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Boundary value problems for nonlinear second-order functional differential equations with piecewise constant arguments

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In this paper, we consider a class of nonlinear second-order functional differential equations with piecewise constant arguments with applications to a thermostat that is controlled by the introduction of functional terms in the temperature and the speed of change of the temperature at some fixed instants. We first prove some comparison results for boundary value problems associated to linear delay differential equations that allow to give a priori bounds for the derivative of the solutions, so that we can control not only the values of the solutions but also their rate of change. Then, we develop the method of upper and lower solutions and the monotone iterative technique in order to deduce the existence of solutions in a certain region (and find their approximations) for a class of boundary value problems, which include the periodic case. In the approximation process, since the sequences of the derivatives for the approximate solutions are, in general, not monotonic, we also give some estimates for these derivatives. We complete the paper with some examples and conclusions.

KEYWORDS

boundary value problems, monotone iterative technique, piecewise constant functional dependence, second-order functional differential equations, upper and lower solutions

MSC CLASSIFICATION

34K10, 34K07, 34K05, 34K12

1 | INTRODUCTION

In the study of nonlinear differential equations, the existence of solutions is sometimes achieved through the development of the method of upper and lower solutions, and the approximation of the extremal solutions in the functional interval defined by those functions is performed by the monotone iterative technique. A fundamental reference on monotone method is Ladde et al[1] (see also previous works[2–6]). To mention some other related works, the application of the monotone method to functional differential equations can be found in Nieto et al,[7] and second-order periodic boundary value problems were considered in Cabada and Nieto.[8]

Dedicated to the memory of Professor J.A. Tenreiro Machado.

The existence of solution to second-order functional differential equations with a functional dependence given by a piecewise constant argument has attracted the attention of many authors. We cite, for instance, previous works.[9–13] For the mentioned works, the delay is given by the integer part function but the nonlinearity in the equation is independent of x'.

Chen and Sun[14] considered a class of linear boundary value problems for nonlinear second-order impulsive functional differential equations with continuous delay function, and they applied the upper and lower solutions method and the monotone iterative technique to obtain the existence of solution. However, the nonlinearity in the equation is also independent of the derivative of x. On the other hand, the class of functional differential equations considered in Corduneanu[15] presents a linear dependence on x'. Some other recent works include a nonlinearity depending on x' but avoid the introduction of delay in the equation.

In Guo and Guo,[16] the authors studied the existence and multiplicity of periodic solutions for a class of second-order delay differential equations with no explicit dependence on x'. More recently, some results based on Avery–Peterson fixed point theorem were provided in Shen et al[17] for a thermostat model including the first-order derivative in the nonlinearity.

Other references relevant to the topic are, for instance, the article by Henríquez and Hernández,[18] which was devoted to the analysis of the approximate controllability of control systems given by second-order semilinear functional differential equations with infinite delay; the work by Sakthivel et al,[19] which is focused on the study of the exact controllability of certain second-order nonlinear impulsive control differential systems; the study of Shoukaku,[20] about the oscillatory behavior of certain hyperbolic equations with continuous distributed deviating arguments; or Liu and Huang,[21] where the coincidence degree theory was applied to obtain results on the existence and uniqueness of T-periodic solutions for a class of second-order neutral functional differential equations. The same approach, coincidence degree theory, was applied in[22] to analyze the existence of periodic solutions for higher-order differential equations with deviating arguments. In this last case, the dependence on the different derivatives $x^{(i)}$ is linear.

In particular, in Zhang,[13] the authors considered the following periodic boundary value problem for second-order functional differential equations:

$$\begin{cases} -x''(t) = f(t, x(t), x([t])), \; t \in J = [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

while previous studies[23,24] were focused on the analysis of similar problems for the case of first-order differential equations.

On the other hand, Nieto and Rodríguez-López[25] presented some results on the study of the existence of solution to second-order functional differential equations with piecewise constant arguments of the type

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma(t), \; t \in J = [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) + \lambda, \end{cases} \tag{1}$$

by proving the existence of solutions to the following linear impulsive periodic boundary value problems:

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = 0, \; t \in \mathbb{R}, \\ x(0) = x(T), \\ x'(0^-) = x'(T^+), \\ x'(s^+) = x'(s^-) + 1, \end{cases} \tag{2}$$

where $s \in J$, $a, b, c, d, \lambda \in \mathbb{R}$, $T > 0$ and σ is a piecewise continuous function. More specifically, it was illustrated how the solution to (1) can be obtained by means of a Green's function which is given by the solution of (2) (see also Yang et al[26]). Further, in Buedo-Fernández et al,[27] conditions were provided in order to prove the existence of solutions to (1) with a constant sign, deducing comparison results which will be useful to prove in this paper the existence of solutions to nonlinear second-order functional differential equations with piecewise constant arguments, for which the nonlinearity depends on the x' term and the delay is also introduced in the derivative.

Our main motivations for the study of this problem are, on one hand, its applicability to the modeling of a thermostat including the dependence on the first-derivative of the state variable, similarly to the one considered in Shen et al,[17] but controlled through the introduction of functional terms in x and x', and on the other hand but also related, the need of determining the behavior of the solutions for such models controlled by the temperature and the speed of change of the temperature at some fixed instants.

Some other models for thermostats have been studied in previous studies[28,29] (see also the study in Webb[30]). For some models with this type of applications from the perspective of fractional calculus, we refer to previous works,[31,32] and also Rezapour et al[33] as example of variable order fractional thermostat models.

The paper is organized as follows. In Section 2, we present the problem of study and recall the comparison results for second-order functional differential equations extracted from Buedo-Fernández et al[27] that will be useful to our purposes, and then, in Section 3, we provide results on the existence of solution for nonlinear second-order functional problems with boundary value conditions by using the upper and lower solutions' method. In Section 4, we include the development of the monotone iterative technique, and finally, in Sections 5 and 6, we present, respectively, some examples and conclusions.

2 | PRELIMINARIES

We consider the problem

$$\begin{cases} x''(t) = g(t, x(t), x'(t), x([t]), x'([t])), \ t \in J = [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) + \lambda, \end{cases} \tag{3}$$

where $\lambda \in \mathbb{R}$, and $g : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous on $(J \backslash \{1, 2, \dots, [T]\}) \times \mathbb{R}^4$ and such that the following limits are finite:

$$\lim_{t \to n^-} g(t, x, y, u, v), \ g(n, x, y, u, v) = \lim_{t \to n^+} g(t, x, y, u, v), \qquad n \in \{1, 2, \dots, [T]\}.$$

This kind of problems is useful in the study of phenomena which are self-regulated at fixed equidistant instants, for instance, at the positive integer numbers. Note that the consideration of this problem allows to consider functional dependence on the derivative x', hence the context is more general than that in other previous works.

We denote by $C(J)$ the space of continuous functions defined on J

$$C(J) = \{u : J \to \mathbb{R} \ : \ u \text{ continuous}\},$$

furnished with the supremum norm.

Definition 1 (Definition 2 of Nieto and Rodríguez-López[25]). Consider the spaces

$$\Lambda := \{y : J \to \mathbb{R} \ : \ y \text{ continuous on } J \backslash \{1, 2, \dots, [T]\}, \text{ and there exist } y(n^-) \in \mathbb{R}, \ y(n^+) = y(n), \ \forall n \in \{1, 2, \dots, [T]\}\}$$

and

$$E := \{x : J \to \mathbb{R} \ : \ x, x' \text{ are continuous and } x'' \in \Lambda\}.$$

Definition 2. A function x is a solution to (3) if $x \in E$ and satisfies the conditions in (3).

Definition 3 (Nieto and Rodríguez-López[25]). Similarly, a function x is a solution to (1) if $x \in E$ and satisfies the conditions in (1), where $x''(n) = x''(n^+)$, $\forall n \in \{0, 1, 2, \dots, [T]\}$.

In the sequel, I denotes the identity mapping,

$$H(z) := \begin{pmatrix} h_1(z) & h_2(z) \\ h_1'(z) & h_2'(z) \end{pmatrix}, \text{for } z \in [0, 1], \qquad \text{and} \qquad C := H(1) = \begin{pmatrix} C_1 & C_2 \\ C_1' & C_2' \end{pmatrix},$$

where $h_1(s)$ is given by

$$
\begin{array}{l}
1-\frac{d}{a}s+\frac{d}{a^2}(1-e^{-as}),\ \text{ if } b=0,\ a\neq 0,\\
1-\frac{d}{2}s^2,\ \text{ if } b=0,\ a=0,\\
\left(1+\frac{d}{b}\right)\left(1+\frac{a}{2}s\right)e^{-\frac{a}{2}s}-\frac{d}{b},\ \text{ if } b\neq 0,\ a^2=4b,\\
\left(1+\frac{d}{b}\right)\frac{\beta e^{\alpha s}-\alpha e^{\beta s}}{\beta-\alpha}-\frac{d}{b},\ \text{ if } b\neq 0,\ a^2>4b,\\
\left(1+\frac{d}{b}\right)e^{-\frac{a}{2}s}\left\{\cos\sqrt{b-\frac{a^2}{4}}s+\frac{a}{2\sqrt{b-\frac{a^2}{4}}}\sin\sqrt{b-\frac{a^2}{4}}s\right\}-\frac{d}{b},\ \text{ if } b\neq 0,\ a^2<4b,
\end{array}
$$

and $h_2(s)$ is given by

$$
\begin{array}{l}
\frac{1}{a}\left(1-e^{-as}-cs+\frac{c}{a}(1-e^{-as})\right),\ \text{ if } b=0,\ a\neq 0,\\
s-\frac{c}{2}s^2,\ \text{ if } b=0,\ a=0,\\
e^{-\frac{a}{2}s}\left[\frac{c}{b}\left(1+\frac{a}{2}s\right)+s\right]-\frac{c}{b},\ \text{ if } b\neq 0,\ a^2=4b,\\
\frac{\left(\frac{\beta c}{b}-1\right)e^{\alpha s}+\left(1-\frac{\alpha c}{b}\right)e^{\beta s}}{\beta-\alpha}-\frac{c}{b},\ \text{ if } b\neq 0,\ a^2>4b,\\
e^{-\frac{a}{2}s}\left\{\frac{c}{b}\cos\sqrt{b-\frac{a^2}{4}}s+\frac{1+\frac{ac}{2b}}{\sqrt{b-\frac{a^2}{4}}}\sin\sqrt{b-\frac{a^2}{4}}s\right\}-\frac{c}{b},\ \text{ if } b\neq 0,\ a^2<4b.
\end{array}
$$

Here, for $b\neq 0,\ a^2>4b$, we denote

$$\alpha=-\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b},\quad \beta=-\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b}. \tag{4}$$

Note that, according to the previous notation,

$$H(T-[T])=\begin{pmatrix} h_1(T-[T]) & h_2(T-[T])\\ h_1'(T-[T]) & h_2'(T-[T])\end{pmatrix}.$$

Theorem 1. *(Theorem 3.2 of Nieto and Rodríguez-López[25]). If hypothesis*

$$det\left(I-H(T-[T])C^{[T]}\right)\neq 0 \tag{5}$$

holds, then problem (1) has a unique solution, for all $\sigma\in\Lambda$ (see Definition 1) and $\lambda\in\mathbb{R}$, which can be obtained by the expression

$$x(t)=\int_0^T K(t,s)\sigma(s)\,ds+\lambda K(t,0),\ t\in J, \tag{6}$$

where, for all $s\in J$, $K(\cdot,s)$ is the unique solution to (2).

See Nieto and Rodríguez-López[25] for the expression of the Green's function K in Theorem 1.

We also fix the notation $\mathbb{N}$ for the set of positive integer numbers and $\mathbb{Z}^+=\mathbb{N}\cup\{0\}$ for the set of nonnegative integer numbers.

Moreover, the following comparison results are useful in the proof of the main results. Consider the set

$$\mathcal{S} := \{(x,y) \in \mathbb{R}^2 \; : \; x \geq 0, \; y \geq -Mx\},$$

where

$$M := \inf_{z \in (0,1)} \frac{h_1(z)}{h_2(z)}.$$

Define also

$$g(z) := \begin{cases} \frac{1-e^{-az}}{a}, & \text{if } b=0,\ a \neq 0, \\ z, & \text{if } b=0,\ a=0, \\ z e^{-\frac{a}{2}z}, & \text{if } b \neq 0,\ a^2 = 4b, \\ \frac{e^{\beta z}-e^{\alpha z}}{\beta-\alpha}, & \text{if } b \neq 0,\ a^2 > 4b, \\ e^{-\frac{a}{2}z} \frac{\sin\sqrt{b-\frac{a^2}{4}}z}{\sqrt{b-\frac{a^2}{4}}}, & \text{if } b \neq 0,\ a^2 < 4b. \end{cases} \tag{7}$$

Theorem 2 (Theorem of Buedo-Fernández et al[27]). *Suppose that the hypothesis (5) holds. Assume that $\sigma \in \Lambda$ is nonnegative on J, $\lambda \geq 0$, and that the following conditions hold:*

(I) $h_1(1) > 0$, and $h_2 > 0$ on $(0,1)$.
(II) The vector V^0 given by

$$V^0 := \left[I - H(T-[T])\; C^{[T]}\right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

satisfies that $C^k\, V^0 \in \mathcal{S}$ for every $k = 0, 1, \ldots, [T]$.
(III) The function g given by (7) is nonnegative on $(0,1)$.
(IV) For each $0 < s < T$ with $s \in (n, n+1)$ for some $n \in \mathbb{Z}^+$, the vector $V_{0,s}$ given, for $T < n+1$, by

$$V_{0,s} := \left[I - H(T-[T])\; C^{[T]}\right]^{-1} \begin{pmatrix} g(T-s) \\ g'(T-s) \end{pmatrix},$$

and for $n+1 \leq T$, by

$$V_{0,s} := \left[I - H(T-[T])\; C^{[T]}\right]^{-1} H(T-[T])\; C^{[T]-n-1} \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix},$$

satisfies that $C^k V_{0,s} \in \mathcal{S}$, for every $k = 0, 1, \ldots, [T]$.
(V) For each $0 < s < T$ with $s \in (n, n+1)$ for some $n \in \mathbb{Z}^+$, if $T \geq n+1$, we also assume that the vector

$$V_{1,s} := \left[C^{n+1} V_{0,s} + \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix}\right] \qquad (\text{with } V_{0,s} \text{ given in IV}),$$

satisfies that $C^k V_{1,s} \in \mathcal{S}$, for every $k = 0, 1, \ldots, [T]-n-1$.

Then the unique solution to problem (1) is nonnegative on J.

Theorem 3 (Theorem 10 of Buedo-Fernández et al[27]). *Suppose that the hypothesis (5) holds. Assume also that the conditions (I)–(V) in Theorem 2 are satisfied. If $\sigma \in \Lambda$ is nonpositive on J and $\lambda \leq 0$, then the unique solution to problem (1) is nonpositive on J.*

Remark 1 (Remark 6 of Buedo-Fernández et al[27]). Condition (I) in Theorems 2–3 is satisfied under the following circumstances:

Case $b = 0$, $a \neq 0$: $1 - \frac{d}{a} + \frac{d}{a^2}(1-e^{-a}) > 0$, and one of the following conditions holds:

- $c = 0$; or
- $c \neq 0$, and $a + c \leq 0$; or
- $c \neq 0$, $a + c > 0$, and $\frac{(a+c)e^{-a}-c}{a} \geq 0$; or
- $c \neq 0$, $a + c > 0$, $\frac{(a+c)e^{-a}-c}{a} < 0$, and $\frac{1}{a}\left(1 - e^{-a} - c + \frac{c}{a}(1-e^{-a})\right) \geq 0$.

Case $a = b = 0$: $d < 2$, and one of the following conditions holds:

- $c \le 1$; or
- $c > 1$, and $c \le 2$.

Case $b \neq 0$, $a^2 = 4b$: $\left(1+\frac{d}{b}\right)\left(1+\frac{a}{2}\right)e^{-\frac{a}{2}} - \frac{d}{b} > 0$, and one of the following conditions holds:

- $\frac{a}{2} + c \le 1$; or
- $\frac{a}{2} + c > 1$, and $e^{-\frac{a}{2}}\left[\frac{c}{b}\left(1+\frac{a}{2}\right)+1\right] - \frac{c}{b} \ge 0$.

Case $b \neq 0$, $a^2 > 4b$: $\left(1+\frac{d}{b}\right)\frac{\beta e^{\alpha}-\alpha e^{\beta}}{\beta-\alpha} - \frac{d}{b} > 0$, and one of the following conditions holds:

- $a + 2c \le \sqrt{a^2-4b}\left(1+\frac{2}{e^{\sqrt{a^2-4b}}-1}\right)$; or
- $a + 2c > \sqrt{a^2-4b}\left(1+\frac{2}{e^{\sqrt{a^2-4b}}-1}\right)$, and $\frac{\left(\frac{\beta c}{b}-1\right)e^{\alpha}+\left(1-\frac{\alpha c}{b}\right)e^{\beta}}{\beta-\alpha} - \frac{c}{b} \ge 0$.

Case $b \neq 0$, $a^2 < 4b$: $\left(1+\frac{d}{b}\right)e^{-\frac{a}{2}}\left\{\cos\hat{R} + \frac{a}{2\hat{R}}\sin\hat{R}\right\} - \frac{d}{b} > 0$, and one of the following conditions holds:

- $\hat{R} \le \frac{\pi}{2}$ and $\frac{a}{2} + c \le 0$; or
- $\hat{R} < \frac{\pi}{2}$, and $0 < \frac{a}{2} + c \le \hat{R}\cot\left(\hat{R}\right)$; or
- $\frac{\pi}{2} < \hat{R} < \pi$, and $\frac{a}{2} + c \le \hat{R}\cot\left(\hat{R}\right) < 0$; or
- $e^{-\frac{a}{2}}\left\{\frac{c}{b}\cos\hat{R} + \frac{1+\frac{ac}{2b}}{\hat{R}}\sin\hat{R}\right\} - \frac{c}{b} \ge 0$, and one of the following restrictions holds:
  - ⋆ $\hat{R} = \frac{\pi}{2}$ and $\frac{a}{2} + c > 0$; or
  - ⋆ $\hat{R} < \frac{\pi}{2}$ and $\frac{a}{2} + c > \hat{R}\cot\left(\hat{R}\right)$; or
  - ⋆ $\hat{R} \in (\frac{\pi}{2}, \pi]$, and $\frac{a}{2} + c > 0$; or
  - ⋆ $\hat{R} \in \left(\pi, \frac{3\pi}{2}\right)$, and $0 < \frac{a}{2} + c \le \hat{R}\cot\left(\hat{R}\right)$; or
  - ⋆ $\frac{3\pi}{2} \ge \hat{R} \ge \pi$, and $\frac{a}{2} + c < 0$; or
  - ⋆ $\frac{\pi}{2} < \hat{R} < \pi$, and $0 > \frac{a}{2} + c > \hat{R}\cot\left(\hat{R}\right)$;
- $e^{-\frac{a}{2}s_1}\left\{\frac{c}{b}\cos\hat{R}s_1 + \frac{1+\frac{ac}{2b}}{\hat{R}}\sin\hat{R}s_1\right\} - \frac{c}{b} > 0$ and one of the following restrictions holds:
  - ⋆ $\hat{R} \in \left(\pi, \frac{3\pi}{2}\right)$, and $\frac{a}{2} + c > \hat{R}\cot\left(\hat{R}\right) > 0$; or
  - ⋆ $\hat{R} = \frac{3\pi}{2}$, and $\frac{a}{2} + c > 0$.
- $\frac{a}{2} + c = 0$, and $\hat{R} \le \frac{\pi}{2}$.

where $\hat{R} := \sqrt{b - \frac{a^2}{4}} > 0$.

Remark 2 (Remark 7 of Buedo-Fernández et al.[27]). Condition (III) in Theorems 2–3 is satisfied if one of the following conditions holds:

- $b = 0$.
- $b \neq 0$, $a^2 \ge 4b$.
- $b \neq 0$, $a^2 < 4b$, and $\hat{R} \le \pi$.

We present a new comparison result which will also be useful in the proof of the main results.

Lemma 1. *Suppose that $x \in C([0,T])$ satisfies that $x' \in \Lambda$ and*

$$\begin{aligned} &x'(t) + Lx(t) + Fx([t]) \le \sigma(t), \\ &x(0) \le x(T), \end{aligned} \tag{9}$$

where $L \neq 0$, $F \in \mathbb{R}$, and assume that

$$e^{-Ls} - \frac{F}{L}(1 - e^{-Ls}) \geq 0, \ \forall s \in (0, 1], \tag{10}$$

and

$$\left(e^{-L} - \frac{F}{L}\left(1 - e^{-L}\right)\right)^{[T]} \left(e^{-L(T-[T])} - \frac{F}{L}\left(1 - e^{-L(T-[T])}\right)\right) < 1. \tag{11}$$

Then, for $t \in [m, m+1) \cap [0, T]$, with $m = 0, \dots, [T]$, we get

$$\begin{aligned} x(t) \leq \frac{1}{1-A} &\left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{F}{L}\left(1 - e^{-L}\right)\right)^{[T]-j} \int_{j-1}^{j} \sigma(s) e^{L(s-j)}\, ds \left(e^{-L(T-[T])} - \frac{F}{L}\left(1 - e^{-L(T-[T])}\right)\right) \right. \\ &+ \left. \int_{[T]}^{T} \sigma(s) e^{L(s-T)}\, ds \right) \left(e^{-L} - \frac{F}{L}\left(1 - e^{-L}\right)\right)^{m} \left(e^{-L(t-m)} - \frac{F}{L}\left(1 - e^{-L(t-m)}\right)\right) \\ &+ \sum_{j=1}^{m} \left(e^{-L} - \frac{F}{L}\left(1 - e^{-L}\right)\right)^{m-j} \int_{j-1}^{j} \sigma(s) e^{L(s-j)}\, ds \left(e^{-L(t-m)} - \frac{F}{L}\left(1 - e^{-L(t-m)}\right)\right) \\ &+ \int_{m}^{t} \sigma(s) e^{L(s-t)}\, ds, \end{aligned}$$

where $A := \left(e^{-L} - \frac{F}{L}\left(1 - e^{-L}\right)\right)^{[T]} \left(e^{-L(T-[T])} - \frac{F}{L}\left(1 - e^{-L(T-[T])}\right)\right)$.

Proof. For $t \in [j, j+1)$, $j = 0, \dots, [T]$, we have

$$x'(t) + Lx(t) + Fx(j) \leq \sigma(t),$$

which implies that

$$(x(t)e^{L(t-j)})' \leq (\sigma(t) - Fx(j))e^{L(t-j)}.$$

Then, by integrating between j and t, we have, for $t \in [j, j+1)$, $j = 0, \dots, [T]$,

$$x(t)e^{L(t-j)} - x(j) \leq \int_{j}^{t} (\sigma(s) - Fx(j))e^{L(s-j)}\, ds;$$

thus, we obtain

$$x(t)e^{L(t-j)} \leq x(j) - Fx(j) \int_{j}^{t} e^{L(s-j)}\, ds + \int_{j}^{t} \sigma(s) e^{L(s-j)}\, ds$$

and

$$x(t) \leq x(j) \left(1 - \frac{F}{L}\left(e^{L(t-j)} - 1\right)\right) e^{-L(t-j)} + \int_{j}^{t} \sigma(s) e^{L(s-t)}\, ds.$$

In particular, for $j = 0, \dots, [T]$, by the continuity of x,

$$x(j+1) \leq x(j) \left(1 - \frac{F}{L}\left(e^{L} - 1\right)\right) e^{-L} + \int_{j}^{j+1} \sigma(s) e^{L(s-j-1)}\, ds.$$

By using the estimate on the constants, taking $j=0$, we get

$$x(1) \leq x(0)\left(1-\frac{F}{L}\left(e^{L}-1\right)\right)e^{-L}+\int_{0}^{1}\sigma(s)e^{L(s-1)}\,ds,$$

and for $j=1$,

$$\begin{aligned} x(2) &\leq x(1)\left(1-\frac{F}{L}\left(e^{L}-1\right)\right)e^{-L}+\int_{1}^{2}\sigma(s)e^{L(s-2)}\,ds \\ &\leq x(0)\left(1-\frac{F}{L}\left(e^{L}-1\right)\right)^{2}e^{-2L}+\left(1-\frac{F}{L}\left(e^{L}-1\right)\right)e^{-L}\int_{0}^{1}\sigma(s)e^{L(s-1)}\,ds+\int_{1}^{2}\sigma(s)e^{L(s-2)}\,ds. \end{aligned}$$

By induction, it is easy to check that, for $k=1,\ \dots\ ,[T]$,

$$\begin{aligned} x(k) &\leq x(0)\left(1-\frac{F}{L}\left(e^{L}-1\right)\right)^{k}e^{-kL}+\sum_{j=1}^{k}\left(1-\frac{F}{L}\left(e^{L}-1\right)\right)^{k-j}e^{-(k-j)L}\int_{j-1}^{j}\sigma(s)e^{L(s-j)}\,ds \\ &= x(0)\left(e^{-L}-\frac{F}{L}\left(1-e^{-L}\right)\right)^{k}+\sum_{j=1}^{k}\left(e^{-L}-\frac{F}{L}\left(1-e^{-L}\right)\right)^{k-j}\int_{j-1}^{j}\sigma(s)e^{L(s-j)}\,ds. \end{aligned}$$

Hence,

$$x([T]) \leq x(0)\left(e^{-L}-\frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]}+\sum_{j=1}^{[T]}\left(e^{-L}-\frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]-j}\int_{j-1}^{j}\sigma(s)e^{L(s-j)}\,ds$$

and for $t\in[[T],T]$, we obtain

$$\begin{aligned} x(t) &\leq x([T])e^{-L(t-[T])}\left(1-\frac{F}{L}\left(e^{L(t-[T])}-1\right)\right)+\int_{[T]}^{t}\sigma(s)e^{L(s-t)}\,ds \\ &= x([T])\left(e^{-L(t-[T])}-\frac{F}{L}\left(1-e^{-L(t-[T])}\right)\right)+\int_{[T]}^{t}\sigma(s)e^{L(s-t)}\,ds. \end{aligned}$$

This proves that

$$x(T) \leq x([T])\left(e^{-L(T-[T])}-\frac{F}{L}\left(1-e^{-L(T-[T])}\right)\right)+\int_{[T]}^{T}\sigma(s)e^{L(s-T)}\,ds.$$

Therefore,

$$\begin{aligned} x(0) \leq\ & x(T) \\ \leq\ & x(0)\left(e^{-L}-\frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]}\left(e^{-L(T-[T])}-\frac{F}{L}\left(1-e^{-L(T-[T])}\right)\right) \\ &+\sum_{j=1}^{[T]}\left(e^{-L}-\frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]-j}\int_{j-1}^{j}\sigma(s)e^{L(s-j)}\,ds\left(e^{-L(T-[T])}-\frac{F}{L}\left(1-e^{-L(T-[T])}\right)\right) \\ &+\int_{[T]}^{T}\sigma(s)e^{L(s-T)}\,ds. \end{aligned}$$

By recalling the expression of A, the previous inequality implies that

$$(1-A)x(0) \leq \sum_{j=1}^{[T]} \left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]-j} \int_{j-1}^{j} \sigma(s)e^{L(s-j)}\,ds \left(e^{-L(T-[T])} - \frac{F}{L}\left(1-e^{-L(T-[T])}\right)\right) + \int_{[T]}^{T} \sigma(s)e^{L(s-T)}\,ds.$$

Since condition (11) is written as $A < 1$, then, for $m = 0, \ldots, [T]$ and $t \in [m, m+1) \cap [0, T]$, we obtain

$$\begin{aligned}
x(t) &\leq x(m)\left(e^{-L(t-m)} - \frac{F}{L}\left(1-e^{-L(t-m)}\right)\right) + \int_{m}^{t} \sigma(s)e^{L(s-t)}\,ds \\
&\leq x(0)\left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{m}\left(e^{-L(t-m)} - \frac{F}{L}\left(1-e^{-L(t-m)}\right)\right) \\
&\quad + \sum_{j=1}^{m} \left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{m-j} \int_{j-1}^{j} \sigma(s)e^{L(s-j)}\,ds \left(e^{-L(t-m)} - \frac{F}{L}\left(1-e^{-L(t-m)}\right)\right) \\
&\quad + \int_{m}^{t} \sigma(s)e^{L(s-t)}\,ds \\
&\leq \frac{1}{1-A}\left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]-j} \int_{j-1}^{j} \sigma(s)e^{L(s-j)}\,ds \left(e^{-L(T-[T])} - \frac{F}{L}\left(1-e^{-L(T-[T])}\right)\right)\right. \\
&\quad \left. + \int_{[T]}^{T} \sigma(s)e^{L(s-T)}\,ds\right)\left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{m}\left(e^{-L(t-m)} - \frac{F}{L}\left(1-e^{-L(t-m)}\right)\right) \\
&\quad + \sum_{j=1}^{m} \left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{m-j} \int_{j-1}^{j} \sigma(s)e^{L(s-j)}\,ds \left(e^{-L(t-m)} - \frac{F}{L}\left(1-e^{-L(t-m)}\right)\right) \\
&\quad + \int_{m}^{t} \sigma(s)e^{L(s-t)}\,ds.
\end{aligned}$$

□

Remark 3. The proof of Lemma 1 has been written by assuming that $T \notin \mathbb{Z}$, so that $[T] < T$. However, there is no contradiction with the case $[T] = T$, where the inequalities are deduced on each interval $[m, m+1)$, for $m = 0, \ldots, [T]-1$. It is obvious that, if $T \in \mathbb{N}$, condition (11) is reduced to

$$A := \left(e^{-L} - \frac{F}{L}\left(1-e^{-L}\right)\right)^{[T]} < 1.$$

Remark 4. It is easily checked that condition (10) is satisfied for $F + L \leq 0$. Besides, (10) is valid for $F + L > 0$, by assuming that

$$e^{-L} - \frac{F}{L}\left(1-e^{-L}\right) \geq 0,$$

that is, (10) holds if

$$F + L > 0, \qquad \frac{F}{L}\left(e^{L}-1\right) \leq 1.$$

Remark 5. Estimate (11) is trivially valid if $L+F>0$. Indeed, the function

$$\varphi(s) := e^{-Ls} - \frac{F}{L}(1-e^{-Ls}) = e^{-Ls}\left(1+\frac{F}{L}\right) - \frac{F}{L}$$

is decreasing on $[0,1]$, since $\varphi'(s) = -(L+F)e^{-Ls} < 0$ and $\varphi(0)=1$. On the other hand, (11) is not fulfilled for $L+F \le 0$.

3 | UPPER AND LOWER SOLUTIONS METHOD

Definition 4. We say that a function $\alpha \in E$ is a lower solution to (3) if the following conditions are satisfied

$$\begin{cases} \alpha''(t) \le g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t])), \ t \in J=[0,T], \\ \alpha(0)=\alpha(T), \\ \alpha'(0) \le \alpha'(T)+\lambda. \end{cases} \tag{12}$$

Similarly, a function $\beta \in E$ is an upper solution to (3) if

$$\begin{cases} \beta''(t) \ge g(t,\beta(t),\beta'(t),\beta([t]),\beta'([t])), \ t \in J=[0,T], \\ \beta(0)=\beta(T), \\ \beta'(0) \ge \beta'(T)+\lambda. \end{cases} \tag{13}$$

In the sequel, we consider the following hypotheses:

- (H_1) There exist $\alpha, \beta \in E$, respectively, lower and upper solutions to problem (3), with $\alpha \le \beta$ on J.
- (H_2) There exist constants $a, b, c, d \in \mathbb{R}$ such that

$$\begin{aligned} &g(t,\beta(t),\beta'(t),\beta([t]),\beta'([t])) - g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t])) \\ &\ge -a(\beta'(t)-\alpha'(t)) - b(\beta(t)-\alpha(t)) - c(\beta'([t])-\alpha'([t])) - d(\beta([t])-\alpha([t])), \ t \in J. \end{aligned}$$

- (H_3) The constants $a, b, c, d \in \mathbb{R}$ are such that the hypothesis (5) and conditions (I)–(V) in Theorem 2 hold.
- (H_4) $\lambda, \mu \in \mathbb{R}$ are fixed with $\lambda \le \mu$.

Theorem 4. *Suppose that there exist $\alpha, \beta \in E$ such that condition (H_2) holds. Assume also that the hypotheses (H_3), and (H_4) are valid. If x fulfills*

$$x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma_\alpha(t), \ t \in J=[0,T],$$

$$x(0)=x(T), \qquad x'(0)=x'(T)+\lambda,$$

and y satisfies

$$y''(t) + ay'(t) + by(t) + cy'([t]) + dy([t]) = \sigma_\beta(t), \ t \in J=[0,T],$$

$$y(0)=y(T), \qquad y'(0)=y'(T)+\mu,$$

where

$$\sigma_\alpha(t) := g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t])) + a\alpha'(t) + b\alpha(t) + c\alpha'([t]) + d\alpha([t]),$$

and

$$\sigma_\beta(t) := g(t,\beta(t),\beta'(t),\beta([t]),\beta'([t])) + a\beta'(t) + b\beta(t) + c\beta'([t]) + d\beta([t]),$$

then $x \le y$ on J.

Proof. Let $w := x-y \in E$. Then, by using (H_2), it is easy to check that, for $t \in J$,

$$\begin{aligned} &w''(t) + aw'(t) + bw(t) + cw'([t]) + dw([t]) \\ &\quad = g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t])) + a\alpha'(t) + b\alpha(t) + c\alpha'([t]) + d\alpha([t]) \\ &\qquad - g(t,\beta(t),\beta'(t),\beta([t]),\beta'([t])) - a\beta'(t) - b\beta(t) - c\beta'([t]) - d\beta([t]) \le 0. \end{aligned}$$

Besides,

$$w(0) = w(T), \qquad w'(0) = x'(T) + \lambda - y'(T) - \mu = w'(T) + \lambda - \mu \leq w'(T).$$

Hence, by hypothesis (H_3) and the application of Theorem 3, we deduce that $w \leq 0$ on J, and thus, $x \leq y$ on J. □

We remark that, in this last theorem, β, α are not required to be, respectively, upper and lower solutions.

Lemma 2. *Assume that hypotheses (H_1) and (H_2) are valid. Suppose also that there exist $L \neq 0, R > 0$ such that $R + L = a$, $LR = b$, $Rc \leq d$,*

$$c + L > 0, \ \text{and} \ e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \geq 0, \tag{14}$$

and define the functions

$$k_1(t) := \alpha'(t) + R(\alpha(t) - \beta(t)) - K_m, \ t \in [m, m+1),$$

$$k_2(t) := \beta'(t) - R(\alpha(t) - \beta(t)) + K_m, \ t \in [m, m+1),$$

where

$$\begin{aligned} K_m := & \frac{(d - Rc)}{1 - A} \\ & \times \left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{[T]-j} \int_{j-1}^{j} (\beta([s]) - \alpha([s])) e^{L(s-j)}\, ds \left(e^{-L(T-[T])} - \frac{c}{L}\left(1 - e^{-L(T-[T])}\right) \right) \right. \\ & \left. + \int_{[T]}^{T} (\beta([s]) - \alpha([s])) e^{L(s-T)}\, ds \right) \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{m} \left(e^{-L(t-m)} - \frac{c}{L}\left(1 - e^{-L(t-m)}\right) \right) \\ & + \sum_{j=1}^{m} \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{m-j} \int_{j-1}^{j} (d - Rc)(\beta([s]) - \alpha([s])) e^{L(s-j)}\, ds \left(e^{-L(t-m)} - \frac{c}{L}\left(1 - e^{-L(t-m)}\right) \right) \\ & + \int_{m}^{t} (d - Rc)(\beta([s]) - \alpha([s])) e^{L(s-t)}\, ds, \end{aligned}$$

for $m = 0, 1, \ldots, S^*$, with S^* being the greatest nonnegative integer less than T (that is, $S^* = [T]$ if $T \notin \mathbb{Z}$, and $S^* = [T] - 1$ if $T \in \mathbb{Z}$), and A being provided in the statement of Lemma 1 taking $F = c$.

Then $K_m \geq 0$, for every m, and $k_1 \leq k_2$ on J.

Proof. Let $x(t) := \alpha'(t) - \beta'(t) + R(\alpha(t) - \beta(t))$, then

$$x(0) = \alpha'(0) - \beta'(0) + R(\alpha(0) - \beta(0)) \leq \alpha'(T) + \lambda - \beta'(T) - \lambda + R(\alpha(T) - \beta(T)) = x(T),$$

and

$$\begin{aligned} x'(t) + Lx(t) = & \alpha''(t) - \beta''(t) + R\alpha'(t) - R\beta'(t) + L\alpha'(t) - L\beta'(t) \\ & + LR\alpha(t) - LR\beta(t) = \alpha''(t) - \beta''(t) + a\alpha'(t) - a\beta'(t) + b\alpha(t) - b\beta(t), \ t \in J. \end{aligned}$$

Besides,

$$\begin{aligned} \alpha''(t) - \beta''(t) \leq & g(t, \alpha(t), \alpha'(t), \alpha([t]), \alpha'([t])) - g(t, \beta(t), \beta'(t), \beta([t]), \beta'([t])) \\ \leq & a(\beta'(t) - \alpha'(t)) + b(\beta(t) - \alpha(t)) + c(\beta'([t]) - \alpha'([t])) + d(\beta([t]) - \alpha([t])). \end{aligned}$$

Hence,

$$\begin{aligned} x'(t) + Lx(t) + cx([t]) \leq & a(\beta'(t) - \alpha'(t)) + b(\beta(t) - \alpha(t)) + c(\beta'([t]) - \alpha'([t])) + d(\beta([t]) - \alpha([t])) \\ & + a\alpha'(t) - a\beta'(t) + b\alpha(t) - b\beta(t) + c\alpha'([t]) - c\beta'([t]) + cR(\alpha([t]) - \beta([t])) \\ = & (d - cR)(\beta([t]) - \alpha([t])), \ t \in J. \end{aligned}$$

By Lemma 1, we deduce that $x(t) \le K_m$, for $t \in [m, m+1)$, that is, $\alpha'(t) - \beta'(t) + R(\alpha(t) - \beta(t)) \le K_m$, for $t \in [m, m+1)$, in consequence, for $t \in [m, m+1)$,

$$\begin{aligned} k_1(t) - k_2(t) &= \alpha'(t) - \beta'(t) + 2R(\alpha(t) - \beta(t)) - 2K_m \\ &\le R(\alpha(t) - \beta(t)) - K_m \le 0, \end{aligned}$$

since it is clear that the constants K_m are nonnegative. □

Remark 6. According to Remarks 4 and 5 and independently of the value of $T > 0$, estimates in (14) represent necessary and sufficient conditions for the validity of

$$e^{-Ls} - \frac{c}{L}(1 - e^{-Ls}) \ge 0, \ \forall s \in (0, 1],$$

simultaneously to

$$A := \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right)\right)^{[T]} \left(e^{-L(T-[T])} - \frac{c}{L}\left(1 - e^{-L(T-[T])}\right)\right) < 1.$$

Remark 7. In the hypotheses of Lemma 2, it is satisfied that $k_1 \le \alpha'$ and $\beta' \le k_2$ on J.

Theorem 5. *Suppose that $g : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous on $(J \setminus \{1, 2, \dots, [T]\}) \times \mathbb{R}^4$ and such that the following limits are finite*

$$\lim_{t \to n^-} g(t, x, y, u, v), \qquad g(n, x, y, u, v) = \lim_{t \to n^+} g(t, x, y, u, v).$$

Assume that condition (H_1) holds. Suppose, further, that the following condition is satisfied: (H_5) There exist constants $a, b, c, d \in \mathbb{R}$, $L \neq 0$, and $R > 0$ satisfying that $R + L = a$, $LR = b$, $Rc \le d$,

$$c + L > 0 \ \text{and} \ e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \ge 0,$$

such that

$$g(t, x, y, u, v) - g(t, \alpha(t), \alpha'(t), \alpha([t]), \alpha'([t])) \ge -a(y - \alpha'(t)) - b(x - \alpha(t)) - c(v - \alpha'([t])) - d(u - \alpha([t]))$$

and

$$g(t, \beta(t), \beta'(t), \beta([t]), \beta'([t])) - g(t, x, y, u, v) \ge -a(\beta'(t) - y) - b(\beta(t) - x) - c(\beta'([t]) - v) - d(\beta([t]) - u),$$

for $t \in J$, and $\alpha(t) \le x \le \beta(t)$, $k_1(t) \le y \le k_2(t)$, $\alpha([t]) \le u \le \beta([t])$, $k_1([t]) \le v \le k_2([t])$, where $k_1(t)$ and $k_2(t)$ are given in the statement of Lemma 2.

Assume also that the constants $a, b, c, d \in \mathbb{R}$ in (H_5) are such that the hypothesis (5) and conditions (I)–(V) in Theorem 2 hold (see condition (H_3)).

Then there exists (at least) one solution u to the second-order differential Equation (3) such that $\alpha \le u \le \beta$ and $k_1 \le u' \le k_2$ on J.

Proof. We consider the following modified problem:

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma_x(t), \ t \in J = [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) + \lambda, \end{cases} \tag{15}$$

where

$$\sigma_x(t) := g(t, p(t,x(t)), q(t,x'(t)), p([t],x([t])), q([t],x'([t]))) \\ + aq(t,x'(t)) + bp(t,x(t)) + cq([t],x'([t])) + dp([t],x([t])),$$

$$p(t,x) := \max\{\alpha(t), \min\{x, \beta(t)\}\} = \begin{cases} \alpha(t), & \text{if } x < \alpha(t), \\ x, & \text{if } \alpha(t) \le x \le \beta(t), \\ \beta(t), & \text{if } x > \beta(t), \end{cases}$$

and

$$q(t,y) := \max\{k_1(t), \min\{y, k_2(t)\}\} = \begin{cases} k_1(t), & \text{if } y < k_1(t), \\ y, & \text{if } k_1(t) \le y \le k_2(t), \\ k_2(t), & \text{if } y > k_2(t). \end{cases}$$

If $x \in E$ is such that $\alpha \le x \le \beta$ and $k_1 \le x' \le k_2$ on J, then x is a solution to (3) if and only if it is a solution to (15). We prove that problem (15) is solvable and that every solution to (15) satisfies that $\alpha \le x \le \beta$ and $k_1 \le x' \le k_2$ on J. Indeed, take $x \in E$ a solution to (15), then we check that $\alpha \le x$ and $k_1 \le x'$. Let $w := \alpha - x \in E$. Then

$$w(0) = \alpha(0) - x(0) = \alpha(T) - x(T) = w(T),$$

$$w'(0) = \alpha'(0) - x'(0) \le \alpha'(T) + \lambda - x'(T) - \lambda = \alpha'(T) - x'(T) = w'(T),$$

that is, $w'(0) = w'(T) + w'(0) - w'(T)$, where $w'(0) - w'(T) \le 0$. Further, by using (H_1), (H_5), and taking into account that $\alpha(t) \le p(t,x(t)) \le \beta(t)$, $k_1(t) \le q(t,x'(t)) \le k_2(t)$, $\alpha([t]) \le p([t],x([t])) \le \beta([t])$, $k_1([t]) \le q([t],x'([t])) \le k_2([t])$, for $t \in J$, we get

$$w''(t) + aw'(t) + bw(t) + cw'([t]) + dw([t]) \\ \le g(t, \alpha(t), \alpha'(t), \alpha([t]), \alpha'([t])) + a\alpha'(t) + b\alpha(t) + c\alpha'([t]) + d\alpha([t]) \\ - g(t, p(t,x(t)), q(t,x'(t)), p([t],x([t])), q([t],x'([t]))) \\ - aq(t,x'(t)) - bp(t,x(t)) - cq([t],x'([t])) - dp([t],x([t])) \le 0.$$

By the comparison result Theorem 3, $w \le 0$, thus $\alpha \le x$ on J. Similarly, we obtain that $x \le \beta$ on J. Take $w := x - \beta \in E$, then

$$w(0) = x(0) - \beta(0) = x(T) - \beta(T) = w(T),$$

$$w'(0) = x'(0) - \beta'(0) \le x'(T) + \lambda - \beta'(T) - \lambda = w'(T),$$

and by using (H_5),

$$w''(t) + aw'(t) + bw(t) + cw'([t]) + dw([t]) \\ \le g(t, p(t,x(t)), q(t,x'(t)), p([t],x([t])), q([t],x'([t]))) \\ + aq(t,x'(t)) + bp(t,x(t)) + cq([t],x'([t])) + dp([t],x([t])) \\ - g(t, \beta(t), \beta'(t), \beta([t]), \beta'([t])) - a\beta'(t) - b\beta(t) - c\beta'([t]) - d\beta([t]) \le 0, \ t \in J.$$

The comparison result Theorem 3 provides that $w \le 0$ on J; thus, $x \le \beta$ on J.

Next, we prove that $k_1 \le x'$ on J by using Lemma 1. Let $L \ne 0$, $R > 0$ be satisfying the conditions in the statement of the theorem and consider $y(t) := \alpha'(t) - x'(t) + R(\alpha(t) - x(t))$, $t \in J$, then, by using (H_1) and (H_5), we have

$$
\begin{aligned}
y'(t)+Ly(t) =\ & \alpha''(t)-x''(t)+R(\alpha'(t)-x'(t))+L(\alpha'(t)-x'(t))+LR(\alpha(t)-x(t))\\
=\ & \alpha''(t)-g(t,p(t,x(t)),q(t,x'(t)),p([t],x([t])),q([t],x'([t])))\\
& -aq(t,x'(t))-bp(t,x(t))-cq([t],x'([t]))-dp([t],x([t]))\\
& +ax'(t)+bx(t)+cx'([t])+dx([t])\\
& +R\alpha'(t)-Rx'(t)+L\alpha'(t)-Lx'(t)+LR\alpha(t)-LRx(t)\\
\le\ & g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t]))-g(t,p(t,x(t)),q(t,x'(t)),p([t],x([t])),q([t],x'([t])))\\
& -aq(t,x'(t))-bp(t,x(t))-cq([t],x'([t]))-dp([t],x([t]))\\
& +cx'([t])+dx([t])+R\alpha'(t)+L\alpha'(t)+LR\alpha(t)\\
\le\ & a(q(t,x'(t))-\alpha'(t))+b(p(t,x(t))-\alpha(t))\\
& +c(q([t],x'([t]))-\alpha'([t]))+d(p([t],x([t]))-\alpha([t]))\\
& -aq(t,x'(t))-bp(t,x(t))-cq([t],x'([t]))-dp([t],x([t]))\\
& +cx'([t])+dx([t])+a\alpha'(t)+b\alpha(t)\\
=\ & c(x'([t])-\alpha'([t]))+d(x([t])-\alpha([t])),\ \ t\in J,
\end{aligned}
$$

and hence, for $t \in J$,

$$
\begin{aligned}
y'(t)+Ly(t)+cy([t]) &\le c(x'([t])-\alpha'([t]))+d(x([t])-\alpha([t]))+c\left(\alpha'([t])-x'([t])+R(\alpha([t])-x([t]))\right)\\
&=(Rc-d)(\alpha([t])-x([t]))=\sigma(t).
\end{aligned}
$$

Besides, since $\alpha'(0) \le \alpha'(T)+\lambda$, $x'(0)=x'(T)+\lambda$, and $\alpha'(0)-x'(0) \le \alpha'(T)+\lambda-x'(T)-\lambda$, then

$$
y(0)=\alpha'(0)-x'(0)+R(\alpha(0)-x(0)) \le \alpha'(T)-x'(T)+R(\alpha(T)-x(T))=y(T).
$$

By using Lemma 1, we obtain that, for $t \in [m, m+1)$,

$$
\begin{aligned}
y(t) \le\ & \frac{(Rc-d)}{1-A}\\
& \times\left(\sum_{j=1}^{[T]}\left(e^{-L}-\frac{c}{L}\left(1-e^{-L}\right)\right)^{[T]-j}\int_{j-1}^{j}(\alpha([s])-x([s]))e^{L(s-j)}\,ds\left(e^{-L(T-[T])}-\frac{c}{L}\left(1-e^{-L(T-[T])}\right)\right)\right.\\
& \left.+\int_{[T]}^{T}(\alpha([s])-x([s]))e^{L(s-T)}\,ds\right)\left(e^{-L}-\frac{c}{L}\left(1-e^{-L}\right)\right)^{m}\left(e^{-L(t-m)}-\frac{c}{L}\left(1-e^{-L(t-m)}\right)\right)\\
& +\sum_{j=1}^{m}\left(e^{-L}-\frac{c}{L}\left(1-e^{-L}\right)\right)^{m-j}\int_{j-1}^{j}(Rc-d)(\alpha([s])-x([s]))e^{L(s-j)}\,ds\left(e^{-L(t-m)}-\frac{c}{L}\left(1-e^{-L(t-m)}\right)\right)\\
& +\int_{m}^{t}(Rc-d)(\alpha([s])-x([s]))e^{L(s-t)}\,ds,
\end{aligned}
$$

where A is given in the statement of Lemma 1 with $F=c$. Note that conditions $e^{-L}-\frac{c}{L}\left(1-e^{-L}\right) \ge 0$ and $c+L>0$ imply that function $\phi(s) := e^{-Ls}-\frac{c}{L}(1-e^{-Ls})$ is decreasing on $[0,1]$ and nonnegative; hence, $\phi(T-[T]) \ge 0$ and $\phi(t-m) \ge 0$. Besides, since $\alpha(s)-\beta(s) \le \alpha(s)-x(s) \le 0$, for every $s \in [0,T]$, and $Rc-d \le 0$, we have that

$$
\begin{aligned}
y(t) \le & \frac{(Rc-d)}{1-A} \\
& \times \left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{c}{L}\left(1-e^{-L}\right)\right)^{[T]-j} \int_{j-1}^{j} (\alpha([s]) - \beta([s]))e^{L(s-j)}\,ds \left(e^{-L(T-[T])} - \frac{c}{L}\left(1-e^{-L(T-[T])}\right)\right) \right. \\
& \left. + \int_{[T]}^{T} (\alpha([s]) - \beta([s]))e^{L(s-T)}\,ds \right) \left(e^{-L} - \frac{c}{L}\left(1-e^{-L}\right)\right)^{m} \left(e^{-L(t-m)} - \frac{c}{L}\left(1-e^{-L(t-m)}\right)\right) \\
& + \sum_{j=1}^{m} \left(e^{-L} - \frac{c}{L}\left(1-e^{-L}\right)\right)^{m-j} \int_{j-1}^{j} (Rc-d)(\alpha([s]) - \beta([s]))e^{L(s-j)}\,ds \left(e^{-L(t-m)} - \frac{c}{L}\left(1-e^{-L(t-m)}\right)\right) \\
& + \int_{m}^{t} (Rc-d)(\alpha([s]) - \beta([s]))e^{L(s-t)}\,ds, \quad \text{for } t \in [m, m+1),
\end{aligned}
$$

that is,

$$
y(t) = \alpha'(t) - x'(t) + R(\alpha(t) - x(t)) \le K_m, \quad \text{for } t \in [m, m+1),
$$

which implies that

$$
k_1(t) = \alpha'(t) + R(\alpha(t) - \beta(t)) - K_m \le \alpha'(t) + R(\alpha(t) - x(t)) - K_m \le x'(t),
$$

for every $t \in [m, m+1)$ and every m.

To prove that $x' \le k_2$ on J, we apply Lemma 1 again to the function

$$
y(t) := x'(t) - \beta'(t) + R(x(t) - \beta(t)), \ t \in J,
$$

which satisfies

$$
y(0) = x'(0) - \beta'(0) + R(x(0) - \beta(0)) \le x'(T) + \lambda - \beta'(T) - \lambda + R(x(T) - \beta(T)) = y(T).
$$

Moreover,

$$
\begin{aligned}
y'(t) + Ly(t) =& x''(t) - \beta''(t) + R(-\beta'(t) + x'(t)) + L(x'(t) - \beta'(t)) + LR(x(t) - \beta(t)) \\
\le& -ax'(t) - bx(t) - cx'([t]) - dx([t]) + g(t, p(t, x(t)), q(t, x'(t)), p([t], x([t])), q([t], x'([t]))) \\
& + aq(t, x'(t)) + bp(t, x(t)) + cq([t], x'([t])) + dp([t], x([t])) \\
& - g(t, \beta(t), \beta'(t), \beta([t]), \beta'([t])) - R\beta'(t) + Rx'(t) + Lx'(t) - L\beta'(t) - LR\beta(t) + LRx(t) \\
\le& -ax'(t) - bx(t) - cx'([t]) - dx([t]) + a(\beta'(t) - q(t, x'(t))) + b(\beta(t) - p(t, x(t))) \\
& + c(\beta'([t]) - q([t], x'([t]))) + d(\beta([t]) - p([t], x([t]))) \\
& + aq(t, x'(t)) + bp(t, x(t)) + cq([t], x'([t])) + dp([t], x([t])) \\
& - R\beta'(t) + Rx'(t) + Lx'(t) - L\beta'(t) - LR\beta(t) + LRx(t) \\
=& c(\beta'([t]) - x'([t])) + d(\beta([t]) - x([t])), \ t \in J;
\end{aligned}
$$

hence,

$$
\begin{aligned}
y'(t) + Ly(t) + cy([t]) \le& c(\beta'([t]) - x'([t])) + d(\beta([t]) - x([t])) \\
& + cx'([t]) - c\beta'([t]) + cRx([t]) - cR\beta([t]) = (d - cR)(\beta([t]) - x([t])).
\end{aligned}
$$

By applying Lemma 1, we get, for $t \in [m, m+1)$,

$$
\begin{aligned}
y(t) \leq\ & \frac{(d-Rc)}{1-A} \\
& \times \left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{c}{L}\left(1-e^{-L}\right) \right)^{[T]-j} \int_{j-1}^{j} (\beta([s]) - x([s])) e^{L(s-j)}\, ds \left(e^{-L(T-[T])} - \frac{c}{L}\left(1 - e^{-L(T-[T])}\right) \right) \right. \\
& \left. + \int_{[T]}^{T} (\beta([s]) - x([s])) e^{L(s-T)}\, ds \right) \left(e^{-L} - \frac{c}{L}\left(1-e^{-L}\right) \right)^{m} \left(e^{-L(t-m)} - \frac{c}{L}\left(1-e^{-L(t-m)}\right) \right) \\
& + \sum_{j=1}^{m} \left(e^{-L} - \frac{c}{L}\left(1-e^{-L}\right) \right)^{m-j} \int_{j-1}^{j} (d-Rc)(\beta([s]) - x([s])) e^{L(s-j)}\, ds \left(e^{-L(t-m)} - \frac{c}{L}\left(1-e^{-L(t-m)}\right) \right) \\
& + \int_{m}^{t} (d-Rc)(\beta([s]) - x([s])) e^{L(s-t)}\, ds.
\end{aligned}
$$

Since all the coefficients in the last expression are greater than or equal to zero and

$$0 \leq \beta(s) - x(s) \leq \beta(s) - \alpha(s), \text{for every } s \in [0, T],$$

we obtain that

$$y(t) = x'(t) - \beta'(t) + R(x(t) - \beta(t)) \leq K_m, \quad \text{for } t \in [m, m+1);$$

therefore,

$$x'(t) \leq \beta'(t) - R(x(t) - \beta(t)) + K_m \leq \beta'(t) - R(\alpha(t) - \beta(t)) + K_m = k_2(t),$$

for every $t \in [m, m+1)$ and every $m = 0, 1, \ldots, S^*$, where S^* is given in Lemma 2.

Finally, we prove that problem (15) is solvable.

Note that, due to the properties of functions α, β, k_1, k_2, g, and the definition of the operators p and q, it is deduced that function $\sigma_x(t) \in \Lambda$, for every $x \in E$. Hence, problem (15) can be written equivalently (see Theorem 1) as

$$x(t) = \int_0^T K(t,s)\sigma_x(s)\, ds + \lambda K(t, 0), \ t \in J,$$

where for each $s \in J$, $K(\cdot, s)$ is the unique solution to an auxiliary problem of the type (2). We define the mapping $\mathcal{B} : C^1(J) \to C^1(J)$ given by

$$[\mathcal{B}x](t) := \int_0^T K(t,s)\sigma_x(s)\, ds + \lambda K(t, 0), \ t \in J,$$

in such a way that the set of solutions of the modified problem (15) is the set of fixed points of $\mathcal{B}$.

Let $M > 0$ be such that $|\alpha(t)| \leq M$, $|\beta(t)| \leq M$, $|\alpha'(t)| \leq M$, $|\beta'(t)| \leq M$, for every $t \in J$ (it is possible since α, $\beta \in E$).

On the other hand, for $t \in [m, m+1)$, $m = 0, 1, \ldots, S^*$,

$$
\begin{aligned}
|k_1(t)| &= |\alpha'(t) + R(\alpha(t) - \beta(t)) - K_m| \le M + 2RM + |K_m| \\
&\le M + 2RM + \frac{(d - Rc)}{1 - A} 2M \\
&\times \left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{[T]-j} \int_{j-1}^{j} e^{L(s-j)}\, ds \left(e^{-L(T-[T])} - \frac{c}{L}\left(1 - e^{-L(T-[T])}\right) \right) \right. \\
&\left. + \int_{[T]}^{T} e^{L(s-T)}\, ds \right) \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{m} \left(e^{-L(t-m)} - \frac{c}{L}\left(1 - e^{-L(t-m)}\right) \right) \\
&+ 2M(d - Rc) \sum_{j=1}^{m} \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{m-j} \int_{j-1}^{j} e^{L(s-j)}\, ds \left(e^{-L(t-m)} - \frac{c}{L}\left(1 - e^{-L(t-m)}\right) \right) \\
&+ 2M(d - Rc) \int_{m}^{t} e^{L(s-t)}\, ds = M_1.
\end{aligned}
$$

Similarly,

$$|k_2(t)| \le M_1, \; t \in [m, m+1), \; m = 0, 1, \ldots, S^*.$$

This proves that $\|k_1\| = \sup_{t \in J} |k_1(t)| \le M_1$ and $\|k_2\| \le M_1$. Consider the compact set in $\mathbb{R}^5$

$$D := \{(t, x, y, z, w) \in \mathbb{R}^5 \;:\; t \in [0, T], \alpha(t) \le x \le \beta(t), \; k_1(t) \le y \le k_2(t), \; \alpha([t]) \le z \le \beta([t]), \; k_1([t]) \le w \le k_2([t])\}.$$

By the hypotheses on g, it is possible to choose $N > 0$ such that

$$|g(t, x, y, z, w)| \le N, \quad \text{for every } (t, x, y, z, w) \in D.$$

Let $\mu \in (0, 1)$ and x be such that $x = \mu \mathcal{B} x$.

$$\|x\|_{C^1(J)} = \mu \|\mathcal{B}x\|_{C^1(J)} = \mu(\|\mathcal{B}x\| + \|(\mathcal{B}x)'\|).$$

From the expression of $\mathcal{B}x$, we deduce that

$$(\mathcal{B}x)'(t) = \int_0^T \frac{\partial}{\partial t} K(t, s)\, \sigma_x(s)\, ds + \lambda \frac{\partial}{\partial t} K(t, 0), \; t \in J.$$

Note that, by definition, $p(t, x(t))$ is between $\alpha(t)$ and $\beta(t)$ and $q(t, x'(t))$ is between $k_1(t)$ and $k_2(t)$. Hence, $(t, p(t, x(t)), q(t, x'(t)), p([t], x([t])), q([t], x'([t]))) \in D$, for every t, and

$$
\begin{aligned}
|\sigma_x(t)| &= |g(t, p(t, x(t)), q(t, x'(t)), p([t], x([t])), q([t], x'([t]))) \\
&\quad + a q(t, x'(t)) + b p(t, x(t)) + c q([t], x'([t])) + d p([t], x([t]))| \\
&\le N + |a| M_1 + |b| M + |c| M_1 + |d| M = N + (|a| + |c|) M_1 + (|b| + |d|) M.
\end{aligned}
$$

This proves that

$$\|\mathcal{B}x\| \le (N + (|a| + |c|) M_1 + (|b| + |d|) M) \sup_{t \in [0,T]} \int_0^T K(t, s)\, ds + |\lambda| \sup_{t \in [0,T]} K(t, 0),$$

and

$$\|(\mathcal{B}x)'\| \leq (N + (|a| + |c|)M_1 + (|b| + |d|)M) \sup_{t\in[0,T]} \int_0^T \left|\frac{\partial}{\partial t}K(t,s)\right| ds + |\lambda| \sup_{t\in[0,T]} \left|\frac{\partial}{\partial t}K(t,0)\right|.$$

Furthermore, K and $\frac{\partial}{\partial t}K$ are bounded, since their expressions depend on the functions h_1, h_2 and function g in Buedo-Fernández et al[27] and their respective derivatives, which are obviously continuous and hence bounded on $[0,1]$. We remark that the nonnegative character of functions $K(\cdot,0)$ and $K(\cdot,s)$ for almost every $s \in (0,T)$ is guaranteed by the conditions in Theorem 2 (see Buedo-Fernández et al[27]).

Then, by Schauder's Fixed Point Theorem, there exists at least one fixed point x of $\mathcal{B}$ which is a solution to (15). This solution x satisfies that $\alpha \leq x \leq \beta$ and $k_1 \leq x' \leq k_2$ on J, and in consequence, it is a solution to (3) and the proof is complete. □

4 | MONOTONE METHOD

In this section, we develop the monotone iterative technique for problem (3).

Theorem 6. *Suppose that $g : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous on $(J\backslash\{1, 2, \dots, [T]\}) \times \mathbb{R}^4$ and such that the following limits are finite*

$$\lim_{t\to n^-} g(t,x,y,u,v), \qquad g(n,x,y,u,v) = \lim_{t\to n^+} g(t,x,y,u,v).$$

Assume that hypothesis (H_1) holds. Suppose that (H_6) There exist constants $a, b, c, d \in \mathbb{R}$, $L \neq 0$, and $R > 0$ with $R + L = a$, $LR = b$, $Rc \leq d$,

$$c + L > 0, \text{ and } e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \geq 0,$$

in such a way that, for those values of $a, b, c, d \in \mathbb{R}$, the following inequality holds:

$$g(t,x,y,u,v) - g(t,\tilde{x},\tilde{y},\tilde{u},\tilde{v}) \geq -a(y-\tilde{y}) - b(x-\tilde{x}) - c(v-\tilde{v}) - d(u-\tilde{u}),$$

for $t \in J$, and

$$\alpha(t) \leq \tilde{x} \leq x \leq \beta(t), \;\; k_1(t) \leq \tilde{y}, y \leq k_2(t), \;\; \alpha([t]) \leq \tilde{u} \leq u \leq \beta([t]), \;\; k_1([t]) \leq \tilde{v}, v \leq k_2([t]),$$

where $k_1(t)$, $k_2(t)$ and K_m, $m = 0, 1, \dots, S^*$, are given in the statement of Lemma 2, $S^* = [T]$ if $T \notin \mathbb{Z}$ and $S^* = [T] - 1$ if $T \in \mathbb{Z}$.

Assume also that the constants $a, b, c, d \in \mathbb{R}$ in (H_6) are such that the hypothesis (5) and conditions (I)–(V) in Theorem 2 hold (see condition (H_3)).

Suppose that

$$\alpha'(t) - \beta'(t) \leq R(\beta(t) - \alpha(t)), \; \forall t, \tag{16}$$

or, more generally,

$$\alpha'(t) - \beta'(t) \leq R(\beta(t) - \alpha(t)) + K_m, \; \forall t. \tag{17}$$

Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ in E such that $\alpha_0 = \alpha$, $\beta_0 = \beta$, and $\{\alpha_n\}$, $\{\beta_n\}$ are uniformly convergent to ρ, γ, which are the extremal solutions to (3) in the set

$$\{\eta \in C^1(J) \mid \alpha \leq \eta \leq \beta \text{ and } k_1 \leq \eta' \leq k_2 \text{ on } J\}.$$

Furthermore, there exist subsequences $\{\alpha'_{n_k}\} \to \rho'$, $\{\beta'_{n_k}\} \to \gamma'$, as $k \to +\infty$.

Proof. For each $\eta \in C^1(J)$, we consider the modified problem

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma_\eta(t), \ t \in J = [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) + \lambda, \end{cases} \tag{18}$$

where $\sigma_\eta(t) := g(t, \eta(t), \eta'(t), \eta([t]), \eta'([t])) + a\eta'(t) + b\eta(t) + c\eta'([t]) + d\eta([t])$, for $t \in J$.

Note that σ_η is continuous on $J\backslash\{1, 2, \dots, [T]\}$, and there exist $\sigma_\eta(n^-) \in \mathbb{R}, \sigma_\eta(n^+) = \sigma_\eta(n)$, for every $n \in \{1, 2, \dots, [T]\}$, hence $\sigma_\eta \in \Lambda$. We define the operator $\mathcal{A}$ as

$$(\mathcal{A}\eta)(t) := \int_0^T K(t, s)\sigma_\eta(s)\,ds + \lambda K(t, 0), \ t \in J,$$

for $\eta \in C^1(J)$.

We choose $\alpha_0 = \alpha$ and α_1 the unique solution to (18), which exists since $\sigma_\alpha \in \Lambda$ and condition (5) holds. Hence,

$$\alpha_1(t) = \int_0^T K(t, s)\sigma_\alpha(s)\,ds + \lambda K(t, 0), \ t \in J.$$

The sequences α_n, β_n are defined as

$$\alpha_n := \mathcal{A}\alpha_{n-1}, \ \beta_n := \mathcal{A}\beta_{n-1}, \ \forall n \geq 1.$$

We check that $\{\alpha_n\}$ is monotone nondecreasing and $\{\beta_n\}$ is monotone nonincreasing.

We proceed in different steps:

(i) $\alpha'(t), \beta'(t) \in [k_1(t), k_2(t)]$, $\forall t \in J$.
(ii) α_n'', β_n'' are bounded.
(iii) If $\eta \in C^1(J)$ is such that $\alpha \leq \eta \leq \beta$, and $k_1 \leq \eta' \leq k_2$ on J, then $\mathcal{A}\eta$ belongs to $[\alpha, \beta]$ and $(\mathcal{A}\eta)'$ is between k_1 and k_2.
(iv) $\mathcal{A}$ is nondecreasing on the set

$$\{\eta \in C^1(J) \mid \alpha \leq \eta \leq \beta \text{ and } k_1 \leq \eta' \leq k_2 \text{ on } J\}.$$

(v) $\{\alpha_n\}$ is uniformly convergent towards ρ and $\{\beta_n\}$ is uniformly convergent towards γ. For the derivatives, we only have convergence of a certain subsequence.
(vi) ρ, γ are extremal solutions to (3) in

$$\{\eta \in C^1(J) \mid \alpha \leq \eta \leq \beta \text{ and } k_1 \leq \eta' \leq k_2 \text{ on } J\}.$$

First, we prove (i). Note that, for every m,

$$k_1(t) = \alpha'(t) + R(\alpha(t) - \beta(t)) - K_m \leq \alpha'(t), \ t \in [m, m+1),$$

$$k_2(t) = \beta'(t) - R(\alpha(t) - \beta(t)) + K_m \geq \beta'(t), \ t \in [m, m+1).$$

If $\alpha'(t) \leq \beta'(t)$, then $k_1 \leq \alpha'(t) \leq \beta'(t) \leq k_2$, but if this condition does not hold, we deduce

$$\alpha'(t) + R(\alpha(t) - \beta(t)) - K_m \leq \beta'(t),$$

and

$$\alpha'(t) \leq \beta'(t) - R(\alpha(t) - \beta(t)) + K_m,$$

for $t \in [m, m+1)$, and every m, from hypothesis (16), or more generally, (17).

We check (ii). Indeed,

$$\alpha_n'(t) = \int_0^T \frac{\partial}{\partial t} K(t,s)\sigma_{\alpha_{n-1}}(s)\,ds + \lambda \frac{\partial}{\partial t} K(t,0),\ t \in J, \tag{19}$$

and

$$\alpha_n''(t) = \int_0^T \frac{\partial^2}{\partial t^2} K(t,s)\sigma_{\alpha_{n-1}}(s)\,ds + \lambda \frac{\partial^2}{\partial t^2} K(t,0),\ t \in J, \tag{20}$$

so that $\alpha_n'' \in \Lambda$, and similarly,

$$\beta_n''(t) = \int_0^T \frac{\partial^2}{\partial t^2} K(t,s)\sigma_{\beta_{n-1}}(s)\,ds + \lambda \frac{\partial^2}{\partial t^2} K(t,0),\ t \in J,$$

and hence, α_n'' and β_n'' are bounded on each $J_k = [k, k+1]$, and thus, bounded.

We prove (iii). Consider $\eta \in C^1(J)$ such that $\alpha \le \eta \le \beta$, and $k_1 \le \eta' \le k_2$ on J, then we prove that $\mathcal{A}\eta$ belongs to $[\alpha, \beta]$ and $(\mathcal{A}\eta)'$ is between k_1 and k_2. Note that $\mathcal{A}\eta$ is the unique solution to (18). Consider $m := \alpha - \mathcal{A}\eta$, then, by using that $\alpha \le \eta \le \beta$, $k_1 \le \eta' \le k_2$, and (H_6), we get, for $t \in [0, T]$,

$$\begin{aligned}
&m''(t) + am'(t) + bm(t) + cm'([t]) + dm([t]) \\
&= \alpha''(t) + a\alpha'(t) + b\alpha(t) + c\alpha'([t]) + d\alpha([t]) \\
&\quad - g(t, \eta(t), \eta'(t), \eta([t]), \eta'([t])) - a\eta'(t) - b\eta(t) - c\eta'([t]) - d\eta([t]) \\
&\le g(t, \alpha(t), \alpha'(t), \alpha([t]), \alpha'([t])) + a\alpha'(t) + b\alpha(t) + c\alpha'([t]) + d\alpha([t]) \\
&\quad - g(t, \eta(t), \eta'(t), \eta([t]), \eta'([t])) - a\eta'(t) - b\eta(t) - c\eta'([t]) - d\eta([t]) \le 0.
\end{aligned}$$

Moreover,

$$m(0) = \alpha(0) - (\mathcal{A}\eta)(0) = \alpha(T) - (\mathcal{A}\eta)(T) = m(T),$$

$$m'(0) = \alpha'(0) - (\mathcal{A}\eta)'(0) \le \alpha'(T) + \lambda - (\mathcal{A}\eta)'(T) - \lambda = m'(T).$$

By the comparison result Theorem 3, $m \le 0$ on $[0, T]$, hence $\alpha \le \mathcal{A}\eta$ on J.

On the other hand, by taking $\tilde{m} := \mathcal{A}\eta - \beta$, we have

$$\tilde{m}''(t) + a\tilde{m}'(t) + b\tilde{m}(t) + c\tilde{m}'([t]) + d\tilde{m}([t]) \le 0,\ t \in [0, T],$$

and $\tilde{m}(0) = \tilde{m}(T)$, $\tilde{m}'(0) \le \tilde{m}'(T)$, hence we deduce again, by the comparison result Theorem 3, that $\tilde{m} \le 0$ on J, and $\mathcal{A}\eta \le \beta$ on J.

Now, we prove that $(\mathcal{A}\eta)' \in [k_1, k_2]$ by using Lemma 1. We consider the function

$$y(t) := \alpha'(t) - (\mathcal{A}\eta)'(t) + R(\alpha(t) - (\mathcal{A}\eta)(t)),\ t \in J,$$

and check that

$$
\begin{aligned}
y'(t)+Ly(t) &= \alpha''(t)-(\mathcal{A}\eta)''(t)\\
&\quad +R(\alpha'(t)-(\mathcal{A}\eta)'(t))+L(\alpha'(t)-(\mathcal{A}\eta)'(t))+LR(\alpha(t)-(\mathcal{A}\eta)(t))\\
&\le g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t]))+R\alpha'(t)+L\alpha'(t)+LR\alpha(t)\\
&\quad -(\mathcal{A}\eta)''(t)-R(\mathcal{A}\eta)'(t)-L(\mathcal{A}\eta)'(t)-LR(\mathcal{A}\eta)(t)\\
&= g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t]))+a\alpha'(t)+b\alpha(t)\\
&\quad -(\mathcal{A}\eta)''(t)-a(\mathcal{A}\eta)'(t)-b(\mathcal{A}\eta)(t)\\
&= g(t,\alpha(t),\alpha'(t),\alpha([t]),\alpha'([t]))+a\alpha'(t)+b\alpha(t)\\
&\quad -g(t,\eta(t),\eta'(t),\eta([t]),\eta'([t]))-a\eta'(t)-b\eta(t)\\
&\quad -c\eta'([t])-d\eta([t])+c(\mathcal{A}\eta)'([t])+d(\mathcal{A}\eta)([t])\\
&\le -c\alpha'([t])-d\alpha([t])+c(\mathcal{A}\eta)'([t])+d(\mathcal{A}\eta)([t])\\
&= c((\mathcal{A}\eta)'([t])-\alpha'([t]))+d((\mathcal{A}\eta)([t])-\alpha([t])),\ \ t\in J,
\end{aligned}
$$

and

$$
\begin{aligned}
y(0) &= \alpha'(0)-(\mathcal{A}\eta)'(0)+R(\alpha(0)-(\mathcal{A}\eta)(0))\\
&\le \alpha'(T)+\lambda-(\mathcal{A}\eta)'(T)-\lambda+R(\alpha(T)-(\mathcal{A}\eta)(T))=y(T).
\end{aligned}
$$

This proves that

$$
y'(t)+Ly(t)+cy([t])\le (d-cR)((\mathcal{A}\eta)([t])-\alpha([t]))\le (d-cR)(\beta([t])-\alpha([t])),
$$

since $(\mathcal{A}\eta)\le\beta$ and $d\ge cR$. By Lemma 1, we get, for $t\in[m,m+1)$, that

$$
\begin{aligned}
y(t)\le{}& \frac{(d-Rc)}{1-A}\\
&\times\left(\sum_{j=1}^{[T]}\left(e^{-L}-\frac{c}{L}\left(1-e^{-L}\right)\right)^{[T]-j}\int_{j-1}^{j}(\beta([s])-\alpha([s]))e^{L(s-j)}\,ds\left(e^{-L(T-[T])}-\frac{c}{L}\left(1-e^{-L(T-[T])}\right)\right)\right.\\
&\left.+\int_{[T]}^{T}(\beta([s])-\alpha([s]))e^{L(s-T)}\,ds\right)\left(e^{-L}-\frac{c}{L}\left(1-e^{-L}\right)\right)^{m}\left(e^{-L(t-m)}-\frac{c}{L}\left(1-e^{-L(t-m)}\right)\right)\\
&+\sum_{j=1}^{m}\left(e^{-L}-\frac{c}{L}\left(1-e^{-L}\right)\right)^{m-j}\int_{j-1}^{j}(d-Rc)(\beta([s])-\alpha([s]))e^{L(s-j)}\,ds\left(e^{-L(t-m)}-\frac{c}{L}\left(1-e^{-L(t-m)}\right)\right)\\
&+\int_{m}^{t}(d-Rc)(\beta([s])-\alpha([s]))e^{L(s-t)}\,ds.
\end{aligned}
$$

Hence,

$$
y(t)=\alpha'(t)-(\mathcal{A}\eta)'(t)+R(\alpha(t)-(\mathcal{A}\eta)(t))\le K_m,\quad \text{for } t\in[m,m+1).
$$

Therefore,

$$
\alpha'(t)-(\mathcal{A}\eta)'(t)+R(\alpha(t)-\beta(t))\le K_m,
$$

and in consequence,

$$
k_1(t)=\alpha'(t)+R(\alpha(t)-\beta(t))-K_m\le(\mathcal{A}\eta)'(t),
$$

for every $t\in[m,m+1)$ and every m.

On the other hand, to prove that $(\mathcal{A}\eta)'\le k_2$, take $y(t):=(\mathcal{A}\eta)'(t)-\beta'(t)+R((\mathcal{A}\eta)(t)-\beta(t))$, $t\in J$, and check the hypotheses of Lemma 1. Indeed,

$$
\begin{aligned}
y(0) &= (\mathcal{A}\eta)'(0)-\beta'(0)+R((\mathcal{A}\eta)(0)-\beta(0))\\
&\le(\mathcal{A}\eta)'(T)+\lambda-\beta'(T)-\lambda+R((\mathcal{A}\eta)(T)-\beta(T))=y(T),
\end{aligned}
$$

and

$$\begin{aligned} y'(t)+Ly(t) &= (\mathcal{A}\eta)''(t)-\beta''(t) \\ &\quad + R((\mathcal{A}\eta)'(t)-\beta'(t)) + L((\mathcal{A}\eta)'(t)-\beta'(t)) + LR((\mathcal{A}\eta)(t)-\beta(t)) \\ &= (\mathcal{A}\eta)''(t)-\beta''(t)+a(\mathcal{A}\eta)'(t)-a\beta'(t)+b(\mathcal{A}\eta)(t)-b\beta(t) \\ &\le g(t,\eta(t),\eta'(t),\eta([t]),\eta'([t]))+a\eta'(t)+b\eta(t)+c\eta'([t])+d\eta([t]) \\ &\quad - c(\mathcal{A}\eta)'([t])-d(\mathcal{A}\eta)([t])-g(t,\beta(t),\beta'(t),\beta([t]),\beta'([t]))-a\beta'(t)-b\beta(t) \\ &\le c\beta'([t])+d\beta([t])-c(\mathcal{A}\eta)'([t])-d(\mathcal{A}\eta)([t]) \\ &= c(\beta'([t])-(\mathcal{A}\eta)'([t]))+d(\beta([t])-(\mathcal{A}\eta)([t])),\ t\in J, \end{aligned}$$

which implies that

$$y'(t)+Ly(t)+cy([t]) \le (d-cR)(\beta([t])-(\mathcal{A}\eta)([t])) \le (d-cR)(\beta([t])-\alpha([t])),\ \text{for}\ t\in[0,T].$$

Since the function σ is the same as in the previous case, then, by Lemma 1, we get

$$y(t)\le K_m, \text{for}\ t\in[m,m+1),$$

and

$$(\mathcal{A}\eta)'(t)-\beta'(t)+R(\alpha(t)-\beta(t)) \le (\mathcal{A}\eta)'(t)-\beta'(t)+R((\mathcal{A}\eta)(t)-\beta(t)) \le K_m, \quad \text{for}\ t\in[m,m+1),$$

thus

$$(\mathcal{A}\eta)'(t)\le \beta'(t)+R(\beta(t)-\alpha(t))+K_m = k_2(t),$$

for every $t\in[m,m+1)$ and every $m=0,1,\dots,[T]$, hence $(\mathcal{A}\eta)'\le k_2$ on J.

To check (iv), consider that $\eta,\xi\in C^1(J)$ are such that $\alpha\le\eta\le\xi\le\beta$, and $k_1\le\eta',\xi'\le k_2$ on J, then $\mathcal{A}\eta\le\mathcal{A}\xi$, which is deduced similarly to Theorem 4, since, by using (H_6), we get, for $t\in J$,

$$\begin{aligned} \sigma_\eta(t)-\sigma_\xi(t) &= g(t,\eta(t),\eta'(t),\eta([t]),\eta'([t]))+a\eta'(t)+b\eta(t)+c\eta'([t])+d\eta([t]) \\ &\quad - g(t,\xi(t),\xi'(t),\xi([t]),\xi'([t]))-a\xi'(t)-b\xi(t)-c\xi'([t])-d\xi([t]) \le 0. \end{aligned}$$

Indeed, $w:=\mathcal{A}\eta-\mathcal{A}\xi$ satisfies that

$$w(0)=w(T), \qquad w'(0)=(\mathcal{A}\eta)'(0)-(\mathcal{A}\xi)'(0)=(\mathcal{A}\eta)'(T)+\lambda-(\mathcal{A}\xi)'(T)-\lambda=w'(T),$$

and

$$w''(t)+aw'(t)+bw(t)+cw'([t])+dw([t])\le 0,\ t\in J.$$

By the comparison result Theorem 3, we deduce that $w\le 0$ and thus, $\mathcal{A}\eta\le\mathcal{A}\xi$ on J.

Now, to prove (v), note that $\{\alpha_n\}$ is nondecreasing, and $\{\beta_n\}$ is nonincreasing. Indeed, it is obvious (by iii)) that $\alpha\le\mathcal{A}\alpha=\alpha_1\le\beta$, $k_1\le(\mathcal{A}\alpha)'=\alpha_1'\le k_2$ and $\alpha\le\mathcal{A}\beta=\beta_1\le\beta$ and $k_1\le(\mathcal{A}\beta)'=\beta_1'\le k_2$ on J. By iv), we have $\alpha_1\le\beta_1$. By applying iv) recursively, we derive the monotonicity of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. On the other hand, by the previous considerations, the integral expressions

$$\alpha_n(t)=\int_0^T K(t,s)\sigma_{\alpha_{n-1}}(s)\,ds+\lambda K(t,0),\ t\in J,$$

$$\beta_n(t)=\int_0^T K(t,s)\sigma_{\beta_{n-1}}(s)\,ds+\lambda K(t,0),\ t\in J,$$

imply that $\{\alpha_n\}$, $\{\beta_n\}\subset C(J)$ are uniformly bounded. We prove that these sets are equicontinuous. Consider the expressions of the corresponding derivatives included in (19)

$$\alpha_n'(t) = \int_0^T \frac{\partial}{\partial t}K(t,s)\sigma_{\alpha_{n-1}}(s)\,ds + \lambda\frac{\partial}{\partial t}K(t,0),\ t \in J,$$

$$\beta_n'(t) = \int_0^T \frac{\partial}{\partial t}K(t,s)\sigma_{\beta_{n-1}}(s)\,ds + \lambda\frac{\partial}{\partial t}K(t,0),\ t \in J.$$

By using the compact set D defined in the proof of Theorem 5, we obtain that the set

$$\{\sigma_\eta(s)\ :\ s \in J, \eta \in C^1(J) \text{ with } \alpha \le \eta \le \beta, \text{ and } k_1 \le \eta' \le k_2\}$$

is bounded, where

$$\sigma_\eta(s) := g(s,\eta(s),\eta'(s),\eta([s]),\eta'([s])) + a\eta'(s) + b\eta(s) + c\eta'([s]) + d\eta([s]),\ s \in J;$$

hence, $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous sets on J. Hence, there exist $\rho, \gamma \in C(J)$ and subsequences $\{\alpha_{n_k}\} \to \rho$, and $\{\beta_{n_k}\} \to \gamma$, uniformly as $k \to +\infty$ (moreover, we can affirm that there exist $\rho, \gamma \in C^1(J)$ and subsequences $\{\alpha_{n_k}\} \to \rho$ and $\{\beta_{n_k}\} \to \gamma$ in $\|\cdot\|_1$, as $k \to +\infty$, that is, $\{\alpha_{n_k}\} \to \rho$, $\{\beta_{n_k}\} \to \gamma$, $\{\alpha_{n_k}'\} \to \rho'$, and $\{\beta_{n_k}'\} \to \gamma'$ uniformly on J).

Since $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone, then $\{\alpha_n\} \to \rho$, and $\{\beta_n\} \to \gamma$, as $n \to +\infty$.

Note that, obviously, $\alpha \le \rho \le \gamma \le \beta$ on J. Furthermore, since $k_1 \le (\alpha_{n_k})' = (\mathcal{A}\alpha_{n_k-1})' \le k_2$, and $k_1 \le (\beta_{n_k})' = (\mathcal{A}\beta_{n_k-1})' \le k_2$, for every k, then $k_1 \le \rho', \gamma' \le k_2$ on J.

To prove (vi), we use that

$$\alpha_n(t) = \int_0^T K(t,s)\sigma_{\alpha_{n-1}}(s)\,ds + \lambda K(t,0),\ t \in J.$$

Since α_{n_k}' is uniformly convergent towards ρ', then, by taking the sequence

$$\alpha_{n_k+1}(t) = \int_0^T K(t,s)\sigma_{\alpha_{n_k}}(s)\,ds + \lambda K(t,0),\ t \in J,$$

and by using that $\sigma_{\alpha_{n_k}}(t) = g(t,\alpha_{n_k}(t),\alpha_{n_k}'(t),\alpha_{n_k}([t]),\alpha_{n_k}'([t])) + a\alpha_{n_k}'(t) + b\alpha_{n_k}(t) + c\alpha_{n_k}'([t]) + d\alpha_{n_k}([t])$, for $t \in J$, is uniformly convergent on J towards

$$\sigma_\rho(t) = g(t,\rho(t),\rho'(t),\rho([t]),\rho'([t])) + a\rho'(t) + b\rho(t) + c\rho'([t]) + d\rho([t]),\ \text{ for } t \in J,$$

we have

$$\rho(t) = \int_0^T K(t,s)\sigma_\rho(s)\,ds + \lambda K(t,0),\ t \in J,$$

hence,

$$\rho''(t) + a\rho'(t) + b\rho(t) + c\rho'([t]) + d\rho([t]) = \sigma_\rho(t),$$

which implies that

$$\rho''(t) = g(t,\rho(t),\rho'(t),\rho([t]),\rho'([t])),\quad \text{for } t \in J,$$

and besides, $\rho(0) = \rho(T)$, and $\rho'(0) = \rho'(T) + \lambda$. Therefore, ρ is a solution to (3). A similar reasoning is valid for the sequence $\{\beta_n\}$ and γ.

On the other hand, if $x \in E$ is a solution such that $\alpha \leq x \leq \beta$ and $k_1 \leq x' \leq k_2$ on J, then $\alpha_1 = \mathcal{A}\alpha \leq \mathcal{A}x = x \leq \mathcal{A}\beta = \beta_1$ and $k_1 \leq (\mathcal{A}\alpha)', (\mathcal{A}x)', (\mathcal{A}\beta)' \leq k_2$ on J. By induction, we have $\alpha_n \leq x \leq \beta_n$ on J, for every n, then $\rho \leq x \leq \gamma$ on J, and the solutions ρ and γ are extremal in

$$\{\eta \in C^1(J) \mid \alpha \leq \eta \leq \beta \text{ and } k_1 \leq \eta' \leq k_2 \text{ on } J\}.$$

□

Remark 8. Once the condition $\alpha \leq \beta$ is satisfied on J, condition (16) is trivially valid at the points $t_0 \in J$ with $\alpha(t_0) = \beta(t_0)$ and $\alpha'(t_0) \leq \beta'(t_0)$, which is consistent with the condition $\alpha \leq \beta$ (if $\alpha(t_0) = \beta(t_0)$ and $\alpha'(t_0) > \beta'(t_0)$, then there would exist a neighborhood where $\alpha > \beta$). On an interval where $\alpha < \beta$, condition (16) is equivalent to $\frac{\alpha'(t)-\beta'(t)}{\beta(t)-\alpha(t)} \leq R$, and integrating, we get $-\ln(\beta(t) - \alpha(t)) \leq Rt + C$, or $\beta(t) \geq Ke^{-Rt} + \alpha(t)$. Hence, if $\alpha(t_0) < \beta(t_0)$, the validity of (16) for $t \geq t_0$ implies that $\beta(t) \geq Ke^{-Rt} + \alpha(t)$ for $t \geq t_0$ (in particular, this implies that $\alpha(t) < \beta(t)$ for all $t \geq t_0$).

Condition (16) is equivalent to the differential inequality $\gamma'(t) \leq -R\gamma(t)$, where $\gamma = \alpha - \beta \leq 0$. If $\gamma(t_0) < 0$, it implies that $\gamma(t) < 0$, $\forall t \geq t_0$. If $\gamma(t_0) = 0$, we have $\gamma'(t_0) \leq 0$.

Remark 9. The constants $a, b, c, d \in \mathbb{R}$ must be chosen in such a way that there exist $L \neq 0$, and $R > 0$ with $R+L = a$, and $LR = b \neq 0$. From the relations specified, once calculated R, L is obtained as $L = \frac{b}{R}$. In consequence, R can be chosen by solving the equation

$$R + \frac{b}{R} = a,$$

which leads to the quadratic equation $R^2 - aR + b = 0$. We easily derive two possibilities:

$$R = \frac{a \pm \sqrt{a^2 - 4b}}{2}.$$

If $a^2 - 4b < 0$, there exists no R; hence, we must assume that $a^2 \geq 4b$. This expression is, in consequence, a restriction on the choice of a, b.

If $a^2 = 4b$, we deduce that $b > 0$ (since $b \neq 0$), and hence, $L > 0$, and $a > 0$. In this case, we have only one possibility for $R = \frac{a}{2}$ (which is consistent with $a > 0$), and $L = \frac{b}{R} = \frac{2b}{a}$.

If $a^2 > 4b$, we have two possibilities for R, and we seek for a positive value of R. Note that, for at least one of those values to be positive, we have to check $a > -\sqrt{a^2 - 4b}$, which is trivial for $a > 0$. Hence, one choice is to assume that $a > 0$ and take $R = \frac{a+\sqrt{a^2-4b}}{2} > 0$. This option is also possible if $a \leq 0$ and $b < 0$.

In the other choice, $R = \frac{a-\sqrt{a^2-4b}}{2}$ is positive if and only if $a > \sqrt{a^2 - 4b}$, which is obviously satisfied for $a > 0$, $b > 0$.

In summary, we have two options:

- $a^2 = 4b > 0$, $R = \frac{a}{2}$ with $a > 0$, and $L = \frac{2b}{a} > 0$ (here, $a, b > 0$).
- $a^2 > 4b$, and two options for R, with the corresponding value of L:
  - $R = \frac{a+\sqrt{a^2-4b}}{2} > 0$, $L = \frac{b}{R} = \frac{2b}{a+\sqrt{a^2-4b}}$ (valid for $a > 0$ independently of the sign of $b \neq 0$, and for $a \leq 0$ and $b < 0$). The sign of L is the sign of b.
  - $R = \frac{a-\sqrt{a^2-4b}}{2} > 0$, $L = \frac{b}{R} = \frac{2b}{a-\sqrt{a^2-4b}} > 0$ (valid for $a > 0$ and $b > 0$).

Note that we need to impose additional restrictions on the constants L, R. These restrictions will give information about the way of choosing the rest of the constants $c, d \in \mathbb{R}$. For instance, $d - cR \geq 0$ is written as

- $a^2 = 4b$: $d - c\frac{a}{2} \geq 0$, that is, $ca \leq 2d$.
- $a^2 > 4b$:
  - If $R = \frac{a+\sqrt{a^2-4b}}{2} > 0$, such condition is written as $c\left(a + \sqrt{a^2 - 4b}\right) \leq 2d$.
  - If $R = \frac{a-\sqrt{a^2-4b}}{2} > 0$, such condition is written as $c\left(a - \sqrt{a^2 - 4b}\right) \leq 2d$.

Note that, taking into account the type of one-sided Lipschitz condition assumed, we are interested in the case $b > 0$ (hence, $L > 0$ and, therefore, $a > 0$).

In general, the sequences $\{\alpha_n'\}$ and $\{\beta_n'\}$ are not monotonic. If $\alpha' \le \beta'$ on J and the sequences $\{\alpha_n'\}$ and $\{\beta_n'\}$ are monotone, then we would deduce that $\{\alpha_n'\} \to \rho'$ and $\{\beta_n'\} \to \gamma'$ uniformly on J, as $n \to +\infty$. Despite of this, we can prove the following estimates for the derivatives $\{\alpha_n'\}$ and $\{\beta_n'\}$.

Theorem 7. *For the sequences given in Theorem 6, the following estimates are valid:*

$$\begin{aligned}
&\alpha_n'(t) - R(\alpha_{n+1}(t) - \alpha_n(t)) - \Gamma_m(\beta - \alpha, t) \\
&\le \alpha_n'(t) - R(\alpha_{n+1}(t) - \alpha_n(t)) - \Gamma_m(S - \alpha_n, t) \\
&\le \alpha_n'(t) - R(\alpha_{n+1}(t) - \alpha_n(t)) - \Gamma_m(\alpha_{n+1} - \alpha_n, t) \le \alpha_{n+1}'(t),
\end{aligned} \tag{21}$$

where S may be either ρ, γ, or β, and

$$\begin{aligned}
\beta_{n+1}'(t) &\le \beta_n'(t) + R(\beta_n(t) - \beta_{n+1}(t)) + \Gamma_m(\beta_n - \beta_{n+1}, t) \\
&\le \beta_n'(t) + R(\beta_n(t) - \beta_{n+1}(t)) + \Gamma_m(\beta_n - S, t) \\
&\le \beta_n'(t) + R(\beta_n(t) - \beta_{n+1}(t)) + \Gamma_m(\beta - \alpha, t),
\end{aligned} \tag{22}$$

where S may be either α, ρ, or γ, for $t \in [m, m+1)$, $m = 0, 1, \dots, [T]$, with

$$\begin{aligned}
\Gamma_m(u,t) :=& \frac{(d - Rc)}{1 - A} \\
&\times \left(\sum_{j=1}^{[T]} \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{[T]-j} \int_{j-1}^{j} u([s]) e^{L(s-j)}\, ds \left(e^{-L(T-[T])} - \frac{c}{L}\left(1 - e^{-L(T-[T])}\right) \right) \right. \\
&\left. + \int_{[T]}^{T} u([s]) e^{L(s-T)}\, ds \right) \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{m} \left(e^{-L(t-m)} - \frac{c}{L}\left(1 - e^{-L(t-m)}\right) \right) \\
&+ \sum_{j=1}^{m} \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right) \right)^{m-j} \int_{j-1}^{j} (d - Rc) u([s]) e^{L(s-j)}\, ds \left(e^{-L(t-m)} - \frac{c}{L}\left(1 - e^{-L(t-m)}\right) \right) \\
&+ \int_{m}^{t} (d - Rc) u([s]) e^{L(s-t)}\, ds.
\end{aligned}$$

Proof. Hereafter, we will also consider $\alpha_{-1} := \alpha$ and $\beta_{-1} := \beta$. Following the reasoning of Theorem 6, assume that n is a nonnegative integer, and take

$$z(t) := \alpha_n'(t) - \alpha_{n+1}'(t) + R(\alpha_n(t) - \alpha_{n+1}(t)), \ t \in J.$$

We deduce that

$$\begin{aligned}
&z'(t) + Lz(t) \\
&= \alpha_n''(t) - \alpha_{n+1}''(t) + R(\alpha_n'(t) - \alpha_{n+1}'(t)) + L(\alpha_n'(t) - \alpha_{n+1}'(t)) + LR(\alpha_n(t) - \alpha_{n+1}(t)) \\
&= \alpha_n''(t) + a\alpha_n'(t) + b\alpha_n(t) - \alpha_{n+1}''(t) - a\alpha_{n+1}'(t) - b\alpha_{n+1}(t) \\
&\le g(t, \alpha_{n-1}(t), \alpha_{n-1}'(t), \alpha_{n-1}([t]), \alpha_{n-1}'([t])) + a\alpha_{n-1}'(t) + b\alpha_{n-1}(t) \\
&\quad + c\alpha_{n-1}'([t]) + d\alpha_{n-1}([t]) - c\alpha_n'([t]) - d\alpha_n([t]) - g(t, \alpha_n(t), \alpha_n'(t), \alpha_n([t]), \alpha_n'([t])) \\
&\quad - a\alpha_n'(t) - b\alpha_n(t) - c\alpha_n'([t]) - d\alpha_n([t]) + c\alpha_{n+1}'([t]) + d\alpha_{n+1}([t]) \\
&\le -c\alpha_n'([t]) - d\alpha_n([t]) + c\alpha_{n+1}'([t]) + d\alpha_{n+1}([t]).
\end{aligned}$$

The first of the previous two inequalities is in fact an equality in case $n \ge 1$, while, if $n = 0$, it comes from the properties of α. We also recall that $\alpha_{-1} = \alpha_0 = \alpha$. Furthermore, the last inequality is an equality provided that $n = 0$, whereas for

$n \geq 1$ we have used that $\alpha \leq \alpha_{n-1} \leq \alpha_n \leq \beta$, and $k_1 \leq \alpha'_{n-1}, \alpha'_n \leq k_2$. Hence, since an inequality eventually appears for every $n \geq 0$, we can write

$$\begin{aligned}
&z'(t) + Lz(t) + cz([t]) \\
&\leq -d\alpha_n([t]) + d\alpha_{n+1}([t]) + Rc\alpha_n([t]) - Rc\alpha_{n+1}([t]) = (d - cR)(\alpha_{n+1}([t]) - \alpha_n([t])) \\
&\leq (d - cR)(\rho([t]) - \alpha_n([t])) \leq (d - cR)(\gamma([t]) - \alpha_n([t])) \\
&\leq (d - cR)(\beta([t]) - \alpha_n([t])) \leq (d - cR)(\beta([t]) - \alpha([t])),
\end{aligned}$$

and

$$\begin{aligned}
z(0) &= \alpha'_n(0) - \alpha'_{n+1}(0) + R(\alpha_n(0) - \alpha_{n+1}(0)) \\
&\leq \alpha'_n(T) + \lambda - \alpha'_{n+1}(T) - \lambda + R(\alpha_n(T) - \alpha_{n+1}(T)) = z(T),
\end{aligned}$$

where the last inequality is an equality if $n \geq 1$. Then, by using Lemma 1, we obtain

$$\begin{aligned}
z(t) &= \alpha'_n(t) - \alpha'_{n+1}(t) + R(\alpha_n(t) - \alpha_{n+1}(t)) \leq \Gamma_m(\alpha_{n+1} - \alpha_n, t) \\
&\leq \Gamma_m(\rho - \alpha_n, t) \leq \Gamma_m(\gamma - \alpha_n, t) \leq \Gamma_m(\beta - \alpha_n, t) \leq \Gamma_m(\beta - \alpha, t),
\end{aligned}$$

for $t \in [m, m+1)$, $m = 0, 1, \ldots, [T]$, which implies condition (21).

Now, for the sequence $\{\beta_n\}$, consider

$$z(t) := \beta'_{n+1}(t) - \beta'_n(t) + R(\beta_{n+1}(t) - \beta_n(t)), \ t \in J.$$

In a similar manner, we obtain

$$\begin{aligned}
z'(t) + Lz(t) &= \beta''_{n+1}(t) + a\beta'_{n+1}(t) + b\beta_{n+1}(t) - \beta''_n(t) - a\beta'_n(t) - b\beta_n(t) \\
&\leq g(t, \beta_n(t), \beta'_n(t), \beta_n([t]), \beta'_n([t])) + a\beta'_n(t) + b\beta_n(t) + c\beta'_n([t]) + d\beta_n([t]) \\
&\quad - c\beta'_{n+1}([t]) - d\beta_{n+1}([t]) - g(t, \beta_{n-1}(t), \beta'_{n-1}(t), \beta_{n-1}([t]), \beta'_{n-1}([t])) \\
&\quad - a\beta'_{n-1}(t) - b\beta_{n-1}(t) - c\beta'_{n-1}([t]) - d\beta_{n-1}([t]) + c\beta'_n([t]) + d\beta_n([t]) \\
&\leq -c\beta'_{n+1}([t]) - d\beta_{n+1}([t]) + c\beta'_n([t]) + d\beta_n([t]).
\end{aligned}$$

Regarding the latter two inequalities, analogous comments to the ones that we have done for α (subcases $n = 0$ and $n \geq 1$) hold too. Thus,

$$\begin{aligned}
&z'(t) + Lz(t) + cz([t]) \\
&\leq -d\beta_{n+1}([t]) + d\beta_n([t]) + Rc\beta_{n+1}([t]) - Rc\beta_n([t]) = (d - cR)(\beta_n([t]) - \beta_{n+1}([t])) \\
&\leq (d - cR)(\beta_n([t]) - \gamma([t])) \leq (d - cR)(\beta_n([t]) - \rho([t])) \\
&\leq (d - cR)(\beta_n([t]) - \alpha([t])) \leq (d - cR)(\beta([t]) - \alpha([t])),
\end{aligned}$$

and, besides,

$$\begin{aligned}
z(0) &= \beta'_{n+1}(0) - \beta'_n(0) + R(\beta_{n+1}(0) - \beta_n(0)) \\
&\leq \beta'_{n+1}(T) + \lambda - \beta'_n(T) - \lambda + R(\beta_{n+1}(T) - \beta_n(T)) = z(T),
\end{aligned}$$

where the last inequality is a equality if $n \geq 1$. By Lemma 1, we have

$$\begin{aligned}
z(t) &= \beta'_{n+1}(t) - \beta'_n(t) + R(\beta_{n+1}(t) - \beta_n(t)) \\
&\leq \Gamma_m(\beta_n - \beta_{n+1}, t) \leq \Gamma_m(\beta_n - \gamma, t) \leq \Gamma_m(\beta_n - \rho, t) \\
&\leq \Gamma_m(\beta_n - \alpha, t) \leq \Gamma_m(\beta - \alpha, t).
\end{aligned}$$

□

Corollary 1. *In the conditions of Theorem 7, we obtain*

$$\begin{aligned}
&\alpha_n'(t) - R(\beta(t) - \alpha(t)) - \Gamma_m(\beta - \alpha, t) \\
&\leq \alpha_n'(t) - R(S_1(t) - \alpha(t)) - \Gamma_m(\beta - \alpha, t) \\
&\leq \alpha_n'(t) - R(S_1(t) - \alpha_n(t)) - \Gamma_m(S_2 - \alpha_n, t) \\
&\leq \alpha_n'(t) - R(S_1(t) - \alpha_n(t)) - \Gamma_m(\alpha_{n+1} - \alpha_n, t) \leq \alpha_{n+1}'(t),
\end{aligned}$$

where S_1, S_2*, may be either* ρ, γ*, or* β*, and*

$$\begin{aligned}
\beta_{n+1}'(t) &\leq \beta_n'(t) + R(\beta_n(t) - S_1(t)) + \Gamma_m(\beta_n - \beta_{n+1}, t) \\
&\leq \beta_n'(t) + R(\beta_n(t) - S_1(t)) + \Gamma_m(\beta_n - S_2, t) \\
&\leq \beta_n'(t) + R(\beta(t) - S_1(t)) + \Gamma_m(\beta - \alpha, t) \\
&\leq \beta_n'(t) + R(\beta(t) - \alpha(t)) + \Gamma_m(\beta - \alpha, t),
\end{aligned}$$

where S_1, S_2 *may be either* α, ρ*, or* γ*, for* $t \in [m, m+1)$, $m = 0, 1, \dots, [T]$.

Proof. It is derived from Theorem 7 and the relation $\alpha \leq \alpha_n \leq \alpha_{n+1} \leq \rho \leq \gamma \leq \beta_{n+1} \leq \beta_n \leq \beta$, where n is any nonnegative integer number. □

5 | EXAMPLES

We present here some examples of application of the main results.

Example 1. Consider the problem

$$\begin{cases} x''(t) = e^t + \frac{1}{16}\cos(x(t)) - \frac{1}{2}x'(t) - \frac{1}{2}x([t]) - \frac{1}{2}x'([t]),\ t \in J = \left[0, \frac{1}{2}\right], \\ x(0) = x(2), \\ x'(0) = x'(2), \end{cases} \tag{23}$$

where $g : J \times \mathbb{R}^4 \to \mathbb{R}$, $g(t, x, y, u, v) = e^t + \frac{1}{16}\cos(x) - \frac{1}{2}y - \frac{1}{2}u - \frac{1}{2}v$ is continuous on $J \times \mathbb{R}^4$. Note that the function g satisfies a one-sided Lipschitz condition of the type

$$g(t, x, y, u, v) - g(t, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \geq -a(y - \tilde{y}) - b(x - \tilde{x}) - c(v - \tilde{v}) - d(u - \tilde{u}),$$

for $t \in J$, $\tilde{x} \leq x$, and $\tilde{u} \leq u$, and the value of the constants $a = c = \frac{1}{2}$ and any arbitrary $b \geq \frac{1}{16}$, $d \geq \frac{1}{2}$, so that (H_2) is satisfied independently of the choices of α, β. We fix $a = c = d = \frac{1}{2}$, $b = \frac{1}{16}$, and $T = \frac{1}{2}$. With this choice, $b \neq 0$, and $a^2 = \frac{1}{4} = 4b$, thus

$$h_1(s) = \left(1 + \frac{d}{b}\right)\left(1 + \frac{a}{2}s\right)e^{-\frac{a}{2}s} - \frac{d}{b} = 9\left(1 + \frac{1}{4}s\right)e^{-\frac{1}{4}s} - 8,$$

$$h_2(s) = e^{-\frac{a}{2}s}\left[\frac{c}{b}\left(1 + \frac{a}{2}s\right) + s\right] - \frac{c}{b} = e^{-\frac{1}{4}s}\left[8\left(1 + \frac{1}{4}s\right) + s\right] - 8,$$

$$h_1'(s) = -\frac{a^2}{4}\left(1 + \frac{d}{b}\right)se^{-\frac{a}{2}s} = -\frac{9}{16}se^{-\frac{1}{4}s}, \quad h_2'(s) = e^{-\frac{a}{2}s}\left[-\frac{ac}{2b}\left(1 + \frac{a}{2}s\right) - \frac{a}{2}s + \frac{ac}{2b} + 1\right] = e^{-\frac{1}{4}s}\left[1 - \frac{3}{4}s\right],$$

and

$$H(T - [T]) = H\left(\frac{1}{2}\right) = \begin{pmatrix} h_1\left(\frac{1}{2}\right) & h_2\left(\frac{1}{2}\right) \\ h_1'\left(\frac{1}{2}\right) & h_2'\left(\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{81}{8}e^{-\frac{1}{8}} - 8 & \frac{19}{2}e^{-\frac{1}{8}} - 8 \\ -\frac{9}{32}e^{-\frac{1}{8}} & \frac{5}{8}e^{-\frac{1}{8}} \end{pmatrix} \approx \begin{pmatrix} 0.9353 & 0.3837 \\ -0.2482 & 0.5516 \end{pmatrix}.$$

It is obvious that the function $\alpha \in E$ defined by $\alpha(t) = 0$, $t \in \left[0, \frac{1}{2}\right]$, is a lower solution to (23). On the other hand, the nonnegative function $\beta \in E$ given by $\beta(t) = 7$, $t \in \left[0, \frac{1}{2}\right]$, is an upper solution to (23) since

$$\beta''(t) = 0 \geq g(t, \beta(t), \beta'(t), \beta([t]), \beta'([t])) = g(t, 7, 0, 7, 0) = e^t + \frac{1}{16}\cos(7) - \frac{7}{2}, \ t \in J = \left[0, \frac{1}{2}\right].$$

Then the hypothesis (H_1) holds.

We must also select $L \neq 0$, and $R > 0$ such that $R + L = \frac{1}{2}$, $LR = \frac{1}{16}$, $R \leq 1$,

$$\frac{1}{2} + L > 0, \text{ and } e^{-L} - \frac{1}{2L}\left(1 - e^{-L}\right) \geq 0,$$

Taking into account that $a^2 = \frac{1}{4} = 4b$, according to Remark 9, we can take $R := \frac{a}{2} = \frac{1}{4}$, and $L := \frac{2b}{a} = \frac{1}{4} > 0$, and all of the conditions mentioned are fulfilled since

$$e^{-L} - \frac{1}{2L}\left(1 - e^{-L}\right) = e^{-\frac{1}{4}} - 2\left(1 - e^{-\frac{1}{4}}\right) = 3e^{-\frac{1}{4}} - 2 \approx 0.3364 > 0.$$

Consider the functions defined in the statement of Lemma 2:

$$k_1(t) := \alpha'(t) + R(\alpha(t) - \beta(t)) - K_0 = -\frac{7}{4} - K_0, \quad t \in \left[0, \frac{1}{2}\right),$$

$$k_2(t) := \beta'(t) - R(\alpha(t) - \beta(t)) + K_0 = \frac{7}{4} + K_0, \quad t \in \left[0, \frac{1}{2}\right),$$

where

$$K_0 := \frac{\left(\frac{1}{2} - \frac{1}{8}\right)}{1 - A}\left(\int_0^{\frac{1}{2}} 7e^{\frac{1}{4}\left(s - \frac{1}{2}\right)}\, ds\right)\left(e^{-\frac{1}{4}t} - 2\left(1 - e^{-\frac{1}{4}t}\right)\right) + \int_0^t 7\left(\frac{1}{2} - \frac{1}{8}\right)e^{\frac{1}{4}(s-t)}\, ds,$$

and A is defined in the statement of Lemma 1 taking $F = c$, that is,

$$A := \left(e^{-L} - \frac{c}{L}\left(1 - e^{-L}\right)\right)^{[T]}\left(e^{-L(T-[T])} - \frac{c}{L}\left(1 - e^{-L(T-[T])}\right)\right) = \left(e^{-\frac{1}{8}} - 2\left(1 - e^{-\frac{1}{8}}\right)\right) = 3e^{-\frac{1}{8}} - 2,$$

so that

$$K_0 := \frac{\frac{3}{8}7e^{-\frac{1}{8}}}{3 - 3e^{-\frac{1}{8}}}\left(\int_0^{\frac{1}{2}} e^{\frac{1}{4}s}\, ds\right)\left(3e^{-\frac{1}{4}t} - 2\right) + \frac{21}{8}e^{-\frac{1}{4}t}\int_0^t e^{\frac{1}{4}s}\, ds = \frac{7}{2}\left(3e^{-\frac{1}{4}t} - 2\right) + \frac{21}{2}\left(1 - e^{-\frac{1}{4}t}\right),$$

Besides, the choices of the constants are such that the hypothesis (5) and conditions (I)–(V) in Theorem 2 hold. Indeed, (5) is reduced to

$$det\left(I - H\left(\frac{1}{2}\right)\right) = det\begin{pmatrix} 9 - \frac{81}{8}e^{-\frac{1}{8}} & -\frac{19}{2}e^{-\frac{1}{8}} + 8 \\ \frac{9}{32}e^{-\frac{1}{8}} & 1 - \frac{5}{8}e^{-\frac{1}{8}} \end{pmatrix} \approx det\begin{pmatrix} 0.064719 & -0.383721 \\ 0.248202 & 0.448439 \end{pmatrix} = 0.1243 \neq 0.$$

Condition (I) is satisfied, due to Remark 1, since $\left(1 + \frac{d}{b}\right)\left(1 + \frac{a}{2}\right)e^{-\frac{a}{2}} - \frac{d}{b} = \frac{45}{4}e^{-\frac{1}{4}} - 8 \approx 0.7615 > 0$, and $\frac{a}{2} + c = \frac{1}{4} + \frac{1}{2} \leq 1$. Condition (III) is trivially valid due to Remark 2. In our case, since $\frac{h_1}{h_2}$ is a decreasing function (see Figure 1), then

$$M := \inf_{z \in (0,1)} \frac{h_1(z)}{h_2(z)} = \inf_{z \in (0,1)} \frac{9\left(1 + \frac{1}{4}z\right)e^{-\frac{1}{4}z} - 8}{e^{-\frac{1}{4}z}\left[8\left(1 + \frac{1}{4}z\right) + z\right] - 8} = \frac{h_1(1)}{h_2(1)} \approx 1.3435,$$

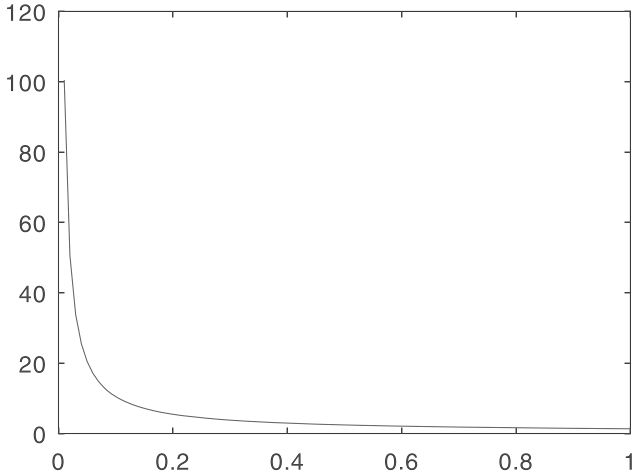

FIGURE 1 Graph of function $\frac{h_1}{h_2}$ on $[0,1]$

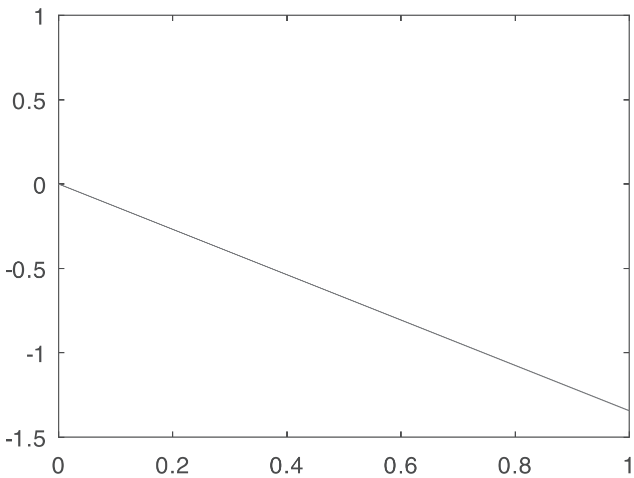

FIGURE 2 Lower bound for the points in the set $\mathcal{S}$

and the set

$$\mathcal{S} := \{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0, \ y \geq -Mx\},$$

is the set of points above the line represented in Figure 2.

For the validity of Condition (II), we must check that V^0 given by

$$V^0 := \left[I - H\left(\frac{1}{2}\right)\right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 3.6088 & 3.0880 \\ -1.9974 & 0.5208 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3.0880 \\ 0.5208 \end{pmatrix}$$

belongs to the set $\mathcal{S}$, but this fact is obvious since its components are positive. Concerning condition (IV), we must prove that $V_{0,s}$, given by

$$\begin{aligned} V_{0,s} &:= \left[I - H\left(\frac{1}{2}\right)\right]^{-1} \begin{pmatrix} g\left(\frac{1}{2}-s\right) \\ g'\left(\frac{1}{2}-s\right) \end{pmatrix} = \left[I - H\left(\frac{1}{2}\right)\right]^{-1} \begin{pmatrix} e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(\frac{1}{2}-s\right) \\ e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(\frac{7}{8}+\frac{1}{4}s\right) \end{pmatrix} \\ &\approx \begin{pmatrix} 3.6088 & 3.0880 \\ -1.9974 & 0.5208 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(\frac{1}{2}-s\right) \\ e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(\frac{7}{8}+\frac{1}{4}s\right) \end{pmatrix} = e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)} \begin{pmatrix} 3.6088\left(\frac{1}{2}-s\right) + 3.0880\left(\frac{7}{8}+\frac{1}{4}s\right) \\ -1.9974\left(\frac{1}{2}-s\right) + 0.5208\left(\frac{7}{8}+\frac{1}{4}s\right) \end{pmatrix}, \end{aligned} \tag{24}$$

belongs to the set $\mathcal{S}$ for every $0 < s < \frac{1}{2}$. Indeed, the first component in each of these (approximated) vectors is given by

$$e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(3.6088\left(\frac{1}{2}-s\right)+3.0880\left(\frac{7}{8}+\frac{1}{4}s\right)\right),$$

and the function $\varphi(s) := 3.6088\left(\frac{1}{2}-s\right)+3.0880\left(\frac{7}{8}+\frac{1}{4}s\right)$, for $s \in \left(0, \frac{1}{2}\right)$, has negative derivative $\varphi'(s) = -3.6088 + \frac{3.0880}{4}$, so that $\varphi(s)$ is greater than or equal to $3.0880 > 0$. This proves that the first component of $V_{0,s}$ is nonnegative for every $s \in \left(0, \frac{1}{2}\right)$. On the other hand, to prove their inclusion in $\mathcal{S}$, it suffices to check that, for each $s \in \left(0, \frac{1}{2}\right)$,

$$-1.9974\left(\frac{1}{2}-s\right)+0.5208\left(\frac{7}{8}+\frac{1}{4}s\right) > -1.3435 * \left(3.6088\left(\frac{1}{2}-s\right)+3.0880\left(\frac{7}{8}+\frac{1}{4}s\right)\right),$$

which is equivalent to the positive character of the function

$$\begin{aligned}\psi(s) &:= (1.3435 * 3.6088 - 1.9974)\left(\frac{1}{2}-s\right)+(0.5208+1.3435 * 3.0880)\left(\frac{7}{8}+\frac{1}{4}s\right)\\ &= 2.8510\left(\frac{1}{2}-s\right)+4.6695\left(\frac{7}{8}+\frac{1}{4}s\right)\end{aligned}$$

for $s \in \left(0, \frac{1}{2}\right)$. This is a decreasing function and its value at $s = \frac{1}{2}$ is clearly positive, so the condition is satisfied. Besides, condition V) is not required since $T \in (0, 1)$. Furthermore, the assumption (H_5) is trivially fulfilled.

In consequence, by Theorem 5, there exists (at least) one solution u to the second-order differential Equation (23) such that $0 \le u \le 7$ and $k_1 \le u' \le k_2$ on $\left[0, \frac{1}{2}\right]$. In Figure 3, we represent the above mentioned estimates for the derivative of u.

Example 2. Consider the problem (23), for which $g : \left[0, \frac{1}{2}\right] \times \mathbb{R}^4 \to \mathbb{R}$ is given by the continuous function

$$g(t, x, y, u, v) = e^t + \frac{1}{16}\cos(x) - \frac{1}{2}y - \frac{1}{2}u - \frac{1}{2}v.$$

From the information in Example 1, the functions $\alpha, \beta \in E$ defined by $\alpha(t) = 0$, $\beta(t) = 7$, $t \in \left[0, \frac{1}{2}\right]$, are, respectively, lower and upper solutions to (23), so that condition (H_1) holds. The one-sided Lipschitz condition observed in Example 1 also guarantees the validity of (H_6) for the value of the constants $a = c = d = \frac{1}{2}$, $b = \frac{1}{16}$, $R = \frac{1}{4}$, and $L = \frac{1}{4} > 0$. Again, the functions k_1 and k_2 defined in the statement of Lemma 2 are, respectively,

$$k_1(t) := -\frac{7}{4} - K_0, \qquad k_2(t) := \frac{7}{4} + K_0, \qquad t \in \left[0, \frac{1}{2}\right),$$

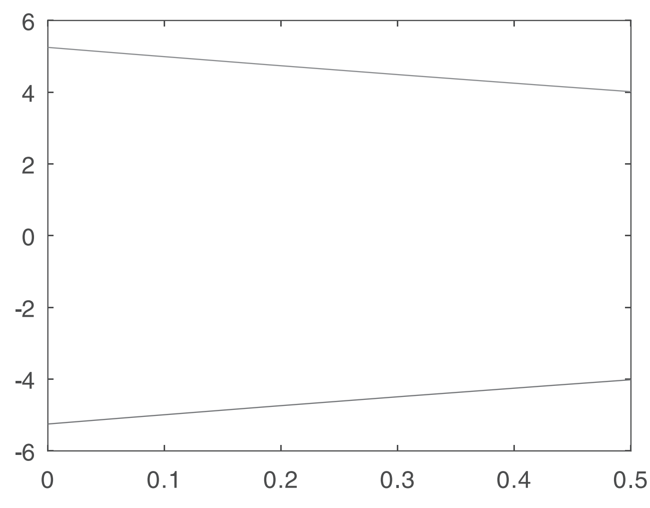

FIGURE 3 Functions k_1 and k_2, which give, respectively, the lower and upper bounds for the derivative of u

where

$$K_0 := \frac{7}{2}\left(3e^{-\frac{1}{4}t}-2\right)+\frac{21}{2}\left(1-e^{-\frac{1}{4}t}\right).$$

We have also checked that the choices of the constants are such that hypothesis (5) and conditions (I)–(V) in Theorem 2 hold.

Since $R>0$, it is easy to check that the inequality (16) holds. Indeed,

$$\alpha'(t)-\beta'(t)=0\le\frac{7}{4}=R(\beta(t)-\alpha(t)),\ \forall t.$$

Then, by Theorem 6, there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ in E such that $\alpha_0=\alpha\equiv 0$, $\beta_0=\beta\equiv 7$, and $\{\alpha_n\}$, $\{\beta_n\}$ are uniformly convergent to ρ, γ, which are the extremal solutions to (23) in the set

$$\left\{\eta\in C^1\left(\left[0,\frac{1}{2}\right]\right)\mid 0\le\eta\le 7 \text{ and } k_1\le\eta'\le k_2 \text{ on } \left[0,\frac{1}{2}\right]\right\}.$$

Furthermore, there exist subsequences $\{\alpha'_{n_k}\}\to\rho'$, $\{\beta'_{n_k}\}\to\gamma'$, as $k\to+\infty$.

The iterative process can be developed in the following way: We define, for each $\eta\in C^1\left(\left[0,\frac{1}{2}\right]\right)$, the auxiliary function

$$\sigma_\eta(t):=g(t,\eta(t),\eta'(t),\eta([t]),\eta'([t]))+a\eta'(t)+b\eta(t)+c\eta'([t])+d\eta([t])=e^t+\frac{1}{16}\cos(\eta(t))+\frac{1}{16}\eta(t),\ \text{ for } t\in\left[0,\frac{1}{2}\right],$$

which clearly belongs to Λ. Since $\lambda=0$, the mapping $\mathcal{A}$ that determines the successive iterations is defined as

$$(\mathcal{A}\eta)(t):=\int_0^{\frac{1}{2}}K(t,s)\sigma_\eta(s)\,ds=\int_0^{\frac{1}{2}}K(t,s)\left(e^s+\frac{1}{16}\cos(\eta(s))+\frac{1}{16}\eta(s)\right)ds,\ t\in\left[0,\frac{1}{2}\right],$$

for $\eta\in C^1\left(\left[0,\frac{1}{2}\right]\right)$, where the Green's function K is given by (see Nieto and Rodríguez-López[25])

$$K(t,s)=\begin{cases}\left(h_1(t)\ h_2(t)\right)V_{0,s}, & t\in\left[0,\frac{1}{2}\right]\cap[0,s),\\[2mm] \left(h_1(t)\ h_2(t)\right)V_{0,s}+g(t-s), & t\in\left[s,\frac{1}{2}\right]\end{cases}$$
$$=\begin{cases}\left(9\left(1+\frac{1}{4}t\right)e^{-\frac{1}{4}t}-8\ \ e^{-\frac{1}{4}t}\left[8\left(1+\frac{1}{4}t\right)+t\right]-8\right)V_{0,s}, & t\in\left[0,\frac{1}{2}\right]\cap[0,s),\\ \left(9\left(1+\frac{1}{4}t\right)e^{-\frac{1}{4}t}-8\ \ e^{-\frac{1}{4}t}\left[8\left(1+\frac{1}{4}t\right)+t\right]-8\right)V_{0,s}+(t-s)e^{-\frac{1}{4}(t-s)}, & t\in\left[s,\frac{1}{2}\right],\end{cases}$$

being $V_{0,s}$ the expression in (24), so that

$$K(t,s)\approx e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\begin{cases}\left(9\left(1+\frac{1}{4}t\right)e^{-\frac{1}{4}t}-8\right)\left(3.6088\left(\frac{1}{2}-s\right)+3.0880\left(\frac{7}{8}+\frac{1}{4}s\right)\right)\\ +\left(e^{-\frac{1}{4}t}\left[8\left(1+\frac{1}{4}t\right)+t\right]-8\right)\left(-1.9974\left(\frac{1}{2}-s\right)+0.5208\left(\frac{7}{8}+\frac{1}{4}s\right)\right), & t\in\left[0,\frac{1}{2}\right]\cap[0,s),\\[2mm] \left(9\left(1+\frac{1}{4}t\right)e^{-\frac{1}{4}t}-8\right)\left(3.6088\left(\frac{1}{2}-s\right)+3.0880\left(\frac{7}{8}+\frac{1}{4}s\right)\right)\\ +\left(e^{-\frac{1}{4}t}\left[8\left(1+\frac{1}{4}t\right)+t\right]-8\right)\left(-1.9974\left(\frac{1}{2}-s\right)+0.5208\left(\frac{7}{8}+\frac{1}{4}s\right)\right)+(t-s)e^{-\frac{1}{4}\left(t-\frac{1}{2}\right)}, & t\in\left[s,\frac{1}{2}\right].\end{cases}$$

The monotonic sequences $\{\alpha_n\}$, $\{\beta_n\}$ are defined as follows:

$$\alpha_0(t)=0,\ \beta_0(t)=7,\qquad t\in\left[0,\frac{1}{2}\right],$$

$$\alpha_n(t) := (\mathcal{A}\alpha_{n-1})(t) = \int_0^{\frac{1}{2}} K(t,s)\left(e^s + \frac{1}{16}\cos(\alpha_{n-1}(s)) + \frac{1}{16}\alpha_{n-1}(s)\right) ds, \qquad t \in \left[0, \frac{1}{2}\right],\ n \geq 1,$$

$$\beta_n(t) := (\mathcal{A}\beta_{n-1})(t) = \int_0^{\frac{1}{2}} K(t,s)\left(e^s + \frac{1}{16}\cos(\beta_{n-1}(s)) + \frac{1}{16}\beta_{n-1}(s)\right) ds, \qquad t \in \left[0, \frac{1}{2}\right],\ n \geq 1.$$

This way, we have $\alpha_0 \equiv 0,\ \beta_0 \equiv 7$, and, for instance, for $t \in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned}
\alpha_1(t) &:= (\mathcal{A}\alpha_0)(t) = \int_0^{\frac{1}{2}} K(t,s)\left(e^s + \frac{1}{16}\cos(\alpha_0(s)) + \frac{1}{16}\alpha_0(s)\right) ds = \int_0^{\frac{1}{2}} K(t,s)\left(e^s + \frac{1}{16}\right) ds \\
&\approx \left(9\left(1+\frac{1}{4}t\right)e^{-\frac{1}{4}t} - 8\right)\int_0^{\frac{1}{2}} e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(3.6088\left(\frac{1}{2}-s\right) + 3.0880\left(\frac{7}{8}+\frac{1}{4}s\right)\right)\left(e^s + \frac{1}{16}\right) ds \\
&+ \left(e^{-\frac{1}{4}t}\left[8\left(1+\frac{1}{4}t\right)+t\right] - 8\right)\int_0^{\frac{1}{2}} e^{-\frac{1}{4}\left(\frac{1}{2}-s\right)}\left(-1.9974\left(\frac{1}{2}-s\right) + 0.5208\left(\frac{7}{8}+\frac{1}{4}s\right)\right)\left(e^s + \frac{1}{16}\right) ds \\
&+ \int_0^t (t-s)e^{-\frac{1}{4}(t-s)}\left(e^s + \frac{1}{16}\right) ds.
\end{aligned}$$

Similarly,

$$\beta_1(t) := (\mathcal{A}\beta_0)(t) = \int_0^{\frac{1}{2}} K(t,s)\left(e^s + \frac{1}{16}\cos(\beta_0(s)) + \frac{1}{16}\beta_0(s)\right) ds = \int_0^{\frac{1}{2}} K(t,s)\left(e^s + \frac{1}{16}\cos(7) + \frac{7}{16}\right) ds, \ \ t \in \left[0, \frac{1}{2}\right],$$

and so on.

For this example, by Theorem 7, we can deduce the following estimates:

$$\begin{aligned}
&\alpha_n'(t) - \frac{1}{4}(\alpha_{n+1}(t) - \alpha_n(t)) - \Gamma_0(7,t) \\
&\leq \alpha_n'(t) - \frac{1}{4}(\alpha_{n+1}(t) - \alpha_n(t)) - \Gamma_0(S-\alpha_n, t) \leq \alpha_n'(t) - \frac{1}{4}(\alpha_{n+1}(t) - \alpha_n(t)) - \Gamma_0(\alpha_{n+1}-\alpha_n, t) \leq \alpha_{n+1}'(t),
\end{aligned}$$

where S may be either ρ, γ, or $\beta \equiv 7$, and

$$\begin{aligned}
\beta_{n+1}'(t) &\leq \beta_n'(t) + \frac{1}{4}(\beta_n(t) - \beta_{n+1}(t)) + \Gamma_0(\beta_n - \beta_{n+1}, t) \\
&\leq \beta_n'(t) + \frac{1}{4}(\beta_n(t) - \beta_{n+1}(t)) + \Gamma_0(\beta_n - S, t) \leq \beta_n'(t) + \frac{1}{4}(\beta_n(t) - \beta_{n+1}(t)) + \Gamma_0(7,t),
\end{aligned}$$

where S may be either $\alpha \equiv 0$, ρ, or γ, for $t \in \left[0, \frac{1}{2}\right)$, and every n, with

$$\Gamma_0(u,t) := \frac{1}{8\left(1-e^{-\frac{1}{8}}\right)}\left(\int_0^{\frac{1}{2}} u([s])e^{\frac{1}{4}\left(s-\frac{1}{2}\right)}\,ds\right)\left(3e^{-\frac{1}{4}t}-2\right)+\int_0^t \frac{3}{8}u([s])e^{\frac{1}{4}(s-t)}\,ds$$
$$= \frac{u(0)}{8\left(1-e^{-\frac{1}{8}}\right)}\left(\int_0^{\frac{1}{2}} e^{\frac{1}{4}\left(s-\frac{1}{2}\right)}\,ds\right)\left(3e^{-\frac{1}{4}t}-2\right)+u(0)\int_0^t \frac{3}{8}e^{\frac{1}{4}(s-t)}\,ds$$
$$= \frac{u(0)}{2}\left(3e^{-\frac{1}{4}t}-2\right)+\frac{3}{2}u(0)\left(1-e^{-\frac{1}{4}t}\right)=\frac{u(0)}{2}.$$

Therefore, the previous inequalities mean

$$\alpha_n'(t)-\frac{1}{4}(\alpha_{n+1}(t)-\alpha_n(t))-\frac{7}{2}$$
$$\leq \alpha_n'(t)-\frac{1}{4}(\alpha_{n+1}(t)-\alpha_n(t))-\frac{S(0)-\alpha_n(0)}{2} \leq \alpha_n'(t)-\frac{1}{4}(\alpha_{n+1}(t)-\alpha_n(t))-\frac{\alpha_{n+1}(0)-\alpha_n(0)}{2} \leq \alpha_{n+1}'(t),$$

where S may be either ρ, γ, or $\beta \equiv 7$, and

$$\beta_{n+1}'(t) \leq \beta_n'(t)+\frac{1}{4}(\beta_n(t)-\beta_{n+1}(t))+\frac{\beta_n(0)-\beta_{n+1}(0)}{2}$$
$$\leq \beta_n'(t)+\frac{1}{4}(\beta_n(t)-\beta_{n+1}(t))+\frac{\beta_n(0)-S(0)}{2} \leq \beta_n'(t)+\frac{1}{4}(\beta_n(t)-\beta_{n+1}(t))+\frac{7}{2},$$

where S may be either $\alpha \equiv 0$, ρ, or γ, for $t \in \left[0, \frac{1}{2}\right)$, and every n.

6 | CONCLUSIONS

In this paper, we presented a study about boundary value problems for nonlinear second-order functional differential equations with piecewise constant arguments of type (3), and obtained the following results:

- We started the research work by recalling several preliminary notions about some suitable spaces of piecewise regular functions, together with some results concerning the expressions of the solutions to the linearized versions of the aforementioned problem (3). In these results, extracted from previous works,[25,27] the deviating argument was based on the integer part function.
- Then, after mentioning some useful results from Buedo-Fernández et al,[27] we obtained some other new comparison results for boundary value problems associated to linear delay differential equations. These results were useful to determine the adequate relation (in terms of order) between two functions and also between their corresponding derivatives.
- Moreover, the notions of upper and lower solutions (β and α) for the problem of interest were presented, and it was proved that, under certain assumptions, the solutions to two comparable linear problems are also comparable.
- Later, in Theorem 5, sufficient conditions were provided in order to prove that the boundary value problem (3) has at least one solution in the functional interval determined by α and β, and whose derivative lies on the functional interval $[k_1, k_2]$ (see Lemma 2, which provided the development of the upper and lower method). The proof of this result was based on the definition of truncation operators by using the functions α, β for the function x, and k_1, k_2 for the derivative x'. Afterwards, the definition of an integral map, whose kernel is given by solutions to linear problems, was a key point to complete the proof by applying comparison-type results, as well as Schauder's Fixed Point Theorem.
- Besides, in Theorem 6, we developed the monotone iterative technique for the nonlinear second-order functional differential problem of interest, by considering certain constraints on the specific upper and lower functions considered. In particular, we proved the existence of monotone sequences $\{\beta_n\}$, $\{\alpha_n\}$ starting at the upper and lower solutions and

converging uniformly, respectively, to the maximal and minimal solutions to (3) in the region determined by $\alpha \leq x \leq \beta$ and $k_1 \leq x' \leq k_2$.

- Finally, since the sequences of the derivatives for the approximate solutions are, in general, not monotonic, we cannot deduce their uniform convergence towards the corresponding derivatives of the extremal solutions. However, in Theorem 7, we obtained some estimates for these derivatives.

The main limitations of the study are the existence of different auxiliary expressions to be used depending on the values of the parameters selected, and the consequent number of regions where it is guaranteed a suitable sign for the solutions to some linear associated problems. Besides, we mention the complexity of the expressions for the solutions of the auxiliary problems considered during the development of the monotone technique. Despite these obstacles, as some of the benefits, we mention that the approach followed allows to present a wide and complete range of cases for the one-sided Lipschitz condition of the nonlinearity to be satisfied in order to develop the procedure, and the use of suitable maximum principles that allow to give a priori bounds for the derivative, so that we can control not only the values of the solutions but also their rate of change. These results are applicable, for instance, to the study of a thermostat that is controlled by the introduction of functional terms in the temperature and the speed of change of the temperature at some fixed instants.

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AUTHOR CONTRIBUTIONS

All the authors contributed equally to the paper.

CONFLICTS OF INTEREST

The authors declare no potential conflicts of interest.

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