

Stability of charged global AdS₄ spacetimes

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ABSTRACT: We study linear and nonlinear stability of asymptotically AdS₄ solutions in Einstein-Maxwell-scalar theory. After summarizing the set of static solutions we first examine thermodynamical stability in the grand canonical ensemble and the phase transitions that occur among them. In the second part of the paper we focus on nonlinear stability in the microcanonical ensemble by evolving radial perturbations numerically. We find hints of an instability corner for vanishingly small perturbations of the same kind as the ones present in the uncharged case. Collapses are avoided, instead, if the charge and mass of the perturbations come to close the line of solitons. Finally we examine the soliton solutions. The linear spectrum of normal modes is not resonant and instability turns on at extrema of the mass curve. Linear stability extends to nonlinear stability up to some threshold for the amplitude of the perturbation. Beyond that, the soliton is destroyed and collapses to a hairy black hole. The relative width of this stability band scales down with the charge Q , and does not survive the blow up limit to a planar geometry.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Classical Theories of Gravity, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

Stability of Anti de Sitter vacua has received much attention in the last years since the numerical experiment in [1]. There it was found that nonlinear evolution of some family of arbitrarily small scalar field perturbations inevitably end up in the collapse and formation of a black hole. This is the so called instability corner [2]. Perturbatively, the problem was also examined [3, 4] in the context of purely gravitational perturbations, and the importance of two ingredients was signalled: the presence of a fully resonant spectrum for the linearized perturbations, and the existence of periodic solutions that acted as centers of some stability islands in the space of initial conditions. Back to the scalar field case, this suggestion received further backup from other contributions [5, 6]. In this case, the periodic solutions were named oscillons, and indeed, the two previous observations became consistent in that the spectrum of linearized perturbations around an oscillon turns out not to be fully resonant. After some years of analytic and numerical work, it has become clear that there is a wealth of situations that one can encounter. One may choose to change either the dynamics (the action) or the kinematics (the boundary conditions). Generically, when departing from the easiest case of perturbations on pure AdS, the resonant property is lost [7, 8]. An odd case is that of AdS₃, where the mass gap (of conical singularities) kills the instability corner despite the fact that the spectrum is fully resonant [9, 10].

In this paper we study both linear and nonlinear stability of asymptotically AdS vacua in Einstein-Maxwell theory interacting with a charged massless scalar. There's little we can add to motivate focussing on this theory. The space of static solutions is rich and involves regular solitons, hairy and Reissner-Nordström (RN) black holes. The first two are interpreted in the context of AdS/CFT as holographic duals of quantum states with spontaneously broken global U(1) symmetry. We will be working in AdS₄ in global coordinates. Our findings build up in parallel with the achievements of [11, 12] where the landscape of static solutions was unraveled for the case of AdS₅. The thermodynamical (linear) stability issue concerning the competition of these three solutions depends upon the ensemble that is considered. We will construct the grand potential to find the correct vacua in the grand canonical ensemble.

For the nonlinear stability we must carry out numerical evolution analysis, searching for endpoints of the evolution in one of the above possible static forms. Performing these simulations with Dirichlet boundary conditions at the boundary, is tantamount to studying the thermalization of the dual quantum system in the microcanonical ensemble.

The Lagrangian that governs the dynamics is given by¹

$$S = \int d^{d+1}x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(\mathcal{R} + \frac{d(d-1)}{l^2} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |D_\mu \phi|^2 \right] \quad (1.1)$$

for an Anti de Sitter space with curvature radius l . Here $D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$ and there is a U(1) gauge symmetry. The coupling e is a free parameter and we can measure all lengths

¹We find it advantageous to keep this somewhat old fashioned normalization, since it allows to make connection with different conventions in the literature. For example, the one used [11, 12] is recovered by setting $\kappa^2 = 1$. In this paper we will set instead $\kappa^2 = (d-1)/2$.

in units of l , hence setting $l = 1$. Analytic expressions for generic d are provided in the appendices, but the numerical analysis will examine only the $d = 3$ case.

RN and hairy black holes are dual to normal and superfluid phases of the dual field theory respectively. In this last case, it is the scalar field that condenses, breaking spontaneously the $U(1)$ global symmetry. The dual state is called an s -wave superfluid [13]. A similar setup was considered in [14], albeit in the probe limit, while our analysis takes full account of the backreaction of the fields on the geometry. In this context, it is known that black holes come typically in pairs, one of which is small and thermodynamically unstable. Moreover, there are solitonic solutions which are completely regular in the bulk. Whenever these are the ground states, they should correspond in the dual field theory to zero entropy condensates. Finally, we also find extremal black holes, both among RN and hairy solutions.

The plan of the paper is the following. Section 2 contains a brief summary of the set of solutions that can be obtained as a function of the electromagnetic coupling e . It is written in the same style as in [11] and [12] for AdS_5 , highlighting differences and similarities. In section 3 we address the issue of the connection of such solutions with superfluid states. This requires identifying the thermodynamically dominant phase in the grand canonical ensemble. We compute the renormalized grand potential and find out both first and second order phase transitions. Nonlinear stability is the topic of concern in section 4. It requires numerically evolving arbitrarily small initial perturbations over AdS_4 . The central question in this section is whether similar conclusions as those obtained in [1] can be extrapolated to the present situation. The answer points in the affirmative, namely, the vacuum exhibits a corner of nonlinear instability even at finite charge. On the other hand, the islands of stability get amplified, probably as a consequence of the electrostatic repulsion. This matches with expectations, since the AdS_4 linear scalar perturbations are still fully resonant in this theory. In section 5 we study the stability of solitonic ground states. In a band above them, oscillating solutions never decay. We found the upper bound of this protected from collapse region and examine the way it scales upon blowing up into a black brane geometry. Our findings indicate that, in such limit, this region does not survive. We also determine the scalar field normal mode spectrum above a solitonic background, finding that it has a dispersive character. In the light of findings in [4] this implies that the weakly turbulent instability present for the AdS_4 vacuum is absent in these geometries.

2 Static solutions

In this section we describe the space of static solutions as a function of the coupling e . We shall start by providing some background material, and relegate to the appendices all the cumbersome expressions.

2.1 Construction

The ansatz for the metric of AdS_{d+1} in global coordinates is the following

$$ds^2 = \frac{l^2}{\cos^2 x} \left(-f(t, x) e^{-2\delta(t, x)} dt^2 + f(t, x)^{-1} dx^2 + \sin^2 x d\Omega_{d-1}^2 \right), \quad (2.1)$$

where $x \in [0, \pi/2]$. The standard Schwarzschild radial coordinate r is related to x as $r = \tan(x)$. Concerning the gauge field, isotropy is maintained by selecting $A = A_t(t, x)dt$. This ansatz leaves as residual gauge transformations

$$\phi \rightarrow e^{i\Lambda(t)}\phi; \quad A_t \rightarrow A_t - ie\partial_t\Lambda(t), \quad (2.2)$$

under which the equations of motion must be covariant. This motivates defining the following U(1) covariant fields²

$$\Phi(t, x) \equiv \phi'(t, x), \quad (2.3)$$

$$\Pi(t, x) \equiv \frac{e^\delta}{f} D_t \phi. \quad (2.4)$$

Equations of motion for the time dependent ansatz can be found in appendix A, eqs. (A.8)–(A.11) and (A.16). Equations for static solutions follow after setting $\Pi = -ie^\delta A_t \phi / f$, $\dot{\Pi} = 0$ and $C = A'_t$. We provide them here for completeness

$$\begin{aligned} \phi''(x) + \left((d-1) \csc(x) \sec(x) + \frac{f'(x)}{f(x)} - \delta'(x) \right) \phi'(x) + e^2 \frac{e^{2\delta(x)} A(x)^2 \phi(x)}{f(x)^2} &= 0, \\ A''(x) + \left((d-1) \cot(x) + (d-3) \tan(x) + \delta'(x) \right) A'(x) - e^2 \frac{2 \sec(x)^2 A(x) \phi(x)^2}{f(x)} &= 0, \\ f'(x) - \frac{d-2+2 \sin(x)^2}{\sin(x) \cos(x)} (1-f(x)) - f(x) \delta'(x) + \frac{1}{2} \cos(x)^3 \sin(x) e^{2\delta(x)} A'(x)^2 &= 0, \\ \delta'(x) + \cos(x) \sin(x) \left(\phi'(x)^2 + e^2 \frac{e^{2\delta(x)} A(x)^2 \phi(x)^2}{f(x)^2} \right) &= 0. \end{aligned} \quad (2.5)$$

The scalar field ϕ can be taken real by an appropriate gauge choice. Solving (2.5) requires, as usual, specifying boundary conditions at both ends of the radial domain. At the boundary, $x_b = \frac{\pi}{2}$, which is holographically related to the UV regime of the field theory, the following asymptotics follow directly from the equations of motion, setting $\rho = x - \frac{\pi}{2}$,

$$\phi(x) = \phi_b + \dots + \phi_{b,3} \rho^3 + \dots, \quad (2.6)$$

$$A_t(x) = \mu + Q\rho + \dots, \quad (2.7)$$

$$f(x) = 1 + \dots + M\rho^3 + \dots, \quad (2.8)$$

$$\delta(x) = \delta_b + \dots + \delta_{b,4} \rho^4 + \dots \quad (2.9)$$

The holographic dictionary identifies the leading term of the scalar field asymptotic series expansion, ϕ_b , with the source of a marginal field theory operator, \mathcal{O} , and the subleading term $\phi_{b,3}$ with its vacuum expectation value, $\langle \mathcal{O} \rangle$. Likewise, from the asymptotic expansion of the gauge field we can read off both the chemical potential, μ , and charge density, $Q \propto \langle J^t \rangle$, of the dual field theory state.

We refer to the infrared end of the radial coordinate as x_o , with either $x_o = 0$ for solitonic solutions, or $x_o = x_h > 0$ for the horizon of a black hole. The infrared series

²Given some function $g = g(t, x)$, we define $\dot{g} \equiv \partial_t g$ and $g' \equiv \partial_x g$.

expansions read

$$\phi(x) = \phi_o + \phi_{o,1}(x - x_o) + \dots, \tag{2.10}$$

$$A_t(x) = A_o + A_{o,1}(x - x_o) + \dots, \tag{2.11}$$

$$\delta(x) = \delta_o + \delta_{o,1}(x - x_o) + \dots, \tag{2.12}$$

$$f(x) = f_o + f_{o,1}(x - x_o) + \dots \tag{2.13}$$

where it should be noted that regularity of solutions at the origin demands $f_o = 1$ and forces every odd term of (2.10)–(2.13) to be zero. In the black hole case, the existence of a horizon and regularity of the gauge field one-form implies

$$f_h = A_h = 0. \tag{2.14}$$

Each static solution of the equations of motion is completely characterized by its infrared series expansion which, in turn, is totally fixed in terms of a finite number of parameters. In the absence of a horizon, these parameters can be the values of the scalar and gauge fields at $x_o = 0$, (ϕ_o, A_o) . A soliton geometry is dual to a field theory state with spontaneously, and not explicitly, broken symmetry. This further demands that the source ϕ_b vanishes, which provides a nonlinear relation between (ϕ_o, A_o) that determines completely A_o in terms of ϕ_o . A family of soliton solutions is therefore uniparametric. The same reasoning goes through to the black hole case. Now, each black hole solution would be totally determined by the triplet $(x_h, \phi_h, A_{h,1})$ and, again, the condition $\phi_b = 0$ would link ϕ_h and $A_{h,1}$ at the given x_h . In this way, a family of black hole solutions is bi-parametric.

To end this subsection, let us note that the system (2.5) enjoys two residual gauge symmetries. The first one is a residual time reparametrisation invariance

$$\delta \rightarrow \delta - \chi \quad A_t \rightarrow e^\chi A_t, \quad t \rightarrow e^{-\chi} t \tag{2.15}$$

that can be employed to set either δ_b or δ_o to zero, fixing the time coordinate t to be the proper time at either $x_b = \pi/2$ or x_o , respectively. The second residual gauge symmetry of (2.5) comes from (2.2), with $\Lambda(t) = \lambda t$. This transformation shifts A_b and A_o by a constant, in such a way that only the difference $A_b - A_o$ is gauge invariant.³

The backreaction of the gauge field on the geometry is controlled by the coupling e and the probe limit corresponds to $e \rightarrow \infty$.

2.2 Summary of phases

Following closely the discussion contained in [11, 12], we parameterise the space of solutions at a given coupling, e , by the charge Q and mass M . Three different regimes appear separated by two threshold values, e_t and e_{sr} , that signal the appearance of two distinct instabilities. The lower one, e_t , marks the threshold for a near-horizon tachyonic instability that affects RN black holes. It is triggered by the fact that the gauge field of an extremal RN black hole makes a negative contribution to the effective scalar field mass, lowering it below

³This symmetry is absent in a black hole background, since it would change the holonomy of the gauge connection around the Euclidean time cycle.

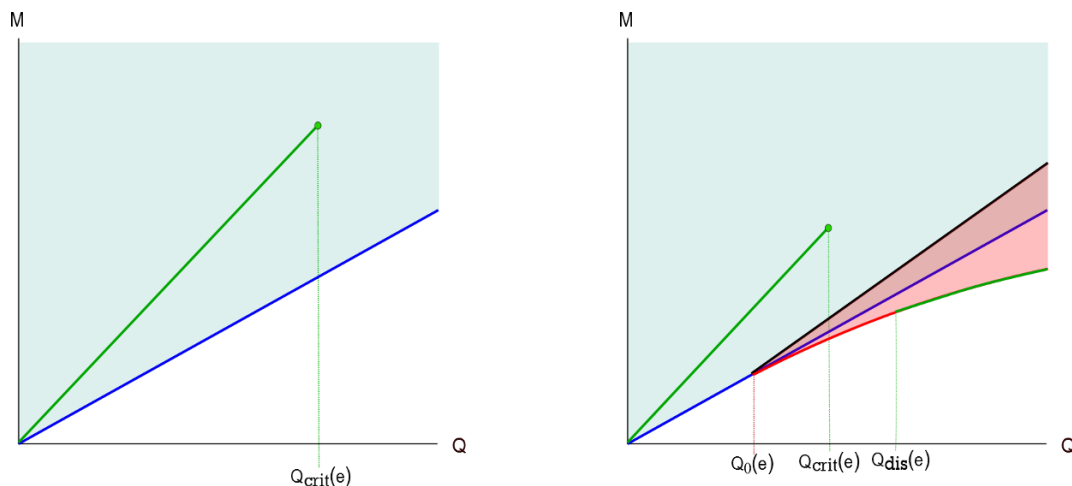


Figure 1. Left plot: phase diagram for $e < e_t$. In this range, there is a single soliton branch that ends up in a limiting value for the charge. The green shaded region are regular RN black holes, and they become extremal at the bottom blue line. Right plot: in this intermediate regime, $e_t < e < e_{sr}$, hairy solutions appear in a band about extremality. The red line stands for extremal hairy black holes, below which there are no solutions. Here we have a critical limiting charge for the vacuum connected solitons (upper green curve) and an unbounded vacuum disconnected soliton line (lower green line).

the Breitenlohner-Freedman bound of the AdS_2 factor of the near-horizon geometry. It is of the same kind as the one found in the context of holographic superconductors [13, 14]. As discussed in appendix B, it is only relevant for large enough black holes, a fact that explains why it was found in the context of planar geometries. On the other hand, for small black holes, there is a superradiant instability at work beyond e_{sr} [11, 12]. In the following table we show the values of these thresholds for AdS_{d+1} with $d = 3, 4$ for $\kappa^2 = 1$

	e_t^2	e_{sr}^2
AdS_4	$3/2$	$9/2$
AdS_5	3	$32/3$

In this table, e_t stands for the tachyonic instability threshold, and e_{sr} for the superradiant one.⁴ Within each interval of possible values for the coupling we have encountered a similar phenomenology.

In the remaining part of this section we summarize the phase diagram for the static solutions of the Einstein-Maxwell-scalar action for values of the coupling lying on each interval and discuss the soliton solutions in those regimes.

- $e < e_t$

Here, the coupling is below the tachyonic instability threshold. The only static solutions present are RN black holes and solitons which exist for values of the charge

⁴General expressions for this thresholds valid at any d can be found in the appendix B.

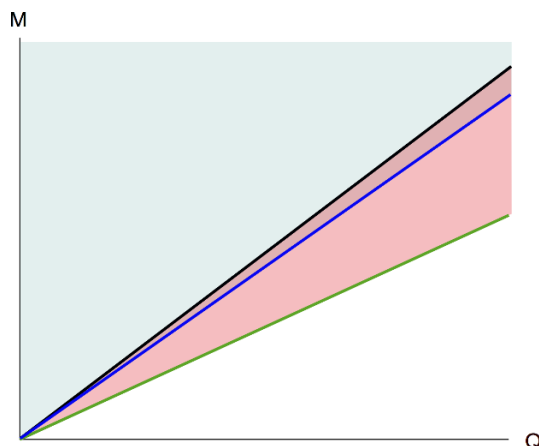


Figure 2. For $e > e_{sr}$ we have two kind of instabilities that lead to the formation of hairy black holes, tachyonic and superradiant. Note that in this regime there exist solitonic solutions for any value of the charge.

smaller than $Q_{\text{crit}}(e)$, a sort of Chandrasekar limit on regular selfgravitating solutions. In figure 1 (left) the green curve corresponds to solitons and the blue curve to the extremal RN black holes. Regular RN black hole solutions exist in the blue shaded region above this curve.

- $e_t < e < e_{sr}$

In this regime, in addition to RN black holes and regular solitons, there exist large and small hairy black holes in a band about extremality for $Q > Q_0(e)$ (red shaded region of figure 1, right). The red curve denotes extremal hairy black hole solutions and the black line is the instability curve for the RN solution. This function has the limiting behaviours $\lim_{e \rightarrow e_t} Q_0(e) = \infty$ and $\lim_{e \rightarrow e_{sr}} Q_0(e) = 0$. Solitons, in green on figure 1, appear in two branches. One is connected with the vacuum and has bounded charge $Q < Q_{\text{crit}}(e)$. The other one has instead a lower bound $Q > Q_{\text{dis}}(e)$. Both bounds merge $Q_{\text{dis}}(e) \rightarrow Q_{\text{crit}}(e)$ in the upper limit $e \rightarrow e_{sr}$. When $Q > Q_{\text{dis}}(e)$ the lowest mass hairy BH is no longer extremal and has non vanishing (possibly diverging) temperature.

- $e_{sr} < e$

Now hairy black holes exist for all values of Q in a band about extremal RN. Concerning solitonic solutions, the two branches seen in figure 1 merge to form a single branch that extends unbounded for all values of $Q \geq 0$. To the best accuracy of our numerics, the soliton line is the limit of vanishingly small hairy black holes with diverging temperature. In this limit, the temperature becomes ill defined for the soliton. The situation is very similar to the one in AdS_5 [12]. However, in that case, a second critical charge was found, beyond which hairy black holes appeared below the soliton line. We have not seen any trace of this behaviour in our numerical scan over the space of static solutions. However we cannot discard a fine structure of

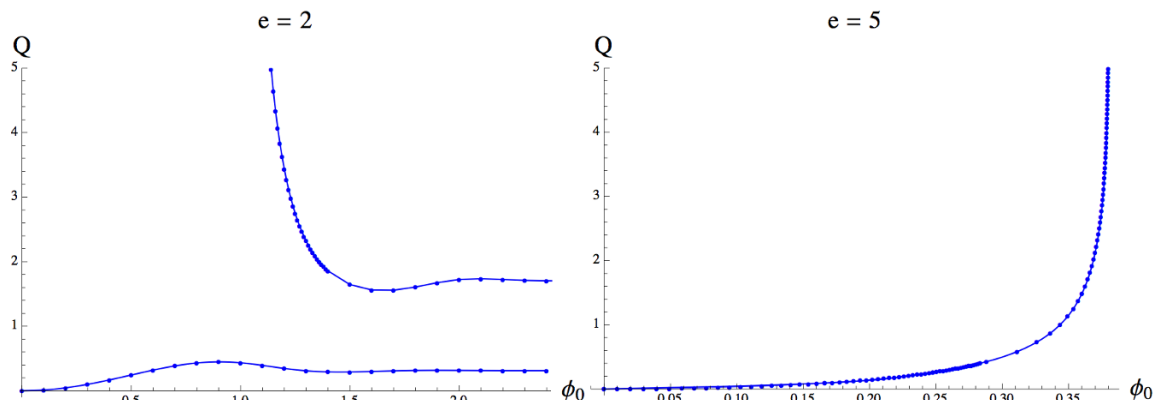


Figure 3. Left: charge of vacuum connected and disconnected soliton branches against ϕ_0 for $e = 2$. Right: soliton charge against ϕ_0 for $e = 5$.

the form observed in AdS_5 beyond the reach of our numerical accuracy. For that, a perturbative analytical study along the lines used in [12] would be needed.

2.2.1 Soliton branches in more detail

Let us describe the soliton solutions in more detail. They are horizonless, fully backreacted, charged solutions, sourced by a normalizable scalar field profile $\phi_s(x)$, where equilibrium is attained by an exact compensation between gravitational and electric forces. Due to the normalizability condition, on the field theory side, a soliton represents a macroscopic Bose-Einstein condensate that spontaneously breaks the $U(1)$ symmetry. They come in one-parameter families that can be parametrized by the central value $\phi_0 \equiv \phi_s(x=0)$. Depending on the value of the coupling e , soliton families display different aspects⁵

- For $e < e_t$, there exists a single soliton branch. It is continuously connected to the AdS_4 vacuum, in the sense that, for $\phi_0 \rightarrow 0$, it reduces to global AdS_4 . Therefore, for $\phi_0 \ll 1$, this solution family admits a perturbative construction, and can be described as a nonlinearly dressed $\omega = 0$ AdS scalar normal mode. Besides this fact, the trademark of this branch is the existence of a critical value $\phi_0 = \phi_{c,1}$ for which both the soliton mass $M(\phi_0)$ and charge $Q(\phi_0)$ attain a maximum. When $\phi_0 > \phi_{c,1}$, M and Q spiral around a limiting value that is reached for $\phi_0 \rightarrow \infty$.
- For $e_t < e < e_{sr}$, there exist two different soliton branches. The first one, connected with the vacuum, was already present in the $e < e_t$ case. The second one, disconnected from the vacuum, is not amenable to a perturbative construction. In this branch, solutions exist for ϕ_0 larger than some critical value $\phi_{c,2a}$ for which the conserved charges (M, Q) diverge. They decrease with $\phi_0 > \phi_{c,2a}$ until they reach a minimum value at some $\phi_0 = \phi_{c,2b} > \phi_{c,2a}$. In parallel with the first soliton branch, for $\phi_0 > \phi_{c,2b}$, M and Q show damped oscillations around a limiting value that is

⁵Our results agree essentially with the ones analyzed in [15] for the case of $m^2 = -2$.

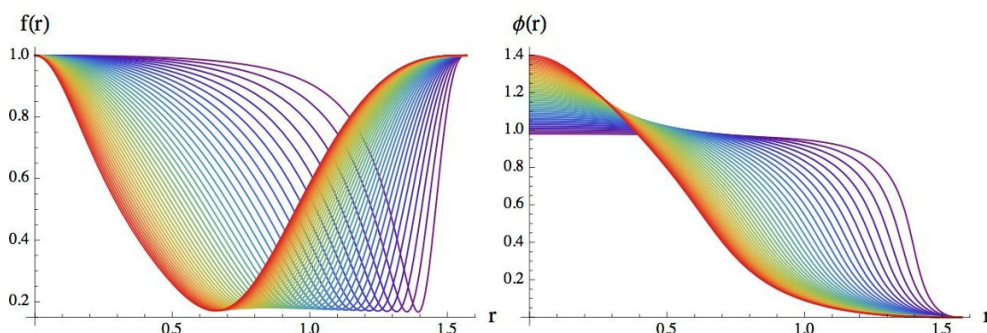


Figure 4. Fields for the vacuum disconnected soliton branch at $e = 2$ from $\phi_0 = 1.4$ (red) to $\phi_0 = 0.98$ (magenta) in steps of $\delta\phi_0 = -0.01$.

attained in the $\phi_0 \rightarrow \infty$ limit. Representative plots of the behavior just described are provided in figure 3 (left).

- For $e \geq e_{sr}$, the two soliton branches described in the previous item fuse into a single solution family that is vacuum connected (see figure 3, right). Again, there exists a critical $\phi_{c,3}$ such that M and Q seem to diverge in the limit $\phi_0 \rightarrow \phi_{c,3}$.

2.2.2 The blow up limit

In figure 4 we plot f_s and ϕ_s for representative solutions of the unbounded branch at $e = 2$. It is clearly appreciated that, as the conserved charges associated with the solution become larger, the field gradients become more localized near the boundary region. As emphasized in [15], the fact that solitons come in one-parameter families with unbounded charge, allows to take a *blow up* limit that maps onto a solution with planar geometry. The procedure starts by looking at the near-boundary expansion

$$\phi(t, r) = \frac{\phi_3}{r^3} + O(r^{-4}) \tag{2.16}$$

$$A = A_t(t, r)dt = \left(\mu - \frac{Q}{r} \right) dt + O(r^{-2}) \tag{2.17}$$

$$ds^2 = - \left(r^2 + 1 - \frac{m}{r} \right) dt^2 + \frac{dr^2}{r^2 + 1 - \frac{m}{r}} + r^2 d\Omega_2^2 + O(r^{-2}). \tag{2.18}$$

Introducing new coordinates

$$r = \lambda \hat{r} \quad t = \frac{\hat{t}}{\lambda} \quad \theta = \frac{\hat{\theta}}{\lambda} \quad \varphi = \hat{\varphi} \tag{2.19}$$

and redefinitions

$$M = \lambda^3 \hat{m}(\lambda) \quad Q = \lambda^2 \hat{q}(\lambda) \quad \mu = \lambda \hat{\mu}(\lambda) \quad \phi_3 = \lambda^3 \hat{\phi}_3(\lambda), \tag{2.20}$$

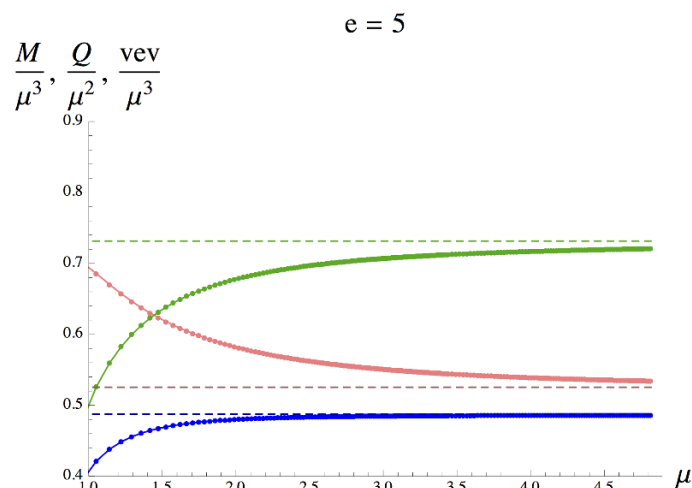


Figure 5. Scale invariant quantities for the $e = 5$ soliton branch. Blue, green and pink colours correspond to the scale invariant ratios of the mass, charge and vev, respectively. Dashed lines correspond to the same ratios computed in the asymptotic planar geometry reached upon blow up.

eqs. (2.16)–(2.18) become

$$\phi = \frac{\hat{\phi}_3}{\hat{r}^3} + O(\hat{r}^{-4}) \quad (2.21)$$

$$A = A_t d\hat{t} = \left(\hat{\mu} - \frac{\hat{q}}{\hat{r}} \right) d\hat{t} + O(\hat{r}^{-2}) \quad (2.22)$$

$$ds^2 = - \left(\hat{r}^2 + \frac{1}{\lambda^2} - \frac{\hat{m}}{\hat{r}} \right) d\hat{t}^2 + \left(\hat{r}^2 + \frac{1}{\lambda^2} - \frac{\hat{m}}{\hat{r}} \right)^{-1} d\hat{r}^2 \quad (2.23)$$

$$+ \hat{r}^2 \left(d\hat{\theta}^2 + \lambda^2 \sin^2 \left(\frac{\hat{\theta}}{\lambda} \right) d\hat{\varphi}^2 \right) + O(\hat{r}^{-2}). \quad (2.24)$$

So far, this is only a reparametrisation of our initial solution. However, if we now take the singular limit $\lambda \rightarrow \infty$ and, simultaneously, move along the soliton branch in such a way that the vector of rescaled quantities $\vec{\chi}(\lambda) \equiv (\hat{m}(\lambda), \hat{q}(\lambda), \hat{\mu}(\lambda), \hat{\phi}_3(\lambda))$ remains finite, we obtain a planar geometry⁶ characterized by the hatted quantities. For example, identifying the parameter λ with μ , the dimensionless ratios

$$\frac{M}{\mu^3}, \quad \frac{Q}{\mu^2}, \quad \frac{\phi_3}{\mu^3} \quad (2.25)$$

must tend to constant quantities when $\mu \rightarrow \infty$ for the planar limiting geometry to have finite energy, charge and vev densities, respectively. In figure 5 the ratios (2.25) are plotted for the soliton branch at $e = 5$. The planar geometry after the blow up limit is taken clearly matches the extremal hairy black brane geometry studied in [16], at the given e .⁷

⁶Note that the boundary sphere maps onto a two-dimensional plane in the $\lambda \rightarrow \infty$ limit, with its metric written in polar coordinates.

⁷We have indeed verified that equations (2.5) reduce to those on [16] after the limit (2.19) if we set $\kappa^2 = 1/2$ so field normalizations agree.

Incidentally, this observation explains why there does not exist a second soliton branch when $e < e_t$. In that case, in planar AdS, there is no near-horizon tachyonic instability that can trigger hair condensation, so there is no limiting extremal solution which this hypothetical branch could map onto.

3 Grand canonical ensemble

The previous section contains a classification of static solutions without examining the issue about their stability. This is not an unambiguous question, as to compare different solutions one must first specify the ensemble. Whereas the plots we have shown are more akin to studying the system in the microcanonical ensemble, physics is more likely related to the grand canonical ensemble, where temperature and chemical potential are feasible knobs. A very generic feature to take into account is that black hole solutions in global AdS (both RN and hairy) split in two kinds: small and large, having negative and positive specific heat respectively. It is the later ones that should be brought into correspondence with stable superfluid quantum states in the dual field theory. In addition to these thermal solutions, there is the one dual to a thermal gas represented by pure Euclidean AdS₄ with a compactified time coordinate. Finally, we have the regular solitons. These static solutions are the building blocks of a rich landscape of first and second order phase transitions. Along this section we are going to report results for $e = 3$. For any coupling $e > e_t$ the situation is qualitatively the same. Below, instead, the transition involves only RN black holes and solitons, since, as evident from figure 1, there are no hairy solutions in this regime of coupling. In this case, our system differs from the one analyzed in [15, 17], where the scalar has a tachyonic mass to start with.

3.1 Small and large hairy black holes

The thermodynamics of AdS-RN solutions, both in the canonical and grand canonical ensemble has been examined with great care in the past [18, 19]. Here we are going to study the thermodynamical behavior of hairy solutions. The Hawking temperature is given by

$$T = \frac{e^{\delta(\pi/2)}}{8\pi} \left((6 - A'_t(x_h))^2 \cos^4(x_h) \tan(x_h) + 2 \cot(x_h) \right) \quad (3.1)$$

and the entropy is

$$S = 2\pi A_h = 8\pi^2 \tan(x_h)^2, \quad (3.2)$$

where A_h is the area of the horizon. In figure 6 (left) we plot the vev of the dual operator \mathcal{O} for fixed chemical potential $\mu = 1.5$ and coupling $e = 3$ as a function of the temperature.

The observed behaviour is typical when the gravitational solution corresponds to a small black hole: condensation appears for temperatures greater than some critical value. We will show that this condensed phase is not physically relevant because it has greater free energy than the RN and soliton solutions. This phenomenon, dubbed retrograde condensation in the literature, has been observed in different contexts [20, 21]. In figure 6 (right) we show the temperature as function of the horizon position for small hairy black holes, where their negative specific heat is manifest.

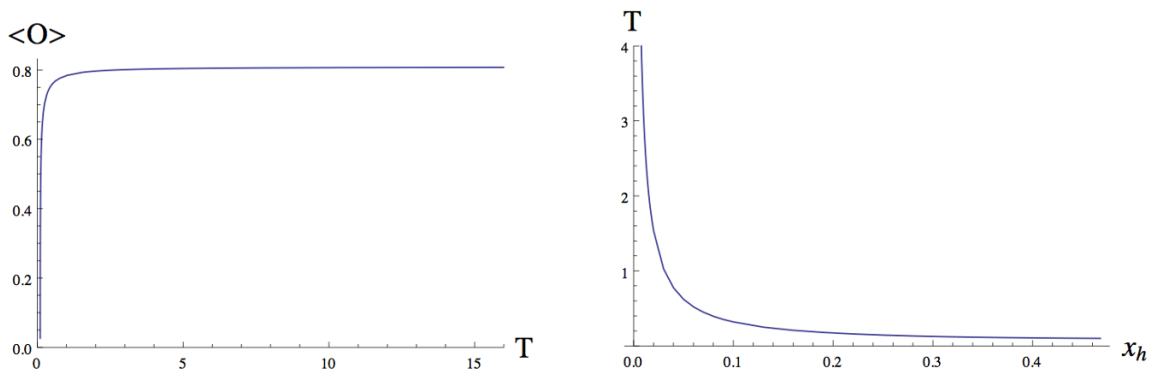


Figure 6. Left: $\text{vev } \langle \mathcal{O} \rangle$ versus temperature T for $\mu = 1.5$ and $e = 3$. Right: temperature T as a function of horizon radius x_h .

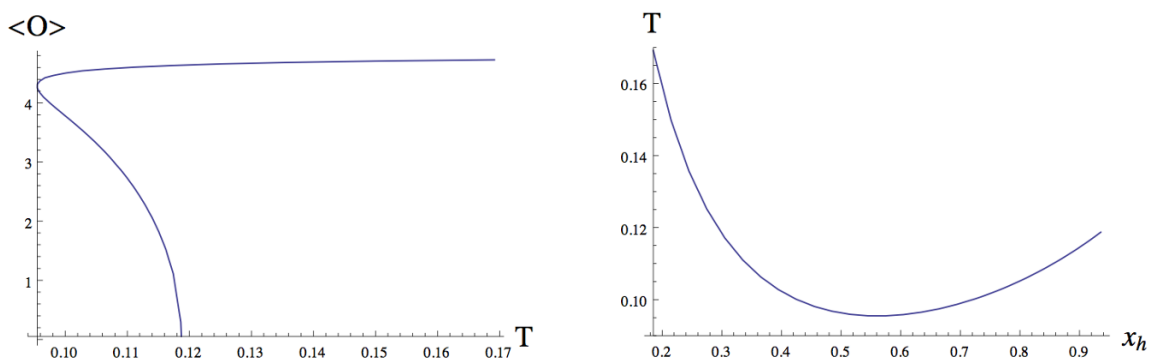


Figure 7. For $e = 3$ and $\mu = 3$, on the left, $\text{vev } \langle \mathcal{O} \rangle$ versus the temperature T . The lower branch corresponds to the large BH solution and the upper curve to the small BH. On the right, temperature as a function of the position of the horizon. Note that there is one value of T for two different x_h .

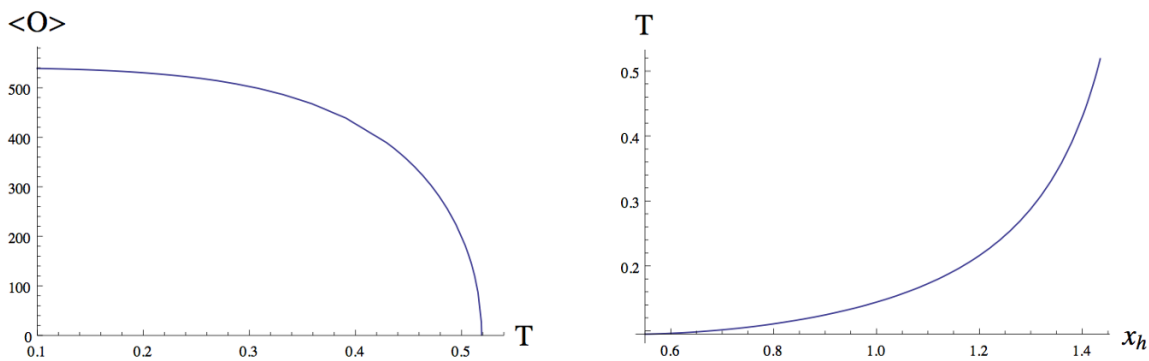


Figure 8. For $e = 3$ and $\mu = 15$, on the left, $\text{vev } \langle \mathcal{O} \rangle$ versus the temperature T . On the right, we observe that the temperature as a function of the horizon position is monotonously growing.

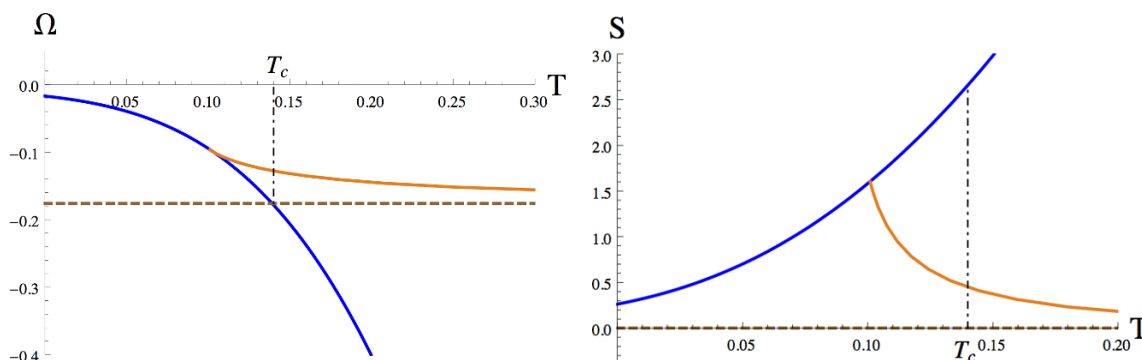


Figure 9. $\mu = 1.5$, $e = 3$. The thermodynamic potential (left) and entropy (right) as function of the temperature. The blue line corresponds to the RN solution, the orange curve is for the small hairy black hole solution and the brown dashed line stands for the soliton. There is a critical temperature, T_c , below which the soliton is the dominant solution. We observe that the entropy is discontinuous at the meeting point between the brown and blue curve showing the appearance of a first order phase transition at T_c .

As an example of a small-large black hole transition we rise the value of the chemical potential up to $\mu = 3$. Figure 7 (left) shows the behavior of the vev as a function of the temperature. Points on the upper (lower) segment are small (large) hairy black hole solutions. Upon lowering the temperature the system undergoes a second order phase transition along the lower (stable) branch from a normal to a superconducting state. This can be appreciated in the typical mean field theory behavior of the condensate near the critical temperature, since it goes to zero as $\langle \mathcal{O} \rangle \propto (T_c - T)^{1/2}$. The denomination large/small comes, as usual, from the double valuedness of the temperature as a function of the horizon radius x_h . This is seen on the right plot of figure 7. Further increasing the chemical potential, we observe the behaviour shown in figure 8 for $\mu = 15$. The same second order phase transition happens here, but the small black hole branch has now disappeared.

3.2 Grand potential

Linear stability of the previous solutions amounts to the minimisation of certain thermodynamical potential that depends on the ensemble. In the grand canonical ensemble this is the grand potential $\Omega(T, \mu)$, which the AdS/CFT correspondence identifies with the on-shell renormalized Euclidean action $\Omega = TS_{\text{on-shell}}$ (see appendix C for calculations).

In figure 9 we plot the grand potential (C.7) and the entropy (3.2) respectively as functions of the temperature. In both figures the blue curve refers to the RN solution and the orange curve to the small hairy black holes for $\mu = 1.5$ and $e = 3$.

The brown dashed line is the free energy of the soliton.⁸ We observe that a first order phase transition occurs, whereby increasing T the solitons switch into RN black holes. The

⁸Since the soliton geometry is regular at $x = 0$, it can be analytically continued to a smooth Euclidean geometry with any period in the time direction. Choosing a finite period corresponds to placing a thermal gas with $O(1)$ entropy and energy above the soliton geometry. A natural way of fixing this period would be to demand continuity of the isotherms in the (Q, M) phase diagram, which would require to assign an infinity temperature to the unbounded soliton branches. However, since they correspond to zero entropy ground states, another natural criterium would be to demand a vanishing temperature. This choice is also consistent with the fact that, in the blow up limit discussed in section 2, they map onto extremal hairy black branes.

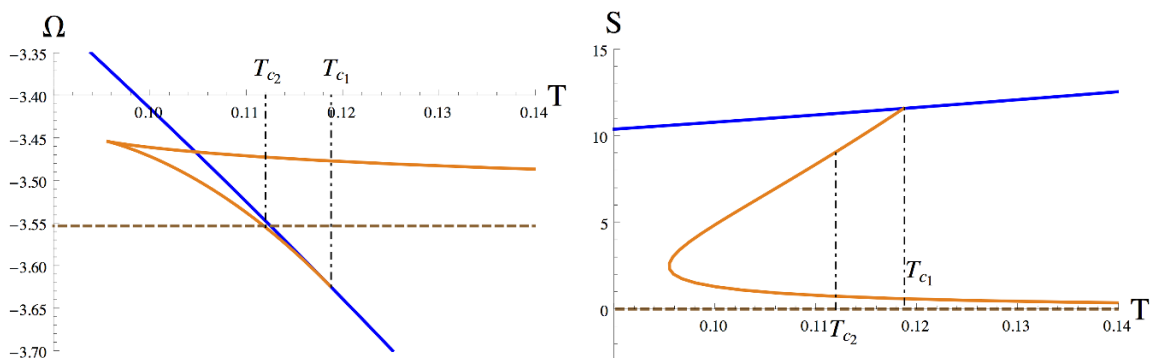


Figure 10. $\mu = 3, e = 3$. The thermodynamic potential (left) and entropy (right) as function of the temperature. The blue line corresponds to the RN solution and the orange curve to the hairy black hole solution. The hairy solution has two branches, the lower one corresponds to the large BH and the upper one to the small BH. The brown dashed curve denotes the soliton free energy.

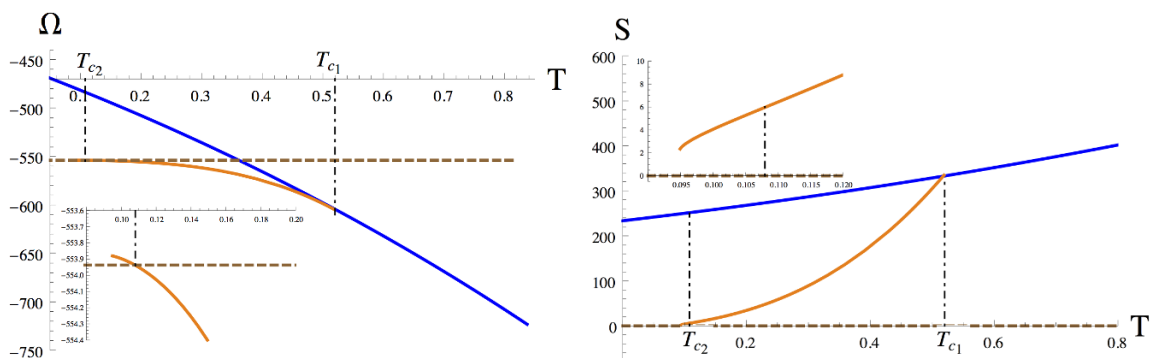


Figure 11. $\mu = 15, e = 3$. Thermodynamic potential (left) and entropy (right) as function of the temperature with the same color coding as in the previous figures. There are no small hairy black holes in this regime.

small hairy black holes are never the ground state at this value of μ . Figure 10 shows the same functions for the $\mu = 3, e = 3$ solutions. The preferred branch is that of large black holes (lower orange segment) instead of small black ones (upper orange line) which have negative specific heat (see figure 7). Again, the derivative of the entropy is discontinuous at a critical temperature T_{c1} and we have a second order phase transition denoting the normal-superconductor transition of the dual QFT. We detect a second first order phase transition temperature T_{c2} , below which solitons are preferred.

Finally, figure 11 corresponds to $\mu = 15$. The phase space is the same, as well as the nature of the phase transitions, the only difference being the disappearance of the small black hole solutions.

4 AdS nonlinear stability

The above sections have relied on a combination of analytical and numerical arguments. The construction of static solutions is performed by solving ODE's and setting up a shooting

procedure. It involves fixing, for example, the radius of the desired solution and the value of the scalar field ϕ_o at such radius. Then the value of $A'_t(x_o)$ is also varied until one obtains a solution with vanishing source $\phi_b = 0$ and non vanishing vev $\phi_{b,3} \neq 0$. The end result is scrutinized to find the actual value of the mass M and the charge Q of the obtained stationary solution. If more than one solution is available, the free energy is to be invoked in order to select the correct ground state. In a sense, this strategy relies on a certain amount of guesswork. Prior to the construction of hairy black holes in [14], the space of known static vacua consisted of either pure AdS or RN black holes. Later on, solitons were first inferred, and then constructed [11] from a limit whereby the hairy black hole's horizon is shrunk to zero.

In this section we will use a complementary approach. A numerical simulation code for time evolution is the closest thing one can have to a real experiment. In this spirit, the approach starts from the other end: one devises a certain initial radial profile for the bulk fields, with a given total mass M and charge Q , and lets it evolve under a scheme that preserves these values.⁹ If the evolution settles down to a certain stationary state, it must necessarily be one in the list above. And if two of them are available with the same values of M and Q , the evolution will select the ground state in the microcanonical ensemble.¹⁰

By evolving an initial condition below the blue curve in figure 4 this would have shown that black holes with abelian hair exist in global AdS, had this paper been written prior to 2008. Moreover, imagine there was any other exotic type of black hole that nobody has constructed yet using static methods (ODE) and, moreover, suppose it has larger entropy. The collapse simulation would *smell* its existence and the fields decay to that solution after exploring large portions of phase space. We must admit we have not found any new such solution using this, admittedly expensive, method.

In figure 12 we have plotted some snapshots of a typical collapse to a hairy black hole. For the values of charge, mass, and coupling e used in this simulation there is no RN black hole available. Nevertheless at $t = 14.8$ the scalar develops a spike at a point where the metric approaches an apparent horizon, signalled by a zero of the function f (blue curve). Even if the zero value is never reached, the dynamics close to this point becomes extremely slowed down in terms of the time at the boundary $x = \pi/2$. At later times, the outer oscillations of the scalar field start piling up on top of the first spike, and the metric function f tries to reach zero at higher values of the coordinate x (see inset). A very high precision and up to 2^{17} grid points are needed to push this numerical evolution safely, and resolve the region close to the collapse with enough accuracy, in particular monitoring the constancy of M and Q values throughout the process. The exponentially decaying ringdown ends up in a static solution where the outside hair profile resembles the ones in figure 4 for the soliton solutions.

⁹The precise implementation of the time evolution code, as well as its convergence properties, are discussed in appendix D.

¹⁰This could proceed in a direct way or through a number of different steps. There are situations where pre-thermalization to some excited intermediate state followed by further relaxation to the true final equilibrium state can be observed, for instance in the latest paper of [22]. It would be interesting to study in detail the possible existence of such metastable attractors in the present context.

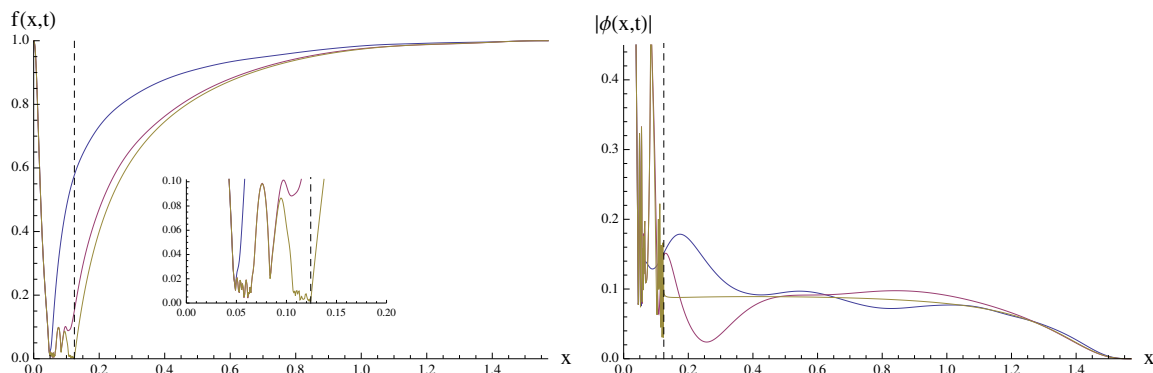


Figure 12. For coupling $e = 20$, we plot three snapshots of a collapse with mass $M = 2$ and charge $Q = 2.9$ at times $t = 14.8, 17.3$ and 28.1 (blue, magenta and yellow). At this time, solution has become an almost static hairy black hole. The evolution exhibits the typical ring down whereby apparent horizon builds up while capturing successive oscillations of the scalar field. They fold behind the accumulation point of the vanishing locus of $f(t, x)$, hence the position of the event horizon.

In ref. [4] a strong claim was made, that the geometries exhibiting a *corner of instability*, were those for which the spectrum of linear perturbations is fully resonant. An important case that shows how sensitive the dynamics is to the resonance property can be found in the case of a scalar field on asymptotically flat space enclosed in a cavity. An instability corner is obtained for Dirichlet boundary conditions [7], where the spectrum is resonant, but not instead in the case of Neumann boundary conditions [6], where it is only asymptotically resonant.¹¹ The upshot of these analysis seems to indicate that, not only an UV cascade is important, but also a non dispersive spectrum, such that the initial wave packet remains focused at all times.

To test this picture further, one would be interested in families of situations that depart smoothly from it and, in this sense, there are two ways to achieve this. One can keep the original lagrangian setup and, say, consider perturbations around equilibrium solutions other than pure AdS. In generic cases, the spectrum of perturbations of the scalar laplacian on this new background will not be resonant and the collapse will not exhibit a corner of instability. This is the proposed mechanism for the stability of broad initial data, as studied in [6]. They can be understood as perturbations of a single oscillon solutions which are known to be stable. A different strategy is to deform the theory by a continuous parameter. Examples of this are Gauss Bonnet [8], or the inclusion of a hard wall [24]. A remarkable example that stands out is AdS₃, which despite being resonant, has a mass gap [9, 10].

The present setup offers another interesting example where the theory is deformed by the presence of additional degrees of freedom, while still preserving the resonant spectrum. At the linear level, scalar perturbations obey the same equations of motion as in Einstein-scalar theory.¹² A perturbation of the (now charged) scalar field of amplitude ϵ sources a gauge field A_μ at the same order, which will backreact on the scalar equation of motion at next order ϵ^2 since it couples through the covariant derivatives. Hence we have another work bench to test whether nonlinear perturbations exhibit a corner of instability.

¹¹See also [23].

¹²We thank Oscar Dias for this observation.

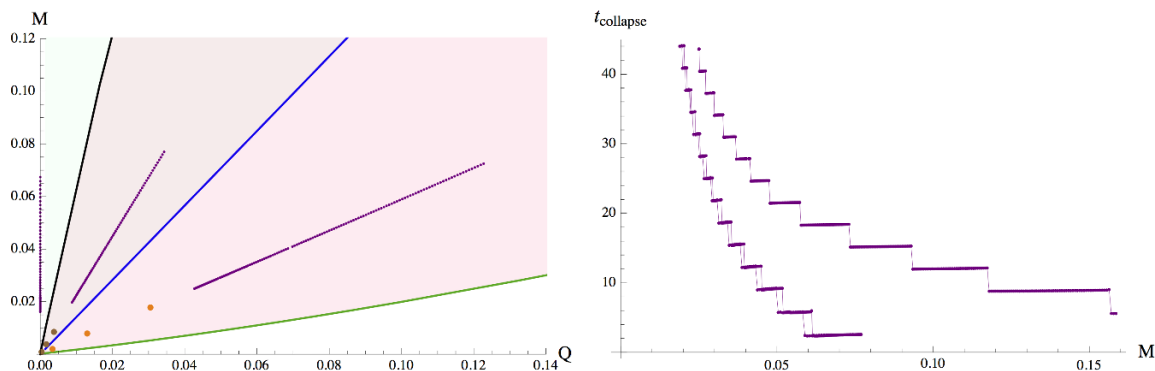


Figure 13. Collapses with $\sigma = 0.1$ and $e = 20$. In the left plot, the phase space and color coding is the same as in figure 3. The dotted purple lines represent three series of initial conditions with $\beta = 0^\circ$, 45° and 82° from left (vertical) to right (most bended). For each line we have taken 3 points deep in the low ϵ limit with $\epsilon = 6, 4$ and 2 , and performed very long time simulations in order to test the scaling hypothesis (see figure 16). Right plot: time for collapse for each of the three lines in the left plot, in the same order from left to right, as a function of the mass M . The time for collapse increases with the closeness to the soliton line, i.e. for fixed mass M , at higher charge Q .

In ref. [25] collapse of a complex scalar was considered and no significant difference was obtained for the phenomenology of charged vs. neutral configurations, apart from a small decrease in collapse time. The results in that paper can be recovered in the limit of vanishing coupling $e \rightarrow 0$ of our work. However, at finite e we find opposite results as we will now see.

4.1 Initial conditions

In this section we will show the results of uniparametric families of collapses that scan across the (Q, M) plane. The first protocol will involve a set of initial conditions, parameterized with some amplitude ϵ . Consider the following family of gaussian initial data that initially fall from the boundary $\Phi = \Phi_1 + i\Phi_2$, and $\Pi = \Pi_1 + i\Pi_2$ with $\Phi_2 = \Pi_1 = 0$ and

$$\Phi_1 = \epsilon \cos \beta \frac{2}{\pi} \cos^2 x \exp\left(-\frac{4 \cot^2(x)}{\pi^2 \sigma^2}\right) \quad (4.1)$$

$$\Pi_2 = \epsilon \sin \beta \frac{2}{\pi} \cos^3 x \exp\left(-\frac{4 \cot^2(x)}{\pi^2 \sigma^2}\right). \quad (4.2)$$

Here the angle β is fixed, and ϵ will decrease monotonically towards zero. The cases $\beta = 0, \pi/2$ corresponds to the uncharged initial conditions studied in [1], albeit with an initial pulse that starts infalling from the boundary [26], a fact that is inspired from the physics of a quench. For $\beta \neq 0, \pi/2$ we are dealing with a shell of charged scalar and gauge field collapsing together.

In figures 13 and 16 we exhibit series of simulations for initial conditions of the form (4.1), (4.2) $\sigma = 0.1$ and 0.2 , and several values of β . The cases examined in [1] would lie on the vertical axis $Q = 0$ (uncharged case). Notice that we have searched for peculiar behaviours in different domains of the phase space in this microcanonical ensemble. Above

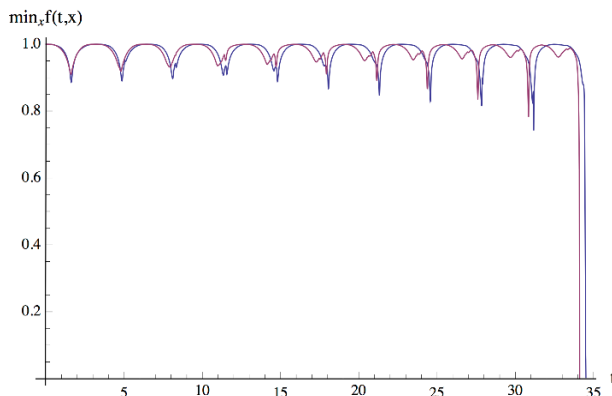


Figure 14. Plot of the minimum of the metric function $f(t,x)$ as a function of time for two simulations with similar histories in the lines $\beta = 45^\circ$ (blue) and $\beta = 82^\circ$ (magenta) in figure 13 left. The roughness of the profile in the second case reveals that the scalar pulse is not smooth but fractures into ripples. This, presumably, is an effect of the electrostatic repulsion in action.

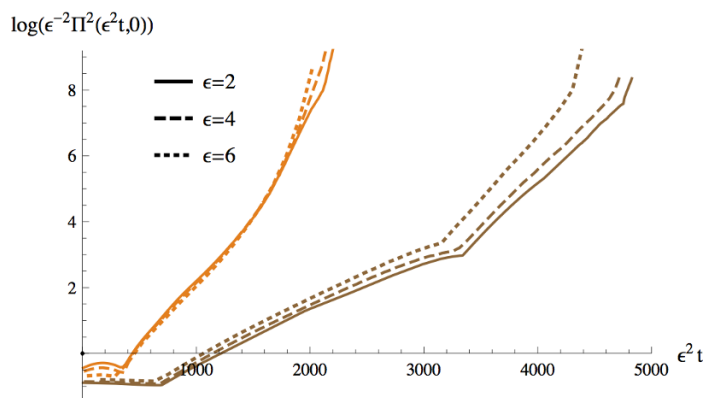


Figure 15. Evolution of the maxima of the Ricci scalar at the origin upon rescaling of the time and the initial amplitude. The six simulations correspond to the six dots in figure 13 with $\sigma = 0.1$ and $\beta = 45^\circ$ (brown) and 82° (orange).

the instability line where only RN solutions exist, we find little or no difference with the case of zero charge. Below this line, the final state of the evolution is a hairy black hole. We have included two lines of collapse that “bend” towards the soliton line (the green line), in the region where hairy black holes exist, both above and below the line of extremal RN (in blue).

We want to stress that it is by no means easy to engineer initial conditions that come close to the line of soliton solutions. In particular, within the family of gaussians spelled out in (4.1) and (4.2), by letting β sweep from 0 to $\pi/2$, the lines incline up to some point, for some β_0 , where the initial conditions approaches maximally the soliton line and then turn back towards the vertical. These values are, for example, $\beta_0 = 82^\circ$ for $\sigma = 0.1$ and $\beta_0 = 75^\circ$ for $\sigma = 0.2$. In principle one can engineer initial conditions that come closer to the soliton line by starting from the other end: namely, by perturbing a soliton, and this will be the subject of the next section. Most remarkable is the fact that it seems impossible to even write initial data whose charge, Q , and mass, M , give a point below the soliton line

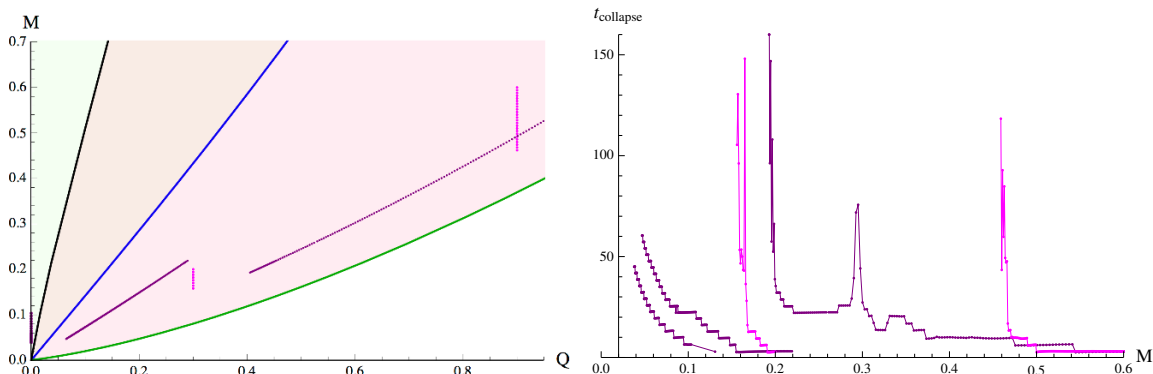


Figure 16. Collapses with $\sigma = 0.2$ and $e = 20$. The dotted purple lines represent three series with $\beta = 0^\circ, 45^\circ$ and 63° . Vertical magenta lines represent collapse series at constant $Q = 0.3$ and 0.9 . On the right plot the associated time curves are in direct correspondence from left to right.

(white region). This seems to point out that soliton solutions extremize certain positive definite functional that can be derived from the action. It would be very interesting to elaborate on this point further.

Time for collapse is one of the important observables in the game. We can see in plot 13 the case of sharp pulses with $\sigma = 0.1$ and $\beta = 0^\circ, 48^\circ$ and 82° . The horizontal axis represents the initial mass, M , which, for fixed σ , grows with ϵ^2 . On the right, the times for collapse for each family are plotted. When moving along an individual series from right to left, the plateaux reflect the number of oscillations that the system undergoes before the final collapse is reached. The fact that electromagnetic repulsion counteracts gravitational attraction is probably behind the fact that oscillations exist at higher values of M when the charge is larger. We see that the behaviour points towards the existence of a corner of instability at the origin even in the charged situation, i.e., no sign of a threshold for stability is appreciated. This seems to confirm the expectation coming from the resonant character of the linearized approximation. From the right figure 13 we draw the important conclusion that charged configurations take longer time to collapse. This is in sharp contrast with the case without gauge field where charged initial conditions were collapsing sooner than neutral ones of the same mass (see figure 6 in [25]).

In the present situation, the charge of the pulse also adds to the defocusing. Still, what figure 13 says is that, for $\sigma = 0.1$, this is not enough to erase the instability corner, even for the most charged gaussians that one can device (the right most blue magenta diagonal). Both plots for collapse time scale with $1/M \sim 1/\epsilon^2$ in the limit $\epsilon \rightarrow 0$. However, a closer look at the evolution of the scalar field reveals that, for $\beta = 89^\circ$, the initial gaussian develops subpulses. A reflection of this can be observed in figure 14 which plots the minimum of the metric function $\min_x f(t, x)$ as a function of time. The evolution still exhibits a quasiperiodic structure where the action of the weak turbulent cascade is apparent in that the minima become sharper and deeper until finally collapse takes over.

The $1/\epsilon^2$ scaling is apparent in the plots of figure 15 for the maxima of the scalar curvature at the origin. The orange curves correspond to the orange dots in figure 13 and the same is true for the brown curves and dots.

After the works in [2, 27, 28], it has become clear that both focusing and defocusing dynamics (i.e. direct and inverse cascade) seem to be in action and in a delicate equilibrium. For very sharp initial data, small σ , focusing wins. In figure 16, times for collapse with initial width $\sigma = 0.2$ are plotted. For vanishing charge $\beta = 0, \pi/2 \Rightarrow Q = 0$, it scales indefinitely as expected as $1/\epsilon^2$ as for sharper pulses. However, as soon as some charge is added, we start seeing a divergence at finite values of t_{collapse} . This is exactly the same effect encountered in [5] but here it appears for smaller values of σ than in that case.

5 Soliton stability

In recent years, configurations analogous to the solitons considered in this paper, known as boson stars, have occupied a prominent role in the study of the AdS nonlinear stability problem. A boson star is a stationary complex scalar field configuration with non zero charge Q that backreacts non trivially on the metric. However, in this case, the bulk $U(1)$ symmetry is global and, therefore, boson stars carry no gauge field A . These solutions provide extended configurations that, once perturbed, help to shed light on how relevant an exactly resonant spectrum of linearized scalar field perturbations is for a weakly turbulent instability to be present on the system. This issue was raised in [4], where the authors showed that just an asymptotically resonant spectrum is not enough to trigger a turbulent cascade along the lines of [1]. Later, it was recognized that boson stars are not endowed with an exactly resonant spectrum and, furthermore, they were shown to be nonlinearly stable [5, 6]. It remains to be seen if the correlation between these two facts survives the present situation. We expect a similar mechanism to be here behind the behaviour of the $\sigma = 0.2$ vertical lines in figure 16 which approach the soliton (green) line at constant charge Q . One of the main aims of this section is to provide evidence for this.

5.1 Linear stability properties

One of the aims of this subsection is to show that whenever the soliton mass $M(\phi_0)$ attains an extremum the solutions become linearly unstable. In order to check this explicitly, in we are going to study linearized radial perturbations of the solitonic solutions with an harmonic time dependence of the form $\cos \omega t$. Before delving into the details it is useful to notice that, as we are considering a time-reversal invariant and, therefore, nondissipative problem, ω^2 is going to be purely real. In this way, an exponentially growing mode that signals an instability appears whenever $\omega^2 < 0$. We start by fixing our perturbations to be of the form¹³

$$\delta(t, x) = \delta_s(x) + \epsilon \delta_1(t, x), \tag{5.1}$$

$$f(t, x) = f_s(x)(1 + \epsilon f_1(t, x)), \tag{5.2}$$

$$A(t, x) = A_s(x) + \epsilon A_1(t, x), \tag{5.3}$$

$$\phi(t, x) = \phi_s(x) + \epsilon(\phi_1(t, x) + i\phi_s(x)\partial_t\phi_2(t, x)), \tag{5.4}$$

with real ϕ_1, ϕ_2 . Since, due to spherical symmetry, the metric carries no degrees of freedom in our setup, the perturbations defined by (5.1)–(5.4) are not independent. In fact, the

¹³The strategy adopted in this paper is an adaptation of the method employed in [5, 29] to study boson star stability.

reason for having chosen this particular form for the scalar field perturbation is that it allows to solve for δ_1 and f_1 in terms of ϕ_1, ϕ_2 and A_1 , by making use of the momentum and Maxwell constraints (A.12), (A.14) linearized in ϵ . Specifically, we get that

$$\delta_1(t, x) = -\frac{\partial_x A_1(t, x)}{A'_s(x)} - \frac{2ef_s(x)e^{-2\delta_s(x)}\phi_s(x)\partial_x\phi_2(t, x)}{\cos(x)^2 A'_s(x)} + C_\delta(x), \quad (5.5)$$

$$f_1(t, x) = -\sin(2x) (\phi'_s(x)\phi_1(t, x) - eA_s(x)\phi_s(x)^2\partial_x\phi_2(t, x)) + C_f(x), \quad (5.6)$$

where C_δ, C_f are integrating functions that must be fixed by the correct choice of boundary conditions. For harmonic perturbations, set

$$(\delta_1(t, x), f_1(t, x), A_1(t, x), \phi_1(t, x), \phi_2(t, x)) = (\hat{\delta}_1(x), \hat{f}_1(x), \hat{A}_1(x), \hat{\phi}_1(x), \hat{\phi}_2(x)) \cos \omega t \quad (5.7)$$

which forces C_δ, C_f to be zero. Then, we can obtain the equations of motion for $\hat{\phi}_2$ and \hat{A}_1 by linearizing in ϵ the equations for δ and A , (A.11) and (A.15) (the linearized equation for f is not independent). The remaining equation for $\hat{\phi}_1$ comes from equation (A.9) for ϕ , linearized in ϵ , after making use of both the $\hat{\phi}_2$ and \hat{A}_1 equations. The final expression of the equations of motion for the perturbations is not particularly illuminating. Defining

$$\vec{Z} = \left(\hat{\phi}_1, \hat{\phi}_2, \hat{A}_1 \right)^T, \quad (5.8)$$

the perturbation equations are of the form

$$\vec{Z}''(x) + M^1(x)\vec{Z}'(x) + (M_a^0(x) + \omega^2 M_b^0(x)) \vec{Z}(x) = 0, \quad (5.9)$$

where $M^1, M_{a,b}^0$ are matrix-valued functions that depend exclusively on the background solution. In order to solve (5.9), we have to choose appropriate boundary conditions for \vec{Z} . At $x = 0$ we demand regularity. As for the background soliton, this forces \vec{Z} to be even at $x = 0$. At $x = \pi/2$, boundary conditions are

$$\hat{\phi}_1(x) = O(\rho^3), \quad (5.10)$$

$$\hat{\phi}_2 = \hat{\phi}_{2,0} + O(\rho^2), \quad (5.11)$$

$$\hat{A}_1 = O(\rho^2). \quad (5.12)$$

The first two conditions come from imposing normalizability on the scalar field perturbation (5.4). The last condition demands a more thoughtful explanation. Let us consider the most general near boundary expansion for \hat{A}_1 ,

$$\hat{A}_1(x) = \hat{A}_{1,0} + \hat{A}_{1,1}\rho + O(\rho^2). \quad (5.13)$$

In this case, it can be shown that we have $\hat{\delta}_1(\pi/2) \propto \hat{A}_{1,1}$ and, in consequence, if we want to maintain our gauge choice for the time coordinate, we must set $\hat{A}_{1,1} = 0$. This is tantamount to demanding that the frequency ω is the one measured by a boundary observer.¹⁴

¹⁴Under a shift $\delta(x) \rightarrow \delta(x) + c$, ω changes as $\omega \rightarrow \omega e^{-c}$ so as to maintain the ωt phase of the perturbation invariant.

On the other hand, nothing prevents us from allowing that $\hat{A}_{1,0} \neq 0$ i.e., perturbations that don't keep fixed the soliton chemical potential. However, examining the explicit form of equation (5.9) we discover that any solution is invariant under the change $\hat{\phi}_1 \rightarrow \hat{\phi}_1, \hat{\phi}_2 \rightarrow \hat{\phi}_2 + \alpha$ and $\hat{A}_1 \rightarrow \hat{A}_1 + \beta$, provided that $\alpha\omega^2 + e\beta = 0$. Therefore, we can employ this residual symmetry¹⁵ to set $\hat{A}_{1,0} = 0$ with no loss of generality, fixing it completely along the way.¹⁶

Before discussing how equation (5.9) was solved numerically, let us make a last general comment. The boundary conditions (5.10)–(5.12) and the relations (5.5)–(5.6) imply that the perturbations here considered don't change the charge and the mass of the soliton at linear order. This observation allows for a better understanding of the relation between the soliton linear stability properties and the fact that the mass curve, $M(\phi_0)$, encounters an extremum at $\phi_0 = \phi_{0,c}$.¹⁷ First, let us mention that, whenever $M'(\phi_{0,c}) = 0$, we also find that $Q'(\phi_{0,c}) = 0$. Therefore, around $\phi_{0,c}$, two infinitesimally close solitons, parameterized respectively by $\phi_{0,c}$ and $\phi_{0,c} + \Delta\phi$, have the same mass and charge, up to $O(\Delta\phi^2)$ corrections. This implies that there must be a time-independent linear radial perturbation that connects these static configurations and, in consequence, equation (5.9) admits a solution with $\omega_1^2(\phi_{0,c}) = 0$, i.e. a zero mode in the soliton spectrum. For $\omega_1^2(\phi_0)$ at least a C^2 function around $\phi_{0,c}$, we find

$$\omega_1^2(\phi_0) = \omega_1^2(\phi_{0,c}) + \partial_{\phi_0}\omega_1^2(\phi_{0,c})(\phi_0 - \phi_{0,c}) + \dots = \partial_{\phi_0}\omega_1^2(\phi_{0,c})(\phi_0 - \phi_{0,c}) + \dots \quad (5.14)$$

which becomes negative one side of the mass curve extremum, signalling an instability. On the remaining part of this section we are going to proceed by solving (5.9) numerically, confirming this expectation. For this task we employed Tchebychev pseudospectral collocation method. Inserting expansions

$$\hat{\phi}_1(x) = \cos(x)^3 \sum_{k=0}^{N-1} c_{1,k} T_k(1 - 4/\pi x), \quad (5.15)$$

$$\hat{\phi}_2(x) = \sum_{k=0}^{N-1} c_{2,k} T_k(1 - 4/\pi x), \quad (5.16)$$

$$\hat{A}(x) = \cos(x)^2 \sum_{k=0}^{N-1} c_{3,k} T_k(1 - 4/\pi x), \quad (5.17)$$

into equation (5.9) and evaluating on a collocation grid $\{t_k, k = 1 \dots N\}$

$$t_k = \frac{\pi}{4} \left(1 - \cos \left(\left(k - \frac{1}{2} \right) \frac{\pi}{N} \right) \right), \quad (5.18)$$

¹⁵This residual symmetry stems from the fact that equation (5.9) is a linear ODE that admits an algebraic solution of the form $\vec{Z} = (0, \alpha, \beta)^T$ with the aforementioned coefficient choice. This algebraic solution, on the other hand, is nothing but the action of a linearized gauge transformation on the trivial $\vec{Z} = 0$ solution.

¹⁶We have checked explicitly that the numerical results presented further on are independent of the particular way this symmetry is fixed. They are also equivalent to the ones obtained when this symmetry is left unbroken.

¹⁷See also the related discussion in [30].

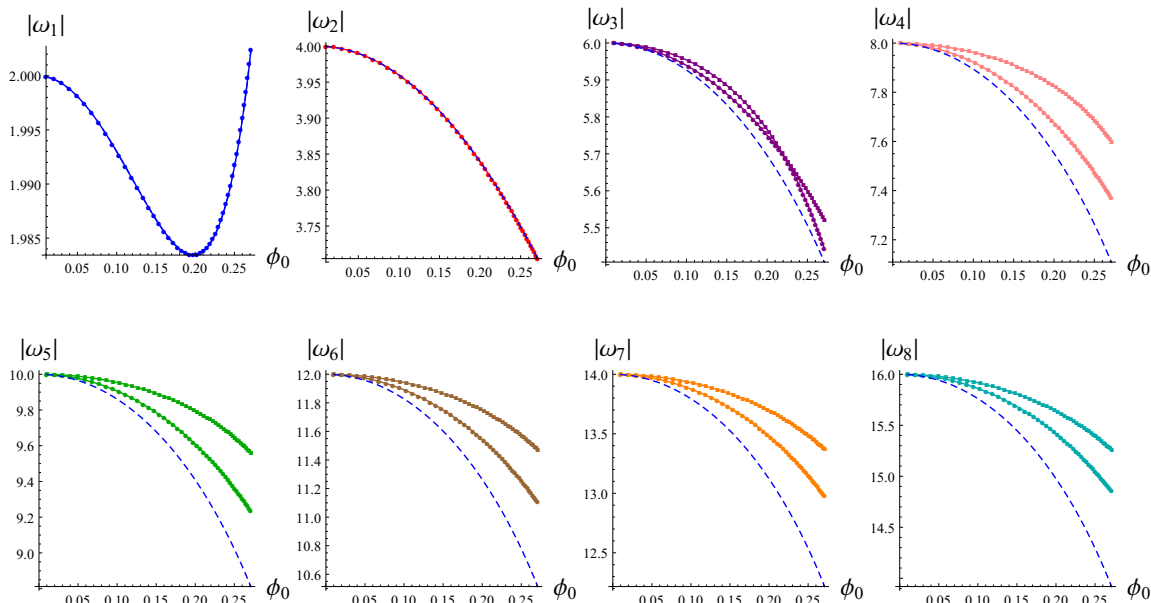


Figure 17. Soliton scalar eigenfrequencies for $e = 5$.

an algebraic generalized eigenvalue problem is to be solved which gives the numerical values of the first soliton normal modes.¹⁸

As for the results, first, the spectrum thus found is not resonant. In the spirit of [4], this should entail the absence of a turbulent cascade in the fully nonlinear regime. In figure 17, we plot the first eight normal frequencies for the soliton branch at coupling $e = 5$. They consistently reduce to their global AdS values as $\phi_0 \rightarrow 0$. A remarkable feature is the mode splitting that occurs for $k \geq 3$.¹⁹ This is common to every perturbative soliton branch we have analyzed. If the spectrum was exactly resonant, we would have that, for $k \geq 2$,

$$|\omega_{k+1}| = |\omega_1| + k(|\omega_2| - |\omega_1|). \quad (5.19)$$

We plot the right hand side of (5.19) for each k in figure 17 (blue-dashed). It is clearly seen that the equality is not satisfied away from $\phi_0 = 0$. A similar exercise can be performed only between the lower or upper splitted eigenfrequencies, choosing as reference the difference between any two consecutive ones, with identical conclusion.

Let us consider the intermediate region $3/2 \leq e^2 \leq 9/2$. In figure 18 (left), we plot $\omega_1^2(\phi_0), \omega_2^2(\phi_0)$ for the vacuum connected soliton branch at $e = 2$, together with the rescaled $Q(\phi_0)$ curve. We clearly see that a zero mode develops precisely at the point where $Q(\phi_0)$ reaches its first maximum and that, past this point, the solutions become linearly unstable.²⁰ The same phenomenon can be clearly appreciated on the vacuum disconnected

¹⁸We discard the values that don't converge when N is increased. A convergence test for the pseudospectral code is provided in appendix D.

¹⁹We remind the reader that a mode splitting was previously found in the perturbative computation of [6].

²⁰With a resolution of $\delta\phi_0 = 10^{-3}$, we have determined that the maximum lies at $\phi_{0,c} = 0.904$. Our pseudospectral code produces the values $\omega_1^2(\phi_0 = 0.903) = 0.0031$ and $\omega_1^2(\phi_0 = 0.905) = -0.0037$, in perfect agreement with expectations.

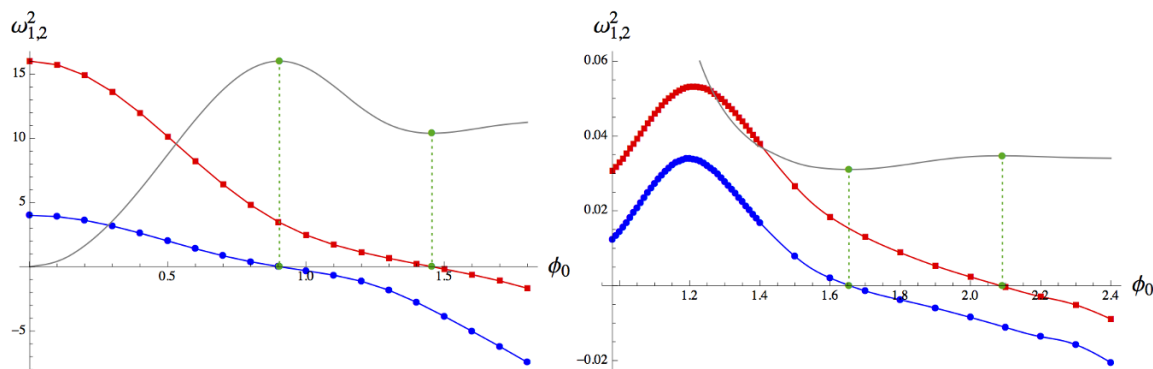


Figure 18. Left: ω_1^2 (blue) and ω_2^2 (red) versus ϕ_0 for the vacuum connected soliton branch at $e = 2$. In grey we show the rescaled charge curve $Q(\phi_0)$. Right: ω_1^2 (blue) and ω_2^2 (red) versus ϕ_0 for the vacuum disconnected soliton branch at $e = 2$. In grey we show the rescaled charge curve $Q(\phi_0)$. Green vertical lines correspond to the position of the charge curve extrema.

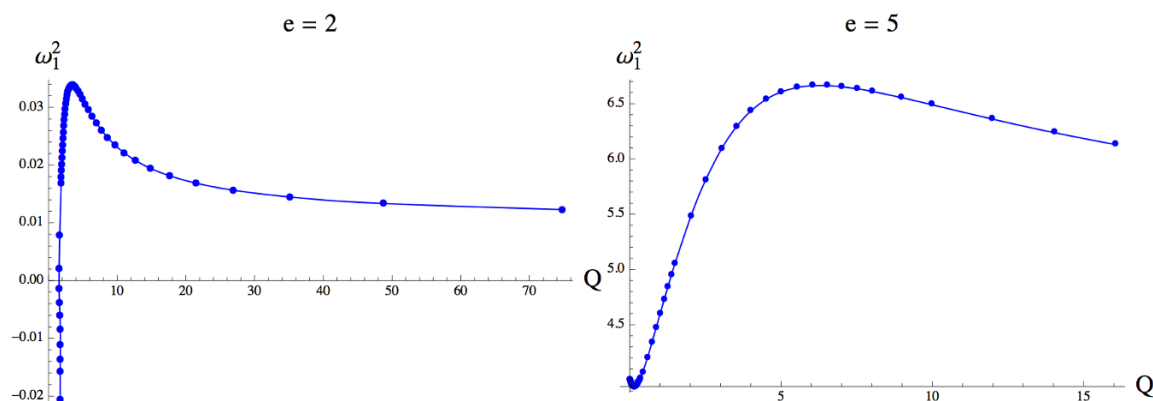


Figure 19. Left: ω_1^2 (blue) versus Q for the vacuum connected soliton branch at $e = 2$. Right: ω_1^2 (blue) versus Q for the vacuum disconnected soliton branch at $e = 5$.

soliton branch after $Q(\phi_0)$ attains its first minimum (figure 18, right). The general pattern we find is that, when $Q(\phi_0)$ hits a new extremum, a new normal mode crosses zero.²¹

As an additional comment, note that, regarding the vacuum disconnected branch, and in the $Q \gg 1$ regime, $\omega_1^2(Q)$ is a decreasing function of Q that stays finite in the $Q \rightarrow \infty$ limit (figure 19, left). Instead, in order for the phase of the harmonic perturbation ωt to remain finite in the blow up limit (2.19), the frequency should scale up as $\omega \sim \mu \sim Q^{\frac{1}{2}}$. We conclude that harmonic linear perturbations die off when the blow up limit is taken. This is consistent with the fact that the soliton branch maps onto a $T = 0$ hairy black brane, for which linearized perturbations correspond to quasinormal rather than normal modes. The discussion goes through in parallel to the regime $e > e_{sr}$ (see figure 19 (right) for $\omega_1^2(Q)$ at $e = 5$).

²¹This is a non trivial result obtained from the computation, since apparently nothing prevents the other option, where some normal mode oscillates up and down, crossing zero at every extremum of the charge curve.

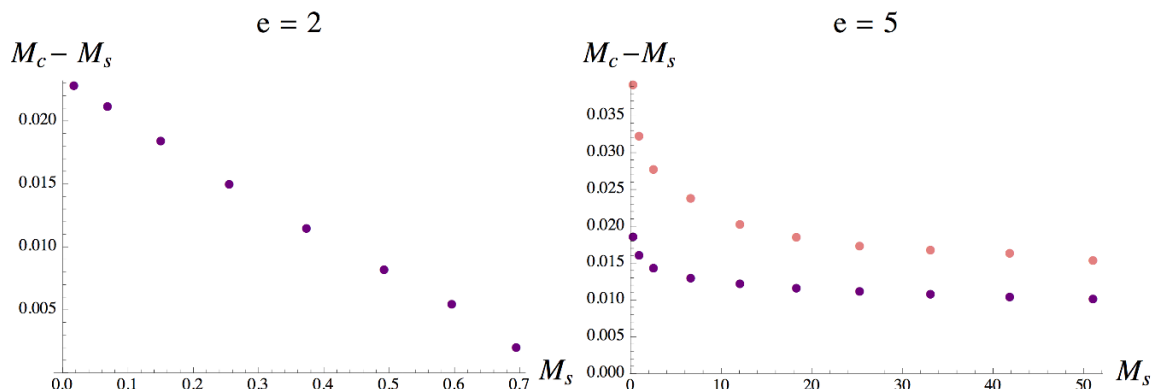


Figure 20. Left: limit of the prompt collapse region for the vacuum connected $e = 2$ soliton branch for $\sigma = 0.05$ (purple). Right: limit of the prompt collapse region for the $e = 5$ soliton branch for $\sigma = 0.1$ (magenta) and $\sigma = 0.05$ (purple).

After having discussed the linear stability properties of the solitons, in the next section we move on to the study of their nonlinear stability.

5.2 Nonlinear stability properties

We consider now the effect that a localized scalar field perturbation has on the soliton. We will stick to the same family of initial conditions that were used to perturb the AdS_4 vacuum (4.1), (4.2) but in a purely real setup. Concretely, our initial condition will be $\phi_s + \phi$ with $\phi(0, x) = 0$ and

$$\Pi_1(0, x) = \epsilon \frac{2}{\pi} \exp\left(-\frac{4 \cot^2(x)}{\pi \sigma^2}\right) \cos^3(x) \quad , \quad \Pi_2(0, x) = 0. \quad (5.20)$$

As this configuration has zero charge (see (A.21)), the family of perturbed solitons that we use to start with spans a vertical line in the (Q, M) plane above the unperturbed soliton solution ϕ_s , like the magenta vertical sets in figure 16 (left). The difference now is that this set of initial conditions explores down to the bottom green line as $\epsilon \rightarrow 0$.

Despite the fact that placing a perturbation like (5.20) on top of the AdS vacuum or on top of a soliton leads to very different initial conditions, the phenomenology we discover is remarkably similar. Namely, the magenta lines found in figure 16 (right) are qualitatively reproduced here. Indeed, for high enough ϵ prompt collapse is observed. Below some threshold mass M_c we have a delayed collapse and a number of oscillations are completed before the system finally undergoes gravitational collapse. This number, and with it the final time for collapse, diverges rapidly at some value of the mass above the soliton curve. Hence we don't see any trace of a nonlinear instability corner centered at the soliton solution (instead of the AdS vacuum). This is presumably again a symptom of the nonresonant character of the spectrum of soliton perturbations.

The possible survival of this oscillating region in the blow up limit is a relevant question and, indeed, was one of the main motivations that started the present work; the answer is negative. More precisely, we are interested in establishing whether its width $M_c - M_s$ has

a finite size relative to M_s

$$M_c - M_s \sim M_s \sim Q_s^{2/3}. \tag{5.21}$$

In figure 20 (right) we plot the location of the mass threshold $M_c - M_s$ for prompt collapse for the $e = 5$ family of solitons of mass M_s . We consider perturbations of the form (5.20) with $\sigma = 0.1$ and $\sigma = 0.05$. We see that, despite having finite width for large mass, the region of oscillations does not have the appropriate scaling to survive in the blow up limit. In this case, for $M_s \gg 1$, entering the oscillating regime requires to fine-tune the initial perturbation (5.20) in such a way that its relative contribution to the system energy goes to zero.

The behaviour just described is fairly natural given that, in the blow up limit, the soliton branch we are perturbing reduces to an extremal hairy black brane with an AdS₄ near-horizon geometry [16]. It is not unreasonable that, above this background, any perturbation localized in the near-boundary region, regardless of its amplitude, leads to direct collapse to a $T \neq 0$ black brane, in the same vein as it happens for zero charge [31]. It remains to be seen if this is also the pattern in other matter models. Our educated guess is that, for any theory that displays an unbounded soliton branch, the blown up extension of the oscillatory regime is finite whenever the theory can support a gapped spectrum of scalar fluctuations in the planar limit. Theories of this kind are not unknown. Consider, for instance, the Improved Holographic QCD models which have recently been analyzed at the dynamical level in [32, 33]. There, the planar geometry dual to the field theory ground state is sourced by a nontrivial scalar field profile that generates a naked singularity in the infrared.²² When considered in the fully nonlinear regime, weak perturbations localized near the boundary may be noncollapsing and forever oscillating, since the singularity would “repel” them from the infrared, so as they never reach their Schwarzschild radius. This behavior has been explicitly seen in more crude models of gapped field theories in planar AdS, such as a scalar field in a hard wall geometry [24] or the AdS-soliton [34]. It would be interesting to classify, in generic terms, which Einstein-Maxwell-scalar theories support a soliton branch with a gapped planar limit and check if this is correlated with a nonvanishing scaling of the oscillatory regime width.

We have also determined the boundary of the prompt collapse region for the vacuum connected soliton branch at coupling $e = 2$, upon scalar fluctuations of the form (5.20) with $\sigma = 0.05$ (figure 20, left). In accordance with the fact that soliton solutions become linearly unstable when the mass curve reaches its maximum, here we find that the width of the oscillation region shrinks to zero in a linear fashion.

6 Conclusions

The original motivation of this paper was to examine the status of the nonlinear instability problem in the Einstein-Maxwell-scalar theory in $d+1 = 4$ dimensions. The work demanded first a thorough unraveling of the landscape of static solutions which showed almost the same features as in one dimension higher [11, 12]. The issue of stability can be examined at different levels. Studying the thermodynamical linear stability demands selecting first

²²This, and what follows in the main text, is true at least for some specific classes of scalar potentials.

a certain ensemble. Working in the grand canonical ensemble we have pinned down the relevant phase transitions.

In the second part of the paper we have studied the nonlinear stability in the microcanonical ensemble. First of all, we have considered perturbations of pure AdS₄ with families of initial conditions of different charges, generalizing the uncharged ones considered in [1]. These initial conditions have an amplitude ϵ and a width σ . We have found that for thin initial pulses $\sigma = 0.1$, the same type of instability sets in at times of order ϵ^{-2} . This fact confirms the expectation that places the origin of the mechanism in the fully resonant character of the linear spectrum, something that is also true in this case. Wider pulses, $\sigma \geq 0.2$, exhibit a divergence in the time for collapse below some critical amplitude. This is also in parallel with the effect detected for wide radial perturbations of AdS₄ in [5]. The role of the oscillons in that situation is probably taken up by the solitons in this setup.

To confirm this picture, we have examined the linear and nonlinear stability of fluctuations placed on top of the soliton solutions. The spectrum of linear normal modes is not resonant. The linear stability, signalled by an imaginary eigenfrequency, is seen to appear, as expected, coinciding with extrema of the mass curve. For higher amplitude perturbations we need to resort to a full fledged simulation of the evolution of the system. In general, linear stability extends to nonlinear stability up to some threshold for the amplitude of the perturbation. Beyond that, the soliton is destroyed and collapses to a hairy black hole. The protection region where oscillations do not decay does not scale properly in the limit of large mass and charge to survive the blow up planar limit. This seems to point to the necessity of having a mass gap, and not just a mass scale, to find such oscillatory behaviours.

While this paper was completed in its last part, namely sections 4 and 5, reference [17] appeared in the arXive. It overlaps substantially with the content of our section 3, and we find agreement with their results whenever they can be compared (their scalar field has negative mass).

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A Ansatz and equations of motion

Let us give the equations for a situation which is slightly more general than the one considered in the main text, and allow for a scalar potential

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} (R - 2\Lambda) - \int d^{d+1}x \sqrt{-g} (D_\mu \phi \bar{D}^\mu \phi^* + V(\phi)) - \frac{1}{4} \int d^{d+1}x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \quad (\text{A.1})$$

with $\kappa^2 = 8\pi G$, $\Lambda = -d(d-1)/2l^2$ for Anti de Sitter. Also, the scalar field is complex and we have $D_\mu \phi \equiv (\partial_\mu - ieA_\mu)\phi$. The equations of motion are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa^2 \left(T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(A)} \right) \quad (\text{A.2})$$

$$\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu\nu} D_\nu \phi) = \frac{\partial V(\phi, \phi^*)}{\partial \phi^*}, \quad (\text{A.3})$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = J^\nu, \quad (\text{A.4})$$

where the energy-momentum and charge currents are

$$T_{\mu\nu}^{(\phi)} = (\bar{D}_\mu \phi^* D_\nu \phi + \mu \leftrightarrow \nu) - g_{\mu\nu} (|D\phi|^2 + V(\phi)), \quad (\text{A.5})$$

$$T_{\mu\nu}^{(A)} = F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F^2, \quad (\text{A.6})$$

$$J^\nu = ie (\phi^* D_\mu \phi - \phi (\bar{D}_\mu \phi^*)) g^{\mu\nu}. \quad (\text{A.7})$$

By employing the ansatz described in section 2, the Klein-Gordon equation can be casted into the first order form

$$\dot{\Phi} = \left(f e^{-\delta} \Pi + ie A_t \phi \right)', \quad (\text{A.8})$$

$$\dot{\Pi} = \frac{1}{\tan^{d-1} x} \left(\tan^{d-1} x f e^{-\delta} \Phi \right)' + ie A_t \Pi - \frac{l^2}{\cos^2 x} e^{-\delta} \partial_{\phi_c} V(\phi), \quad (\text{A.9})$$

while, from the Einstein equations, we obtain,

$$f' = \frac{d-2+2\sin^2 x}{\sin x \cos x} (1-f) - \frac{2\kappa^2}{d-1} \sin x \cos x f (|\Phi|^2 + |\Pi|^2) \quad (\text{A.10})$$

$$- \frac{\kappa}{(d-1)l^2} e^{2\delta} \cos^3 x \sin x A_t'(t, x)^2 - \frac{2\kappa^2 l^2}{d-1} \tan x V(|\phi|),$$

$$\delta' = -\frac{2\kappa}{d-1} \sin x \cos x (|\Phi|^2 + |\Pi|^2). \quad (\text{A.11})$$

There is one additional equation coming from the (t, x) component of Einstein equations that yields the *momentum constraint*

$$\dot{f} = -\frac{4\kappa}{d-1} \sin x \cos x f^2 e^{-\delta} \text{Re}(\Phi \Pi_c). \quad (\text{A.12})$$

Concerning Maxwell's equations, define $C = A_t'(t, x)$, then from (A.4) we can derive the two following equations

$$(e^\delta C)' = 2e \frac{l^2}{\cos^2 x} \text{Im}(\phi \Pi_c) - e^\delta C ((d-3) \tan x + (d-1) \cot x) \quad (\text{A.13})$$

$$(e^\delta C) = 2e \frac{l^2 e^{-\delta} f}{\cos^2 x} \text{Im}(\phi \Phi_c) \quad (\text{A.14})$$

which can be easily shown to be compatible. The first one can be recasted as follows

$$\left(e^\delta C \sin^{d-1} x \cos^{3-d} x \right)' = 2el^2 \tan^{d-1} x \operatorname{Im}(\phi \Pi_c) \quad (\text{A.15})$$

and the second is the *Maxwell constraint*. With the condition that A'_t be bounded at the origin we can integrate (A.15) to find

$$A'_t = 2el^2 e^{-\delta} \sin^{1-d} x \cos^{d-3} x \int_{x_0}^x \tan^{d-1} x \operatorname{Im}(\phi \Pi_c) \quad (\text{A.16})$$

The electromagnetic current is given by the following expression

$$J^\mu(t, x) = \frac{2e \cos^2 x}{l^2} (-e^\delta \operatorname{Im}(\phi \Pi_c), f \operatorname{Im}(\phi \Phi_c), 0, 0)$$

and the charge density leads to the definition of the conserved *physical* charge

$$\mathcal{Q} = \int_{S_{d-1}} d\Omega_{S_{d-1}} \int_0^{\pi/2} dx \sqrt{-g} J^0(x) \quad (\text{A.17})$$

$$= -2el^{d-1} V_{S_{d-1}} \int_0^{\pi/2} dy \tan^{d-1} y \operatorname{Im}(\phi \Pi_c) \quad (\text{A.18})$$

Take (A.16) and notice that the following expression holds

$$\frac{(-1)^{d-3}}{(d-3)!} A_t^{(d-2)}(\pi/2) = 2el^2 e^{-\delta(\pi/2)} \int_0^{\pi/2} \tan^{d-1} y \operatorname{Im}(\phi \Pi_c) = \frac{-\mathcal{Q} e^{-\delta(\pi/2)}}{l^{d-3} V_{S_{d-1}}} \quad (\text{A.19})$$

or, after dividing by $(2-d)$ and multiplying by $e^{\delta(\pi/2)}$, because \mathcal{Q} has to be invariant under $\delta \rightarrow \delta + c$,

$$e^{\delta(\pi/2)} \frac{(-1)^{d-2}}{(d-2)!} A_t^{(d-2)}(\pi/2) = \frac{\mathcal{Q}}{(d-2)l^{d-3} V_{S_{d-1}}} \equiv Q. \quad (\text{A.20})$$

In other words

$$Q = -\frac{2el^2}{d-2} V_{S_{d-1}} \int_0^{\pi/2} \tan^{d-1} y \operatorname{Im}(\phi \Pi_c). \quad (\text{A.21})$$

The Reissner-Nordström black hole solution can be recovered by setting $\phi = \Phi = \Pi = \delta = 0$. Specializing to the $d = 3$ case, the stationary solution to (A.10) and (A.15) is

$$f_{RN}(x) = 1 - M \frac{\cos^3 x}{\sin x} + \kappa \frac{1}{2} Q^2 \frac{\cos^4 x}{\sin^2 x}, \quad (\text{A.22})$$

$$A_{tRN}(x) = \mu + Q \frac{\cos x}{\sin x}. \quad (\text{A.23})$$

This expression gives the mass and charge of the black hole as

$$M = \frac{1}{6} f_{RN}^{(3)}(\pi/2), \quad (\text{A.24})$$

$$Q = -A'_t(\pi/2) e^{\delta(\pi/2)}. \quad (\text{A.25})$$

The charge is always proportional to the derivative of the gauge field on the boundary, i.e. the charge density, which agrees with (A.20). More generally, the mass of a black hole in the time dependent background (2.1) can be written

$$M = (\sin x_h)(\sec x_h)^3 e^{\delta(\pi/2) - \delta(x_h)} + \quad (\text{A.26})$$

$$+ \kappa e^{\delta(\frac{\pi}{2})} \int_{x_h}^{\pi/2} \left(\tan^2 y \left(\Phi^2 + \Pi^2 + \frac{V(\phi)}{\cos^2 x} \right) e^{-\delta(y)} + \frac{1}{2l^2} \sin^2 y A'_t(t, y)^2 e^{\delta(y)} \right)$$

B Instabilities

Take the RN metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2 \quad (\text{B.1})$$

with

$$f(r) = 1 + \frac{r^2}{l^2} - M \left(\frac{l}{r}\right)^{d-2} + \kappa^2 \frac{d-2}{d-1} Q^2 \left(\frac{l}{r}\right)^{2d-4}, \quad (\text{B.2})$$

$$A_t(r) = \mu \left(1 - \frac{r_h^{d-2}}{r^{d-2}}\right) = \mu - \frac{Q}{r^{d-2}}. \quad (\text{B.3})$$

We will show that extremal RN black holes suffer from two potential instabilities.

B.1 Tachyonic instability

This is sourced by the lowest scalar mode close to the horizon. The near horizon limit is obtained by expanding

$$r \rightarrow r_h(1 + \lambda/z), \quad t \rightarrow \left(\frac{f''(r_h)}{2} r_h \lambda\right)^{-1} t \quad (\text{B.4})$$

and taking the limit $\lambda \rightarrow 0$. In the extremal case $f(r_h) = f'(r_h) = 0$ and, therefore, the limit has the form of $\text{AdS}_2 \times S^{d-1}$

$$ds^2 \rightarrow \left(\frac{f''(r_h)}{2}\right)^{-1} \left(\frac{-dt^2 + dz^2}{z^2}\right) + r_h^2 d\Omega_{d-1}^2$$

and also

$$A_t \rightarrow \frac{\lambda \mu (d-2)}{z}.$$

One may take the near horizon limit in the equation of motion of the scalar field and extract the mass from the coefficient of ϕ

$$m^2 = -\frac{4e^2(d-2)^2 \mu^2}{f''(r_h)^2 r_h^2}. \quad (\text{B.5})$$

We observe explicitly that the mass gap is set by the chemical potential. The scalar field will be unstable if its mass violates the BF bound of AdS_2 , hence if $m^2 < -1/4$. This implies

$$e > e_t = \frac{1}{4} \frac{f''(r_h) r_h}{(d-2)\mu}$$

The values of M and Q at extremality are given by

$$M_{\text{ext}} = 2r_h^{d-2} \left(1 + \frac{d-1}{d-2} r_h^2\right); \quad Q_{\text{ext}} = \mu_{\text{ext}} r_h^{d-2} = \frac{r_h^{d-2}}{d-2} \left((r_h^2 d + d-2)(d-1)\right)^{1/2}$$

Evaluating $f''(r_h)$ with these values results in the following expression

$$e_t^2 = \frac{4 + d(d-4 + r_h^2(d-1))}{4r_h^2(d-1)(d(r_h^2+1) - 2)},$$

which diverges in the limit $r_h \rightarrow 0$. The tachyonic instability is thus suppressed for infinitesimally small RN black holes. Instead, for large r_h , $e_t(r_h)$ admits the following expansion

$$e_t^2 = \frac{d(d-1)}{4} + \frac{6-5d+d^2}{4r_h^2} + \mathcal{O}(r_h^{-4}) + \dots \tag{B.6}$$

In summary, for $e_t^2 < 3/2$ in $d = 3$ and for $e_t^2 < 3$ in $d = 4$, there is no tachyonic instability. As this threshold is surpassed, large black holes are unstable first. Small black holes do not suffer from this instability.

B.2 Superradiant instability

If $e\mu > \Delta_0$, with Δ_0 being the lowest eigenvalue of the scalar linearized equation, RN black holes becomes superradiant in the limit $r_h \rightarrow 0$, meaning that this mode scatters off with a reflection coefficient $|R| > 1$. The final state should be a hairy black hole with greater entropy than the RN one, which dominates the micro-canonical ensemble. For a massless scalar, $\Delta_0 = d$. Substituting the extremal value

$$\mu_{\text{ext}} = Q_{\text{ext}} r_h^{2-d} = \frac{1}{d-2} \left((r_h^2 d + d - 2)(d - 1) \right)^{1/2}$$

gives, for $r_h \ll 1$,

$$e_{sr}^2 = \frac{d^2(d-2)}{d-1} \left(1 - \frac{d}{2(d-2)} r_h^2 + \mathcal{O}(r_h^{3/2}) + \dots \right), \tag{B.7}$$

which constitutes the relevant lower bound on the coupling, since $\mu < \mu_{\text{ext}}$. The precise superradiance threshold is

- in $d = 3$,

$$e_{sr}^2 = \frac{9}{2} - \frac{27r_h^2}{2} + \dots$$

Hence all extremal RN black holes above $e^2 = 9/2$ are unstable.

- in $d = 4$,

$$e_c^2 = \frac{32}{3} - \frac{64r_h^2}{3} + \dots$$

This is the result of [12]. For $e^2 > 32/2$ all extremal RN black holes are superradiantly unstable.

It should be noted that the condition $e\mu > \Delta_0$ is obtained in a perturbative expansion in r_h and, in consequence, it can't be extrapolated to sufficiently large RN black holes.

C Renormalized action

The central quantity in the grand canonical ensemble is the grand potential $\Omega(T, \mu)$. The AdS/CFT correspondence identifies it with the on-shell regularized Euclidean action. To

compute it we continue to Euclidean signature and compactify time with a period $\frac{1}{T}$. Writing $S_{\text{on-shell}} = \frac{\hat{S}_{\text{bulk}}}{T}$ one has

$$\hat{S}_{\text{bulk}} = - \int d^3x \sqrt{-g} \mathcal{L} \quad (\text{C.1})$$

where

$$\mathcal{L} = \frac{1}{2\kappa^2} (R - 2\Lambda) - D_\mu \phi \bar{D}^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (\text{C.2})$$

Using the equations of motion one can show that $2\mathcal{L} = -(G_t^t + G_x^x)$ whence

$$\hat{S}_{\text{bulk}} = 4\pi \left(\int_{x_h}^{x_b} dx \sec^2(x) e^{-\delta(x)} - e^{-\delta(x)} \tan(x) \sec^2(x) f(x) \Big|_{x=x_b} \right) \quad (\text{C.3})$$

where the (diverging) result has been regularized at some boundary value $x_b = \pi/2 - \epsilon$.

To have a well defined variational problem we must add the Gibbons-Hawking term

$$\hat{S}_{GH} = \int d\theta d\phi \sqrt{\bar{g}} K \Big|_{x=x_b}, \quad (\text{C.4})$$

where \bar{g} stands for the induced metric at x_b , $K = g^{\mu\nu} \nabla_{\mu\nu} n_\nu$ is the extrinsic curvature and $n^\mu = \cos(x) \sqrt{f(x)} \delta^\mu_x$ is the outward pointing unit normal vector to the boundary

$$\hat{S}_{GH} = -2\pi \left[\tan(x) e^{-\delta(x)} (\tan(x) f'(x) + f(x) (-2 \tan(x) \delta'(x) + 6 \sec^2(x) - 2)) \right]_{x=x_b}. \quad (\text{C.5})$$

The sum $\hat{S}_{\text{bulk}} + \hat{S}_{GH}$ diverges when $x_b \rightarrow \pi/2$. This can be handled by the following counterterm

$$\hat{S}_{ct} = -4 \int_{x=\pi/2-\epsilon} d\theta d\phi \sqrt{-g_{bdy}} = 16\pi \left[\frac{\tan(x)^2}{\cos(x)} \sqrt{f(x)} e^{-\delta(x)} \right]_{x=x_b}. \quad (\text{C.6})$$

The grand canonical thermodynamic potential Ω is obtained from the limit $\epsilon \rightarrow 0$

$$\Omega = \lim_{x \rightarrow \frac{\pi}{2}} \left(\hat{S}_{\text{bulk}} + \hat{S}_{GH} + \hat{S}_{ct} \right). \quad (\text{C.7})$$

We checked the Smarr relations for this expression which, in particular, imply that, indeed, $\Omega = M - TS - \mu Q$, as demanded by thermodynamic consistency. If we wish to compute the thermodynamic properties in the canonical ensemble (fixed Q) instead the relevant quantity to consider would be the free energy $F = M - TS$.

D Numerics

D.1 Time-evolution code

It is a virtue of the coordinate system (2.1) that the Einstein and Maxwell equations appear as constraints (A.10), (A.11) and (A.16) that can be solved at each instant of time. The evolution of the system is then driven by the scalar field equations (A.8) and (A.9). Starting from given nonequilibrium initial data, such as (4.1) and (4.2), we have solved these equations numerically by resorting to a fourth-order accurate finite-difference evolution code.

Time evolution is performed by an explicit Runge-Kutta method. In order to deal with the high-frequency noise generated due to the finiteness of the discretization grid, we implement standard Kreiss-Oliger dissipation. By setting $\delta(\pi/2) = 0$ we obtain stable evolutions with a constant Courant factor λ with no need of local mesh refinement in time. On the other hand, spatial derivatives are discretized by employing a centered finite-difference stencil, while integrations are handled by a specifically designed routine, based on local polynomial interpolation. To deal with boundary conditions and numerical stability both at $x = 0, \pi/2$ requires some detailed procedures that can be found in [35].

The major difficulty in the present setup stems from the fact that, upon evolution, the scalar profile develops very spiky features that demand a high resolution. To resolve these sharp features, which are apparent in figure 12, we used global mesh refinement in space, eventually reaching $2^{17} + 1$ grid points to discretize the interval $x \in [0, \pi/2]$. This has required a parallel implementation that employs the MPI infrastructure to run the code on the SVG cluster at the CESGA facility (www.cesga.es). Optimal results have been obtained for ~ 30 nodes running in parallel. Smarter solutions involving local space mesh refinement are left for the future.

The quality control parameters employed to activate the refinement process are both the norm of the momentum constraint (A.12), as well as the relative mass loss at each time step. As a matter of fact, only at late times in the simulation are such mentioned fine resolutions required. The code stops at a time t_f when the minimum value of $A(t, x)$ reaches below an user defined cutoff A_c .²³ This is the time that is meant in the right figures 13 and 16. Of course, mathematically speaking, the apparent horizon will only form in the infinite future $\lim_{t_f \rightarrow \infty} \min[A(t_f, x)] = 0$ in the chosen coordinate gauge.

D.2 Convergence tests

D.2.1 Time evolution code

To illustrate the convergence properties of our time evolution code, we consider the functional

$$\Delta_n[g](t) \equiv \|g_n(t, x) - g_{n+1}(t, x)\| = \left(\int_0^{\pi/2} \tan(x)^2 (g_n(t, x) - g_{n+1}(t, x))^2 \right)^{1/2}, \quad (\text{D.1})$$

where g_n refers to any function computed on a discretization grid of spatial resolution $h = \pi/2^{n+1}$. Fourth-order convergence then implies that

$$\Delta_{n+1}[g](t) = 2^{-4} \Delta_n[g](t). \quad (\text{D.2})$$

In figure 21, we plot $\Delta_n[\Phi_1](t), \Delta_n[\Phi_2](t)$ at $n = 11, 12, 13$ for simulations with initial data

$$\Phi = \epsilon_1 \frac{2}{\pi} \sin x \exp\left(-\frac{4 \tan^2(x)}{\pi^2 \sigma^2}\right), \quad (\text{D.3})$$

$$\Pi = i\epsilon_2 \frac{2}{\pi} \exp\left(-\frac{4 \tan^2(x)}{\pi^2 \sigma^2}\right), \quad (\text{D.4})$$

²³For the simulations shown in the main text, we have set $A_c = 0.02$.

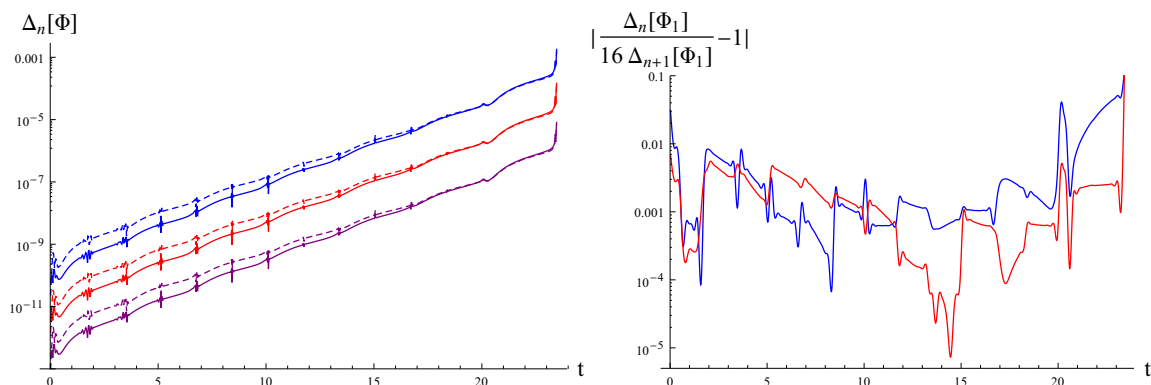


Figure 21. Left: $\Delta_n[\Phi]$ for $n = 11$ (blue), $n = 12$ (red) and $n = 13$ (purple). Solid lines correspond to Φ_1 , while dashed ones to Φ_2 . The error norm decreases with increasing grid resolution. Right: absolute value deviation of the quotient $\frac{\Delta_n[\Phi_1]}{16 \Delta_{n+1}[\Phi_1]}$ from 1 for $n = 11$ (blue) and $n = 12$ (red). The error norm converges to zero at the right order.

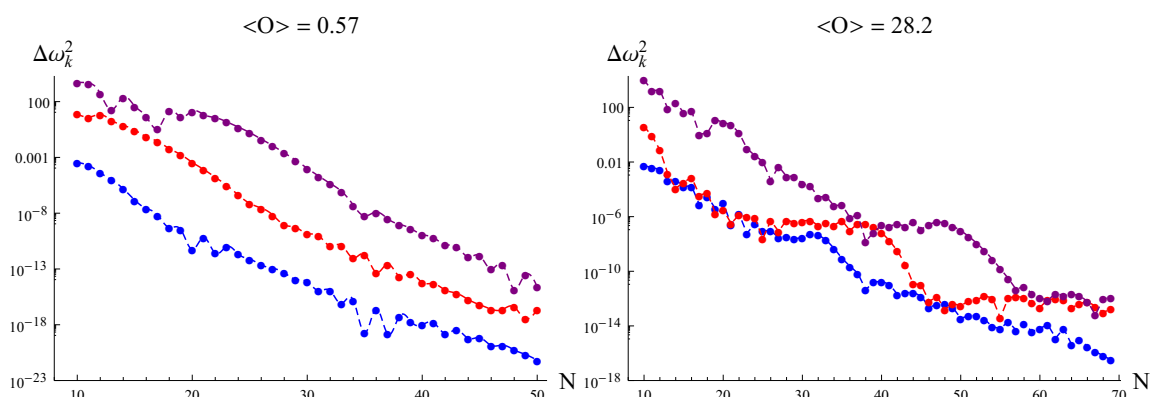


Figure 22. Left: at $e = 5$, and for a soliton with $\langle \mathcal{O} \rangle = 0.57$, we plot the errors $\Delta\omega_1^2$ (blue), $\Delta\omega_{10}^2$ (red), $\Delta\omega_{20}^2$ (purple) on a collocation grid ranging from $N = 10$ to $N = 50$. It is seen that the error tends exponentially to zero as the resolution N is increased. Right: the same quantities for the $e = 5$, $\langle \mathcal{O} \rangle = 28.2$ soliton, on a collocation grid ranging from $N = 10$ to $N = 70$.

where $\epsilon_1 = \epsilon_2 = 12$, $\sigma = 0.1$ and $e = 5$. The scalar field completes seven bounces before collapse is achieved at $t_f = 23.47$. We see precisely that relation (D.2) is fulfilled.²⁴

D.2.2 Soliton eigenfrequencies pseudospectral code

For the computation of the normal modes on top of a soliton background, we have resorted to the pseudospectral method described in the main text. As mentioned, the output of this procedure are the first N soliton normal modes. For pseudospectral methods, we expect exponential convergence, since we are approximating the analytical scalar field eigenmodes. In order to determine the convergence properties of the method, we show in figure 22 the

²⁴We have deactivated the global mesh refinement algorithm, and thus we observe deviations from exact fourth-order convergence right before collapse.

quantity $\Delta\omega_k^2(N)$, defined as

$$\Delta\omega_k^2(N) = |\omega_k^2(N+1) - \omega_k^2(N)|. \quad (\text{D.5})$$

The fact that $\Delta\omega_k^2(N) \rightarrow 0$ exponentially as $N \rightarrow \infty$ implies that the sequence $\{\omega_k^2(N), N = N_0, N_0 + 1, \dots\}$ converges as anticipated.

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