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RESEARCH

On Commutative Tensor Factors of Group Algebras

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Abstract
We prove that any tensor product factorization with a commutative tensor factor of a modular group algebra over a prime field comes from a direct product decomposition of the group basis. This extends previous work by Carlson and Kovács for the commutative case and answers one of their questions in certain cases.

Keywords Group algebras · Tensor factors · Positive characteristic

Mathematics Subject Classification (2010) 16S34 · 20C05 · 16L30

The aim of this paper is to investigate how the group algebra kG of a group G over a field k may decompose as a tensor product of subalgebras. The most obvious factorizations come from direct product decompositions of G, but in general, these are not the only ones. For example, if k has an element of multiplicative order 4, then $kC_4 \cong k(C_2 \times C_2) \cong kC_2 \otimes_k kC_2$, even though C_4 is directly indecomposable. More elaborate examples in the semisimple case are provided at the end of this paper. However, no tensor product factorization is known that does not come from a direct product decomposition of the group basis in the modular case, i.e. when G is a finite p-group with p the characteristic of k.

If it were true that every tensor product factorization of a modular group algebra comes from a direct product decomposition of the group basis, then there would be a Krull–Schmidt-type theorem for tensor product factorizations of modular group algebras.Observe

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that modular group algebras are finite-dimensional, local and augmented; and for artinian local augmented algebras of characteristic zero, such Krull–Schmidt property holds [5].

A first attempt to study tensor product factorizations of modular group algebras was carried out by J. F. Carlson and L. G. Kovács, who proved that for commutative modular group algebras any tensor product factorization comes from a direct product decomposition of the group basis. More precisely, they proved the following theorem.

Theorem 1 [2, Theorem 2.2] *Let p be a prime, k a field of characteristic p and G a finite abelian p-group. To each tensor product factorization $kG = \bigotimes_{i=1}^{n} A_i$ of kG, there is a direct product decomposition $G = \prod_{i=1}^{n} G_i$ such that for each i:*

(1) $A_i \cong kG_i$ and
(2) $kG = A_i \otimes \left(\bigotimes_{j \neq i} kG_j\right)$.

We continue this study by proving the following result, which extends the first item in Theorem 1 for group algebras over the prime field (see also Remark 7).

Theorem 2 *Let G be a finite p-group, let $\mathbb{F}_p$ be the field with p elements, and suppose that $\mathbb{F}_p G = B \otimes_{\mathbb{F}_p} C$ for some subalgebras B and C of $\mathbb{F}_p G$ such that B is commutative. Then $G = \mathcal{B} \times \mathcal{C}$ for some subgroups $\mathcal{B}$ and $\mathcal{C}$ of G such that $\mathbb{F}_p\mathcal{B} \cong B$ and $\mathbb{F}_p\mathcal{C} \cong C$.*

As the "simplest" question that arises from Theorem 1, Carlson and Kovács asked the following:

Question 1 [2, Section 5] Does the group algebra of a directly indecomposable p-group over a field of characteristic p ever admit a nontrivial tensor product factorization (as algebras)?

In [2, Section 5] they, in an initial attempt to tackle this question in the noncommutative case, answered their question for the nonabelian groups of order 8. To the best of our knowledge, no further progress on this topic has been made since. Their argument relies on the product formula of centers $\mathrm{Z}(B \otimes C) = \mathrm{Z}(B) \otimes \mathrm{Z}(C)$. Another product formula $[B \otimes C, B \otimes C] = B \otimes [C, C] + [B, B] \otimes C$ as submodules of $B \otimes C$ may be used instead here (see [8]). As an application of Theorem 2, we give an answer to the Carlson–Kovács question for groups generated by at most three elements and for groups with cyclic derived subgroup.

Corollary 3 *Let G be a finite directly indecomposable p-group and let $\mathbb{F}_p$ be the field with p elements. Then $\mathbb{F}_p G$ is indecomposable as a tensor product of proper subalgebras provided at least one of the following holds:*

(1) G can be generated by three elements,
(2) G' is cyclic.

Observe that our result includes many finite p-groups, such as those that are metacyclic, extraspecial, of maximal class, or of order at most p^5.

We start by setting basic notation. Throughout, p is a prime number, k is a field of characteristic p, G is a finite p-group and C_n denotes a cyclic group of order n. All tensor products are taken over the coefficient field unless otherwise specified.

Let A be a k-algebra. The group of units of A is denoted by $\mathcal{U}(A)$. If X and Y are subsets of a k-algebra A, then XA denotes the right ideal of A generated by X and $[X, Y]$ denotes the k-span of the set of Lie commutators $[x, y] = xy - yx$ with $x \in X$ and $y \in Y$.

We use the notation $\mathrm{Z}(G)$ for the center of G, and for a positive integer i,

$$\Omega_i(G) = \left\langle g \in G : g^{p^i} = 1 \right\rangle, \quad \mho_i(G) = \left\langle g^{p^i} : g \in G \right\rangle$$

and

$$R_i(G) = \frac{\Omega_i(\mathrm{Z}(G))\mho_i(G)G'}{\mho_i(G)G'}.$$

We also use analogous versions of the first three notations in the context of algebras, namely, for a not necessarily unital k-algebra A, $\mathrm{Z}(A)$ denotes the center of A, $\Omega_i(A)$ denotes the subalgebra of A generated by the elements $a \in A$ such that $a^{p^i} = 0$, and $\mho_i(A)$ denotes the subalgebra of A generated by the elements of the form a^{p^i} with $a \in A$.

Let $H_i(G)$ denote a homocyclic component of G of exponent p^i, in the sense of [4], i.e. $H_i(G)$ is maximal among the direct factors of G that are isomorphic to a direct product of cyclic groups of order p^i. This is unique up to isomorphism and we abuse notation by referring to it as "the" homocyclic component of G.

Suppose that A is an augmented algebra with augmentation ideal $I(A)$, i.e. $I(A)$ is an ideal of A of codimension 1. We write $V(A) = \mathcal{U}(A) \cap (1 + I(A))$, a normal subgroup of $\mathcal{U}(A)$. If $I(A)$ is nilpotent, then $V(A)$ is a p-group of exponent p^s, where s is the smallest nonnegative integer with $a^{p^s} = 0$ for every $a \in I(A)$. Moreover, every subalgebra B of A is augmented with augmentation ideal $I(B) = B \cap I(A)$, and if $A = B \otimes C$ with B and C subalgebras of A, then

$$A = k \oplus I(B) \oplus I(C) \oplus I(B)I(C) \quad \text{and} \quad I(A) = I(B) \oplus I(C) \oplus I(B)I(C). \qquad (*)$$

For example, the group algebra kG is an augmented algebra and its augmentation ideal is its Jacobson radical, which is nilpotent and generated by the elements of the form $g - 1$ with $g \in G$. We abbreviate $I(kG)$ by $I(G)$. Observe that if N is a normal subgroup of G, then $I(N)kG$ is the kernel of the natural homomorphism $kG \to k(G/N)$.

The proof of Theorem 2 relies on the following three results. The first two are valid for arbitrary coefficient fields of characteristic p, but the third, which is taken from [4], is only known for $k = \mathbb{F}_p$.

Lemma 4 *For every pair of positive integers i and j we have*

(1) $I(\mho_i(G)\, G')kG = \mho_i(I(G))kG + I(G')kG.$
(2) $I(\Omega_i(\mathrm{Z}(G))G')kG = \Omega_i(\mathrm{Z}(I(G)))kG + I(G')kG.$
(3) $I(\Omega_i(\mathrm{Z}(G))\, \mho_j(G)\, G')kG = \Omega_i(\mathrm{Z}(I(G)))kG + \mho_j(I(G))kG + I(G')kG.$
(4) G has a cyclic direct factor of order p^i if and only if

$$\exp\left(1 + I\left(R_i(G)\right) k \left(\frac{G}{\mho_i(G)G'}\right)\right) \geq p^i.$$

Proof (1) is [9, Lemma 2.6] and (2) is a particular case of [9, Lemma 2.3]. (3) Let $\pi : kG \to kG/(I(\mho_j(G))kG + I(G')kG)$ be the natural homomorphism and $\lambda\colon k\mathrm{Z}(G) \to k\mathrm{Z}(G)$ be the map given by $\lambda(x) = x^{p^i}$. By [11, Lemma 6.10],

$$\mathrm{Z}(I(G)) = I(\mathrm{Z}(G)) \oplus [kG, kG] \cap \mathrm{Z}(kG)$$

and, by [12, Proposition III.6.1],

$$\ker(\lambda) = I(\Omega_i(\mathrm{Z}(G)))k\mathrm{Z}(G).$$

Thus

$$\pi(\Omega_i(Z(I(G)))) = \pi(\Omega_i(I(\mathrm{Z}(G)) \oplus [kG, kG] \cap \mathrm{Z}(kG))) = \pi(\Omega_i(I(\mathrm{Z}(G))) \oplus \Omega_i([kG, kG] \cap \mathrm{Z}(k(G))))$$
$$= \pi(\Omega_i(I(\mathrm{Z}(G)))) = \pi(\ker \lambda) = \pi(I(\Omega_i(\mathrm{Z}(G)))k\mathrm{Z}(G)),$$

and the result follows.

(4) By [4, Lemma 4.3], $H_i(G) \cong H_i(R_i(G))$. Observe that the group $R_i(G)$ is abelian of exponent at most p^i. Hence G has an abelian direct factor of exponent p^i if and only if $\exp(R_i(G)) \geq p^i$. Now, since $G/\mho_i(G)G'$ is abelian, we have that

$$\exp\left(1 + I\left(R_i(G)\right) k\left(\frac{G}{\mho_i(G)G'}\right)\right) = \exp(R_i(G)).$$

□

Proposition 5 *Suppose that $kG = B \otimes C$, where B and C are proper subalgebras of kG with B commutative. Then*

(a) $[kG, kG] = [I(C), I(C)] + [I(C), I(C)]I(B) \subseteq I(C) \oplus I(B)I(C)$.
(b) *B, $C/[C, C]C$ and $kG/[C, C]kG$ are group algebras.*
(c) *If p^s is the exponent of $V(B)$, then $I(\mho_s(G)G')kG \subseteq I(C) \oplus I(B)I(C)$.*
(d) *G has a cyclic direct factor of order* $\exp(V(B))$.

Proof (a) Since B is central in kG, by Eq. (∗),

$$[kG, kG] = [I(C), I(C)] + [I(C), I(B)I(C)] + [I(B)I(C), I(B)I(C)]$$
$$= [I(C), I(C)] + [I(C), I(C)]I(B) \subseteq I(C) \oplus I(B)I(C).$$

(b) It is well known that $k(G/G') \cong kG/I(G')kG$ and $[kG, kG]kG = I(G')kG$. On the other hand, there is a natural epimorphism $B \otimes C \to B \otimes (C/[C, C]C)$ given by $b \otimes c \mapsto b \otimes (c + [C, C]C)$. Since $[kG, kG] = B \otimes [C, C]$, we have $[kG, kG]kG = [C, C]kG$. Thus $k(G/G') \cong kG/[C, C]kG \cong B \otimes (C/[C, C]C)$. Then the result follows from Theorem 1.

(c) Let p^s be the exponent of $V(B)$. Since B is central in kG and $I(C) \oplus I(B)I(C)$ is an ideal of kG, $z^{p^s} \in I(C) \oplus I(B)I(C)$ for every $z \in I(G)$. This, (a) and Lemma 4(1) show that $I(\mho_s(G)G')kG \subseteq I(C) \oplus I(B)I(C)$.

(d) Let $x \in I(B)$ such that $1 + x$ has order p^s, the exponent of $V(B)$. By Eq. (∗) and (c), $B \cap I(\mho_s(G)G')kG = 0$, so

$$|1 + x + I(\mho_s(G)G')kG| = p^s.$$

On the other hand,

$$x \in \{z \in \mathrm{Z}(kG) \cap I(G) : z^{p^s} = 0\} \subseteq I(\Omega_s(\mathrm{Z}(G))\mho_s(G)G')kG,$$

by Lemma 4(3), so

$$1 + x + I(\mho_s(G)G')kG \in 1 + \frac{I(\Omega_s(\mathrm{Z}(G))\mho_s(G)G')kG}{I(\mho_s(G)G')kG} \cong 1 + I\left(R_s(G)\right) k\left(\frac{G}{\mho_s(G)G'}\right).$$

Thus the latter group has an element of order p^s, hence Lemma 4(4) yields the result. □

In the remainder of the paper $k = \mathbb{F}_p$ and consequently $I(H)$ denotes the augmentation ideal of $\mathbb{F}_p H$, for each finite p-group H.

Lemma 6 [4, Lemma 4.11] *Let V be a subspace of $I(G)$ containing $I(G)^2$, and consider the map*

$$\Lambda_G^{s-1}: \frac{I(\Omega_s(\mathrm{Z}(G))G')\mathbb{F}_pG + I(G)^2}{I(G)^2} \to \frac{I(G)^{p^{s-1}} + I(\mho_s(G)G')\mathbb{F}_pG}{I(G)^{p^{s-1}+1} + I(\mho_s(G)G')\mathbb{F}_pG}$$

$$z + I(G)^2 \mapsto z^{p^{s-1}} + I(G)^{p^{s-1}+1} + I(\mho_s(G)G')\mathbb{F}_pG.$$

Then the following conditions are equivalent:

(1) There is a direct sum decomposition

$$\frac{I(\Omega_s(\mathrm{Z}(G))G')\mathbb{F}_pG + I(G)^2}{I(G)^2} = \frac{V}{I(G)^2} \oplus \ker \Lambda_G^{s-1}.$$

(2) There exists a decomposition $G = H \times K$ with H homocyclic of exponent p^s and K not admitting any direct product cyclic direct factor of order p^s such that

$$V = I(H)\mathbb{F}_pG + I(G)^2.$$

We are finally ready to prove the main theorem.

Proof of Theorem 2 By Proposition 5(b), there is a group basis $\mathcal{B}$ of B. Let p^s be the exponent of $\mathcal{B}$. Since B is commutative, $V(B)$ has exponent p^s too. As B is central in $\mathbb{F}_pG$, Lemma 4(2) yields $I(B) \subseteq I(\Omega_s(\mathrm{Z}(G))G')\mathbb{F}_pG$.

Let $b \in \mathcal{B}$ be an element of order p^s in $\mathcal{B}$. So $\mathcal{B} = \langle b \rangle \times \mathcal{B}_0$ for some subgroup $\mathcal{B}_0$ of $\mathcal{B}$. Then $b-1 \in I(B) \setminus I(B)^2$ and $(b-1)^{p^{s-1}} \in I(B)^{p^{s-1}} \setminus I(B)^{p^{s-1}+1}$, by Jennings' Theorem (see [7] or [10, Theorem 11.1.20]).

We claim that $(b-1)^{p^{s-1}} \notin I(G)^{p^{s-1}+1} + I(\mho_s(G)G')\mathbb{F}_pG$. Indeed, the decomposition $I(G) = I(B) \oplus I(C) \oplus I(B)I(C)$ yields a surjective algebra homomorphism $\pi : I(G) \to I(B)$ with kernel $I(C) \oplus I(B)I(C)$ and $\pi(I(G)^m) = I(B)^m$ for any positive integer m. If $(b-1)^{p^{s-1}} \in I(G)^{p^{s-1}+1} + I(\mho_s(G)G')\mathbb{F}_pG$, then there is $b_1 \in I(B)^{p^{s-1}+1}$ such that $(b-1)^{p^{s-1}} - b_1 \in I(C) \oplus I(B)I(C) + I(\mho_s(G)G')\mathbb{F}_pG \subseteq I(C) \oplus I(B)I(C)$, by Proposition 5(c). Then $(b-1)^{p^{s-1}} = b_1 \in I(B)^{p^{s-1}+1}$, in contradiction with the previous paragraph.

Then $b - 1 + I(G)^2 \notin \ker \Lambda_G^{s-1}$. Therefore there is a subspace V of $I(G)$ containing $b-1$ and $I(G)^2$ such that

$$\frac{I(\Omega_s(\mathrm{Z}(G))G')\mathbb{F}_pG + I(G)^2}{I(G)^2} = \frac{V}{I(G)^2} \oplus \ker \Lambda_G^{s-1}.$$

Then Lemma 6 yields that $G = H \times K$, with H homocyclic of exponent p^s and $I(H)\mathbb{F}_pG + I(G)^2 = V$. Let $\Phi(G)$ denote the Frattini subgroup of G. As $g\Phi(G) \mapsto g - 1 + I(G)^2$ defines an isomorphism from $G/\Phi(G)$ to the additive group of $I(G)/I(G)^2$ [12, Proposition III.1.15], there is $h \in H$ such that $h - 1 + I(G)^2 = b - 1 + I(G)^2$. Then $h - 1 + I(G)^2 \notin \ker \Lambda_G^{s-1}$ and therefore h has order p^s. Thus $G = \langle h \rangle \times G_0$ for some subgroup G_0 of G.

Let J be the ideal of $\mathbb{F}_pG$ generated by $b-1$. Thus $J + I(G)^2 = I(\langle h \rangle)\mathbb{F}_pG + I(G)^2$. Write $C_0 = \mathbb{F}_p\mathcal{B}_0 \otimes C$, so

$$\mathbb{F}_pG = \mathbb{F}_p\langle b \rangle \otimes C_0 = \mathbb{F}_p \oplus I(\langle b \rangle) \oplus I(C_0) \oplus I(\langle b \rangle)I(C_0) = C_0 \oplus J.$$

Thus the codimension of J in $\mathbb{F}_pG$ is $\dim(C_0) = |G|/|b| = |G|/|h| = |G_0|$. Now it follows from [4, Lemma 4.9] that

$$\mathbb{F}_pG = J \oplus \mathbb{F}_pG_0.$$

Thus $C_0 \cong \mathbb{F}_p G_0$. Proceeding by induction on the size of $\mathcal{B}$, we derive that $C \cong \mathbb{F}_p \mathcal{C}$ for some subgroup $\mathcal{C}$ of G such that $G \cong \mathcal{B} \times \mathcal{C}$. This finishes the proof of Theorem 2. □

Remark 7 Observe that the assumption $k = \mathbb{F}_p$ has been used to apply Lemma 6 and to use the isomorphism $G/\Phi(G) \cong I(G)/I(G)^2$. We wonder whether Theorem 2 holds for arbitrary coefficient fields of characteristic p.

We prove the two statements of Corollary 3 separately.

Corollary 8 *Let G be a finite directly indecomposable p-group. If G can be generated by three elements, then $\mathbb{F}_p G$ is indecomposable as tensor product of proper subalgebras.*

Proof Suppose that a nontrivial tensor product factorization $\mathbb{F}_p G = B \otimes C$ exists. As $I(B) \cap I(G)^2 = I(B)^2$, and similarly for C,

$$I(G)/I(G)^2 = (I(B) + I(G)^2)/I(G)^2 \oplus (I(C) + I(G)^2)/I(G)^2 \cong I(B)/I(B)^2 \oplus I(C)/I(C)^2.$$

Moreover, $I(B)^2 \neq I(B)$, and similarly for C, so the two summands are nontrivial. As G can be generated by three elements, $|G/\Phi(G)| = |I(G)/I(G)^2| \leq p^3$, and hence one of the summands in the above decomposition has dimension 1, say the first one. Then $B = B_0 + I(B)^2$ for any unital algebra B_0 generated by one element in $I(B) \setminus I(B)^2$. Then $B = B_0$, by [6, Proposition 5.2]. So applying Theorem 2 we derive that B is isomorphic to some group algebra $\mathbb{F}_p H$, where H is a direct factor of G, a contradiction. □

Corollary 9 *Let G be a finite directly indecomposable p-group. If G' is cyclic, then $\mathbb{F}_p G$ is indecomposable as tensor product of proper subalgebras.*

Proof Suppose by contradiction that $\mathbb{F}_p G = B \otimes C$, with both B and C nontrivial subalgebras of $\mathbb{F}_p G$. As G' is cyclic, it follows from [12, Proposition III.1.15(ii)] that $I(G')\mathbb{F}_p G/I(G')I(G)$ is one-dimensional. Moreover, $I(G')\mathbb{F}_p G = [I(G), I(G)]\mathbb{F}_p G$ and hence $I(G')\mathbb{F}_p G/I(G')I(G)$ is spanned by any nonzero element of the form $[a_1, a_2] + I(G')I(G)$ with $a_1, a_2 \in I(G)$. Since $I(G) = I(B) \oplus I(C) \oplus I(B)I(C)$, $[I(G), I(G)^2] \subseteq I(G')I(G)$ and $[I(B), I(C)] = 0$, we can take a_1 and a_2 as above, with both in $I(B)$ or both in $I(C)$. Without loss of generality, assume that they belong to $I(C)$. Then $I(G')\mathbb{F}_p G = [a_1, a_2]\mathbb{F}_p G \subseteq I(C)\mathbb{F}_p G = I(C) \oplus I(B)I(C)$. It follows that $[I(B), I(B)] \subseteq I(B) \cap I(G')\mathbb{F}_p G \subseteq I(B) \cap (I(C) \oplus I(B)I(C)) = 0$, so B is commutative. Hence G is decomposable by Theorem 2, a contradiction. □

We wonder whether the commutativity assumptions in both Theorem 2 and Proposition 5(b) are really needed.

Question 2 Suppose that $\mathbb{F}_p G = B \otimes C$.

(1) Is B a group algebra?
(2) Is $\mathbb{F}_p G/[C, C]\mathbb{F}_p G$ a group algebra?

Actually, using Theorem 2 it is easy to see that the two questions are equivalent. Indeed, if the answer to the first question is positive, then B and C are group algebras and hence so are $C/[C, C]C$ and $B \otimes (C/[C, C]) \cong \mathbb{F}_p G/[C, C]\mathbb{F}_p G$. Conversely, if the second question has a positive answer, then $\mathbb{F}_p G/[C, C]\mathbb{F}_p G$ is a group algebra, which is isomorphic to $B \otimes (C/[C, C]C)$. Hence, by Theorem 2, B is also a group algebra.

In the non-modular case over fields, it is easy to find nontrivial tensor product factorizations of group algebras of directly indecomposable groups. For example, if q is a divisor of $p-1$, then all the group algebras of abelian groups of order q over $\mathbb{F}_p$ are isomorphic. In particular, if $q = r^n$ with r prime and $n \geq 2$, then C_q is directly indecomposable while $\mathbb{F}_p C_q$ has nontrivial tensor product factorizations. This example also shows that Theorem 2 fails in the non-modular case. Observe that the tensor factors are group algebras since the example is built upon the failure of the isomorphism problem in this setting, even existing a directly indecomposable group and a decomposable one with isomorphic group algebras. This may suggest that the tensor factors of group algebras should be group algebras again, and in that case group algebras of indecomposable groups with positive answer for the isomorphism problem should be tensor indecomposable. Unfortunately this is not the case. For example, the isomorphism problem has a positive solution for rational group algebras of groups of order 16. This can be seen by simply examining the following Wedderburn decompositions of the rational group algebras of the nonabelian groups of order 16, where Γ_i represents the i-th group in the GAP Small Groups Library [3]:

$$\begin{aligned}
\mathbb{Q}\Gamma_3 &= 4\mathbb{Q} \oplus 2\mathbb{Q}(\sqrt{-1}) \oplus 2M_2(\mathbb{Q}),\\
\mathbb{Q}\Gamma_4 &= 4\mathbb{Q} \oplus 2\mathbb{Q}(\sqrt{-1}) \oplus 2\mathbb{H}(\mathbb{Q}),\\
\mathbb{Q}\Gamma_6 &= 4\mathbb{Q} \oplus 2\mathbb{Q}(\sqrt{-1}) \oplus M_2(\mathbb{Q}(\sqrt{-1})),\\
\mathbb{Q}\Gamma_7 &= 4\mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus M_2\mathbb{Q}(\sqrt{2}),\\
\mathbb{Q}\Gamma_8 &= 4\mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus M_2\mathbb{Q}(\sqrt{-2}),\\
\mathbb{Q}\Gamma_9 &= 4\mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus \mathbb{H}(\mathbb{Q}(\sqrt{2})),\\
\mathbb{Q}\Gamma_{11} &= 8\mathbb{Q} \oplus 2M_2(\mathbb{Q}),\\
\mathbb{Q}\Gamma_{12} &= 8\mathbb{Q} \oplus 2\mathbb{H}(\mathbb{Q}),\\
\mathbb{Q}\Gamma_{13} &= 8\mathbb{Q} \oplus M_2(\mathbb{Q}(\sqrt{-1})).
\end{aligned}$$

These Wedderburn decompositions can be computed with the command `WedderburnDecompositionInfo` of the GAP package Wedderga [1]. The first two decompositions show that

$$\begin{aligned}
\mathbb{Q}\Gamma_3 &\cong (2\mathbb{Q}) \otimes_{\mathbb{Q}} (2\mathbb{Q} \oplus \mathbb{Q}(\sqrt{-1}) \oplus M_2(\mathbb{Q})) \text{ and}\\
\mathbb{Q}\Gamma_4 &\cong (2\mathbb{Q}) \otimes_{\mathbb{Q}} (2\mathbb{Q} \oplus \mathbb{Q}(\sqrt{-1}) \oplus \mathbb{H}(\mathbb{Q}))
\end{aligned}$$

While $2\mathbb{Q} \cong \mathbb{Q}C_2$, neither of the right-hand tensor factors is isomorphic to a group algebra, because the Wedderburn decompositions of the nonabelian groups of order 8 are

$$\mathbb{Q}D_8 = 4\mathbb{Q} \oplus M_2(\mathbb{Q}) \quad \text{and} \quad \mathbb{Q}Q_8 = 4\mathbb{Q} \oplus \mathbb{H}(\mathbb{Q}).$$

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Declarations

Competing interests The authors declare no competing interests.

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