 3: assume that there exists $k \in \mathbb{Z}$ s.t. $x(k) = 0$. So,

$$-\Delta_p x(k-1) = \lambda f(k, 0) = 0.$$

This implies

$$\Delta x(k-1) = \Delta x(k),$$

that is,

$$x(k+1) = -x(k-1).$$

Since $x$ has non-nregative values, we get

$$x(k-1) = x(k) = x(k+1) = 0,$$

hence $x = 0$, a contradiction.

We prove that $x : [k_2, +\infty[ \to \mathbb{R}^-$ is nondecreasing and concave. Indeed, for all $k \geq k_2$, we choose $n > k$ and have

$$x_n(k-1) < x_n(k) \text{ and } \Delta x_n(k-1) \geq \Delta x_n(k),$$

hence, passing to the limit,

$$x(k-1) \leq x(k) \text{ and } \Delta x(k-1) \geq \Delta x(k).$$

In a similar way, we have that $x :] - \infty, -k_2] \to \mathbb{R}^-$ is nonincreasing and concave.

From these two properties, we deduce that for all $n \geq k_2$ the minimum of $x_n$ is attained at some $k \in [-k_2, k_2]$. As consequence, we deduce that

$$\min_{k \in \mathbb{Z}} x(k) = \min_{k \in [-k_2, k_2]} x(k) \leq a < 0,$$

and, in particular, the limit function $x$ cannot be identically zero.

Moreover, we have

(4) $\quad \lim_{|k| \to +\infty} \Delta x(k) = 0.$

Indeed, the mapping $\Delta x : [k_2, +\infty[ \to \mathbb{R}^+$ is nonincreasing, so there exists

$$\lim_{k \to +\infty} \Delta x(k) = l \in \mathbb{R}^+ \cup \{+\infty\}.$$

Arguing by contradiction, assume $l > 0$. Then, we would have

$$\lim_{k\to+\infty} x(k) = +\infty,$$

against the fact that $x(k) \leq 0$ for all $k \in \mathbb{Z}$. So, $l = 0$. Similarly, we have that $\Delta x \to 0$ as $k \to -\infty$. Finally,

(5) $$\lim_{|k|\to+\infty} x(k) = 0.$$

Indeed, since $x : [k_2, +\infty[\to \mathbb{R}^-$ is nondecreasing and bounded, there exists

$$\lim_{k\to+\infty} x(k) = m \in [\bar{a}, 0].$$

Arguing by contradiction, we assume $m < 0$. Then, $A = [\bar{a}, m]$ is a compact subset of $[\bar{a}, 0[$ and there exist $k_3$ and a mapping $\psi : \mathbb{Z}\setminus[-k_3+1, k_3-1] \to \mathbb{R}$ as in hypothesis $\mathbf{H}'(iv)$ (we may choose $k_3 \geq k_2$). For all $k > k_3$, by (3) we have

$$-\Delta_p x(k-1) = \lambda f(k, x(k)) \geq \lambda\psi_A(k) \text{ (recall that } \lambda > 0\text{)},$$

which implies

$$\begin{aligned}
\phi_p(\Delta x(k)) &\leq \phi_p(\Delta x(k-1)) - \lambda\psi_A(k) \\
&\leq \phi_p(\Delta x(k-2)) - \lambda\left[\psi_A(k-1) + \psi_A(k)\right] \\
&\leq (\ldots) \\
&\leq \phi_p(\Delta x(k_3-1)) - \lambda\sum_{h=k_3}^{k}\psi_a(h).
\end{aligned}$$

By the properties of $\psi_A$, this implies

$$\lim_{k\to+\infty} \Delta x(k) = -\infty,$$

which contradicts (4).

In a similar way we study the behavior of $x$ at $-\infty$ and get (5).

By (3) and (5), we have that $x : \mathbb{Z} \to \mathbb{R}_0^-$ is a solution of $(P^\lambda)$.

The proof of $(ii)$ is analogous. □

**Remark 8** If only $\mathbf{H}$ holds, we still have solutions for the difference equation

$$-\Delta x(k-1) = \lambda f(k, x(k)) \text{ for all } k \in \mathbb{Z}.$$

Precisely, if $F(k_1, \bar{a}_{k_1}) > 0$, then there is a bounded negative solution and if $F(k_1, \bar{b}_{k_1}) > 0$, then there is a bounded positive solution (with undetermined asymptotic behaviour).

Finally, we discuss further Example 5:

**Example 9** Let $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be defined as in Example 5. We already know that $f$ satisfies $\mathbf{H}(i) - (iii)$, in particular $\bar{a} = -3$, $a = -1$, $b = 1$ and $\bar{b} = 3$. Moreover, it satisfies condition $\mathbf{H}'(v)$. Indeed, for all compact interval $B = [l_1, l_2] \subset ]0, 3]$, we can set $k_4 = 2$ and $\psi_B(k) = -l_1^3$ for all $k \in \mathbb{Z}$, $|k| \geq 2$ and get

$$f(k, t) \leq \psi_B(k) \text{ for all } k \in \mathbb{Z},\ |k| \geq 2 \text{ and all } t \in B.$$

So, by Theorem 7, for all $\lambda > 3^{p-2}8/p$ there exists a positive homoclinic solution $x : \mathbb{Z} \to \mathbb{R}^+$ of problem $(P^\lambda)$ s.t. $x(k) \leq 3$ for all $k \in \mathbb{Z}$ and $\max_{k\in\mathbb{Z}} x(k) \geq 1$.
However, $f$ does not satisfy condition $\mathbf{H}'(iv)$. Indeed, for any compact set $A \subset [-3, 0[$ s.t. $1 \in B$, we have

$$\min_{t\in B} f(k,t) = 0 \text{ for all } k \in \mathbb{Z},\ |k| \geq 2.$$

So, the best we can say about negative solutions is that for all $\lambda > 3^{p-2}8/p$ there is a mapping $x : \mathbb{Z} \to \mathbb{R}_0^-$ s.t.

$$-\Delta_p x(k-1) = \lambda f(k, x(k)) \text{ for all } k \in \mathbb{Z},$$

$x(k) \geq -3$ for all $k \in \mathbb{Z}$ and $\min_{k\in\mathbb{Z}} x(k) \leq -1$.

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