Hybrid localization for nonlinear systems: lower/upper solution and Krasnosel'skiĭ fixed point theorem techniques

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Abstract. We present a novel localization of the solutions of a system of two differential equations. It combines, in a component-wise manner, the method of lower and upper solutions with the localization provided by compression–expansion type fixed point theorems in cones. The main result is based on a recent fixed point theorem for operator systems.

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1 Introduction

Consider the following system of second–order equations with Dirichlet boundary conditions

$$\begin{cases} -u'' = f(t,u,v), & t \in [0,1], \\ -v'' = g(t,u,v), & t \in [0,1], \\ u(0) = u(1) = 0 = v(0) = v(1), \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and $g : [0,1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The method of lower and upper solutions has been widely employed in order to establish existence results to different types of differential problems. In the case of two-point boundary value problems for second–order differential equations, we refer the reader to the overview paper [1] and the monograph [2]. On the other hand, compression–expansion type fixed point theorems in cones are a powerful tool to obtain and localize positive solutions of boundary value problems, see for instance [3] and the references therein.

Our aim is to obtain solutions (u, v) to problem (1.1) with a component-wise localization in the following way: u is non-negative and its sup-norm satisfies some estimation of the form $0 < r \leq \|u\|_\infty \leq R$, whereas v is located in a functional interval $[\alpha, \beta]$ with $\alpha, \beta : [0,1] \to \mathbb{R}$ continuous functions. Clearly, the localization of u recalls that obtained as an application of Krasnosel'skiĭ compression–expansion fixed point theorem in cones and v seems to be located between a couple of well–ordered lower and upper solutions. To the best of our knowledge, this type of hybrid approach is new in the literature and complements that due to the *vector version of Krasnosel'skiĭ fixed point theorem* established by Precup in [5]. We highlight that our technique is not restricted to problem (1.1) and its range of applicability is only limited by that of the method of lower/upper solutions and of Krasnosel'skiĭ theorem.

Our results are based on the following fixed point principle for operator systems which combines the assumptions of the classical Krasnosel'skiĭ and Schauder fixed point theorems. It was recently proved in [4], by means of the fixed point index theory.

Theorem 1.1 *Let U and V be bounded and relatively open subsets of a cone K of a Banach space X such that $0 \in V \subset \overline{V} \subset U$ and D be a closed convex subset of a Banach space Y.*

Assume that $T = (T_1, T_2) : \left(\overline{U} \setminus V\right) \times D \to K \times D$ is a compact map and there exists $h \in K \setminus \{0\}$ such that either of the following conditions holds in $\left(\overline{U} \setminus V\right) \times D$:

(a) $T_1(u_1, u_2) + \mu\, h \neq u_1$ if $u_1 \in \partial V$ and $\mu > 0$, and $T_1(u_1, u_2) \neq \lambda\, u_1$ if $u_1 \in \partial U$ and $\lambda > 1$; or

(b) $T_1(u_1, u_2) \neq \lambda\, u_1$ if $u_1 \in \partial V$ and $\lambda > 1$, and $T_1(u_1, u_2) + \mu\, h \neq u_1$ if $u_1 \in \partial U$ and $\mu > 0$.

Then T has at least a fixed point $u = (u_1, u_2) \in K \times D$ with $u_1 \in \overline{U} \setminus V$.

2 Auxiliary problem

It is well–known that problem (1.1) can be equivalently rewritten as a system of Hammerstein equations of the form

$$\begin{cases} u(t) = \displaystyle\int_0^1 G(t,s) f(s, u(s), v(s))\, ds =: T_1(u,v)(t), \\ v(t) = \displaystyle\int_0^1 G(t,s) g(s, u(s), v(s))\, ds =: T_2(u,v)(t), \end{cases} \tag{2.2}$$

where $G : [0,1] \times [0,1] \to \mathbb{R}$ denotes the corresponding Green's function, that is,

$$G(t,s) = \begin{cases} s\,(1-t), & \text{if } s \leq t, \\ t\,(1-s), & \text{if } s > t. \end{cases}$$

Clearly, it satisfies that

$$\begin{aligned} G(t,s) &\leq s\,(1-s) && \text{for all } t, s \in [0,1], \\ s\,(1-s)/4 &\leq G(t,s) && \text{for all } t \in [1/4, 3/4],\ s \in [0,1]. \end{aligned} \tag{2.3}$$

In relation with the Green's function G, we will employ the following notations:

$$A := \min_{[1/4,3/4]} \int_{1/4}^{3/4} G(t,s)\,ds, \qquad B := \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds.$$

Observe that a simple computation leads to $A = 1/16$ and $B = 1/8$.

These properties of the Green's function allow us to work with the following cone K in the Banach space of continuous functions $X = \mathcal{C}([0,1])$ endowed with the sup-norm $\|w\|_\infty := \max_{t\in[0,1]} |w(t)|$, namely,

$$K := \left\{ w \in X : w(t) \geq 0 \text{ for all } t \in [0,1] \text{ and } \min_{t\in[1/4,3/4]} w(t) \geq \|w\|_\infty /4 \right\}.$$

It is standard to check that the operator $T = (T_1, T_2) : K \times X \to X \times X$ defined as in (2.2) satisfies that $T_1(K \times X) \subset K$ and T is completely continuous (i.e., it is continuous and maps bounded sets into relatively compact ones), see for instance [3].

Here, we assume that

(A_1) g is bounded, i.e., there exists $k > 0$ such that $|g(t,x,y)| \leq k$ for all $(t,x,y) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}$;

(A_2) there exist $\rho_1, \rho_2 > 0$, $\rho_1 \neq \rho_2$, such that

$$m_f(\rho_1) := \min\{f(t,x,y) : t \in [1/4,3/4],\ \rho_1/4 \leq x \leq \rho_1,\ y \in \mathbb{R}\} \geq \frac{\rho_1}{A};$$
$$M_f(\rho_2) := \max\{f(t,x,y) : t \in [0,1],\ 0 \leq x \leq \rho_2,\ y \in \mathbb{R}\} \leq \frac{\rho_2}{B}.$$

Under these conditions we obtain the following existence result.

Theorem 2.1 *Assume that f and g are continuous functions and assumptions (A_1) and (A_2) hold. Then the Hammerstein type system* (2.2) *has at least one solution $(u,v) \in K \times X$ such that $r \leq \|u\|_\infty \leq R$, with $r := \min\{\rho_1, \rho_2\}$ and $R := \max\{\rho_1, \rho_2\}$, and $\|v\|_\infty \leq R_2$ with $R_2 := B\,k$.*

Equivalently, the system (1.1) *has at least one solution (u,v) such that $u \geq 0$, $r \leq \|u\|_\infty \leq R$ and $\|v\|_\infty \leq R_2$.*

Proof. Let us apply Theorem 1.1 to the operator $T = (T_1, T_2)$ defined as in (2.2). To do so, choose $U := \{w \in K : \|w\|_\infty < R\}$, $V := \{w \in K : \|w\|_\infty < r\}$ and $D = \{w \in X : \|w\|_\infty \leq R_2\}$. Observe that the operator $T = (T_1,T_2) : \left(\overline{U} \setminus V\right) \times D \to K \times D$ is well-defined since $T_2\left(\left(\overline{U} \setminus V\right) \times D\right) \subset D$. Indeed, for every $t \in [0,1]$,

$$|T_2(u,v)(t)| \leq \int_0^1 |G(t,s)|\,|g(s,u(s),v(s))|\,ds \leq k \int_0^1 |G(t,s)|\,ds \leq k\,B = R_2,$$

and so $\|T_2(u,v)\|_\infty \leq R_2$ for all $(u,v) \in \left(\overline{U} \setminus V\right) \times D$.

Now, let us prove that $T_1(u,v)+\mu\,\mathbf{1}\neq u$ for all $(u,v)\in K\times D$ with $\|u\|_\infty=\rho_1$ and all $\mu>0$ (where $\mathbf{1}$ stands for the constant function equal to one). By *reductio ad absurdum*, we suppose that there exist $(u,v)\in K\times D$ with $\|u\|_\infty=\rho_1$ and $\mu>0$ such that

$$u(t)=\int_0^1 G(t,s)f(s,u(s),v(s))\,ds+\mu,\qquad t\in[0,1].$$

Since $u\in K$ and $\|u\|_\infty=\rho_1$, we have that $\rho_1/4\leq u(t)\leq\rho_1$ for all $t\in[1/4,3/4]$. Hence, by assumption (A_2), we have that for $t\in[1/4,3/4]$,

$$u(t)>\int_0^1 G(t,s)f(s,u(s),v(s))\,ds\geq\int_{1/4}^{3/4} G(t,s)f(s,u(s),v(s))\,ds\geq m_f(\rho_1)\,A\geq\rho_1,$$

a contradiction.

Finally, let us show that $\|T_1(u,v)\|_\infty\leq\rho_2$ for all $(u,v)\in K\times D$ with $\|u\|_\infty=\rho_2$, which clearly ensures that $T_1(u,v)\neq\lambda\,u$ for all $(u,v)\in K\times D$ with $\|u\|_\infty=\rho_2$ and $\lambda>1$. By hypothesis (A_2), for $t\in[0,1]$,

$$T_1(u,v)(t)=\int_0^1 G(t,s)f(s,u(s),v(s))\,ds\leq M_f(\rho_2)\int_0^1 G(t,s)\,ds\leq M_f(\rho_2)\,B\leq\rho_2,$$

and thus, taking the maximum in $[0,1]$, $\|T_1(u,v)\|_\infty\leq\rho_2$ for all $(u,v)\in K\times D$ with $\|u\|_\infty=\rho_2$.

Therefore, one of the alternatives (a) or (b) in Theorem 1.1 holds and, in conclusion, T has a fixed point $(u,v)\in\left(\overline{U}\setminus V\right)\times D$. $\square$

3 Main results

Let us introduce the concept of u–uniform lower and upper solutions for the second equation of (1.1), namely, the differential problem

$$-v''=g(t,u,v),\quad t\in[0,1],\quad v(0)=v(1)=0. \tag{3.4}$$

Definition 3.1 *A function* $\alpha:[0,1]\to\mathbb{R}$, $\alpha\in\mathcal{C}([0,1])\cap\mathcal{C}^2((0,1))$, *is said to be a u–uniform lower solution for* (3.4) *if*

$$-\alpha''(t)\leq g(t,u,\alpha(t)),\quad t\in(0,1),\quad \alpha(0)\leq 0,\quad \alpha(1)\leq 0,$$

whenever $u\in\mathbb{R}_+$.

Similarly, a function $\beta\in\mathcal{C}([0,1])\cap\mathcal{C}^2((0,1))$ *is an u–uniform upper solution for* (3.4) *if it satisfies the previous inequalities in the reverse order.*

Now we present the main result of this note.

THEOREM 3.1 *Assume that* f *and* g *are continuous functions and there exist* α *and* β u*–uniform lower and upper solutions for* (3.4)*, respectively, such that* $\alpha(t) \leq \beta(t)$ *for all* $t \in [0,1]$*. Moreover, assume that there exist* $\rho_1, \rho_2 > 0$*,* $\rho_1 \neq \rho_2$*, such that*

$$m_f^{\alpha,\beta}(\rho_1) := \min\{f(t,x,y) : t \in [1/4,3/4],\ \rho_1/4 \leq x \leq \rho_1,\ \alpha(t) \leq y \leq \beta(t)\} \geq \frac{\rho_1}{A};$$

$$M_f^{\alpha,\beta}(\rho_2) := \max\{f(t,x,y) : t \in [0,1],\ 0 \leq x \leq \rho_2,\ \alpha(t) \leq y \leq \beta(t)\} \leq \frac{\rho_2}{B}.$$

Then the system (1.1) *has at least one solution* (u,v) *such that* $u(t) \geq 0$ *for all* $t \in [0,1]$*,* $\min\{\rho_1,\rho_2\} \leq \|u\|_\infty \leq \max\{\rho_1,\rho_2\}$ *and* $\alpha(t) \leq v(t) \leq \beta(t)$ *for all* $t \in [0,1]$*.*

Proof. Let us consider the modified problem

$$\begin{cases} -u'' = f^*(t,u,v), & t \in [0,1], \\ -v'' = g^*(t,u,v), & t \in [0,1], \\ u(0) = u(1) = 0 = v(0) = v(1), \end{cases} \tag{3.5}$$

where

$$f^*(t,x,y) = f(t,x,\gamma(t,y)), \qquad g^*(t,x,y) = g(t,\delta_R(x),\gamma(t,y)),$$

and $\delta_R : \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ are the continuous functions given by

$$\delta_R(x) = \min\{x,R\} \quad \text{and} \quad \gamma(t,y) = \max\{\min\{y,\beta(t)\},\alpha(t)\},$$

with $R := \max\{\rho_1,\rho_2\}$.

Since f^* and g^* are continuous functions satisfying assumptions (A_1) and (A_2), Theorem 2.1 ensures that the modified problem (3.5) has a solution (u,v) such that u is non-negative on $[0,1]$ and $\min\{\rho_1,\rho_2\} \leq \|u\|_\infty \leq \max\{\rho_1,\rho_2\}$.

Finally, let us see that $\alpha(t) \leq v(t) \leq \beta(t)$ for all $t \in [0,1]$. Suppose that there exist $t_0 \in (0,1)$ such that

$$\min_{t\in[0,1]} (v-\alpha)(t) = v(t_0) - \alpha(t_0) < 0$$

and

$$v(t_0) - \alpha(t_0) < v(t) - \alpha(t) \quad \text{for all } t \in (t_0, 1).$$

Then we have that $v'(t_0) - \alpha'(t_0) = 0$ and, by the continuity of $v - \alpha$, there exists $\varepsilon > 0$ such that $v(t) - \alpha(t) < 0$ for all $t \in (t_0, t_0+\varepsilon)$. Being (u,v) a solution of (3.5), we deduce that

$$-v''(t) = g(t,u(t),\alpha(t)), \quad t \in (t_0, t_0+\varepsilon).$$

By integration and the definition of α, we obtain that

$$\alpha'(t) - v'(t) = \int_{t_0}^{t} (\alpha''(s) + g(s,u(s),\alpha(s)))\, ds \geq 0 \quad \text{on } (t_0, t_0+\varepsilon),$$

so $v(t) - \alpha(t) \leq v(t_0) - \alpha(t_0)$ for all $t \in (t_0, t_0+\varepsilon)$, which contradicts the choice of t_0. Therefore, $\alpha \leq v$ on $[0,1]$ and the inequality $v \leq \beta$ can be proven in a similar way.

In conclusion, (u,v) solves (1.1) and $\alpha(t) \leq v(t) \leq \beta(t)$ for all $t \in [0,1]$, as wished. □

Example 3.1 *Consider the problem* (1.1) *with the nonlinearities*

$$f(t,x,y)=\sqrt{x}\,e^{|y|},\quad g(t,x,y)=-t^2\cos^2(x)-y\,e^{|y|},\quad (t,x,y)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}.$$

It is easy to check that the functions $\alpha(t)=-t$ *and* $\beta(t)=0$, $t\in[0,1]$, *are, respectively,* u*-uniform lower and upper solutions for the second equation of the system.*

Further, observe that the choices of $\rho_1=1/1024$ *and* $\rho_2=9/64$ *allow us to prove that*

$$m_f^{\alpha,\beta}(\rho_1)\geq 16\,\rho_1 \quad \text{and} \quad M_f^{\alpha,\beta}(\rho_2)\leq 8\,\rho_2,$$

as required in Theorem 3.1.

Therefore, the system has a solution (u,v) *such that* u *is a nonnegative function satisfying the norm estimation* $1/1024\leq\|u\|_\infty\leq 9/64$ *and, in addition,* $-t\leq v(t)\leq 0$ *for all* $t\in[0,1]$.

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