A UNIFIED VARIATIONAL APPROACH TO DISCONTINUOUS DIFFERENTIAL EQUATIONS

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Abstract. A direct variational technique involving Clarke generalized gradient is used to treat general boundary value problems with discontinuous nonlinearities. Based on the theory of positive definite symmetric operators it is established the nonsmooth variational form of the regularized inclusions which give the Fillipov solutions of the discontinuous problems. These solutions reduce to classical solutions in case that a transversality condition on the set of discontinuities is satisfied. The results apply to a wide class of concrete boundary value problems of different orders. Two illustrative examples are given.

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1. Introduction

Variational methods are frequently employed in the study of ordinary or partial differential equations with discontinuous nonlinearities, see for instance [3–5, 9, 12, 15] and the references therein. These existence results are mainly based on critical point theory for nonsmooth functionals, which allow to obtain weak solutions. There is another significant line of research in the study of discontinuous differential equations in which the solutions are given in the sense of set–valued analysis, as Filippov solutions, see for example [1, 6].

Here we consider a general problem with discontinuous nonlinearities and we develop a technique which allow us to localize the critical points of the energy functional in bounded sets. We achieve the existence of classical solutions to this class of problems. Obviously, it is necessary to impose a transversality condition on the set of discontinuities, which in our case is given by a countable number of time–dependent curves, following the technique in [10, 14, 17, 21, 22]. We highlight that our transversality condition is more general than similar assumptions employed in the related literature [3, 4], as discussed in Remark 4.1, and no monotonicity conditions are imposed on the nonlinear parts of the considered problems, cf. [8, 16].

We study the existence of solutions for an operator problem of the form

$$\begin{cases} Lu = f(t,u) & \text{for a.a. } t \in (0,1), \\ u \in D(L) \cap \mathcal{B}, \end{cases} \tag{1.1}$$

where the set $\mathcal{B} \subset C[0,1]$ incorporates "boundary conditions", $L : D(L) \subset C[0,1] \to L^2(0,1)$ is an operator whose restriction to $D := D(L) \cap \mathcal{B}$ is an invertible operator, and $f : I \times \mathbb{R} \to \mathbb{R}$ is a function which may be discontinuous with respect to the second variable. Any function $u \in D$ with $f(\cdot, u) \in L^2(0,1)$ and $(Lu)(t) = f(t, u(t))$ for a.a. $t \in (0,1)$ is called a *classical solution* of problem (1.1).

Then the problem is equivalent to the fixed point equation

$$u = N(u),$$

where $N = L^{-1}\mathcal{N}_f$ and $\mathcal{N}_f(u) = f(\cdot, u)$ is the Nemytskii operator associated to f. A fixed point approach to the general problem (1.1) has been considered in [21].The aim of this paper is to present an alternative variational approach. We shall succeed this in the framework of nonsmooth analysis having as key ingredient the relationship of the Clarke generalized gradient associated to f with the multi-valued Nemytskii operator of the regularization of f (Lemma 3.2 below). We believe that this variational approach could further be developed to implement for discontinuous problems many other techniques of solution localization and multiplicity, already known in the continuous case [20].

Notice that, in particular, problem (1.1) can be a two-point boundary value problem for a differential operator of a given order. Thus our result on the general problem (1.1) can be applied to many concrete boundary value problems. Two such examples are presented in the last section.

This paper is organized as follows. In Section 2, we present some concepts and properties from nonsmooth analysis about the generalized gradient in the sense of Clarke. In Section 3, we give the variational framework of the problem (1.1) based on the variational theory of positive-definite symmetric linear operators. Also the main existence result, Theorem 3.4, is stated and proved. Finally, in Section 4, two examples of application of the general theory are included: second-order problems with Dirichlet boundary conditions and fourth-order boundary value problems with cantilever boundary conditions.

2. Preliminaries

Here we introduce some concepts of nonsmooth analysis that we use throughout the text. For more details about generalized gradients we refer to [11] or [19, Ch. 3].

Let X be a real Banach space and X' its dual. We say that a given functional $\phi : X \to \mathbb{R}$ is *locally Lipschitz* if, for each $u \in X$, there exists an open neighborhood $U \subset X$ of u and a positive constant $k > 0$ such that

$$|\phi(v) - \phi(w)| \leq k \, \|v - w\| \qquad \text{for all } v, w \in U.$$

For such a function ϕ, the *generalized directional derivative* at $u \in X$ in the direction of $v \in X$ is defined by

$$\phi^0(u; v) = \limsup_{w \to u,\, t \downarrow 0} \frac{\phi(w + tv) - \phi(w)}{t}.$$

Moreover, the *generalized gradient* (in the sense of Clarke [11]) at $u \in X$ of ϕ is the subset $\partial\phi(u) \subset X'$ defined by

$$\partial\phi(u) = \{\xi \in X' : \ \phi^0(u; v) \geq (\xi, v) \quad \text{for all } v \in X\},$$

where $(\cdot, \cdot)$ stands for the duality pairing between X' and X.

Note that

$$\partial(\lambda\phi)(u) = \lambda\partial\phi(u)$$

and

$$\partial(\phi + \psi)(u) \subset \partial\phi(u) + \partial\psi(u) \tag{2.1}$$

for any locally Lipschitz functionals ϕ and ψ, any $\lambda \in \mathbb{R}$ and any $u \in X$. The equality in (2.1) holds if one of the sets $\partial\phi(u)$, $\partial\psi(u)$ is a singleton. If ψ is Fréchet differentiable in a neighborhood of $u \in X$, then $\partial\psi(u) = \{\psi'(u)\}$. Hence, if ψ is Fréchet differentiable and ϕ is locally Lipschitz, then

$$\partial(\phi + \psi)(u) = \partial\phi(u) + \{\psi'(u)\}$$

for any $u \in X$.

Moreover, if ϕ is locally Lipschitz, then $\partial\phi : X \to \mathcal{P}(X')$ is an upper semicontinuous mapping with respect to the weak* topology on X'. This implies that its graph, $Gr(\partial\phi)$, is closed.

A point $u_0 \in X$ is said to be a *critical point* of ϕ if $0 \in \partial\phi(u_0)$, or equivalently

$$\phi^0(u_0; v) \geq 0 \quad \text{for all } v \in X.$$

In particular, a point of local minimum is a critical point.

3. Main results

We begin by given the variational form of the problem (1.1). To this aim we use the variational theory of positive-definite symmetric linear operators (see [13], [18, Ch.4], [24, Ch. 5]) adapted to our problem (1.1).

In addition to the assumption that $D \subset C\,[0,1]$ and L is invertible as an operator from D to $L^2(0,1)$, hence

$$L^{-1}\left(L^2\,(0,1)\right) = D, \tag{3.1}$$

we assume the following condition:

- **(H1):** $L : D \subset L^2\,(0,1) \to L^2\,(0,1)$ is a linear densely defined operator, symmetric and positive-definite.

Note that L is *symmetric* if $(Lu, v)_{L^2} = (u, Lv)_{L^2}$ for every $u, v \in D$, and *positive-definite* if there exists a constant $c > 0$ such that

$$(Lu, u)_{L^2} \geq c^2\,|u|_{L^2}^2 \quad \text{for every } u \in D.$$

For such a linear operator, we endow the dense linear subspace D of $L^2\,(0,1)$ with the bilinear functional

$$\langle u, v\rangle := (Lu, v)_{L^2} \quad (u, v \in D)\,.$$

The completion of $(D,\ \langle\cdot,\cdot\rangle)$ is denoted by X and is called the *energetic space* of L. By the construction, we have $D \subset X \subset L^2\,(0,1)$ with dense inclusions. We use the same symbol $\langle\cdot,\cdot\rangle$ to denote the extended inner product on X. The corresponding norm $\|u\| = \sqrt{\langle u, u\rangle}$ is called the *energetic norm* associated to L. If $u \in D$, then in view of the positivity of L, one has the *Poincaré inequality*

$$|u|_{L^2} \leq c^{-1}\,\|u\| \quad \text{for every } u \in D.$$

By density, the above inequality extends to the whole X. Let X' be the dual space of X. We shall identify $L^2\,(0,1)$ to its dual via Riesz's representation theorem, and then from $X \subset L^2\,(0,1)\,,$ we have $L^2\,(0,1) \subset X'$. According to Riesz's representation theorem one can define the operator $S : X' \to X$ by

$$\langle Sh, v\rangle = (h, v)\,, \quad v \in X,$$

where the notation (h, v) stands for the value of the functional $h \in X'$ on the element v. Note that if $h \in L^2\,(0,1)\,,$ then (h, v) is the inner product in $L^2\,(0,1)$ of h and v. It is easy to see that S is a bounded linear operator which realizes an isometry between X' and X, and whose inverse $S^{-1} : X \to X'$ extends L from D to X and is called the *Friedrichs* *extension* of L. We use the same symbol L to denote this extension of L to X, and then $S = L^{-1}$. For the given $h \in X'$, the element $u := L^{-1}h \in X$ is said to be the *weak solution* of the equation

$$Lu = h. \tag{3.2}$$

Hence $u \in X$ is a weak solution of equation (3.2) if the following identity holds:

$$\langle u, v \rangle = (h, v), \quad v \in X.$$

In case that the weak solution u belongs to D, equivalently $h \in L^2(0,1)$, we say that u is a *classical solution* of equation (3.2).

Analogously, by a *weak solution* of problem (1.1) we mean a function $u \in X$ such that $f(\cdot, u) \in X'$ and $u = L^{-1} f(\cdot, u)$, that is the following identity holds:

$$\langle u, v \rangle = (f(\cdot, u), v), \quad v \in X. \tag{3.3}$$

If u is a weak solution and $f(\cdot, u) \in L^2(0,1)$, then from assumption (3.1), one has $u \in D$, and identity (3.3) gives

$$(Lu, v)_{L^2} = (f(\cdot, u), v)_{L^2}, \quad v \in D,$$

which in virtue of the density of D in $L^2(0,1)$ implies the equality $Lu = f(\cdot, u)$ in $L^2(0,1)$, i.e., u is a classical solution of (1.1).

A sufficient condition under which weak solutions reduce to classical solutions is the following growth property of f guaranteing that the Nemytskii operator $\mathcal{N}_f$ associated to f maps $L^2(0,1)$ into $L^2(0,1)$.

(H2): $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is such that the composed function $f(\cdot, u(\cdot))$ is measurable for every $u \in L^2(0,1)$ and there are a constant $a \geq 0$ and a function $h \in L^2(0,1;\mathbb{R}_+)$ with

$$|f(t, \tau)| \leq a|\tau| + h(t) \tag{3.4}$$

for all $\tau \in \mathbb{R}$ and a.a. $t \in (0,1)$.

We associate to problem (1.1) the *energy functional*

$$E : X \to \mathbb{R}, \quad E(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F(s, u(s))\, ds,$$

where $F(t,\tau) = \int_0^\tau f(t,s)\, ds$. The functional E can be expressed as

$$E(u) = \frac{1}{2}\|u\|^2 - J(u),$$

where $J : L^2(0,1) \to \mathbb{R}$ is given by

$$J(u) = \int_0^1 F(s, u(s))\, ds.$$

Note that J is locally Lipschitz both on $L^2(0,1)$ and X. Indeed, for every bounded set $U \subset L^2(0,1)$ and any $u, v \in U$, by (H2) and Hölder's inequality, one has

$$\begin{aligned} |J(u)-J(v)| &\leq \int_0^1 \left| \int_{v(s)}^{u(s)} f(s,\tau)\, d\tau \right| ds \leq \int_0^1 \left| \int_{v(s)}^{u(s)} (a\,|\tau| + h(s)\,)\, d\tau \right| ds \\ &\leq \int_0^1 \left(\frac{a}{2} \left| u(s)^2 - v(s)^2 \right| + h(s)\, |u(s)-v(s)| \right) ds \\ &\leq \left(a \sup_{w\in U} |w|_{L^2} + |h|_{L^2} \right) |u-v|_{L^2}\,. \end{aligned}$$

Also, if U is bounded in X, then for every $u, v \in U$, one has

$$|J(u)-J(v)| \leq \frac{1}{c} \left(a \sup_{w\in U} |w|_{L^2} + |h|_{L^2} \right) \|u-v\|\,.$$

In the sequel, let X_R denote the closed ball of X centered at the origin, of radius $R>0$, and let ∂X_R be its boundary.

Lemma 3.1. *Under assumptions* (H1) *and* (H2), *the functional E is bounded from below on each ball X_R of X. If in addition $a < c^2$, then*

$$\inf_{X_R} E < \inf_{\partial X_R} E \tag{3.5}$$

for every $R > 2c\,|h|_{L^2} / \left(c^2 - a\right)$.

Proof. From (3.4), we have

$$|F(t,\tau)| \leq \frac{a}{2}\tau^2 + h(t)\,|\tau|$$

for all $t \in [0,1]$ and $\tau \in \mathbb{R}$. Then for any $u \in X_R$, one has

$$\begin{aligned} E(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,u(t))\,dt \geq -\int_0^1 F(t,u(t))\,dt \\ &\geq -\frac{a}{2}|u|_{L^2}^2 - |h|_{L^2}|u|_{L^2} \geq -\frac{a}{2c^2}\|u\|^2 - \frac{1}{c}|h|_{L^2}\|u\| \\ &\geq -\frac{a}{2c^2}R^2 - \frac{1}{c}|h|_{L^2} R > -\infty. \end{aligned}$$

Thus, for any $R>0$, E is bounded from below on X_R.

Let $a < c^2$ and $R > 2c\,|h|_{L^2} / \left(c^2 - a\right)$. For any $u \in \partial X_R$, we then have

$$\begin{aligned} E(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,u(t))\,dt \geq \frac{1}{2}\|u\|^2 - \frac{a}{2}|u|_{L^2}^2 - |h|_{L^2}|u|_{L^2} \\ &\geq \frac{1}{2}\left(1-\frac{a}{c^2}\right)\|u\|^2 - \frac{1}{c}|h|_{L^2}\|u\| = \frac{1}{2}\left(1-\frac{a}{c^2}\right)R^2 - \frac{1}{c}|h|_{L^2} R \\ &> 0 = E(0) \geq \inf_{X_R} E, \end{aligned}$$

which proves (3.5). □

Our second condition on L is a compactness hypothesis guaranteeing the Palais-Smale condition.

(H3): The set $\Sigma := \{u \in D :\ (Lu,u)_{L^2} = 1\}$ is relatively compact in $L^2(0,1)$.

This condition implies that the embedding $X \subset L^2(0,1)$ is compact, and in consequence, the embedding $L^2(0,1) \subset X'$ is also compact. Indeed, if (v_n) is any bounded sequence in X, there is (u_n) a sequence of elements from D such that $\|u_n - v_n\| \to 0$. Then $|u_n - v_n|_{L^2} \to 0$ too. Clearly this implies that (u_n) is bounded in X. Let $\rho_n = \|u_n\| = (Lu_n, u_n)_{L^2}^{1/2}$. Passing if necessary to subsequences, we may assume that (ρ_n) is convergent to some ρ. One has $u_n/\sqrt{\rho_n} \in \Sigma$. Then from (H3), up to a subsequence, $u_n/\sqrt{\rho_n} \to u$ in $L^2(0,1)$. As a result, $u_n \to \rho u$ in $L^2(0,1)$. This together with $|u_n - v_n|_{L^2} \to 0$ implies $v_n \to \rho u$ in $L^2(0,1)$. Thus we have proved that any bounded sequence in X has a convergent subsequence in $L^2(0,1)$.

Since f may be discontinuous in the state variable, we will consider the regularization given by the multivalued function $\widetilde{f} : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ defined as

$$\widetilde{f}(t,x) = \left[\underline{f}(t,x), \overline{f}(t,x)\right],$$

where

$$\underline{f}(t,x) := \liminf_{y \to x} f(t,y) \quad \text{and} \quad \overline{f}(t,x) := \limsup_{y \to x} f(t,y).$$

Observe that $\widetilde{f}(t,x) = \{f(t,x)\}$ provided that $f(t,\cdot)$ is continuous at x.

Note that if f satisfies (3.4), then $\widetilde{f}$ also has this property.

Define the Nemytskii operator associated to $\widetilde{f}$ by

$$\mathcal{N}_{\widetilde{f}}(u) = \{v \in L^2(0,1) \, : \, v(t) \in \widetilde{f}(t,u(t)) \text{ for a.a. } t \in (0,1)\}.$$

We have the following computation of ∂J, see [12] or [15, Theorem 6.3].

Lemma 3.2. *Assume that f satisfies (H2) and $\underline{f}(\cdot,u(\cdot))$, $\overline{f}(\cdot,u(\cdot))$ are measurable for every $u \in L^2(0,1)$. Then*

$$\partial J(u) \subset \partial_x F(\cdot,u(\cdot)) = \tilde{f}(\cdot,u(\cdot)).$$

The following local property of L together with a transversality condition on f are the key ingredient to prove that the fixed points of $L^{-1}\mathcal{N}_{\widetilde{f}}$ are in fact fixed points of $L^{-1}\mathcal{N}_f$. Note that in particular, the local property holds in case of differential operators, think for example to the operator $Lu = -u''$. Also note that the transversality condition was previously employed in [21, 22] and it is based on the assumptions used in the earlier paper [10].

- **(H4):** If $u, v \in D(L)$ and $u(t) = v(t)$ for all t in some set $I \subset [0,1]$ of positive measure, then $(Lu)(t) = (Lv)(t)$ for a.a. $t \in I$.
- **(H5):** There is a countable number of functions $\gamma_n \in D(L)$ and a countable number of subintervals $I_n \subset [0,1]$ such that

$$\{(L\gamma_n)(t)\} \cap \widetilde{f}(t,\gamma_n(t)) \subset \{f(t,\gamma_n(t))\} \quad \text{for a.a. } t \in I_n, \tag{3.6}$$

and

$$f(t,\cdot) \text{ is continuous on } \mathbb{R} \setminus \bigcup_{\{n:\; t \in I_n\}} \{\gamma_n(t)\} \quad \text{for a.a. } t \in [0,1].$$

Lemma 3.3. *Let conditions (H1), (H2), (H4) and (H5) hold. If $u \in L^{-1}\mathcal{N}_{\widetilde{f}}(u)$, then $u = L^{-1}\mathcal{N}_f(u)$.*

Proof. Let $u \in X$ be such that $u \in L^{-1}\mathcal{N}_{\widetilde{f}}(u)$. Then, since by definition $\mathcal{N}_{\widetilde{f}}(u) \subset L^2(0,1)$, one has $u \in D$ and

$$(Lu)(t) \in \widetilde{f}(t,u(t)) \quad \text{for a.a. } t \in (0,1)\,. \tag{3.7}$$

Consider the sets

$$A_n := \{t \in I_n : u(t) = \gamma_n(t)\}.$$

If the measure of A_n is positive, then according to (H4) we have $(Lu)(t) = (L\gamma_n)(t)$ for a.a. $t \in A_n$ and thus

$$(L\gamma_n)(t) \in \widetilde{f}(t,u(t)) = \widetilde{f}(t,\gamma_n(t)) \quad \text{for a.a. } t \in A_n.$$

This in virtue of (3.6) implies that $(L\gamma_n)(t) = f(t,\gamma_n(t))$ for a.a. $t \in A_n$, equivalently,

$$(Lu)(t) = f(t,u(t)) \quad \text{for a.a. } t \in A_n.$$

Hence, u satisfies the equation $Lu = f(t,u)$ a.e. in $A = \bigcup_n A_n$. For $t \in [0,1] \setminus A$, we have $\widetilde{f}(t,u(t)) = \{f(t,u(t))\}$ and then (3.7) is $(Lu)(t) = f(t,u(t))$. Thus $Lu = f(t,u)$ a.e. in $(0,1)$, that is $u = L^{-1}\mathcal{N}_f(u)$. □

Now we are in a position to state and prove our main result.

Theorem 3.4. *If conditions (H1)-(H5) hold, then problem (1.1) has at least one solution $u \in X_{R_0}$ which minimizes the functional E on X_{R_0}, where*

$$R_0 = \frac{2c\,|h|_{L^2}}{c^2 - a}.$$

Proof. Let $R > R_0$ be arbitrary. By Lemma 3.1, E is bounded from below on X_R. Let (u_n) be any minimizing sequence in X_R, that is,

$$E(u_n) \to m := \inf_{X_R} E.$$

Since the embedding $X \subset L^2(0,1)$ is compact, passing to a subsequence we may assume that $u_n \to u$ in L^2. Then $J(u_n) \to J(u)$, as J is a locally Lipschitz continuous functional.

Moreover, (u_n) being bounded in X, we have that $u_n \rightharpoonup u$ weakly in X. Hence, $\|u\| \leq \liminf_{n\to\infty} \|u_n\|$, whence $u \in X_R$ and

$$\frac{1}{2}\|u\|^2 \leq \liminf_{n\to\infty} \frac{1}{2}\|u_n\|^2\,.$$

It follows that

$$E(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F(s,u(s))\,ds \leq \liminf_{n\to\infty} E(u_n) = m.$$

Therefore, $E(u) = m$. Since $\inf_{X_R} E < \inf_{\partial X_R} E$, u is interior to X_R and then, as a minimum point, u is a critical point of E in the sense of Clarke. Thus

$$0 \in \partial E(u) = Lu - \partial J(u).$$

By Lemma 3.2, $\partial J(u) \subset \mathcal{N}_{\widetilde{f}}(u)$. Hence, we have

$$0 \in Lu - \mathcal{N}_{\widetilde{f}}(u),$$

or

$$u \in L^{-1}\mathcal{N}_{\widetilde{f}}(u),$$

that is, u is a fixed point of $L^{-1}\mathcal{N}_{\widetilde{f}}$. Finally Lemma 3.3 guarantees that $u = L^{-1}\mathcal{N}_f(u)$, that is u is a solution of (1.1).

It remains to show that the same result holds in X_{R_0}. To this aim, take $R = R_n := R_0 + 1/n$. As above, there is $u_n \in X_{R_n}$ with

$$E(u_n) = \inf_{X_{R_n}} E, \quad 0 \in \partial E(u_n). \tag{3.8}$$

We may assume up to a subsequence that $u_n \to u_0$ in X. Indeed, one has $u_n = L^{-1} w_n$, where $w_n = \mathcal{N}_f(u_n)$, and since (u_n) is bounded in X, it is also bounded in $L^2(0,1)$, whence (w_n) is bounded in $L^2(0,1)$. Now the embedding of $L^2(0,1)$ into X' being compact, (w_n) is relatively compact in X'. Next $\left(L^{-1} w_n\right)$ is relatively compact in X, finally, from $u_n = L^{-1} w_n$ it follows that (u_n) is relatively compact in X. From $\|u_n\| \leq R_n$ we have $\|u_0\| \leq R_0$. Next taking the limit in (3.8) by using that $Gr(\partial E)$ has closed graph we obtain

$$E(u_0) = \inf_{X_{R_0}} E, \quad 0 \in \partial E(u_0).$$

Finally, $u_0 \in L^{-1} \mathcal{N}_{\widetilde{f}}(u_0)$ and Lemma 3.3 implies that u_0 is a solution of (1.1). □

The following remark is useful in order to check condition (3.6) in examples.

Remark 3.1. *Condition* (3.6) *is satisfied if there exist* $\delta, \varepsilon > 0$ *such that*

(i) $(L\gamma_n)(t) + \delta \leq f(t,\tau)$ *for a.a.* $t \in I_n$ *and all* $\tau \in [\gamma_n(t) - \varepsilon, \gamma_n(t) + \varepsilon]$*;*

or

(ii) $(L\gamma_n)(t) - \delta \geq f(t,\tau)$ *for a.a.* $t \in I_n$ *and all* $\tau \in [\gamma_n(t) - \varepsilon, \gamma_n(t) + \varepsilon]$*.*

These conditions recall the notion of discontinuity admissible curves presented in [17].

4. Examples

4.1. **A second-order problem with Dirichlet boundary conditions.** As an application of the previous theory, we study the existence of Carathéodory solutions to the two–point boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ may have discontinuities with respect to the spatial variable.

Here,

$$Lu := -u'', \quad D(L) = H^2(0,1), \quad \text{and} \quad \mathcal{B} = \{u \in C[0,1] \, : \, u(0) = u(1) = 0\}.$$

In this case, the energetic space is $X = H_0^1(0,1)$ with inner product and norm

$$\langle u, v \rangle = \int_0^1 u'(s) v'(s) \, ds, \quad \|u\| = \left(\int_0^1 u'^2(s) \, ds \right)^{\frac{1}{2}}.$$

The operator L satisfies condition (H1). Indeed, if we integrate by parts twice, we obtain that $(Lu, v)_{L^2} = (u, Lv)_{L^2}$ for every $u, v \in D$, and thus L is symmetric. Moreover, if $u \in D$, then

$$|u(t)| = \left| \int_0^t u'(s) \, ds \right| \leq \int_0^1 |u'(s)| \, ds \leq \left(\int_0^1 1^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^1 |u'(s)|^2 \, ds \right)^{\frac{1}{2}} = |u'|_{L^2},$$

and thus

$$(Lu, u)_{L^2} = |u'|_{L^2}^2 \geq |u|_{L^2}^2,$$

so L is positive–definite.

In addition, condition (H3) is easily deduced from the compactness of the embedding $H_0^1(0,1) \subset L^2(0,1)$.

Next, we state a technical result (see [2, Lemma 5.8.13]), which is crucial to guarantee that assumption (H4) holds.

Lemma 4.1. *Let $a, b \in \mathbb{R}$, $a < b$. If $\varphi : [a,b] \to \mathbb{R}$ is almost everywhere differentiable on $[a,b]$, then to any null measure set $A \subset \mathbb{R}$ there exists a null measure set $B \subset \varphi^{-1}(A)$ such that*

$$\varphi'(t) = 0 \quad \text{for all } t \in \varphi^{-1}(A) \setminus B.$$

As a consequence, if $u, v \in H^2(0,1) \subset W^{2,1}(0,1)$ and $u(t) = v(t)$ for all $t \in I \subset [0,1]$, by applying twice Lemma 4.1 to $\varphi(t) := u(t) - v(t)$, we deduce that there exists a null measure set $B \subset \varphi^{-1}(\{0\})$ such that

$$\varphi''(t) = 0 \quad \text{for all } t \in \varphi^{-1}(\{0\}) \setminus B.$$

Hence, $-u''(t) = -v''(t)$ for a.a. $t \in I$, as desired.

Now we are in a position to establish the existence of solutions to problem (4.1). By a weak solution of (4.1), we mean a function $u \in H_0^1(0,1)$ such that

$$\int_0^1 u'(s)v'(s)\, ds = \int_0^1 f(s, u(s))v(s)\, ds \quad \text{for all } v \in H_0^1(0,1).$$

Clearly, in virtue of the equality $L^{-1}\left(L^2(0,1)\right) = D$, the weak solutions of (4.1) are also classical solutions in the sense of Carathéodory.

Therefore, as an straightforward consequence of Theorem 3.4, we obtain the following existence result for problem (4.1), which complements those in [17].

Theorem 4.2. *Assume that f fulfills the following two conditions:*

(h1) The composed functions $f(\cdot, u(\cdot))$, $\underline{f}(\cdot, u(\cdot))$ and $\overline{f}(\cdot, u(\cdot))$ are measurable for every $u \in L^2(0,1)$ and there are a constant $a \in [0,1)$ and a function $h \in L^2(0,1;\mathbb{R}_+)$ with

$$|f(t,\tau)| \le a|\tau| + h(t)$$

for all $\tau \in \mathbb{R}$ and a.a. $t \in (0,1)$.

(h2) There is a countable number of functions $\gamma_n \in H^2(0,1)$ and a countable number of subintervals $I_n \subset [0,1]$ such that

$$\{-\gamma_n''(t)\} \cap \widetilde{f}(t, \gamma_n(t)) \subset \{f(t, \gamma_n(t))\} \quad \text{for a.a. } t \in I_n, \tag{4.2}$$

and

$$f(t,\cdot) \text{ is continuous on } \mathbb{R} \setminus \bigcup_{\{n:\ t \in I_n\}} \{\gamma_n(t)\} \quad \text{for a.a. } t \in [0,1].$$

Then problem (4.1) has at least one solution $u \in X_{R_0}$ which minimizes the functional E on X_{R_0}, where $R_0 = 2|h|_{L^2}/(1-a)$.

We emphasize that (4.2) is related to other existent transversality conditions in the literature.

Remark 4.1. *Note that if γ_n are constant functions, $\gamma_n \equiv k_n$, then $-\gamma_n''(t) = 0$ and so condition (4.2) can be rewritten as*

$$0 \in \widetilde{f}(t, k_n) \quad \text{implies} \quad f(t, k_n) = 0 \quad \text{for a.a. } t \in [0,1],$$

which is basically condition (3.18) in [3]. A similar hypothesis is presented in [4, Theorem 3.3], but again no time-dependent discontinuities are considered.

Observe that in [3, 4] the set of discontinuities is a null measure set, while that in our condition it is limited to a countable set. In the case of considering only constant functions γ (as in [3, 4]), then our result can be easily improved, based on Lemma 4.1, in order to allow a non countable number of discontinuity points, as done in [21, Theorem 3.5].

Let us illustrate the previous result with a concrete example. We consider positive piecewise continuous nonlinearities which are discontinuous over a convex time–dependent function γ.

Example 4.3. *Consider the second–order problem* (4.1) *with*

$$f(t,x)=\begin{cases} f_1(t,x), & \text{if } x\geq \gamma(t),\\ f_2(t,x), & \text{if } x<\gamma(t),\end{cases}$$

where $f_1,f_2:[0,1]\times\mathbb{R}\to(0,\infty)$ are continuous and bounded functions, and $\gamma:[0,1]\to\mathbb{R}$, $\gamma\in C^2[0,1]$, such that $\gamma''(t)\geq 0$ for all $t\in[0,1]$.

First, observe one can easily verify that condition (h1) in Theorem 4.2 holds, since the functions f_1 and f_2 are continuous and bounded.

Next, notice that

$$\underline{f}(t,x)=\begin{cases} f_1(t,x), & \text{if } x>\gamma(t),\\ \min\{f_1(t,x),f_2(t,x)\}, & \text{if } x=\gamma(t),\\ f_2(t,x), & \text{if } x<\gamma(t),\end{cases}$$

and

$$\overline{f}(t,x)=\begin{cases} f_1(t,x), & \text{if } x>\gamma(t),\\ \max\{f_1(t,x),f_2(t,x)\}, & \text{if } x=\gamma(t),\\ f_2(t,x), & \text{if } x<\gamma(t).\end{cases}$$

Hence, $\widetilde{f}(t,\gamma(t))=[\min\{f_1(t,\gamma(t)),f_2(t,\gamma(t))\},\max\{f_1(t,\gamma(t)),f_2(t,\gamma(t))\}]$ and the fact that f_1 and f_2 are positive and continuous functions implies $\widetilde{f}(t,\gamma(t))\subset(0,+\infty)$ for all $t\in[0,1]$.

For all $t\in[0,1]$, $f(t,\cdot)$ is continuous on $\mathbb{R}\setminus\{\gamma(t)\}$ and

$$\{-\gamma''(t)\}\cap\widetilde{f}(t,\gamma(t))\subset\{-\gamma''(t)\}\cap(0,+\infty)=\emptyset,$$

so condition (h2) is also satisfied and therefore Theorem 4.2 ensures the existence of a Carathéodory solution.

For instance, we can choose $f_1(t,x)=e^{-x^2}$, $f_2(t,x)=2-\cos(x+t)$ and $\gamma(t)=-2t+1$.

Note that the results can be easily adapted for the more general Sturm-Liouville boundary value problem

$$\begin{cases} -\left(p\left(t\right)u'\right)'+r\left(t\right)u=f(t,u(t)), & t\in[0,1],\\ \alpha u'(0)-\beta u\left(0\right)=\gamma u'(1)+\delta u\left(1\right)=0,\end{cases}$$

where condition (H1) for the associated operator L holds (see [13, Example 3.2]) provided that $\alpha,\beta,\gamma,\delta$ are nonnegative constants such that α,β and γ,δ are not simultaneously zero, respectively; $p\in C^1\left[0,1\right]$, $p\geq 0$ and $1/p\in L^1\left(0,1\right)$; $r\in C\left[0,1\right]$, $r\geq 0$.

4.2. **A fourth-order boundary value problem.** Consider the two-point boundary value problem with cantilever boundary conditions

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{4.3}$$

Now,

$$Lu := u^{(4)}, \quad D(L) = H^4(0,1), \quad \mathcal{B} = \{u \in C^3[0,1] \,:\, u(0) = u'(0) = u''(1) = u'''(1) = 0\}$$

and as shown in [7, 23],

$$X = \{u \in H^2(0,1) \,:\, u(0) = u'(0) = 0\}$$

with inner product and norm

$$\langle u, v \rangle = \int_0^1 u''(s)v''(s)\,ds, \quad \|u\| = \left(\int_0^1 \left(u''(s)\right)^2\,ds\right)^{\frac{1}{2}}.$$

It can be checked in a similar way than in the previous case that L is symmetric and positive-definite with $c = 1$, so condition (H1) holds. Also condition (H3) is consequence of Rellich-Kondrachov embedding theorem.

Therefore, Theorem 3.4 gives a sufficient condition for the existence of one weak solution in the space X to problem (4.3). Note that if u is a weak solution, then the boundary conditions $u''(1) = u'''(1) = 0$ hold (see [7] or [23]) and thus u is a solution in the Carathéodory sense.

A similar result can be established for the boundary value problem

$$\begin{cases} \sum\limits_{k=0}^{m} (-1)^k \frac{d^k}{dt^k}\left[p_k(t)\frac{d^k u}{dt^k}\right] = f(t, u(t)) \\ u^{(j)}(0) = u^{(j)}(1) = 0, \quad j = 0, \cdots, m-1, \end{cases}$$

for which condition (H1) is fullfiled (see [13, Example 3.6]) if $p_k \in C^k[0,1]$, $p_k \geq 0$ for $k = 0, \cdots, m$ and there exists k_0 such that $p_{k_0}(t) \geq \rho > 0$ for all $t \in [0,1]$.

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References

[1] C. Bereanu, P. Jebelean and C. Şerban, The Dirichlet problem for discontinuous perturbations of the mean curvature operator in Minkowski space, *Electron. J. Qual. Theory Differ. Equ.*, No. 35 (2015), 1–7.
[2] V. I. Bogachev, *Measure theory*, Vol. I, Springer, 2007.
[3] G. Bonanno and G. M. Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* (2009) 2009:670675.
[4] G. Bonanno and S. M. Buccellato, Two point boundary value problems for the Sturm–Liouville equation with highly discontinuous nonlinearities, *Taiwan. J. Math.*, **14** 5 (2010), 2059–2072.
[5] G. Bonanno, A. Iannizzotto and M. Marras, On ordinary differential inclusions with mixed boundary conditions, *Differential Integral Equations*, **30** (2017), 273–288.
[6] G. Bonanno, P. Jebelean and C. Şerban, Three periodic solutions for discontinuous perturbations of the vector p-Laplacian operator, *Proc. Roy. Soc. Edinburgh Sect. A*, **147** (2017), 673–681.
[7] A. Cabada, R. Precup, S. Tersian and L. Saavedra, Multiple positive solutions to a fourth-order boundary-value problem, *Electron. J. Differential Equations*, No. 254 (2016), 1–18.
[8] S. Carl and S. Heikkilä, *Nonlinear Differential Equations in Ordered Spaces,* Chapman & Hall/CRC, Boca Raton, 2000.

[9] K.-C. Chang, Variational methods for non–differentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.*, **80** (1981), 102–129.
[10] J. Á. Cid and R. López Pouso, Ordinary differential equations and systems with time–dependent discontinuity sets, *Proc. Roy. Soc. Edinburgh Sect. A*, **134** (2004), 617–637.
[11] F. H. Clarke, *Optimization and nonsmooth analysis*, John Wiley and Sons, New York, 1983.
[12] D. Costa and J. Gonçalves, Critical point theory for nondifferentiable functionals and applications, *J. Math. Anal. Appl.*, **153** (1990), 470–485.
[13] G. Dincă, *Metode variaţionale şi aplicaţii* [Variational Methods and Aplications] (in Romanian), Editura Tehnică, Bucharest, 1980.
[14] R. Figueroa, R. López Pouso and J. Rodríguez–López, A version of Krasnosel'skiĭ's compression–expansion fixed point theorem in cones for discontinuous operators with applications, *Topol. Methods Nonlinear Anal.*, **51** 2 (2018), 493–510.
[15] M. R. Grossinho and S. Tersian, *An Introduction to Minimax Theorems and Their Applications to Differential Equations*, Kluwer, Dordrecht, 2001.
[16] S. Heikkilä and V. Lakshmikantham, *Monotone iterative techniques for discontinuous nonlinear differential equations,* Marcel Dekker, New York, 1994.
[17] R. López Pouso, Schauder's fixed–point theorem: new applications and a new version for discontinuous operators, *Bound. Value Probl.*, **2012**:92 (2012).
[18] S. G. Mikhlin, *Linear Partial Differential Equations* (in Russian), Vysshaya Shkola, Moscow, 1977.
[19] D. Motreanu, V. V. Motreanu and N. Papageorgiou, *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, 2014.
[20] R. Precup, A variational analogue of Krasnoselskii's cone fixed point theory, in *Nonlinear Analysis and Boundary Value Problems*, Springer Proceedings in Mathematics & Statistics 292, Springer, 2019, pp. 1-18.
[21] R. Precup and J. Rodríguez–López, Positive solutions for discontinuous problems with applications to ϕ-Laplacian equations, *J. Fixed Point Theory Appl.*, **20**:156 (2018), 1–17.
[22] R. Precup and J. Rodríguez–López, Positive solutions for ϕ-Laplace equations with discontinuous state-dependent forcing terms, *Nonlinear Anal. Model. Control*, **24** (2019), 447–461.
[23] L. Yang, H. Chen and X. Yang, The multiplicity of solutions for fourth-order equations generated from a boundary condition, *Appl. Math. Lett.*, **24** (2011), 1599–1603.
[24] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Springer, Berlin, 1995.

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