

# Multiplicity of solutions of a two point boundary value problem for a fourth-order equation

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## Abstract

We study the existence of multiple solutions for semi linear fourth-order differential equation describing elastic deflections. The proof of the main result is based on a three critical point theorem.

**Key words:** Three critical point theorem, two-points BVP, fourth-order equation

## 1 Introduction

A multiplicity result for critical points of  $C^1$  functionals on finite-dimensional spaces was proved by Ricceri [13] as follows:

**Theorem 1** *Let  $X$  be a finite-dimensional real Hilbert space, and let  $J : X \rightarrow \mathbb{R}$  be a  $C^1$ -function such that*

$$\liminf_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2} \geq 0.$$

*Moreover, let  $x_0 \in X$  and  $r, s \in \mathbb{R}$  with  $0 < r < s$ , be such that*

$$\inf_{x \in X} J(x) \leq J(x_0) \leq \inf\{J(x) : r \leq \|x - x_0\| \leq s\}.$$

*Then, there exists  $\lambda^* > 0$  such that the equation*

$$x + \lambda^* J'(x) = x_0$$

*has at least three solutions.*

The last theorem was extended to a wider class of functionals in finite-dimensional spaces in [4, Theorem 3]. The proof of Theorem 1 is based on earlier results of Ricceri on the well-posedness of some minimization problems. Cabada and Iannizzotto in [3] extended Theorem 1 removing the assumption on the finite dimension of space  $X$  and condition on annulus around point  $x_0$ . Their given result is the following

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**Theorem 2** Let  $(X, \|\cdot\|)$  be a uniformly convex Banach space with strictly convex dual space,  $J \in C^1(X)$  be a functional with compact derivative,  $x_0, x_1 \in X, p, r \in \mathbb{R}$  be such that  $p > 1, r > 0$  and  $\|x_1 - x_0\| < r$ . Let the following conditions be satisfied:

- (i)  $\liminf_{\|x\| \rightarrow \infty} \frac{J(x)}{\|x\|^p} \geq 0$ ;
- (ii)  $\inf_{x \in X} J(x) < \inf\{J(x) : \|x - x_0\| \leq r\}$ ;
- (iii)  $J(x_1) \leq \inf\{J(x) : \|x - x_0\| = r\}$ .

Then, there exists a nonempty open set  $A \subseteq ]0, +\infty[$  such that for all  $\lambda \in A$  the functional  $x \mapsto \frac{\|x - x_0\|^p}{p} + \lambda J(x)$  has at least three critical points in  $X$ .

As a consequence of this result we can deduce the following one.

**Theorem 3** Suppose that the conditions (ii) and (iii) of Theorem 2 are trivially satisfied and the functional  $J$  is bounded from below on  $X$ , i.e.

- (i<sub>1</sub>)  $\exists C \in \mathbb{R}$ , such that  $J(x) \geq C, \forall x \in X$ .

Then, the assertion of Theorem 2 is still valid.

**Proof** Let us consider the functional

$$J_C(x) = \frac{\|x - x_0\|^p}{p} + \lambda(J(x) - C).$$

Since

$$\inf_{x \in X} (J(x) - C) = \inf_{x \in X} J(x) - C$$

and

$$\inf\{J(x) - C : \|x - x_0\| \leq r\} = \inf\{J(x) : \|x - x_0\| \leq r\} - C,$$

the conditions (ii) and (iii) of Theorem 2 are satisfied for the functional  $J_C$ .

Clearly  $J_C(x) \geq 0$  and therefore (i) is also satisfied for  $J_C$ . Then, by Theorem 2 there exists a set  $A_C \subseteq ]0, +\infty[$  such that for all  $\lambda \in A_C$  the functional  $J_C$  has at least three critical points, which are also critical points of the functional  $J$ .  $\square$

Theorem 2 improves Theorem 1 since the space  $X$  is supposed to be infinite dimensional and the geometric conditions are more general and easy to verify in the different applications. Another three critical points theorems were proved and presented in P. Pucci & J.Serrin [5], D. G. Figueredo [7], Moroz, A. Vignoli & P. Zabreiko [12], J. Mawhin & M. Willem [11], M. R. Grossinho and S. Tersian [8], G. Bonanno [1], F. Faraci, A. Iannizzotto [6].

In this paper, we apply Theorem 2 to a two-point boundary value problem for fourth-order differential equation describing elastic deflections of a rod with clamped ends. As a model problem we consider the following one for the nonlinear fourth-order equation

$$(P_\lambda) : \begin{cases} u^{(4)}(x) + \lambda f(x, u(x)) = 0, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = \lambda g(u(1)). \end{cases}$$

where the functions  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and  $\lambda \geq 0$  is a real parameter.

Assume that the following conditions on the functions  $f$  and  $g$  are fulfilled:

- (f<sub>1</sub>)  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function;  
(f<sub>2</sub>) There exist  $k_2 > k_1 > 0$ , such that  $t f(x, t) \geq 0$  for every  $x \in [0, 1]$  and for all  $t$ , such that  $|t| \leq k_1$  or  $|t| \geq k_2$ ;  
(g<sub>1</sub>)  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function;  
(g<sub>2</sub>)  $t g(t) \geq 0$  for all  $t$ , such that  $|t| \leq k_1$  or  $|t| \geq k_2$ ;  
(h) There exist  $c_1 < -k_1$  or  $c_1 > k_1$  and  $\alpha > \beta > 0$  such that:  

$$\int_0^{c_1} f(x, t) dt \leq -\alpha, \quad \forall x \in [0, 1] \quad \text{and} \quad \int_0^{c_1} g(t) dt \leq \beta.$$

Our main result is the following

**Theorem 4** *Assume that the conditions (f<sub>1</sub>), (f<sub>2</sub>), (g<sub>1</sub>), (g<sub>2</sub>) and (h) on the functions  $f$  and  $g$  are satisfied. Then, there exists a set  $A \subseteq ]0, +\infty[$  such that for each  $\lambda \in A$ , the problem  $(P_\lambda)$  has at least two nontrivial solutions.*

Theorem 4 is proved in Section 3. Further comments and examples are also given. The proof is based on the variational approach presented in Section 2. We refer to the reader for similar problems, obtained by topological and variational methods, given in G. Bonanno, B. Di Bella [2], T.F. Ma [10], F. Li, Q. Zhang, Z. Liang [9], M. R. Grossinho and S. Tersian [8] and references therein.

## 2 Preliminaries

In this section we give some definitions and auxiliary results for the variational formulation and treatment of the problem  $(P_\lambda)$ .

Let  $X_p, p \geq 2$  be the space

$$X_p := \{u \in W^{2,p}(0, 1) : u(0) = u'(0) = 0\}$$

with the norm

$$\|u\|_p^p := \int_0^1 |u''(t)|^p dt.$$

Here  $W^{2,p}(0, 1)$  is the Sobolev space of functions  $u : [0, 1] \rightarrow \mathbb{R}$  such that  $u$  and its generalized derivative  $u'$  are absolutely continuous functions and  $u'' \in L^p(0, 1)$ . If  $p = 2$  we denote

$$X := X_2 = \{u \in W^{2,2}(0, 1) = H^2(0, 1) : u(0) = u'(0) = 0\}$$

which is a Hilbert space with inner product

$$\langle u, v \rangle := \int_0^1 u''(t)v''(t) dt$$

and norm

$$\|u\|^2 := \int_0^1 (u''(t))^2 dt.$$

By a standard way one can prove the following

**Lemma 5** *The following assertions are fulfilled:*

(a) *The norm*

$$\|u\|_p^p = \int_0^1 |u''(t)|^p dt$$

is equivalent to the usual norm in  $X_p$

$$\|u\|_{X_p}^p = \int_0^1 (|u''(t)|^p + |u'(t)|^p + |u(t)|^p) dt.$$

(b) The inclusion  $X_p \subset C^1([0, 1])$  is compact.

To study the problem  $(P_\lambda)$ , we consider the functional  $I : X \rightarrow \mathbb{R}$  defined by

$$I(u) := \frac{\|u\|^2}{2} + \lambda J(u),$$

where

$$J(u) := \int_0^1 F(x, u(x)) dx + G(u(1))$$

and

$$F(x, t) = \int_0^t f(x, s) ds, \quad G(t) = \int_0^t g(s) ds.$$

The functional  $I : X \rightarrow \mathbb{R}$  is Fréchet differentiable and, for any  $u, v \in X$ , the following identity holds

$$\langle I'(u), v \rangle = \int_0^1 u''(x) v''(x) dx + \lambda \left( \int_0^1 f(x, u(x)) v(x) dx + g(u(1)) v(1) \right).$$

Following the framework of Tersian and Chaparova [15] and L. Yang, H. Chen and H. Yang [16], we arrive at the standard regularity property

**Lemma 6** *If  $u$  is a critical point of  $I$  in  $X$ , then  $u$  is a classical solution of the problem  $(P_\lambda)$ .*

### 3 Proof of the main result and comments

This section is devoted to proof the main result of this paper. Moreover, we present an example in which we apply the general result to a particular case of problem  $(P_\lambda)$ .

**Proof of Theorem 4.** By Lemma 5 for  $p = 2$  and the continuity of functions  $f$  and  $g$  we deduce that the derivative of the functional  $J$  is compact.

Let's see that  $J$  is bounded from below:

From  $(f_1)$  and  $(f_2)$  we have that the function  $F$  attains its finite minimum at a point of  $[0, 1] \times [-k_2, -k_1]$ , if  $c_1 < 0$ , or in  $[0, 1] \times [k_1, k_2]$ , if  $c_1 > 0$ . By  $(g_1)$  and  $(g_2)$  it follows that  $G$  is bounded from below and then,  $J$  is bounded from below too.

To verify assumption (ii) of Theorem 2, let us define the function

$$u_n(x) = \begin{cases} 0, & 0 \leq x \leq 1/n, \\ c_1 \cos^2(\frac{n\pi x}{2}), & 1/n \leq x \leq 2/n, \\ c_1, & 2/n \leq x \leq 1, \end{cases}$$

where  $c_1$  satisfies assumption (h).

It is clear that  $u_n \in X \subset C^1([0, 1])$ ,  $\forall n \in \mathbb{N}$ .

We have

$$J(u_n) = \int_0^{1/n} F(x, 0)dx + \int_{1/n}^{2/n} F\left(x, c_1 \cos^2\left(\frac{n\pi x}{2}\right)\right) dx + \int_{2/n}^1 F(x, c_1)dx + G(c_1). \quad (1)$$

Since  $|c_1 \cos^2(\frac{n\pi x}{2})| \leq |c_1|$ , by  $(f_1)$  there exists  $A > 0$  such that

$$\left|F\left(x, c_1 \cos^2\left(\frac{n\pi x}{2}\right)\right)\right| \leq A.$$

Then, by (1) and  $(h)$ , we have that

$$J(u_n) \leq \frac{A}{n} - \alpha \left(1 - \frac{2}{n}\right) + \beta = \frac{A + 2\alpha}{n} + \beta - \alpha.$$

Moreover, for sufficiently large  $n$ , it is clear that  $(A + 2\alpha)/n < \alpha - \beta$  and therefore  $J(u_n) < 0$  for such  $n$ . As consequence,  $\inf_{u \in X} J(u) < 0$ .

On the other hand, since  $F(x, t) \geq 0$  for all  $(x, t) \in [0, 1] \times [-k_1, k_1]$  and  $G(t) \geq 0$  for  $|t| \leq k_1$ , we have that

$$\inf\{J(u) : u \in X, \|u\|_\infty \leq k_1\} \geq 0.$$

From the fact that  $\|u\|_\infty \leq \|u\|$ , we deduce that

$$\{u \in X, \|u\| \leq k_1\} \subseteq \{u \in X, \|u\|_\infty \leq k_1\}$$

and, as a consequence

$$\inf\{J(u) : u \in X, \|u\| \leq k_1\} \geq \inf\{J(u) : u \in X, \|u\|_\infty \leq k_1\} \geq 0.$$

So, we deduce condition  $(ii)$  with  $r = k_1$  and  $x_0 = 0$ .

In order to verify  $(iii)$  we take  $r = k_1$  and  $x_0 = x_1 = 0$  and deduce that

$$0 = J(0) \leq \inf\{J(u) : u \in X, \|u\| \leq k_1\} \leq \inf\{J(u) : u \in X, \|u\| = k_1\}.$$

The result is proved.  $\square$

We finish this section by the following example for a particular case of functions  $f$  and  $g$ .

**Example 7** Define the functions  $f$  and  $g$  as follows:  $f(x, t) = a(x) f_1(t)$  where  $a(x) = 2 + \sin(\pi x)$ ,

$$f_1(t) = \begin{cases} (t - 2\pi)^2 & t \geq 2\pi, \\ 3(t - \pi)(t - 2\pi), & \pi \leq t \leq 2\pi, \\ \sin t, & -\pi \leq t \leq \pi, \\ -3(t + \pi)(t + 2\pi), & -2\pi \leq t \leq -\pi, \\ -(t + 2\pi)^2 & t \leq -2\pi, \end{cases}$$

and

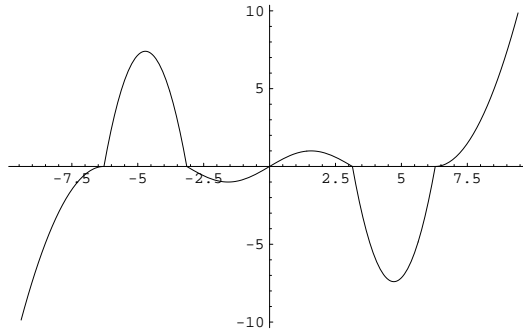


Figure 1: Graph of the function  $f_1(t)$  for  $-3\pi \leq t \leq 3\pi$ .

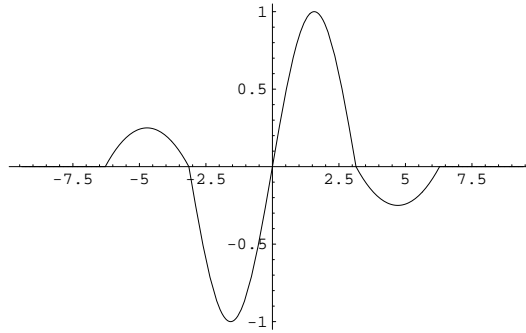


Figure 2: Graph of the function  $g(t)$  for  $-3\pi \leq t \leq 3\pi$ .

$$g(t) = \begin{cases} 0 & t \geq 2\pi, \\ \frac{1}{\pi^2}(t - \pi)(t - 2\pi), & \pi \leq t \leq 2\pi, \\ \sin t, & -\pi \leq t \leq \pi, \\ -\frac{1}{\pi^2}(t + \pi)(t + 2\pi), & -2\pi \leq t \leq -\pi, \\ 0 & t \leq -2\pi. \end{cases}$$

Taking  $c_1 = -3\pi$  we have

$$\begin{aligned} \int_0^{c_1} f(x, t) dt &= a(x) \left( \int_0^{-\pi} \sin t dt - 3 \int_{-\pi}^{-2\pi} (t + \pi)(t + 2\pi) dt - \int_{-2\pi}^{-3\pi} (t + 2\pi)^2 dt \right) \\ &= a(x) \left( 2 - \frac{\pi^3}{6} \right) \leq 2 - \frac{\pi^3}{6} \approx -3.1677, \end{aligned}$$

and

$$\int_0^{c_1} g(t) dt = \int_0^{-\pi} \sin t dt - \frac{1}{\pi^2} \int_{-\pi}^{-2\pi} (t + \pi)(t + 2\pi) dt = 2 - \frac{\pi}{6} \approx 1.4764.$$

So, assumptions  $(f_1) - (h)$  are satisfied with  $k_1 = \pi, k_2 = 2\pi, c_1 = -3\pi, \alpha = \frac{1}{6}\pi^3 - 2$  and  $\beta = 2 - \frac{1}{6}\pi$ . In consequence there exists a set  $A \subseteq ]0, +\infty[$  such that for each  $\lambda \in A$ , the problem  $(P_\lambda)$  has at least two nontrivial solutions. The graphs of functions  $f_1(t)$  and  $g(t)$  for  $-3\pi \leq t \leq 3\pi$  are presented on Figure 1 and Figure 2 respectively.

One can verify that the variational approach based on Theorem 2 can be also applied to boundary value problem for fourth-order p-Laplacian equations of the type

$$(P_{p,\lambda}) : \begin{cases} (\varphi_p(u''(x)))'' + \lambda f(x, u(x)) = 0, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = 0, \\ (\varphi_p(u''))'(1) = \lambda g(u(1)), \end{cases}$$

where  $p > 2$  and  $\varphi_p(t) = t|t|^{p-2}$ .

In this case we consider the functional  $I_p : X_p \rightarrow \mathbb{R}$

$$I_p(u) := \frac{\|u\|_p^p}{p} + \lambda J(u).$$

The functional  $I_p : X_p \rightarrow \mathbb{R}$  is Fréchet differentiable and its critical points are weak solutions of the problem  $(P_{p,\lambda})$ . Note that a function  $u \in X_p$  is a weak solution of problem  $(P_{p,\lambda})$  iff for any  $v \in X_p$ , the following identity holds

$$\int_0^1 \varphi_p(u''(x)) v''(x) dx + \lambda \left( \int_0^1 f(x, u(x)) v(x) dx + g(u(1)) v(1) \right) = 0.$$

By a solution of the problem  $(P_{p,\lambda})$  we mean a function  $u \in C^1([0, 1])$  such that  $(\varphi_p(u''(x)))'' \in C([0, 1])$  and the boundary conditions and the equation are satisfied in  $[0, 1]$ . By a standard way it follows that the weak solutions are solutions of the problem.

Using Theorem 2 and Lemma 5 we arrive at the following existence and multiplicity result

**Theorem 8** *Assume that the conditions  $(f_1), (f_2), (g_1), (g_2)$  and  $(h)$  on the functions  $f$  and  $g$  are satisfied. Then, there exists a set  $A \subseteq ]0, +\infty[$  such that for each  $\lambda \in A$ , the problem  $(P_{p,\lambda})$  has at least two nontrivial solutions.*

Finally, we point out that the conclusion of Theorem 4 is still valid if we suppose assumptions  $(f_1) - (h)$  for functions  $f$  and  $g$  considering the problem

$$(P_{\lambda,\mu}) : \begin{cases} u^{(4)}(x) + \lambda f(x, u(x)) = 0, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = \mu g(u(1)), \end{cases}$$

The corresponding functional for the problem  $(P_{\lambda,\mu})$ ,  $I_\mu : X \rightarrow \mathbb{R}$ , is defined in this case by

$$I_\mu(u) = \frac{\|u\|^2}{2} + \lambda J_\mu(u),$$

where

$$J_\mu(u) = \int_0^1 F(x, u(x)) dx + \frac{\mu}{\lambda} G(u(1))$$

and

$$F(x, t) = \int_0^t f(x, s) ds, \quad G(t) = \int_0^t g(s) ds.$$

Since condition (h) is fulfilled, it is obvious that, if  $\int_0^{c_1} g(t) dt \leq 0$  then  $\frac{\mu}{\lambda} \int_0^{c_1} g(t) dt \leq 0 \leq \beta$  for all  $\mu > 0$ . On the contrary, if  $\int_0^{c_1} g(t) dt > 0$ , we have, provided that  $0 < \mu \leq \lambda$ , that the following inequalities are fulfilled:

$$\int_0^{c_1} \frac{\mu}{\lambda} g(t) dt = \frac{\mu}{\lambda} \int_0^{c_1} g(t) dt \leq \int_0^{c_1} g(t) dt \leq \beta.$$

Thus, following the steps of the proof of Theorem 4, we verify that functional  $J_\mu$  satisfies assumptions (i) – (iii) of Theorem 2, and so we arrive at the next existence and multiplicity result for problem  $(P_{\lambda,\mu})$ :

**Theorem 9** *Assume that the conditions  $(f_1), (f_2), (g_1), (g_2)$  and (h) on the functions  $f$  and  $g$  are satisfied. Then, there exists a set  $A \subseteq ]0, +\infty[$  such that for each  $\lambda \in A$  and for every  $\mu \in (0, \lambda]$  if  $\int_0^{c_1} g(t) dt > 0$  and for all  $\mu > 0$  if  $\int_0^{c_1} g(t) dt \leq 0$ , the problem  $(P_{\lambda,\mu})$  has at least two nontrivial solutions.*

Combining this last result we can deduce the following existence and multiplicity result for the  $p$  – laplacian case.

**Theorem 10** *Assume that the conditions  $(f_1), (f_2), (g_1), (g_2)$  and (h) on the functions  $f$  and  $g$  are satisfied. Then, there exists a set  $A \subseteq ]0, +\infty[$  such that for each  $\lambda \in A$  and for every  $\mu \in (0, \lambda]$  if  $\int_0^{c_1} g(t) dt > 0$  and for all  $\mu > 0$  if  $\int_0^{c_1} g(t) dt \leq 0$ , the problem*

$$(P_{p,\lambda,\mu}) : \begin{cases} (\varphi_p(u''(x)))'' + \lambda f(x, u(x)) = 0, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = 0, \\ (\varphi_p(u''))'(1) = \mu g(u(1)), \end{cases}$$

*has at least two nontrivial solutions.*

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