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Second–order discontinuous ODEs and billiard problems

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A B S T R A C T

We present an existence principle for boundary value problems involving discontinuous ordinary differential equations of the second order using the Krasovskii regularization technique. Especially we obtain sufficient conditions of transversality type for Krasovskii solutions to be also Carathéodory solutions of the original problem. This result is applied on a certain billiard problem, which can be thought as an ordinary differential equation with state-dependent impulses that is equivalent to certain discontinuous differential equation. In particular, we obtain new existence and multiplicity results for Dirichlet problems in billiard spaces with time-varying boundaries.

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1. Introduction

In this paper, we deal with second–order ordinary differential equations with discontinuous nonlinearities, i.e., *discontinuous differential equations*. It is well-known that if the right-hand side in the differential equation is discontinuous with respect to the spatial variable, then existence of Carathéodory solutions is not guaranteed. Here, in order to obtain existence results, we follow the line of the previous papers [4,11,13]: firstly, we look for solutions of a differential inclusion, which can be seen as a regularization of the former problem, and secondly, we provide conditions that ensure that solutions of the differential inclusion are also solutions of the differential equation. We will refer to such condition as a *transversality condition* and, in our case, it was inspired by that for first–order systems due to Bressan and Shen [3], later relaxed in [13]. To the best knowledge of the authors, this one is the first paper in which this type of transversality condition is adapted to second-order ordinary differential equations.

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Discontinuous differential equations appear in a natural way in many physical processes (see, for instance, the recent paper [2]). As an application of our theory, here we focus on Dirichlet problems in one-dimensional billiard spaces with possible uneven surfaces, which will be reduced to second–order problems where the nonlinearity has jump discontinuities.

Classical billiard spaces are some compact connected sets with sufficiently nice boundary and the structure of trajectories of a ball moving inside of the table in a uniform linear motion having absolutely elastic impact with the boundary is investigated, see e.g. [12]. Each such trajectory corresponds to a solution of the system of ODEs $x'' = 0$. On the other hand, less is known when the ball does not move uniformly along the straight lines, i.e. the trajectories correspond to the more general differential equations $x'' = f(t, x)$, where the ball is subjected to some force (caused by uneven surface or some external force). Some recent results in finite-dimensional billiard spaces can be found in [7–9,14].

Here we investigate Dirichlet problems for one-dimensional billiard table *with time-changing boundary*, which can be understood as boundary value problems for differential equations of the second order with state-dependent impulses in the form

$$x'' = h(t, x), \quad \text{for a.a. } t \in I := [0, T], \text{ such that } x(t) \in (\alpha(t), \beta(t)), \tag{1.1}$$

$$x'(s+) - \gamma'(s) = \gamma'(s) - x'(s-), \quad \text{if } s \in (0, T), \ x(s) = \gamma(s), \text{ for } \gamma \in \{\alpha, \beta\}, \tag{1.2}$$

$$x(0) = A, \quad x(T) = B, \tag{1.3}$$

where $\alpha, \beta : I \to \mathbb{R}$, $\alpha, \beta \in W^{2,1}(I; \mathbb{R})$, $\alpha < \beta$ on I, $A \in (\alpha(0), \beta(0))$, $B \in (\alpha(T), \beta(T))$ and $h \in Car(E_{\alpha,\beta}; \mathbb{R})$ where $E_{\alpha,\beta} := \{(t, x) \in I \times \mathbb{R} : \alpha(t) \le x \le \beta(t)\}$. Problem (1.1)–(1.3) can be transformed into a Dirichlet second–order discontinuous problem and so using the results from the first part of this paper, we give sufficient conditions for the existence and multiplicity of solutions to the impulsive BVP. As it is shown, we generalize the results from [14].

The paper is organized as follows. First part is devoted to a general discontinuous differential equations with some boundary conditions. It is shown the Krasovskii and Filippov regularization, which convert the discontinuous differential equation into a regular differential inclusion. So called transversality conditions guarantee that solutions of this regularization are also solutions to the original problem – see Theorem 2.2 and 2.3 and Corollary 2.5. The main existence results are stated in Theorem 2.7 and 2.8. In the second part we are interested in impulsive problems of type (1.1)–(1.3), which are then converted into nonimpulsive (but discontinuous) problems. Depending on the way how the boundary of the billiard table is changing we investigate the linear and also nonlinear cases. Existence and multiplicity results are given in Theorem 3.11, 3.12, 3.18 and 3.19.

2. Second-order discontinuous ODEs

Consider the second order problem

$$x'' = f(t, x), \quad \text{for a.a. } t \in I := [0, T], \quad x \in \mathcal{B}, \tag{2.4}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ may be discontinuous with respect to both variables and the set $\mathcal{B} \subset \mathcal{C}(I; \mathbb{R}^n)$ denotes initial or boundary conditions. We will say that a function $x : I \to \mathbb{R}^n$ is a *Carathéodory solution* of (2.4) if $x \in W^{2,1}(I; \mathbb{R}^n)$, $x''(t) = f(t, x(t))$ for a.a. $t \in I$ and it belongs to the set $\mathcal{B}$. Let us note that $W^{2,1}(I; \mathbb{R}^n)$ stands for all functions with values in $\mathbb{R}^n$ having absolutely continuous first derivatives and Lebesgue integrable second derivatives on I.

When the right-hand side of the differential equation in (2.4) is discontinuous with respect to the state variable, the classical technique in the literature consists in replacing the differential equation by an inclusion and so considering a problem of the following type

$$x'' \in F(t,x), \quad \text{for a.a. } t \in I := [0,T], \quad x \in \mathcal{B}, \tag{2.5}$$

where $F : I \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map which *regularizes* the discontinuous function f (i.e. $\mathcal{P}(\mathbb{R}^n)$ is the set of all subsets of $\mathbb{R}^n$).

If F is defined as

$$F(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\left(t, \overline{B}_\varepsilon(x)\right) =: \mathcal{K}(f)(t,x), \quad (t,x) \in I \times \mathbb{R}^n, \tag{2.6}$$

where $\overline{\text{co}}$ means closed convex hull and $\overline{B}_\varepsilon(x)$ is the ball centered at x and radius ε, Carathéodory solutions of the differential inclusion (2.5) are called *Krasovskii solutions* of (2.4). On the other hand, in case that F is defined as

$$F(t,x) = \bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\left(t, \overline{B}_\varepsilon(x) \setminus N\right) =: \mathcal{F}(f)(t,x), \quad (t,x) \in I \times \mathbb{R}^n, \tag{2.7}$$

where m denotes the Lebesgue measure, Carathéodory solutions of the differential inclusion (2.5) are called *Filippov solutions* of (2.4). Following with the same terminology, the maps F defined in (2.6) and (2.7) are, respectively, the *Krasovskii* and *Filippov envelopes* of f.

In the sequel, we assume that the following condition holds:

(H_F) if $f(t,\cdot)$ is continuous at the point x, then $F(t,x) = \{f(t,x)\}$.

Observe that both Krasovskii and Filippov envelopes satisfy condition (H_F).

Now, we shall discuss under what additional conditions on f, Carathéodory solutions of the differential inclusion (2.5) are also Carathéodory solutions of (2.4). To do that, we need the following technical result, see [1, Lemma 5.8.13].

Lemma 2.1. *Let $a, b \in \mathbb{R}$, $a < b$. If $\varphi : [a,b] \longrightarrow \mathbb{R}$ is almost everywhere differentiable on $[a,b]$, then for each null measure set $A \subset \mathbb{R}$ there exists a null measure set $B \subset \varphi^{-1}(A)$ such that*

$$\varphi'(t) = 0 \quad \text{for all } t \in \varphi^{-1}(A) \setminus B.$$

We are in a position to give sufficient conditions in order to ensure that the solutions of (2.5) are also solutions of (2.4).

Theorem 2.2. *Assume that condition (H_F) holds and that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C}$ with at most countable set $\mathcal{C}$, and differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \to \mathbb{R}$, $[a_k, b_k] \subset I$, such that for a.a. $t \in I$,*

$$f(t,\cdot) \text{ is continuous in } \mathbb{R}^n \setminus \bigcup_{k \in \mathcal{C}\,:\, t \in [a_k,b_k]} \{x \in \mathbb{R}^n : \tau_k(t,x) \in A_k\}$$

and for each $k \in \mathcal{C}$ and each $(t,x) \in \tau_k^{-1}(A_k)$ we have

$$\nabla \tau_k(t,x) \cdot (1,z) \neq 0 \quad \text{for all } z \in K, \tag{2.8}$$

where $K \subset \mathbb{R}^n$ is a compact set such that $x'(t) \in K$ for all $t \in I$ and every solution x of (2.5).

Then, if x is a Carathéodory solution of the inclusion (2.5), it is also a Carathéodory solution of the discontinuous problem (2.4).

Proof. Let x be a Carathéodory solution of the differential inclusion (2.5). Let us show that

$$m\left(\{t \in [a_k, b_k] : \tau_k(t, x(t)) \in A_k\}\right) = 0$$

for all $k \in \mathcal{C}$. For an arbitrarily fixed $k \in \mathcal{C}$, we define $\varphi(t) = \tau_k(t, x(t))$ for all $t \in [a_k, b_k]$. Since $m(A_k) = 0$, Lemma 2.1 guarantees the existence of a set $B \subset \varphi^{-1}(A_k)$, with $m(B) = 0$, such that for every $t \in \varphi^{-1}(A_k) \setminus B$ we have $\varphi'(t) = 0$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_k(t, x(t)) = 0.$$

By the chain rule, we can rewrite the previous expression as

$$\nabla\tau_k(t, x(t)) \cdot (1, x'(t)) = 0 \quad \text{for all } t \in \varphi^{-1}(A_k) \setminus B.$$

Since $x'(t) \in K$ for all $t \in I$, it follows from condition (2.8) that $\varphi^{-1}(A_k)$ is a null-measure set.

Therefore, for a.a. $t \in I$ the function $f(t, \cdot)$ is continuous at $x(t)$, which due to hypothesis (H_F) implies that $F(t, x(t)) = \{f(t, x(t))\}$ for a.a. $t \in I$. Since x is a solution of the differential inclusion (2.5), $x''(t) \in F(t, x(t)) = \{f(t, x(t))\}$ for a.a. $t \in I$, thus x is a Carathéodory solution of (2.4). □

Note that in order to apply Theorem 2.2 we need to construct the set K and thus we need to have some *a priori* estimates on the derivatives of any solution of (2.5). This drawback is avoided by the following result.

Theorem 2.3. *Assume that condition (H_F) holds and that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C}$ with at most countable set $\mathcal{C}$, and two times differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \to \mathbb{R}$, $[a_k, b_k] \subset I$, such that for a.a. $t \in I$,*

$$f(t, \cdot) \text{ is continuous in } \mathbb{R}^n \setminus \bigcup_{k \in \mathcal{C} \,:\, t \in [a_k, b_k]} \{x \in \mathbb{R}^n : \tau_k(t, x) \in A_k\}$$

and for each $k \in \mathcal{C}$ and each $(t, x) \in \tau_k^{-1}(A_k)$ we have

$$(1, v)\, H(\tau_k)(t, x)\, (1, v)^T + \nabla_x\tau_k(t, x) \cdot z \neq 0 \tag{2.9}$$

for all $v \in \mathbb{R}^n$ such that $\nabla\tau_k(t, x) \cdot (1, v) = 0$ and all $z \in F(t, x)$ (where $H(\tau_k)$ denotes the Hessian matrix of τ_k).

Then, if x is a Carathéodory solution of the inclusion (2.5), it is also a Carathéodory solution of the discontinuous problem (2.4).

Proof. Let x be a Carathéodory solution of the differential inclusion (2.5). Again it suffices to see that for each $k \in \mathcal{C}$ the set

$$J_k := \{t \in [a_k, b_k] : \tau_k(t, x(t)) \in A_k\}$$

has Lebesgue null measure. For a fixed $k \in \mathcal{C}$, by Lemma 2.1 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_k(t, x(t)) = \nabla\tau_k(t, x(t)) \cdot (1, x'(t)) = 0 \quad \text{for a.a. } t \in J_k.$$

Applying again Lemma 2.1 we deduce that for a.a. $t \in J_k$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tau_k(t,x(t)) = (1,x'(t))\,H(\tau_k)(t,x(t))\,(1,x'(t))^T + \nabla_x\tau_k(t,x(t))\cdot x''(t) = 0.$$

This fact joint with condition (2.9) implies that $m(J_k) = 0$. □

Remark 2.4. If τ_k is linear, i.e., $\tau_k : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\tau_k(x_1, x_2, \ldots, x_n) = a_1\,x_1 + a_2\,x_2 + \cdots + a_n\,x_n,$$

with $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, then the transversality condition (2.9) reads simply as

$$\nabla\tau_k(x)\cdot z \neq 0 \quad \text{for all } z \in F(t,x).$$

Equivalently,

$$\nu \cdot z \neq 0 \quad \text{for all } z \in F(t,x),$$

where ν denotes the vector $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, that is, ν is a normal vector of the hyperplane $\tau_k(x) = c$ where f may be discontinuous.

Let us now focus on the scalar case of (2.4), i.e., with $n = 1$. By the implicit function theorem, if τ is regular enough, the discontinuity hypersurfaces of type

$$\tau(t,x) = c$$

can be seen, at least locally, as the graphs of time-dependent curves of type $x = \gamma(t)$.

Corollary 2.5. *Assume that $n = 1$, condition (H_F) holds and there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{Z}$, and two times differentiable mappings $\gamma_k : [a_k, b_k] \subset I \to \mathbb{R}$ such that for a.a. $t \in I$,*

$$f(t,\cdot) \text{ is continuous in } \mathbb{R} \setminus \bigcup_{k\in\mathcal{C}\,:\,t\in[a_k,b_k]}\ \bigcup_{c_k\in A_k} \{\gamma_k(t) + c_k\}$$

and for each $k \in \mathcal{C}$, the function γ_k satisfies either

(i) $\gamma_k''(t) \notin F(t, \gamma_k(t) + c_k)$ for a.a. $t \in [a_k, b_k]$ and each $c_k \in A_k$; or
(ii) $\gamma_k'(t) \notin K$ for a.a. $t \in [a_k, b_k]$, where $K \subset \mathbb{R}$ is a compact set such that $x'(t) \in K$ for all $t \in I$ and every solution x of (2.5).

Then, if x is a Carathéodory solution of the inclusion (2.5), it is also a Carathéodory solution of the discontinuous problem (2.4).

Proof. It suffices to apply Theorems 2.2 and 2.3 with $\tau_k(t,x) = x - \gamma_k(t)$. Indeed, for $(t,x) \in [a_k, b_k] \times \mathbb{R}$ we have

$$\nabla\tau_k(t,x) = (-\gamma_k'(t), 1)$$

and

$$H(\tau_k)(t,x) = \begin{pmatrix} -\gamma_k''(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, condition (2.9) reads as follows: for each $k \in \mathcal{C}$ and each $x = \gamma_k(t) + c_k$, with $c_k \in A_k$, we have

$$-\gamma_k''(t) + z \neq 0 \quad \text{for all } z \in F(t,x).$$

This is exactly condition (i) here. Note that alternative (ii) implies (2.8), so the conclusion follows. □

Notice that there is a variety of existence results in the literature concerning differential inclusion (2.5) with different initial or boundary conditions, see for instance [5,6]. By means of them and the previous conditions which ensure that solutions of (2.5) are in fact Carathéodory solutions of (2.4), we are able to establish existence results for (2.4).

Let us now deal with second order equations subject to Dirichlet boundary conditions

$$x'' = f(t,x), \quad \text{for a.a. } t \in I, \quad x(0) = x_0, \quad x(T) = x_T, \tag{2.10}$$

where $x_0, x_T \in \mathbb{R}^n$, that is, problem (2.4) with $\mathcal{B} = \{x \in \mathcal{C}(I;\mathbb{R}^n) : x(0) = x_0,\ x(T) = x_T\}$.

Theorem 2.6. *Assume that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:*

(C_1) *for all $x \in \mathbb{R}^n$, $f(\cdot, x)$ is measurable;*
(C_2) *there exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}^n$, we have*

$$\|f(t,x)\| \leq M(t);$$

(C_3) *there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{Z}$, and two times differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \to \mathbb{R}$, $[a_k, b_k] \subset I$, such that for a.a. $t \in I$,*

$$f(t,\cdot) \text{ is continuous in } \mathbb{R}^n \setminus \bigcup_{k \in \mathcal{C}\,:\, t \in [a_k,b_k]} \{x \in \mathbb{R}^n : \tau_k(t,x) \in A_k\}$$

and for each $k \in \mathcal{C}$ and each $(t,x) \in \tau_k^{-1}(A_k)$ we have

$$(1,v)\, H(\tau_k)(t,x)\, (1,v)^T + \nabla_x \tau_k(t,x) \cdot z \neq 0$$

for all $v \in \mathbb{R}^n$ such that $\nabla \tau_k(t,x) \cdot (1,v) = 0$ and all $z \in \mathcal{K}(f)(t,x)$.

Then problem (2.10) has at least one Carathéodory solution.

Proof. By [5, Theorem 12.2], conditions (C_1) and (C_2) ensure that the differential inclusion

$$x'' \in \mathcal{K}(f)(t,x), \quad \text{for a.a. } t \in I, \quad x(0) = x_0, \quad x(T) = x_T,$$

has a solution.

Then condition (C_3) guarantees that it is also a Carathéodory type solution to the Dirichlet problem (2.10), as a consequence of Theorem 2.3. □

In the scalar case, we have the following existence result for the Dirichlet problem (2.10).

Theorem 2.7. *Assume that $n = 1$ and $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (C_1), (C_2) and*

(C_3^*) *there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{Z}$, and two times differentiable mappings $\gamma_k : [a_k, b_k] \subset I \to \mathbb{R}$ such that for a.a. $t \in I$,*

$$f(t, \cdot) \text{ is continuous in } \mathbb{R} \setminus \bigcup_{k \in \mathcal{C}\,:\, t \in [a_k, b_k]} \bigcup_{c_k \in A_k} \{\gamma_k(t) + c_k\}$$

and for each $k \in \mathcal{C}$, the function γ_k satisfies either

(i) $\gamma_k''(t) \notin \mathcal{K}(f)(t, \gamma_k(t) + c_k)$ for a.a. $t \in [a_k, b_k]$ and each $c_k \in A_k$; or

(ii) $\gamma_k'(t) \notin \left[\dfrac{x_T - x_0}{T} - \|M\|_{L^1}, \dfrac{x_T - x_0}{T} + \|M\|_{L^1}\right]$ for a.a. $t \in [a_k, b_k]$.

Then problem (2.10) has at least one Carathéodory solution.

Proof. Note that again conditions (C_1) and (C_2) ensure that problem (2.10) has a Krasovskii solution. We need to see that it is a Carathéodory solution of (2.10). It follows as a direct application of Corollary 2.5 since the compact interval

$$K = \left[\frac{x_T - x_0}{T} - \|M\|_{L^1}, \frac{x_T - x_0}{T} + \|M\|_{L^1}\right]$$

satisfies that $x'(t) \in K$ for all $t \in I$ and every Krasovskii solution of (2.10). □

Observe that the previous result can be even improved allowing f to be discontinuous over the graphs of a countable number of solutions of the differential equation $x'' = f(t, x)$.

Theorem 2.8. *Assume that $n = 1$ and $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (C_1), (C_2) and*

$(\tilde{C}_3)$ *there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C}$, $j \in \mathcal{D}$ (with $\mathcal{C}$ and $\mathcal{D}$ at most countable sets) and two times differentiable mappings $\gamma_k : [a_k, b_k] \subset I \to \mathbb{R}$ and $\varphi_j : [\tilde{a}_j, \tilde{b}_j] \subset I \to \mathbb{R}$ such that for a.a. $t \in I$,*

$$f(t, \cdot) \text{ is continuous in } \mathbb{R} \setminus N(t),$$

where

$$N(t) = N_1(t) \cup N_2(t), \quad N_1(t) = \bigcup_{k \in \mathcal{C}\,:\, t \in [a_k, b_k]} \bigcup_{c_k \in A_k} \{\gamma_k(t) + c_k\}, \quad N_2(t) = \bigcup_{j \in \mathcal{D}\,:\, t \in [\tilde{a}_j, \tilde{b}_j]} \{\varphi_j(t)\},$$

for each $k \in \mathcal{C}$, the function γ_k satisfies either

(i) $\gamma_k''(t) \notin \mathcal{K}(f)(t, \gamma_k(t) + c_k)$ for a.a. $t \in [a_k, b_k]$ and each $c_k \in A_k$; or

(ii) $\gamma_k'(t) \notin \left[\dfrac{x_T - x_0}{T} - \|M\|_{L^1}, \dfrac{x_T - x_0}{T} + \|M\|_{L^1}\right]$ for a.a. $t \in [a_k, b_k]$;

and for each $j \in \mathcal{D}$, $\varphi_j''(t) = f(t, \varphi_j(t))$ for a.a. $t \in [\tilde{a}_j, \tilde{b}_j]$.

Then problem (2.10) has at least one Carathéodory solution.

Proof. By conditions (C_1) and (C_2), problem (2.10) has a Krasovskii solution x, so we will show that x is also a Carathéodory solution of (2.10).

It can be proven (just as above) that the set

$$J^\gamma = \bigcup_{k \in \mathcal{C}} \{t \in [a_k, b_k] : x(t) - \gamma_k(t) \in A_k\}$$

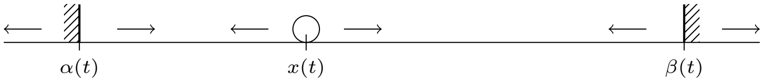

Fig. 1. A ball inside a one-dimensional billiard table with moving borders.

has Lebesgue measure zero. Suppose that there exists $j \in \mathcal{D}$ such that $m(J_j^\varphi) > 0$, with

$$J_j^\varphi = \{t \in [\tilde{a}_j, \tilde{b}_j] : x(t) = \varphi_j(t)\} \quad \text{and} \quad J^\varphi = \bigcup_{j \in \mathcal{D}} J_j^\varphi.$$

Then $x''(t) = \varphi_j''(t)$ for a.a. $t \in J_j^\varphi$. By the definition of φ_j, we have that $\varphi_j''(t) = f(t, \varphi_j(t))$ for a.a. $t \in J_j^\varphi$ and thus $x''(t) = f(t, x(t))$ for a.a. $t \in J_j^\varphi$. Hence, $x''(t) = f(t, x(t))$ for a.a. $t \in J^\varphi$.

Finally, since $f(t, \cdot)$ is continuous at $x(t)$ for a.a. $t \in I \setminus (J^\gamma \cup J^\varphi)$, we conclude that x is a Carathéodory solution of (2.10). □

3. Applications to differential problems in time–dependent billiard spaces

In this section, we apply the developed tools to the impulsive boundary value problem (1.1)–(1.3).

We will use the following concept of solution for problem (1.1)-(1.2).

Definition 3.1. We say that a function $x \in C(I; \mathbb{R})$ is a Carathéodory solution of the billiard problem (1.1)-(1.2) if

- $\operatorname{graph} x \subset E_{\alpha,\beta}$,
- $x \in W^{2,1}(J; \mathbb{R})$ and x satisfies the differential equation in (1.1) for a.e. $t \in J$, for every interval $J \subset I$ for which $x(t) \in (\alpha(t), \beta(t))$ for every $t \in J$,
- if $x(s) = \alpha(s)$, for some $s \in (0, T)$, then there exist $x'(s+)$ and $x'(s-)$ satisfying $x'(s+) - \alpha'(s) = \alpha'(s) - x'(s-)$,
- if $x(s) = \beta(s)$, for some $s \in (0, T)$, then there exist $x'(s+)$ and $x'(s-)$ satisfying $x'(s+) - \beta'(s) = \beta'(s) - x'(s-)$.

Remark 3.2. Note that the impulsive problem (1.1)–(1.2) is called "a billiard problem", because it is a mathematical model of a ball moving in a line segment (=billiard table) between the "walls" changing their positions in time – see Fig. 1. Moreover, if the ball is inside of the segment, its motion is determined by the differential equation (1.1). And if the ball "hits the boundary", the bounce is determined by (1.2), i.e. it is absolutely elastic.

Without any loss of generality (see Lemma 3.3), we assume a special case of BVP for the lower barrier α identically equal to zero, i.e.

$$x'' = f(t, x), \quad \text{for a.a. } t \in I := [0, T], \text{ such that } x(t) \in (0, \gamma(t)), \tag{3.11}$$

$$x'(s+) = -x'(s-), \quad \text{if } s \in (0, T),\ x(s) = 0, \tag{3.12}$$

$$x'(s+) - \gamma'(s) = \gamma'(s) - x'(s-), \quad \text{if } s \in (0, T),\ x(s) = \gamma(s), \tag{3.13}$$

$$x(0) = A, \quad x(T) = B, \tag{3.14}$$

where $\gamma \in W^{2,1}(I; \mathbb{R})$, $\gamma > 0$ on I, $A \in (0, \gamma(0))$, $B \in (0, \gamma(T))$ and $f \in Car(E_{0,\gamma}; \mathbb{R})$ with $E_{0,\gamma} := \{(t, x) \in I \times \mathbb{R} : 0 \leq x \leq \gamma(t)\}$.

Lemma 3.3. *Let α, $\beta \in W^{2,1}(I;\mathbb{R})$, $\alpha < \beta$ on I, $h \in Car(E_{\alpha,\beta};\mathbb{R})$. If x is a Carathéodory solution of the billiard problem* (3.11)–(3.13) *with*

$$\gamma := \beta - \alpha \quad \text{and} \quad f(t,x) := h(t, x + \alpha(t)) - \alpha''(t), \tag{3.15}$$

then the function

$$y(t) = x(t) + \alpha(t), \quad t \in I,$$

is a solution of (1.1)–(1.2) *with the same number of impacts with the boundary. If, moreover, x satisfies boundary conditions $x(0) = A - \alpha(0)$, $x(T) = B - \alpha(T)$ with $A \in (\alpha(0), \beta(0))$ and $B \in (\alpha(T), \beta(T))$, then y is a solution of boundary value problem* (1.1)–(1.3)*.*

Proof. Let x be a solution of (3.11)–(3.13). First, let us note that $x(t) \in (0, \gamma(t))$ iff $y(t) \in (\alpha(t), \beta(t))$, $x(t) = 0$ iff $y(t) = \alpha(t)$ and $x(t) = \gamma(t)$ iff $y(t) = \beta(t)$. Let $J \subset I$ be an interval such that $y(t) \in (\alpha(t), \beta(t))$ for each $t \in J$. Then $x(t) \in (0, \gamma(t))$ for each $t \in J$, $x \in W^{2,1}(J;\mathbb{R})$ and

$$y''(t) = x''(t) + \alpha''(t) = f(t, x(t)) + \alpha''(t) = h(t, x(t) + \alpha(t)) = h(t, y(t))$$

for each $t \in J$. Let $y(s) = \alpha(s)$ for some $s \in I$. Then $x(s) = 0$ and

$$y'(s+) - \alpha'(s) = x'(s+) + \alpha'(s) - \alpha'(s) = x'(s+) = -x'(s-) = -(y'(s-) - \alpha'(s)) = \alpha'(s) - y'(s-).$$

Finally, if $y(s) = \beta(s)$ for some $s \in I$, then $x(s) = \gamma(s)$ and similarly $y'(s+) - \beta'(s) = \beta'(s) - y'(s-)$. The rest of the proof is trivial. □

Remark 3.4. Let us note that if $h \in Car(E_{\alpha,\beta};\mathbb{R})$ and $\alpha, \beta \in W^{2,1}(I;\mathbb{R})$, then for f and γ defined in (3.15) we have $f \in Car(E_{0,\gamma};\mathbb{R})$ and $\gamma \in W^{2,1}(I;\mathbb{R})$. Therefore, from Lemma 3.3 we can see that in order to investigate BVP (1.1)–(1.3), it is sufficient to give results for the problem (3.11)–(3.14). Also note that there is a one-to-one correspondence between solutions of problem (3.11)–(3.13) with (3.15) and problem (1.1), (1.2).

Now, let us focus our attention on the billiard problem (3.11)–(3.13). Consider the map $\Delta : I \times \mathbb{R} \to \mathbb{R}$ defined as

$$\Delta(t,x) = \begin{cases} x - 2k\gamma(t), & \text{if } x \in [2k\gamma(t), (2k+1)\gamma(t)), \text{ for some } k \in \mathbb{Z}, \\ 2(1+k)\gamma(t) - x, & \text{if } x \in [(2k+1)\gamma(t), 2(k+1)\gamma(t)), \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Remark 3.5. Note that Δ is continuous and $\Delta(t,x) \in [0, \gamma(t)]$ for all $(t,x) \in I \times \mathbb{R}$. Moreover

(a) $\Delta(t,x) \in (0, \gamma(t))$ if and only if $x \in (\ell\gamma(t), (\ell+1)\gamma(t))$ for some integer ℓ,
(b) $\Delta(t,x) = 0$ if and only if $x = 2k\gamma(t)$ for some integer k and
(c) $\Delta(t,x) = \gamma(t)$ if and only if $x = (2k+1)\gamma(t)$ for some integer k.

For each $t \in I$, the function $x \mapsto \Delta(t,x)$ is Lipschitz continuous with the Lipschitz constant equal to 1, i.e. for each $(t,x), (t,y) \in [0,T] \times \mathbb{R}$ we have

$$|\Delta(t,x) - \Delta(t,y)| \le |x - y|.$$

Let us consider the modified problem

$$x'' = f^*(t,x), \tag{3.16}$$

with $f^* : I \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f^*(t,y) = \begin{cases} f(t, y - 2k\gamma(t)) + 2k\gamma''(t), & \text{if } y \in (2k\gamma(t), (2k+1)\gamma(t)), \text{ for some } k \in \mathbb{Z}, \\ -f(t, 2(1+k)\gamma(t) - y) + 2(k+1)\gamma''(t), & \text{if } y \in ((2k+1)\gamma(t), 2(k+1)\gamma(t)), \text{ for some } k \in \mathbb{Z}, \\ k\gamma''(t), & \text{if } y = k\gamma(t) \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Lemma 3.6. *If y is a Carathéodory solution of equation (3.16), then the function*

$$x(t) = \Delta(t, y(t)), \ t \in [0,T] \tag{3.17}$$

is a solution of the impulsive problem (3.11)–(3.13).

Proof. Let y be a Carathéodory solution of (3.16). Observe that $x(t) \in (0, \gamma(t))$ if and only if $y(t) \bmod \gamma(t) \neq 0$, i.e. $y(t)/\gamma(t)$ is not an integer.
Let $J \subset [0,T]$ be an interval such that $x(t) \in (0, \gamma(t))$ for each $t \in J$, i.e. $\Delta(t, y(t)) \in (0, \gamma(t))$ for each $t \in J$. Then according to Remark 3.5 and the fact that y and γ are continuous, there exists $\ell \in \mathbb{Z}$ such that $y(t) \in (\ell\gamma(t), (\ell+1)\gamma(t))$ for each $t \in J$. Let $\ell = 2k$, $k \in \mathbb{Z}$. Then

$$x(t) = \Delta(t, y(t)) = y(t) - 2k\gamma(t)$$

and

$$\begin{aligned} x''(t) &= y''(t) - 2k\gamma''(t) = f^*(t, y(t)) - 2k\gamma''(t) \\ &= f(t, y(t) - 2k\gamma(t)) = f(t, x(t)) \quad \text{for a.a. } t \in J. \end{aligned}$$

Let $\ell = 2k+1$, $k \in \mathbb{Z}$. Then

$$x(t) = \Delta(t, y(t)) = 2(1+k)\gamma(t) - y(t)$$

and

$$\begin{aligned} x''(t) &= 2(1+k)\gamma''(t) - y''(t) = 2(1+k)\gamma''(t) - f^*(t, y(t)) \\ &= f(t, 2(1+k)\gamma(t) - y(t)) = f(t, x(t)) \quad \text{for a.a. } t \in J. \end{aligned}$$

Let $x(s) = 0$ for some $s \in (0,T)$. We will prove that $x'(s+) = -x'(s-)$. Since $\Delta(s, y(s)) = 0$, it follows from Remark 3.5 that there exists $k \in \mathbb{Z}$ such that $y(s) = 2k\gamma(s)$. We consider three cases:

(a) $y'(s) > 2k\gamma'(s)$: Then there exists $\delta > 0$ such that

$$\forall t \in (s, s+\delta) \ : \ y(t) \in (2k\gamma(t), (2k+1)\gamma(t)) \quad \text{and} \quad \forall t \in (s-\delta, s) \ : \ y(t) \in ((2k-1)\gamma(t), 2k\gamma(t)).$$

Therefore,

$$x'(s+) = \lim_{t \to s+} x'(t) = \lim_{t \to s+} y'(t) - 2k\gamma'(t) = y'(s) - 2k\gamma'(s)$$

and

$$x'(s-) = \lim_{t\to s+} x'(t) = \lim_{t\to s+} 2k\gamma'(t) - y'(t) = 2k\gamma'(s) - y'(s).$$

(b) $y'(s) < 2k\gamma'(s)$: Then there exists $\delta > 0$ such that

$$\forall t \in (s, s+\delta) \; : \; y(t) \in ((2k-1)\gamma(t), 2k\gamma(t)) \quad \text{and} \quad \forall t \in (s-\delta, s) \; : \; y(t) \in (2k\gamma(t), (2k+1)\gamma(t)).$$

Similarly to the case (a) we get

$$x'(s+) = \lim_{t\to s+} x'(t) = \lim_{t\to s+} 2k\gamma'(t) - y'(t) = 2k\gamma'(s) - y'(s)$$

and

$$x'(s-) = \lim_{t\to s+} x'(t) = \lim_{t\to s+} y'(t) - 2k\gamma'(t) = y'(s) - 2k\gamma'(s).$$

(c) $y'(s) = 2k\gamma'(s)$: For sufficiently small $h \in \mathbb{R}$ we have

$$\begin{aligned} |x(s+h) - x(s)| &= |\Delta(s+h, y(s+h)) - \Delta(s, y(s))| \\ &= |\Delta(s+h, y(s+h)) - \Delta(s+h, 2k\gamma(s+h))| \le |y(s+h) - 2k\gamma(s+h)| \\ &= |y(s) - 2k\gamma(s) + (y'(s) - 2k\gamma'(s))h + \eta(h)| = |\eta(h)| \end{aligned}$$

where $\lim_{h\to 0} \frac{|\eta(h)|}{|h|} = 0$. This implies that the differential of x at s is zero, i.e. $x'(s) = 0$ and so $x'(s+) = 0 = -x'(s-)$.

Let $x(s) = \gamma(s)$ for some $s \in (0, T)$. Since $\Delta(s, y(s)) = \gamma(s)$, it follows that there exists $k \in \mathbb{Z}$ such that $y(s) = (2k+1)\gamma(s)$. We can show that $x'(s+) - \gamma'(s) = \gamma'(s) - x'(s-)$ separating into three cases in a similar way to the previous discussion when $x(s) = 0$ for some $s \in (0, T)$. Therefore, x is a solution of the problem (3.11)–(3.13). □

3.1. The case of a linear γ

If the mapping γ describing the change of length of the billiard table in time is linear, then we are able to obtain multiplicity results.

Lemma 3.7. *Let y be a Carathéodory solution of equation* (3.16) *with $y(0) \bmod \gamma(0) \neq 0$, $y(T) \bmod \gamma(T) \neq 0$, such that y/γ is strictly monotone. Then the function x defined by* (3.17) *is a solution of the impulsive problem* (3.11)–(3.13) *having exactly*

$$\left| \left\lfloor \frac{y(0)}{\gamma(0)} \right\rfloor - \left\lfloor \frac{y(T)}{\gamma(T)} \right\rfloor \right|$$

impacts with the boundary.

Proof. From Lemma 3.6 it follows that x is a solution of (3.11)–(3.13). It remains to compute the number of impacts. Let us note that x has an impact with the boundary at the instant s if and only if $y(s)/\gamma(s)$ is an integer.

(a) Let y/γ be strictly increasing. Let us denote

$$k = \left\lfloor \frac{y(0)}{\gamma(0)} \right\rfloor, \quad \ell = \left\lfloor \frac{y(T)}{\gamma(T)} \right\rfloor, \quad p = \ell - k.$$

From continuity of y/γ it follows that the image of y/γ is an interval $[y(0)/\gamma(0), y(T)/\gamma(T)]$. The only integers from this interval are $k+1, \dots, \ell$. Since $\#\{k+1, \dots, \ell\} = p$, the solution x has exactly p impacts with the boundary.

(b) The case for y/γ strictly decreasing can be done similarly. □

Lemma 3.8. *Let $y, \gamma : [0, T] \to \mathbb{R}$ be positive functions.*

(a) *If y is strictly increasing and γ is decreasing, then y/γ is strictly increasing.*
(b) *If y is strictly decreasing and γ is increasing, then y/γ is strictly decreasing.*

Proof. Almost trivial. □

Now we give an existence result for the auxiliary boundary value problem if the function γ is linear, i.e. $\gamma(t) = R + rt$, $t \in [0, T]$, where $r, R \in \mathbb{R}$.

Lemma 3.9. *Let γ be a linear function which is positive on $[0, T]$ and $A, B \in \mathbb{R}$.*

(a) *If γ is decreasing and $A \in (0, \gamma(0))$, $B \bmod \gamma(T) \neq 0$ satisfy that*

$$\frac{B - A}{T} > \|M\|_{L^1},$$

then (3.16), (3.14) has a strictly increasing solution.

(b) *If γ is increasing and $A \bmod \gamma(0) \neq 0$, $B \in (0, \gamma(T))$ satisfy that*

$$\frac{A - B}{T} > \|M\|_{L^1},$$

then (3.16), (3.14) has a strictly decreasing solution.

(c) *If γ is decreasing and $A \in (0, \gamma(0))$, $B \bmod \gamma(T) \neq 0$ satisfy that*

$$\frac{A - B}{T} > \|M\|_{L^1} - r,$$

then (3.16), (3.14) has a strictly decreasing solution y such that $y(t) < \gamma(t)$ for all $t \in [0, T]$.

(d) *If γ is increasing and $A \bmod \gamma(0) \neq 0$, $B \in (0, \gamma(T))$ satisfy that*

$$\frac{B - A}{T} > \|M\|_{L^1} + r,$$

then (3.16), (3.14) has a strictly increasing solution y such that $y(t) < \gamma(t)$ for all $t \in [0, T]$.

Proof. According to the linearity of γ we see that

$$f^*(t, y) = \begin{cases} f(t, y - 2k\gamma(t)), & \text{if } y \in (2k\gamma(t), (2k+1)\gamma(t)), \text{ for some } k \in \mathbb{Z}, \\ -f(t, 2(1+k)\gamma(t) - y), & \text{if } y \in ((2k+1)\gamma(t), 2(k+1)\gamma(t)), \text{ for some } k \in \mathbb{Z}, \\ 0, & \text{if } y = k\gamma(t) \text{ for some } k \in \mathbb{Z}, \end{cases}$$

which implies that f^* satisfies (C_2). Moreover, observe that for a.a. $t \in I$,

$$f^*(t,\cdot) \text{ is continuous in } \mathbb{R} \setminus \bigcup_{j\in\mathbb{Z}} \{j\gamma(t)\}.$$

Hence, choose $\mathcal{C} = \emptyset$, $\mathcal{D} = \mathbb{Z}$ and $\varphi_j : I \to \mathbb{R}$ the function defined as $\varphi_j(t) = j\gamma(t)$, $j \in \mathcal{D}$. Since

$$\varphi_j''(t) = j\gamma''(t) = f^*(t, j\gamma(t)) = f^*(t, \varphi_j(t)), \quad t \in I,$$

we deduce that f^* satisfies condition $(\tilde{C}_3)$. From Theorem 2.8 we see that there exists at least one solution y of the problem (3.16), (3.14).

Suppose that alternative (a) holds, that is, γ is decreasing and $A \in (0, \gamma(0))$, $B \bmod \gamma(T) \neq 0$ satisfy that

$$\frac{B-A}{T} > \|M\|_{L^1}.$$

Let us show that y is strictly increasing. Indeed, it follows as a straightforward consequence of the Mean Value Theorem that any solution y of (3.16), (3.14) satisfies that

$$y'(t) \in \left[\frac{B-A}{T} - \|M\|_{L^1}, \frac{B-A}{T} + \|M\|_{L^1}\right] \quad \text{for all } t \in I, \tag{3.18}$$

and thus $y' > 0$ on I, which clearly implies that y is strictly increasing and subsequently positive. Similarly, in case (b), we get a strictly decreasing solution which is also positive.

In case (c), we have again that any solution y of (3.16), (3.14) satisfies (3.18) and so inequality

$$\frac{A-B}{T} > \|M\|_{L^1} - r$$

implies that

$$y'(t) \leq \frac{B-A}{T} + \|M\|_{L^1} < r = \gamma'(t) \leq 0 \quad \text{for all } t \in I.$$

Therefore, y is strictly decreasing and, moreover, $y' < \gamma'$ on $[0,T]$ what, joint with $y(0) = A \in (0, \gamma(0))$, ensures that $y < \gamma$ on $[0,T]$. Similarly, in case (d), we get a strictly increasing solution y satisfying that $y < \gamma$ on $[0,T]$. □

Remark 3.10. Observe that the existence of solutions of the auxiliary problem in the proof of Lemma 3.9 can also be proven via Theorem 2.7 instead of Theorem 2.8, but with slight modifications in the definition of f^*. For instance, in order to prove the case (a), we can define $f^* : I \times \mathbb{R} \to \mathbb{R}$ as

$$f^*(t,y) = \begin{cases} f(t, y - 2k\gamma(t)), & \text{if } y \in (2k\gamma(t), (2k+1)\gamma(t)), \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ -f(t, 2(1+k)\gamma(t) - y), & \text{if } y \in ((2k+1)\gamma(t), 2(k+1)\gamma(t)), \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for a.a. $t \in I$,

$$f^*(t,\cdot) \text{ is continuous in } \mathbb{R} \setminus \bigcup_{k\in\mathbb{N}\cup\{0\}} \{k\,\gamma(t)\}.$$

Define $\gamma_k(t) := k\,\gamma(t)$, for $t \in [0,T]$ and $k \in \mathbb{N} \cup \{0\}$. Then

$$\gamma_k'(t) = k\gamma'(t) = kr \leq 0 < \frac{B-A}{T} - \|M\|_{L^1} \quad \text{for all } t \in [0,T].$$

Therefore, f^* satisfies (C_3^*) as well.

Theorem 3.11. *Let γ be a linear function, i.e. $\gamma(t) = R + rt$, $t \in [0,T]$, with $r, R \in \mathbb{R}$. Then for each integer p satisfying*

$$p \geq \frac{\max_{[0,T]} \gamma + T\|M\|_{L^1}}{\min_{[0,T]} \gamma}, \tag{3.19}$$

the impulsive boundary value problem (3.11)–(3.14) has at least one solution with exactly p impacts.

Furthermore, for each integer p such that

$$p \geq \frac{\max_{[0,T]} \gamma + T(\|M\|_{L^1} + |r|)}{\min_{[0,T]} \gamma}, \tag{3.20}$$

the impulsive boundary value problem (3.11)–(3.14) has an additional solution with exactly p impacts.

Proof. Let γ be decreasing. For each $i \in \mathbb{N}$ we consider a boundary value problem (3.16),

$$y(0) = A, \quad y(T) = i\gamma(T) + \theta_i, \tag{3.21}$$

where

$$\theta_i = \begin{cases} B & \text{if } i \text{ is even,} \\ \gamma(T) - B & \text{if } i \text{ is odd.} \end{cases} \tag{3.22}$$

Then for p satisfying (3.19) we have

$$\frac{p\gamma(T) + \theta_p - A}{T} \geq \frac{\gamma(0) + T\|M\|_{L^1} + \theta_p - A}{T} = \frac{\gamma(0) - A + \theta_p + T\|M\|_{L^1}}{T} > \|M\|_{L^1}.$$

According to Lemma 3.9 there exists a strictly increasing and positive solution y of the boundary value problem (3.16), (3.21) for $i = p$. According to Lemma 3.8, the function y/γ is strictly increasing. From Lemma 3.7 it follows that the function x defined by (3.17) is a solution of (3.11)–(3.13) having exactly

$$\left| \left\lfloor \frac{y(0)}{\gamma(0)} \right\rfloor - \left\lfloor \frac{y(T)}{\gamma(T)} \right\rfloor \right| = \left| \left\lfloor \frac{A}{\gamma(0)} \right\rfloor - \left\lfloor \frac{p\gamma(T) + \theta_p}{\gamma(T)} \right\rfloor \right| = |0 - p| = p$$

impacts. Since $x(0) = \Delta(0, y(0)) = A$ and $x(T) = \Delta(T, y(T)) = \Delta(T, p\gamma(T) + \theta_p) = B$, x satisfies (3.14) as well.

Now, for each $i \in \mathbb{N}$ we consider the boundary value problem (3.16) coupled with Dirichlet conditions

$$y(0) = A, \quad y(T) = -i\gamma(T) + \theta_i, \tag{3.23}$$

where θ_i is defined as in (3.22). Then for p satisfying (3.20) we have

$$\frac{A - (-p\gamma(T) + \theta_p)}{T} \geq \frac{A + \gamma(0) + T(\|M\|_{L^1} - r) - \theta_p}{T} \geq \frac{\gamma(0) - |\theta_p - A|}{T} + \|M\|_{L^1} - r > \|M\|_{L^1} - r.$$

By Lemma 3.9 there exists a strictly decreasing solution y of (3.16), (3.23) for $i = p$ such that $y < \gamma$. Then Bolzano theorem ensures that there exists a unique $t_1 \in (0,T)$ such that $y(t_1) = 0$. Note that y is negative and strictly decreasing in (t_1, T), so y/γ is strictly decreasing. By Lemma 3.7 again we obtain that the function x defined by (3.17) is a second solution of (3.11)–(3.13) with exactly p impacts.

Let γ be increasing. For each $i \in \mathbb{Z}$ we consider a boundary value problem (3.16),

$$y(0) = i\gamma(0) + \theta_i, \quad y(T) = B, \tag{3.24}$$

where

$$\theta_i = \begin{cases} A & \text{if } i \text{ is even,} \\ \gamma(0) - A & \text{if } i \text{ is odd.} \end{cases}$$

Similarly, as for γ decreasing, we prove that there exists a strictly decreasing solution of this auxiliary problem and using Lemma 3.8 and 3.7 we prove that there exists a solution of the original problem with p impacts. Moreover, for p satisfying the inequality (3.20), the auxiliary problem has a strictly increasing solution what ensures the existence of a second solution of the original problem with p impacts. □

Now, we are ready to prove the main existence and multiplicity result for the billiard problem (1.1)–(1.3).

Theorem 3.12. *Let $\beta - \alpha$ be a linear function, i.e. $\beta(t) = R + rt + \alpha(t)$, $t \in [0, T]$ with $r, R \in \mathbb{R}$ and $M \in L^1(I)$ be such that for a.a. $t \in I$ and all $x \in [\alpha(t), \beta(t)]$ we have*

$$\|h(t, x + \alpha(t)) - \alpha''(t)\| \leq M(t). \tag{3.25}$$

Then for each integer p satisfying

$$p \geq \frac{\max_{[0,T]}(\beta - \alpha) + T\|M\|_{L^1}}{\min_{[0,T]}(\beta - \alpha)}, \tag{3.26}$$

the impulsive boundary value problem (1.1)–(1.3) has at least one solution with exactly p impacts.

Furthermore, for each integer p such that

$$p \geq \frac{\max_{[0,T]}(\beta - \alpha) + T(\|M\|_{L^1} + |r|)}{\min_{[0,T]}(\beta - \alpha)}, \tag{3.27}$$

the impulsive boundary value problem (1.1)–(1.3) has an additional solution with exactly p impacts.

Proof. We consider the problem (3.11)–(3.14) with (3.15) and

$$A := A - \alpha(0), \quad B := B - \alpha(T).$$

If p satisfies (3.26), then it also satisfies (3.19) and so there exists at least one solution x of problem (3.11)–(3.14) with exactly p impacts. According to Lemma 3.3 the function $y = x + \alpha$ is a solution of problem (1.1)–(1.3) with the same number of impacts. The existence of two solutions can be proved similarly. □

Remark 3.13. Clearly, f does not have to be a Carathéodory function. Indeed, it may be discontinuous with respect to both variables. We may assume that f satisfies the following: there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C}$, with $\mathcal{C}$ a countable set, and two times differentiable mappings $\gamma_k : [a_k, b_k] \subset I \to \mathbb{R}$ such that for a.a. $t \in I$,

$$f(t, \cdot) \text{ is continuous in } [0, \gamma(t)] \setminus \bigcup_{k \in \mathcal{C}} \bigcup_{c_k \in A_k} \{\gamma_k(t) + c_k\}$$

and for each $k \in \mathcal{C}$,

$$\gamma_k''(t) \notin \mathcal{K}(f)(t, \gamma_k(t) + c_k) \text{ for a.a. } t \in [a_k, b_k] \text{ and each } c_k \in A_k.$$

Lemma 3.9 can be proven again as an application of Theorem 2.7.

In the particular case in which γ is constant, that is, $\gamma(t) = R$ for all $t \in [0, T]$, with $R > 0$, we obtain the following multiplicity result [14].

Corollary 3.14. *Let $A, B \in (0, R)$ and $p \in \mathbb{N}$ be such that*

$$p \geq \frac{T}{R} \|M\|_{L^1} + 1.$$

Then there exist at least two solutions of

$$x'' = f(t, x), \quad \textit{for a.a. } t \in I := [0, T], \ \textit{such that } x(t) \in (0, R), \tag{3.28}$$

$$x'(s+) = -x'(s-), \quad \textit{if } s \in (0, T), \ x(s) \in \{0, R\}, \tag{3.29}$$

$$x(0) = A, \quad x(T) = B, \tag{3.30}$$

having exactly p impacts.

Remark 3.15. If the right-hand side f is identically equal to zero, the problem (3.28)-(3.30) becomes trivially solvable (it is a “classical” billiard table with uniform linear motion of the ball). In this case, there exists exactly one solution with no impact, and for each positive integer p there exist exactly two solutions having exactly p impacts with the boundary. In general, the existence of impact-free solution is not guaranteed. For example, let us put

$$\gamma(t) = R, \ t \in [0, T], \quad f(t, x) = a > 0, \ (t, x) \in [0, T] \times (0, R), \ A, B = R/2. \tag{3.31}$$

Then the function

$$x(t) = \frac{a}{2} t^2 - \frac{aT}{2} t + \frac{R}{2}, \quad t \in [0, T]$$

is the only function $x \in W^{2,1}([0, T])$ satisfying $x''(t) = a$ for a.e. $t \in [0, T]$ and $x(0) = R/2$, $x(T) = R/2$. Its values stay in the interval $(0, R)$ if and only if

$$x\left(\frac{T}{2}\right) = -\frac{a}{2}\left(\frac{T}{2}\right)^2 + \frac{R}{2} > 0.$$

Therefore if $R \leq a(T/2)^2$, then the problem (3.28)-(3.30) for (3.31) doesn’t have any impact-free solution.

Let us note that if moreover $(3 + 2\sqrt{2})R \leq a(T/2)^2$, then the problem has no solution with exactly one impact.

3.2. The case of a nonlinear γ

Some existence result concerning (3.11)–(3.14) can be obtained even in the case in which γ is nonlinear.

Lemma 3.16. *Let $A, B \in \mathbb{R}$ and assume that there exists $N \in \mathbb{N}$ such that*

$$M(t) \leq N\gamma''(t) \quad \textit{for a.a. } t \in I. \tag{3.32}$$

Then the modified problem (3.16), (3.14) has at least one Carathéodory solution.

Proof. Let $C := \max\{|A|, |B|\} + N \max_{t\in[0,T]} \gamma(t)$ and consider the auxiliary problem

$$x'' = \tilde{f}(t,x), \quad t \in I, \quad x(0) = A, \; x(T) = B, \tag{3.33}$$

where

$$\tilde{f}(t,y) = \begin{cases} f^*(t,C), & \text{if } y > C, \\ f^*(t,y), & \text{if } |y| \le C, \\ f^*(t,-C), & \text{if } y < -C. \end{cases}$$

Note that since f is a Carathéodory function, then

- for all $y \in \mathbb{R}$, $\tilde{f}(\cdot, y)$ is measurable on I;
- there exists $\tilde{M} \in L^1(I)$ such that for a.a. $t \in I$ and all $y \in \mathbb{R}$ we have $\left|\tilde{f}(t,y)\right| \le \tilde{M}(t)$;

and thus $\tilde{f}$ satisfies conditions (C_1) and (C_2). Moreover, observe that for a.a. $t \in I$,

$$\tilde{f}(t,\cdot) \text{ is continuous in } [-C,C] \setminus \bigcup_{j\in\mathbb{Z}} \{j\gamma(t)\}.$$

Clearly, $\tilde{f}(t,\cdot)$ is continuous in $\mathbb{R} \setminus [-C,C]$.

Hence, choose $\mathcal{C} = \emptyset$, $\mathcal{D} = \mathbb{Z}$, $\varphi_j : I \to \mathbb{R}$ the function defined as $\varphi_j(t) = j\gamma(t)$, $j \in \mathcal{D}$, and without loss of generality the subintervals $[\tilde{a}_j, \tilde{b}_j] = \varphi_j^{-1}([-C,C])$ (if $j\gamma^{-1}([-C,C])$ is not connected just consider as many functions φ_j as components have). Since

$$\varphi_j''(t) = j\gamma''(t) = f^*(t, j\gamma(t)) = \tilde{f}(t, \varphi_j(t)), \quad t \in [\tilde{a}_j, \tilde{b}_j],$$

we deduce that $\tilde{f}$ satisfies condition (C_3^*). Therefore, Theorem 2.8 guarantees that (3.33) has a Carathéodory solution.

Let us see that if y is a Carathéodory solution of the modified problem (3.33), then $|y(t)| \le C$ for all $t \in I$. To prove it, note that condition (3.32) implies that $f^*(t,y) \ge 0$ if $y > N\gamma(t)$ and $f^*(t,y) \le 0$ if $y < -N\gamma(t)$. Then, since $C \ge N \max_{t\in[0,T]} \gamma(t)$, it follows that

$$\tilde{f}(t,y) \ge 0 \;\; \text{if } y > C \;\; \text{and} \;\; \tilde{f}(t,y) \le 0 \;\; \text{if } y < -C.$$

Assume, to the contrary, that there exists $t_0 \in (0,R)$ such that

$$y(t_0) = \max_{t\in[0,T]} y(t) > C$$

and $y(t_0) > y(t)$ for all $t \in (t_0, R)$. By the continuity of y, we deduce that there exists $r > 0$ such that $y(t) > C$ for all $t \in (t_0 - r, t_0 + r)$ and so

$$y''(t) = \tilde{f}(t, y(t)) > 0 \qquad \text{for a.a. } t \in (t_0 - r, t_0 + r).$$

By integration, taking into account that $y'(t_0) = 0$, we have

$$y'(t) = \int_{t_0}^{t} y''(s)\,ds = \int_{t_0}^{t} \tilde{f}(s, y(s))\,ds > 0 \qquad \text{for } t \in (t_0, t_0 + r).$$

Hence, y is increasing in the interval $(t_0, t_0 + r)$, which contradicts the choice of t_0. Analogously, we can show that $y(t) > -C$ for all $t \in I$.

Therefore, if y is a Carathéodory solution of the modified problem (3.33), then it is also a Carathéodory solution of (3.16), (3.14). □

Remark 3.17. Clearly, condition (3.32) is satisfied if f is a continuous function in $E_{0,\gamma}$ and γ is a two times continuously differentiable function such that $\gamma''(t) > 0$ for all $t \in I$.

Condition (3.32) provides "a priori" bounds for the solutions of the modified problem (3.16), (3.14). Our reasoning goes in the line of [10] and reminds the method of lower and upper solutions. Indeed, the constant functions $\alpha \equiv -C$ and $\beta \equiv C$ can be seen, respectively, as a lower and an upper solution for (3.16), (3.14).

As a direct consequence of Lemmas 3.6 and 3.16, we derive the following existence result for the time-dependent billiard problem (3.11)–(3.14).

Theorem 3.18. *Assume that there exists $N \in \mathbb{N}$ such that condition (3.32) holds. Then the impulsive problem (3.11)–(3.14) has at least one solution.*

Theorem 3.19. *Let $M \in L^1(I)$ be such that for a.a. $t \in I$ and $x \in [\alpha(t), \beta(t)]$ we have (3.25). Moreover, assume that there exists $N \in \mathbb{N}$ such that*

$$M(t) \leq N(\beta''(t) - \alpha''(t)) \quad \textit{for a.a. } t \in I.$$

Then the impulsive problem (1.1)–(1.3) has at least one solution.

Proof. As in the proof of Theorem 3.12, we consider the problem (3.11)–(3.14) with (3.15) and $A := A - \alpha(0)$, $B := B - \alpha(T)$ and use Lemma 3.3 together with Theorem 3.18. □

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