CENTRO INTERNACIONAL DE ESTUDOS
DE DOUTORAMENTO E AVANZADOS
DA USC (CIEDUS)

TESE DE DOUTORAMENTO

Some nonlocal operators in porous medium equations: the extension problem and regularity theory

Jean-Daniel Djida

ESCOLA DE DOUTORAMENTO INTERNACIONAL

PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA

ANO 2019

DECLARACIÓN DO AUTOR DA TESE

Some nonlocal operators in porous medium equations: the extension problem and regularity theory

D. Jean-Daniel Djida

Presento a miña tese, seguindo o procedemento adecuado ao Regulamento, e declaro que:

1) A tese abarca os resultados da elaboración do meu traballo.
2) No seu caso, na tese faise referencia ás colaboracións que tivo este traballo.
3) A tese é a versión definitiva presentada para a súa defensa e coincide coa versión enviada en formato electrónico.
4) Confirmo que a tese non incorre en ningún tipo de plaxio doutros autores nin de traballos presentados por min para a obtención doutros títulos.

En Santiago de Compostela, 23 de marzo de 2019

Asdo. Jean-Daniel Djida

DECLARACIÓN DO DIRECTOR DA TESE

Some nonlocal operators in porous medium equations: the extension problem and regularity theory

Prof. Juan José Nieto Roig e Prof. Iván Carlos Area Carracedo

INFORMAN:

Que a presente tese se corresponde co traballo realizado por D. Jean-Daniel Djida, baixo a nosa dirección, e autorizamos a súa presentación, considerando que reúne os requisitos esixidos no Regulamento de Estudos de Doutoramento da USC, e que como directores desta non incorremos nas causas de abstención establecidas na lei 40/2015.

En Santiago de Compostela, 23 de marzo de 2019

Asdo. Prof. Juan José Nieto Roig e Prof. Iván Carlos Area Carracedo

To my lovely wife Pauline Donn and my daughter Anita

Summary

The aim of this work is to investigate some nonlocal nonlinear equations of porous type of the form

$$\mathbb{D}_t^\alpha u + LF(u) = 0,$$

and to obtain new regularity results for these equations. Here, the nonlocal operator $\mathbb{D}_t^\alpha$ with $0 < \alpha \leq 1$ is the fractional time-derivative operator, L is a general nonlocal operator associated to the particular Lévy process and F is a monotone non-decreasing function, satisfying some conditions that allow for degeneracies, like $F(0) = 0$ and $F'(0) = 0$.

Such equations model various anomalous diffusion phenomena. The prototype is the classical porous medium equation

$$u_t + \Delta u^m = 0, \qquad \text{(PME)}$$

which models the flow of gasses through porous media. Starting from the classical equation (PME), different directions can be considered, due to their recent interest for the mathematical community and their applicability in modelling real world problems.

My contribution to the material presented in this thesis is contained in the papers

- P1 J-D. Djida and A. Fernandez. *Interior Regularity Estimates for a Degenerate Elliptic Equation with Mixed Boundary Conditions*, 7(65): 1–16, 2018;
- P2 J-D. Djida. *Wellposedness and boundary regularity for nonlocal parabolic problem with fractional derivative*, 2019, Submitted;
- P3 J-D. Djida, J. J. Nieto and I. Area, *Nonlocal time-porous medium equation: weak solutions and finite speed of propagation*, Discrete Continuous Dyn. Syst. Ser. B, 22:1, 2019;
- P4 J-D. Djida, I. Area and J. J. Nieto. *A De Giorgi-Nash type theorem for nonlocal time porous medium equations*, 2018, Submitted;
- P5 J-D. Djida, J. J. Nieto and I. Area, *Nonlocal time porous medium equation with fractional time derivative*, Revista Matemática Complutense, 2019, in press;

in collaboration with my advisors (Professor Juan José Nieto and Professor Iván Area), coauthored also with my collaborator Doctor Arran Fernandez.

This thesis gathers some recent research on the general nonlocal operator of Lévy type, the fractional Laplacian and the fractional derivative. Starting from the theory of these nonlocal operators, we will collect recent results and observations and enrich the material with some original contributions on existence, interior and boundary regularity results in the linear setting for some integro-differential problems, which will be of interest for the nonlinear problem. Also, we will see that some nonlocal effects registered by the fractional Laplacian find correspondence for other fractional time-derivative operators. This is the case for the extended Caputo type or the Marchaud definitions. Later, the extension problem (Dirichlet to Neumann map) related to the fractional derivative will be also analysed. Furthermore, we will present known recent results on regularity theory of some nonlocal porous medium equation with fractional Laplacian and fractional derivative and discuss in detail new results on the regularity for the fully nonlocal nonlinear time-porous medium .

The main topics of the thesis are the following:

1. *Interior regularity for a degenerate elliptic equation with mixed boundary conditions.* We start with the proof of existence and uniqueness of the weak solution of the elliptic problem $(\mathscr{D}_{right})^s v = f$, for $s \in (0,1)$. Here the nonlocal operator $(\mathscr{D}_{right})^s$ denotes the Marchaud fractional derivative (or extended Caputo derivative). As a main result, we prove interior Schauder regularity estimates for a degenerate elliptic equation with mixed Dirichlet–Neumann boundary conditions. The degenerate elliptic equation arises from the Bernardis–Reyes–Stinga–Torrea extension of the Dirichlet problem for the Marchaud derivative (*Regularity theory for the fractional harmonic oscillator*, 2016), which is the analogue of the one involving the fractional Laplacian introduced by Caffarelli and Silvestre (*An extension problem related to the fractional Laplacian*, 2016).

2. *Boundary regularity for Parabolic problem with nonlocal operators.* $\mathbb{D}_t^\alpha u + Lu = 0$. Here, L is the general nonlocal operator of Lévy type. We present the results for the boundary regularity of the solution to the elliptic and parabolic problem obtained by Ros-Oton, Serra and Fernández-Real (*The Dirichlet problem for the fractional Laplacian: regularity up to the boundary* and *Boundary regularity for the fractional heat equation*) and use them to derive the boundary regularity results for a parabolic problem with general nonlocal operator and fractional derivative. The result obtained shows that a solution u of the above homogeneous fractional parabolic equation on a bounded domain Ω ss thatidfiest $u \in \mathcal{C}^s(\mathbb{R}^d)$ and that u/δ^s can be extended Hölder continuously up to $\overline{\Omega}$, where $\delta(x) = dist(x, \partial\Omega)$. Comments on the interior and boundary regularity results for the general nonlinear parabolic equation of porous type $\mathbb{D}_t^\alpha u + LF(u) = 0$ are also given.

3. *Wellposedness for the nonlinear nonlocal diffusion equation* $\mathbb{D}_t^\alpha u + (-\Delta)^s \varphi(u) = f$. In the first stage $\alpha \in (0,1)$, $s \in (0,1)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, smooth and increasing function. $\mathbb{D}_t^\alpha$ and $(-\Delta)^s$ denote the Caputo fractional time derivative and fractional Laplacian respectively. The second stage deals with the *wellposedness of the nonlinear nonlocal diffusion equation* $\mathbb{D}_t^\alpha u - \nabla \cdot (u^{m-1} \nabla P) = f$, with $m > 1$, $P = \mathcal{K}(u)$, $\mathcal{K}$ being a Riesz potential. The problem has been recently studied

by Caffarelli and Vázquez (*Nonlinear porous medium flow with fractional potential pressure*, 2011) and by Biler, Karch and Monneau (*Nonlinear Diffusion of Dislocation Density and Self-Similar Solutions*, 2009) in the particular case $m = 2$ and $\alpha = 1$, in which they show the relevance of this model for applications. Similarly Stan, Teso and Vázquez (*Finite and infinite speed of propagation for porous medium equations with fractional pressure*, 2014) investigate the effect of the nonlinearity on the finite speed of propagation of the solution when $\alpha = 1$. Finally, regularity results have been obtained by Allen, Vasseur and Caffarelli (*A parabolic problem with a fractional time derivative*, 2016) for the case $m = 2$. We investigate the existence and finite speed of propagation of weak solutions when the exponent $m \geq 2$, $0 < \alpha \leq 1$ and $0 < s \leq 1$. This part ends with some comments on self-similarity solutions of the above mentioned models of porous type.

4. *De Giorgi–Nash–type theorem for the time-porous medium equation.* $\mathbb{D}_t^\alpha u + \Delta\varphi(u) = f$. We prove a De Giorgi–Nash–Moser Hölder type regularity theorem for the time porous medium equation, which gives an interior Hölder estimate for bounded weak solutions. The proof relies on a priori estimates for time-nonlocal operators and uses De Giorgi's technique (*Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari*).

5. *Regularity theory for the nonlocal time-porous medium equation with fractional Laplacian* $\mathbb{D}_t^\alpha u + \mathcal{L}^s\varphi(u) = f$. This nonlinear nonlocal diffusive evolution equation, governed by a Lévy-type nonlocal operator (e.g., fractional Laplacian), fractional time-derivative and involving porous medium-type nonlinearities is the extension of the work titled *De Giorgi–Nash-type theorem for nonlocal time porous medium equations*. The result shows that the solution is bounded and Hölder continuous for all positive time. This result is resumed in (*Nonlocal time porous medium equation with fractional time derivative*, 2019).

Resumo

O obxecto deste traballo é investigar algunhas ecuacións non lineares de tipo poroso da forma

$$\mathbb{D}_t^\alpha u + LF(u) = 0,$$

e obter novos resultados de regularidade para as ditas ecuacións. O operador non local $\mathbb{D}_t^\alpha$ con $0 < \alpha \leq 1$ é a derivada fraccionaria no tempo, L é un operador xeral nonlocal no espazo asociado ao proceso de Lévy correspondente e F é unha función monótona non decrecente, que satisface certas condicións para as dexeneracións como $F(0) = 0$ e $F'(0) = 0$.

Este tipo de ecuacións modelan distintos fenónemos de difusión anómala. A ecuación tipo é a (clásica) ecuación en medio poroso

$$u_t + \Delta u^m = 0, \tag{PME}$$

que modela o fluxo de gases a través de medio poroso. Comezando pola ecuación clásica (PME), poden ser considerados distintos enfoques, debido ao seu interese recente para a comunidade matemática e á súa aplicabilidade para modelar problemas reais.

As miñas achegas están contidas nos artigos:

- P1 J-D. Djida and A. Fernandez. *Interior Regularity Estimates for a Degenerate Elliptic Equation with Mixed Boundary Conditions*, 7(65): 1–16, 2018;
- P2 J-D. Djida. *Wellposedness and boundary regularity for nonlocal parabolic problem with fractional derivative*, 2019, submetido a publicación;
- P3 J-D. Djida, J. J. Nieto and I. Area, *Nonlocal time-porous medium equation: weak solutions and finite speed of propagation*, Discrete Continuous Dyn. Syst. Ser. B, 22:1, 2019;
- P4 J-D. Djida, I. Area and J. J. Nieto. *A De Giorgi-Nash type theorem for nonlocal time porous medium equations*, 2018, submetido a publicación;
- P5 J-D. Djida, J. J. Nieto and I. Area, *Nonlocal time porous medium equation with fractional time derivative*, Revista Matemática Complutense, 2019, en prensa;

en colaboración cos meus co-directores (Profesor Juan José Nieto e Profesor Iván Area), e un en colaboración co Doutor Arran Fernandez.

A presente tese abrangue investigacións recentes sobre un operador xeral de tipo Lévy, laplaciano fraccionario e derivada fraccionaria. Comezando pola teoría destes operadores non locais, recompilamos resultados recentes e distintas observacións, así como enriquecemos o material con algunhas achegas orixinais sobre existencia, regularidade no interior e na fronteira no caso linear para algúns problemas integro-diferenciais, que serán de axuda para analizar o caso non linear. Ademais, analizaremos como determinados efectos que se presentan no laplaciano fraccionario teñen correspondencia para outros operadores de diferenciación fraccionarios no tempo. Máis aínda tamén analizaremos o problema de extensión (de Dirichlet a Neumann) relacionado coa derivada fraccionaria. Por outra banda tamén presentaremos resultados recentes sobre teoría de regularidade de certas ecuacións en medio poroso non locais con laplaciano fraccionario e derivada fraccionaria, e analizamos novos resultados sobre a regularidade para modelos xerais non lineares en tempo en medio poroso.

Os principais aspectos da presente tese son:

1. *Regularidade interior para unha ecuación elíptica dexenerada con condicións de fronteira mixtas*. Comezamos coa existencia e unicidade da solución fraca do problema elíptico $(\mathscr{D}_{right})^s v = f$, para $s \in (0,1)$. Nesta ecuación o operador non local $(\mathscr{D}_{right})^s$ denota a derivada fraccionaria de Marchaud (ou derivada de Caputo extendida). Como resultado principal, demostraremos estimacións de regularidade interior de tipo Schauder para unha ecuación elíptica dexenerada con condicións de fronteira mixtas de tipo Dirichlet-Neumann. A ecuación elíptica xorde da extensión de Bernardis-Reyes-Stinga-Torrea do problema de Dirichlet para a derivada de Marchaud (*Regularity theory for the fractional harmonic oscillator*, 2016), que é o análogo do problema co laplaciano fraccionario introducido por Caffarelli e Silvestre (*An extension problem related to the fractional Laplacian*, 2016).

2. *Regularidade na fronteira para un problema parabólico con operadores non lineares.* $\mathbb{D}_t^\alpha u + Lu = 0$. Neste contexto, L é o operador non local xeral de tipo Lévy. Presentamos os resultados para a regularidade na fronteira da solución do problema parabólico e elíptico obtido por Ros-Oton, Serra e Fernández-Real (*The Dirichlet problem for the fractional Laplacian: regularity up to the boundary* e *Boundary regularity for the fractional heat equation*), e os empregamos para obter resultados de regularidade na fronteira para un problema parabólico cun operador non local xeral e derivada fraccionaria. O resultado obtido mostra que a solución u da ecuación parabólica fraccionaria homoxénea sobre un dominio limitado Ω verifica que $u \in C^s(\mathbb{R}^d)$ e que u/δ^s pode ser extendida continuamente (Hölder) ata $\overline{\Omega}$, onde $\delta(x) = dist(x, \partial\Omega)$. Tamén incorporamos algúns comentarios sobre resultados de regularidade interior e na fronteira para a ecuación xeral parabólica non lnear de tipo poroso $\mathbb{D}_t^\alpha u + LF(u) = 0$.

3. *Formulación da ecuación de difusión non linear* $\mathbb{D}_t^\alpha u + (-\Delta)^s \varphi(u) = f$. Nun primeiro nivel $\alpha \in (0,1)$, $s \in (0,1)$ e $\varphi : \mathbb{R} \to \mathbb{R}$ é unha función continua, crecente e regular. $\mathbb{D}_t^\alpha$ e $(-\Delta)^s$ denotan a derivada de Caputo fracionaria en tempo e o laplaciano fraccionario, respectivamente. Nun segundo nivel analizaremos a *formulación da ecuación de difusión non linear* $\mathbb{D}_t^\alpha u - \nabla \cdot (u^{m-1} \nabla P) = f$, con $m > 1$,

$P = \mathcal{K}(u)$, onde $\mathcal{K}$ é un potencial de Riesz. O problema foi estudado recentemente por Caffarelli e Vázquez (*Nonlinear porous medium flow with fractional potential pressure*, 2011) e por Biler, Karch, Monneau (*Nonlinear Diffusion of Dislocation Density and Self-Similar Solutions*, 2009) no caso particular $m = 2$ e $\alpha = 1$, onde amosan a relevancia do modelo nas aplicacións. De xeito similar Stan, Teso e Vázquez (*Finite and infinite speed of propagation for porous medium equations with fractional pressure*, 2014), investigan o efecto da non linearidade sobre a velocidade finita de propagación da solución cando $\alpha = 1$. Finalmente, Allen, Vasseur and Caffarelli (*A parabolic problem with a fractional time derivative*, 2016) obtiveron resultados de regularidade no caso $m = 2$. Nós investigamos a existencia e velocidade finita de propagación de solucións fracas (ou febles) cando o exponente $m \geq 2$, $0 < \alpha \leq 1$ e $0 < s \leq 1$. Esta parte remata con algunhas achegas sobre solucións auto-semellantes dos xa indicados modelos de tipo poroso.

4. *Un teorema de tipo De Giorgi-Nash para a ecuación en medio poroso no tempo.* $\mathbb{D}_t^\alpha u + \Delta\varphi(u) = f$. Demostramos un resultado de regularidade de tipo De Giorgi-Nash-Moser Hölder para a ecuación en medio poroso no tempo, que dá unha estimación interior de tipo Hölder para as solucións fracas limitadas. A demostración baéase nunha estimación a priori para problemas de operadores fraccionarios en tempo e emprega a ténica de De Giorgi (*Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari*).

5. *Regularidade para a ecuación non local en tempo de tipo poroso con laplaciano fraccionario* $\mathbb{D}_t^\alpha u + (-\Delta)^s\varphi(u) = f$. Estas ecuacións evolutivas de difusión non locais e non lineares, rexidas por un operador non local de tipo Lévy, derivada fraccinaria no tempo e involucrando nonlinearidades de tipo poroso é unha extensión do traballo *A De Giorgi-Nash type theorem for nonlocal time porous medium equations*. O resultado demostra que a solución está limitada e é Hölder continua para todos os tempos positivos. Este resultado aparece en (*Nonlocal time porous medium equation with fractional time derivative*, 2019).

Acknowledgements

First, but of course not least, God.

I am greatly indebted to Prof. Juan José Nieto and Prof. Iván Area, who have been the optimal advisors during these years. Most of the mathematics that I know, I have learned from them. Furthermore, they have constantly encouraged me and have been always available. I really appreciate their extraordinary dedication.

My friend and colleagues both at the departments of Mathematics at USC and also at AIMS Cameroon, whom I have been very lucky to share the environment with, have also been quite helpful. I have enjoyed and learned much when discussing mathematics and other things with them.

I am thankful to AIMS Cameroon and all his staff (Prof. Dr. Mama Fouapouagnigni, Prof. Dr. Marco Garuti, Prof. Dr. Gisèle Mophou, Mme. Mary Fultang, and others) for the support and the facility that they have provided to me during the course of my study my study.

Other very important people in my relationship with mathematics and other fields have been Prof. Moustapha Fall, Prof. Abdon Atangana, Prof. Dr. Gisèle Mophou, Dr. Arran Fernandez, Dr. Kouakep Yannick, Prof. Doka Serge and Prof. Nana Engo. I am also indebted to them.

Finaly, I am indebted to my wife Donn Pauline and daughter Anita, to my mother Djoumba Anita, my late father Djidéré André, to my late uncles Ali Jean Marc and Joseph Djida, to my brothers and sisters (Armand, Guillaume, Fréderic, Darline, Nathalie, Hélène, Simeon, Simon, and Barbara), the rest of my family, and to Cristina and Mariña who are my second family in Spain. Their unconditional support and help during these years have been extremely important to me.

Contents

Summary **ix**

Resumo **xiii**

Acknowledgements **xvii**

Introduction and summary of the results **1**

0.1 Nonlocal operators 2
0.2 Nonlinear diffusion 4
0.3 Main Results 6
0.3.1 Interior regularity for a degenerate elliptic equation with mixed boundary conditions 6
0.3.2 Boundary regularity for Parabolic problem with nonlocal operators . 7
0.3.3 Nonlocal time-porous medium equation: weak solutions and finite speed of propagation 8
0.3.4 A De Giorgi–Nash-type theorem for the time-porous medium equation 10
0.3.5 Regularity theory for the nonlocal time-porous medium equation with fractional Laplacian 12

1 Fractional Sobolev spaces and fractional nonlocal operators **13**

1.1 Fourier Transform of tempered distributions 13
1.2 Sobolev spaces of integer order 14
1.3 Sobolev space of fractional order 15
1.4 Weighted spaces 17
1.5 Some fractional nonlocal operators 18
1.5.1 The fractional derivatives and integrals operators 18
1.5.2 Nonlocal operators and Lévy processes 22
1.5.3 Stable processes 25
1.5.4 The fractional Laplacian and inverse Laplacian 25

2 Interior regularity for a degenerate elliptic equation with mixed boundary conditions **29**

2.1 Introduction 29
2.2 Notations and preliminary results 31

2.2.1 Variational formulation of the fractional elliptic equation with right Marchaud fractional derivative . . . 34
2.3 Existence and Uniqueness of the weak solution of the fractional elliptic equation with right Marchaud fractional derivative . . . 36
2.4 Regularity estimate up to the boundary for the degenerated equation with the newmann boundary condition . . . 40
2.5 Conclusion . . . 48

3 Boundary regularity for parabolic problem with nonlocal operators 49
3.1 Introduction and results . . . 49
3.2 Preliminaries and known results . . . 52
3.2.1 Existence of eigenfunctions and asymptotic behavior of eigenvalues . 55
3.2.2 Regularity of eigenfunctions . . . 57
3.2.3 Solution representation . . . 58
3.3 Boundary regularity results : proof of Theorem 3.1.1 . . . 60
3.4 Conclusion . . . 62

4 Nonlocal fractional porous medium equation: weak solutions and finite speed of propagation 65
4.1 Introduction . . . 65
4.2 Preliminary results . . . 70
4.3 Existence and uniqueness of solutions of (TFPME) . . . 73
4.3.1 Existence and uniqueness results. Proof of Theorem 4.1.1 . . . 73
4.4 Existence of weak solutions for (TPMEFP) . . . 80
4.4.1 A regularized problem . . . 80
4.4.2 Solution representation of the regularized problem . . . 81
4.4.3 Contraction property for the map $\mathcal{P}$ . . . 84
4.4.4 Exponential tail control . . . 86
4.4.5 Some Sobolev estimates . . . 88
4.5 Finite speed of propagation and free Boundary property. Proof of Theorem 4.1.3 93
4.5.1 Finite speed of propagation properties . . . 93
4.5.2 Topological boundary of the support of the solution: growth estimate of the support . . . 96
4.6 Conclusion . . . 97

5 A De Giorgi-Nash type theorem for the time porous medium equation 99
5.1 Introduction . . . 99
5.2 Preliminaries . . . 101
5.2.1 Fractional time derivative properties and its discretization . . . 101
5.3 Background on the De Giorgi technique for regularity of solution . . . 106
5.4 The First De Giorgi Lemma . . . 111
5.4.1 Energy inequalities estimate . . . 111
5.4.2 The oscillation reduction lemmas . . . 115
5.5 The Second De Giorgi Lemma . . . 120
5.6 Proof of the Hölder regularity . . . 121

5.7 Conclusion 125

6 Regularity theory for the nonlocal time-porous medium equation with fractional laplacian 127

6.1 Introduction 127
6.2 Preliminaries 131
6.3 The First De Giorgi Lemma 132
6.3.1 Energy inequalities estimate 132
6.3.2 The oscillation reduction lemmas 138
6.4 The Second De Giorgi Lemma 143
6.5 Proof of the Hölder regularity 148
6.6 Conclusion and comment on boundary regularity 152

Bibliography 163

Introduction and summary of the results

The aim of this work is to investigate some nonlocal fractional nonlinear equations (analogue to nonlinear equations with local operators) and obtain new regularity results for these equations. More precisely, we extend the results [5, 6, 43, 45, 130] from nonlocal fractional problems in space to nonlocal fractional problems in space and in time.

This thesis is concerned with the regularity of solutions to nonlocal equations. Some of the most important contributions of the recent years are obtained by Caffarelli, Serra, Sylvestre, Ros-Oton and Vázquez [41–43,45,130] and references therein, where they studied the regularity of solutions to nonlinear nonlocal equations. Therefore other important works in the field [5, 80, 81, 139, 158].

One of the reasons for the growing interest of the above-mentioned results in the nonlinear setting is due to the fact that local linear diffusion theory has enjoyed much progress and is now solidly established in the theory and applications in the field of analysis. However, it has been observed that many of the equations modelling physical phenomena without excessive assumptions (simplification of the problems) are essentially nonlinear and their salient characteristics are not reflected by the linear theories that had been developed, notwithstanding the fact that such linear theories had been-and continue to be-very efficient for a number of applications in various domains. Unfortunately, the mathematical difficulties of building theories for suitable nonlinear versions of the three classical partial differential equations (Laplace's equation, heat equation and wave equation) made it impossible to make significant progress in the rigorous treatment of these nonlinear problems until the 20th century was well advanced [155].

This observation also applies to other important nonlinear PDEs or systems of PDEs, like the Navier–Stokes equations, nonlinear Schrödinger equations, the Vlasov equation and the Fisher equation. In his survey notes [155], Juan Luis Vázquez reminded us that the main obstacle to the systematic study of the nonlinear PDE theory was the perceived difficulty and the lack of tools. He quoted that these difficulties were reflected in a passage by John Nash (1958) who said in his seminal paper [124], *"The open problems in the area of nonlinear PDE are very relevant to applied mathematics and science as a whole, perhaps more than the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed".*

This is a grand project in pure and applied science, and it is still going on. In order to start, and following the mathematical style that cares first about foundations, John Nash set about the task of proving the regularity of the weak solutions of the PDEs he was going to deal with. More precisely, the problem was to prove continuity (Hölder regularity) of the weak solutions of elliptic and parabolic equations assuming, the coefficients a_{ij} to be uniformly elliptic (positive definite matrices) but only bounded and measurable as functions of $x \in \mathbb{R}^d$. This was done in parallel by John Nash [124], De Giorgi [89] and later on, the results were then taken up and given a new proof by Moser [122], who went on to establish the Harnack inequality [121], a very useful tool.

Once the tools were ready to start nonlinear PDEs in a rigorous way, it was discovered that the resulting mathematics are quite different from their linear counterparts, they are often difficult and complex, they turn out to be more realistic than the linearized models in the applications to real-world phenomena, and finally they give rise to a whole new set of phenomena unknown in the local linear setting, such as **long-range interaction and "memory effect" driven by nonlocal operators**. Hence the aim of this thesis, which deals with these type of operators.

0.1 Nonlocal operators

The interest in nonlocal operators has increased in the last decades, given their numerous applications in many branches of engineering, physics and biology. Nonlocal operators have the peculiarity of capturing long-range interactions i.e., events that happen far away, either in time or in space.

Diffusion processes with long-range effects are modeled in many situations using nonlocal operators. We exemplify this by supposing that an object that follos a Lévy process $\{X_t,\ t \geq 0\}$ either dies or disappears when exiting a domain Ω. The notion of Lévy process and memory effect will be introduced and explained in Chapter 1; For now, we can think of it as a random walk with possibly long jumps at random times. Furthermore, [1, 31], if we think of a function depending on time, nonlocal time operators (fractional derivatives) exhibit a "memory effect" that is, they "see past events", providing a model in which the state of a system at a given time depends on the past. They describe a causal system, also called a non-anticipative system. Hence, under these circumstances, combining these effects (long-range interaction and memory effect in time), the probability distribution of the expected position of the particle after a time t is a function $v(x,t)$ that solves the problem

$$\begin{cases} \mathbb{D}_t^\alpha v + Lv = 0 & \text{in } \Omega,\ t > 0, \\ v = 0 & \text{in } \mathbb{R}^d \setminus \Omega,\ t > 0, \end{cases} \tag{0.1.1}$$

where $\mathbb{D}_t^\alpha$, with $0 < \alpha \leq 1$ is the fractional nonlocal time operator and L is a spatial nonlocal operator (explained below) associated to the particular Lévy process.

We now present what "nonlocal operators" means, intuitively [1]. Nonlocal operators are operators such that the value of the image of a function at a certain point depends on other points rather than just a neighbourhood of the selected point. To illustrate it, if L is a general nonlocal operator, being $v : \mathbb{R}^d \to R$ a function, and fixing a vector $x_0 \in \mathbb{R}^d$, then the value of $Lv(x_0)$ depends on the value of $v(x)$ in other points outside a neighbourhood of x_0. That is in contrast with typical local operators, where the value of the image of the operator at a given point depends only on the value of the function near the point. The reader should remember that this first approximation of the notion of nonlocal operators is purely intuitive. By near we mean that we could take any neighbourhood of the point.

Of special interest concerning nonlocal operators in this thesis are the fractional Laplacian operator [147] (which can be derived from the more general operator L, as we shall see in Chapter 1) and the fractional derivative (which in extended form becomes Marchaud derivatives up to a certain constant) [84, 116, 134], usually defined via the Fourier Transform

$$\widehat{(-\Delta)^s} v(x) = |\xi|^{2s} \widehat{v}(\xi)$$

and

$$\widehat{\partial_x^\alpha v}(x) = (\pm i\xi)^\alpha \, \widehat{v}(\xi),$$

for functions v is the Schwartz class. When $s \in (0,1)$ and $\alpha \in (0,1)$, the fractional Laplacian and the fractional derivative (extended Caputo or Marchaud) operator can also be defined by the integral formula using hyper-singular kernels

$$(-\Delta)^s v(x) = C_{d,s} \text{P.V.} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x-y|^{N+2s}} \, dy$$

and

$$\partial_t^\alpha v(t) = C(\alpha) \int_{-\infty}^t \frac{v(t) - v(\tau)}{(t-\tau)^{1+\alpha}} \, d\tau,$$

where P.V. stands for the principal value and $C_{d,s} = \pi^{-(2s+d/2)} \Gamma(s + d/2)/\Gamma(-s)$ is a normalization constant. The inverse operators are given respectively by the Riesz Potential

$$(-\Delta)^{-s} v(x) = C_{d,-s} \text{P.V.} \int_{\mathbb{R}^d} \frac{v(y)}{|x-y|^{d-2s}} \, dy$$

and

$$\partial_t^{-\alpha} v(t) := \int_{\mathbb{R}} \frac{v(\tau)}{|t-\tau|^{1-\alpha}} \, d\tau,$$

where the Riesz potential ([134, 149]) is defined as

$$\mathcal{I}_\alpha = C_\alpha |x-y|^{\alpha-1} \quad \text{for } \alpha < 1 \tag{0.1.2}$$

with the constant $C_\alpha = \frac{1}{\pi}\Gamma(1-\alpha)\sin\frac{\pi\alpha}{2}$.

It is important to recall that fractional time-derivatives are the most elementary objects of Fractional Calculus, a branch of mathematical analysis that studies the possibility of taking real number fractional powers or complex number powers of the differentiation operator d/dt and the integration operator.

The foundations of the theory of fractional derivatives were laid down by Liouville in a paper from 1832. As one can expect [119], fractional operators generalize classical (integer) ones, in the sense that if the order of the fractional operator is given by the parameter $\alpha \in (0,1)$, then letting $\alpha \to 0^+$ one obtains the identity, and letting $\alpha \to 1^-$, one gets the classical (integer order) operator. In the literature, there are several definitions of fractional operators, like the Riemann–Liouville, the Caputo, the Riesz, the Marchaud and the Atangana–Baleanu fractional derivative (see [8, 11, 48, 84, 116, 119, 134] for more details on fractional integrals, derivatives and applications). The rational reason for these multiple definitions is due to the fact that their kernels take different representations in different functional spaces. Different definitions use different kernels, but all of them make weighted averages in time.

0.2 Nonlinear diffusion

The motivation for studying the effect of these nonlocal operators in the nonlinear setting comes from rich properties embedded in the nonlinear analysis equation. This thesis deals with a class of nonlinear diffusion equations whose prototype is

$$u_t = \Delta u^m, \tag{PME}$$

which is known as *porous medium equation* (PME) for $m > 1$. Equations of type (PME) and the p-Laplacian evolution equation, with $p > 1$,

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \tag{PLE}$$

are among the most popular nonlinear equations of parabolic type belonging to the class

$$u_t = \nabla \cdot (A(u, \nabla u) \nabla u).$$

The (PLE) equation belongs to the class of gradient-dependent diffusion equations: $A(\nabla u) = |\nabla u|^{p-2}$, which is degenerate for $p > 2$ and is singular for $p < 2$. In the particular case $m = 1$, $p = 2$, we recover the classical heat equation. But here in this thesis, we focus in the (PME).

The (PME) has density-dependent diffusivity: the diffusion coefficient $A(u) = mu^{m-1}$ makes the equation degenerate for $m > 1$ and singular for $m < 1$. When $m < 1$ the (PME) is called the Fast Diffusion Equation with applications in plasma physics. The terminology slow/fast diffusion refers to the property of finite speed of propagation, which depends of the range $m > 1$ and $m < 1$ for the (PME) [146].

Evolution equations of porous medium type have a long history, both in terms the applications that originally motivated their consideration and from the mathematical point of view. We refer in particular to the monograph by Juan Luis Vázquez [153], where a very detailed and comprehensive survey of the state of the art is given. Here we are concerned with a number of (nonlocal) variants and extensions of (PME), for which we address several fundamental issues such as existence, uniqueness of solutions and their regularity properties. Our analysis aims to link (as far as possible) the properties of nonlocal operators to the solutions of nonlinear PDEs of porous type with suitable functional inequalities associated with the underlying functional spaces, thus giving a certain amount of generality to the discussion.

The combination of fractional nonlocal operators and porous medium nonlinearities gives rise to interesting mathematical models that have been studied in the last decade both because of a number of scientific applications and for their mathematical properties. The one under study here is in the in the form

$$\begin{cases} \mathbb{D}_t^\alpha v + L\,F(v) &= 0 \quad \text{in } \Omega,\ t>0,\\ v &= 0 \quad \text{in } \mathbb{R}^d \setminus \Omega,\ t>0. \end{cases} \tag{0.2.1}$$

Here, F is a monotone non-decreasing function, satisfying some conditions that allow for degeneracies, like $F(0)=0$ and $F'(0)=0$.

As we shall see, two main models for flows in porous media involving nonlocal fractional diffusion effects that could be derived from (0.2.1) will be of interest. For convenience, we will call them here the *time-fractional porous medium equation* (TFPME) and *time-fractional porous medium equation with fractional potential pressure* (TPMEFP).

(i) Nonlocal time-fractional porous medium equation. The first model presented is the Time-Fractional Porous Medium Equation, for $m>1$ and $s,\ \alpha \in (0,1)$:

$$\partial_t^\alpha u + (-\Delta)^s u^m = f, \tag{TFPME}$$

where ∂_t^α and $(-\Delta)^s$ are, respectively, the fractional derivative in the Marchaud sense or the extended Caputo derivative, and fractional Laplacian operator described above. The model (TFPME) can be interpreted as the nonlocal analogue of the classical (PME) ($\alpha=1$, $s=1$, $m>1$). Major differences in terms of qualitative properties appear at the level of speed of propagation. The problem (PME) has finite speed of propagation, which is usually expressed by showing that compactly supported initial data produce compactly supported solutions (consequently, free boundaries could be observed). The model (TFPME) has infinite speed of propagation i.e., even if u_0 is compactly supported, the solution is strictly positive for all times $u(x,t)>0$, $t>0$. This latter property is a consequence of the nonlocal character of the fractional operator when $\alpha=1$ in (TFPME). We note that, when $\alpha=1$ in (TFPME), some important properties like self-similarity of solutions, existence, regularity and asymptotic behaviour of the solutions have been obtained by Juan Luis Vázquez and collaborators [26, 61–63] and many others.

(ii) Nonlocal time porous medium equation with fractional pressure. Consider the case when the pressure P is related to u by a nonlocal operator of the fractional type

$$P = \mathcal{K}(u),$$

where $\mathcal{K} = (-\Delta)^{-s}$ is the inverse of the fractional Laplacian operator, with $0<s<1$. The diffusion model with nonlocal effects and memory effects is the following

$$\partial_t^\alpha u(x,t) = \nabla\cdot(u\nabla P) + f, \quad P = (-\Delta)^{-s}(u). \tag{ACV}$$

This has been studied by Allen, Caffarelli and Vasseur [6]. The case when $\alpha=1$ has been introduced by Caffarelli and Vázquez [45] and its properties have been studied in a series of papers [43, 44].

Starting from equation (ACV), more general models can be considered that maintain the core features of the (PME). We discuss here the following problem:

$$\partial_t^\alpha u(x,t) = \nabla \cdot (u^{m-1} \nabla p) + f, \quad p = (-\Delta)^{-s}(u), \tag{TPMEFP}$$

where $m \geq 2$. We call this problem *time-porous medium with fractional pressure* (TPMEFP). The analogue of this problem has been studied in the case $\alpha = 1$ [146]. The nonlinearity has surprising effects on the propagation property of the solution, as we will describe in Section 0.3.3.

0.3 Main Results

We now present the results obtained by the author in this thesis.

0.3.1 Interior regularity for a degenerate elliptic equation with mixed boundary conditions

In Chapter 2, we deal with the interior regularity for a degenerate elliptic equation with mixed boundary conditions involving the fractional derivative (extended Caputo or Marchaud) through extended problems. It is a remarkable fact that in some occasions nonlocal operators can be equivalently represented as local (though possibly degenerate or singular) operators in one dimension more. Moreover, as a counterpart, several models arising in a local framework give rise to nonlocal equations, due to boundary effects.

In this context, the nonlocality of the fractional Marchaud derivative prevents us from applying local PDE techniques to treat linear and nonlinear nonlocal problems. To overcome this difficulty, Bernardis, Reyes, Stinga and Torrea [16] showed that the right fractional Marchaud derivative can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. Similar extension properties have been found for the fractional Laplacian by Caffarelli and Silvestre [40].

To recall the extension problem in the framework of Bernardis, Reyes, Stinga and Torrea [16] and give a concrete intuition of it, we will present some known results for some problems that are naturally set on an extended space to start with, and we will show their relation with the fractional Laplacian on a trace space. We will also give a detailed justification of this extension procedure.

An interesting application that follows from this extension procedure is a Harnack inequality for Marchaud-stationary functions in an interval $J \subseteq R$ (namely for functions that satisfy $(\mathscr{D}_{right})^s v = 0$ in J) [16]. This result is obtained from the Harnack inequality for some degenerate parabolic operators by "looking at it" on the trace. Indeed, using the extension operator, it is quite easy to obtain this type of result.

Our approach relies on constructing a parabolic local operator by adding an extra variable, say the time variable, on the positive half-plane, and working on the extended plane $[0,\infty) \times$

$\mathbb{R}$. Namely, we prove [71] that $\mathcal{U}$, the harmonic extension of v into $2-2\alpha$ extra dimensions, is a solution to

$$\begin{cases} \mathcal{M}_\alpha\,\mathcal{U} = 0 & \text{in } \mathbb{R}\times[0,\infty), \\ \lim_{t\to 0}\mathcal{N}_\alpha\,\mathcal{U}(t,\cdot) = f & \text{on } \Omega, \\ \mathcal{U} = 0 & \text{on } \mathbb{R}\setminus\Omega, \end{cases}$$

and v is $\mathcal{C}^{0,\alpha-\gamma-\frac{1}{p}}$ and Hölder continuous of order $0<\alpha-\gamma-\frac{1}{p}<1$. We define

$$\lim_{t\to 0}\mathcal{N}_\alpha\mathcal{U}(t,x) = c_s(\mathscr{D}_{right})^s v(x),$$

where $c_s := \frac{4^{s-1/2}\Gamma(s)}{\Gamma(1-s)}$ is a positive multiplicative constant depending only on $s\in(0,1)$.

Here the differential operators $\mathcal{M}_s$ and $\mathcal{N}_s$ are given respectively by:

$$\begin{cases} \mathcal{M}_\alpha\mathcal{U} & := -(\mathscr{D}_{right})\mathcal{U} + \frac{1-2s}{t}\mathcal{U}_t + \mathcal{U}_{tt}; \\ \mathcal{N}_\alpha\mathcal{U} & := -t^{1-2s}\mathcal{U}_t. \end{cases}$$

We use the notation $(\mathscr{D}_{right})$ for the derivative *from the right* at the point $x\in\mathbb{R}$ that is:

$$(\mathscr{D}_{right})v(x) = \lim_{t\to 0^+}\frac{v(x)-v(x+t)}{t},$$

for good enough functions v. Observe that $(\mathscr{D}_{right})$ equals the negative of the lateral derivative $\left(\frac{d}{dx^+}\right)$ as usually defined in calculus [16].

0.3.2 Boundary regularity for Parabolic problem with nonlocal operators

In some cases, nonlocal problems usually present a similar structure to PDE problems, requiring boundary conditions and specifying the equation in a bounded domain. For example, the Dirichlet problem for the fractional Laplacian is given as

$$\begin{cases} (-\Delta)^s v &=& f(x) & \text{in } \Omega, \\ v &=& 0 & \text{in } \mathbb{R}^d\setminus\Omega, \end{cases} \tag{0.3.1}$$

and for the parabolic case as:

$$\begin{cases} \partial_t v + (-\Delta)^s v &=& 0 & \text{in } \Omega,\ t>0, \\ v &=& 0 & \text{in } \mathbb{R}^d\setminus\Omega,\ t>0, \\ v(x,0) &=& v_0(x) & \text{in } \Omega,\ \text{for } t=0. \end{cases} \tag{0.3.2}$$

In these two problems, the “boundary conditions” are prescribed outside the domain, in contrast with classical PDE problems, where the boundary conditions are prescribed only on the boundary of the domain. Few results in the literature deal with the regularity up to the boundary of problem of type (0.3.1) and (0.3.2) [22, 83, 130, 161]. The difficulty is that it is not trivial to find an explicit solution for problem (0.3.1). A simple expression is obtained for a given $C^{1,1}$ domain $\Omega = B_1$ and a well-defined function $f(x) \equiv 1$ in Ω as:

$$v(x) = \sigma \left(1 - |x|^2\right)^s,$$

for $\sigma > 0$. This solution is not of class $\mathcal{C}^\infty$ up to the boundary, but it is only $\mathcal{C}^s(\overline{\Omega})$. As observed [83, 130], this boundary behaviour is the same for all solutions, in the sense that any solution to (0.3.1) satisfies

$$-C\delta^s \le v \le C\delta^s \qquad in \qquad \Omega, \tag{0.3.3}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Ros-Oton and Serra found [130] that $v \in \mathcal{C}^s(\overline{\Omega})$. Furthermore, for any $f \in L^\infty(\Omega)$, the solution v to the elliptic problem (0.3.1) satisfies that v/δ^s is Hölder continuous up to the boundary and

$$\|v/\delta^s\|_{C^\beta(\overline{\Omega})} \le C\|f\|_{L^\infty(\Omega)}, \tag{0.3.4}$$

for some $\beta > 0$ small. For problems involving the general nonlocal operator of the form of L in (0.1.1) and (0.3.2), the same authors in [83] proved that, for each $t_0 > 0$,

$$\sup_{t \ge t_0} \left\| \frac{v(\cdot, t)}{\delta^s} \right\|_{C^s(\overline{\Omega})} \le C(t_0)\|v_0\|_{L^2(\Omega)}. \tag{0.3.5}$$

The method consists of bounding the L^∞ norm of the eigenfunctions in $\mathbb{R}^d$ by its L^2 norm and a multiplicative constant, giving explicitly its dependence on the eigenvalue. After that finds the expression of the solution of the fractional nonlocal equation in terms of the eigenfunctions and ends with the proof of the uniqueness of the solution.

In Chapter 3, we will provide similar boundary regularity results for the parabolic problem with general nonlocal operators of Lévy type, following the strategy presented above [83, 130]. The main problem under consideration is

$$\begin{cases} \mathbb{D}_t^\alpha u + Lu &= 0 & \text{in } \Omega,\ t > 0, \\ u &= 0 & \text{in } \mathbb{R}^d \setminus \Omega,\ t > 0, \\ u(x,0) &= u_0(x) & \text{in } \Omega, \text{ for } t = 0, \end{cases}$$

where $\mathbb{D}_t^\alpha$ stands for the fractional derivative in the sense of Caputo and L is the general nonlocal operator of Lévy type as we shall see in Chapter 1.

0.3.3 Nonlocal time-porous medium equation: weak solutions and finite speed of propagation

Chapter 4 is devoted to the study of the existence and uniqueness of the general nonlinear nonlocal time-evolution equation of porous medium type and to the question of existence of approximate solutions and finite speed of propagation of the time-porous medium equation with fractional pressure. We present those results in two parts:

• **Existence and uniqueness of the nonlocal time-porous medium equation**. The problem of existence with standard nonlinear diffusion goes back to the work of BrezisBk [29, 30]. We prove [73] an existence and uniqueness result for the homogeneous Dirichlet problem in the form

$$\begin{cases} \partial_t^\alpha u + (-\Delta)^s(\varphi(u)) = f & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, $m > 1$, $0 < s < 1$ and $d \geq 1$. For the general theory, we use signed solutions $u(t,x)$. Actually, the result holds for more general nonlinearities than the power one, so we will adopt the more general context and consider the problem where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, smooth and increasing function. We assume moreover that $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$. The leading example will be $\varphi(u) = |u|^{m-1}u$ with $m > 0$.

For the sake of simplicity and generality, we rewrite the above system of the following nonlinear evolution problem

$$\begin{cases} \mathscr{A}u(t) + \mathscr{B}(u(t)) \ni f & \text{in} \quad \mathscr{H} \quad \text{for a.e.} \quad t \in (0,T) \\ u(0) = u_0, \end{cases}$$

where $\mathscr{B}$ is the subdifferential of a proper, convex and lower semicontinuous function $j : \mathscr{H} \to (-\infty,+\infty]$ with compact sublevels in $\mathscr{H}$ satisfying suitable compatibility conditions. The nonlinear and possibly multivalued operator $\mathscr{B}$ acts from $\mathscr{H}$ to $2^{\mathscr{H}}$, the space of all subsets of $\mathscr{H}$.

Our result holds under suitable assumptions on the structure of the linear and nonlinear maximal monotone operators $\mathscr{A}$ and $\mathscr{B}$ respectively. More precisely, in our analysis, we suppose that $\mathscr{A}$ is bounded, so that the domain $D(\mathscr{A}) \equiv \mathscr{H}$. Later on, as the next step, we prove that the nonlocal fractional time-derivative in the Caputo sense ∂_t^α can also be represented by the maximal monotone operator $\mathscr{A}$. Thus, using the Bénilan–Brezis–Crandall functional semigroup approach [29, 30, 56, 58], a unique solution is constructed, and $u \in \mathcal{C}([0,T] : \mathcal{H}^*(\Omega))$.

We note that a similar approaches have been adopted in the case $\alpha = 1$ [24, 62, 63] and when the exponent $s = 1$ [15].

• **Existence and finite speed of propagation for the time porous medium equation with fractional pressure.** Here we consider the nonlinear nonlocal evolution problem

$$\partial_t^\alpha u(x,t) = \nabla\cdot(u^{m-1}\nabla p) + f, \quad p = (-\Delta)^{-s}(u), \qquad \text{(TPMEFP)}$$

for exponents $m \geq 2$, $0 < s < 1$. Both u and f are nonnegative. The problem is posed for $x \in \mathbb{R}^d$, $d \geq 1$ and $t > 0$, and we give initial conditions

$$u(0,x) = u_0(x) \quad \text{for } x \in \mathbb{R}^d.$$

The initial data $u_0 : \mathbb{R}^d \to [0,\infty)$ and the forcing term f are bounded with compact support or fast decay at infinity. The pressure p is related to u through a linear fractional potential operator $p = \mathcal{K}(u)$. To be more specific, $\mathcal{K} = (-\Delta)^{-s}$ for $0 < s < 1$ with kernel $K(x,y) = c|x-y|^{-(d-2s)}$ (i.e., a Riesz operator). This equation was first introduced by Luis Caffarelli and Juan Luis Vázquez [45] for $m = 2$ as a model for nonlinear diffusion of porous medium type with nonlocal diffusion effects and later extended [146].

When $m = 2$ and $\alpha = 1$ [43–45] the properties of finite speed of propagation, a priori estimates for the solutions, C^α regularity, existence of self-similar solutions and asymptotic behaviour have been established. Additional results on Hölder regularity estimates have been developed [3].

We established existence results for a certain class of weak solutions for $m \geq 2$, for which we determine whether the property of compact support is conserved in time or not, with explicit dependence on the parameter m [72]. This result is motivated by the finite speed of propagation that happens for $m = 2$. Indeed, we discovered that the case $m = 2$ is a borderline case: when $m \in [1,2)$ the problem has infinite speed of propagation, while for $m \in [2,\infty)$ it has finite speed of propagation. These similar properties have also been observed when $\alpha = 1$ [146].

A major difficulty in this work is the lack of uniqueness and comparison of the solutions [45]. The existence is proved by approximating the problem (TPMEFP) through regularization, elimination of the degeneracy and reduction of the spatial domain. The approximated problem can be solved by deriving suitable energy estimates and then passing to the limit using a parabolic compactness argument. For all $m \geq 2$ and $\alpha \in (0,1)$, we prove an exponential tail behaviour in space. This is essential information to be able to control the energy terms when $m < 3$ [145].

When $m \geq 2$, we prove the property of finite speed of propagation for solutions to (TPMEFP). More precisely, we show that compactly supported initial data u_0 produce solutions $u(x,t)$ that have compact support in space for all $t > 0$. Indeed, we can construct an explicit function $\mathcal{U}(x,t)$ with compact support in space, which represents an upper barrier for u. Finding such $\mathcal{U}(x,t)$ is a task that is not trivial in view of the lack of comparison principle; the proof is based on delicate contradiction arguments [45, 146] at the first point in space and time where $u(x,t)$ touches the barrier $\mathcal{U}(x,t)$ from below. The proof uses ideas from the case $m = 2$ [45, 146].

0.3.4 A De Giorgi–Nash-type theorem for the time-porous medium equation

Our concern in Chapter 5 is the regularity of weak solutions for the nonlinear and nonlocal Cauchy problem involving the fractional derivative

$$\partial_t^\alpha u + \Delta\varphi(u) = f \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \qquad u(\cdot,0) = u_0,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary, $m > 1$, $\alpha \in (0,1)$ and $d \geq 1$. The function $\varphi(u)$ is nondecreasing and continuous. The initial datum is assumed to be bounded. For a wide class of nonlinearities, including the porous media case, these solutions turn out to be bounded and Hölder continuous for all positive time. Our main result is a De Giorgi–Nash–Moser Hölder type regularity theorem for the time porous medium equation, which gives an interior Hölder estimate for bounded weak solutions of the above nonlocal nonlinear Cauchy problem [69].

Let us review here the strategy and main tools. Since the equation is space and time-invariant, we may assume that $t_1 < t_2 = 0$, and then we may study the regularity around $x = 0$ and $t = 0$. The basic ideas of the proof of regularity (Hölder continuity) [38, 43–45] are as follows: we will prove some basic De Giorgi-type oscillation lemmas that say that the oscillation of the solution u shrinks in a certain way when we restrict the domain with a certain scale. To be precise, we will rely on the iterated application of three lemmas, so-called *oscillation reduction lemmas*. These technical results need only be proved for bounded nonnegative weak solutions u defined in a cylinder $[-R, 0] \times \mathbb{R}^d$ for a geometric sequence of cylinders $\Gamma_k = [-R^{-k}, 0] \times B_{R^{-k}}$, with $k \geq 0$. We denote by Γ the parabolic cylinder $[-R, 0] \times B_R(0)$. By parabolic we mean at this point a space-time subset of $[-R, 0] \times \mathbb{R}^d$. The oscillation of u,

$$\gamma_k = \sup_{\Gamma_k} u - \inf_{\Gamma_k} u,$$

decreases geometrically, i.e.,

$$\gamma_{k+1} \leq \nu \gamma_k \quad \text{for} \quad \nu < 1.$$

This is proved in several steps, following the L^2 to L^∞ and oscillation lemmas discussed previously.

The idea is the following: Suppose that, on the cylinder $\Gamma_0 = [-1, 0] \times B_1$, u lies between -1 and 1. Then at least half of the time it will be below or above zero. Let us say that it is below zero. Then, because of the diffusion process will nonlocality effects, by the time we are at the top of the cylinder and near zero, u should have gone uniformly strictly below 1, so now $-1 \leq u \leq 1 - \lambda$ and the oscillation γ has been reduced.

If we achieve this above result, we renormalize and repeat. How do we achieve this oscillation reduction ?

Here, following De Giorgi, we proceed in three steps (splited in three lemmas). First, One of the lemmas controls the decrease of the supremum of the solution once we restrict the size of the parabolic neighborhood of $(0, 0)$, from say Γ_1 into a smaller cylinder like $\Gamma_{1/4}$. Indeed, we show that if u is "most of the time negative" or very tiny in the trip $\Gamma_1 = [-1, 0] \times B_1$, then, indeed, it cannot stick to the value 1 close to the top of the cylinder and so it goes strictly below 1 in, say, $\Gamma_{1/4} = [-1/4, 0] \times B_{1/4}$.

Second, another lemma implies that, under suitable assumptions, the solution separates from zero in the same type of cylinders. This is done by closing the gap between "being negative most of the time" and "being negative half of the time". This takes a finite sequence of cut-offs and renormalizations, exploiting the fact that for u to go from a level (say 0) to another (say 1), some minimal amount of energy is necessary (known as the De Giorgi isoperimetric inequality).

Finally, the third lemma improves the first result so as to obtain a real alternative between going a bit down and a bit up. This is what is needed to make the iteration possible and efficient.

In our case, the arguments are complicated by the global nonlocal character of the diffusion that may cancel the local effect that we described above. Luckily, we may encode the nonlocal effect locally into the harmonic extension, which requires some careful treatment.

One of the technical complication is that we must truncate not only in u and time t but also in space, yet this does not have the effect of fully localizing the energy inequality, as a global term remains.But in the light of the iterative interaction between the Sobolev and energy inequalities, we overcome this difficulty.

As a result, for u being a bounded weak solution to the above Cauchy problem and u being continuous in the $(\mathbb{R}^d \times (-\infty, T))$, under some assuptions on the nonlinearity φ, as we shall see in Chapter 5, there exists $\beta \in (0,1)$ such that $u \in C^\beta(\mathbb{R}^d \times (-\infty, T))$.

0.3.5 Regularity theory for the nonlocal time-porous medium equation with fractional Laplacian

Chapter 6 is devoted to the study of the nonlinear and nonlocal Cauchy problem

$$\partial_t^\alpha u + \mathcal{L}^s \varphi(u) = f \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \qquad u(\cdot, 0) = u_0,$$

where ∂_t^α is a nonlocal fractional time operator (e.g., the Caputo derivative). The nonlocal operator $\mathcal{L}^s$ is a Lévy-type nonlocal operator with a kernel having the same singularity at the origin as that of the fractional Laplacian. The nonlinearity φ is nondecreasing and continuous, and the initial datum u_0 is assumed to be in $L^1(\mathbb{R}^d)$. For a wide class of nonlinearities, including the porous media case, $\varphi(u) = |u|^{m-1}u$, $m > 1$, the solution u turns out to be bounded and Hölder continuous for $t > 0$. This type of problem has been studied in the case $\alpha = 1$ [61]. As a novelty we studied the case where the fractional derivative is involved [73].

Our main result is a De Giorgi–Nash–Moser Hölder regularity theorem for the time-porous medium equation, which says (as in Subsection 0.3.4) that u is continuous in $(\mathbb{R}^d \times (-\infty, T))$ under some assuptions on the nonlinearity φ, as we shall see in Chapter 5 and Chapter 6. Then, there is some $\beta \in (0,1)$ such that $u \in C^\beta(\mathbb{R}^d \times (-\infty, T))$. The strategy of the proof follows the one presented in Subsection 0.3.4. We will end the chapter with some comments on the boundary regularity for the general-porous medium equation of the type (0.2.1).

Chapter 1
Fractional Sobolev spaces and fractional nonlocal operators

In this chapter, we sketch some of the concepts on fractional Sobolev spaces and fractional nonlocal operators in general in the form in which they will be exploited later on. First, we recall some notions on Sobolev and fractional Sobolev spaces later on, we will provide notions on fractional calculus. Secondly, the Lévy processes and their relations with nonlocal operators (in space), as well as the definition of stable process will be presented. We will define the fractional Laplacian in two different ways, its definition in the integral form and in the Fourier approach. Finally, we shall recall the Caffarelli–Sylvestre extension problem related to the Fracctional Laplacian operator. Our treatment is mostly self-contained, and we tacitly assume that the reader has some knowledge of the basic objects discussed here. Since this is an introductory chapter to convey the framework we work in, the rigorous proofs will be kept to a minimum. Some extra reading of the references may be necessary to truly learn the material. The results of this chapter are based on a variety of papers and books [11, 19, 20, 49, 76, 84, 116, 119, 125, 134].

1.1 Fourier Transform of tempered distributions

First, let us fix our notation and terminology for the spaces of test functions and distributions we shall use below. Throughout this chapter, whenever we write Ω, we mean a bounded domain in $\mathbb{R}^d$. We start by recalling the notion of a Fourier transform of a tempered distribution.

We consider the Schwartz space of rapidly decaying functions defined as

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \;\middle|\; \forall \alpha,\, \beta \in \mathbf{N}_0^d,\ \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_\beta f(x)| < \infty \right\}. \tag{1.1.1}$$

In other words, the Schwartz space consists of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power of x. Endowed with the family of seminorms

$$[f]^{\alpha,d}_{\mathcal{S}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} (1+|x|)^d \sum_{|\alpha|\le d} |D^\alpha f(x)|, \tag{1.1.2}$$

the Schwartz space is a locally convex topological space. We denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, the topological dual of $\mathcal{S}(\mathbb{R}^d)$.

Definition 1.1.1. *Let us consider a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, denoting the space variable $x \in \mathbb{R}^d$ and the frequency variable $\xi \in \mathbb{R}^d$. The Fourier transform of a function φ is given by*

$$\mathscr{F}\varphi(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)\,dx \tag{1.1.3}$$

and the inverse Fourier transform of a function φ, is given by

$$\mathscr{F}^{-1}\varphi(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x}\varphi(\xi)\,d\xi. \tag{1.1.4}$$

Both Fourier transform and inverse Fourier transform are continuous on $\mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}(\mathbb{R}^d)$. Note that for every $\varphi \in \mathscr{S}$, one has that $\mathscr{F}\varphi \in \mathscr{S}$. Moreover, since

$$\mathscr{F}^{-1}\mathscr{F}\varphi = \mathscr{F}\mathscr{F}^{-1}\varphi = \varphi,$$

each of them is, in fact, an isomorphism and a homeomorphism from $\mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}(\mathbb{R}^d)$.

By using definition (1.1.3), we have

$$u \in L^2(\mathbb{R}^d) \quad \text{if and only if} \quad \mathscr{F}u \in L^2(\mathbb{R}^d)$$

and

$$\|u\|_{L^2(\mathbb{R}^d)} = \|\mathscr{F}u\|_{L^2(\mathbb{R}^d)}, \tag{1.1.5}$$

for every $u \in L^2(\mathbb{R}^d)$. The identity in (1.1.5) is the so-called Parseval–Plancherel formula, which will be crucial in what follows for proving the equivalence between the fractional spaces $\mathcal{H}^s(\mathbb{R}^d)$ and $\widehat{\mathcal{H}}^s(\mathbb{R}^d)$, with $s \in (0,1)$ [147]. For several applications to elliptic problems of linear and nonlinear functional analysis we refer to the book [54].

1.2 Sobolev spaces of integer order

We briefly recall the definition of Sobolev space of integer order, which, in general, arises in a natural way as spaces of weak solutions to partial differential equations. These spaces consist of functions that belong together with all generalized partial derivatives up to a certain order to some L^p-space. We shall define integer-order Sobolev spaces as spaces of functions with a domain being an arbitrary open set $\Omega \subset \mathbb{R}^d, d > 1$.

We consider functions with values in $\mathbb{R}$ and linear spaces (Banach spaces). In the following, we recall definitions and properties from [76, §5.2.2].

Definition 1.2.1 (Sobolev space)**.** *Let $1 \le p \le \infty$ and $k \in \mathbb{N}_0$. The Sobolev space $W^{k,p}(\Omega)$ consists of all functions $v \in L^p(\Omega)$ such that for each multi-index $\alpha \in N_0^d$ with $|\alpha| \le k$ the generalized derivative $\partial^\alpha v$ exists and belongs to $L^p(\Omega)$. In the other words, this means*

$$W^{k,p}(\Omega) = \left\{ v \in L^p(\Omega) : \partial^\alpha v \in L^p(\Omega) \ \textit{for all} \ \alpha \in N_0^d \ \textit{with} \ |\alpha| \le k \right\}.$$

Furthermore, for $v \in W^{1,p}(\Omega)$ we set

$$\nabla v = (\partial_1 g, \ldots, \partial_d v).$$

Remark 1.2.2. *The reader should note that* $W^{0,p}(\Omega) = L^p(\Omega)$.

Next we recall some elementary properties of Sobolev spaces, from which the proof can be found in Evans [76].

Proposition 1.2.3. *Let* $1 \leq p \leq \infty$ *and* $k \in \mathbb{N}_0$. *Endowed with the norm*

$$\|v\|_{W^{k,p}}(\Omega) = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{for} \quad 1 \leq p \leq \infty,$$

$$\|v\|_{W^{k,\infty}}(\Omega) = \sum_{|\alpha| \leq k} \|\partial^\alpha v\|_{L^\infty(\Omega)},$$

the linear space $W^{k,p}(\Omega)$ *is a Banach space.*

Remark 1.2.4. *The space* $W^{k,p}(\Omega)$ *is reflexive if* $1 < p < \infty$ *and is separable if* $1 \leq p < \infty$. *Furthermore, we note that, for all values* $k \in \mathbb{N}_0$, *the norms on* $W^{k,2}(\Omega)$ *are separable Hilbert spaces , and we define*

$$\mathcal{H}^k(\Omega) = W^{k,2}(\Omega).$$

The space $\mathcal{H}^k(\Omega)$ *is also defined as the completion of* $\mathcal{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ *with respect to the norm* $\|\cdot\|_{W^{k,p}(\Omega)}$.

The associated scalar product on these spaces is defined by

$$(u\ ,\ v)_{\mathcal{H}^k(\Omega)} = \sum_{|\alpha| \leq k} (\partial^\alpha u\ ,\ \partial^\alpha v)_{L^2(\Omega)}.$$

1.3 Sobolev space of fractional order

In Section 1.2, we have seen how the Sobolev space is defined for integers order. In the same manner, the Sobolev space of fractional order can be defined. There are several ways of defining the Sobolev space of fractional order [125] among these different strategies, we choose the one that involves the Fourier transform (see Section 1.1).

Definition 1.3.1 (Sobolev space of fractional order)**.** *Let* Ω *be an arbitrary open set in* $\mathbb{R}^d$ *and* $1 \leq p < \infty$. *The space* $W^{s,p}(\Omega)$, $0 < s < 1$, *is defined by*

$$W^{s,p}(\Omega) = \left\{ v \in L^2(p) : \frac{|v(x) - v(y)|}{|x-y|^{\frac{d}{p}+s}} \in \quad L^p(\Omega \times \Omega) \right\}.$$

For the $r = s + k$, *with* $k \in \mathbb{N}_0$ *and* $0 \leq s < 1$, *the linear space* $W^{r,p}(\Omega)$, $0 \leq r < \infty$,, *is defined by*

$$W^{r,p}(\Omega) = \left\{ v \in W^{k,p}(\Omega) : \partial^\alpha v \in W^{s,p}(\Omega) \ \text{ for all } \ \alpha \in \mathbb{N}_0^d \ \text{ with } \ |\alpha| = k \right\}.$$

Remark 1.3.2. *One can notice that if $r = k \in \mathbb{N}_0$, then the space $W^{r,p}(\Omega)$ given by Definition 1.3.1 coincides with $W^{k,p}(\Omega)$ in Definition 1.2.1.*

Fractional Sobolev Inequality. Fractional Sobolev spaces enjoy quite a number of important functional inequalities [125]. It is impossible to list all the results and the possible applications. Therefore, we will only present one important inequality which has a simple and nice proof, namely the Fractional Sobolev Inequality.

The Fractional Sobolev Inequality can be written as follows:

Theorem 1.3.3 ([125]). *For any $s \in (0,1)$, $p \in \left(1, \frac{d}{s}\right)$ and $u \in C_0^\infty(\mathbb{R}^d)$,*

$$\|u\|_{L^{\frac{dp}{d-sp}}(\mathbb{R}^d)} \leq C \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{\frac{1}{p}}, \tag{1.3.1}$$

for some $C > 0$, depending only on d and p.

Proof. For brevity, we recall the following proof [125].

We have that

$$|u(x)| \leq |u(x)-u(y)| + |u(y)|.$$

For a fixed R (that will be given later on), we integrate over the ball $B_R(x)$ and have that

$$|B_R(x)||u(x)| \leq \int_{B_R(x)} |u(x)-u(y)| \, dy + \int_{B_R(x)} |u(y)| \, dy = I_1 + I_2. \tag{1.3.2}$$

We apply the Hölder inequality for the exponents p and $p/(p-1)$ in the first integral and obtain that

$$\begin{aligned} I_1 &= \alpha \int_{B_R(x)} \frac{|u(x)-u(y)|}{|x-y|^{\frac{d+sp}{p}}} |x-y|^{\frac{d+sp}{p}} \, dy \\ &\leq \alpha R^{\frac{d+sp}{p}} \left(\int_{B_R(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \, dy \right)^{\frac{1}{p}} \left(\int_{B_R(x)} dy \right)^{\frac{p-1}{p}} \\ &= \alpha C R^{d+s} \left(\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \, dy \right)^{\frac{1}{p}}. \end{aligned}$$

The Hölder inequality for $\frac{dp}{d-sp}$ and $\frac{dp}{d(p-1)+sp}$ gives, in the second integral,

$$\begin{aligned} I_2 &\leq \alpha \left(\int_{B_R(x)} |u(y)|^{\frac{dp}{d-sp}} \, dy \right)^{\frac{d-sp}{dp}} \left(\int_{B_R(x)} dy \right)^{\frac{d(p-1)+sp}{dp}} \\ &\leq \alpha R^{\frac{d(p-1)+sp}{p}} \left(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}} \, dy \right)^{\frac{d-sp}{dp}}. \end{aligned}$$

Dividing by R^d in (1.3.2) and renaming the constants, it follows that

$$|u(x)| \leq C R^s \left[\left(\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \, dy \right)^{\frac{1}{p}} + R^{-\frac{d}{p}} \left(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}} \, dy \right)^{\frac{d-sp}{dp}} \right],$$

where $C = C(d,p) > 0$. We take now R such that

$$\left(\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dy\right)^{\frac{1}{p}} = R^{-\frac{d}{p}}\left(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}}\,dy\right)^{\frac{d-sp}{dp}}$$

and we obtain

$$|u(x)| \leq C\left(\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dy\right)^{\frac{d-sp}{dp}}\left(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}}\,dy\right)^{\frac{s(d-sp)}{d^2}}.$$

Raising to the power $\frac{dp}{d-sp}$ and integrating over $\mathbb{R}^d$, we get that

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{dp}{d-sp}}\,dx \leq C\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dx\,dy\right)\left(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}}\,dy\right)^{\frac{ps}{d}}.$$

After a simplification, we obtain that

$$\left(\int_{\mathbb{R}^d} |u(x)|^{\frac{dp}{d-sp}}\,dx\right)^{\frac{d-sp}{dp}} \leq C\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dx\,dy\right)^{\frac{1}{p}},$$

which is (1.3.1). □

1.4 Weighted spaces

Weighted spaces of smooth functions play an important role in the context of partial differential equations (PDEs). They are widely used, for instance, to treat PDEs with degenerate coefficients or domains with a nonsmooth geometry [106, 117, 118, 151], as is the case here. For evolution equations, power weights in time play an important role in order to obtain results for rough initial data [46, 105]. This subsection dedicated to weighted spaces is motivated by the appearance of the Muckenhoupt weight $w := t^{1-2s}$, which appears in Chapter 2 ((2.1.5) and (2.1.6)). For general literature on weighted spaces we refer the reader to [33, 55, 117, 118, 123, 127, 150, 151] and references therein.

In a general framework, a function $w : \mathbb{R}^d \to [0,\infty)$, for an integer $d \geq 1$, is called a *weight* if w is locally integrable and the zero set $\{x : w(x) = 0\}$ has Lebesgue measure zero. For $p \in [1,\infty]$ we denote by A_p the *Muckenhoupt class* of weights. In the case $p \in (1,\infty)$, we say that $w \in A_p$ if

$$\sup_{\mathcal{B} \text{ cubes in } \mathbb{R}^d} \left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} w(x)\,dx\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty.$$

In the case $p = 1$, we say that $w : \mathbb{R} \to [0,\infty)$ belongs to A_1 if there exists some constant C such that

$$\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} w(y)\,dy \leq Cw(x)$$

for all $x \in \mathcal{B}$ and all balls $\mathcal{B} \subset \mathbb{R}^d$. In the case $p = \infty$, we define $A_\infty = \bigcup_{1 \le p < \infty} A_p$. Note that, for functions with support contained in $(-\infty, 0)$ or $(0, \infty)$, the class of weights is denoted by A_p^+ or A_p^- respectively. We refer the reader to [46, 123, 148] for the general properties of these classes.

Next, for a strongly measurable function f and a number $p \in [1, \infty)$, we define the weighted L^p norm by

$$\|f\|_{L^p(\mathbb{R}^d,w)} := \left(\int_{\mathbb{R}^d} \|f(x)\|^p w(x)\, dx \right)^{1/p},$$

and we define the weighted L^p space to be the following Banach space:

$$L^p(\mathbb{R}^d, w) := \{f \text{ strongly measurable } : \|f\|_{L^p(\mathbb{R}^d,w)} < \infty\}.$$

1.5 Some fractional nonlocal operators

We describe here the basics of some different fractional operators, the well-known fractional Laplacian [1, 125] and fractional derivatives. We recall here the basics of some different fractional derivative operators. The notion (or, better still, several possible notions) of fractional derivatives attracted the attention of many distinguished mathematicians, such as Leibniz, Bernoulli, Euler, Fourier, Abel, Liouville, Riemann, Hadamard, Riesz and Caputo, among others [134, pages xxvii–xxxvi]. The fractional exponent will be denoted by $\alpha \in (0, 1)$. For more exhaustive discussions and comparisons [11, 50, 51, 101, 113, 128, 134]. For simplicity, we consider only the Caputo or Marchaud fractional derivative due to their similarity properties with the fractional Laplacian [1, 3, 5, 6, 71].

1.5.1 The fractional derivatives and integrals operators

In fractional calculus, the orders of differentiation and integration are extended beyond the integer domain to the real line and even the complex plane. This field of study has a long history, having been considered by Leibniz, Riemann and Hardy, among others [119]. It also has a wide variety of applications, including in bioengineering [90, 112], chaos theory [160], drug transport [75, 132, 144], epidemiology [52], geohydrology [10], random walks [157], thermodynamics [156] and viscoelasticity [104].

Fractional derivatives and integrals can be defined in several different ways, not all of which agree with each other, and thus it must always be clear which definition is being used. In fact, new models of fractional calculus are being developed all the time [11, 51] for some fractional models developed only in the last few years.

Definition 1.5.1 (Riemann–Liouville fractional integral [82]). *Let x and α be complex variables, and a be a constant in the extended complex plane (usually taken to be either* 0 *or* $-\infty$*). For* $\mathrm{Re}(\alpha) < 0$*, the αth derivative, or $(-\alpha)$th integral, of a function v is*

$$I_{a+}^{\alpha} v(x) := \frac{1}{\Gamma(-\alpha)} \int_a^x (x - y)^{-\alpha - 1} v(y)\, \mathrm{d}y,$$

provided that this expression is well-defined. (If $a = -\infty$, the operator is denoted by I_{+}^{α} instead of I_{a+}^{α}.)

Since x, α and a are defined to be in the complex plane, we must consider the issue of which branch to use when defining the function $(x-y)^{-\alpha-1}$ and which contour from a to x to use for the integration. Clearly $\arg(x-y)$ can be fixed to be always equal to $\arg(x-a)$ i.e., by taking the contour of integration to be the straight line-segment $[a,x]$ [134, §22].

Definition 1.5.2 (Riemann–Liouville fractional derivative [82])**.** *Let x, α, a be as in Definition 1.5.1 except with $\mathrm{Re}(\alpha) \geq 0$. The αth derivative of a function v is*

$$D_{a+}^{\alpha} v(x) := \tfrac{d^n}{dx^n}\left(I_{a+}^{\alpha-n} v(x)\right),$$

where $n := \lfloor \mathrm{Re}(\alpha) \rfloor + 1$, provided that this expression is well-defined. (Again, if $a = -\infty$, the operator is denoted by simply D_{+}^{α} instead of D_{a+}^{α}.)

For functions v such that $D_{a+}^{\alpha} v(x)$ is analytic in α, Definition 1.5.2 is the analytic continuation in α of Definition 1.5.1. This provides some motivation for why this formula should be used [82].

When the order of differentiation and integration becomes continuous, the term *differintegration* is often used to cover both. When the order of differintegration lies in the complex plane, its real part is what defines the difference between differentiation and integration.

Definition 1.5.3 (Caputo fractional derivative)**.** *Let x, α, a be as in Definition 1.5.1 except with $\mathrm{Re}(\alpha) \geq 0$. The αth derivative of a function v is*

$$D_{a+}^{\alpha} v(x) := D_{a+}^{\alpha-n}\left(\tfrac{d^n f}{dx^n}\right),$$

where $n := \lfloor \mathrm{Re}(\alpha) \rfloor + 1$, provided that this expression is well-defined.

If a is a real number and $n = 1$, the usual Caputo fractional derivative can be written as:

$$\mathbb{D}_t^{\alpha} := {}_a^c D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \tag{1.5.1}$$

Fractional *integrals* in the Caputo context are exactly Riemann–Liouville integrals, so a new definition is not needed for them.

The constant a used in the above definitions can be thought of as a constant of integration. However, in the fractional context it appears in the formulae for derivatives as well as those for integrals. It is almost always assumed to be either 0 or $-\infty$.

Some standard properties of integer-order differintegrals extend to the fractional case: for instance, D_{a+}^{α} is still a linear operator for any α and a. But other standard theorems of calculus no longer hold in the fractional case, or hold in a more complicated way. For instance, the fractional derivative of a fractional derivative is not always a fractional derivative [82].

Note that when a is infinite and f has sufficient decay conditions, the series term disappears. In this case, the Riemann–Liouville and Caputo fractional derivatives (Definitions 1.5.2 and 1.5.3) are equivalent.

Lemma 1.5.4 (Fourier transforms of fractional differintegrals [82])**.** *If $v(x)$ is a function with well-defined Fourier transform $\widehat{v}(\lambda)$ and $\alpha \in \mathbb{C}$ is such that $D_+^\alpha v(x)$ is well-defined, then the Fourier transform of $D_+^\alpha v(x)$ is $(-i\lambda)^\alpha \widehat{v}(\lambda)$.*

Proof. If $\mathrm{Re}(\alpha) < 0$, then Definition 1.5.1 can be rewritten as a convolution: $D_+^\alpha f = f * \Phi$ where $\Phi(x) = \frac{x^{-\alpha-1}}{\Gamma(-\alpha)}$ when $x > 0$ and $\Phi(x) = 0$ otherwise. Convolutions transform to products under the Fourier transform, so the result follows.

If $\mathrm{Re}(\alpha) \geq 0$, the result follows from the fractional integral case (proved above) and the $\alpha \in \mathbb{N}$ case (which is standard). □

Definition 1.5.5 (Marchaud derivative)**.** *The right Marchaud derivative of a well defined function v is given by*

$$(\mathscr{D}_{right})^\alpha v(x) = \lim_{\delta \to 0^+} \frac{C}{\Gamma(-\alpha)} \int_{x+\delta}^{\infty} \frac{v(y) - v(x)}{(y-x)^{1+\alpha}}\, dy, \tag{1.5.2}$$

with C_α a positive normalisation constant.

Remark 1.5.6 ([71])**.** *The one-sided nonlocal derivative in the sense of Marchaud can also be obtained by extending the Caputo derivative. Indeed, by integrating by parts equation* (1.5.1)*, we obtain an equivalent definition [5, 84] as follows:*

$$\partial_t^\alpha f(t) = C(\alpha) \int_{-\infty}^{t} \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}}\, d\tau, \tag{1.5.3}$$

for all $t < a$, so that $f(t) = f(a)$, where $C(\alpha)$ is a constant depending on α. Actually, for sufficiently regular functions f, we have:

$$\begin{aligned}
\Gamma(1-\alpha)\,{}_a^cD_t^\alpha f(t) = \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau &= \int_a^t \left(\frac{d}{d\tau} \frac{\big(f(\tau) - f(t)\big)}{(t-\tau)^\alpha} - \alpha \frac{\big(f(\tau) - f(t)\big)}{(t-\tau)^{1+\alpha}} \right) d\tau \\
&= \frac{f(t) - f(a)}{t^\alpha} - \lim_{\tau \to t} \frac{f(t) - f(\tau)}{(t-\tau)^\alpha} - \alpha \int_a^t \frac{f(\tau) - f(t)}{(t-\tau)^{1+\alpha}}\, d\tau \\
&= -\frac{f(a)}{t^\alpha} - \lim_{\tau \to t} \frac{f(t)}{(t-\tau)^\alpha} - \alpha \int_a^t \frac{f(\tau) - f(t)}{(t-\tau)^{1+\alpha}}\, d\tau - \alpha \int_a^t \frac{f(t)}{(t-\tau)^{1+\alpha}}\, d\tau \\
&= \frac{f(t) - f(a)}{t^\alpha} + \alpha \int_a^t \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}}\, d\tau.
\end{aligned} \tag{1.5.4}$$

Hence, we take the convention that $f(t) = f(a)$ for any $t \leq a$. With this extension, one has that, for any $t > a$,

$$\alpha \int_{-\infty}^{a} \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}}\, d\tau = \alpha \int_{-\infty}^{a} \frac{f(t) - f(a)}{(t-\tau)^{1+\alpha}}\, d\tau = \frac{f(t) - f(a)}{t^\alpha}.$$

We can write (1.5.4) *as*

$$\partial_t^\alpha f(t) = C(\alpha) \int_{-\infty}^{t} \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}}\, d\tau.$$

This type of formula also relates the Caputo derivative to the so-called Marchaud derivative [84, 134]. Therefore the results obtained in this thesis could also be applied for the Marchaud derivative.

Lemma 1.5.7 (Laplace transforms of fractional integrals)**.** *If $f(x)$ is a function with well-defined Laplace transform $\tilde{f}(\lambda)$, and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) < 0$ is such that $D^{\alpha}_{0+}f(x)$ is well-defined, then the Laplace transform of $D^{\alpha}_{0+}f(x)$ is $(-i\lambda)^{\alpha}\tilde{f}(\lambda)$.*

Proof. Similar to the proof of Lemma 1.5.4 or [119, Chapter III]. □

The corresponding result for Laplace transforms of fractional *derivatives* is more complicated, because of the initial-value terms [119, Chapter IV].

The extension problem with Marchaud fractional derivative. In the next statement, we recall the results obtained [16], which show that the fractional derivatives on the line are Dirichlet-to-Neumann operators for an extension degenerate PDE problem in $\mathbb{R} \times (0, \infty)$, where the data f have been taken in the more general setting: more precisely, a weighted $L^p(w)$ space, where w satisfies the one-sided version A^+_p [123] of the familiar A_p condition of Muckenhoupt.

Fix $0 < s < 1$. Given a semigroup $\{T_t\}_{t\geq 0}$ acting on real functions, the *generalized Poisson integral* of f is given by [16, Equation (1.9)]

$$P^s_t f(x) = \frac{t^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-t^2/(4s)} T_s f(x) \, \frac{ds}{s^{1+s}}, \quad x \in \mathbb{R}. \tag{1.5.5}$$

By considering the semigroup of translations $T_s f(x) = f(x+s)$, $s \geq 0$, we find

$$P^s_t f(x) = f * P^s_t(x) := \int_{\mathbb{R}} f(s) P^s_t(x-s) \, ds,$$

where

$$P^s_t(x) := \frac{t^{2s} e^{t^2/4x}}{4^s\Gamma(s)(-x)^{1+s}} \chi_{(-\infty,0)}(x). \tag{1.5.6}$$

Since the kernel P^s_t is increasing and integrable in $(-\infty, 0)$, it is well known that the function

$$P^s_* f(x) := \sup_{t>0} |f| * P^s_t(x) = \int_{\mathbb{R}} |f(t)| P^s_t(x-t) \, dt$$

is pointwise controlled by the usual Hardy–Littlewood maximal operator. However, since the support of P^s_t is $(-\infty, 0)$, a sharper control can be obtained by using the one-sided Hardy–Littlewood maximal operator. This control and the behavior of P^s_* in weighted L^p-spaces will be used in the results. We revise briefly the two fundamental theorems [16].

Theorem 1.5.8 ([16])**.** *Consider the semigroup of translations $T_t f(x) = f(x+t)$, $t \geq 0$, initially acting on functions $f \in \mathcal{S}$. Let $P^s_t f$, $0 < s < 1$, be as in* (1.5.5)*. Then:*

1. *For $1 \le p \le \infty$, P_t^s is a bounded linear operator from $L^p(\mathbb{R})$ into itself and*

$$\|P_t^s f\|_{L^p(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}.$$

2. *When $f \in \mathcal{S}$, the Fourier transform of $P_t^s f$ is given by*

$$\widehat{P_t^s f}(\xi) = \frac{2^{1-s}}{\Gamma(s)} (-it\xi^{1/2})^s \|_s(-it\xi^{1/2}) \widehat{f}(\xi), \quad \xi \in \mathbb{R},$$

where $\|_\alpha(z)$ is the modified Bessel function of the third kind or Macdonald's function, which is defined for arbitrary α and $z \in \mathbb{C}$ [109, Chapter 5]. In particular,

$$\widehat{P_t f}(\xi) = e^{-t(-i\xi)^{1/2}} \widehat{f}(\xi).$$

3. *The maximal operator P_*^s defined by $P_*^s f(x) = \sup_{t>0} |P_t^s f(x)|$ is bounded from $L^p(\mathbb{R}, w)$ into itself, for $w \in A_p^+, 1 < p < \infty$, and from $L^1(\mathbb{R}, w)$ into weak-$L^1(\mathbb{R}, w)$, for $w \in A_1^+$.*

4. *Let $f \in L^p(w)$, for $w \in A_p^+$, $1 \le p < \infty$. The function $\mathcal{U}(x,t) \equiv P_t^s f(x)$ is a classical solution to the extension problem* (2.1.4).

Theorem 1.5.9 (Extension problem)**.** *Let $f \in L^p(w)$, $w \in A_p^+$, $1 < p < \infty$. Then the function*

$$\mathcal{U}(x,t) := \frac{t^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-t^2/(4\tau)} T_\tau f(x) \frac{d\tau}{\tau^{1+s}}, \quad x \in \mathbb{R},\ t > 0,$$

is a classical solution to the extension problem

$$\begin{cases} -(\mathscr{D}_{right})\mathcal{U} + \frac{1-2s}{t}\mathcal{U}_t + \mathcal{U}_{tt} = 0, & \text{in } \mathbb{R} \times (0,\infty), \\ \lim_{t\to 0^+} U(x,t) = f(x), & \text{a.e. and in } L^p(w). \end{cases}$$

Moreover, for $c_s := \frac{4^{s-1/2}\Gamma(s)}{\Gamma(1-s)} > 0$, we have

$$-c_s \lim_{t\to 0^+} t^{1-2s}\mathcal{U}_t(x,t) = (\mathscr{D}_{right})^s f(x) \quad \text{in the distributional sense.}$$

Remark 1.5.10. *This parallel result regarding the extension problem in the case of the Marchaud fractional time derivative has been derived as well [31, 84].*

1.5.2 Nonlocal operators and Lévy processes

In general, nonlocal equations arise naturally from the properties of Lévy processes. Based on an intuitive approach, Lévy processes are stochastic processes with independent and stationary increments that represent the motion of a point object as an extension of Brownian motion, with independent displacements and statistically identical over time intervals with the same length. Examples of Lévy processes include Brownian motion, the Poisson process and stable processes. They consist of paths that can present jump discontinuities of random size at random moments, and they can be used to model many physical systems, as well as in engineering, economics, ecology and some social and natural phenomena.

Given a Probability space $(\Omega, \mathcal{A}, \mathcal{P})$, the definition of a Lévy process is as follows.

Definition 1.5.11 (Lévy process)**.** *A Lévy process $X = (X_t, t \geq 0)$ is a real-valued (or $\mathbb{R}^d$-valued) stochastic process that fulfils the following requirements:*

1. $P(X_0 = 0) = 1$ *almost surely.*
2. *The random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent for all $n \geq 1$, for $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$.*
3. *The random variable $X_{t+s} - X_t$ has the same distribution as X_s, for $s, t \geq 0$.*
4. *For all $\varepsilon > 0$ and for all $s \geq 0$,*

$$\lim_{t \to s} P(|X_t - X_s| > \varepsilon) = 0.$$

In order to determine the relations in between Lévy processes and nonlocal equations, it is necessary to introduce the Lévy–Khintchine formula. To do so, recall that the characteristic function of a random variable, X, is defined by $\phi_X : \mathbb{R}^d \to \mathbb{C}$,

$$\phi_X(z) := \mathbb{E}\left[e^{iz \cdot X}\right],$$

and for the Lévy–Khintchine formula as

$$\phi_{X_t}(z) := \mathbb{E}\left[e^{iz \cdot X_t}\right].$$

The Lévy–Khintchine formula states that:

Theorem 1.5.12 (Lévy-Khintchine formula)**.** *If $X = (X_t, t \geq 0)$ is a Lévy process, then*

$$\phi_{X_t}(z) = e^{t\eta(z)}, \qquad z \in \mathbb{R}^d,$$

where $\eta(z)$ is a function given by

$$\eta(z) = ib \cdot z - \frac{1}{2} z \cdot Az + \int_{\mathbb{R}^d} \left(e^{iz \cdot y} - 1 - iz \cdot y \chi_{B_1}(y)\right) \nu(dy), \tag{1.5.7}$$

where B_1 is the unit ball, b is a vector, A is a nonnegative definite matrix and ν is a Lévy measure.

The proof of the theorem uses the fact that the Lévy processes are infinitely divisible, so that one can express $X_t = Y_1 + \cdots + Y_k$, where Y_j are independent, identically distributed random variables, for any $k \in \mathbb{N}$ [1].

Next, recall that, for $\nu \in \mathbb{R}^d$ denoted the measure, the well-known Lévy measure is defined as follows.

Definition 1.5.13 (Lévy measure)**.** *Let $X = (X_t, t \geq 0)$ be a Lévy process on $\mathbb{R}^d$. Then, the measure ν on $\mathbb{R}^d$ defined as follows is the Lévy measure associated to X:*

$$\nu(\Omega) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0,\ \Delta X_t \in \Omega\right\}\right], \Omega \in \mathcal{B}(\mathbb{R}^d), \tag{1.5.8}$$

where

$$\Delta X_t = X_t - \lim_{s \uparrow t} X_s. \tag{1.5.9}$$

That is, the measure of a set Ω is the expected number of jumps with size in Ω per unit time.

We also recall that a Lévy process almost surely defines a semigroup $\{W_t, t \geq 0\}$ acting on functions $u : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$(W_t u)(x) = \mathbb{E}\left[u(x + X_t)\right], \tag{1.5.10}$$

which is a semigroup thanks to the properties of a Lévy process, since W_0 is the identity and $W_{t+s} = W_t \circ U_s$, because $X_t - X_s$ is distributed as X_{t-s}. From that property of semigroups, it has an infinitessimal generator given denoted by L and defined by

$$Lu = \lim_{t \downarrow 0} \frac{\mathbb{E}\left[u(x + X_t)\right] - u(x)}{t}. \tag{1.5.11}$$

The linear operator L defined in (1.5.11) describes the semigroup, since the Feller property is satisfied ($\lim_{t\downarrow 0} W_t u(x) = u(x), \quad \forall\, x$). Moreover, it is possible to express an evolution of the function u over the time by means of this operator, defining $u(x,t) := (W_t u)(x)$, and, assuming $u(x,t)$ is regular enough, it follows that, for all $t \geq 0$, $x \in \mathbb{R}^d$,

$$\partial_t u = Lu. \tag{1.5.12}$$

Therefore, using the Lévy–Khintchine formula, it is possible to express the infinitesimal generator as

$$Lu = b \cdot \nabla u(x) + \mathrm{tr}\left(B \cdot \Delta u\right) + \int_{\mathbb{R}^d} \left\{u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y)\right\} \nu(y). \tag{1.5.13}$$

The first term corresponds to the drift, the second term to the diffusion and the third term to the jump part. From the Lévy processes, it has naturally arisen, as a nonlocal equation. The process $\{X_t, t \geq 0\}$ is said to be a pure jump process when $B = 0$ and $b = 0$. The infinitesimal generator can be written as

$$Lu = \int_{\mathbb{R}^d} \left(u(x+y) - u(x)\right) \mathcal{K}(y)\, dy, \tag{1.5.14}$$

where the following assumptions where made:

$$\nu(dy) = \mathcal{K}(y)\, dy \tag{1.5.15}$$

and

$$\mathcal{K}(y) = \mathcal{K}(-y), \tag{1.5.16}$$

providing $\mathcal{K} \geq 0$ and $\int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) \mathcal{K}(y)\, dy < \infty$, with $a \wedge b = \min\{a, b\}$. When the kernel $\mathcal{K}(y) = c|y|^{-d-2s}$, then L is the fractional Laplacian, as will be seen in Section 1.5.4.

1.5.3 Stable processes

Let us introduce a specially relevant kind of Lévy process, the α-stable Lévy processes. These processes are the ones satisfying certain scaling properties. For $\alpha \in (0,2)$, stable processes are equivalent to Gaussian random processes when dealing with infinite-variance random variables.

Definition 1.5.14 (α-stable Lévy process [130])**.** *Let $X = (X_t, t \geq 0)$ be a Lévy process on $\mathbb{R}^d$. It is called α-stable if*

$$X_1 = \frac{1}{t^{1/\alpha}} X_t \quad \textit{for all} \;\; t \geq 0 \tag{1.5.17}$$

that is, there exists a property of self-similarity saying that X_b is distributed like $\frac{1}{t^{1/\alpha}} X_{bt}$ for $b, t > 0$.

The infinitesimal generator of α-stable Lévy processes are uniquely determined by a finite measure on the unit sphere S^{d-1}, which is often referred as the spectral measure of the process [81]. When this measure is absolutely continuous with respect to the classical measure of the sphere and is symmetric, stable processes have nonlocal operators associated as infinitesimal generators, say when $\alpha < 2$, of the form

$$Lu(x) = \int_{\mathbb{R}^d} (u(x) - u(x+y)) \frac{a(y/|y|)}{|y|^{d+2s}} \, dy, \tag{1.5.18}$$

where $s \in (0,1)$ the stability exponent is $\alpha = 2s$. Here, a is any nonnegative function $a \in L^1(S^{d-1})$ satisfying $a(y) = a(-y)$, providing a is symmetric.

The most simple and important case of a stable process, as it has been already introduced, is when it is radially symmetric (isotropic). In this case, the infinitesimal generator is a multiple of the fractional Laplacian.

1.5.4 The fractional Laplacian and inverse Laplacian

The fractional Laplacian, $(-\Delta)^s$, is a pseudo-differential operator that can be derived naturally from the standard Laplacian in terms of the integral formulation and the use of the Fourier transform. In this section, we will recall the two approaches and provide the inverse fractional Laplacian $(-\Delta)^{-s}$ as well.

The fractional Laplacian, $(-\Delta)^s$, for $s \in (0,1)$, is a nonlocal operator that can be written as:

$$(-\Delta)^s u(x) := \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \, dy. \tag{1.5.19}$$

Here, the notation " P.V. " stands for "in the Principal Value sense", that is

$$(-\Delta)^s u(x) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \, dy.$$

The definition in (1.5.19) differs from others available in the literature since a normalizing factor has been omitted for the sake of simplicity: this multiplicative constant is only important in the limits as $s \nearrow 1$ and $s \searrow 0$, but plays no essential role for a fixed fractional parameter $s \in (0,1)$.

The operator in (1.5.19) can be also conveniently written in the form

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\,dy. \tag{1.5.20}$$

The expression in (1.5.20) reveals that the fractional Laplacian is a sort of second order difference operator, weighted by a measure supported in the whole of $\mathbb{R}^d$ and with a polynomial decay, namely

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^d} \delta_u(x,y)\,d\mu(y),$$

$$\text{where} \quad \delta_u(x,y) := u(x+y)+u(x-y)-2u(x) \quad \text{and} \quad d\mu(y) := \frac{dy}{|y|^{d+2s}}. \tag{1.5.21}$$

Of course, one can give a pointwise meaning of (1.5.19) and (1.5.20) if u is sufficiently smooth and with a controlled growth at infinity [1]. Besides, it is possible to provide a functional framework to define such an operator in the weak sense [139] and a viscosity solution approach is often appropriate to construct general regularity theories [41].

From the point of view of the Fourier transform, denoted, as usual, by $\widehat{\cdot}$ or by $\mathcal{F}$ (depending on the typographical convenience), an instructive computation [125, Proposition 3.3] shows that

$$\widehat{(-\Delta)^s u}(\xi) = c\,|\xi|^{2s}\,\widehat{u}(\xi),$$

for some $c > 0$. An appropriate choice of the normalization constant in (1.5.19) (also independent of n and s) allows us to take $c = 1$, and we will take this normalization for the sake of simplicity (and with the slight abuse of notation of dropping constants here and there). With this choice, the fractional Laplacian in Fourier space is simply multiplication by the symbol $|\xi|^{2s}$, consistent with the fact that the classical Laplacian corresponds to multiplication by $|\xi|^2$. In particular, the fractional Laplacian recovers the classical Laplacian as $s \nearrow 1$.

Remark 1.5.15 ([1]). *The operator obtained by the inverse Fourier transform of $|\xi|^2\,\widehat{u}$, the classical Laplacian, reduces to a local operator. This is not true for the inverse Fourier transform of $|\xi|^{2s}\,\widehat{u}$. Consequently, it is interesting to note that the fact that the classical Laplacian is a local operator is not immediate from its definition in Fourier space, since computing Fourier transforms is always a nonlocal operation.*

In addition, it satisfies the semigroup property, for any $s, s' \in (0,1)$ with $s+s' \leq 1$,

$$\mathcal{F}(-\Delta)^s(-\Delta)^{s'}u = |\xi|^{2s}\,\mathcal{F}((-\Delta)^{s'}u) = |\xi|^{2s}\,|\xi|^{2s'}\,\widehat{u} = |\xi|^{2(s+s')}\,\widehat{u} = \mathcal{F}(-\Delta)^{s+s'}u$$

that is

$$(-\Delta)^s(-\Delta)^{s'}u = (-\Delta)^{s'}(-\Delta)^s u = (-\Delta)^{s+s'}u. \tag{1.5.22}$$

As a special case of (1.5.22), when $s = s' = 1/2$, we have that the square root of the Laplacian raised to power two give the classical Laplacian, namely

$$\left((-\Delta)^{1/2}\right)^2 = -\Delta. \tag{1.5.23}$$

This observation shows that $U : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ is the harmonic extension.

If $d > 2s$, the inverse operator $(-\Delta)^{-s}$ coincides with the Riesz potential of order $2s$. It can be represented by convolution with the Riesz kernel K_s:

$$(-\Delta)^{-s}u = K_s * u, \quad K_s(x) = \frac{1}{c(d,s)}|x|^{-(d-2s)},$$

where $c(d,s) = \pi^{d/2-2s}\Gamma(s)/\Gamma((d-2s)/2)$. Notice that $K_s \in L^1_{\text{loc}}(\mathbb{R}^d)$. When $d = 1$ and $s \in [1/2, 1)$ we have to consider the composed operator $\nabla(-\Delta)^{-s}$. This operator is called *nonlocal gradient* and is denoted by ∇^{1-2s} [17, 125, 146].

The harmonic extension of the fractional Laplacian. The harmonic extension of the fractional Laplacian in the framework considered here is due to Luis Caffarelli and Luis Silvestre [40]. We also recall this extension procedure. The idea of this extension procedure is that the nonlocal operator $(-\Delta)^s$ acting on functions defined on $\mathbb{R}^d$ may be reduced to a local operator, acting on functions defined in the higher-dimensional half-space $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0, +\infty)$. Indeed, take $U\colon \mathbb{R}^{d+1}_+ \to \mathbb{R}$ to be the solution to the equation

$$\begin{cases} \operatorname{div}\left(y^{1-2s}\nabla U(x,y)\right) = 0 & \text{in} \quad \mathbb{R}^{d+1}_+, \\ U(x,0) = u(x) & \text{in} \quad \mathbb{R}^d. \end{cases} \tag{1.5.24}$$

Then, up to constants, we have that

$$-\lim_{y\to 0^+}\left(y^{1-2s}\partial_y U(x,y)\right) = (-\Delta)^s u(x).$$

Chapter 2
Interior regularity for a degenerate elliptic equation with mixed boundary conditions

The Marchaud derivative can be obtained as a Dirichlet-to-Neumann map via an extension problem to the upper half space. In this chapter, we start with the proof of the weak formulation and prove the existence and uniqueness of the weak solution of the Elliptic problem $(\mathscr{D}_{right})^s v = f$. As a main result, we prove interior Schauder regularity estimates for a degenerate elliptic equation with mixed Dirichlet-Neumann boundary conditions. The degenerate elliptic equation arises from the Bernardis–Reyes–Stinga–Torrea extension of the Dirichlet problem for the Marchaud derivative.

The results presented in this chapter have been published [71].

2.1 Introduction

In recent years, there has been a growing interest in the study of fractional elliptic equations involving the right Marchaud derivative $(\mathscr{D}_{right})^s$, such as equations of the form

$$(\mathscr{D}_{right})^s v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{in} \quad [b, \infty), \tag{2.1.1}$$

where without loss of generality $\Omega := [a, b) \subset \mathbb{R}$, with $a < b$ and $0 < s < 1$.

The right Marchaud derivative of a function $w : \mathbb{R} \to \mathbb{R}$ is defined via Fourier transforms as

$$\widehat{(\mathscr{D}_{right})}^s w(x) = (\pm i\xi)^s \, \widehat{w}(\xi), \tag{2.1.2}$$

and it can also be expressed by the pointwise formula

$$(\mathscr{D}_{right})^s v(x) = \frac{c_s}{\Gamma(-s)} \int_x^\infty \frac{v(y) - v(x)}{(y - x)^{1+s}} \, dy, \tag{2.1.3}$$

where c_s is a positive normalization constant. We observe from (2.1.3) that the right Marchaud derivative is a nonlocal operator (see Chapter 1 for the general definition of the fractional nonlocal operators in general and right Marchaud derivative in particular). This fact prevents us from applying local PDE techniques to treat nonlinear problems for $(\mathscr{D}_{right})^s$. To overcome this difficulty, Bernardis, Reyes, Stinga and Torrea showed in [16] that the right fractional Marchaud derivative can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. Similar extension properties have been found for the fractional Laplacian by Caffarelli and Silvestre [40].

To be more precise, consider the function $\mathcal{U} : \mathbb{R}^2_+ := \mathbb{R} \times [0, \infty) \to \mathbb{R}$ that solves the boundary-value problem

$$\begin{cases} \mathcal{M}_\alpha \mathcal{U}(t,x) & = 0 \quad \text{for } (t,x) \in \mathbb{R}^2_+, \\ \lim_{t\to 0} \mathcal{N}_\alpha \mathcal{U}(t,x) & = f(x) \quad \text{for } x \in \Omega, \\ \mathcal{U}(t,x) & = 0 \quad \text{for } x \in \mathbb{R} \setminus \Omega, \end{cases} \tag{2.1.4}$$

Then we have [16]

$$\lim_{t\to 0} \mathcal{N}_\alpha \mathcal{U}(t,x) = c_s (\mathscr{D}_{right})^s v(x),$$

where $c_s := \frac{4^{s-1/2}\Gamma(s)}{\Gamma(1-s)}$ is a positive multiplicative constant depending only on $s \in (0,1)$. Here the differential operators $\mathcal{M}_s$ and $\mathcal{N}_s$ are given by:

$$\mathcal{M}_\alpha \mathcal{U} := -(\mathscr{D}_{right})\mathcal{U} + \frac{1-2s}{t}\mathcal{U}_t + \mathcal{U}_{tt}; \tag{2.1.5}$$

$$\mathcal{N}_\alpha \mathcal{U} := -t^{1-2s}\mathcal{U}_t. \tag{2.1.6}$$

We use the notation $(\mathscr{D}_{right})$ for the derivative *from the right* at the point $x \in \mathbb{R}$ that is:

$$(\mathscr{D}_{right})v(x) = \lim_{t\to 0^+} \frac{v(x) - v(x+t)}{t}, \tag{2.1.7}$$

for good enough functions v. Observe that $(\mathscr{D}_{right})$ equals the negative of the lateral derivative $\left(\frac{d}{dx^+}\right)$ as usually defined in calculus [16].

This characterization of $(\mathscr{D}_{right})^s v$ via the local (degenerate) PDE (2.1.5) was used for the first time [16] to get maximum principles. To solve (2.1.4), Stinga and Torrea noted that (2.1.5) can be thought of as the harmonic extension of v into $2-2s$ extra dimensions [16]. Furthermore, taking advantage of the general theory of degenerate elliptic equations developed by Fabes, Jerison, Kenig and Serapioni in 1982–83, they proved comparison principles for $\mathcal{U}$ (and thus for v).

The aim of this chapter is to prove an interior Schauder estimate for the problem (2.1.4), involving any fractional power of the derivative $(\mathscr{D}_{right})^s$ as an operator that maps a Dirichlet condition to a Neumann-type condition via an extension problem [16]. We will start by providing some results on the existence and uniqueness of weak solutions of problem (2.1.1) and recall some well known results on the maximum principle [16].

Coming back to the problem (2.1.4), a significant contribution of the above extension problem is to provide a way of applying classical analysis methods to partial differential equations containing one-sided Marchaud derivative operators. By means of such extension techniques, a series of important results such as comparison principles, Harnack inequalities and regularity estimates for solutions to degenerate elliptic equations involving the fractional Laplacian have been studied by many authors [34, 35, 40, 77, 78, 93, 95, 110, 126, 136, 149, 159]. The same analysis was done for the one-sided fractional derivative operator in the sense of Marchaud [16, Theorem 1.1 and Corollary 1.2].

In view of these results, we immediately observe that interior regularity and boundary regularity for the degenerate elliptic equation with mixed boundary conditions involving the one-sided Marchaud derivative is missing in the literature. Indeed, from the pioneering work [16, 31, 149] on the analogue extension problem for nonlocal operators that map Dirichlet to Neumann, one can reduce a nonlocal problem involving fractional derivatives to a local one by keeping their qualitative properties. Using this technique, as presented in Chapter 1, one can study interior and boundary regularity. Hence, the raison d'être for this chapter.

Presentation of the main results. The purpose of this work is to analyze completely existence for the fractional elliptic equations involving the right Marchaud derivative (2.1.1) and the interior Schauder regularity estimates for a degenerate elliptic equation with mixed Dirichlet–Neumann boundary conditions for (2.1.4). We will present them separately, since these results involve different techniques.

Ia. Existence and uniqueness of the weak solution of (2.1.1)**.** Throughout the study, we will fix the notation J to be the functional associated to the weak formulation of the problem (2.1.1). By means of the variational approach, the existence and uniqueness of the problem is better understood via the well-known method of minimising sequences associated to the functional J passing through the continuity of J. In Section 2.3 we prove Theorem 2.3.1, which shows the main details of the proof, passing through the variational formulation of (2.1.1) in Section 2.2.1.

Ib. Interior Schauder regularity estimates for a degenerate elliptic equation with mixed Dirichlet–Neumann boundary conditions for (2.1.4)**.** Our strategy consists of proving first a Liouville-type theorem for solutions associated to (2.1.1), which says that they are of class $C^{0,\sigma}$ for $\sigma \in (0, \min(1,s))$. After that, we deduce the interior estimates from this Liouville theorem, by making an appropriate change of variables, and, using extension technique argument developed [16, 149], we will use this result and estimate to provide Hölder regularity estimate for (2.1.4). These results, formulated in Theorem 2.4.3, are fully detailed in Section 2.4.

2.2 Notations and preliminary results

In this section, we introduce some notations, definitions and preliminary results used throughout this chapter.

Here and in the following pages in this chapter, we consider $s \in (0,1)$, $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R} = \{z = (t,x) : t > 0\}$ and $\Omega \subset \mathbb{R}$ a bounded Lipschitz domain. For an open set Ω, an integer $k \geq 1$ and a real number $\lambda \in (0,1]$, the Hölder spaces $C^{k,\lambda}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order derivatives are uniformly Hölder continuous with exponent λ in Ω.

Furthermore, we introduce the following notation for intervals, boxes and balls:

$$\begin{aligned} B_r(x_0) &:= \{x \in \mathbb{R} : |x - x_0| < r\}, \\ \mathcal{B}_r^+(x_0) &:= [0, r) \times B_r(x_0), \\ \mathcal{B}_r(z_0) &:= \{z = (t, x) \in \mathbb{R} \times \mathbb{R} : |z - z_0| < r\}. \end{aligned} \tag{2.2.1}$$

For $\Omega \subset \mathbb{R}$ an open set, we say $v : \Omega \to \mathbb{R}$ is in $C^{0,\gamma}(\Omega)$ i.e., Hölder continuous with exponent $\gamma \in (0, 1)$, if

$$\|v\|_{C^{0,\gamma}(\Omega)} := \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|^{\gamma}} < +\infty.$$

The one-sided nonlocal derivative in the sense of Marchaud can also be obtained by extending the Caputo derivative given in Chapter 1 (Definition 1.5.3). Indeed, for α real, and by making an integration by parts in Definition 1.5.3, we obtain an equivalent definition (see, Definition (2.2.2) or see also [5, 84]). We recall the definition for the completeness of the reader.

Definition 2.2.1. *The right Marchaud derivative of a well defined function v is given by*

$$(\mathscr{D}_{right})^{\alpha} v(x) = \lim_{\delta \to 0^+} \frac{C}{\Gamma(-\alpha)} \int_{x+\delta}^{\infty} \frac{v(y) - v(x)}{(y - x)^{1+\alpha}} \, dy, \tag{2.2.2}$$

with C_α a positive normalisation constant.

Note that the integral in (2.2.2) is absolutely convergent for functions in the Schwartz class $\mathcal{S}$. Furthermore, one should notice that the nonlocal operators $(\mathscr{D}_{right})^s$ and $(\mathscr{D}_{right})^{-s}$ depend on the values of v on the whole half line (x, ∞).

We recall that the inverse of the right- fractional Marchaud derivative $(\mathscr{D}_{right})^{-s}$ is defined as

$$(\mathscr{D}_{right})^{-s} v(x) := \int_{\mathbb{R}} \frac{v(y)}{|x - y|^{1-s}} \, dy = \mathcal{I}_s * v(x) \tag{2.2.3}$$

where the Riesz potential [134, 149] is defined as

$$\mathcal{I}_s = C_s |x - y|^{s-1} \quad \text{for } s < 1, \tag{2.2.4}$$

with the constant $C_s = \frac{1}{\pi} \Gamma(1 - s) \sin \frac{\pi s}{2}$.

For $u \in \mathcal{S}$, $(\mathscr{D}_{right})^s u \in \mathcal{S}_s$, where

$$\mathcal{S}_s := \left\{ f \in C^{\infty}(\mathbb{R}) : (1 + |x|^{1+s}) f^k(x) \in L^{\infty}(\mathbb{R}), \text{ for each } k \geq 0 \right\}.$$

The topology in $\mathcal{S}_s$ is given by the family of seminorms $[f]_k := \sup_{x \in \mathbb{R}} \left| (1 + |x|^{1+s}) f^{(k)}(x) \right|$, for $k \geq 0$. Let $\mathcal{S}_s'$ be the dual space of $\mathcal{S}_s$; then $(\mathscr{D}_{right})^s$ defines a continuous operator from $\mathcal{S}_s'$ into $\mathcal{S}'$ [16].

We saw briefly in Chapter 1 the notion of weighted Sobolev space. We recall the weighted spaces here due to the appearance of the Muckenhoupt weight $w := t^{1-2s}$, which appears in (2.1.5) and (2.1.6).

Example 2.2.2. *Problem* (2.1.4) *is a weighted—singular or degenerate, depending on the value of* $s \in (0,1)$*— elliptic equation on* $\mathbb{R}^2_+$ *with mixed boundary conditions. The weight* $w := t^{1-2s}$ *belongs to the Muckenhoupt class* A_2^+ *i.e., there exists a constant* C *such that, for any* $\mathcal{B} \subset \mathbb{R}^2_+$*,*

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |t|^{1-2s} dt\, dx\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |t|^{2s-1} dt\, dx\right) \leq C.$$

For this reason, when working with one-sided weights, we can assume without loss of generality that $\Omega := [a,b) = \mathbb{R}$ *[123].*

Definition 2.2.3 (see [126])**.** *Given* $s \in (0,1)$*,* $\mu = 1-2s \in (-1,1)$*, and an open set* $\mathcal{B} \subset \overline{\mathbb{R}^2_+}$*, we denote*

$$L^2(t^\mu;\mathcal{B}) := \left\{\mathcal{U} : \mathbb{R}^2_+ \to \mathbb{R},\ \int_{\mathcal{B}} t^\mu |\mathcal{U}|^2 dt\, dx < +\infty\right\},$$

endowed with the norm

$$\|\mathcal{U}\|_{L^2(t^\mu;\mathcal{B})} := \left(\int_{\mathcal{B}} t^\mu |\mathcal{U}|^2 dt\, dx\right)^{1/2}.$$

We also denote

$$H^1(t^\mu;\mathcal{B}) := \left\{\mathcal{U} \in L^2(t^\mu;\mathcal{B}) : \nabla\mathcal{U} \in L^2(t^\mu;\mathcal{B})\right\},$$

with the induced norm

$$\|\mathcal{U}\|_{H^1(t^\mu;\mathcal{B})} := \left(\int_{\mathcal{B}} t^\mu \left(|\mathcal{U}|^2 + |\nabla\mathcal{U}|^2\right) dt\, dx\right)^{1/2}.$$

Using the variable $(t,x) \in \mathbb{R}^2_+$, the space $H^s(\mathbb{R})$ coincides with the trace on $\partial\mathbb{R}^2_+$ of

$$\dot{H}^1(t^\mu;\mathcal{B}) := \left\{\mathcal{U} \in L^2_{loc}(\mathbb{R}^2_+) :\ \int_{\mathbb{R}^2_+} t^\mu \left(\mathcal{U}^2 + |\nabla\mathcal{U}|^2\right) dt\, dx < +\infty\right\}.$$

In other words [35, 126], for any given function $\mathcal{U} \in \dot{H}^1(t^\mu;\mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$, we have that $v := \mathcal{U}|_{\partial\mathbb{R}^2_+} \in H^s(\mathbb{R})$, and there exists a constant $C = C(s) > 0$ such that

$$\| v \|_{H^s(\mathbb{R})} \leq C \| \mathcal{U} \|_{H^1(t^\mu;\mathcal{B})}.$$

By a density argument, every function $\mathcal{U} \in \dot{H}^1(t^\mu;\mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$ has a well-defined trace $v \in H^s(\mathbb{R})$. Conversely, any $v \in H^s(\mathbb{R})$ is the trace (restriction to $t = 0$) of a function $\mathcal{U} \in \dot{H}^1(t^\mu;\mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$.

Definition 2.2.4. *We say that a function $\mathcal{U} \in \dot{H}^1(t^\mu;\mathcal{B})$ is a weak solution of* (2.1.4) *if*

$$\int_{\mathcal{B}} t^\mu \nabla\mathcal{U}(t,x)\nabla\Psi(t,x)dt\,dx - c_s^{-1}\int_\Omega f(x)\mathrm{Tr}(\Psi)(x)\,dx = 0, \tag{2.2.5}$$

where f is as in (2.1.1), *$\mathrm{Tr}(\Psi)$ denotes the trace $\Psi\big|_{\{0\}\times\mathbb{R}}$, and $\Psi \in \dot{H}^1(t^\mu;\mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$ is an arbitrary test function.*

2.2.1 Variational formulation of the fractional elliptic equation with right Marchaud fractional derivative

Let $\Omega := [a,b) \subset \mathbb{R}$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$. Note here that, for the space of smooth functions compactly supported, we adopt the notation $\mathcal{C}_c^\infty$ instead of $\mathcal{C}_0^\infty$.

We consider the function space

$$\mathcal{L}_s^1 := \left\{\varphi : \mathbb{R} \to \mathbb{R} \quad : \quad \|\varphi\|_{\mathcal{L}_s^1} := \int_{\mathbb{R}} \frac{|\varphi(x)|}{1+|x|^{1+s}}\,dx < \infty\right\}.$$

Definition 2.2.5. *Let Ω be an open set. Provided $v \in \mathcal{L}_s^1$, the distribution $(\mathscr{D}_{right})^s v$ is defined by*

$$\left\langle \left(\mathscr{D}_{right}\right)^s v, \phi \right\rangle = \int_{\mathbb{R}^2} v\left(\mathscr{D}_{left}\right)^s \phi\,dx = \int_\Omega f\,\phi\,dx, \quad \text{for all } \phi \in \quad \mathcal{C}_c^\infty(\Omega).$$

Saying that $\left(\mathscr{D}_{right}\right)^s v = f$ in $\mathcal{D}'(\Omega)$ is equivalent to the very weak formulation

$$\int_{\mathbb{R}} v\left(\mathscr{D}_{left}\right)^s \phi\,dx = \int_\Omega f\,\phi\,dx, \quad \text{for all } \phi \in \quad \mathcal{C}_c^\infty(\Omega).$$

We consider the bilinear form

$$\begin{aligned} \langle\ ,\ \rangle_{\mathcal{D}^{s,2}(\Omega)} : \mathcal{C}_c^\infty(\Omega) \times \mathcal{C}_c^\infty(\Omega) &\Longrightarrow \mathbb{R} \\ (v\ ,\ \phi) &\longmapsto \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v(x)-v(y))\,(\phi(x)-\phi(y))}{|x-y|^{1+s}}dx\,dy, \end{aligned}$$

which is a scalar product on $\mathcal{C}_c^\infty(\Omega)$. We recall that the Hilbert space $\mathcal{D}^{s,2}(\Omega)$ is the completion of $\mathcal{C}_c^\infty(\Omega)$.

Lemma 2.2.6. *If Ω is a bounded Lipschitz open set,*

$$\mathcal{D}^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}), \quad \text{such that} \quad u = 0 \quad \text{in} \quad \mathbb{R}\setminus\Omega\}.$$

Proof. Since Ω is Lipschitz, $\mathcal{C}_c^\infty(\Omega)$ is dense in $H_0^s(\Omega)$ with respect to the $\mathcal{D}^{s,2}(\Omega)$-norm. Then $H_0^s(\Omega) \subseteq \mathcal{D}^{s,2}(\Omega)$.

Let $u \in \mathcal{D}^{s,2}(\Omega)$, since $\mathcal{D}^{s,2}(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{\|\ \|_{[]_{s,2}}}$, there exists a sequence $(u_n) \subset \mathcal{C}_c^\infty(\Omega)$ such that $u_n \longrightarrow u$ a.e. in $\mathbb{R}$. Thus for a.e. $x \in \mathbb{R} \setminus \Omega$ and $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that, for any $n > n_\varepsilon$,

$$|u(x)| = |u_n(x) - u(x)| < \varepsilon, \quad \text{for all} \quad \varepsilon > 0,$$

and then $u = 0$ in $x \in \mathbb{R} \setminus \Omega$. Furthermore, we have also that Ω is bounded, so $u \in H^s(\mathbb{R})$. Therefore, $u \in H_0^s(\Omega)$, which implies that $\mathcal{D}^{s,2}(\Omega) \subseteq H_0^s(\Omega)$. □

For $s = 1$, the space $\mathcal{D}^{1,2}(\Omega)$ coincides with $H_0^1(\Omega)$.

We recall that the norm in the space $\mathcal{D}^{s,2}(\Omega)$ is given by:

$$\|v\|_{\mathcal{D}^{s,2}(\Omega)} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|v(x) - v(y)\|^2}{|x-y|^{1+s}} dx\, dy \right)^{\frac{1}{2}},$$

and this norm is equivalent to the H^s-norm according to the following proposition.

Proposition 2.2.7. *Let $s \in (0,1)$ and Ω be a bounded open subset of $\mathbb{R}$. Let $v : \Omega \to \mathbb{R}$ be a measurable function compactly supported in Ω. Then there exists a positive constant $C := C(s,\Omega)$ depending only on s and Ω such that*

$$\|v\|_{L^2(\Omega)} \leq C \|v\|_{\mathcal{D}^{s,2}(\Omega)}. \tag{2.2.6}$$

Proof. Consider the following estimate,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+s}} dx\, dy \geq \int_\Omega u^2(x) \int_{\mathbb{R}\setminus\Omega} |x-y|^{-1-s} dx\, dy. \tag{2.2.7}$$

Since

$$\int_{\mathbb{R}\setminus\Omega} |x-y|^{-1-s} dx\, dy \geq C(s)|\Omega|^{-s}, \tag{2.2.8}$$

we set

$$\rho := \left(\frac{|\Omega|}{w} \right).$$

We notice that Ω and the ball B_ρ of radius ρ have the same measure. Thus it follows that

$$\begin{aligned} \left|(\mathbb{R} \setminus \Omega) \cap B_\rho(x)\right| &= \left|B_\rho(x)\right| - \left|\Omega \cap B_\rho(x)\right| = |\Omega \cap B_\rho(x)| \\ &= \left|\Omega \cap \left(\mathbb{R} \setminus B_\rho(x)\right)\right|. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}\setminus\Omega} \frac{dy}{|x-y|^{1+s}}\,dy &= \int_{(\mathbb{R}\setminus\Omega)\cap B_\rho(x)} \frac{dy}{|x-y|^{1+s}} + \int_{(\mathbb{R}\setminus\Omega)\cap\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}} \\
&\geq \int_{(\mathbb{R}\setminus\Omega)\cap B_\rho(x)} \frac{dy}{\rho^{1+s}} + \int_{(\mathbb{R}\setminus\Omega)\cap\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}} \\
&= \frac{\left|(\mathbb{R}\setminus\Omega)\cap B_\rho(x)\right|}{\rho^{1+s}} + \int_{(\mathbb{R}\setminus\Omega)\cap\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}} \\
&= \frac{\left|\Omega\cap\left(\mathbb{R}\setminus B_\rho(x)\right)\right|}{\rho^{1+s}} + \int_{(\mathbb{R}\setminus\Omega)\cap\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}} \\
&\geq \int_{\Omega\cap\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}} + \int_{(\mathbb{R}\setminus\Omega)\cap\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}} \\
&= \int_{\left(\mathbb{R}\setminus B_\rho(x)\right)} \frac{dy}{|x-y|^{1+s}}.
\end{aligned}$$

If we use polar coordinates centered at x, i.e., $rz = y - x$ with $r = |x-y|$, then

$$\int_{(\mathbb{R}\setminus\Omega)} \frac{dy}{|x-y|^{1+s}} = |S_1| \int_\rho^\infty r^{-1-s} dr = |S_1| \frac{\rho^{-s}}{s} = C(s)|\Omega|^{-s}.$$

□

Now let Ω be a bounded open set with Lipschitz boundary, $v \in \mathcal{D}^{s,2}(\Omega)$ and $f \in L^\infty(\Omega)$. We consider (2.1.1). In the following section, we provide the proof of existence and uniqueness of the weak solution.

2.3 Existence and Uniqueness of the weak solution of the fractional elliptic equation with right Marchaud fractional derivative

In this section, we are interested in the existence and the uniqueness of the weak solution of the fractional elliptic problem above that is, the well-posedness of the equation.

The problem (2.1.1) is weakly formulated as follows:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v(x)-v(y))\,(\phi(x)-\phi(y))}{|x-y|^{1+s}} dx\,dy - \int_\Omega f(x)\phi(x)\,dx = 0, \quad \phi \in \mathcal{D}^{s,2}(\Omega). \tag{2.3.1}$$

Theorem 2.3.1. *Consider the following problem $\mathcal{P}$:*

$$\begin{aligned}
&\textit{Find} \quad v \in \mathcal{D}^{s,2}(\Omega) \quad \textit{such that} \\
&J(v,v) = \min_{\phi\in\mathcal{D}^{s,2}(\Omega)} J(\phi,\phi),
\end{aligned} \tag{2.3.2}$$

where $J(v,\phi) = \langle v, \phi\rangle_{\mathcal{D}^{s,2}(\Omega)} - \int_\Omega f\,\phi\,dx.$

Then $\mathcal{P}$ *admits a unique solution.*

We need to prove some results before presenting the proof of the Theorem 2.3.1.

Lemma 2.3.2. *Let* J *be defined as in above Theorem 2.3.1. Then* J *is continuous.*

Proof. Let $J(v,\phi) = a(v,\phi) - \ell(\phi)$ such that

$$a(v,\phi) := \int_{\mathbb{R}^2} \frac{(v(x)-v(y))\,(\phi(x)-\phi(y))}{|x-y|^{1+s}} dx\,dy,$$

and

$$\ell(\phi) = \int_\Omega f\,\phi\,dx.$$

We have

$$|\ell(\phi)| = \left|\int_\Omega f\,\phi\,dx\right| \le \|f\|_{L^2(\Omega)}\,\|\phi\|_{L^2(\Omega)},$$

so the linear form ℓ is continuous.

Using Cauchy-Schwartz inequality and by Proposition 2.2.7, we have

$$\begin{aligned} |a(v,\phi)| &= \left|\int_{\mathbb{R}^2} \frac{(v(x)-v(y))\,(\phi(x)-\phi(y))}{|x-y|^{1+s}} dx\,dy\right| \\ &\le \int_{\mathbb{R}^2} \frac{|v(x)-v(y)|\,|\phi(x)-\phi(y)|}{|x-y|^{1+s}} dx\,dy \\ &\le C(s,\Omega)\|v\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\ &\le C(s,\Omega)\|v\|_{\mathcal{D}^{s,2}(\Omega)}\|\phi\|_{\mathcal{D}^{s,2}(\Omega)}, \end{aligned}$$

up to relabeling the constant $C(s,\Omega)$ depending only on s, and Ω. Hence the bilinear form is continuous. Therefore J is continuous. □

Lemma 2.3.3. *Let* $(u_n)_{n\ge 1} \subset \mathcal{D}^{s,2}(\Omega)$ *be a minimizing sequence i.e.,*

$$\lim_{k\to+\infty} J(v_k, v_k) = \inf_{\phi\in\mathcal{D}^{s,2}(\Omega)} J(\phi,\phi) = I(\Omega) \tag{2.3.3}$$

then, $(u_k)_{k\ge 1}$ *is bounded in* $\mathcal{D}^{s,2}(\Omega)$.

Proof. It is worthy of note that (2.3.3) implies the following

$$I(\Omega) \le J(\zeta,\zeta) < I(\Omega) + \frac{1}{k}, \quad k \ge 1, \quad \zeta \in \mathcal{D}^{s,2}(\Omega). \tag{2.3.4}$$

Let $\zeta \in \mathcal{C}_c^\infty(\Omega)$, with Cauchy-Schwartz inequality and (2.3.4), we get that

$$\begin{aligned} I(\Omega) &\leq J(\zeta,\zeta) = \|\zeta\|^2_{\mathcal{D}^{s,2}(\Omega)} - \int_\Omega f\zeta\, dx \\ &\leq \|\zeta\|^2_{\mathcal{D}^{s,2}(\Omega)} + \|f\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)} \\ I(\Omega) &< +\infty. \end{aligned} \tag{2.3.5}$$

Next we define the following $I(\Omega) \leq J(v_k, v_k) \leq I(\Omega) + \frac{1}{k}, \quad k \geq 1$, with

$$J(v_k, v_k) = \|v_k\|^2_{\mathcal{D}^{s,2}(\Omega)} - \int_\Omega f\, v_k\, dx. \tag{2.3.6}$$

Using Cauchy-Schwartz inequality, we have the estimate

$$\int_\Omega f v_k\, dx \leq \|f\|_{L^2(\Omega)}\|v_k\|_{L^2(\Omega)} \leq \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_k\|^2_{L^2(\Omega)}. \tag{2.3.7}$$

Combining (2.3.6) and (2.3.7), we get

$$J(v_k, v_k) = \|v_k\|^2_{\mathcal{D}^{s,2}(\Omega)} - \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|v_k\|^2_{L^2(\Omega)}\right),$$

hence

$$\|v_k\|^2_{\mathcal{D}^{s,2}(\Omega)} \leq J(v_k, v_k) + \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|v_k\|^2_{L^2(\Omega)}\right).$$

Then using (2.3.4) and (2.3.5) with Proposition 2.2.7 we get:

$$\begin{aligned} \|v_k\|^2_{\mathcal{D}^{s,2}(\Omega)} &\leq I(\Omega) + \frac{1}{k} + \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|v_k\|^2_{L^2(\Omega)}\right) \\ &\leq I(\Omega) + \frac{1}{k} + \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + C(s,\Omega)\|v_k\|^2_{\mathcal{D}^{s,2}(\Omega)}\right), \end{aligned}$$

where $C(s,\Omega)$ is a positive constant depending only on s and Ω. So,

$$C(s,\Omega)\|v_k\|^2_{\mathcal{D}^{s,2}(\Omega)} \leq I(\Omega) + \frac{1}{k} + \frac{1}{2}\|f\|^2_{L^2(\Omega)} \longrightarrow K \in \mathbb{R}, \quad \text{as} \quad k \to +\infty,$$

where $K = I(\Omega) + \frac{1}{2}\|f\|^2_{L^2(\Omega)}$, up to relabeling the constant $C(s,\Omega)$. Then the sequence $(v_k)_{k\geq 1}$ is bounded in $\mathcal{D}^{s,2}(\Omega)$. □

Proof of Theorem 2.3.1. Under the regularity assumptions of Ω, the compact embedding with the fractional sobolev norm H^s tells us that $\mathcal{D}^{s,2}(\Omega)$ is compactly embedded (precompact) in $L^2(\Omega)$.

Making use of Lemma 2.3.3, the sequence $(v_k)_{k\geq 1}$ is bounded in $\mathcal{D}^{s,2}(\Omega)$, then there exists a subsequence $(v_{k_m})_{m\geq 1}$ of $(v_k)_{k\geq 1}$ such that

$v_{k_m} \rightharpoonup v \in \mathcal{D}^{s,2}(\Omega)$ (weak convergence) and also

$v_{k_m} \to v \in L^2(\Omega)$ (that implies a pointwise convergence in) Ω and a weak one, i.e.,

$v_{k_m} \rightharpoonup v \in L^2(\Omega)$ as $l \to +\infty$

One should notice that we have here three different convergences with different subsequence, but we keep the same notation. With the pointwise convergence, we get the same limit by unicity of limit.

Therefore we get that

$$J(v_{k_m}, v_{k_m}) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v_{k_m}(x) - v_{k_m}(y)|^2}{|x-y|^{1+s}} dy\, dx - \int_{\Omega} f\, v_{k_m}\, dx \leq I(\Omega) + \varepsilon, \quad \text{for all } \varepsilon > 0.$$

So,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v_{k_m}(x) - v_{k_m}(y)|^2}{|x-y|^{1+s}} dy\, dx \leq \int_{\Omega} f\, v_{k_m}\, dx + I(\Omega) + \varepsilon, \quad \text{for all } \varepsilon > 0.$$

Now by Fatou's lemma, we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \liminf_{m\to+\infty} \frac{|v_{k_m}(x) - v_{k_m}(y)|^2}{|x-y|^{1+s}} dy\, dx \leq \liminf_{m\to+\infty} \int_{\Omega} f\, v_{k_m}\, dx + I(\Omega) + \varepsilon, \quad \text{for all} \varepsilon > 0.$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x) - v(y)|^2}{|x-y|^{1+s}} dy\, dx \leq \liminf_{m\to+\infty} \int_{\Omega} f\, v_{k_m}\, dx + I(\Omega) + \varepsilon, \quad \text{for all} \varepsilon > 0.$$

Since $(v_{k_m})_{\geq 1}$ converges weakly to v in $L^2(\Omega)$, we have

$$\liminf_{m\to+\infty} \int_{\Omega} f\, v_{k_m}\, dx = \lim_{m\to+\infty} \int_{\Omega} f\, v_{k_m}\, dx = \int_{\Omega} f\, v\, dx,$$

then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x) - v(y)|^2}{|x-y|^{1+s}} dy\, dx \leq \int_{\Omega} f\, v\, dx + I(\Omega) + \varepsilon, \quad \text{for all} \varepsilon > 0.$$

Therefore,

$$J(v,v) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x) - v(y)|^2}{|x-y|^{1+s}} dy\, dx - \int_{\Omega} f\, v\, dx \leq I(\Omega) + \varepsilon, \quad \text{for all} \varepsilon > 0.$$

So,

$$J(v,v) \leq I(\Omega) \Longrightarrow J(v,v) = I(\Omega).$$

J has a minimizer $v \in \mathcal{D}^{s,2}(\Omega)$. This proves the existence result.

To prove the uniqueness of the solution, let $v_1, v_2 \in \mathcal{D}^{s,2}(\Omega)$ be the solution of (2.1.1)

$$(\mathscr{D}_{right})^s v_1 = f \quad \text{in } \Omega, \quad v_1 = 0 \quad \text{in} \quad \mathbb{R} \setminus \Omega,$$

and

$$(\mathscr{D}_{right})^s v_2 = f \quad \text{in } \Omega, \quad v_2 = 0 \quad \text{in} \quad \mathbb{R} \setminus \Omega,$$

set $w = v_1 - v_2$, then we have

$$(\mathscr{D}_{right})^s w = 0 \quad \text{in}\, \Omega, \quad w = 0 \quad \text{in} \quad \mathbb{R} \setminus \Omega. \tag{2.3.8}$$

The weak formulation of the equation (2.3.8), with w the test function, is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|v(x) - v(y)|^2}{|x - y|^{1+s}} dy\, dx = 0, \quad w \in \mathcal{D}^{s,2}(\Omega),$$

which implies with inequality (2.2.6) that

$$\|w\|^2_{L^2(\Omega)} \leq C(s,\Omega)\|w\|^2_{\mathcal{D}^{s,2}(\Omega)} = 0 \Longrightarrow \|w\|^2_{L^2(\Omega)} = 0,$$

hence $w = 0$ q.e. in Ω, $v_1 = v_2$ a.e. in Ω. □

Proposition 2.3.4. *Problem* (2.1.1) *has a unique solution that depends continuously on the data* $f \in L^2(\Omega)$.

Proof. Thanks to Theorem 2.3.1, the equation (2.1.1) has a unique solution. In order to show that the solution depends continuously on the data f, let v_1,$v_2 \in \mathcal{D}^{s,2}(\Omega)$ be solutions of the same type of equation as (2.1.1) with data f_1,$f_2 \in \mathcal{D}^{s,2}(\Omega)$ respectively. Note that

$$\langle v, \phi \rangle_{\mathcal{D}^{s,2}(\Omega)} = \int_{\Omega} f\, \phi\, dx = \ell(\phi),$$

given by the weak formulation. Using inequality (2.2.6), we have that

$$\begin{aligned}
\|v_1 - v_2\|_{\mathcal{D}^{s,2}(\Omega)} &= \frac{\langle v_1 - v_2, v_1 - v_2 \rangle_{\mathcal{D}^{s,2}(\Omega)}}{\|v_1 - v_2\|_{\mathcal{D}^{s,2}(\Omega)}} = \frac{\ell(v_1 - v_2)}{\|v_1 - v_2\|_{\mathcal{D}^{s,2}(\Omega)}} \\
&\leq \frac{\|v_1 - v_2\|_{L^2(\Omega)} \|f_1 - f_2\|_{L^2(\Omega)}}{\|v_1 - v_2\|_{\mathcal{D}^{s,2}(\Omega)}} \\
C(s,\Omega) &\leq \frac{\|v_1 - v_2\|_{\mathcal{D}^{s,2}(\Omega)} \|f_1 - f_2\|_{L^2(\Omega)}}{\|v_1 - v_2\|_{\mathcal{D}^{s,2}(\Omega)}} \\
&\leq C(s,\Omega)\|f_1 - f_2\|_{L^2(\Omega)}.
\end{aligned}$$

□

2.4 Regularity estimate up to the boundary for the degenerated equation with the newmann boundary condition

In this section, we study the interior regularity estimate up to the boundary for the degenerated equation with the Neumann boundary condition associated to problem (2.1.4). Namely we provide the proof of Theorem 2.4.3. But before we get into that, it is necessary to explain the main ideas in the proof of interior regularity provided by Theorem 2.4.3. The proof of

Theorem 2.4.3 is inspired [16, 79, 126, 149]. The method for this proof differs substantially from interior regularity methods for second-order equations, but is similar to the proof for the fractional Laplacian. Recall that for second-order equations, one first shows that D^2u is bounded, and then the estimate for equations with bounded measurable coefficients implies a $C^{2,\sigma}$ estimate for $\sigma \in (0, \min(1,s))$. This is also true for the boundary regularity for solutions to fully nonlinear equations [36].

We shall start by the regularity property of the problem (2.1.1). We show in Proposition 2.4.2 that the solution of the problem (2.1.1) is of class $\mathcal{C}^{0,\sigma}$. To the best of the authors' knowledge, the proofs available in the literature are those dealing with the case of the fractional Laplacian (see for instance [149] and [140, Proposition 2.1.9]). With this result in hand, and by making an appropriate change of variables, we will use this result and estimate to prove our main theorem.

We start by recalling the following lemma from [31], which gives a Liouville-type theorem for (2.1.1) in the case $f = 0$.

Lemma 2.4.1. *Let $u \in \mathcal{C}(\mathbb{R})$ be a function satisfying $(\mathscr{D}_{right})^s u = 0$ in $\mathbb{R}_+$, $u = 0$ in $\mathbb{R}_-$, and $|u(x)| \le C(1+|x|^\gamma)$ for some $\gamma < s$. Then $u(x) = kx^s$.*

The proof of this lemma relies on similar reasoning as the proof of [31, Theorem 2.2.3] for the Caputo density function.

In the case where we have a non-vanishing right hand side ($f \neq 0$) as in (2.1.1), we state the following Liouville-type theorem for the one-sided Marchaud derivative.

Proposition 2.4.2. *Let $s \in (0,1)$ and let $u \in \mathcal{L}^1_s \cap L^\infty_{loc}(B_1)$ be the solution to*

$$(\mathscr{D}_{right})^s u = f \quad in \quad B_1.$$

(a) For $1 < p < \infty$, if $f \in L^p(B_1, w)$ and $r \in (0,1)$ and $\gamma \in (0, \min(1,s))$ is such that $0 < \gamma - \frac{1}{p} < 1$, then $u \in \mathcal{C}^{0,\gamma-\frac{1}{p}}(B_1, w)$ and there exists a positive constant $C := C(s, r, \gamma, p) > 0$ such that

$$\|u\|_{\mathcal{C}^{0,\gamma-\frac{1}{p}}(B_r,w)} \le C\left(\|u\|_{L^p(B_r,w)} + \|f\|_{L^p(B_1,w)}\right). \tag{2.4.1}$$

(b) If $f \in L^\infty(B_1)$ and $r \in (0,1)$ and $\gamma \in (0, \min(1,s))$, then $u \in \mathcal{C}^{0,\gamma}(B_1)$ and there exists a positive constant $C := C(s,r,\gamma) > 0$ such that

$$\|u\|_{\mathcal{C}^{0,\gamma}(B_r)} \le C\left(\|u\|_{L^\infty(B_r)} + \|f\|_{L^\infty(B_1)}\right). \tag{2.4.2}$$

Proof. We will show that u has the corresponding regularity in a neighbourhood of the origin. We split the proof into two parts, as follows.

Proof of (a): $f \in L^p(B_1, w)$. Let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function such that $\eta = 1$ on B_r, $\eta = 0$ on $\mathbb{R} \setminus B_1$, and $0 \le \eta \le 1$ on $\mathbb{R}$. Consider the Riesz potential as defined in (2.2.4). Then the function

$$v(x) := \int_{\mathbb{R}} \mathcal{I}_s(x,y)(\eta f)(y)dy, \quad \text{for all} \quad x \in \mathbb{R},$$

satisfies

$$(\mathscr{D}_{right})^s v(x) = \eta(x) f(x) \quad \text{for all } x \in \mathbb{R}_+. \tag{2.4.3}$$

We first estimate the L^p norm of v for $s < 1$. Since the kernel $(\mathscr{D}_{right})^{-s}$ is positive and $\eta \geq 0$ is a smooth function with compact support in B_r, we write

$$v = (\mathscr{D}_{right})^{-s}(\eta f) = (\mathscr{D}_{right})^{1-s} \circ (\mathscr{D}_{right})^{-1}(\eta f).$$

We note that, by using a similar argument as for the Poisson equation for the fractional Laplacian, we find that $(\mathscr{D}_{right})^{-1}(\eta f)$ is an element of $\mathcal{C}^{1,\gamma}$ with norm depending only on $\|f\|_{\mathcal{C}^{0,\gamma}}$.

Since ηf is compactly supported, we get

$$\|v\|_{C^{0,\gamma}(\mathbb{R})} \leq C_{s,\gamma} \|\eta f\|_{L^p(B,w^p)} + C\|v\|_{L^p(\mathbb{R},w^p)} \leq C_{s,\gamma,B'} \|f\|_{L^p(B,w^p)}.$$

For $s < 1$ and $\gamma \in (0, \min(s,1))$ and $x, y \in B_r$, we have

$$\begin{aligned} v(x) - v(y) &= \int_{\mathbb{R}} (\mathcal{I}_s(x,z) - \mathcal{I}_s(y,z))\,(\eta f)(z) dz \\ &= C_{1,s} \int_{\mathbb{R}} \left(|x-z|^{s-1} - |y-z|^{s-1} \right) (\eta f)\,(z)\, dz. \end{aligned}$$

Next we consider the following inequalities [80], valid for $\gamma \in (0, \min(1,s))$, $m \in \mathbb{R}$ with $m + \gamma > 0$ and for every $x, y, z \in B_r$:

$$\left| \left(|x-z|^{-m} - |y-z|^{-m} \right) \right| \leq \frac{|m|}{m+\gamma} |x-y|^{\gamma} \left(|x-z|^{-(m+\gamma)} + |y-z|^{-(m+\gamma)} \right). \tag{2.4.4}$$

For $m = 1 - s$, and for $1 < p < \infty$, we can write

$$\begin{aligned} |v(x) - v(y)| &< \frac{(1-s)}{(1-s+\gamma)} \int_{\mathbb{R}} |x-y|^{\gamma} \left(|x-z|^{s-1-\gamma} + |y-z|^{s-1-\gamma} \right) |(\eta f)(z)| dz \\ &\leq C_{s,\gamma} |x-y|^{\gamma - \frac{1}{p}} \int_{\mathbb{R}} |x-y|^{\frac{1}{p}} |x-z|^{s-1-\gamma} |(\eta f)(z)| dz, \end{aligned}$$

since $\frac{|y-z|}{|x-z|} > 1$. Using the fact that the support of η is always contained in the ball of radius

2 centred at $x \in B_r$, we have that

$$\begin{aligned}
|v(x)-v(y)| &\leq C(1,s,\gamma)|x-y|^{\gamma}\int_{\mathbb{R}}|x-y|^{s-1-\gamma+\frac{1}{p}}|(\eta f)(y)|dy,\\
&\leq C(s,\gamma)|x-y|^{\gamma-\frac{1}{p}}\left(\int_{\mathbb{R}} w^{-1}\,|x-y|^{p(s-1-\gamma)+1}\,w\,|(\eta f)(y)|^p\,dy\right)^{1/p}\\
&\leq \frac{C(s,\gamma)}{|x-y|^{-\gamma+\frac{1}{p}}}\left(\int_{B(x,2)} w\,|(\eta f)(y)|^p\,dy\right)^{\frac{1}{p}}\left(\int_{B(x,2)}\frac{w^{-\frac{1}{p-1}}}{|x-y|^{-q}}dy\right)^{\frac{p-1}{p}}\\
&\leq \frac{C(s,\gamma)}{|x-y|^{-\gamma+\frac{1}{p}}}\left(\int_{B_1} w\,|f(y)|^p\,dy\right)^{\frac{1}{p}}\left(\int_{B_1}\frac{w^{-\frac{1}{p-1}}}{|x-y|^{-q}}dy\right)^{\frac{p-1}{p}}\\
&\leq 2C(s,\gamma)|x-y|^{\gamma-\frac{1}{p}}\,\|f\|_{L^p(B_1,w)}\left(\int_{B_1} w^{-\frac{1}{p-1}}\,|y|^{\frac{p(s-1-\gamma)+1}{p-1}}\,dy\right)^{\frac{p-1}{p}},
\end{aligned}$$

up to relabelling of the positive constant $C(s,\gamma)$ that depends on s and γ. We have defined $q=\frac{p(s-1-\gamma)+1}{p-1}$.

Replacing $w(y)$ by its value $|y|^{1-2s}$ and by the use of the polar coordinates $y = rx$, $r > 0$, we get that

$$\begin{aligned}
\int_{B_1} w(y)^{-\frac{1}{p-1}}\,|y|^{\frac{p(s-1-\gamma)+1}{p-1}}\,dy &:= \int_{S_1}\int_0^1 |r|^{\frac{1}{p-1}(2s+p(s-1-\gamma))}\,dr\,d\sigma(x)\\
&\leq \frac{(p-1)}{p(s-1-\gamma)+p-1)}\,|S_1|\\
&\leq C(p,\gamma,s).
\end{aligned}$$

Then,

$$|v(x)-v(y)| \leq C(p,\gamma,s)|x-y|^{\gamma-\frac{1}{p}}\,\|f\|_{L^p(B_1,w)}\,.$$

Hence, we conclude that

$$\|v\|_{C^{0,\gamma-\frac{1}{p}}(B_r)} \leq C_s\|f\|_{L^p(B_1,w)}, \tag{2.4.5}$$

for every $\gamma \in (0,\min(1,s))$.

Next, by change of variables, the function $\xi := u - v$ satisfies $(\mathscr{D}_{right})^s\xi = 0$ in B_r by (2.4.3). Then, thanks to the derivative estimate, for every $r' \in (0,r)$,

$$\|\nabla\xi\|_{L^p(B_{r'},w)} \leq C_{s,r'}\|\xi\|_{L^p(B_1,w)} \leq C_{s,r'}\left(\|u\|_{L^p(B_r,w)}+\|v\|_{L^p(B_r,w)}\right).$$

The difference function $\xi = u - v$ is smooth in B_1 and is bounded. From this observation, together with (2.4.5), we have that

$$\|u\|_{C^{0,\gamma-\frac{1}{p}}(B_{r'})} = \|\xi+v\|_{C^{0,\gamma}(B_{r'})} \leq C_{s,r'}\left(\|u\|_{L^p(B_r,w)}+\|f\|_{L^p(B_1,w)}\right),$$

for every $\gamma \in (0, \min(1,s))$ with $0 < \gamma - \frac{1}{p} < 1$, as required.

Proof of (b): $f \in L^\infty(B_1)$. The proof in this case is similar to the previous one. We consider as above a smooth cutoff function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta = 1$ on B_r, $\eta = 0$ on $\mathbb{R} \setminus B_1$, and $0 \le \eta \le 1$ on $\mathbb{R}$. Then we consider the Riesz potential as defined in (2.2.4), so that we can estimate the L^∞ norm of v for $s < 1$. Since the kernel $\mathcal{I}_s$ is positive and $\eta \ge 0$ is a smooth function with compact support in B_r, we get

$$\|v\|_{C^{0,\gamma}(\mathbb{R})} \le C_{s,\gamma}\|\eta f\|_{L^\infty(\mathbb{R})} + C\|v\|_{L^\infty(\mathbb{R})} \le C_{s,\gamma,B'}\|f\|_{L^\infty(B)}.$$

Next, by using the inequality stated in (2.4.4), we get that for $\gamma \in (0, \min(s,1))$ and $x, y \in B_r$,

$$\begin{aligned}|v(x) - v(y)| &\le C_{s,\gamma}|x-y|^\gamma \int_{\mathbb{R}} |x-y|^{s-1-\gamma}\eta(y)|f(y)|dy \\ &\le C_{s,\gamma}|x-y|^\gamma \|f\|_{L^\infty(B_1)} \int_{B_2} |x-y|^{s-1-\gamma}dy \\ &\le C_{s,\gamma}\|f\|_{L^\infty(B_1)}|x-y|^\gamma.\end{aligned}$$

Hence, we conclude that

$$\|v\|_{C^{0,\gamma}(B_r)} \le C_s\|f\|_{L^\infty(B_1)}, \tag{2.4.6}$$

for every $\gamma \in (0, \min(1,s))$.

Next, by change of variables, the function $\xi := u - v$ satisfies $(\mathscr{D}_{right})^s\xi = 0$ in B_r by (2.4.3). Therefore, thanks to [149, Corollary 1.13], we have the derivative estimate for every $r' \in (0,r)$:

$$\|\nabla w\|_{L^\infty(B'_r)} \le C_{s,r'}\|w\|_{L^\infty(B_1)} \le C_{s,r'}(\|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}).$$

The difference function $\xi = u - v$ is smooth in B_1 and is bounded. From this, together with (2.4.6), we conclude that

$$\|u\|_{C^{0,\gamma}(B_{r'})} = \|w+v\|_{C^{0,\gamma}(B_{r'})} \le C_{s,r'}\left(\|u\|_{L^\infty(B_r)} + \|f\|_{L^\infty(B_1)}\right),$$

for every $\gamma \in (0, \min(1,s))$. □

Now we are in a position to state and prove our main result on the interior Schauder estimate for the solution function $\mathcal{U}$ on the set $\overline{\mathcal{B}_1^+}$.

Theorem 2.4.3. *Let $s \in (0,1)$ and let $\mathcal{U} \in L^\infty(\mathbb{R}_2^+) \cap H^1(t^\mu; \mathcal{B}_2^+)$ be a weak solution to*

$$\begin{cases}\mathcal{M}_\alpha\mathcal{U} = 0 & \text{in } \mathcal{B}_2^+, \\ \lim_{t\to 0}\mathcal{N}_\alpha\,\mathcal{U}(t,\cdot) = f & \text{on } B_2, \\ \mathcal{U} = 0 & \text{on } \mathbb{R}\setminus\Omega.\end{cases}$$

(a) *For* $1 < p < \infty$*, if* $f \in L^p(B_1, w)$ *and* $\gamma \in (0, \min(1,s))$ *is such that* $0 < s - \gamma - \frac{1}{p} < 1$*, then* $\mathcal{U} \in \mathcal{C}^{0,s-\gamma-\frac{1}{p}}\left(\overline{\mathcal{B}_1^+}, w\right)$*. Moreover,*

$$\| \mathcal{U} \|_{C^{s-\gamma-\frac{1}{p}}(\overline{\mathcal{B}_1^+},w)} \le C\left(\| \mathcal{U} \|_{L^p(\mathbb{R}_2^+,w)} + \| f \|_{L^p(\mathbb{R},w)}\right),$$

where C is a positive constant depending only on s, γ*, and p.*

(b) *If* $f \in L^\infty(B_1)$ *and* $\gamma \in (0, \min(1,s))$*, then* $\mathcal{U} \in C^{s-\gamma}\left(\overline{\mathcal{B}_1^+}\right)$*. Moreover,*

$$\|\mathcal{U}\|_{C^{s-\gamma}\left(\overline{\mathcal{B}_1^+}\right)} \le C\left(\|\mathcal{U}\|_{\mathbb{R}^2_+} + \|f\|_{L^\infty(\mathbb{R})}\right),$$

where C is a positive constant depending only on s and γ*.*

Proof. Again, we present the two parts of the proof separately.

Proof of (a): $f \in L^p(\mathbb{R}, w)$**.** We choose a cut-off function $\eta \in \mathcal{C}_c^\infty(B_2)$ such that $\eta \equiv 1$ on B_2 and $0 \le \eta \le 1$ on $\mathbb{R}$. Let $\overline{v}$ be the unique solution to the equation

$$(\mathscr{D}_{right})^s \overline{v} = \overline{f} \qquad \text{in} \quad \mathbb{R},$$

where $\overline{f} := \eta f$. Making use of the previous result Proposition 2.4.2, we know that $\overline{v} \in C^{s-\gamma-\frac{1}{p}}(\mathbb{R}, w)$ and

$$\|\overline{v}\|_{C^{s-\gamma-\frac{1}{p}}(\mathbb{R},w)} \le C\left(\|\overline{v}\|_{L^p(\mathbb{R},w)} + \left\|\overline{f}\right\|_{L^p(\mathbb{R},w)}\right),$$

where $C > 0$ is a constant that depends only on s, γ, and p.

The next step is to consider the Bernardis–Reyes–Stinga–Torrea extension $\overline{\mathcal{U}}$ of $\overline{v}$ i.e., the function

$$\overline{\mathcal{U}}(t, \cdot) = P_t^s(t, \cdot) \star \overline{v},$$

which satisfies the equations

$$\begin{cases} \mathcal{M}_\alpha \overline{\mathcal{U}}(t,x) = 0 & \text{in } \mathbb{R}^2_+, \\ \lim_{t\to 0} \mathcal{N}_\alpha \overline{\mathcal{U}}(t,x) = (\mathscr{D}_{right})^s \overline{v}(x) = \overline{f}(x) & \text{on } \mathbb{R}. \end{cases}$$

By a change of variables, we have

$$\overline{\mathcal{U}}(t,x) = \left(P_t^s(t,\cdot) \star \overline{v}\right)(x) = \int_{\mathbb{R}} \overline{v}(x - ty) H_s(y) dy, \tag{2.4.7}$$

where

$$H_s(y) = P_t^s(1,y) = \frac{C_{1,s} e^{-1/(4(-y))}}{(-y)^{1+s}} \chi_{(-\infty,0)}(y). \tag{2.4.8}$$

Then, if we set $z_1 = (t_1, x_1), z_2 = (t_2, x_2) \in \overline{\mathbb{R}^2_+}$, we have the estimate

$$|\overline{\mathcal{U}}(z_1) - \overline{\mathcal{U}}(z_2)| \leq |z_1 - z_2|^{s-\gamma-\frac{1}{p}} \parallel \overline{v} \parallel_{C^{s-\gamma-\frac{1}{p}}(\mathbb{R},w)} \int_{\mathbb{R}} \max\{|y|^{s-\gamma-\frac{1}{p}}, 1\} H_s(y) dy,$$
$$\leq C|z_1 - z_2|^{s-\gamma-\frac{1}{p}} \left(\|\overline{v}\|_{L^p(\mathbb{R},w)} + \left\|\overline{f}\right\|_{L^p(\mathbb{R},w)} \right).$$

By direct computation from (2.4.7), and using Theorem 1.5.8, we have:

$$\|\overline{\mathcal{U}}\|_{L^p(\mathbb{R}^2_+,w)} \leq \|\overline{v}\|_{L^p(\mathbb{R},w)} \leq C \left\|\overline{f}\right\|_{L^p(\mathbb{R},w)}.$$

Therefore,

$$\|\overline{\mathcal{U}}\|_{C^{s-\gamma-\frac{1}{p}}\left(\overline{\mathbb{R}^2_+}\right)} \leq C \left(\|\overline{V}\|_{L^\infty(\mathbb{R}^2_+)} + \left\|\overline{f}\right\|_{L^\infty(\mathbb{R})} \right), \tag{2.4.9}$$

for a positive constant $C > 0$ depending only on s, p and γ.

Next we put $\widetilde{\mathcal{U}} = \mathcal{U} - \overline{\mathcal{U}}$, so that $\widetilde{\mathcal{U}}$ satisfies

$$\begin{cases} \mathcal{M}_\alpha \widetilde{\mathcal{U}} = 0 & \text{in } \mathcal{B}_2^+, \\ \lim_{t\to 0} \mathcal{N}_\alpha \widetilde{\mathcal{U}}(t, \cdot) = (1-\eta) f = 0 & \text{on } B_2. \end{cases}$$

Considering the even reflection $\widetilde{Z}$ of $\widetilde{\mathcal{U}}$ in the variable t, as described in [31, Lemma 4.1], we have that

$$\mathcal{M}_s \widetilde{Z} = 0 \text{ in } \mathcal{B}_2.$$

From the definition (2.1.7) of $\left(\mathscr{D}_{right}\right)$, and using [16, Corollary 1.13] or [47, Corollary 1.5], we have that for $x \in B_1$ and $t \in (-1, 1)$ fixed,

$$\left|(\mathscr{D}_{right})\widetilde{Z}(t, x)\right| \leq C \parallel \widetilde{Z}(t, \cdot) \parallel_{L^p(B_2,w)}. \tag{2.4.10}$$

Next, from the fact that

$$(\mathscr{D}_{right})\widetilde{Z} = \widetilde{Z}_{tt} + \frac{\mu}{|t|}\widetilde{Z}_t$$

and from the inequality (2.4.10), we obtain

$$\left|\widetilde{Z}_{tt} + \frac{\mu}{|t|}\widetilde{Z}_t\right| \leq C \parallel \widetilde{Z} \parallel_{L^p(\mathcal{B}_2,w)}.$$

Therefore,

$$\left|\left(|t|^\mu \widetilde{Z}_t\right)_t\right| \leq C|t|^\mu \parallel \widetilde{Z} \parallel_{L^p(\mathcal{B}_2,w)}.$$

Hence,

$$\left|\widetilde{Z}_{tt}\right| \leq C \parallel \widetilde{Z} \parallel_{L^p(\mathcal{B}_2,w)}.$$

For any point $(t, x) \in \overline{\mathcal{B}}_1$, we have, by (2.4.10), that

$$\left|(\mathscr{D}_{right})\widetilde{Z}(t, x)\right| + \left|\widetilde{Z}_{tt}(t, x)\right| \leq C \parallel \widetilde{Z} \parallel_{L^p(\mathcal{B}_2,w)},$$

which implies that

$$\widetilde{Z} \in C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}}_1, w).$$

Thus, we have that $\widetilde{\mathcal{U}} \in C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}_1^+}, w)$ such that

$$\begin{aligned} \| \widetilde{\mathcal{U}} \|_{C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}_1^+}, w)} &\leq C \left(\| \mathcal{U} \|_{L^p(\mathcal{B}_2^+, w)} + \| \overline{v} \|_{L^p(\mathbb{R})} \right) \\ &\leq C \left(\| \mathcal{U} \|_{L^p(\mathcal{B}_2^+, w)} + \| f \|_{L^p(B_2, w)} \right). \end{aligned}$$

We finally obtain

$$\begin{aligned} \| \mathcal{U} \|_{C^{s-\gamma-\frac{1}{p}}(\overline{\mathcal{B}_1^+}, w)} &\leq C \left(\| \widetilde{\mathcal{U}} \|_{C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}_1^+}, w)} + \| \overline{\mathcal{U}} \|_{C^{s-\gamma-\frac{1}{p}}(\overline{\mathbb{R}^2_+}, w)} \right) \\ &\leq C \left(\| \mathcal{U} \|_{L^p(\mathcal{B}_2^+, w)} + \| f \|_{L^p(B_2), w} \right), \end{aligned}$$

since $\widetilde{\mathcal{U}} = \mathcal{U} - \overline{\mathcal{U}}$. This ends the proof of the first case.

Proof of (b): $f \in L^\infty(\mathbb{R})$. The proof here is similar to the first case above. By considering the same cut-off function $\eta \in \mathcal{C}_c^\infty(B_2)$ with $\eta \equiv 1$ on B_2 and $0 \leq \eta \leq 1$ on $\mathbb{R}$, we let $\overline{v}$ be the unique solution to the equation

$$(\mathscr{D}_{right})^s \overline{v} = \overline{f} \qquad \text{in} \quad \mathbb{R},$$

where $\overline{f} := \eta f$. Making use of the previous result of Proposition 2.4.2, we have that $\overline{v} \in C^{s-\gamma}(\mathbb{R})$ and

$$\|\overline{v}\|_{C^{s-\gamma}(\mathbb{R})} \leq C \left(\|\overline{v}\|_{L^\infty(\mathbb{R})} + \left\|\overline{f}\right\|_{L^\infty(\mathbb{R})} \right),$$

where $C > 0$ is a constant that depends only on s and γ.

The next step is to consider the Bernardis–Reyes–Stinga–Torrea extension $\overline{\mathcal{U}}$ of $\overline{v}$ i.e.,

$$\overline{\mathcal{U}}(t, \cdot) = P_t^s(t, \cdot) \star \overline{v},$$

which satisfies the equation

$$\begin{cases} \mathcal{M}_\alpha \overline{\mathcal{U}} = 0 & \text{in } \mathbb{R}^2_+, \\ \lim_{t \to 0} \mathcal{N}_\alpha \overline{\mathcal{U}}(t, x) = (\mathscr{D}_{right})^s \overline{v}(x) = \overline{f}(x) & \text{on } \mathbb{R}. \end{cases}$$

Proceeding as in the previous case, it follows that

$$\|\overline{\mathcal{U}}\|_{C^{s-\gamma}\left(\overline{\mathbb{R}}^2\right)} \leq C \left(\|\overline{\mathcal{U}}\|_{L^\infty(\mathbb{R}^2)} + \left\|\overline{f}\right\|_{L^\infty(\mathbb{R})} \right), \tag{2.4.11}$$

for a positive constant $C > 0$ depending only on s and γ.

Next we put $\widetilde{\mathcal{U}} = \mathcal{U} - \overline{\mathcal{U}}$, so that $\widetilde{\mathcal{U}}$ satisfies

$$\begin{cases} \mathcal{M}_\alpha \widetilde{\mathcal{U}} = 0 & \text{in } \mathcal{B}_2^+, \\ \lim_{t \to 0} \mathcal{N}_\alpha \widetilde{\mathcal{U}}(t, \cdot) = (1 - \eta) f = 0 & \text{on } B_2. \end{cases}$$

We finally obtain

$$\begin{aligned}\| \mathcal{U} \|_{C^{s-\gamma}(\overline{\mathcal{B}_1^+})} &\leq C \left(\| \widetilde{\mathcal{U}} \|_{C^{1-\gamma}(\overline{\mathcal{B}_1^+})} + \| \overline{\mathcal{U}} \|_{C^{s-\gamma}(\overline{\mathbb{R}_+^2})} \right) \\ &\leq C \left(\| \mathcal{U} \|_{L^\infty(\mathcal{B}_2^+)} + \| f \|_{L^\infty(B_2)} \right),\end{aligned}$$

□

2.5 Conclusion

Regularity theorems are an important result in the theory of PDEs, and their fractional counterparts also play a significance role in the study of problems involving nonlocal behaviour. As already observed in various papers [35, 40, 67, 79, 82, 126, 141] in the theory of fractional nonlocal PDEs, it is possible to find the qualitative behaviour of a solution. In this chapter, we prove the existence and uniqueness of weak solution of the fractional elliptic equation with right Marchaud fractional derivative. Furthermore we show that the degenerate elliptic equation with mixed boundary conditions for a problem with fractional Marchaud derivative admits an interior regularity estimate. The current work fits in with some results obtained in the case of fractional Laplacians with Caffareli–Silvestre extensions. This opens the door to a possible approach to treat nonlinear nonlocal problems, of the porous medium type. We stress that the type of regularity result proved herein is only one of many possible versions of regularity theorems.

Chapter 3

Boundary regularity for parabolic problem with nonlocal operators

In this chapter, we study the boundary regularity for the parabolic problem with fractional derivative and general nonlocal stable operators of Lévy type. We start by recalling results on the elliptic problem for the fnonlocal stable operators of Lévy type and present the results for the boundary regularity of the solution to the elliptic problem obtained by Ros-Oton, Serra and Fernández Real. We use them to derive the main original result of this chapter. We show that a solution u of the homogeneous fractional parabolic problem with fractional derivative on a bounded domain Ω fulfils that $u \in C^s(\mathbb{R}^d)$ and that u/δ^s can be extended Hölder continuously up to $\overline{\Omega}$, where $\delta(x) = dist(x, \partial\Omega)$. Our results are similar to the one obtained by Ros-Oton, Serra and Fernández Real under certain conditions.

The results presented in this chapter have been published in [68].

3.1 Introduction and results

This chapter is devoted to the study of the boundary regularity of solutions to the general nonlocal parabolic equation in time and in space of the form

$$\begin{cases} \mathbb{D}_t^\alpha u + Lu &= 0 \quad \text{in } \Omega,\ t > 0, \\ u &= 0 \quad \text{in } \mathbb{R}^d \setminus \Omega,\ t > 0, \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \text{ for } t = 0, \end{cases} \tag{3.1.1}$$

in bounded $C^{1,1}$ domains Ω. Here, $\mathbb{D}_t^\alpha$ is the fractional derivative in the sense of Caputo defined in Chapter 1 which reads as

$$\mathbb{D}_t^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u(\tau)\, d\tau, \tag{3.1.2}$$

for $0 < \alpha < 1$. The general nonlocal operator L, also defined in Chapter 1 is an infinitessimal generator of stable and symmetric Lévy processes. This operator is uniquely determined by a finite measure on the unit sphere S^{d-1}, often referred as the spectral measure of the process. When this measure is absolutely continuous, symmetric stable processes have generators of the form

$$Lu(x) = \int_{\mathbb{R}^d} \big(2u(x) - u(x+y) - u(x-y)\big) \frac{a(y/|y|)}{|y|^{d+2s}}\, dy, \tag{3.1.3}$$

where $s \in (0,1)$ and a is any nonnegative function in $L^1(S^{d-1})$ satisfying $a(\theta) = a(-\theta)$ for $\theta \in S^{d-1}$.

As it has been shown in Chapter 1, the most simple and important case of stable process is the fractional Laplacian, whose spectral measure is constant on S^{d-1}, so that the process is isotropic.

The regularity theory for general operators of the form (3.1.3) has been recently developed by Fernández Real, Ros-Oton and Serra [83, 129, 131]. In the case the $\alpha = 1$, regularity of solutions to parabolic problems like (3.1.1) or related ones have been studied [22, 38, 39, 43, 45, 63, 81, 96, 108, 137, 161]. Still, most of these works deal with the interior regularity of solutions (barely, how regular it is when the points considered are far enough from the boundary, so that it does not interfere), and there are few results concerning the regularity up to the boundary.

Results regarding this regularity are quite new. Actually, the results here presented obtained by Fernández Real, Ros-Oton and Serra [83, 129, 131], are the main motivation of this chapter, and are the ones that will be extended to the fractional nonlocal parabolic problem with fractional derivative. This extension is heavily based on the eigenfuntions of the elliptic part of (3.1.1). This has been observed by the Ros-Oton, Serra and Fernández Real in [83, 131] in the case of the fractional heat equation. Indeed, for the elliptic problem with the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \tag{3.1.4}$$

when $\Omega = B_1$ and $h \equiv 1$, the solution of problem (3.1.4) has a simple explicit expression given by $u(x) = c(1-|x|^2)^s$ in B_1, for some constant $c > 0$ [83, 130]. But this solution is not smooth up to the boundary, and it is only $C^s(\overline{\Omega})$.

This boundary behavior is the same for all solutions, in the sense that any solution to (3.1.4) satisfies

$$-C\delta^s \leq u \leq C\delta^s \text{ in } \Omega, \tag{3.1.5}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Combining this bound with the well known interior regularity estimates, it has been shown that $u \in C^s(\overline{\Omega})$ [130]. Also, for any $h \in L^\infty(\Omega)$, the solution u to the elliptic problem (3.1.4) satisfies that u/δ^s is Hölder continuous up to the boundary, and

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C\|h\|_{L^\infty(\Omega)}, \tag{3.1.6}$$

for some $\alpha > 0$ small.

In the case $\alpha = 1$, (3.1.1) reduces to the fractional heat equation, for which Chen–Kim–Song [161] established sharp two-sided estimates for the heat kernel of in $C^{1,1}$ domains [22]. Their estimates yield in particular a bound of the form (3.1.5) for solutions to (3.1.1) for positive times, and this implies that solutions $u(x,t)$ satisfy $u(\cdot,t) \in C^s(\overline{\Omega})$ for $t > 0$. Nevertheless, the results [22, 161] do not yield any estimate like (3.1.6) developed [83, 130], which says, for $t > 0$, and $\alpha = 1$, solutions to (3.1.1) satisfy (3.1.6) so that for any fixed $t_0 > 0$, solutions $u(x,t)$ satisfy

$$\sup_{t \geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C(t_0) \|u_0\|_{L^2(\Omega)}.$$

This result has been generalised by the same authors [83, 130] using the general nonlocal operator L.

The main idea of the proof is to write the solution of the fractional heat equation in terms of the eigenfunctions, later, check that these eigenfunctions fulfil bounds of the form (3.1.10)-(3.1.9), and deduce the desired result. Next we have to prove first that the eigenfunctions belong to L^∞, with an explicit bound of the form $\|\phi_k\|_{L^\infty} \leq Ck^\sigma$ for some fixed exponent σ, and being ϕ_k the k-th eigenfunction.

Inspired by their techniques, we adapt it to the problem (3.1.1). Barely, we show that the the boundary regularity results of (3.1.1) satisfy the same bound established [83, 130], with constant depending on the parameter α for the fractional derivative.

The main result which says that u is Hölder regular of the form u/δ^s up to the boundary reads as:

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^d$ be any bounded $C^{1,1}$ domain, and $s \in (0,1)$. Let L be any operator of the form* (3.1.3) *and satisfying the ellipticity conditions*

$$\Lambda_1 \leq \inf_{\nu \in S^{n-1}} \int_{S^{d-1}} |\nu \cdot \theta|^{2s} a(\theta) d\theta, \qquad 0 \leq a(\theta) \leq \Lambda_2 \quad \textit{for all } \theta \in S^{d-1}, \tag{3.1.7}$$

where Λ_1 and Λ_2 are positive constants.

Let $u_0 \in L^2(\Omega)$, and let u be the solution to

$$\begin{cases} \mathbb{D}_t^\alpha u + Lu &= 0 & \text{in } \Omega,\ t > 0, \\ u &= 0 & \text{in } \mathbb{R}^d \setminus \Omega,\ t > 0, \\ u(x,0) &= u_0(x) & \text{in } \Omega, \text{ for } t = 0. \end{cases} \tag{3.1.8}$$

Then,

(a) For each $t_0 > 0$,

$$\sup_{t \geq t_0} \left\| \frac{u(\cdot,t)}{\delta^s} \right\|_{C^{s-\varepsilon}(\overline{\Omega})} \leq C_2(t_0) \|u_0\|_{L^2(\Omega)}, \tag{3.1.9}$$

(b) For each $t_0 > 0$,

$$\sup_{t \geq t_0} \|u(\cdot,t)\|_{C^s(\mathbb{R}^d)} \leq C_1(t_0) \|u_0\|_{L^2(\Omega)}. \tag{3.1.10}$$

where $\delta(x) = dist(x, \partial\Omega)$, and for any $\varepsilon > 0$.

The constants C_1 and C_2 depend only on t_0, d, s, α, ε, Ω, and the ellipticity constants Λ_1 and Λ_2. Moreover, these constants C_1 and C_2 blow up as $t_0 \downarrow 0$.

We also provide the following corollary.

Corollary 3.1.2. *The solution to the fractional heat equation for general nonlocal operators* (3.1.8)*, u, where L is of the form* (3.1.3) *and* (3.1.7) *for an initial condition at $t = 0$, $u(x,0) = u_0(x) \in L^2(\Omega)$, and for a $C^{1,1}$ bounded domain Ω, satisfies that*

1. *For each $t_0 > 0$,*
$$\sup_{t \geq t_0} \left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|_{C^s(\mathbb{R}^d)} \leq C_1 \|u_0\|_{L^2(\Omega)},$$
2. *For each $t_0 > 0$,*
$$\sup_{t \geq t_0} \left\| \frac{1}{\delta^s} \frac{\partial u}{\partial t}(\cdot, t) \right\|_{C^{s-\varepsilon}(\overline{\Omega})} \leq C_2 \|u_0\|_{L^2(\Omega)},$$
where $\delta(x) = dist(x, \partial\Omega)$, and for any $\varepsilon > 0$.

The constants C_1 and C_2 depend only on t_0, d, s, α, the ellipticity constants Λ_1 and Λ_2 and Ω.

This chapter is organised as follows: In Section 3.2, we introduce some notations and definitions, some well known results on the existence, regularity of eigenfunctions and asymptotic behavior of eigenvalues which will be needed in this work. We also provide the solution representation of the problem (3.1.1). In Section 3.3, we provide the proof of our main result on boundary regularity stated as Theorem 3.1.1 and Section 3.4 is devoted to the conclusion.

3.2 Preliminaries and known results

In the following we recall some known results on the existense of eigenfunctions for the elliptic problem involving the general nonlocal operator L and give some properties for the fractional derivatives of Caputo sense.

We start by recalling the following nonlocal operator of a Lévy type that has been well defined in Chapter 1 as

$$Lu(x) = \int_{\mathbb{R}^d} \big(2u(x) - u(x+y) - u(x-y)\big) \frac{a(y/|y|)}{|y|^{d+2s}}\, dy,$$

where $s \in (0,1)$ and a is any nonnegative function in $L^1(S^{d-1})$ satisfying $a(\theta) = a(-\theta)$ for $\theta \in S^{d-1}$. When a is radially symmetric (isotropic), we have that the infinitesimal generator L is a multiple of the fractional Laplacian

$$(-\Delta)^s u = c_{s,d} \int_{\mathbb{R}^d} \frac{(u(x) - u(x+y))}{|y|^{d+2s}}\, dy.$$

We provide the following result which on the Gagliardo seminorm associated to the operator L.

Lemma 3.2.1 ([130]). *Let $a \in L^1(S^{d-1})$. Then, the Gagliardo seminorm associated to the operator L in* (3.1.3),

$$[u]_{H^s_L} = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(x+y))^2}{|y|^{d+2s}} a\left(\frac{y}{|y|}\right) dx\, dy, \tag{3.2.1}$$

is equivalent to the Gagliardo seminorm in $H^s(\mathbb{R}^d)$,

$$[u]_{H^s} = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(x+y))^2}{|y|^{d+2s}} dx\, dy. \tag{3.2.2}$$

Proof. To prove the above equivalence, we can notice that, norms associated to the operator L are equivalent to norms associated to the fractional Laplacian and also,

$$[u]_{H^s_L} = \langle Lu, u\rangle, \tag{3.2.3}$$

for the standard inner product in $L^2(\mathbb{R}^d)$. This can be seen writing explicitly the scalar product and Lu in its nonlocal form. Now, we use Plancherel's theorem, thus obtaining

$$\langle Lu, u\rangle = \langle \widehat{Lu}, \hat{u}\rangle = \int_{\mathbb{R}^d} A(\xi)\hat{u}^2(\xi) d\xi. \tag{3.2.4}$$

Next, using the bounds on $A(\xi)$ and $A_{(-\Delta)^s}(\xi) = |\xi|^{2s}$ we have

$$\mu_1 \langle \widehat{(-\Delta)^s u}, \hat{u}\rangle \leq \langle \widehat{Lu}, \hat{u}\rangle \leq \mu_2 \langle \widehat{(-\Delta)^s u}, \hat{u}\rangle, \tag{3.2.5}$$

and using Plancherel's theorem again, the desired result is obtained. □

This equivalence, in the Fourier side, can be written as

$$[u]^2_{H^s_L} = \int_{\mathbb{R}^d} A(\xi)\hat{u}^2(\xi)\, d\xi. \tag{3.2.6}$$

Proposition 3.2.2 ([83]). *Let $\Omega \subset \mathbb{R}^d$ be any bounded domain, $s \in (0,1)$, $g \in L^2(\Omega)$, and u be the weak solution of*

$$\left\{ \begin{array}{rcll} Lu & = & g & \text{in } \Omega, \\ u & = & 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{array} \right. \tag{3.2.7}$$

where L is a nonlocal operator of the form (3.1.3) *and* (3.1.7). *Assume in addition that $g \in L^p(\Omega)$. Then,*

(a) If $1 < p < \frac{d}{2s}$,

$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad q = \frac{dp}{d-2ps},$$

where C is a constant depending only on d, s, p and Λ_2.

(b) If $p = \frac{d}{2s}$,

$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad \forall q < \infty,$$

where C is a constant depending only on d, s, q, Ω and Λ_2.

(c) If $\frac{d}{2s} < p < \infty$,

$$\|u\|_{L^\infty(\Omega)} \le C\|g\|_{L^p(\Omega)},$$

where C is a constant depending only on d, s, p, Ω and Λ_2.

The proof of the above proposition relies on the use of the results of [91] and compare the fundamental solution associated to the operator L with the one of the fractional Laplacian $(-\Delta)^s$.

For the fractional derivative in the Caputo sense $\mathbb{D}_t^\alpha$, see (3.1.2) also Chapter 1 for some properties related to this operator. Here we are interested in the solvability of the fractional initial value problem involving the Caputo fractional derivative $\mathbb{D}_t^\alpha$. To this end, we consider the following problem

$$\mathbb{D}_t^\alpha q(t) = \lambda q(t), \qquad t > 0. \tag{3.2.8}$$

The solvability of this fractional differential equation, is done through the Laplace transform [99]. By applying the Laplace transform in both sides of (3.2.8), we get

$$\widetilde{q}(\omega) = \frac{\omega^\alpha}{\omega^\alpha + \lambda} q(0). \tag{3.2.9}$$

The inverse Laplace transform of $\widetilde{q}(\omega)$ is given by the well known Mittag-Leffler function with two parameters defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \ \alpha > 0, \ \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{3.2.10}$$

It is clear that $E_{1,1}(z) = e^z$ and that $E_{\alpha,\beta}(z)$ is an entire function. The following estimate of $E_{\alpha,\beta}(z)$ will be useful. Let $0 < \alpha \le 1$, $\beta \in \mathbb{R}$ and μ be such that $\frac{\alpha\pi}{2} < \mu < \min\{\pi, \alpha\pi\}$. Then there is a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1 + |z|}, \ \ \mu \le |\arg(z)| \le \pi. \tag{3.2.11}$$

In the literature, frequently the notation $E_\alpha := E_{\alpha,1}$ is used.

The Laplace transform of the Mittag-Leffler function is given by:

$$\int_0^\infty e^{-\omega t} t^{\alpha+\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) dt = \frac{\omega^{\alpha-\beta}}{(\omega^\alpha + \lambda)^2}, \quad \mathrm{Re}(\omega) > |\lambda|^{\frac{1}{\alpha}}. \tag{3.2.12}$$

If $\lambda > 0$, then

$$\frac{d}{dt}\Big(E_{\alpha,1}(-\lambda t^\alpha)\Big) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \ \ t > 0. \tag{3.2.13}$$

The proofs of (3.2.12) is contained [99].

Using the above properties of the Mittag-Leffler functions, the inverse Laplace transform of (3.2.9) reads as

$$q(t) = E_{\alpha,1}\left(-\lambda_k t^\alpha\right) q(0),$$

where $E_{\alpha,1}$ is the Mittag-Leffler function defined in (3.2.10).

To show the regularity of strong solutions of (3.1.1), we shall frequently use the following estimates that follow from (3.2.11) and a straightforward computation.

Lemma 3.2.3 ([100]). *Let $0 < \alpha \leq 1$, $0 \leq \gamma < 1$, $\lambda > 0$ and $\beta > 0$ be real numbers. Then the following assertions hold.*

1. *There is a constant $C > 0$, such that for every $t > 0$,*

$$\left|\lambda^{1-\gamma} t^{\alpha-1} E_{\alpha,\beta}(-\lambda t^{\alpha})\right| \leq C t^{\alpha\gamma-1}. \tag{3.2.14}$$

2. *There is a constant $C > 0$, such that for every $t > 0$,*

$$\left|\lambda^{1-\gamma} E_{\alpha,1}(-\lambda t^{\alpha})\right| \leq C t^{\alpha(\gamma-1)}. \tag{3.2.15}$$

For more details on fractional order derivatives, integrals and properties for the Mittag-Leffler functions [15].

3.2.1 Existence of eigenfunctions and asymptotic behavior of eigenvalues

In this section, we briefly recall results collected from [83, 130], on the existence of eigenfunctions and asymptotic behavior of eigenvalues, proved for general stable operators with any spectral measure $a \in L^1(S^{d-1})$. We then consider the elliptic problem with the general nonlocal operator of Lévy type L as:

$$\begin{cases} L\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{3.2.16}$$

Proposition 3.2.4 ([83]). *Let $\Omega \subset \mathbb{R}^d$ be any bounded domain, and L an operator of the form* (3.1.3)*, with $a \in L^1(S^{d-1})$. Then,*

(a) There exist a sequence of eigenfunctions forming a Hilbert basis of L^2.

(b) If $\{\lambda_k\}_{k\in\mathbb{N}}$ is the sequence of eigenvalues associated to the eigenfunctions of L in increasing order, then

$$\lim_{k\to\infty} \lambda_k k^{-\frac{2s}{d}} = C_0,$$

for some constant C_0 depending only on d, s, Ω and L. Moreover,

$$C(\mu_2) \leq C_0 \leq C(\mu_1),$$

for $C(\mu_1)$ and $C(\mu_2)$ positive constants depending only on d, s, Ω and μ_1 and μ_2 respectively; where $\mu_1 > 0$ and $\mu_2 > 0$ are given by the expressions

$$\mu_2 = \int_{S^{d-1}} a(\theta)d\theta, \qquad \mu_1 = \inf_{\nu\in S^{d-1}} \int_{S^{d-1}} |\nu\cdot\theta|^{2s} a(\theta)d\theta. \tag{3.2.17}$$

For general stable operators L, the asymptotic behavior of its eigenvalues follows from the following result by Geisinger [86].

Given an operator L, we denote by $A(\xi)$ its Fourier symbol (i.e., $\widehat{Lu}(\xi) = A(\xi)\hat{u}(\xi)$). For the fractional Laplacian, the symbol $A(\xi)$ is given as $A(\xi) = |\xi|^{2s}$. The result by Geisinger, introduce two main hypotheses on A [86]. Namely,

1. There is a function $A_0 : \mathbb{R}^d \to \mathbb{R}$ with the following three properties. A_0 is homogeneous of degree $\alpha > 0$: $A_0(\nu\xi) = \nu^\alpha A_0(\xi)$ for $\xi \in \mathbb{R}^d$ and $\nu > 0$. The set of $\xi \in \mathbb{R}^d$ with $A_0(\xi) < 1$ has finite Lebesgue-measure, and the function A_0 fulfils

$$\lim_{\nu\to\infty} \nu^{-\alpha} A(\nu\xi) = A_0(\xi),$$

with uniform convergence in ξ.

2. There are constants $C_* > 0$ and $M \in \mathbb{N}$ such that for all $\eta \in \mathbb{R}^d$,

$$\sup_{\xi\in\mathbb{R}^d} \left(\frac{1}{2}(A(\xi+\eta) + A(\xi-\eta)) - A(\xi) \right) \leq C_*(1+|\eta|)^M.$$

Under these two conditions, we recall the following theorem.

Theorem 3.2.5 ([86])**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite volume and assume that A is the symbol of a differential operator L that satisfies the previous two conditions. Then,*

$$\lim_{k\to\infty} \lambda_k k^{-\frac{2s}{d}} = C_{L,\Omega},$$

where

$$C_{L,\Omega} = (2\pi)^{2s} |\Omega|^{-2s/d} V_L^{-2s/d},$$

and

$$V_L = \left| \left\{ \xi \in \mathbb{R}^d : A_0(\xi) < 1 \right\} \right|.$$

To prove Proposition 3.2.4, we only have to check that any stable operator (3.1.3) fulfils the previous conditions. To do so, we use that the Fourier symbol $A(\xi)$ of L can be explicitly written in terms of s and the spectral measure a, as

$$A(\xi) = \int_{S^{d-1}} |\xi\cdot\theta|^{2s} a(\theta) d\theta;$$

see [135]. Using that expression, it is clear that

$$0 < \mu_1 |\xi|^{2s} \leq A(\xi) \leq \mu_2 |\xi|^{2s},$$

where μ_1 and μ_2 are given by (3.2.17). By means of Lemma 3.2.1, we have the following proof of Proposition 3.2.4 developed in [83], for which we recall below for the brevity of the reader.

Proof of Proposition 3.2.4 (see [83]). The first statement follows by seeing that the norms H^s and H^s_L are equivalent in $\mathbb{R}^d$ (see, Lemma 3.2.1). For the second statement we use Theorem 3.2.5.

Let A be the Fourier symbol of the stable operator L. We need to check the two conditions of the theorem by Geisinger. The first condition holds taking $A_0 = A$, since it is homogeneous, and we have the previous bounds.

For the second condition, we claim that, for any $a \geq b \geq 0$ and $s \in (0,1)$, we have

$$2a^{2s} + 2b^{2s} \geq (a+b)^{2s} + (a-b)^{2s}. \tag{3.2.18}$$

For, since $s \in (0,1)$, by concavity we have

$$\begin{aligned}(a+b)^{2s} + (a-b)^{2s} &= (a^2+b^2+2ab)^s + (a^2+b^2-2ab)^s \\ &\leq 2(a^2+b^2)^s \\ &\leq 2(a^{2s}+b^{2s}),\end{aligned}$$

hence the claim. Thus, using (3.2.18), we have that

$$|\xi\cdot\theta + \eta\cdot\theta|^{2s} + |\xi\cdot\theta - \eta\cdot\theta|^{2s} \leq 2|\xi\cdot\theta|^{2s} + 2|\eta\cdot\theta|^{2s},$$

and therefore,

$$\begin{aligned}&A(\xi+\eta) + A(\xi-\eta) - 2A(\xi) \\ &\quad= \int_{S^{d-1}} \left\{ |\xi\cdot\theta+\eta\cdot\theta|^{2s} + |\xi\cdot\theta-\eta\cdot\theta|^{2s} - 2|\xi\cdot\theta|^{2s} \right\} a(\theta)d\theta \\ &\qquad\qquad \leq 2\int_{S^{d-1}} |\eta\cdot\theta|^{2s} a(\theta)d\theta \leq 2|\eta|^{2s}\mu_2,\end{aligned}$$

which is the desired result.

Finally, since it has been shown that

$$C_0 = (2\pi)^{2s}|\Omega|^{-\frac{2s}{d}} \left| \left\{ \xi \in \mathbb{R}^d : A(\xi) < 1 \right\} \right|^{-\frac{2s}{d}},$$

and using the bounds on A, then we can obtain

$$(2\pi)^{2s}|\Omega|^{-\frac{2s}{d}} V_d(\mu_2^{-\frac{1}{2s}}) \leq C_0 \leq (2\pi)^{2s}|\Omega|^{-\frac{2s}{d}} V_d(\mu_1^{-\frac{1}{2s}}),$$

where $V_d(R)$ denotes the volume of an d-ball with radius R.

□

3.2.2 Regularity of eigenfunctions

The aim of this section is to recall the regularity conditions up to the boundary on the eigenfunctions discussed [83]

$$\begin{cases} L\phi = \lambda\phi & \text{in } \Omega \\ \phi = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{3.2.19}$$

We recall that, by construction, the eigenfunctions belong to $L^2(\Omega)$. The proof shows that they are, in $L^\infty(\Omega)$. This enable to obtain a bound for its L^∞ norm. This bound is then obtained in terms of the eigenvalue λ of the eigenfunctions, since this expression will be needed to prove Theorem 3.1.1. We recall the known result on regularity of eigenfunctions in the following proposition and the posterior corollary, where, the eigenfunctions in $C^{1,1}$ bounded domains have C^s regularity in R^d, and the ratio of the function and the term δ^s is Hölder continuous up to the boundary.

Proposition 3.2.6 ([83]). *Let $\Omega \subset \mathbb{R}^d$ be any bounded domain, $s \in (0,1)$, and L an operator of the form* (3.1.3)-(3.1.7). *Let ϕ be any solution to the eigenvale problem* (3.2.19). *Then, $\phi \in L^\infty(\Omega)$, and*

$$\|\phi\|_{L^\infty(\Omega)} \leq C\lambda^{w-1}\|\phi\|_{L^2(\Omega)},$$

for some constant C depending only on d, s, Ω and the ellipticity constant Λ_2, and some $w \in \mathbb{N}$ depending only on d and s.

Proposition 3.2.6 follows from Proposition 3.2.2 and a bootstrap argument. Combining Proposition 3.2.6 with the boundary regularity results in [131], the following corollary is obtained.

Corollary 3.2.7 ([83]). *Let $\Omega \subset \mathbb{R}^d$ be any bounded $C^{1,1}$ domain, $s \in (0,1)$, and L an operator of the form* (3.1.3)-(3.1.7).

Let ϕ be any solution to (3.2.19). *Then, $\phi \in C^s(\mathbb{R}^d)$ and $\phi/\delta^s \in C^{s-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$, with the estimates*

$$\|\phi\|_{C^s(\mathbb{R}^d)} \leq C\lambda^w\|\phi\|_{L^2(\Omega)},$$
$$\|\phi/\delta^s\|_{C^{s-\varepsilon}(\overline{\Omega})} \leq C\lambda^w\|\phi\|_{L^2(\Omega)}.$$

The constant C depends only on d, s, Ω, ε and the ellipticity constants Λ_1 and Λ_2, and $w \in \mathbb{N}$, depends only on d and s.

3.2.3 Solution representation

We have recall from [83] the well known resuls on the regularity of eigenfunctions for the general nonlocal operator L. Let us use them to find explicit expression for the solutions of the equation (3.1.1) in terms of these eigenfunctions. This is achieved through a procedure which is similar to the one usually performed for the fractional ordinary differential equation: separation of variables. We proceed to prove the following proposition, refering to the solution representation, its uniqueness and a first step towards its regularity.

Proposition 3.2.8. *Consider the fractional nonlocal problem* (3.1.1) *with general nonlocal operator L for an initial condition at $t = 0$, $u(x,0) = u_0(x) \in L^2(\Omega)$, being u_k its coefficients in the orthonormal basis $\{\phi_k\}_k$, and for a bounded domain Ω. Then*

1. *The solution of the problem* (3.1.1) *is of the form*

$$u(x,t) = \sum_{k>0} u_k\phi_k(x)E_{\alpha,1}[-\lambda_k t^\alpha].$$

2. *$u(\cdot,t) \to u_0(x)$ as $t \downarrow 0$ in $L^2(\Omega)$ norm.*

Proof. We make the proof of this proposition in two steps:

Step I. We proceed assuming $u(x,t) = v(x)q(t)$, with $v(x) = 0$, $x \in \mathbb{R}^d \setminus \Omega$. The equation (3.1.1) becomes,

$$v(x)\mathbb{D}_t^\alpha q(t) + q(t)\, Lv(x) = 0.$$

This implies

$$\begin{cases} L\,v(x) &= \lambda v(x) \quad \text{in } \Omega, \\ \mathbb{D}_t^\alpha q(t) &= -\lambda q(t) \quad \text{for } t > 0, \end{cases} \tag{3.2.20}$$

for some constant $\lambda \in \mathbb{R}$. The first expression corresponds to the known eigenvalues-eigenfunctions problem for the general nonlocal operator L in Proposition 3.2.4. The only solutions are for $v(x) = \phi_k(x)$ and $\lambda = \lambda_k > 0$ for some k, being $\phi_k(x)$ the only eigenfunctions of L, and λ_k the corresponding eigenvalues. The second expression for (3.2.20) is a fractional differential equation given in (3.2.9), which is solved by means of the Laplace transform. So that

$$q(t) = E_{\alpha,1}\left(-\lambda_k t^\alpha\right) q(0),$$

where $E_{\alpha,1}$ is the Mittag-Leffler function defined in (3.2.10). A general solution will be written as linear combinations of all the possibilities

$$u(x,t) = \sum_{k>0} b_k \phi_k(x) E_{\alpha,1}\left(-\lambda_k t^\alpha\right).$$

The coefficients are found thanks to the initial value condition. For $t = 0$, $u(x,0) = u_0(x) \in L^2(\Omega)$ is simply the linear combination with b_k coefficients and the corresponding eigenfunctions, which will be normalized to unitary norm in $L^2(\Omega)$. Thanks to the spectral theorem we already know the eigenfunctions are a basis of $L^2(\Omega)$. In this way, if $u_0(x) \in L^2(\Omega)$, we can express it as linear combinations of the eigenfunctions,

$$u_0 = \sum_{k=1}^{\infty} u_k \phi_k(x), \quad \|\phi\|_{L^2(\Omega)} = 1.$$

Finally, in all, the solution is given as

$$u(x,t) = \sum_{k>0} u_k \phi_k(x) E_{\alpha,1}\left(-\lambda_k t^\alpha\right). \tag{3.2.21}$$

Next from the complete monotonicity and the convergence properties of the Mittag-Leffler function, the solution (3.2.21) converges in L^2 for all $t \geq 0$. Since for $t = 0$ does,

$$\|u(x,t)\|_{L^2(\Omega)} = \left\| \sum_{k>0} u_k \phi_k(x) E_{\alpha,1}\left(-\lambda_k t^\alpha\right) \right\|_{L^2(\Omega)} = \sum_{k>0} u_k^2 \left| E_{\alpha,1}\left(-\lambda_k t^\alpha\right) \right|^2.$$

Step II. $u(x,t)$ is a solution by construction. We use Parseval's identity,

$$\begin{aligned} \|u(x,t) - u_0\|_{L^2(\Omega)}^2 &= \left\| \sum_{k>0} u_k \phi_k(x) E_{\alpha,1}\left(-\lambda_k t^\alpha\right) \right\|_{L^2(\Omega)}^2 \\ &= \sum_{k>0} u_k^2 \left| E_{\alpha,1}\left(-\lambda_k t^\alpha\right) - 1 \right|^2. \end{aligned}$$

Notice that each of the last term of the convergence series converges

$$u_k^2 \left| E_{\alpha,1}\left(-\lambda_k t^\alpha\right) - 1\right|^2 < u_k^2, \text{ since } \sum_{k>0} u_k^2 < \infty.$$

Therefore $\|u(x,t) - u_0\|^2_{L^2(\Omega)} \to 0$ as $t \downarrow 0$.

Next consider two functions u_1, u_2, both solutions of the problem (3.1.1) and let $Z := u_1 - u_2$; then Z is a solution of the system

$$\begin{cases} \mathbb{D}_t^\alpha Z + LZ &= 0 \quad \text{in } \Omega,\ t > 0, \\ Z &= 0 \quad \text{in } \mathbb{R}^d \setminus \Omega,\ t > 0, \\ Z(x,0) &= 0 \quad \text{in } \Omega,\ \text{for } t = 0, \end{cases} \tag{3.2.22}$$

This equation can be rewritten as the following Cauchy problem

$$\begin{cases} \mathbb{D}_t^\alpha Z + LZ &= 0 \quad \text{in } \Omega,\ t > 0, \\ \\ Z(x,0) &= 0 \quad \text{in } \Omega,\ \text{for } t = 0. \end{cases} \tag{3.2.23}$$

Thus, the unique strong solution of (3.2.22) is given by $Z = 0$. Hence, $u_1 = u_2$ and we have shown uniqueness. □

Corollary 3.2.9. *The solution u to the fractional nonlocal problem* (3.1.1) *with general nonlocal operator L, for an initial condition at $t = 0$, $u(x,0) = u_0(x) \in L^2(\Omega)$, and for $C^{1,1}$ bounded domain Ω, satisfying that for any fixed $x_0 \in \mathbb{R}^d$, $u(x_0,\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+)$.*

Proof. As it has been shown previously, the expression of the solution representation $u(x_0,t) = \sum_{k>0} u_k\phi_k(x_0)E_{\alpha,1}\left(-\lambda_k t^\alpha\right)$ converges uniformly over any compact subset $K \subset \mathbb{R}^+$, for $w = 0$. Since $\phi \in \mathcal{C}^s(\mathbb{R}^d)$ and the continuity of every terms, so it is $u(x_0,t)$. Next we compute the derivative of u as

$$\frac{\partial u}{\partial t}(x_0,t) = -\sum_{k>0} u_k\phi_k(x_0)\lambda_k t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right).$$

For $0 \le \gamma < 1$, we recall from Lemma 3.2.3

$$\left|\lambda_k^{1-\gamma} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha)\right| \le C t^{\alpha\gamma-1},$$

which converges uniformly as seen before, for $w = 1$, repeat the iterative process, to see that $u(x_0,\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+)$. □

3.3 Boundary regularity results : proof of Theorem 3.1.1

This section is devoted to the proof of the main result on boundary regularity of (3.1.1) given in Theorem 3.1.1. The proof is based of the well known results regularity of eifenfunction for the elliptic problem (3.2.19) (results collected from [83, 130]) in Section 3.2.2, and from Proposition 3.2.8. After that, we will also prove Corollary 3.1.2 which provide the regularity up to the boundary of the solution $u(x,t)$ in the t-variable.

Proof of Theorem 3.1.1. we start with first claim.

• **Claim (a):** First, by Proposition 3.2.4, which ensures the existence of sequences of eigenfunctions of L, and from Proposition 3.2.8, the solution u of (3.1.1) can be expressed as

$$u(x,t) = \sum_{k>0} u_k \phi_k(x) E_{\alpha,1}\left(-\lambda_k t^\alpha\right).$$

We recall here that the initial condition $u_0 = \sum_{k>0} u_k \phi_k$ and ϕ_k denote the unitary eigenfunctions of the Dirichlet elliptic problem (3.2.19), and λ_k the corresponding eigenvalues, in increasing order. Proposition 3.2.4 ensures that the solution will be always in $L^2(\Omega)$, due to the decreasing in time property of the L^2 norm. We can then try to bound $\|u(\cdot,t)/\delta^s\|_{C^{s-\varepsilon}(\overline{\Omega})}$, through the expression found in Corollary 3.2.7, and noticing that the sequence $|u_k|$ has a maximum which satisfies $\max_{k>0}|u_k| \leq \|u_0\|_{L^2(\Omega)}$,

$$\begin{aligned}
\left\| \frac{u(\cdot,t)}{\delta^s} \Big(\right\|_{C^{s-\varepsilon}(\overline{\Omega})} &= \left\| \sum_{k>0} u_k \frac{\phi_k}{\delta^s} E_{\alpha,1}\left(-\lambda_k t^\alpha\right) \right\|_{C^{s-\varepsilon}(\overline{\Omega})} \\
&\leq \sum_{k>0} |u_k| \left\| \frac{\phi_k}{\delta^s} \right\|_{C^{s-\varepsilon}(\overline{\Omega})} \left| E_{\alpha,1}\left(-\lambda_k t^\alpha\right) \right| \\
&\leq \|u_0\|_{L^2(\Omega)} \sum_{k>0} C_{\lambda_k} \|\phi_k\|_{L^2(\Omega)} \lambda_k^{\gamma-1} \left| \lambda_k^{1-\gamma} E_{\alpha,1}\left(-\lambda_k t^\alpha\right) \right| \\
&= \|u_0\|_{L^2(\Omega)} C \sum_{k>0} \lambda_k^{w+\gamma-1} t^{\alpha(\gamma-1)},
\end{aligned}$$

where we applied in the last line the estimate of the Mittag-Leffler function given in (3.2.15). We have bounded $\|u(\cdot,t)/\delta^s\|_{C^{s-\varepsilon}(\overline{\Omega})}$ by an expression decreasing with time, and where C depends only on d, s, Ω, α, ε, Λ_1 and Λ_2. We need only to consider $\|u(\cdot,t_0)/\delta^s\|_{C^{s-\varepsilon}(\overline{\Omega})}$ and bound it using the previous expression.

It is enough to prove that the series $\sum_{k>0} \lambda_k^{w+\gamma-1} t_0^{\alpha(\gamma-1)}$ converges, considering the asymptotics previously introduced Proposition 3.2.4. This leads us to study the convergence of the tail.

There exists k_0 such that

$$\begin{aligned}
\sum_{k>0} \lambda_k^{w+\gamma-1} t_0^{\alpha(\gamma-1)} &< \sum_{k\geq k_0} \left(\frac{3}{2} C_0 k^{\frac{2s}{d}}\right)^{w+\gamma-1} t_0^{\alpha(\gamma-1)} \\
&= \left(\frac{3}{2} C_0\right)^{w+\gamma-1} t_0^{\alpha(\gamma-1)} \sum_{k\geq k_0} k^{\frac{2s}{d}(w+\gamma-1)} < \infty,
\end{aligned}$$

where it has been used the notation and the results from Proposition 3.2.4. Hence for fixed $t_0 > 0$,

$$\|u(\cdot,t_0)/\delta^s\|_{C^{s-\varepsilon}(\overline{\Omega})} \leq C(t_0)\|u_0\|_{L^2(\Omega)},$$

where $C(t_0)$ depends only on d, s, Ω, α, the ellipticity constants Λ_1 and Λ_2 and t_0.

• **Claim (b):** To prove the claim (b), we use the same argument as used in **claim (a)**, but now we consider

$$\|u(\cdot,t)\|_{C^s(\mathbb{R}^d)} = \left\|\sum_{k>0} u_k\phi_k(x)E_{\alpha,1}\left(-\lambda_k t^\alpha\right)\right\|_{C^s(\mathbb{R}^d},$$

and follow the same procedure, as the previous one, to see that $\|u(\cdot,t)\|_{C^s(\mathbb{R}^d)}$ is bounded by an expression equivalent to the one found for **Claim (a)**. □

Proof of Corollary 3.1.2. To prove this corollary we can simply use the same argument used in the proof of Theorem 3.1.1, but now considering the expression of $\frac{\partial u}{\partial t}(\cdot,t)$ obtained computing the derivative of the solution representation (3.2.21) as:

$$\frac{\partial u}{\partial t}(\cdot,t) = -\sum_{k>0} u_k\phi_k(x)\lambda_k t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right),$$

since the series is uniformly convergent with respect to time over compact subsets in $\mathbb{R}^+$. So norm of time derivative u over $C^s(\mathbb{R}^d)$ is given as

$$\begin{aligned}
\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{C^s(\mathbb{R}^d)} &= \left\|\sum_{k>0} u_k\phi_k(x)\lambda_k t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right)\right\|_{C^s(\mathbb{R}^d)} \\
&\leq \sum_{k>0} u_k\|\phi_k\|_{C^s(\mathbb{R}^d)}\lambda_k^\gamma\left|\lambda_k^{1-\gamma}t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right)\right| \\
&\leq \sum_{k>0} u_k C_{\lambda_k}\|\phi_k\|_{L^2(\Omega)}\lambda_k^\gamma\left|\lambda_k^{1-\gamma}t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right)\right| \\
&\leq u_0 C\sum_{k>0}\lambda_k^{w+\gamma}\left|\lambda_k^{1-\gamma}t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right)\right| \\
&\leq u_0 C t^{\alpha\gamma-1}\sum_{k>0}\lambda_k^{w+\gamma}.
\end{aligned}$$

From here on, the remaining of the proof is similar to the one of Theorem 3.1.1. One can find that $\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{C^s(\mathbb{R}^d)}$ is bounded by an expression equivalent to the one found. □

3.4 Conclusion

As already observed in various papers [21, 83, 130] and references therein, regularity of the some linear PDEs with general nonlocal operators needs to be investiguated more deeply. In this chapter, we adapted the technique developed [83, 130], to prove boundary regularity for the parabolic problem with time and space nonlocal operators. The current work fits in with some results obtained in the case of fractional Laplacians for the linear fractional heat equation [83] and the ratio u/δ^s is bounded almost in the similar ways for a fixed time t_0. This method can be extended to more general settings, to treat nonlinear nonlocal problems, of the porous medium type.

Chapter 4
Nonlocal fractional porous medium equation: weak solutions and finite speed of propagation

This chapter is devoted to the study of the existence and uniqueness of the general nonlinear nonlocal time evolution equation of porous medium type. And also to the existence of approximate solution and finite speed of propagation of the time porous medium equation with fractional pressure. The initial data is assumed to be a bounded function with compact support and fast decay at infinity. In the latest case, we determine whether the property of compact support is conserved in time, depending on some parameters of the problem. Special attention is paid to the property of finite propagation for specific values of the parameters.

The results presented in this chapter have been published [72, 73].

4.1 Introduction

In this chapter, we study the existence and uniqueness of the general nonlinear nonlocal time evolution equation of porous medium type of the form

$$\partial_t^\alpha u + \mathcal{L}^s \varphi(u) = f.$$

For the general theory, we use signed solutions $u(t,x)$. Actually, the result holds for more general nonlinearities than the power one at almost no extra cost so that we will adopt the more general context and consider the problem where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, smooth and increasing function. We assume moreover that $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$. In this chapter, the leading example of nonlinear nonlocal operators will be $\mathcal{L}^s \varphi(u) := |u|^{m-1}u$ with $m > 1$ and $\mathcal{L}^s \varphi(u) := \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s}(u))$ for $m \geq 2$. Hence the two main models we consider here:

$$\partial_t^\alpha u + (-\Delta)^s |u|^{m-1} u = f, \qquad \text{(TFPME)}$$

and

$$\partial_t^\alpha u(x,t) = \nabla \cdot (u^{m-1} \nabla p) + f, \quad p = (-\Delta)^{-s}(u). \qquad \text{(TPMEFP)}$$

In these two problems (TFPME) and (TPMEFP), both functions u and f are nonnegative. The problem is posed for $x \in \mathbb{R}^d$, $d \geq 1$ and $t > 0$, and we give initial conditions

$$u(0,x) = u_0(x) \quad \text{for } x \in \mathbb{R}^d.$$

The initial data $u_0 : \mathbb{R}^d \to [0,\infty)$ and the forcing term f are bounded with compact support or fast decay at infinity. The pressure p is related to u through a linear fractional potential operator $p = \mathcal{K}(u)$. The pressure p takes into consideration nonlocal effects through the inverse fractional laplacian operator $\mathcal{K} = (-\Delta)^{-s}$, for $0 < s < 1$ with kernel $K(x,y) = c|x-y|^{-(d-2s)}$, the Riesz potential of order $2s$ (see Chapter 1 or [147, 154]), which depends linearly on the density function u according to the Darcy law.

The motivation of studying these two models (TFPME) and (TPMEFP) comes first from their applications in various aspects of real life problems and from their qualitative properties in terms of speed of propagation.

This phenomenon could take place when $u_0 = 0$; then the pressure $p = 0$ and the equation becomes degenerate. This degeneracy result is the phenomenon of the finite speed of propagation. This of course is also related to the notion of free boundary. So we can show that under certain assumptions on the initial data, the free boundary $\Gamma = \partial\{u(x,t) > 0\}$, is a smooth surface when $0 < t < T$, for some $T > 0$. If u_0 is nonnegative, integrable and with compact support, then the Cauchy problem for the (TPMEFP) admits a unique solution on $(-\infty, T) \times \mathbb{R}^d$ which has constant mass. Furthermore, since the equation becomes degenerate when $u_0 = 0$, the solution is not expected to be smooth. The physical interpretation of the equation points out that under ideal conditions, the free boundary should be a smooth surface and the pressure a smooth function up to the interface.

As already observed [146], when $\alpha = 1$ in (TFPME), the speed of propagation is infinite. This can be proved for all $\alpha \in (0,1)$. This is the opposite phenomenon which occurs for the model (TPMEFP). That means, the speed of propagation is finite for all $\alpha, s \in (0,1)$ and for all $m \geq 2$, as we shall see in this chapter.

It turns out that our problem (TPMEFP) has intriguing properties like finite speed of propagation for $m \geq 2$ which was left out [6]. It is important to recall that the model (TPMEFP) was first introduced by Caffarelli and Vázquez [45], when $m = 2$, as a model for nonlinear diffusion of porous medium type with nonlocal diffusion effects and lat.er extended [146]. When $m = 2$, and $\alpha = 1$, in the papers [43–45] the authors have established the properties of finite speed of propagation, a priori estimates for the solutions, C^α regularity, existence of self-similar solutions and asymptotic behaviour. When $\alpha \in (0,1)$ and $m = 2$, additional results on Hölder regularity estimates can be found [3].

We are specially interested in better understanding well-posedness and velocity of propagation when the fractional time derivative which takes into account the “memory” effect of the solution is involved. We point out that some materials have the so-called thermal memory [102, 111]. On the contrary, when the permeability of the medium changes over time such as in porous medium equation, it might be interesting to use a fractional time derivative.

In the model (TPMEFP), we have chosen the Caputo fractional derivative, first because of its well established theory and moreover for its resemblance with the Marchaud derivative and fractional Laplacian [3–5, 71]. Indeed several different classes of fractional differintegral operators have been defined: from the classical Riemann–Liouville, Caputo, Marchaud, Weyl, and Grünwald–Letnikov formulae [84, 116, 119, 134] to more recently developed models such as Caputo–Fabrizio [51] or Atangana–Baleanu [11], just to cite some of them. Each of these various nonlocal fractional models have properties and applications which the others do not, and they all present different problems and challenges in their analysis [9, 13, 14, 70, 74, 114, 133].

Since the scope of the chapter is based on the well-posedness and finite speed of propagation, we will not discuss about the asymptotic behaviour of the solution which is also of great importance [66]. This chapter combines several compactness techniques, based on the method introduced [44, 45]. The main difficulties of the construction are: the nonlocal and nonlinear character of the equation, absence of comparison principle, as well as absence of explicit self-similar solutions when the fractional derivative is involved.

As one of the novelties of this work, we provide the proof of existence and uniqueness of the solution to (TFPME), establish existence of approximate solution and provide properties of finite speed of propagation for the model (TPMEFP).

Presentation of the main results. The purpose of this work is to analyze completely the existence and uniqueness of solutions to (TFPME), and establish existence of approximate solution and provide properties of finite speed of propagation for the model (TPMEFP). We will present them separately, since these results involve different techniques.

I. Existence and uniqueness of solutions for (TFPME)**.**

The problem of existence with standard nonlinear diffusion goes back [29, 30]. We prove an existence and uniqueness result for the homogeneous Dirichlet problem in the form of (TFPME) [73]. For the sake of simplicity and generality, we rewrite (TFPME) to the following nonlinear evolution problem

$$\begin{cases} \mathscr{A}u(t) + \mathscr{B}(u(t)) \ni f \quad \text{in} \quad \mathscr{H} \quad \text{for a.e.} \quad t \in (0,T), \\ u(0) = u_0, \end{cases}$$

where $\mathscr{B}$ is the subdifferential of a proper, convex and lower semicontinuous function $j{:}\mathscr{H} \to (-\infty, +\infty]$ with compact sublevels in $\mathscr{H}$ satisfying suitable compatibility conditions. The nonlinear and possibly multivalued operator $\mathscr{B}$ acts from $\mathscr{H}$ to $2^{\mathscr{H}}$, the space of all subsets of $\mathscr{H}$. Moreover, u_0 and f are given data.

Our result holds under suitable assumptions on the structure of the linear and nonlinear maximal monotone operators $\mathscr{A}$ and $\mathscr{B}$ respectively. More precisely, in our analysis we suppose that $\mathscr{A}$ is bounded, so that the domain $D(\mathscr{A}) \equiv \mathscr{H}$. Later on, as the next step, we prove that the nonlocal fractional time derivative in the Caputo sense ∂_t^α can also be represented by the maximal monotone operator $\mathscr{A}$. Thus, using the Bénilan–Brezis–Crandall functional semigroup approach [29, 30, 56, 58], a unique solution is constructed, and $u \in \mathcal{C}([0,T] : \mathcal{H}^*(\Omega))$. Here we use the concept of $\mathcal{H}^*$ solution as presented [24].

We note similar approach have been adopted [24, 62, 63] in the case $\alpha = 1$, and [15] when the exponent $s = 1$.

The main result on existence and uniqueness is as follows.

Theorem 4.1.1. *For every $u_0 \in \mathcal{H}^*(\Omega)$ there exists a unique solution*

$$u \in L^2([0,T]; D(\mathscr{B})) \cap \mathcal{H}^{\frac{\alpha}{2}}([0,T]; \mathcal{H}^*(\Omega)),$$

of Problem (TFPME) *for every $T > 0$, which is global in time. We have*

$$t\,\varphi(u) \in \mathrm{L}^\infty(0,T : \mathcal{H}^*(\Omega)), \quad t\,\partial_t^\alpha u \in \mathrm{L}^\infty(0,T : \mathcal{H}^*(\Omega)), \tag{4.1.1}$$

and $u\varphi(u) \in L^1((0,T) \times \Omega))$. The solution map $S_t : u_0 \mapsto u(t)$ defines a semigroup of (non-strict) contractions in $\mathcal{H}^*(\Omega)$, i.e., for two solutions u and v,

$$\|u(t) - v(t)\|_{\mathcal{H}^*(\Omega)} \le \|u(0) - v(0)\|_{\mathcal{H}^*(\Omega)}. \tag{4.1.2}$$

which turns out to be also compact in $\mathcal{H}^*(\Omega)$.

II. Existence of approximate solutions and finite speed of propagation for (TPMEFP).

Next, we study the existence of approximate solutions and finite speed of propagation of the for (TPMEFP). We present those results in two parts:

IIa. Existence of approximate solutions for (TPMEFP).

In [72] we successfully adapted the technique of "true exaggerated supersolution" introduced [44, 45] and used [146]. The main result on the existence of weak solution fully presented in Section 4.4 reads as follows:

Theorem 4.1.2. *Let $0 \le u_0(x) \le Ae^{-a|x|}$ and $0 \le f(t,x) \le Ae^{-|x|}$ both satisfy the exponential bounds decay for some $a, A \ge 0$, and let us assume that $u_0 \in \mathcal{C}^2$. Then, there exists a weak solution u to* (TPMEFP) *in $(0,\infty) \times \mathbb{R}^d$ with right hand side f and $u(0,x) = u_0$.*

IIb. Property of finite speed of propagation for (TPMEFP).

In order to state and prove the result regarding the finite speed of propagation for (TPMEFP), we shall consider the method developed by Caffarelli and Vázquez [45]. A similar method has also been used [146] to establish existence of a class of weak solutions for which the properties of compact support is conserved. This was for the model (TPMEFP) when $\alpha = 1$ only.

In [72], we prove that for $m \in [2,3)$, whenever the parameter $0 < \alpha < 1$ and $0 < s < 1/2$, the solution becomes compactly supported for all $t \ge 0$ for a given compacted initial data u_0. Our quantitative estimate does not involve any control of the L^1 norm and the speed of propagation obtained is influenced by the time parameter α. We would like to point out here that in the limit as $\alpha \nearrow 1$, we recover the speed of propagation known for the case of porous medium equation with long-range interaction and nonlocal potential pressure, that is, the model (TPMEFP) with $\alpha = 1$. Furthermore the solution is bounded for all the time and supported in the complement of the ball $B_{r(t)}$. Full details of the proof are given in Section 4.5. This is summarized in the following.

Theorem 4.1.3. *Let $0 < \alpha < 1$ and $0 < s < 1/2$. Assume u is a bounded weak solution $0 \leq u \leq L$ to* (TPMEFP) *with the nonlocal fractional potential pressure $p = (-\Delta)^{-s}u$. Assume that $u_0(x) = u(x,0)$ has compact support. Then $u(\cdot,t)$ is compactly supported for all $t \geq 0$. In other words, if u_0 is below the parabola like function*

$$\widetilde{u}_0 = a(|x| - b)^2,$$

for some constant $a, b > 0$, with support in the ball $B_r(0)$ then there exists a large enough constant C, such that

$$u(x,t) \leq a\left(Ct - (|x| - b)\right)^2,$$

with the finite speed of propagation

$$C(L,a) = C(1,1)L^{\frac{2m-3+2s}{2\alpha}}\, a^{\frac{1-2s}{\alpha}}.$$

Next, as a consequence of this result the free boundary properties of the positivity set in Q is also obtained. We show that the growth of the support is bounded in finite time. We concentrated on giving only a lower bound for the radius of the ball $R(t)$ where the solution u vanishes. This result is given in Corollary 4.1.4.

Corollary 4.1.4. *Let $u \geq 0$ be a bounded weak solution of* (TPMEFP) *posed in Q and let us assume that it takes initial data given by a bounded function $u_0 \in L^\infty(B_R)$ such that $u_0 \leq L$ as above. Let us also assume that u_0 vanishes a.e. in a ball $B_r(x_0) \subset B_R$. Then, there exists a time t_1 such that for every $0 < t < t_1$ the solution $u(t)$ vanishes at least in a smaller ball $B_{r(t)}(x_0)$ with $0 < r(t) \leq R$, providing $|x| \leq R$ and the function $r(t)$ is monotone non-increasing. In these conditions we obtain an estimate for the free boundary point of the form*

$$|x(t)| \leq R + C_2 t^{\alpha/(2-2s)},$$

if $0 < \alpha < 1$ and $0 < s < 1/2$.

The chapter is organized as follows: In Section 4.2, we give some definitions and properties of the fractional derivative, fractional Laplacian and inverse fractional Laplacian. Next, we provide the weak formulation of the problem and recall some functional inequalities and some useful lemmas. Section 4.3 is devoted to the proof of existence and uniqueness of solutions for the problem (TFPME). Next in Section 4.4, we start with a regularized version of the problem (TPMEFP). Later on, we provide an exponential tail control for $m \geq 2$. We end the section with the proof of existence and uniqueness of solutions stated in Theorem 4.1.2. Section 4.5 is devoted to the proof of finite speed of propagation and free boundary property solution for the model (TPMEFP) given by Theorem 4.1.3.

We recall some notation that will be intensively used throughout the paper:

- α denotes the order the extended Caputo derivative or Marchaud derivative.
- s denotes the order of the spatial fractional operator associated to the fractional Laplacian.
- a stands for the initial time for which our equation is defined.

- d refers to the space dimension.
- Λ_1,Λ_2 denotes the elliptic positive constants which gives the bound of the kernel of the fractional derivative.
- ε refers to the time length of the discrete approximation.
- t,τ denote time variables.
- B_R denotes the ball with radius R.

4.2 Preliminary results

In this section we recall some previous results as well as we prove some new results that will be useful in our further analysis.
The fractional time derivative. Among the different fractional derivatives existing in the literature, in this chapter we consider the extended Caputo or Marchaud derivative [49, 116]. As stated in Chapter 1, the usual Caputo derivative for $0 < \alpha < 1$ is defined by

$$ {}_a^cD_t^\alpha v(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} v'(\tau) d\tau. $$

Using integration by parts and proceed as defined [3, 5, 31, 71, 84], defining $v(t) \equiv v(a)$ for $t < a$, the extended form or the Marchaud derivative is defined as

$$ \partial_t^\alpha v(t,\cdot) = \alpha \int_{-\infty}^t [v(t,\cdot) - v(\tau,\cdot)] \mathscr{K}(t,\tau)\, d\tau. \tag{4.2.1} $$

The kernel $\mathscr{K}$ also satisfies the conditions

$$ \mathscr{K}(t,t-\tau) = \mathscr{K}(t+\tau,t) \qquad \text{and} \qquad \frac{\Lambda_1}{(t-\tau)^{1+\alpha}} \leq \mathscr{K}(t,\tau) \leq \frac{\Lambda_2}{(t-\tau)^{1+\alpha}}. \tag{4.2.2} $$

The formulation (4.2.1) is also known as the Marchaud derivative (see, Chapter 1 or [116, 134]). The reason of working with formulation (4.2.1) is that it allows one to easily utilize the nonlocal nature of the fractional time derivative for regularity purposes. This was succesfully accomplished for divergence problems [5] as well as for nondivergence problems [3, 4].

Proposition 4.2.1. *Let $g,h \in C^1(a,T)$. Then*

$$ \int_a^T u\partial_t^\alpha h + h\partial_t^\alpha u = \int_a^T u(t)h(t) \left[\frac{1}{(T-t)^\alpha} + \frac{1}{(t-a)^\alpha} \right] dt $$
$$ + \alpha \int_a^T \int_a^t \frac{[u(t)-u(\tau)][h(t)-h(\tau)]}{(t-\tau)^{1+\alpha}}\, d\tau\, dt - \int_a^T \frac{u(t)h(a)+h(t)u(a)}{(t-a)^\alpha}\, dt. \tag{4.2.3} $$

We now give the exact formulation of our weak solutions associated (TPMEFP).

We say that u is a weak solution of (TPMEFP) if for any $\zeta \in C_0^\infty(-\infty,T)\times\mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d}\int_{-\infty}^{T}\int_{-\infty}^{t}[u(t,x)-u(\tau,x)][\zeta(t,x)-\zeta(\tau,x)]\mathscr{K}(t,\tau,x)\,d\tau\,dt\,dx$$
$$+\int_{\mathbb{R}^d}\int_{-\infty}^{T}\int_{-\infty}^{2t-T}u(t,x)\zeta(t,x)\mathscr{K}(t,\tau,x)\,d\tau\,dt\,dx-\int_{\mathbb{R}^d}\int_{-\infty}^{T}u(t,x)\partial_t^\alpha\zeta(t,x)\,dt\,dx$$
$$+\int_{-\infty}^{T}\int_{\mathbb{R}^d}\nabla\zeta(t,x)u(t,x)\nabla(-\Delta)^{-s}u\,dx\,dt=\int_{-\infty}^{T}\int_{B_R}f(t,x)\zeta(t,x). \quad (4.2.4)$$

We will also utilize a fractional Sobolev norm that arises from the fractional derivative.

Lemma 4.2.2. *Let u be defined on $[a,T]$. We have for two constants c_1,c_2 depending on $\alpha,|T-a|$*

$$\|u\|_{L^{\frac{2}{1-\alpha}}}(a,T)\le c_1\|u\|^2_{H^{\alpha/2}}(a,T)$$
$$\le c_2\left(\alpha\int_a^T\int_a^t\frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\,d\tau\,dt+\int_a^T\frac{u^2(t)}{(T-a)^\alpha}\right).$$

We point out that if $u=u_+-u_-$ (positive and negative parts, respectively), then

$$\int_a^T u_\pm(t)\partial_t^\alpha u_\mp(t)\ge 0. \quad (4.2.5)$$

Next we recall the following lemma from [5]. For a given convex function F with $F''\ge\alpha$, $F'\ge 0$, $F(0)=0$ and under the assumption that $u\ge 0, u(0)=0$, then there exists a constant c depending on α and Λ such that

$$\varepsilon\sum_{j\le k}F(u(\varepsilon j))\partial_\varepsilon^\alpha u(\varepsilon j)\ge c\varepsilon\sum_{j\le k}\frac{F(u(\varepsilon j))}{(\varepsilon(j-i))^{1+\alpha}}+c\frac{\alpha}{2}\varepsilon^2\sum_{j\le k}\sum_{i<j}\frac{[u(\varepsilon j)-u(\varepsilon i)]^2}{(\varepsilon(j-i))^{1+\alpha}}.$$

In the continuous version, if u is a limit of u_ε, then it follows that

$$\int_a^T F(u(t))\partial_t^\alpha u(t)\,dt\ge c\int_a^T\frac{F(u(t))}{(T-a)^\alpha}+c\frac{\alpha}{2}\int_a^T\int_a^t\frac{[u(t)-u(\tau)]^2}{(t-\tau)^{1+\alpha}}\,d\tau\,dt. \quad (4.2.6)$$

Next we recall the so-called Mainardi function which is a particular Wright function [115]

$$\mathcal{M}_\alpha(z)=\sum_0^{+\infty}\frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)}=\frac{1}{2i\pi}\int_G\lambda^{\alpha-1}e^{(\lambda-z\lambda^\alpha)}d\lambda,\quad 0<\alpha<1, \quad (4.2.7)$$

where G is a contour which starts and ends at $-\infty$ and encircles the origin once clockwise. We also have the following relation between the Wright function and the Mittag-Leffler function [134]:

$$E_\alpha(z)=\int_0^\infty\mathcal{M}_\alpha(t)e^{zt}\,dt,\quad 0<\alpha<1. \quad (4.2.8)$$

The fractional Laplacian and the inverse operator. In this part, we recall some definitions and basic notions for the functional setting of the problem related to the fractional Laplacian and its inverse. All the definitions, lemmas and theorems we recall here are basically borrowed from [125, 146, 154].

Let $\mathcal{F}$ denote the Fourier transform as in Chapter 1. For given $s \in (0,1)$ we consider the space

$$H^s(\mathbb{R}^d) := \left\{ u : L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\},$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^d)} := \|u\|_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi.$$

For functions $u \in H^s(\mathbb{R}^d)$, the fractional Laplacian operator is defined by

$$(-\Delta)^s u(x) = C_{d,s} \, \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)),$$

for $x \in \mathbb{R}^d$, where $C_{d,s} = \pi^{-(2s+d/2)} \Gamma(d/2 + s)/\Gamma(-s)$. Then,

$$\|u\|_{H^s(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)} + C\|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^d)}.$$

The inverse operator $\mathcal{K}_s := (-\Delta)^{-s}$ coincides with the Riesz potential of order $2s$ (see Chapter 1.

Let $\varepsilon > 0$ and $u : \mathbb{R}^d \to \mathbb{R}$. The approximated fractional Laplacian operator is defined by

$$\mathcal{K}^s_\varepsilon[u](x) := C_{d,s} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{(|x - y|^2 + \varepsilon^2)^{\frac{d+2s}{2}}} dy. \tag{4.2.9}$$

For any $\varepsilon > 0$, $\mathcal{K}^s_\varepsilon$ is an integral operator with non-singular kernel and $\mathcal{K}_\varepsilon[u] \to (-\Delta)^s u$ pointwise in $\mathbb{R}^d$ as $\varepsilon \to 0$ for suitable functions u.

Next we define the bilinear form

$$\mathcal{E}_\varepsilon(u, v) = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))\,(v(x) - v(y))}{(|x - y|^2 + \varepsilon^2)^{\frac{d+2s}{2}}} dxdy \quad \text{for } u, v \in D(\mathcal{K}_\varepsilon),$$

and the quadratic form

$$\overline{\mathcal{E}}_\varepsilon(u) := \mathcal{E}_\varepsilon(u, u) = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{[u(x) - u(y)]^2}{(|x - y|^2 + \varepsilon^2)^{\frac{d+2s}{2}}} dxdy.$$

The bilinear form $\mathcal{E}_\varepsilon$ is well defined for functions in the space $\dot{H}^s_\varepsilon(\mathbb{R}^d)$, which is the closure of $C^\infty_c(\mathbb{R}^d)$ with respect to the Gagliardo seminorm given by $\overline{\mathcal{E}}_\varepsilon$. We define

$$H^s_\varepsilon(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \overline{\mathcal{E}}_\varepsilon(u) < \infty \right\}. \tag{4.2.10}$$

The space $H^s_\varepsilon(\mathbb{R}^d)$ is endowed with the standard norm

$$\|u\|^2_{H^s_\varepsilon} = \|u\|^2_{L^2(\mathbb{R}^d)} + \overline{\mathcal{E}}_\varepsilon(u).$$

Furthermore, we recall that

$$H^s(\mathbb{R}^d) \subset H^s_{\varepsilon_1}(\mathbb{R}^d) \subset H^s_{\varepsilon_2}(\mathbb{R}^d) \quad \text{for} \quad 0 < \varepsilon_1 < \varepsilon_2. \tag{4.2.11}$$

We refer to [146] for a precise discussion of these spaces in a more general framework.

Next we state the following generalized Stroock-Varopoulos inequality [146, Lemma 2.2].

Lemma 4.2.3 (Generalized Stroock-Varopoulos Inequality for $\mathcal{K}^s_\varepsilon$). *Let us assume that $u \in H^s_\varepsilon(\mathbb{R}^d)$, and let $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi \in \mathcal{C}^1(\mathbb{R})$ and $\psi' \geq 0$. Then*

$$\int_{\mathbb{R}^d} \psi(u)\mathcal{K}^s_\varepsilon[u]dx \geq \int_{\mathbb{R}^d} \left|(\mathcal{K}^s_\varepsilon)^{\frac{1}{2}}[\Psi(u)]\right|^2 dx, \tag{4.2.12}$$

where $\psi' = (\Psi')^2$.

4.3 Existence and uniqueness of solutions of (TFPME)

In this section, we will consider the homogeneous Dirichlet problem for equation (TFPME) on a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. For the general theory, we will use signed solutions $u(t,x)$. In this, case we will still use the notation $u^m = |u|^{m-1}u$, $m > 1$ for the sake of brevity. The definition of solution to be introduced below includes zero Dirichlet boundary conditions which are implicitly taken as a consequence of the definition of both nonlocal space and time operators.

Our first goal is to prove an existence and uniqueness result. Actually, the result holds for more general nonlinearities than the power one at almost no extra cost so that we will adopt the more general context and consider the problem

$$\begin{cases} \partial^\alpha_t u + \mathcal{L}^s \varphi(u) = f & \text{in } (0,+\infty) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.3.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, smooth and increasing function. We assume moreover that $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$. The leading example will be $\varphi(u) = |u|^{m-1}u$ with $m > 0$.

Choosing the correct type of generalized solutions will be critical in the proofs. Here we use the concept of $\mathcal{H}^*$ solution as presented in [24].

Definition 4.3.1. *$u \in C([0,T], \mathcal{H}^*(\Omega))$ is an $\mathcal{H}^*$-solution if $\varphi(u) \in L^1([0,T], \mathcal{H}(\Omega))$ such that*

$$\int_0^T \int_\Omega \psi\, \partial^\alpha_t u\, dx\, dt + \int_0^T \int_\Omega \varphi(u)\mathcal{L}^s \psi\, dx\, dt = \int_0^T \int_\Omega f\, \psi\, dx\, dt, \qquad \forall\, \psi \in C^1_c([0,T], \mathcal{H}(\Omega)). \tag{4.3.2}$$

Notice that since $\mathcal{A}$ is an isomorphism from $\mathcal{H}$ into $\mathcal{H}^*$, then equation (4.3.2) is equivalent to $\psi = (\mathcal{A})^{-1}\tilde{\psi}$

$$\int_0^T \int_\Omega u\partial_t((\mathcal{A})^{-1}\tilde{\psi}) = \int_0^T \int_\Omega \varphi(u)\tilde{\psi} \quad \forall\, \tilde{\psi} \in C_c^1([0,T], \mathcal{H}^*(\Omega)). \tag{4.3.3}$$

This is a formulation of weak solutions for the potential equation $\partial_t \mathcal{L}^{-1}u + \varphi(u) = 0$.

4.3.1 Existence and uniqueness results. Proof of Theorem 4.1.1

The proof of Theorem 4.1.1 follows essentially by applying the techniques of monotone operators in Hilbert spaces. More precisely our Theorem 4.1.1 is the $\mathscr{H}^*$ version of Brezis' result in $\mathscr{H}^{-1}(\Omega)$, namely [29, Corollary 31]. The proof is adapted to our operators since the proper functional analysis have been settled [25]. Next, we will review and prove some of the main steps. Let us begin by rewriting the problem (TFPME) and setting up some notations.

We rewrite the problem (TFPME) to the following nonlinear evolution problem

$$\begin{cases} \partial_t^\alpha u(t) + \mathscr{B}(u(t)) \ni f \quad \text{in} \quad \mathscr{H} \quad \text{for a.e.} \quad t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{4.3.4}$$

where $\mathscr{B}$ is the subdifferential of a proper, convex and lower semicontinuous function j: $\mathscr{H} \to (-\infty, +\infty]$ with compact sublevels in $\mathscr{H}$ satisfying suitable compatibility conditions. The nonlinear and possibly multivalued operator $\mathscr{B}$ act from $\mathscr{H}$ to $2^{\mathscr{H}}$, the space of all subsets of $\mathscr{H}$. Moreover, u_0 and f are given data. In this chapter, we aim to analyze the existence and uniqueness of (4.3.4) under suitable assumptions on the structure of the linear and nonlinear maximal monotone operators $\mathscr{A}$ and $\mathscr{B}$ respectively. More precisely, in our analysis we suppose that $\mathscr{A}$ is bounded, so that the domain $D(\mathscr{A}) \equiv \mathscr{H}$. Later on, we will be prove that the nonlocal fractional time derivative in the Caputo sense ∂_t^α can be also represented by the maximal monotone operator $\mathscr{A}$.

As regard to the other operator, we ask $\mathscr{B}$ to be the subdifferential of a convex, proper and lower semicontinuous function $\varphi : \mathscr{H} \to (-\infty, +\infty]$ and such that $j(r)/|r| \to \infty$ as $|r| \to \infty$. We let $\varphi = \partial j$ be the sub-differential of j. Furthermore, for $u \in \mathcal{H}^*(\Omega)$ we define

$$\Psi(u) = \int_\Omega j(u)\, dx$$

whenever $u \in \mathrm{L}^1(\Omega)$ and $j(u) \in \mathrm{L}^1(\Omega)$, and define $\Psi(u) = +\infty$ otherwise. More precisely we consider the function $\varphi(u) = |u|^{m-1}u$ and $j(u) = |u|^{m+1}/(m+1)$, so that $\Psi(u) = \|u\|_{\mathrm{L}^{m+1}(\Omega)}/(m+1)$, in view of the problem we deal with. More general wider range of powers, namely any $m > 0$ can be considered instead.

In this context, staying in the more general case of j and φ, while taking $\alpha = 1$, some results on existence and uniqueness of the problem (4.3.4) using the concept of maximal monotonicity theory and accretive operators have been proposed [25, 27, 63]. We split the proof of Theorem 4.1.1 in three steps.

(i)– Maximal monotonotonicity of the sub-differential $\partial\Psi$ in $\mathcal{H}^*(\Omega)$. We need to prove the following Proposition.

Proposition 4.3.2. *Ψ is convex and lower semi-continuous function in $\mathcal{H}^*(\Omega)$, so that its sub-differential $\partial\Psi$ is a maximal monotone operator in $\mathcal{H}^*(\Omega)$. Moreover, this sub-differential $\partial\Psi$ can be characterized as follows:*

$$f \in \partial\Psi(u) \qquad \textit{if and only if} \qquad \mathcal{L}^{-1}f(x) \in \varphi(u(x)) \qquad \textit{a.e. in } \Omega. \tag{4.3.5}$$

Proof. Following Brezis' arguments, the proof of Theorem 4.1.1 will be splitted in several steps. It is worth recalling here that an essential point in the proof is that we can identify the space $\mathcal{H}$ with the usual Sobolev spaces, (see Section 1.2 in Chapter 1) hence the usual Sobolev imbeddings and inequalities of $H_0^s(\Omega) = W_0^{2,s}(\Omega)$ in L^p spaces hold for $\mathcal{H}$ as well ; we refer to Chapter 1 for further details on these issues.

The proof of this proposition is exactly the same as the one provided in [25, Proposition 3.1]. The reader can also find similar reasoning in [29, Theorem 17], pages. 123–124, just by replacing $\mathcal{H}^{-1}$ with $\mathcal{H}^*$. Nevertheless we provide the main steps of the proof [25, Proposition 3.1].

• STEP 1. First, we need to prove that the functional $u \mapsto \Psi(u)$ is l.s.c. on $\mathrm{L}^1(\Omega)$. By Fatou's Lemma it is sufficient to see that this functional is convex l.s.c. on $\mathrm{L}^1(\Omega)$.

• STEP 2. In the second stage, we define the operator $\mathscr{B}$ on $\mathcal{H}^*$ to be

$$\mathscr{B}u = \{\mathcal{L}^s w \mid w \in \mathcal{H} \text{ and } w(x) \in \varphi(u(x)) \text{ a.e. on } \Omega\}$$

with $u \in \mathrm{dom}(\mathscr{B})$ if and only if there is some $w \in \mathcal{H}$ such that $w \in \varphi(u(x))$ a.e. on Ω. We prove that $\mathscr{B} \subset \partial\Psi$. This means we show that for $f \in \mathscr{B}u$, i.e. $u \in \mathcal{H}^* \cap \mathrm{L}^1(\Omega)$, $f = \mathcal{L}^s w$ with $w \in \mathcal{H}$, $w(x) \in \varphi(u(x))$ a.e. on Ω. As proven in [25], under these assumptions $\Psi(u)$ is bounded and for every $v \in \mathcal{H}^* \cap \mathrm{L}^1(\Omega)$. Furthermore we have $j(v) \in \mathrm{L}^1(\Omega)$ so that the identity holds true

$$\Psi(v) - \Psi(u) \geq \langle f, v-u \rangle_{\mathcal{H}^* \times \mathcal{H}^*} = \langle w, v-u \rangle_{\mathcal{H} \times \mathcal{H}^*} ,$$

where we have used the identification of $\mathcal{H}$ with $\mathcal{H}^*$ via the isomorphism $\mathcal{L}^s$.

□

(ii)– Existence of mild solution by Yoshida approximation. We state the following Proposition.

Proposition 4.3.3. *Assume f is of bounded variation and Hölder continuous of order $1-\alpha+\gamma$, $\gamma > 0$ on $[0,T]$ so that $|\psi(0)| \leq \Lambda\varepsilon^\alpha \, |f(t_1) - f(t_2)|$. Then, $\ell(t) = \lim \prod_{i=1}^{[t/\varepsilon]} J_\varepsilon\left(t_i^\varepsilon\right)\ell(0)$ exists for all $\ell(0) \in \mathcal{X}$. Moreover for $\ell(0) \in \mathrm{dom}\left(\mathscr{B}(0)\right)$*

$$|\ell_\varepsilon(t) - \ell(t)| \leq C\varepsilon^\alpha.$$

Proof. The proof of this theorem follows line by line [97, Theorem 4.1]. For the sake of completeness we recall taking into consideration our nonlinear operator $\mathscr{B}$. The proof can be easily adapted to our operator $\mathscr{B}$ whence it is defined as in Proposition 4.3.2.

For $\varepsilon > 0$ define the resolvent

$$J_\varepsilon(t_i)\ell = \left(I - \varepsilon\mathscr{B}(t_i)\right)^{-1}\ell, \quad \ell \in \mathcal{Z}$$

and $w_i = J_\varepsilon(\mathscr{B}(t_i))\,\ell$. Then, $\psi = w_1 - w_2 \in \mathcal{Z}$ satisfies $\Delta\left(\frac{1}{\varepsilon}\right)\psi(0) = y_1 - y_2$ for $y_i \in \mathscr{B}w_i(0)$.

We assume there exist a continuous function $f : [0,T] \to \mathcal{X}$ and a positive constant Λ such that

$$|\psi(0)| \leq \Lambda\varepsilon^{\alpha} |f(t_1) - f(t_2)|. \tag{4.3.6}$$

Since $\varepsilon\psi - \psi' = 0$, it follows from (4.3.16) that

$$|\mathscr{B}_\varepsilon(t_1)\ell - \mathscr{B}_\varepsilon(t_2)\ell| \leq \Lambda\varepsilon^{\alpha-1} |f(t_1) - f(t_2)| \tag{4.3.7}$$

where we made used of the Yoshida approximation

$$\mathscr{B}_\varepsilon(t_i)\ell = \frac{1}{\varepsilon}\left(J_\varepsilon(t_i)\ell - \ell\right).$$

For $\varepsilon > 0$, let $\left\{\ell_k^\varepsilon\right\}$ be the sequence generated by

$$\ell_k^\varepsilon = J_\varepsilon(t_k^\varepsilon)\ell_{k-1}^\varepsilon, \quad \ell_0^\varepsilon = w \in \mathcal{Z}.$$

The product formula $\ell_k^\varepsilon = \prod_{i=1}^{m} J_\varepsilon(t_i)\ell$ defines an approximation sequence and satisfies the contraction

$$\left|\prod_{i=1}^{m} J_\varepsilon(t_i)\ell_1 - \prod_{i=1}^{m} J_\varepsilon(t_i)\ell_2\right| \leq |\ell_1 - \ell_2|. \tag{4.3.8}$$

Next from the equation (4.3.7), the following estimation follows

$$\begin{aligned} |\mathcal{B}_\varepsilon(t_k^\varepsilon)\ell_k^\varepsilon| &= \left|\mathcal{B}_\varepsilon(t_k^\varepsilon)J_\varepsilon(t_k^\varepsilon)\ell_{k-1}^\varepsilon\right| \\ &\leq \left|\mathcal{B}_\varepsilon(t_k^\varepsilon)\ell_{k-1}^\varepsilon\right| \\ &\leq \left|\mathcal{B}_\varepsilon(t_{k-1}^\varepsilon)\ell_{k-1}^\varepsilon\right| + \Lambda\varepsilon^{\alpha-1}\left|f(t_k^\varepsilon) - f(t_{k-1}^\varepsilon)\right|. \end{aligned}$$

Since the function f is of bounded variation on $[0,T]$, then if we set

$$a_k = |\mathcal{B}_\varepsilon(t_k^\varepsilon)\ell_k^\varepsilon|, \quad b_k = \Lambda\left|f(t_k^\varepsilon) - f(t_{k-1}^\varepsilon)\right|$$

we get

$$a_k - a_{k-1} = \varepsilon^{\alpha-1}b_k.$$

From the Hölder continuous property of b_k, we have that $\left|\mathcal{B}_\varepsilon(t_k^\varepsilon)\right| \leq K_0\varepsilon^{\alpha-1}$ for all k and ε.

Next we define $\varepsilon = 2^{-n}$, $\mu = 2^{-m}$ and $\kappa = 2^{m-n}$, with $t_k^\varepsilon = k\varepsilon$, so that for $1 \leq j \leq \kappa$:

$$\widehat{\ell}_{i\kappa+j}^{\mu} = J_\mu\left(t_{(i+1)\kappa}^{\varepsilon}\right)\widehat{\ell}_{i\kappa+j-1}^{\mu}.$$

Now from the inequality (4.3.6) we have that

$$\begin{aligned} \left|\ell_{i\kappa+j}^{\mu} - \widehat{\ell}_{i\kappa+j}^{\mu}\right| &\leq \left|J_\mu\left(t_{(i+1)\kappa}^{\varepsilon}\right)\ell_{i\kappa+j-1}^{\mu} - J_\mu\left(t_{(i+1)\kappa}^{\varepsilon}\right)\widehat{\ell}_{i\kappa+j-1}^{\mu}\right| \\ &\leq \left|\ell_{i\kappa+j-1}^{\mu} - \widehat{\ell}_{i\kappa+j-1}^{\mu}\right| + \Lambda\mu^{\alpha}\left|f\left(t_{(i+1)\kappa}^{\mu}\right) - f\left(t_{(i\kappa+j)}^{\mu}\right)\right|. \end{aligned}$$

We assume the function f to be Hölder continuous with order $1-\alpha+\gamma$, $\gamma>0$, then

$$\sum_{j=1}^{N}\mu^{\alpha}\left|f\left(t^{\varepsilon}_{(i+1)\kappa}\right)-f\left(t^{\varepsilon}_{(i\kappa+j)}\right)\right|\leq\varepsilon 2^{m-n-\gamma m}.$$

If we define the piecewise constant function by

$$\ell_{\mu}(t)=\ell^{\mu}_{i\kappa+j},\quad \widehat{\ell}_{\mu}(t)=\widehat{\ell}^{\mu}_{i\kappa+j},\quad \text{for}\quad t\in[t_{i\kappa+j},[t_{i\kappa+j+1}],$$

then the following inequality holds true

$$\left|\ell_{\mu}(t)-\widehat{\ell}_{\mu}(t)\right|\leq\varepsilon\Lambda 2^{(1-\gamma)m-n}. \tag{4.3.9}$$

It then follows from the Crandall-Liggett theorem that

$$\left|\ell_{\varepsilon}(t)-\widehat{\ell}_{\varepsilon}(t)\right|\leq\varepsilon C\left|\mathscr{B}\left(t^{\varepsilon}_{i-1}\right)\ell^{\varepsilon}_{i-1}\right|.$$

Using (4.3.7), we get the estimate

$$\left|\ell_{\varepsilon}(t)-\widehat{\ell}_{\mu}(t)\right|\leq\widetilde{C}\varepsilon^{\alpha}. \tag{4.3.10}$$

Combining both inequalities (4.3.9) and (4.3.10)

$$\begin{aligned}\left|\ell_{\varepsilon}(t)-\ell_{\mu}(t)\right| &\leq\left|\ell_{\varepsilon}(t)-\widehat{\ell}_{\mu}(t)+\widehat{\ell}_{\mu}(t)-\ell_{\mu}(t)\right|\leq\left|\ell_{\varepsilon}(t)-\widehat{\ell}_{\mu}(t)\right|+\left|\widehat{\ell}_{\mu}(t)-\ell_{\mu}(t)\right|\\ &\leq\varepsilon\Lambda 2^{(1-\gamma)m-n}+\widetilde{C}\varepsilon^{\alpha}\leq\varepsilon\Lambda 2^{(1-\gamma)m-(2-\alpha)n}+\widetilde{C}\varepsilon^{\alpha}\\ &\leq C\varepsilon^{\alpha},\end{aligned} \tag{4.3.11}$$

for $m\leq\frac{(2-\alpha)n}{(1-\gamma)}$. We defined the sequence $\{m_k\}$ by $m_k=\left[\frac{(2-\alpha)}{(1-\gamma)}m_{k-1}\right]$, $m_0=n$.

Then, by induction argument with respect to k,

$$\left|\ell_{\mu_k}(t)-\ell_{\mu_{k-1}}(t)\right|\leq C2^{-\alpha m_{k-1}},$$

where $\mu_k=2^{-m_k}$. From the relation

$$m_k-m_{k-1}\geq\frac{1-\alpha+\gamma}{1-\gamma}m_0>0.$$

$$\left|\ell_{\mu_k}(t)-\ell_{\mu_0}(t)\right|\leq M\varepsilon^{\alpha},$$

for some $M>0$ and thus the sequence $\{\ell_{\varepsilon}\}$, $\varepsilon=2^{-n}$ is a Cauchy sequence. As a consequence $\ell(t)=\lim_{\varepsilon\to 0}\ell_{\varepsilon}(t)$ exists in $\mathcal{C}\left(0,T;\mathcal{Z}\right)$ which defines the solution to (4.3.4) with $\ell(0)=w\in\text{dom}\left(\mathcal{B}(0)\right)$ and hence $\text{dom}\left(\mathcal{B}(0)\right)$ is dense in $\mathcal{Z}$. Thus (4.3.4) has the unique mild solution for all $x\in\mathcal{X}$. □

(iii)– $\mathscr{B}$ generates the nonlinear semigroup of contraction on $\mathcal{Z}$. We consider the Caputo fractional derivative given by Definition 1.5.3 or see also Chapter 1. Setting

$$g(t) = g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha},$$

and by a change of variables, $t - \tau = -\eta$, we can write

$$\int_0^t g(t-\tau)u'(\tau)d\tau = \int_{-t}^0 g(\eta)u'(t+\eta)d\eta = \int_{-\infty}^0 g(\eta)u'(t+\eta)d\eta \qquad (4.3.12)$$

with initial value $u(\eta) = w(\eta), \ \ \eta \leq 0$.

By the same change of variables $t - \tau = -\eta$, we write

$$g(\eta) = \frac{1}{\Gamma(1-\alpha)} |\eta|^{-\alpha},$$

so that (4.3.12) reduces to Definition 1.5.3. In the other words, if $\eta \to u(t+\eta)$ is absolute continuous, it follows that

$$\int_{-\infty}^0 g(\eta)u'(t+\eta)d\eta = \int_0^t g(t-\tau)u'(\tau)d\tau.$$

We then embed the solution $u(t) = \ell(t,0)$ in the state "history" space

$$\ell(t,\eta) = u(t+\eta) \in \mathcal{Z} = \mathcal{C}\left((-\infty,0],\mathcal{X}\right).$$

Then the equation of the form (4.3.4) has the Markovian form as the evolution in $\mathcal{Z}$:

$$\frac{d}{dt}\ell(t) = \mathcal{B}(t)\ell(t),$$

where the operator $\mathcal{B}(t)$ is defined by

$$\mathcal{B}(t)w = w'(\eta), \quad \eta \in (-\infty,0] \qquad (4.3.13)$$

in $\mathcal{Z}$ with domain

$$\text{dom}\,(\mathcal{B}) = \left\{ w' \in \mathcal{Z} : w(0) \in \text{dom}\,(\mathscr{B}), \quad \int_{-\infty}^0 g(\eta)w'(\eta)d\eta \in \mathscr{B}w(0) \right\}.$$

The dynamics of (4.3.4) is embedded in (4.3.13) as the nonlocal boundary value condition as $\eta = 0^+$ for the first order differential operator $\mathcal{B}(t)$. We analyse the well-posedness and the property of solution to (4.3.4) based on the semigroup generated by (4.3.13). In other words, we show that the solution map

$$(u_0, f) \in \mathcal{X} \times \mathcal{C}(0,T;\mathcal{X}) \to u(t) \in \mathcal{C}(0,T;\mathcal{X})$$

exists and is continuous. It will be shown that if $\mathscr{B}$ is dissipative and maximal monotone in $\mathcal{X}$, then $\mathscr{B}$ is dissipative and maximal monotone in $\mathcal{Z}$. In this way we then use the semigroup generation theory to define the solution $\ell(t) \in \mathcal{C}(0,T;\mathcal{X})$ to equation (4.3.13) and the solution to (4.3.4) by $u(t) = \ell(t,0) \in \mathcal{C}(0,T;\mathcal{X})$.

We consider the nonlinear fractional inclusion of the form of (4.3.4). Let a graph $\mathscr{B} \subset \mathcal{X} \times \mathcal{X}$ be dissipative, i.e, for any $[x_i, y_j] \in \mathscr{B}$ for $i, j = 1, 2, \ldots, N$ with $i \neq j$, there exists $x^* \in \mathscr{F}(x_1 - x_2)$ such that $Re\,\langle y_1 - y_2\,,\, x^* \rangle \leq 0$, where $\mathscr{F} : \mathcal{X} \to \mathcal{X}^*$ is the duality mapping. Equivalently $|x - \varepsilon y| \geq |x|$ for all $\varepsilon > 0$ and $[x, y] \in \mathscr{B}$.

Next we define $\mathscr{B}$ in $\mathcal{Z} = \mathcal{C}\left((-\infty, 0]; \mathcal{X}\right)$ by $\mathscr{B}w = w'$ with domain

$$\operatorname{dom}(\mathcal{B}) = \left\{ w' \in \mathcal{Z} : w(0) \in \operatorname{dom}(\mathscr{B}), \quad \int_{-\infty}^{0} g(\eta) w'(\eta) d\eta \in \mathscr{B}w(0) \right\}.$$

Theorem 4.3.4. *Assume $\mathscr{B}$ is dissipative and $\mathcal{R}\left(\varepsilon I - \mathscr{B}\right)$ for all sufficiently small $\varepsilon > 0$. Then, $\mathscr{B}$ is dissipative and $\mathcal{R}\left(\varepsilon I - \mathscr{B}\right) = \mathcal{Z}$ for all sufficiently small $\varepsilon > 0$. Thus $\mathscr{B}$ generates the nonlinear semigroup of contraction on $\mathcal{Z}$.*

Proof. For $w_1, w_2 \in \operatorname{dom}(\mathscr{B})$, suppose for all $\eta < 0$ that

$$|w_1(0) - w_2(0)| > |w_1(\eta) - w_2(\eta)|.$$

For all $x^* \in \mathcal{F}\left(w_1(0) - w_2(0)\right)$, we can write

$$\begin{aligned}
&\left\langle \int_{-\infty}^{0} g_\varepsilon(\eta) \left(w_1' - w_2'\right) d\eta\,,\, x^* \right\rangle \\
&= \int_{-\infty}^{0} \frac{\left(g_\varepsilon(\eta) - g_\varepsilon(\eta - \varepsilon)\right)}{\varepsilon} \left\langle (w_1(\eta) - w_2(\eta)) - (w_1(0) - w_2(0))\,,\, x^* \right\rangle d\eta \leq 0.
\end{aligned}$$

Observe that for $\eta < 0$

$$\begin{aligned}
&\left\langle (w_1(\eta) - w_2(\eta)) - (w_1(0) - w_2(0))\,,\, x^* \right\rangle \\
&\leq \left(|w_1(\eta) - w_2(\eta)| - |w_1(0) - w_2(0)|\right) |w_1(0) - w_2(0)| < 0.
\end{aligned}$$

Thus

$$\left\langle \int_{-\infty}^{0} g_\varepsilon(\eta) \left(w_1' - w_2'\right) d\eta\,,\, x^* \right\rangle < 0. \tag{4.3.14}$$

But, it comes out that, for

$$y_1 \in \mathscr{B}\, w_1(0), \qquad y_2 \in \mathscr{B}\, w_2(0)$$

there exists an $x^* \in F\left(w_1(0) - w_2(0)\right)$ such that

$$\langle y_1 - y_2, x^* \rangle \geq 0. \tag{4.3.15}$$

This is in contradiction with (4.3.14). As a consequence, there exists η_0 such that

$$|w_1(\eta_0) - w_2(\eta_0)| = |w_1 - w_2|_{\mathcal{Z}},$$

and thus $\langle w'(\eta_0)\ ,\ x^* \rangle = 0$ for all $x^* \in F\left(w_1(\eta_0) - w_2(\eta_0)\right)$.

Hence the following inequality holds true

$$\begin{aligned}\left|\varepsilon\left(w_1 - w_2\right) - \left(w_1' - w_2'\right)\right|_{\mathcal{Z}} &\geq \left\langle \varepsilon\left(w_1(\eta_0) - w_2(\eta_0)\right) - \left(w_1'(\eta_0) - w_2'(\eta_0)\right)\ ,\ x^* \right\rangle \\ &= \varepsilon\left|w_1(\eta_0) - w_2(\eta_0)\right| = \left|w_1 - w_2\right|_{\mathcal{Z}}\,. \end{aligned} \tag{4.3.16}$$

For the range condition

$$\varepsilon w - w' = f, \quad \int_{-\infty}^{0} g(\eta) w'(\eta) d\eta \in \mathscr{B} w(0).$$

We have

$$w = e^{\varepsilon\eta} w(0) + \psi$$

and

$$\Delta(\varepsilon) w(0) - \int_{-\infty}^{0} g(\eta)\psi'(\eta) d\eta \in \mathscr{B}\, w(0),$$

where the function $\psi(\eta)$ is given by

$$\psi(\eta) = \int_{\eta}^{0} e^{\varepsilon(\eta-\xi)} f(\xi) d\xi.$$

Since $\mathscr{B}$ is **m-dissipative**,

$$w(0) = \left(\Delta(\varepsilon) I - \mathscr{B}\right)^{-1} \int_{-\infty}^{0} g(\eta)\psi'(\eta) d\eta$$

exists and $\mathcal{R}\left(\varepsilon I - \mathscr{B}\right) = \mathcal{Z}$.

Thus, the theorem follows from the Crandall-Liggett theorem. □

4.4 Existence of weak solutions for (TPMEFP)

In order to construct existence of weak solutions of (TPMEFP) for the general initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and for a well behaved forcing term $f \in L^\infty(Q)$ having exponential decay at infinity, we proceed by the regularization method, which will ensure compactness, together with an $L^1 - L^\infty$ smoothing effect.

4.4.1 A regularized problem

The main idea is to consider a regularized version of (TPMEFP) where all the terms that might cause a blow up are approximated. We call u the regularized solution. We add a vanishing viscosity term $\delta\Delta u$ to (TPMEFP) that ensures good properties of regularity for solution and we eliminate the degeneracy at the zero level set by putting $u^{m-1} \sim (u + \mu)^{m-1}$. Next we eliminate the singularity character of the fractional derivative by discrete approximation. Restricting our problem (TPMEFP) in a bounded domain B_R with radius R, the regularized approximated problem reads

$$\begin{cases} {}_0^cD_t^\alpha u = \delta\Delta u + \nabla \cdot \left(d_\mu(u)\nabla\mathcal{K}_s^\varepsilon u\right) + f & \text{in } B_R \times (0,T), \\ u(x,0) = \widehat{u}_0(x) & \text{in } B_R, \\ u(x,t) = 0 & \text{in } B_R^c \times (0,T), \end{cases} \tag{4.4.1}$$

depending on the parameters $\delta, \mu, R > 0$, and we define $d_\mu(u) = (u+\mu)^{m-1} : \mathbb{R} \to \mathbb{R}$.

We say that u is a weak solution of (4.4.1) if

$$\int_0^T \int_{B_R} \zeta \, {}_0^cD_t^\alpha u \, dx\, dt - \int_0^T \int_{B_R} \left(\delta\nabla u + (u+\mu)^{m-1}\nabla\mathcal{K}_s^\varepsilon u\right) \cdot \nabla\zeta \, dx\, dt = \int_0^T \int_{B_R} f\zeta \, dx\, dt \tag{4.4.2}$$

for smooth test functions $\zeta \in \mathcal{C}_0^\infty$ that vanish on the spatial boundary ∂B_R and for large t.

4.4.2 Solution representation of the regularized problem

Indeed, existence of smooth weak solutions of (4.4.1) is proved via mild solutions, i.e, u is the fixed point of (4.4.1). We look for the solution representation of the abstract fractional differential equation associated to (4.4.1). More precisely, we consider the fractional Cauchy problem for initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $f \in L^\infty(Q)$ and having exponential decay at infinity as

$$\begin{cases} {}_0^cD_t^\alpha u(t) = \mathcal{A}u(t) + \nabla \cdot \Psi(u)(t) + f(t), & t \in (0,T], \\ u(x,0) = u_0(x), \end{cases} \tag{4.4.3}$$

where $\mathcal{A}u := \delta\Delta u$, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a Banach space, $\alpha \in (0,1)$, $u_0 \in \mathcal{D}(\mathcal{A})$ and $\Psi(u) = (u+\mu)^{m-1}\nabla\mathcal{K}_s^\varepsilon u$. The operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ is a densely defined linear operator which fulfils the following assumption [120]:

- (H_1) The operator $\mathcal{A}$ is m-accretive in $\mathbb{X}$; this means maximal monotone.

- (H_2) The operator $\mathcal{A}$ is the generator of a semigroup of contractions $(Q(t))_{t\geq 0}$, i.e.

$$\sup_{t\geq 0} \|Q(t)\|_{B\mathbb{X}} \leq 1, \tag{4.4.4}$$

  where $(B(\mathbb{X}), \|\cdot\|_{\mathbb{X}})$ is a Banach space of all linear bounded operators on $\mathbb{X}$.

Note that $\mathcal{A}$ is an m-accretive operator in L^2 and $(\mathcal{A}, \mathcal{D}(\mathcal{A})) = \left(\delta\Delta, H^2(B_R) \cap H_0^1(B_R)\right)$, for $\delta > 0$ small enough. Hence $\mathcal{A}$ is a generator of contraction semigroup [59, 120].

Remark 4.4.1. *It is worthy of mention that for $\alpha = 1$ and $f = 0$ in* (4.4.3)*, we fall in the classical case from which by Duhamel's principle (see also [17, Proposition 5.2]), the solution representation is given by*

$$u(t) = \mathrm{e}^{\delta t\Delta}u_0 + \int_0^t \nabla \mathrm{e}^{\delta(t-\tau)\Delta} \cdot \Psi(u(\tau))\, d\tau \qquad \text{with} \quad \Psi(u) = (u+\mu)^{m-1}\nabla \mathcal{K}_s^{\varepsilon} u, \tag{4.4.5}$$

in $(\mathcal{C}[0,T], L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d)$ where $\mathrm{e}^{t\Delta}$ denotes the heat semigroup. The map

$$\mathcal{T} : u \mapsto \mathrm{e}^{\delta t\Delta}u_0 + \int_0^t \nabla \mathrm{e}^{\delta(t-s)\Delta} \cdot \Psi(u(s))\, ds,$$

with $\mathbb{X} = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ has a fixed point by the Banach contraction principle as soon as $T = T(\|u_0\|_X) > 0$ is sufficiently small [17].

For $f \neq 0$, one can consider the classical elliptic problem

$$gu - \delta\Delta u - \nabla \cdot \Psi(u) = f, \quad \text{on } B_R \tag{4.4.6}$$

with $u \equiv 0$ on ∂B_R, $g, f \geq 0$ are smooth in $L^\infty(Q)$. As shown in [6, 88] for $h \in \mathcal{C}_0^{0,\beta}$ with $h_1 \geq 0$ we can apply Schauder's estimate theory and by bootstrapping argument to conclude that

$$\|u\|_{\mathcal{C}_0^{1,\beta}} \leq C(m,\delta,s)\, \|h\|_{\mathcal{C}_0^{0,\beta}}.$$

This implies that the map $\mathcal{T} : h_1 \to u$ is a compact map. So the set $\{h_1\}$ is a closed convex set. Hence we can apply the fixed point theorem [88] to conclude that there exists a solution to (4.4.6)*. Hence, the existence of the classical case for $\alpha = 1$ is recovered.*

Proposition 4.4.2. *Let u_0 in $\mathbb{X}$ and $f \in L^\infty(Q)$. Assume that (H_1) and (H_2) hold true. Let u be the solution representation to* (4.4.3)*. Then*

$$u(t) = \mathbb{P}_1(t)u_0 + \int_0^t \nabla \cdot (\mathbb{P}_2(t-\tau)\Psi(u(\tau)))\, d\tau + \int_0^t \mathbb{P}_2(t-\tau)f(\tau)d\tau, \tag{4.4.7}$$

where

$$\mathbb{P}_1(t) = \int_0^\infty \mathcal{M}_\alpha(\zeta)\mathcal{Q}(t^\alpha\zeta)d\zeta, \quad \text{and} \quad \mathbb{P}_2(t) = \alpha\int_0^\infty \zeta t^{\alpha-1}\mathcal{M}_\alpha(\zeta)\mathcal{Q}(t^\alpha\zeta)d\zeta. \tag{4.4.8}$$

Proof. Using the fact that the Laplace transform of ${}_0^cD_t^\alpha u$ is given by

$$\mathscr{L}\left\{{}_0^cD_t^\alpha u\right\}(p) = p^\alpha\widetilde{u}(p) - p^{\alpha-1}u_0,$$

we can write the Laplace transform of the fractional differential equation (4.4.3) as

$$\begin{aligned} p^\alpha\widetilde{u}(p) &= p^{\alpha-1}u_0 + \mathcal{A}\,\widetilde{u}(p) + \nabla\cdot\widetilde{\Psi}(p) + \widetilde{f}(p), \\ \widetilde{u}(p) &= (p^\alpha I - \mathcal{A})^{-1}p^{\alpha-1}u_0 + \nabla\cdot(p^\alpha I - \mathcal{A})^{-1}\widetilde{\Psi}(p) + (p^\alpha\mathbb{I} - \mathcal{A})^{-1}\widetilde{f}(p), \end{aligned} \tag{4.4.9}$$

where I is the identity operator.

Next we compute each of the terms $(p^\alpha I - \mathcal{A})^{-1}$ and $(p^\alpha I - \mathcal{A})^{-1} p^{\alpha-1}$.

• **Computation of** $(p^\alpha I - \mathcal{A})^{-1}$: Observe that for any $g \in \mathbb{X}$, we have that

$$(p^\alpha I - \mathcal{A})^{-1} g = \int_0^\infty e^{-p^\alpha \theta} \mathcal{Q}(\theta) g \, d\theta.$$

Since $\|\mathcal{Q}(t)\| \leq K$, it follows that the integral above is absolutely convergent for all $p > 0$. Hence

$$\begin{aligned}(p^\alpha I - \mathcal{A})^{-1} g = \int_0^\infty e^{-p^\alpha \theta} \mathcal{Q}(\theta) g \, d\theta &= \int_0^\infty \int_0^\infty e^{-p\theta^{1/\alpha}} \rho_\alpha(v) \mathcal{Q}(\theta) g \, dv \, d\theta \\ &= \int_0^\infty e^{-pt} \left(\int_0^\infty \theta^{-1/\alpha} \rho_\alpha(t\theta^{-1/\alpha}) \mathcal{Q}(\theta) g \, d\theta \right) dt,\end{aligned}$$

where ρ_α is the one-side stable probability density, whose Laplace transform is given by [120]

$$\int_0^\infty \rho_\alpha(t) e^{-pt} \, dt = e^{-p^\alpha}, \tag{4.4.10}$$

and which satisfies

$$\alpha \mathcal{M}_\alpha(\zeta) = \zeta^{-1-1/\alpha} \rho_\alpha(\zeta^{-1/\alpha}), \tag{4.4.11}$$

which is in terms of the Mainardi function defined in (4.2.7). By using (4.4.11) it yields

$$\begin{aligned}(p^\alpha I - \mathcal{A})^{-1} g &= \int_0^\infty e^{-pt} \left(\int_0^\infty \theta^{-1/\alpha} \rho_\alpha(t\theta^{-1/\alpha}) \mathcal{Q}(\theta) g \, d\theta \right) dt \\ &= \int_0^\infty e^{-pt} \left(\int_0^\infty \alpha \zeta t^{\alpha-1} \mathcal{M}_\alpha(\zeta) \mathcal{Q}(t^\alpha \zeta) g \, d\zeta \right) dt.\end{aligned}$$

Hence

$$(p^\alpha I - \mathcal{A})^{-1} g = \widetilde{\mathbb{P}}_2(p) g,$$

with

$$\mathbb{P}_2(t) = \int_0^\infty \alpha \zeta t^{\alpha-1} \mathcal{M}_\alpha(\zeta) \mathcal{Q}(t^\alpha \zeta) \, d\zeta.$$

• **Computation of** $p^{\alpha-1} (p^\alpha I - \mathcal{A})^{-1}$: Similarly as in the previous computation,

$$p^{\alpha-1} (p^\alpha I - \mathcal{A})^{-1} g = p^{\alpha-1} \int_0^\infty e^{-p^\alpha \theta} \mathcal{Q}(\theta) g \, d\theta.$$

From (4.4.10) we have

$$\begin{aligned}&- \int_0^\infty e^{-pt} \left(\int_0^\infty \rho_\alpha(\theta) \mathcal{Q}(t^\alpha \theta^{-\alpha}) g \, d\theta \right) dt \\ &\quad = \int_0^\infty e^{-pt} \left(\int_0^\infty \frac{1}{\alpha} \zeta^{-1-1/\alpha} \rho_\alpha(\zeta^{-1/\alpha}) \mathcal{Q}(t^\alpha \zeta) g \, d\zeta \right) dt \\ &\quad = \int_0^\infty e^{-pt} \left(\mathcal{M}_\alpha(\zeta) \mathcal{Q}(t^\alpha \zeta) g \, d\zeta \right) dt.\end{aligned} \tag{4.4.12}$$

Hence we get that

$$p^{\alpha-1}\left(p^{\alpha}I-\mathcal{A}\right)^{-1}g=\widetilde{\mathbb{P}}_1(p)g,$$

with

$$\mathbb{P}_1(t)=\int_0^{\infty}\mathcal{M}_{\alpha}(\zeta)\mathcal{Q}(t^{\alpha}\zeta)\,d\zeta.$$

Thus replacing $p^{\alpha-1}\left(p^{\alpha}I-\mathcal{A}\right)^{-1}$ and $\left(p^{\alpha}I-\mathcal{A}\right)^{-1}$ in (4.4.9), we get

$$\widetilde{u}(p)=\widetilde{\mathbb{P}}_1(p)u_0+\nabla\cdot\widetilde{\mathbb{P}}_2(p)*\widetilde{\Psi}(p)+\widetilde{\mathbb{P}}_2(p)*\widetilde{f}(p).$$

Consequently applying the inverse Laplace transform, we get

$$u(t)=\mathbb{P}_1(t)u_0+\int_0^t\nabla\cdot\left(\mathbb{P}_2(t-\tau)\Psi(u(\tau))\right)\,d\tau+\int_0^t\mathbb{P}_2(t-\tau)f(\tau)\,d\tau.$$

□

The next step is to show that the solution $u\in\mathcal{C}([0,T],X)$ can be seen as a fixed point of

$$\mathcal{P}:w\mapsto\mathbb{P}_1(t)w_0+\int_0^t\nabla\cdot\left(\mathbb{P}_2(t-\tau)\Psi(w(\tau))\right)d\tau+\int_0^t\mathbb{P}_2(t-\tau)f(\tau)d\tau. \quad (4.4.13)$$

which maps $\overline{B}(0,R)$ into itself, and is a contraction.

4.4.3 Contraction property for the map $\mathcal{P}$

Next we state the following lemma which specifies the contraction property for the map $\mathcal{P}$ defined in (4.4.13).

Proposition 4.4.3. *For all $\alpha,s\in(0,1)$, there exist a positive $T\in(0,1)$ depending only on the initial data u_0, and a function $u\in\mathcal{C}([0,T],X)$ such that* (4.4.13) *holds true. Moreover the operator $\mathcal{P}$ maps $\mathcal{C}([0,T],X)$ into itself, and there exist $C>0$ and $\kappa>0$ such that for all $u,v\in\overline{B}(0,R)\subset\mathcal{C}([0,T],X)$,*

$$\|\mathcal{P}(u)-\mathcal{P}(v)\|_{\mathcal{C}([0,T],X)}\leq C_2(R)T^{\kappa}\|u-v\|_{\mathcal{C}([0,T],X)}, \quad (4.4.14)$$

where $C_2(R)$ is a constant which depends on α,s,d,m,δ.

To prove this lemma, we adapted the proof of [17, Lemma 5.4].

Proof. From (4.4.13) we write the difference of $\mathcal{P}$'s as

$$\begin{aligned}\mathcal{P}(u)(t)-\mathcal{P}(v)(t)&=\mathbb{P}_1(t)\left(u_0-v_0\right)+\int_0^t\nabla\cdot\mathbb{P}_2(t-\tau)\left(\Psi(u(\tau))-\Psi(v(\tau))\right)(\tau)d\tau\\&=\int_0^t\nabla\cdot\mathbb{P}_2(t-\tau)\left(\Psi(u(\tau))-\Psi(v(\tau))\right)(\tau)d\tau.\end{aligned}$$

We consider the estimate of $\|\nabla \mathbb{P}_2 v\|_p$ (see for example [98, 103] where this type of estimate was successfully computed)

$$\|\nabla \mathbb{P}_2 v\|_p = C_1(\alpha, p, r) t^{-(1-\alpha)-\frac{\alpha d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{\alpha}{2}} \|v\|_r, \tag{4.4.15}$$

for $1 \le r \le p \le \infty$.

Now using the inequality (4.4.15), for $\langle p, r\rangle = \langle 1, 1\rangle$ and $\langle p, r\rangle = \langle \infty, q\rangle$, respectively, we have that

$$\begin{aligned} \|\mathcal{P}(u) - \mathcal{P}(v)\|_{\mathcal{C}([0,T],L^1(\mathbb{R}^d))} &\le C_0(R) C_1(\alpha, p, r) \|u - v\|_{\mathcal{C}([0,T],X)} \int_0^T t^{-1+\frac{\alpha}{2}} dt \\ &\le C_1(\alpha, p, d, r, C_0(R)) T^{\frac{\alpha}{2}} \|u - v\|_{\mathcal{C}([0,T],X)}, \end{aligned} \tag{4.4.16}$$

and

$$\begin{aligned} \|\mathcal{P}(u) - \mathcal{P}(v)\|_{\mathcal{C}([0,T],L^\infty(\mathbb{R}^d))} &\le C_0(R) C_1(\alpha, p, r) \|u - v\|_{\mathcal{C}([0,T],X)} \int_0^T t^{-1+\frac{\alpha}{2}\left(1-\frac{d}{q}\right)} dt \\ &\le C_3(\alpha, p, d, r, C_0(R)) T^{\frac{\alpha}{2}\left(1-\frac{d}{q^\#}\right)} \|u - v\|_{\mathcal{C}([0,T],X)}. \end{aligned} \tag{4.4.17}$$

Finally, combining (4.4.13), (4.4.16) and (4.4.17), we obtain the desired estimate (4.4.14) for $s, \alpha \in (0, 1)$, with $\kappa = \frac{\alpha}{2} - \frac{\alpha d}{2q^\#}$, now with a new constant $C_2(R) = C(\alpha, s, d, q^\#, R, G'(2R))$, where $q^\# > d$ has been chosen to ensure $\frac{\alpha d}{2q^\#} < \frac{\alpha}{2}$. □

Corollary 4.4.4. *For $v = 0$ in the estimate* (4.4.14) *we have that*

$$\|\mathcal{P}(u)\|_{\mathcal{C}([0,T],X)} \le \|u_0\|_X + R C_2(R) T^\kappa. \tag{4.4.18}$$

Therefore, we can choose $R = 2\|u_0\|_X$ and $T > 0$ such that $C_2(R)T^\kappa \le \frac{1}{2}$ in order to ensure that $\mathcal{P}$ maps $\overline{B}(0, R)$ into itself, hence a contraction.

Now we will use the existence of solutions to (4.4.6) to obtain —via recursion arguments— solutions to the discretized problem

$$\partial_\varepsilon^\alpha u - \delta \Delta u - \nabla \cdot \left((u + \mu)^{m-1} \nabla \mathcal{K}_s^\varepsilon \right) = f \text{ on } [0, T] \times B_R, \tag{4.4.19}$$

with $u(0, x) = u_0(x)$ an initially defined smooth function with compact support, $\varepsilon = T/k$ for some $k \in \mathbb{N}$. We will eventually let $k \to \infty$, so that $\varepsilon \to 0$.

We first rewrite the problem (4.4.3) in the form

$$\partial_t^\alpha Y(t) = cY(t) + h(t), \tag{4.4.20}$$

with $h(t)$ a nonlinear term. Hence the solution (4.4.3) takes the form

$$Y(t) = Y(0) E_\alpha(ct^\alpha) + \alpha \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(c(t-\tau)^\alpha) h(\tau)\, d\tau,$$

where E_α and $E_{\alpha,\alpha}$ are respectively the Mittag-Leffler and the parametric Mittag-Leffler function of order α.

We will utilize in the next lemma one specific instance of (4.4.20). We define $Y_1(t)$ to be the solution to (4.4.20) with $Y(0) = \sup u(0,x),\ \ c = 0,\ \ h = 2\Lambda f$.

Lemma 4.4.5. *Let u be a solution to* (4.4.19). *Then, there exists ε_0 depending only on $T, \alpha, \|f\|_{L^\infty}$ such that if $\varepsilon \le \varepsilon_0$, then*

$$u(\varepsilon j) \le Y_1(\varepsilon j).$$

Proof. We have from Proposition 4.4.3 that $Y_1(t)$ is an increasing and bounded function. This means that we can write

$$\partial_\varepsilon^\alpha Y_1(\varepsilon j) \ge \Lambda^{-1}\partial_\varepsilon^\alpha Y_1(\varepsilon j) \ge \frac{2}{3\Lambda} D_t^\alpha Y_1(\varepsilon j) = \frac{4}{3} f.$$

Next we use $(u(t,x) - Y_1(t))_+$ as a test function. Since $(u - Y_1)_+(0) = 0$, it follows from the weak formulation that

$$\begin{aligned}
&\varepsilon \sum_{j<k} \int_{B_R} f(\varepsilon j, x)\,(u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \; dx \\
&\qquad = \varepsilon \sum_{j<k} \int_{B_R} (u - Y_1)_+ \partial_\varepsilon^\alpha [(u - Y_1)_+ - (u - Y_1)_- + Y_1] dx \\
&\qquad\quad + \delta\varepsilon \sum_{j<k} \int_{B_R} \nabla u \nabla\,(u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \; dx \\
&\qquad\quad + \varepsilon \sum_{j<k} \int_{B_R} \Big((u+\mu)^{m-1} \nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]\Big)\,(u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \; dx
\end{aligned}$$

Then for ε small enough and $j > 0$, we have the following estimate

$$\begin{aligned}
&\varepsilon \sum_{j\le k} \int_{B_R} f(\varepsilon j, x)(u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \\
&\ge \int_{B_R} \varepsilon \sum_{j\le k} \frac{4}{3}(u - Y_1)_+ f + \delta\varepsilon \sum_{j\le k} \int_{B_R} \chi_{\{u>Y_1\}} \|\nabla u\|^2 dx \\
&+ \Big((u+\mu)^{m-1} \chi_{\{u>Y_1\}} \nabla u \nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]\Big)\,(u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \, dx \\
&\ge \frac{4}{3}\varepsilon \sum_{j\le k} \int_{B_R} f(\varepsilon j, x)(u(\varepsilon j, x) - Y_1(\varepsilon j))_+.
\end{aligned}$$

Thus $(u - Y_1)_+ \equiv 0$. □

4.4.4 Exponential tail control

In order to prove existence and finite speed of propagation properties, the estimation of certain rate of decay of our solutions as $|x| \to \infty$ is necessary. This method is known as a comparison method with suitable family of barrier functions, that in [45] received the name of "true supersolutions".

Lemma 4.4.6. *Let* $0 < \alpha < 1$, $0 < s < 1/2$, $m \geq 2$ *and let* u *be the solution to problem* (4.4.1). *We assume that* u *is bounded* $0 \leq u(x,t) \leq L$ *and that* u_0 *and* f *lies below a function of the form*

$$u_0(x) = Ae^{-a|x|}, \quad f = Ae^{-|x|}, \quad A, a > 0. \tag{4.4.21}$$

If A *is large enough, then there exists a positive constant* C *depending on* d,α, s, a, L, A,$\|u(0,x)\|_{L^\infty}$, $\|f\|_{L^\infty}$, *such that for any* $T > 0$ *we will have the comparison*

$$u(x,t) \leq AY_2(\varepsilon j)e^{-|x|},$$

for all $x \in \mathbb{R}^d$ *and all* $0 < t \leq T$ *provided that* $t = \varepsilon j$.

Proof. We basically adapt the technique of tail control [45] by constructing some kind of "exaggerated supersolutions" Y_2 solution of (4.4.20), with the properties $Y_2(0) = 2$, $h = 0$, and $c = C\Lambda^{-1}$.

• **Reduction.** By scaling we may put $a = L = 1$. This is done by considering instead of u, the function $\tilde{u}$ defined as

$$u(x,t) = L\tilde{u}(ax, bt), \quad b^\alpha = L^{m-1}a^{2-2s}, \tag{4.4.22}$$

which satisfies the equation

$${}_0^cD_t^\alpha \tilde{u} = \delta_1 \Delta \tilde{u} + \nabla.(d_{\frac{\mu}{L}}(\tilde{u})\nabla \mathcal{K}_s^{\varepsilon a}(\tilde{u})) + \delta_2 \widetilde{f},$$

with $\delta_1 = a^{2s}\delta/L^{m-1}$ and $\delta_2 = a^{2s-2}/L^m$.

Notice that $\tilde{u}(x,0) \leq A_1 e^{-|x|}$ with $A_1 = A/L$. The corresponding bound for $\tilde{u}(x,t)$ will be $\tilde{u}(x,t) \leq A/L\ Y_2(\varepsilon j)e^{-|x|}$ with a new constant C_1 embedded in $Y_2(\varepsilon j)$ as $C_1 = C/b = C\left(L^{(m-1)/\alpha}a^{(2-2s)/\alpha}\right)^{-1}$.

• **Contact analysis.** We assume that $0 \leq u(x,0) \leq 1$ and also that

$$u(x,0) \leq Y_2(0,x)e^{-r} \leq Ae^{-r}, \quad r = |x| > 0,$$

where $A > 0$ is a constant that will be chosen to be say larger than 2 (as done below).

Next, we consider a radially symmetric candidate for the upper barrier function of the form

$$\hat{u}(x,\varepsilon j) = AY_2(\varepsilon j)e^{-r}.$$

The constant C embedded in $Y_2(\varepsilon j)$, will be determined in terms of A to satisfy a true supersolution condition which is obtained by contradiction at the first point (x_c, t_c) of possible contact of u and $\widehat{u}$.

Since u is smooth and continuous, and satisfies a "true supersolution", then

$$u(x,\varepsilon j) \leq \widehat{u}(x,\varepsilon j) = LY_2(\varepsilon j)e^{-r}$$

for some $L > A$, which could be determined in order for $LY_2(\varepsilon j)e^{-r}$ to satisfy a "true supersolution" condition. We lower $L \geq A$ until it touchs u for the first time at the contact point $u(x_c, t_c) \in Q$.

Note that if there exists a contact point, it cannot happen at the boundary ∂B_R since $u = 0$. Also since u is smooth, the contact cannot happen at a point $(\varepsilon j, 0)$. Furthermore this cannot happen at the initial time due to the fact that $LY_2(0, x) \geq 2A \geq 2u(0, x)$.

The equation satisfied by u can be written in the form

$$ {}_0^cD_t^\alpha u = \delta\Delta u + (m-1)(u+\mu)^{m-2}\nabla u \cdot \nabla p + (u+\mu)^{m-1}\Delta p + f, \tag{4.4.23} $$

with $p = \mathcal{K}_s^\varepsilon[u]$. We will obtain necessary conditions so that (4.4.23) holds at the contact point (x_c, t_c). Then, we prove that there exists a suitable choice of parameters C, A, μ such that the contact can not hold.

• **Estimates on u and p at the first contact point.**

For $0 < \alpha < 1, 0 < s < 1/2$, at the first contact point (x_c, t_c), we have the estimates

$$ \partial_r u = -AY_2(t_c)e^{-r_c}, \quad \Delta u \leq AY_2(t_c)e^{-r_c}, \partial_{t_c}^\alpha u(t_c) \geq \frac{2}{3\Lambda} {}_0^cD_t^\alpha Y_2(t_c). $$

From the assumption that our solution u is bounded by $0 \leq u \leq 1$, then

$$ u(x_c, t_c) = AY_2(t_c)e^{-r_c} \leq Y_1(T)K \leq 1. \tag{4.4.24} $$

Moreover, from [45] we have the following upper bounds for the pressure term at the contact point for $0 < s < 1/2$:

$$ \Delta p(x_c, t_c) \leq K_1 \leq Y_1(T)K_1, \quad (-\partial_r p)(x_c, t_c) \leq K_2 \leq Y_1(T)K_2. \tag{4.4.25} $$

Now assume that there exists a first contact for $t > 0$, at a space and time of contact point (x_c, t_c). At the contact point (x_c, t_c) with $r_c = |x_c|$, equation (4.4.23) implies that

$$ \begin{aligned} \frac{2}{3}CY_2(t_c)e^{-r_c} = \frac{\Lambda}{3}L{}_0^cD_t^\alpha Y_2(t_c)e^{-r_c} &\leq \partial_\varepsilon^\alpha LY_2(t_c)e^{-r_c} \\ &\leq \delta Y_2(t_c)e^{-r_c} + (m-1)\left(u(x_c, t_c)+\mu\right)^{m-2}(-Y_2(t_c)e^{-r_c})(\partial_r p) \\ &\qquad + \left(u(x_c, t_c)+\mu\right)^{m-1}\Delta p + f(t_c, r_c). \end{aligned} \tag{4.4.26} $$

From (4.4.25) with $K = \max\{K_1Y_1(T), K_2Y_1(T)\}$, we obtain from the previous inequality, after simplification by $LY_2(t_c)e^{-r_c}$,

$$ \begin{aligned} \frac{2}{3}C \leq \delta &+ (m-1)\left(u(x_c, t_c)+\mu\right)^{m-2}K \\ &+ \left(u(x_c, t_c)+\mu\right)^{m-2}\left(1 + \frac{\mu}{LY_2(t_c)}e^{r_c}\right)K + \frac{f(t_c, r_c)}{LY_2(t_c)}e^{r_c}. \end{aligned} $$

Thus,

$$ C \leq 2\delta + 2K\left(u(x_c, t_c)+\mu\right)^{m-2}\left(m + \frac{\mu e^{r_c}}{LY_2(t_c)}\right) + 2 + \frac{f(t_c, r_c)}{LY_2(t_c)}e^{r_c}. $$

Moreover, by (4.4.24) we have that $\mu < u(x_c, t_c) + \mu < 1 + \mu$ and $f(t_c, r_c) \leq Le^{-r_c}$. Since $m \geq 2$, then

$$C \leq 2\delta + 2K(1+\mu)^{m-2}(m+\mu) + 2.$$

Choosing δ, μ small enough, we get from the above inequality

$$C \leq 2mK^{m-2} + 2.$$

If we now choose $C > mK^{m-2} + 1$, we obtain a contradiction since C depends on d, $\|u_0\|_{L^\infty}$ and $\|f\|_{L^\infty}$.

□

4.4.5 Some Sobolev estimates

In what follows, we perform some estimations on the solution of the problem (4.4.19). Due to the fact that the tail of the fractional time derivative naturally introduces a right hand side, we cannot choose the natural test function $\ln(u)$ as a test function, because this introduces integrability issues since u can evaluate to zero. To overcome this difficulty, as shown in [6], we will make use of a test function of the form

$$F(t) = \frac{1}{\sigma+1}(t+\mu)^{\sigma} - \mu^{\sigma} t,$$

where $0 < \sigma < 1$, which satisfies the hypothesis of Lemma 4.4.6, for u being a solution of (4.4.19) and satisfying the condition $|u| \leq Ke^{-|x|}$ for some large K. The use of the extension of $u(\varepsilon j, x) = u(0, x)$ for $j < 0$, shows that, u is the solution to (4.4.19) with nonnegative right hand side

$$\delta\Delta\left(u(0,x)\right) + \nabla \cdot \mathcal{K}_s^{\varepsilon}[u(0,x)], \quad \text{for } j \leq 0.$$

We fix cut-off function $\zeta(t)$ with $\zeta(t) \geq K$ for $t \leq -2$ and $\zeta(t) = 0$ for $t \geq -1$ and define

$$u = \zeta + (u-\zeta)_+ - (u-\zeta)_- =: u_\zeta^+ - u_\zeta^- + \zeta.$$

Following the strategy used in [6], we take our test function as $\varepsilon F'\left([u(t,x) - \zeta(t)]\right)$.

On the other hand, u is weak solution of (4.4.19) and satisfies

$$\varepsilon \sum_{j\leq k} \int_{B_R} \zeta \partial_\varepsilon^\alpha u(\varepsilon j) dx + \varepsilon\delta \sum_{j\leq k} \int_{B_R} \nabla\zeta \cdot \nabla u(\varepsilon j)\, dx \\ + \varepsilon \sum_{j\leq k} \int_{B_R} \nabla\zeta \left(u(\varepsilon j) + \mu\right)^{m-1} \nabla \mathcal{K}_s^{\varepsilon}[u] dx = \varepsilon \sum_{j\leq k} f(\varepsilon j, x)\zeta\, dx. \quad (4.4.27)$$

Next we proceed with the a priori estimate of each terms of (4.4.27).

• Estimates of the nonlocal operator in time

We have defined $\tilde{u} = u(t)$ for $\varepsilon j - 1 < t \le \varepsilon j$. From [6, Lemma 4.3] there exist two constants c and c_1 depending on α, T, Λ such that for $\varepsilon < 1$, the following relation holds true

$$\varepsilon \sum_{j\le k} F'\big(u_\zeta^+(\varepsilon j)\big)\partial_\varepsilon^\alpha u(\varepsilon j) \ge c \int_{-\infty}^{T}\int_{-\infty}^{t} [\tilde{u}_\zeta^+(t) - \tilde{u}_\zeta^+(\tau)]^2 (t-\tau)^{-\alpha-1} d\tau\, dt$$
$$+ c\int_{-\infty}^{T} F\big(u_\zeta^+(t)\big)(T-t)^{-\alpha} dt - c_1 \int_{-\infty}^{T} F'\big(\tilde{u}_\zeta^+(t)\big) {}_0^cD_t^\alpha \zeta(t) dt.$$

So we have that

$$\varepsilon \sum_{j\le k} F'\big(u_\zeta^+(\varepsilon j)\big)\partial_\varepsilon^\alpha u(\varepsilon j) \approx \|u\|_{W^{\alpha/2,2}(0,T)}.$$

• **Estimates of the local and nonlocal spatial terms**. For the local spatial term we have

$$\delta\varepsilon \sum_{j\le k}\int_{B_R} \nabla F'\left(u_\zeta^+(\varepsilon j, x)\right)\nabla u\, dx$$
$$= \delta\varepsilon\sigma \sum_{j\le k}\int_{B_R} \left(u_\zeta^+(\varepsilon j, x) + \mu\right)^{\sigma-1} \nabla u_\zeta^+(\varepsilon j, x)\nabla u\, dx$$
$$\ge \delta\sigma \int_0^T \int_{B_R} (\tilde{u} + \mu)^{\sigma-1} |\nabla \tilde{u}|^2\, dx\, dt. \quad (4.4.28)$$

Next we provide the estimate for the nonlocal spatial term

$$\varepsilon \sum_{j\le k}\int_{B_R} \nabla\zeta\,(u+\mu)^{m-1}\,\nabla\mathcal{K}_s^\varepsilon[u]dx$$
$$= \varepsilon \sum_{j\le k}\int_{B_R} \nabla F'(u_\zeta^+)\,(u+\mu)^{m-1}\,\nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]dx$$
$$= \varepsilon \sum_{j\le k}\int_{B_R} \nabla F'(u_\zeta^+)\left(u_\zeta^+ + \zeta + \mu\right)^{m-1}\,\nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]dx.$$

But since $\nabla F'(u_\zeta^+(\varepsilon j, x)) = \sigma\left(u_\zeta^+ + \mu\right)^{\sigma-1}\nabla u_\zeta^+$, then we have

$$\varepsilon \sum_{j\le k}\int_{B_R} \nabla\zeta\,(u+\mu)^{m-1}\,\nabla\mathcal{K}_s^\varepsilon[u]dx$$
$$= \varepsilon \sum_{j\le k}\int_{B_R} \nabla F'(u_\zeta^+)\,(u+\mu)^{m-1}\,\nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]dx$$
$$\ge \varepsilon\sigma \sum_{j\le k}\int_{B_R} \nabla u_\zeta^+\left(u_\zeta^+ + \mu\right)^{m+\sigma-2}\nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]dx$$
$$\ge \frac{\varepsilon\sigma}{m+\sigma-1}\sum_{j\le k}\int_{B_R} \nabla\left(u_\zeta^+ + \mu\right)^{m+\sigma-1}\nabla(-\Delta)^{-1}\mathcal{K}_{1-s}^\varepsilon[u]dx$$
$$\ge \frac{\sigma}{m+\sigma-1}\int_{-2}^{T} \mathcal{E}_\zeta\left(\left(u_\zeta^+ + \mu\right)^{m+\sigma-1}, u\right) dt.$$

The last line involving the bilinear form is well defined. Indeed, if we set

$$\nabla Z(u) = \nabla \left(u_\zeta^+ + \mu\right)^{m+\sigma-1},$$

then

$$\varepsilon \sum_{j\le k} \int_{B_R} \nabla Z(u) \nabla (-\Delta)^{-1} \mathcal{K}_{1-s}^{\varepsilon}[u] dx = \varepsilon \sum_{j\le k} \int_{B_R} Z(u)(-\Delta)(-\Delta)^{-1} \mathcal{K}_{1-s}^{\varepsilon}[u] dx$$
$$= \varepsilon \sum_{j\le k} \int_{B_R} Z(u) \mathcal{K}_{1-s}^{\varepsilon}[u] dx.$$

Now using the generalized Stroock-Varopoulos inequality (4.2.12) with $Z' = (V')^2$ and $\nabla Z(u) = \nabla F'\left(u_\zeta^+(\varepsilon j, x)\right)(u+\mu)^{m-1}$, we get that

$$\varepsilon \sum_{j\le k} \int_{B_R} \nabla F'(u_\zeta^+)\,(u+\mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{K}_{1-s}^{\varepsilon}[u] dx \ge \varepsilon \sum_{j\le k} \int_{B_R} \left| V(u) \left(\mathcal{K}_{1-s}^{\varepsilon}[u]\right)^{\frac{1}{2}} \right|^2 dx,$$

where

$$Z(z) = \sigma \int_0^z (y+\mu)^{\sigma+m-2} dy = \frac{\sigma}{\sigma+m-1} [z+\mu]^{\sigma+m-1},$$

and

$$V(z) = \int_0^z \left[Z'(y)\right]^{1/2} dy.$$

As a consequence $V(u) \in L^2\left((-\infty, T)\,; H_\varepsilon^{1-s}(\mathbb{R}^d)\right)$.

Next we recall from [6, Proposition 10.1], that if $u_\zeta^+(x) - u_\zeta^+(y) \ge 0$, then

$$\left(u_\zeta^+ + \mu\right)^{m+\sigma-1}(x) - \left(u_\zeta^+ + \mu\right)^{m+\sigma-1}(y) \ge \left(u_\zeta^+(x) - u_\zeta^+(y)\right)^{m+\sigma-1}.$$

Hence

$$\varepsilon \sum_{j\le k} \int_{B_R} \nabla F'\left(u_\zeta^+(\varepsilon j, x)\right)(u+\mu)^{m-1} \nabla p\, dx$$
$$\ge \frac{\sigma c}{m+\sigma+1} \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{K}_{1-s}^{\varepsilon}(x,y) |\tilde{u}(x) - \tilde{u}(y)|^{m+\sigma}\, dx\, dy\, dt. \quad (4.4.29)$$

• **Estimate of the whole problem**. Combining these three previous estimates with the right hand side term f we have for a certain constant C depending on $\alpha, s, \Lambda, \sigma, K, T, d$

$$
\begin{aligned}
&c\int_{B_R}\int_0^T\int_0^t[\tilde{u}(t)-\tilde{u}(\tau)]^2(t-\tau)^{-\alpha-1}d\tau\, dt\, dx+\delta\sigma\int_0^T\int_{B_R}(\tilde{u}+\mu)^{\sigma-1}|\nabla\tilde{u}|^2dx\, dt\\
&\quad+\frac{c\sigma}{m-1+\sigma}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathcal{K}^{\varepsilon}_{1-s}(x,y)|\tilde{u}(x)-\tilde{u}(y)|^{m+\sigma}\,dx\,dy\,dt\\
&\quad+c\int_{B_R}\int_0^T F\left(\tilde{u}(t)\right)(T-t)^{-\alpha}dt\,dx\\
&\leq c\int_{B_R}\int_{-2}^T F'\left(\tilde{u}_\zeta^+(t)\right){}_0^cD_t^\alpha\zeta(t)dt\,dx+\int_{-2}^T\int_{B_R}f(x,t)F'\left(u_\zeta^+\right)dx\,dt\\
&\leq C\int_{-2}^T\int_{B_R}\left[\left(u_\zeta^++\mu\right)^\sigma-\mu^\sigma\right]dx\,dt\leq C\int_{-2}^T\int_{B_R}\left[\left(Ke^{-|x|}+\mu\right)^\sigma-\mu^\sigma\right]dx\,dt\\
&\leq C\int_{-2}^T\int_{B_R}K^\sigma e^{-\sigma|x|}dx\,dt\leq C(\alpha,s,\Lambda,\sigma,K,T,d). \qquad (4.4.30)
\end{aligned}
$$

From [6, Proposition 10.2],

$$\left(u_\zeta^++\mu\right)^\sigma-\mu^\sigma\leq 2^\sigma\left(u_\zeta^+\right)^\sigma.$$

So

$$
\begin{aligned}
C\int_{-2}^T\int_{B_R}\left[\left(u_\zeta^++\mu\right)^\sigma-\mu^\sigma\right]dx\,dt&=\int_{-2}^T\int_{B_R}\left[\left(Ke^{-|x|}+\mu\right)^\sigma-\mu^\sigma\right]dx\,dt\\
&\leq 2^\sigma C\int_{-2}^T\int_{B_R}K^\sigma e^{-\sigma|x|}\,dx\,dt\leq C.
\end{aligned}
$$

We then obtain an estimate which remain uniform as $\sigma, \mu \to 0$

$$
\begin{aligned}
&\delta\sigma\int_0^T\int_{B_R}(\tilde{u}+\mu)^{\sigma-1}|\nabla\tilde{u}|^2dx\,dt\\
&\quad+\int_0^T\|\tilde{u}\|^{m+\sigma}_{W^{(2-2s)/(m+\sigma),m+\sigma}(B_R)}dt+\int_{B_R}\|\tilde{u}\|^2_{W^{\alpha/2,2}(0,T)}dx\leq C, \qquad (4.4.31)
\end{aligned}
$$

with the constant C depending only on the exponential decay of f,u_0, and on α, s, d, T, but not on δ, R.

Note that the existence of a weak solution of problem (TPMEFP) is done by passing to the limit step-by-step in the approximating problems. With the estimate (4.4.31) we take $u = \lim_{\mu,\varepsilon} \to 0$, in order to obtain

$$\partial_\varepsilon^\alpha u-\delta\Delta u-\nabla\cdot\left(u^{m-1}\nabla\left(-\Delta\right)^{-s}u\right)=f \qquad [0,T]\times B_R. \qquad (4.4.32)$$

Now the next step will be to provide a compactness result. We state the two following lemmas from which we refer to [6] for similar proofs with slight differences.

Lemma 4.4.7. *Assume for any $v \in \mathcal{F}$,*

$$\int_0^T \|v(t,x)\|^{m+\sigma}_{W^{(2-2s)/(m+\sigma),m+\sigma}(B_R)} + \int_{B_r} \|v(t,x)\|^2_{W^{\alpha/2,2}(0,T)} \leq C. \tag{4.4.33}$$

Then $\mathcal{F}$ is totally bounded in $L^p([0,T]\times B_R)$ for $m \geq 2$ and for $1 \leq p \leq 2$.

The second lemma guarantees that $\nabla(-\Delta)^{-s}u \in L^p$ as $\delta \to 0, R \to \infty$.

Lemma 4.4.8 ([6]). *Let u be a solution to* (4.4.32) *with right hand side f and u_0 both satisfying the exponential bound* (4.4.21). *Then*

$$\int_0^T \|(-\Delta)^{-s}u(t,\cdot)\|^{m+\sigma}_{W^{(2-2s)/(m+\sigma)+2s,m+\sigma}}\, dt \leq C$$

with the constant C depending only on the exponential bounds in (4.4.21), *d,σ,T,m.*

Corollary 4.4.9. *Let u_k be a sequence of solutions to* (4.4.32) *with $R \to \infty$ and $\delta \to 0$. For fixed $\rho > 0$, there exist a subsequence and limit with*

$$u_k \to u_0 \in L^p(B_\rho) \text{ for } 1 \leq p \leq 2 \text{ and } u_k \rightharpoonup u_0 \in L^{m+\sigma}\left(0,T; W^{(2-2s)/(m+\sigma),m+\sigma}\right).$$

Furthermore, for any compactly supported ζ

$$\varepsilon \sum_{j\leq k} \int_{\mathbb{R}^d} \left[\zeta(x,\varepsilon j)\partial^\alpha_\varepsilon u_0(\varepsilon j,x) + u_0^{m-1}\nabla\zeta\nabla(-\Delta)^{-s}u_0\right] = \varepsilon \sum_{j\leq k} \int_{\mathbb{R}^d} f\zeta\, dx. \tag{4.4.34}$$

Proof. The proof of this corollary follows from (4.4.31) and Lemma 4.4.7. On the other hand, from Lemma 4.4.8, and for σ small enough depending on s, then

$$\frac{2-2s}{m+\sigma} + 2s > 1, \quad 0 < \sigma < 1,$$

we have that

$$\nabla(-\Delta)^{-1}\mathcal{K}_{1-s}[u_k] \rightharpoonup \nabla(-\Delta)^{-s}u_0 \in L^{m+\sigma}\left(0,T; W^{\frac{(2-2s)}{(m+\sigma)}+2s-1,m+\sigma}\right).$$

Hence as in [6]

$$\nabla(-\Delta)^{-s}u_k \rightharpoonup \nabla(-\Delta)^{-s}u_0 \in L^{m+\sigma}\left((0,T)\times\mathbb{R}^d\right), \tag{4.4.35}$$

which shows that u_0 is a solution. □

Now we give the proof of Theorem 4.1.2

Proof of Theorem 4.1.2. The proof of the theorem is based on passing to the limit of the regularized solution obtained in Section 4.4.1 and it follows also the original technique presented in [6]. We first assume f, u_0 smooth and satisfying the exponential bounds (4.4.21). Next consider solutions u_ε to (4.4.34) over a finite interval $(0,T)$. The first limit is done as $\varepsilon \to 0$

and it is based on the compactness criteria of type Simon-Aubin-Lions [142] so that there exist a subsequence and a limit $u_\varepsilon \to u_0$ with the weak convergence as in (4.4.35) and strong convergence over compact sets for $1 \le p \le 2$ just as in Lemma 4.4.7 and in [6]. We now consider a sequence of solution $\{u_j\}$ with $\{f_j\}, \{(u_0)_j\} \in \mathcal{C}^\infty$ with $f_j \to f$ and $(u_0)_j \to u_0$ in weak sense to L^∞. Then, there exists a limit solution u with right hand side f satisfying (4.4.21). From Lemma 4.4.5 we can let $T \to \infty$, which ends the proof. □

4.5 Finite speed of propagation and free Boundary property. Proof of Theorem 4.1.3

4.5.1 Finite speed of propagation properties

We start by recalling the definition of finite speed of propagation [153].

Definition 4.5.1. *We say that finite propagation holds for a certain class of solutions u of an evolution equation if*

(a) for given two times $0 \le t_1, t_2 \le T$, the support of the solution at time t_2 is included in a neighbourhood of radius $g(|t_2 - t_1|)$ of the support of $u(t_1)$, where g is a continuous function $\mathbb{R}_+ \to \mathbb{R}_+$ with $g(0+) = 0$,

(b) the function g is independent of the solution under consideration, —we call it uniform finite propagation,

(c) $g(t) \le Ct$ for some constant $C > 0$ —we say that propagation has finite speed.

We now prove Theorem 4.1.3, by using a method similar to the exponential tail control section and the original idea introduced in [45], (also used for further models [18, 146]) but with some technical adaptation to our model.

Proof of Theorem 4.1.3. We assume a nonnegative solution u of (TPMEFP) having bounded initial data $u(x,0) = u_0(x) \le Ae^{-|x|} \le L$. We also assume that u_0 is below the parabola like function $\widetilde{u}_0(x) = a(|x| - b)^2$, $a, b > 0$. The support of $\widetilde{u}_0$ is the ball of radius b and the graphs of u_0 and $\widetilde{u}_0$ are strictly separated in that ball. If we take as a comparison function $\tilde{u}(x,t) = a\left(Ct - (|x| - b)\right)^2$, the goal is to argue the fact that the first point in time and space where the function $u(x,t)$ touches the parabola $\widetilde{u}$ from below happens for $x \ne \infty$ and for $t > 0$. As in the exponential tail control section 4.4.4, we called (x_c, t_c) the contact point for which

$$u(x_c, t_c) = \widetilde{u}(x_c, t_c) = a\left(Ct_c - (|x_c| - b)\right)^2.$$

Similarly as shown in [146, Lemma 7.2] the contact can not happen at the vanishing point $|x_f(t_c)| = (b + Ct_c)$ of the barrier or the boundary of the support of the parabola $\widetilde{u}$ at time t_c.

Let us assume that u is of class $\mathcal{C}^2$. We consider two cases.

- **Case where** $a = L = 1$:

At the first contact point (x_c, t_c), since $u \leq 1$ we have

$$u(x_c, t_c) = h^2, \quad \partial_r u(x_c, t_c) = -2h, \quad \Delta u(x_c, t_c) \leq 2d,$$

where we defined $h =: b + Ct_c - |x_c|$, to be the distance from which x_c lies from $|x_f(t_c)|$. Furthermore

$$\begin{aligned} {}_0^cD_t^\alpha u(x_c, t_c) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_c} (t_c - \tau)^{-\alpha} u'(x_c, \tau) d\tau \\ &= \frac{2aC}{\Gamma(2-\alpha)} \left[(-|x_c| + b) t_c^{1-\alpha} + \frac{C}{(2-\alpha)} t_c^{2-\alpha} \right] \\ &= \frac{2aC}{\Gamma(2-\alpha)} \left[\frac{C}{(2-\alpha)} t_c - (|x_c| - b) \right] t_c^{1-\alpha} \leq 2c_\alpha C \left[Ct_c - (|x_c| - b) \right] t_c^{1-\alpha}, \end{aligned}$$

${}_0^cD_t^\alpha u(x_c, t_c) \geq 2Cht_c^{-\alpha}$.

For $p = \mathcal{K}_s(u)$ and using the following equation

$${}_0^cD_t^\alpha u = (m-1)u^{m-2} \nabla u \cdot \nabla p + u^{m-1} \Delta p + f,$$

we get the inequality

$$2Cht_c^{-\alpha} \leq 2(m-1)h^{2m-3} \left(-\overline{\partial_r} p + \frac{h}{2} \overline{\Delta p} + Ae^{-r_c} \right), \tag{4.5.1}$$

where $\overline{\partial_r} p$ and $\overline{\Delta p}$ are the values of ∇p and Δp at the point (x_c, t_c) as in section 4.4.4. We used also the fact that the function $f(x_c, t_c)$ is controlled by Ae^{-r_c}.

So in order to get a contradiction, we use the estimate for the value of $\overline{\partial_r} p$ and $\overline{\Delta p}$ already obtained in [45] which reads as

$$-\overline{\partial_r} p \leq K_1 + K_2 h^{1+2s} + K_3 h \quad \text{and} \quad \overline{\Delta p} \leq K_4.$$

We set $K = \max\{K_1, K_2, K_3, K_4\}$, we can rewrite the previous inequality as

$$-\overline{\partial_r} p \leq \left(1 + h^{1+2s} + h \right) K \quad \text{and} \quad \overline{\Delta p} \leq K. \tag{4.5.2}$$

Therefore, combining the inequalities (4.5.1) and (4.5.2) we get

$$2Cht_c^{-\alpha} \leq 2(m-1)h^{2m-3} K \left(1 + \frac{3}{2}h + h^{1+2s} \right) + Ae^{-r_c} \tag{4.5.3}$$

$$C \leq (m-1)h^{2m-4} \left(K \left(1 + \frac{3}{2}h + h^{1+2s} \right) + \frac{A}{2h} \right) t_c^\alpha, \tag{4.5.4}$$

which is impossible for C large and independent of the distance h of the boundary of the support of the parabola $\widetilde{u}$ at time t_c, since $m > 2$ and $|h| \leq 1$. Hence for the minimal constant

$$C = C(\alpha, s, d) = \min \left\{ (m-1)h^{2m-4} \left(K \left(1 + \frac{3}{2}h + h^{1+2s} \right) + \frac{A}{2h} \right) t_c^\alpha \right\},$$

it cannot be a contact point point with $h \neq 0$, proving that $m > 2$. For $m < 2$, we do not obtain a contradiction in the estimate (4.5.3), since the term Kh^{2m-4} can be very large for small values of $|h|$.

- **Case where** $a \neq 1$ and $L \neq 1$:

In this part, the goal is to show that the equation is invariant under the scaling. For this purpose we use the scaling property that if u is solution to (TPMEFP), then

$$\widehat{u}(x,t) = Au(Bx, Tt), \tag{4.5.5}$$

is also solution to (TPMEFP) if $T^{\alpha} = A^{m-1}B^{2-2s}$, for $A, B, T > 0$.

We search the parameters A, B, T for which $\widehat{u}$ defined by (4.5.5) satisfies

$$0 \le \widehat{u}(x,t) \le L, \quad \widehat{u}(x,0) \le \widehat{a}(|x| - \widehat{b})^2.$$

In fact at the initial point, the function satisfies $Au(Bx,0) \le A\,(B|x| - b)^2$, then $u(x,t) \le \widetilde{u}(x,t) = (Ct - (|x| - b))^2$ for all $t > 0$. From this observation we have that

$$Au(x,0) \le A\,(B|x| - b)^2 \le AB^2 \left(|x| - \frac{b}{B}\right)^2 \le \widehat{a}\left(|x| - \widehat{b}\right)^2,$$

where $\widehat{a} = AB^2$, $\widehat{b} = \frac{b}{B}$, $A = L$. Using the relation between A, B and T, we get that $B = \left(\widehat{a}/L\right)^{\frac{1}{2}}$ and $T = L^{\frac{m-2+s}{\alpha}}\,\widehat{a}^{\frac{1-s}{\alpha}}$.

Then $\widehat{u}(x,t)$ is below the upper barrier $\widehat{\widetilde{u}} = \widehat{a}\left(\widehat{C}t - (|x| - \widehat{b})\right)^2$, where the new speed is given by

$$\widehat{C} = C(1,1)T/B = C(1,1)L^{\frac{2m-3+2s}{2\alpha}}\;\widehat{a}^{\frac{1-2s}{\alpha}}.$$

□

Remark 4.5.2. *The previous result about the finite speed propagation shows the dependency of the parameter α. This speed increases in the limit when $\alpha \to 0$, for $0 < s < 1/2$. However we recover the result of speed of propagation for PME with potential pressure [146] in the limit when $\alpha \nearrow 1$ as*

$$\widehat{C} = C(1,1)L^{m-\frac{3}{2}+s}\;a^{-s+\frac{1}{2}}.$$

4.5.2 Topological boundary of the support of the solution: growth estimate of the support

As a consequence of the results of finite speed of propagation, we assert the existence of free boundary points by showing that the growth of the support is bounded in a finite time which was stated in Corollary 4.1.4, proved below.

Proof of Corollary 4.1.4. We follow the idea of the proof to the one given in [153]. From section 4.4.5, the support of $u(\cdot,t)$ is bounded and is contained in a ball of radius $r(t)$. For a given time t_1, we look for a barrier as in the previous section with positive constant a. For this purpose as in [45] we choose $L = ar_1^2$ and we use the formula of the parabolic barrier so that $b = r(t_1) + r_1 + \varepsilon$, for $\varepsilon \in (0,1)$. In the limit as $\varepsilon \to 0$ and using the speed estimate in Theorem 4.1.3 we get the inequality

$$r(t) - r(t_1) - r_1 \leq \widehat{C}(t - t_1) = CL^{\frac{2m-3+2s}{2\alpha}}(t_1)\, a^{\frac{1-2s}{2\alpha}}(t - t_1).$$

Using the L^∞ bound $L(t_1) \leq L(0) = L$ and the fact that $L = ar_1^2$, we get

$$r(t) - r(t_1) - r_1 \leq CL^{\frac{2m-3+2s}{2\alpha}}\, L^{\frac{1-2s}{2\alpha}}\, r_1^{\frac{2s-1}{\alpha}}(t - t_1),$$
$$r(t) \leq r(t_1) + r_1 + CL^{\frac{m-1}{\alpha}} r_1^{\frac{2s-1}{\alpha}}(t - t_1).$$

By setting $t_1 = 0$ and estimating the right-hand side in r_1, it yields

$$|x(t)| \leq R + C_2 t^{\alpha/(2-2s)},$$

which implies the desired result. □

4.6 Conclusion

We studied the existence and uniqueness of the general nonlinear nonlocal time evolution equation of porous medium type passing through the existence of time porous porous medium equation with potential pressure and finite speed of propagation. Having obtained the existence of solution for the problem (TFPME), we can start thinking on the question of Hölder regularity theory for the model (TFPME) when $s \to 1$. Since regularity theory is an important notion in the theory of PDEs, their fractional counterparts also play a significance role in the study of problems involving nonlocal behaviour. Hence the motivation for studying regularity properties for (TFPME) when $s \to 1$ in Chapter 4 and for all $\alpha \in (0,1)$ in Chapter 5. Furthermore, regarding the model (TPMEFP), we note that our estimates do not give the exact vanishing set, but only a lower bound $R(t)$ for the radius of the ball where the solution u vanishes. Furthermore in the limit $\alpha \nearrow 1$ and $s \to 1/2$ we can also show that we get the linear growth. In a similar way, for $s \to 0$ and $\alpha \nearrow 1$ we get the standard $t^{1/2}$ growth of the (PME). The difficulty with this problem is to get the Hölder regularity estimate for the problem (TPMEFP).

Chapter 5

A De Giorgi-Nash type theorem for the time porous medium equation

We study the regularity of weak solutions to the nonlinear and nonlocal Cauchy problem involving the fractional derivative. The nonlinearity involved is nondecreasing and continuous, and the initial datum is assumed to be bounded. For a wide class of nonlinearities, including the porous media case, these solutions turn out to be bounded and Hölder continuous for all positive time. Our main result is a De Giorgi-Nash-Moser Hölder type regularity theorem for the time porous medium equation, which gives an interior Hölder estimate for bounded weak solutions. The proof relies on a priori estimates for time fractional operators problems and uses De Giorgi's technique.

The results presented in this chapter have been published in [69].

5.1 Introduction

In this chapter, we focus our attention on problems posed on bounded space domain. More precisely, we consider the nonlinear fractional diffusion equation

$$\partial_t^\alpha u + \Delta\varphi(u) = f \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \qquad u(\cdot,0) = u_0, \tag{TPME}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary, $m > 1$, $\alpha \in (0,1)$ and $d \geq 1$. We point out that the proper definition of nonlocal operator $\partial_t^\alpha u$, which refers to the Caputo fractional derivative of u, has been clearly defined in Chapter 1. When the permeability of the medium changes over time such as in porous medium equation, it might be interesting to use a fractional time derivative. In what follows, we shall consider the extended Caputo or the Marchaud derivative, due to divergence form of the problem to be studied.

The linear operator $\partial_t^\alpha - \Delta$ is often described in the literature as a nonlocal time diffusion operator with memory effects. As for the regularity estimate, a beautiful result has been obtained in [158] for the linear counterparts of problem (TPME). The same is true for the problem (TPME). In the literature (see [23,61] and references therein) one can find analogues of this type of problem where the nonlocal operator involved is the well known fractional Laplacian operator $(-\Delta)^s$ of order $s \in (0,2)$ which leads to the nonlocal nonlinear operator of the form $\partial_t + (-\Delta)^s\varphi(\cdot)$.

The nonlinearity φ is continuous and nondecreasing, and it may be assumed without loss of generality that it satisfies $\varphi(0) = 0$. The local analogue $\partial_t u - \Delta\varphi(u) = 0$ is known as the filtration equation and has been well studied in [12, 60, 64, 153]. The typical example is that of powers, $\varphi(s) = |s|^{m-1}s$, which includes both the case of nonlocal porous media, $m > 1$, and nonlocal fast diffusion, $0 < m < 1$ [12, 56, 61, 137]. Following [61] (where these following conditions were stated), we prove the continuity of bounded solutions when the nonlinearity satisfies

$$\varphi \in C^1(\mathbb{R}), \qquad \varphi(0) = 0, \qquad \varphi'(s) > 0 \quad \text{for } s \neq 0. \tag{5.1.1}$$

Notice that φ can be degenerate at the level 0. However, in this work, we do not consider the case when the nonlinearities are too degenerate, like the Stefan one, $\varphi(s) = (s-1)_+$, or singular, like the one corresponding to fast diffusion. Furthermore, if

$$\begin{aligned} & C_1 \frac{\varphi(r)}{r} \leq \varphi'(s) \leq C_2 \frac{\varphi(r)}{r}, \quad \text{for } 0 < |r| < 1, \ |s| \in (|r|/4, 3|r|), \\ & \sup_{[A,B]} \varphi' \leq D_M \frac{\varphi(B) - \varphi(A)}{B - A}, \qquad \text{if } -M \leq A < B \leq M, \end{aligned} \tag{5.1.2}$$

for some constants $0 < C_1 < C_2$ and $D_M > 0$, we get Hölder regularity. These conditions control the oscillation of the nonlinearity close to the origin, and are only needed to deal with points at which the equation is degenerate. They are satisfied for example if

$$c_1 |r|^{m-1} \leq \varphi'(r) \leq c_2 |r|^{m-1} \quad \text{for } 0 \leq |r| \leq 1,$$

for some constants $0 < c_1 \leq c_2$, $m \geq 1$.

In view of these results, we immediately observe that interior regularity for the time porous medium equation with fractional derivative is missing in the literature. Indeed, from the pioneering work in the local paradigm [64], or in the case of the fractional Laplacian [61], one can find similar results, where the same technique has been used. We note in the linear setting some results in [5, 6, 158]. So, using the De Giorgi technique, we can study interior Hölder regularity estimate for the time porous medium equation. Hence, the main aim of this work.

Presentation of the main results. The purpose of this work is to give the interior regularity estimate for the nonlinear Cauchy problem (TPME) by means of the De Giorgi approach.

In the proof, performed in Section 5.6, we will use De Giorgi's method (that we recalled in Section 5.3)—see [89]. Thus, we will prove that the oscillation of the solution in space-time α-cylinders of radius R,

$$\mathcal{C}_R = \{|x - x_0| < R^\alpha, \, |t - t_0| < R^2\} \subset Q,$$

is reduced to a fraction of the cylinder $\mathcal{C}_{\gamma R}$, $\gamma < 1$, at least by a constant factor ϖ_*. This implies α–Hölder continuity,

$$|u(x,t) - u(x_0,t_0)| \leq C\left(|x - x_0|^{\beta/\alpha} + |t - t_0|^{\beta/2}\right),$$

with an exponent $\beta = \log \varpi_* / \log \gamma^{2/\alpha}$.

As we will see in Section 5.6, Theorem 5.6.2 gives an interior Hölder estimate for bounded weak solutions of the nonlocal time porous medium equation (TPME) in terms of the data and the L^∞-bound of the solution. It can be viewed as the time fractional analogue of the classical parabolic version ($\alpha = 1$) of the well-known De Giorgi-Nash theorem on the Hölder continuity of weak solutions to elliptic equations in divergence form (De Giorgi [89], Nash [124]), see also Moser [121] for the elliptic, as well as the seminal contribution by DiBenedetto [64] for the porous medium equation.

The control of the oscillation in our nonlocal setting follows the procedure developed in [5,6,38] for a linear problem with a rough kernel (which has applications to certain nonlinear problems), combined with some ideas to deal with the nonlinearity borrowed from [12,61].

This chapter is organised as follows: In Section 5.2, we introduce some notations and definitions of function spaces and their associated norms which will be needed in this work. We also provide some preliminary notions on the weak solutions associated to the problem (TPME). In Section 5.3, we recall the main philosophy behind the De Giorgi technique on regularity solution through a simple problem. Next, Sections 5.4 and 5.5 respectively, are devoted to the proofs of some intermediate results on the energy inequalities and their oscillations estimates commonly known as the " L^2, L^∞ " estimates. Finally in Section 5.6 we provide the proof of our main result stated in Theorem 5.6.2 and conclusion in Section 5.7.

5.2 Preliminaries

Let us introduce the following notation for boxes, and balls:

$$\begin{aligned} B_R(x_0) &:= \{x \in \mathbb{R} : |x - x_0| < R\}, \\ \Gamma_R(x_0) &:= [-R, 0] \times B_R(x_0). \end{aligned} \tag{5.2.1}$$

5.2.1 Fractional time derivative properties and its discretization

In this section, we recall must of the results obtained in [5] and recall some of their proofs for the brevity of the readers.

As shown in the Chapter 1, the fractional derivative involved in (TPME) is given as

$$\partial_t^\alpha u(t) = {}_{-\infty}\partial_t^\alpha u(t) = \alpha \int_{-\infty}^t \frac{u(t) - u(\tau)}{(t-\tau)^{1+\alpha}}.$$

This is obtained by an integration by parts of the Caputo derivative

$${}_a^cD_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(\tau)}{(t-\tau)^\alpha}\, d\tau,$$

for $0 < \alpha < 1$, so that

$$\Gamma(1-\alpha)\, {}_aD_t^\alpha u(t) = \frac{u(t) - u(a)}{(t-a)^\alpha} + \alpha \int_a^t \frac{u(t) - u(\tau)}{(t-\tau)^{\alpha+1}}\, d\tau.$$

For notational simplicity, we use the rescaled Caputo derivative $\partial_t^\alpha := \Gamma(1-\alpha)\, {}_aD_t^\alpha$ to avoid writing $\Gamma(1-\alpha)^{-1}$ in all of our formulas. So letting $u(t) \equiv u(a)$ for $t < a$, we get ∂_t^α.

For a function $u(t)$ defined on $[a,t]$ and working with ∂_t^α, there are two advantageous ways of defining $u(t)$ for $t < a$. The first way is to define $u(t) \equiv 0$ for $t < a$. The second way is to define $u(t) \equiv u(a)$ for $t < a$.

We now define the concept of weak solution to the problem (TPME).

Definition 5.2.1. *A weak solution of the time porous medium equation* (TPME) *is a bounded measurable function* $u \in \mathcal{C}(-\infty, T; L^1(\mathbb{R}^d))$ *with* $\varphi(u) \in L^2(-\infty, T; \dot{\mathcal{H}}(\mathbb{R}^d))$ *such that*

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{-\infty}^{T} \frac{\eta(t)u(t)}{(T-t)^\alpha} dt\, dx + \alpha \int_{\mathbb{R}^d}\int_0^T\int_{-\infty}^t \frac{(u(t)-u(\tau), \eta(t)-\eta(\tau))}{(t-\tau)^{1+\alpha}}\, d\tau\, dt\, dx \\ - \int_{\mathbb{R}^d}\int_{-\infty}^{T} u(t)\partial_t^\alpha \eta(t)\, dt + \int_{\mathbb{R}^d}\int_{-\infty}^{0} v(t)\eta(t)\left[\frac{1}{(T-t)^\alpha} + \frac{1}{(t-a)^\alpha}\right] dt\, dx \\ + \int_{-\infty}^{T} \mathcal{B}[\varphi(u), \eta]\, dt = \int_{\mathbb{R}^d}\int_0^T f\eta\, dxdt, \quad (5.2.2)\end{aligned}$$

for every $\eta \in \mathcal{C}_0^\infty(Q)m$ *and the bilinear form* $\mathcal{B}[f,g] = \nabla f \cdot \nabla g$.

For a function $u(t)$ defined on (a,T), defining $u(t) = 0$ for $t < a$ is useful when obtaining an energy estimate related to the weak formulation (5.2.2).

Lemma 5.2.2 ([5])**.** *Let* $u \in C([a,T])$*. If we extend* u *to all of* $\mathbb{R}$ *by having* $u(t) = 0$ *for* $t < a$ *and then reflecting evenly across* T*, we obtain*

$$\alpha \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt \le 8\left(\alpha \int_a^T\int_a^t \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt + \int_a^T \frac{u^2(t)}{(t-a)^\alpha}\right).$$

Proof. The proof is borrowed from [5]. We note that since the integrand is symmetric in τ and t we have

$$\int_a^T\int_a^T \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt = 2\int_a^T\int_a^t \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt.$$

By the even reflection across the point T, we then have

$$\int_a^{2T-a}\int_a^{2T-a} \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt \le 8\int_a^T\int_a^t \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt.$$

Since $u(t) = 0$ if $|t| \le |a|$, we only have to consider when $t \in (a, 2T-a), \tau \notin (a, 2T-a)$ and vice-versa.

$$\begin{aligned}\alpha \int_a^{2T-a}\int_{-\infty}^{a} \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{1+\alpha}}\, d\tau\, dt &= \alpha \int_a^{2T-a} u^2(t)\int_{-\infty}^{a} |t-\tau|^{-1-\alpha}\, d\tau\, dt \\ &= \int_a^{2T-a} \frac{u^2(t)}{(t-a)^\alpha}\, dt \le 2\int_a^T \frac{u^2(t)}{(t-a)^\alpha}\, dt.\end{aligned}$$

The other three remaining pieces of integration are bounded exactly in the same manner. □

Next we recall the following estimate

Lemma 5.2.3 ([5]). *Let*

$$h(t) := \max\{|t|^{\nu} - 1, 0\}$$

with $\nu < \alpha$. *Then*

$$\partial_t^{\alpha} h \geq -c_{\nu,\alpha}$$

for $t \in \mathbb{R}$. *Here,* $c_{\nu,\alpha}$ *is a constant depending only on* α, ν.

Proof. We recall this proof from [5]. By definition

$$\partial_t^{\alpha} h = \int_a^t \frac{-\nu|\tau|^{\nu-1}}{(t-\tau)^{\alpha}}\, d\tau \geq \int_{-\infty}^t \frac{-\nu|\tau|^{\nu-1}}{(t-\tau)^{\alpha}}\, d\tau.$$

Since $|\tau|^{\nu-1}$ and $(t-\tau)^{-\alpha}$ are both increasing functions of τ for $\tau < 0$, it follows that

$$\int_{-\infty}^t \frac{\nu|\tau|^{\nu-1}}{(t-\tau)^{\alpha}}\, d\tau,$$

is an increasing function of t. If $t \leq -1$, then

$$\partial_t^{\alpha} h \geq \int_{-\infty}^t \frac{-\nu|\tau|^{\nu-1}}{(t-\tau)^{\alpha}}\, d\tau \geq \int_{-\infty}^{-1} \frac{-\nu|\tau|^{\nu-1}}{(-1-\tau)^{\alpha}}\, d\tau \geq -c_{\alpha,\nu}.$$

If $t > -1$, then

$$\partial_t^{\alpha} h(t) \geq \int_{-\infty}^{-1} \frac{-\nu|\tau|^{\nu-1}}{(t-\tau)^{\alpha}}\, d\tau \geq \int_{-\infty}^{-1} \frac{-\nu|\tau|^{\nu-1}}{(-1-\tau)^{\alpha}}\, d\tau \geq -c_{\alpha,\nu}.$$

□

Next we proceed with the discretization in time. This discretization will also be useful when proving the Hölder continuity. Since we do not know a priori if the weak formulation problem will have the required regularity property. To find a solution, we subdivide the interval (a, T) into k intervals and let $\varepsilon = T/k$. For each fixed k, $-\infty < j \leq k$, and for all $t < a$, so that $u(a+\varepsilon j, x) = u(a, x)$ for $j < 0$, the discrete fractional derivative is read as

$$\partial_{\varepsilon}^{\alpha} u(a+\varepsilon j) := \varepsilon\alpha \sum_{-\infty<i<j} \frac{u(a+\varepsilon j) - u(a+\varepsilon i)}{(\varepsilon(j-i))^{1+\alpha}}. \tag{5.2.3}$$

The following integration by parts type estimate will be useful

Lemma 5.2.4 ([5]).

$$\begin{aligned}\alpha \sum_{0<j\leq k} \varepsilon u(a+\varepsilon j)\partial_{\varepsilon}^{\alpha} u(a+\varepsilon j) \geq{}& \varepsilon^{1-\alpha} \sum\sum_{0\leq i<j\leq k} \frac{u^2(a+\varepsilon j) - u^2(a+\varepsilon i)}{(j-i)^{1+\alpha}} \\ &+ \frac{\varepsilon^{1-\alpha}}{2} \sum_{0<j<k} \frac{u^2(a+\varepsilon j)}{2^{1+\alpha}(k-j)^{\alpha}} + \frac{\varepsilon^{1-\alpha}}{2} \sum_{0<j\leq k} \frac{u^2(a+\varepsilon j)}{2j^{\alpha}} \\ &- \varepsilon^{1-\alpha} \sum_{0<j\leq k} \frac{u(a)u(a+\varepsilon j)}{j^{\alpha}}.\end{aligned} \tag{5.2.4}$$

Proof. For notational simplicity, we assume throughout this proof that $a = 0$. For $i > 0$ we write

$$u(\varepsilon j)[u(\varepsilon j) - u(\varepsilon i)] = [u(\varepsilon j) - u(\varepsilon i)]^2/2 + [u^2(\varepsilon j) - u^2(\varepsilon i)]/2.$$

We note that

$$\sum_{2j-k \leq i < j} \frac{u^2(j)}{(j-i)^{1+\alpha}} = \sum_{j < i \leq k} \frac{u^2(j)}{(i-j)^{1+\alpha}}.$$

We thus conclude that

$$\begin{aligned} \alpha \sum_{0<j\leq k} \varepsilon u(\varepsilon j)\partial_\varepsilon^\alpha u(\varepsilon j) &= \alpha \varepsilon^{1-\alpha} \sum\sum_{0\leq i<j\leq k} \frac{[u(\varepsilon j) - u(\varepsilon i)]^2}{2(j-i)^{1+\alpha}} \\ &+ \varepsilon^{1-\alpha} \sum_{0<j\leq k} \sum_{-\infty<i<2j-k} \frac{u^2(\varepsilon j)}{2(j-i)^{1+\alpha}} \\ &+ \varepsilon^{1-\alpha} \sum_{0<j\leq k} \sum_{-\infty<i<j} \frac{u^2(\varepsilon j)/2 - u(\varepsilon j)u(0)}{(j-i)^{1+\alpha}}. \end{aligned} \tag{5.2.5}$$

We also have the following bound for $l < j$

$$\sum_{-\infty<i<l} (j-i)^{-(1+\alpha)} \geq 2^{-(1+\alpha)} \int_{-\infty}^{l} (j-t)^{-(1+\alpha)} \geq 2^{-(1+\alpha)} (j-l)^{1+\alpha}. \tag{5.2.6}$$

Applying (5.2.6) to the appropriate terms in (5.2.5), and ignoring the appropriate positive term when $j = k$, we obtain (5.2.4). □

The next lemma is analogous to Lemma 5.2.3 for the discrete Caputo derivative.

Lemma 5.2.5 ([5]). *Let h be as in Lemma 5.2.3. Then for $0 < \varepsilon < 1$, there exists $c_{\nu,\alpha}$ depending on α and ν but independent of a such that*

$$\partial_\varepsilon^\alpha h(t) \geq -c_{\nu,\alpha},$$

for $t \in \varepsilon\mathbb{Z}$ and $a < t < 0$.

The next lemma gives a fractional Sobolev bound for an extension of discrete functions. Throughout this chapter whenever we have a function u defined on $\varepsilon\mathbb{Z}$, we extend u to all of $\mathbb{R}$ by

$$u(t) = u(\varepsilon j) \quad \text{for } j-1 < t \leq j.$$

This extension works particularly well for the Caputo derivative.

Lemma 5.2.6. *Let $j \geq 1$.*

$$\frac{1}{j^\alpha} \leq \int_{j-1}^{j} \frac{1}{t^\alpha}\, dt \leq \frac{1}{(1-\alpha)j^\alpha}.$$

Proof. The first inequality is immediate since $j^{-\alpha} \leq t^{-\alpha}$ for $t > j$. We now note that

$$\int_{j-1}^{j} t^{-\alpha} = \frac{1}{1-\alpha}\left[j^{1-\alpha} - (j-1)^{1-\alpha}\right]$$
$$= \frac{j^{-\alpha}}{1-\alpha}\left[j - (j-1)\left(\frac{j}{j-1}\right)^{\alpha}\right] \leq \frac{j^{-\alpha}}{1-\alpha}\left[j-(j-1)\right] = \frac{1}{(1-\alpha)j^{\alpha}}.$$

□

Lemma 5.2.7. *If u_ε is the appropriate extension of u, then there exists c depending on α, but independent of a such that if $a < -1$, then with $a + \varepsilon k = T$*

$$\alpha \int_a^T \int_{-\infty}^t \frac{(u(t)-u(s))^2}{(t-s)^{1+\alpha}}\, ds\, dt \leq c\varepsilon^2 \alpha \sum\sum_{i<j\leq k} \frac{(u(a+\varepsilon j) - u(a+\varepsilon i))^2}{(\varepsilon(j-i))^{1+\alpha}}$$

and

$$\int_a^T \frac{u(t)}{(t-a)^{\alpha}} \leq c\varepsilon \sum_{0<j\leq k} \frac{u^2(a+\varepsilon j)}{(a+\varepsilon j)^{\alpha}}.$$

Proof. We note that for $i < j$

$$\int_{\varepsilon(j-1)}^{\varepsilon j} \int_{\varepsilon(i-1)}^{\varepsilon i} \frac{1}{(t-s)^{1+\alpha}}\, ds\, dt \leq \varepsilon^2 \max\left\{2^{1+\alpha}, \frac{2-2^{1+\alpha}}{1-\alpha}\right\} \frac{1}{(\varepsilon(j-i))^{1+\alpha}}.$$

Then

$$\alpha \int_a^T \int_a^t \frac{(u(t)-u(s))^2}{(t-s)^{1+\alpha}}\, ds\, dt \leq c\varepsilon^{1-\alpha}\alpha \sum\sum_{0\leq i<j\leq k} \frac{(u(\varepsilon j) - u(\varepsilon i))^2}{(j-i)^{1+\alpha}}.$$

The first conclusion then follows.

We now claim from Lemma 5.2.6,

$$\frac{1}{j^{\alpha}} \leq \int_{j-1}^{j} t^{-\alpha}\, dt \leq \frac{1}{(1-\alpha)j^{\alpha}}. \tag{5.2.7}$$

The first inequality is trivial. To prove the second inequality we compute

$$\int_{j-1}^{j} t^{-\alpha} = \frac{1}{1-\alpha}\left[j^{1-\alpha} - (j-1)^{1-\alpha}\right] = \frac{j^{-\alpha}}{1-\alpha}\left[j - (j-1)\left(\frac{j}{j-1}\right)^{\alpha}\right]$$
$$\leq \frac{j^{-\alpha}}{1-\alpha}.$$

□

5.3 Background on the De Giorgi technique for regularity of solution

We dedicate this part to the brief description of the technique introduced by De Giorgi to study the regularity of solutions to elliptic equations with rough coefficients. Most of the material to expose this background part are borrowed from [38, 42, 53, 152].

To study the regularity of solutions to elliptic equations with rough coefficients, De Giorgi introduced his technique in 1957 [89] to solve the 19th Hilbert problem. In his work, De Giorgi showed the regularity and analyticity of variational ("energy minimizing weak") solutions to nonlinear elliptic variational problems, based on a very geometric method. Later on, John Nash independently introduced similar techniques in 1958 [124]. Subsequently, Moser provided a new formulation of the proof in [121]. Those methods are now usually called De Giorgi-Nash-Moser techniques. These methods have been extended to degenerated cases, like the p-Laplacian, first in the elliptic case by Ladyzhenskaya and Uralt'seva [107]. The degenerated parabolic cases were covered later by DiBenedetto [64] (see also DiBenedetto, Gianazza and Vespri [65]. More recently, these techniques have been extended to integral operators, like the fractional diffusion with fractional Laplacian [5, 38, 44], Kassmann [94] and Felsinger and Kassmann [81]. Furthermore these methods have also been successfully applied for nonlocal parabolic problems with fractional derivative [3–6, 158]. Further application to fluid mechanics and relatives can be found in [42, 152].

In general, the De Giorgi techniques say that any weak solution of a given linear elliptic equation with rough diffusion coefficients is Hölder continuous over any closed balls compactly included in the defining region of the given equation. In this section, we first state a common version of the famous De Giorgi's theorem. Next, we present the first half of its proof which deals with the L^∞-boundness of weak solutions over closed balls compactly included in the defining region of the given elliptic equation. The first half of the proof for the De Giorgi's theorem is strong enough in illustrating those basic ideas that can be applied to deduce the local L^∞-boundness of weak solutions for the incompressible Navier-Stokes equations under some suitable assumptions made on the weak solutions under consideration. Once the local L^∞-boundedness of our weak solutions for the nonlocal porous medium equations is established under some suitable assumption, we can at once invoke the famous theorem of Serrin [138] to deduce the C^∞-smoothness of our given weak solutions.

Before we give the statement of the De Giorgi's theorem and the first half of its proof, let us mention that the present background is based on [38, 42, 53, 152], and the treatment as presented in [152] will have an equal influence on the presentation given below.

We begin by recalling the following definition

Definition 5.3.1 ([53]). *Let $a_{ij} \in L^\infty(\Omega)$ be measurable functions of class L^∞ defined over some bounded open set Ω of $\mathbb{R}^d$, which are symmetric, that means $a_{ij} = a_{ji}$ and enjoy the uniform elliptic bounded condition*

$$\Lambda^{-1}|y|^2 \leq \sum_{i,j} a_{ij} y_i y_j \leq \Lambda |y|^2.$$

We say that $u \in W^{1,2}(\Omega)$ *is a weak solution of the elliptic equation* $\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$ *on* Ω*, if it happens that the following equality is valid for all test functions* $\zeta \in W_0^{1,2}(\Omega)$.

$$\int_\Omega \sum_{ij} a_{ij}\partial_j u \partial_i \zeta \, dx = 0.$$

Next with the above Definition 5.3.1, we now recall the following De Giorgi's theorem.

Theorem 5.3.2 (De Giorgi [89]). *Consider an elliptic equation* $\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$ *defined on the unit ball* B_1 *centered at the origin of* $\mathbb{R}^d$*, where the coefficients* $a_{ij} \in L^\infty(B_1)$ *are some bounded measurable functions verifying the following two conditions*

- $a_{ij} = a_{ji}$*, for any* $1 \le i, j \le N$*;*
- *the functions* a_{ij} *verify the uniform elliptic condition in the sense that there exists some positive elliptic positive constant* Λ *for which*

$$\Lambda^{-1}|y|^2 \le \sum_{i,j} a_{ij} y_i y_j \le \Lambda |y|^2$$

is valid for all $y \in \mathbb{R}^d$.

Then, there exists some positive constants C *and* β*, depending only on* Λ*,* d*, and* $\|a_{ij}\|_{L^\infty(B_1)}$ *such that every weak solution* $u \in W^{1,2}(B_1)$ *of the elliptic equation*

$$\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$$

will satisfy the following inequality estimation

$$\|u\|_{C^\beta(B_{1/2})} \le C\left[\|u\|_{L^2(B_1)} + \|\nabla u\|_{B_1}\right].$$

The proof is based on the interplay between the Sobolev inequality, which says that $\|u\|_{L^{2+\varepsilon}}$ is controlled by $\|\nabla u\|_{L^2}$, and the energy inequality, which basically says that, in turn, since u is a solution of equation, $\nabla u_\sigma\|_{L^2}$ is controlled by $u_\sigma\|_{L^2}$ for every truncation $u_\sigma = [u - \sigma]_+$.

We next recall from [53] the first half of the proof for the De Giorgi's theorem. More precisely, we will show that every weak solution $u \in W^{1,2}(B_1)$ of the elliptic equation $\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$ must satisfy the following estimation

$$\|u\|_{L^\infty(B_{1/2})} \le C\|u\|_{W^{1,2}_{B_1}},$$

where C is some universal positive constant depending only on Λ, d, and $\|a_{ij}\|_{L^\infty(B_1)}$.

Next we will define some truncated functions which will be handy for further analysis.

For any given weak solution $u \in W^{1,2}(B_1)$ to the given linear elliptic equation

$$\sum_{i,j} \partial_i (a_{ij} \partial_j u) = 0,$$

we define

- For each integer $k \geq 0$, let $u_k = \left[u - \left(1 - \frac{1}{2^k}\right) \right]_+$. That is, u_k is the positive part of the function $u - \left(1 - \frac{1}{2^k}\right)$, or, we say that u_k is the truncation of u at the level $1 - \frac{1}{2^k}$.

- For each $k \geq 0$, let $B_k = B_{\left(\frac{1}{2} + \frac{1}{2^{k+1}}\right)}$ be the ball with radius $\left(\frac{1}{2} + \frac{1}{2^{k+1}}\right)$ centered at the origin of $\mathbb{R}^d$.

We also need a sequence $\{\phi_k\}_{k=1}^{\infty}$ of test functions in $C_0^{\infty}(B_1)$ which verifies the following properties.

- For each $k \geq 1$, we have $0 \leq \phi_k \leq 1$ on B_1.

- For each $k \geq 1$, we have $\phi_k = 1$ on B_k, and that $\phi_k = 0$ on $B_1 - B_{k-1}$.

- For each $k \geq 1$, we have $|\nabla \phi_k\| \leq C 2^k$ on B_1, where C is some constant depending only on the dimension of $\mathbb{R}^d$.

Next step is to present the energy of the truncation of our weak solution u at the level $\left(1 - 2^{-k}\right)$.

The proof is based on energy. We consider a sequence of level sets of energy on shrinking balls. As mentioned above, we have a regularization effect. So we expect to have a layer close to ∂B_1. In the recursive process, we want to escape from this layer.

We consider the family of balls $\widetilde{B}_k$ centered at zero with radius $\left(1 + 2^{-k}\right)$. We notice that $\widetilde{B}_0 = B_1$ and $\widetilde{B}_k$ converges to $B_{1/2}$ when k converges to infinity. The sequence of ball $\widetilde{B}$ goes in a dyadic way from B_1 to $B_{1/2}$.

We consider in the same way, a family of "energy levels" C_k going in a dyadic way from 0 to $1/2$:

$$C_k = \left(1 - 2^{-k}\right).$$

We now define the sequence of energy above the level set C_k in the ball $\widetilde{B}_k$:

$$A_k = \int_{B_k} |u_k|^2 + |\nabla u_k|^2,$$

for every $k \geq 1$.

The main point of the De Giorgi's method is to build up a nonlinear recurrence relation for the sequence A_k. More precisely, we need to construct some universal positive constant C_1 and some positive index $\nu > 1$ such that the nonlinear recurrence relation

$$A_k \leq C_1^k A_{k-1}^{\nu}, \tag{5.3.1}$$

is valid for any given weak solution $u \in W^{1,2}(B_1)$ of the elliptic equation $\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$. Hence, this leads to the conclusion that $\lim_{k\to\infty} A_k = 0$, provided the given initial value A_0 is small enough, then the nonlinear term will be even smaller.

Consequently, in the ball of radius $1/2$, the energy above the level $1/2$ vanishes. And so, u is smaller than $1/2$ in this ball. The nonlinear inequality (5.3.1) is obtained via rather simple tools. We used linear tools as the energy estimate (which is linear since the equation itself is linear) and the Sobolev imbedding.

Next by taking the definition of ϕ_k into account, and by employing the uniform elliptic condition, we have the inequality

$$\begin{aligned}\int_{B_k} |\nabla u_k|^2 &\leq \int_{B_1} \phi_k^2 |\nabla u_k|^2 \leq \Lambda \int_{B_1} \sum_{i,j} a_{ij} \left(\partial_j u_k\right) \left(\partial_i u_k\right) \phi_k^2 \\ &\leq 4\Lambda \int_{B_1} \sum_{i,j} a_{ij} \left(\partial_i \phi_k\right) \left(\partial_j \phi_k\right) u_k^2 \leq 4\Lambda^2 \int_{B_1} |\nabla \phi_k|^2 u_k^2 \\ &\leq 4^{k+1}\Lambda^2 C^2 \int_{B_{k-1}} u_k^2.\end{aligned}$$

That is, we have,

$$\int_{B_k} |\nabla u_k|^2 \leq 4^k \Lambda^2 C \int_{B_{k-1}} u_k^2.$$

By making use of the Sobolev's embedding theorem for a given bounded open region with smooth boundary, we can find a constant C_2, depending only on the dimension of $\mathbb{R}^d$, such that, for every positive k, and every $f \in W^{1,2}\left(B_k\right)$, we have

$$\|f\|_{\frac{2d}{d-2} B_k} \leq C_2 \left[\|f\|_{L^2_{B_k}} + \|\nabla f\|_{L^2(B_k)} \right].$$

We note that such a universal constant C_2 exists because the ball $B_{1/2}$ is included in B_k, for every $k \geq 0$,and so the family of open balls B_k will not shrink down to zero, as k becomes large.

Now, by applying the Chebyshev inequality and the above Sobolev inequality, we can at once carry out the following estimation for all $k \geq 1$.

$$\begin{aligned}\int_{B_{(k-1)}} u_k^2 &= \int_{B_{(k-1)}} u_k^2 \chi_{\{u_k>0\}} \leq \int_{B_{(k-1)}} u_k^2 \chi_{\{u_{(k-1)}>2^{-k}\}} \\ &\leq 2^{k\left(\frac{2d}{d-2}-2\right)} \int_{B_{(k-1)}} u_k^2 u_{(k-1)}^{\frac{2d}{d-2}-2} \leq 16^{\frac{k}{d-2}} \int_{B_{(k-1)}} u_{(k-1)}^{\frac{2d}{d-2}} \\ &16^{\frac{k}{d-2}} C_2 \left[\|u_{(k-1)}\|_{L^2_{B_{(k-1)}}}^{\frac{2d}{d-2}} + \|\nabla u_{(k-1)}\|_{L^2(B_{(k-1)})}^{\frac{2d}{d-2}} \right] \\ &\leq 2C_2 16^{\frac{k}{d-2}} A_{(k-1)}^{\frac{2d}{d-2}}.\end{aligned}$$

We also have that

$$\int_{B_k} |\nabla u_k|^2 \leq C\Lambda^2 4^k \int_{B_{k-1}} u_k^2 \leq 2C_2 C\Lambda^2 4^k 16^{\frac{k}{d-2}} A_{(k-1)}^{\frac{2d}{d-2}}.$$

Combining our last two inequalities, we may deduce that, for every $k \geq 1$, we have

$$A_k \leq C_0^k A_{(k-1)}^{\frac{2d}{d-2}},$$

where C_0 stands for some universal constant depending only on Λ and d. The expository of the argument will be achieved, when the following technical lemma in the theory of recurrence relation [152] holds true.

Lemma 5.3.3. *For any positive constant $C > 0$ and any positive $\beta > 1$, there exists some sufficiently small $\delta_0 > 0$ so that whenever we have a sequence $\{a_k\}_{k\geq 0}$ of nonnegative numbers verifying $0 < a_0 \leq \delta_0$ and $a_k \leq C^k a_{k-1}^{\beta}$, for all $k \geq 1$, we will have that $\lim_{k\to\infty} a_k =$ 0.*

Indeed, by applying the above Lemma 5.3.3 in the case when $\beta = \frac{d}{d-2}$. It comes out that there exists some sufficiently small $\delta_0 > 0$, depending only on Λ and d, such that for any weak solution $u \in W^{1,2}(B_1)$ of the elliptic equation

$$\sum_{i,j} \partial_i \left(a_{ij} \partial_j u\right) = 0,$$

which verifies

$$\|u\|_{L^2(B_1)}^2 + \|\nabla\|_{L^2(B_1)}^2 = A_0 \leq \delta_0,$$

we have $\lim_{k\to\infty} A_k = 0$, which in particular tells us that

$$\int_{B_{1/2}} [u-1]_+^2 \leq \lim_{k\to\infty} A_k = 0,$$

and hence $u \leq 1$, almost everywhere on $B_{1/2}$.

The above argument shows that if our weak solution u to the given linear elliptic equation satisfies $\|u\|_{W^{1,2}(B_1)} \leq \delta_0$, then we must have $u \leq 1$ almost everywhere on $B_{1/2}$. Next, if we multiply our weak solution u by -1, $-u$ is just another weak solution to the same elliptic equation $\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$. Hence, if u is a weak solution of $\sum_{i,j} \partial_i \left(a_{ij}\partial_j u\right) = 0$ which verifies the condition that $\|u\|_{W^{1,2}(B_1)} \leq \delta_0$, then we will also have $(-u) \leq 1$ almost everywhere on $B_{1/2}$, which means the same thing as $u \geq -1$ almost everywhere on $B_{1/2}$.

So, in conclusion, every weak solution $u \in W^{1,2}(B_1)$, with $\|u\|_{W^{1,2}(B_1)} \leq \delta_0$, must satisfy $\|u\|_{L^\infty(B_{1/2})} \leq 1$. Hence, that completes the first half of the proof of the famous De Giorgi's theorem.

Remark 5.3.4. *We have gone through this background on the De Girogi's technique for the elliptic case. In the forthcoming sections, we adapt this strategy to the nonlinear parabolic problem as it is in the linear case [5, 38] and nonlinear setting in [6, 61].*

5.4 The First De Giorgi Lemma

In this section, we prove Theorem 5.6.2. The main ingredient of the proof are the De Giorgi lemmas, commonly known as the " L^2, L^∞ " estimates. These techniques have been previously used [5,38,61]. The nonlinearity part being confined into the fractional time derivative operators, we will need to establish some energy estimates for the weak solutions of

$$\partial_t^\alpha \vartheta(u) + \Delta u = f, \tag{5.4.1}$$

for different functions ϑ and operator ∂_t^α related, respectively, to our original function $\beta = \varphi^{-1}$ and our original nonlocal operator ∂_t^α. To be more precise, ϑ will have the form $\vartheta(s) = a\beta(bs + c)$ for some a, $b > 0$ [61]. Hence, in the sequel we always assume without further mention that

$$\vartheta \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{s_0\}), \quad \vartheta'(s) > 0 \quad \text{for } s \neq s_0, \qquad \text{for some } s_0 \in \mathbb{R}.$$

Indeed the first step is to derive a priori estimates for weak solutions. We need integral inequalities that measure the behaviour of the weak solutions for (5.4.1) near its infimum and its supremum in the interior of an appropriate cylinder. At this point, we can ignore the equation and it remains to show that functions that satisfy these inequalities are Hölder continuous.

5.4.1 Energy inequalities estimate

We begin by developing the necessary energy inequalities associated to the structure of equation (5.4.1). The first inequality that needs to be proven is, somehow, a generalization of the famous Caccioppoli inequality [87] that is often used for elliptic equations. We use this inequality to control, in some sense, the norm of the fractional derivative $\partial_t^\alpha \vartheta(u)$, the norm of the gradient ∇u with the norm of the function u itself.

We first introduce the following Lipschitz function ψ:

$$\psi(t,x) := (|t|^{\alpha/2} - 1)_+ + (|x| - 1)_+ ,$$

so that for $L \geq 0$, we define

$$\psi_L(t,x) = L + \psi(t,x). \tag{5.4.2}$$

Let us introduce (see e.g., [61]) $\ell = \inf_{\{u-\psi_L \geq 0\}} u \geq 0$ and $M = \sup_{\{u-\psi_L \geq 0\}} u < \infty$.

Lemma 5.4.1 (Energy estimates)**.** *Let u be a weak solution of the nonlocal time porous medium equation (TPME) in some finite time interval I including (t_1,t_2). Then,*

$$\begin{aligned}\Lambda_1 \int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\, dx + \int_{t_1}^{t_2} \|(u-\psi_L)_+\|^2_{H^1(\mathbb{R}^d)} \\ - (\Lambda_2 - \Lambda_1) \int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\, dx\Big|_{t=t_1} \\ \leq C_{\alpha,\Lambda_1,\Lambda_2} \int_{\mathbb{R}^d}\int_{t_1}^{t_2} \Big[(u-\psi_L)_+^2 + (u-\psi_L)_+ + \chi_{\{(u>\psi_L\}}\Big],\end{aligned} \tag{5.4.3}$$

where $\Lambda_1 = \inf_{\ell\leq\tau\leq M} \vartheta'(\tau)$ and $\Lambda_2 = \vartheta(M) - \vartheta(\ell)$.

Proof. If we multiply (5.4.1) by the function $(u-\psi_L)_+$ and integrate over some finite time interval $(t_1,t_2) \subset I$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{t_1}^{t_2} \partial_t\big[(u-\psi_L)_+(\partial_t^{\alpha-1}\vartheta)\big]\, dx\, dt - \int_{\mathbb{R}^d}\int_{t_1}^{t_2} \partial_t(u-\psi_L)_+(\partial_t^{\alpha-1}\vartheta)\, dx\, dt \\ + \int_{\mathbb{R}^d}\int_{t_1}^{t_2} \nabla(u-\psi_L)_+ \cdot \nabla u\, dx\, dt = \int_{\mathbb{R}^d}\int_{t_1}^{t_2} (u-\psi_L)_+ f(x,t)\, dx\, dt,\end{aligned} \tag{5.4.4}$$

where the bilinear form $\mathcal{B}$ is defined as in Definition (5.2.2), so that

$$\mathcal{B}[u,(u-\psi_L)_+] = \nabla u \cdot \nabla(u-\psi_L)_+.$$

Using the properties of ϑ, $\partial_t\vartheta$ and $\partial_t^\alpha\vartheta$, if $\ell = \inf_{\{u\geq\psi\}} u \geq 0$ and $M = \sup_{\{u\geq\psi\}} u < \infty$, then the above weak formulation (5.4.4) takes the form

$$\begin{aligned}\Lambda_1 \int_{\mathbb{R}^d}\int_{t_1}^{t_2} (u-\psi_L)_+(\partial_t^\alpha u)\, dx\, dt + \int_{t_1}^{t_2} \mathcal{B}\,[u,(u-\psi_L)_+]\, dt \\ \leq (\Lambda_2 - \Lambda_1) \int_{\mathbb{R}^d} (u-\psi_L)_+(\partial_t^{\alpha-1} u)\, dx\Big|_{t=t_1} + \int_{\mathbb{R}^d}\int_{t_1}^{t_2} (u-\psi_L)_+ f(x,t)\, dx\, dt,\end{aligned} \tag{5.4.5}$$

where

$$\Lambda_1 = \inf_{\ell\leq\tau\leq M} \vartheta'(\tau), \qquad \Lambda_2 = \vartheta(M) - \vartheta(\ell). \tag{5.4.6}$$

Since we do not know a priori if the cut-offs of u were valid test functions, we prove the lemmas for the sequence of approximating functions u_ε and obtain the results of the Lemmas for the solution u. In the next two sections, we will abuse notation for convenience and also to make the proofs more transparent. We will write u to mean a solution of (5.4.1) and assume that ε is understood.

We subdivide the interval (a,T) into k intervals and let $\varepsilon = T/k$. We extend $u(t,x) = u(a+\varepsilon j,x)$, $\vartheta(t,x) = \vartheta(a+\varepsilon j,x)$, and $\psi(t,x) = \psi(a+\varepsilon j,x)$ for $a+\varepsilon(j-1) < t \leq a+\varepsilon j$. In this setting, for each fixed k we may solve via recursion

$$\alpha\varepsilon\sum_{i<j}\frac{\vartheta(a+\varepsilon j,x)-\vartheta(a+\varepsilon i,x)}{(\varepsilon(j-i))^{1+\alpha}} - \Delta u(a+\varepsilon j,x) = f(a+\varepsilon j,x), \tag{5.4.7}$$

for each $-\infty < j \leq k$. Here $f(t,x) = f(a,x)$ for $t < a$, so that $u(a+\varepsilon j,x) = u(a,x)$ for $j < 0$.

If u is a solution to (5.4.7), we consider the truncated function $\varepsilon[u-\psi_L]_+$ as a test function, where ψ_L is defined as in (5.4.2). We will only consider $L \leq 1/2$, so that the assumption on the initial condition will apply. We add in j and integrate over $\mathbb{R}^d$ to obtain

$$\begin{aligned}\Lambda_1\int_{\mathbb{R}^d}\sum_{0<j_1<j_2\leq k}&\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx\\ &+\sum_{0<j_1<j_2\leq k}\varepsilon\mathcal{B}\left[u,(u-\psi_L)_+\right](a+\varepsilon j,x)\\ \leq (\Lambda_2-\Lambda_1)&\int_{\mathbb{R}^d}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx\Big|_{t=t_1}\\ &+\int_{\mathbb{R}^d}\sum_{0<j\leq k}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)f(a+\varepsilon j,x)\,dx. \end{aligned}\tag{5.4.8}$$

We write $v = (u-\psi_L)_+$. We also define v and f for non-integer values as we did for u and ψ. Then

$$\sum_{0<j\leq k}\varepsilon\mathcal{B}[u,v] = \int_a^T\mathcal{B}[u,v],$$

and

$$\int_{\mathbb{R}^d}\sum_{0<j\leq k}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)f(a+\varepsilon j,x) = \int_{\mathbb{R}^d}\int_a^T fv.$$

We start by writing write $u = (u-\psi_L)_+ + (u-\psi_L)_- + \psi_L$.

• **Control of the elliptic portion**. The elliptic portion $\mathcal{B}[u,v]$ can be controlled as follows

$$\int_{\mathbb{R}^d}\mathcal{B}[u,v] = \mathcal{B}[(u-\psi_L)_+,(u-\psi_L)_+]+\mathcal{B}[(u-\psi_L)_-,(u-\psi_L)_+]+\mathcal{B}[\psi_L,(u-\psi_L)_+].$$

Now, due to the observation that $(u-\psi_L)_+ \cdot (u-\psi_L)_- = 0$, we have that

$$\begin{aligned}\mathcal{B}[(u-\psi_L)_-,(u-\psi_L)_+] &= \int_{\mathbb{R}^d} \nabla(u-\psi_L)_+ \cdot \nabla(u-\psi_L)_- \, dx \\ &= \int_{\mathbb{R}^d} \nabla(u-\psi_L)_+ \cdot \nabla(u-\psi_L)_{\mathsf{neg}},\end{aligned}$$

where we denote $(u-\psi_L)_{\mathsf{neg}} = -(u-\psi_L)_- \geq 0$. In particular,

$$\mathcal{B}[(u-\psi_L)_-,(u-\psi_L)_+] \geq 0.$$

This "good term" is not fully exploited in this section. It will be used in a crucial way in the second De Giorgi lemma section. The remainder can be written as:

$$\mathcal{B}[\psi_L,(u-\psi_L)_+] = \int_{\mathbb{R}^d} \nabla(\psi_L) \cdot \nabla(u-\psi_L)_+ \, dx = 0.$$

The other part of the remainder can be controlled in the following way where $C_{d,\Lambda_1,\alpha}$ is some universal constant depending only on d and Λ_1 and α.

• **Control of the piece in time**. We now control the piece in time. The term

$$(u-\psi_L)_+(a+\varepsilon j)\partial_\varepsilon^\alpha(u-\psi_L)_-(a+\varepsilon j) \geq 0,$$

so we may ignore this term on the left hand side of the equation. We will however utilize it in a crucial way in the second De Giorgi lemma. We also recognize that $(u-\psi_L)_+(x,a) = 0$ for all $x \in \mathbb{R}^d$ by assumption, so that

$$\sum_{0<j\leq k} \varepsilon v(a+\varepsilon j,x)\partial_\varepsilon^\alpha v(a+\varepsilon j,x) = \sum_{-\infty<j\leq k} \varepsilon v(a+\varepsilon j,x)\partial_\varepsilon^\alpha v(a+\varepsilon j,x).$$

We move the term involving $\partial_\varepsilon^\alpha \psi_L$ to the right hand side of the equation and use [5, Lemma 2.3] to control this term by the L^1 norm of v.

For the last term in the right hand side, we have $|f|v$ is controlled by v, since $|f| \leq 1$. Adding all the terms together and now utilizing Lemma 5.2.4 and Lemma 5.2.7 respectively, we have that

$$\begin{aligned}&\Lambda_1 \int_{\mathbb{R}^d} \left(\int_a^T \frac{v^2(t,x)}{(t-a)^\alpha} \, dt + \int_a^t \int_a^t \frac{(v(t,x)-v(\tau,x))^2}{(t-\tau)^{1+\alpha}} \, d\tau \, dt \right) dx + \int_{t_1}^{t_2} \|\nabla v\|_2^2 \, dt \\ &\leq (\Lambda_2-\Lambda_1) \int_{\mathbb{R}^d} \left(\int_a^T \frac{v^2(t,x)}{(t-a)^\alpha} \, dt + \int_a^t \int_a^t \frac{(v(t,x)-v(\tau,x))^2}{(t-\tau)^{1+\alpha}} \, d\tau \, dt \right) dx \Bigg|_{t=t_1} \\ &+ C_{d,\Lambda,\alpha} \left[\int_a^t \int_{\mathbb{R}^d} v + \chi_{\{v>0\}} \right].\end{aligned} \tag{5.4.9}$$

Next, in order to employ the Sobolev embedding theorem, we need to compare $\mathcal{B}[(u-\psi_L)_+,(u-\psi_L)_+]$ with $\|(u-\psi_L)_+\|_{H^1(\mathbb{R}^d)}^2$ as follows:

$$\|(u-\psi_L)_+\|_{H^1(\mathbb{R}^d)}^2 = \|(u-\psi_L)_+\|_2^2 + \|\nabla(u-\psi_L)_+\|_2^2.$$

Hence

$$\mathcal{B}[(u-\psi_L)_+,(u-\psi_L)_+] = \|(u-\psi_L)_+\|^2_{H^1(\mathbb{R}^d)} - \|(u-\psi_L)_+\|_2^2.$$

Thus replacing the term $\|\nabla(u-\psi_L)_+\|_2^2$ by its value in (5.4.9) and using Lemma 5.2.2 we get the desired energy inequality

$$\begin{aligned}\Lambda_1 \int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\, dx + \int_{t_1}^{t_2} \|(u-\psi_L)_+\|^2_{H^1(\mathbb{R}^d)} dt \\ - (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\, dx\Big|_{t=t_1} \\ \leq C_{\alpha,\Lambda_1,\Lambda_2}\int_{\mathbb{R}^d}\int_{t_1}^{t_2}\left[(u-\psi_L)_+^2 + (u-\psi_L)_+ + \chi_{\{(u>\psi_L\}}\right].\end{aligned}$$

□

5.4.2 The oscillation reduction lemmas

The next step is to obtain a first De Giorgi type oscillation reduction lemma. These technical results need only be proved for bounded nonnegative weak solutions defined in a strip $\Gamma_R = B_R \times [-R,0]$. If u is mostly negative in space-time measure in a certain parabolic cylinder, then the supremum goes down if we restrict to a smaller nested cylinder (in order keep the natural homogeneity between the space and time variables, since the problem involves the nonlocal time variables). Due to the nonlocal character of the operator ∂_t^α, it is necessary to have some control of the far away behaviour of the solution. This is done, as in [5, 38, 43, 61], via a barrier function. In order to simplify our approach, we work with normalised cylinders. The general case is treated by scaling. Indeed, one of the lemmas controls the decrease of the supremum of the solution once we restrict the size of the parabolic neighbourhood of $(0,0)$, the other one implies that under suitable assumptions the solution separates from zero. A third one improves the first result so as to obtain a real alternative between going a bit down and a bit up, which leads to the proof of regularity [43]. Here is the first basic lemma.

Lemma 5.4.2 (De Giorgi's first lemma)**.** *There exists a positive constant $\delta(\vartheta) \in (0,1)$*

$$\delta(\vartheta) := \frac{\inf\limits_{0\leq\tau\leq 2} \vartheta'(\tau)}{1+\vartheta(2)-\vartheta(0)-\inf\limits_{0\leq\tau\leq 2}\vartheta'(\tau)}, \tag{5.4.10}$$

depending only on d and α – but independent of ε and a– such that for any solution $u : [a,0]\times\mathbb{R}^d \hookrightarrow \mathbb{R}$ to (5.4.7) with $\|f\|_{L^\infty(Q)} \leq 1$ and $a \leq -1$, the following implication for u holds true.

If it is verified that

$$\int_a^T \int_{\mathbb{R}^d} [u(t,x)-\psi]_+^2\, dx\, dt \leq \delta(\vartheta)^{\left(1-\frac{2\alpha}{2+\alpha d}\right)},$$

and

$$u(a,x) \leq \psi(a,x) + \frac{1}{2} \qquad \textit{for all} x \in \mathbb{R}^d,$$

then we have

$$u(t,x) \leq \frac{1}{2} + \psi(x,t) \qquad \textit{for}(x,t) \in [a,T] \times \mathbb{R}^d.$$

Hence, we have in particular that $u(t,x) \leq \frac{1}{2}$ *on* $[-1,0] \times B_1(0)$.

Proof. Let

$$L_k = \frac{1}{2}(1 - \frac{1}{2^k}) \text{ and } t_k = -1 - \frac{1}{2^k}.$$

Moreover, we will use the abbreviation $Q_k = [a,0] \times \mathbb{R}^d$, for $t_k \in [a,0]$. We take

$$\psi_{L_k}(t) = L_k + (|t|^{\alpha/2} - 1)_+,$$

in (5.4.2) and $u_k(t) = (u - \psi_{L_k})_+(\cdot,t)$. Since $\psi_k \geq 0$, we take $\ell = 0$ in (5.4.10). Observe that if we start the iteration from $k = 1$ we may take $\ell = 1/4$ and $M = 2$ in (5.4.6) to simplify.

We split the proof in two parts.

First step: Short form of the energy inequality. Let us consider two variables σ, t that satisfy $a \leq t_{k-1} \leq \sigma \leq t_k \leq 0$. By taking the time integral over $[\sigma, t]$ in inequality (5.4.3), we obtain

$$\begin{aligned} \Lambda_1 \int_{\mathbb{R}^d} \|(u - \psi_{L_k})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})} \, dx + \int_\sigma^t \|(u - \psi_{L_k})_+\|^2_{H^1(\mathbb{R}^d)} d\tau \\ \leq (\Lambda_2 - \Lambda_1) \int_{\mathbb{R}^d} \|(u - \psi_{L_k})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})} \, dx \Big|_{t=2} \\ + \, C \int_{\mathbb{R}^d} \int_2^t \Big[(u - \psi_{L_k})^2_+ + (u - \psi_{L_k})_+ + \chi_{\{(u > \psi_{L_k}\}} \Big], \end{aligned}$$

where $C \equiv C(\alpha, d, \Lambda_1, \Lambda_2)$.

Next, by first taking the average over $\sigma \in [t_{k-1}, t_k]$, and then taking the sup over $t \in [a,0]$ in the above inequality, we deduce from the above inequality that

$$\begin{aligned} \int_{\mathbb{R}^d} \|(u - \psi_{L_k})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})} + \int_a^T \|(u - \psi_{L_k})_+\|^2_{H^1(\mathbb{R}^d)} \\ \leq C_{\alpha,d,\Lambda_1,\Lambda_2} \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \left[\int_{\mathbb{R}^d} \int_a^t (u - \psi_{L_k})^2_+ + (u - \psi_{L_k})_+ + \chi_{\{(u > \psi_{L_k}\}} \right]. \end{aligned} \tag{5.4.11}$$

We now use the Sobolev embedding $H^{\frac{\alpha}{2}}(\mathbb{R}) \subset L^{\frac{2}{1-\alpha}}(\mathbb{R})$ to obtain from (5.4.11)

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\int_a^T v^{\frac{2}{1-\alpha}} \right)^{1-\alpha} + \int_a^T \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}} \\ \leq C_{\alpha,d,\Lambda_1,\Lambda_2} \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \int_a^T \int_{\mathbb{R}^d} (v^2 + v + \chi_{\{v>0\}}) \, dt \, dx, \end{aligned} \tag{5.4.12}$$

where we have settled $v = (u - \psi_{L_k})_+$ for simplicity purposes.

In the following computation we use Hölder's inequality twice with

$$\frac{\beta}{p_1} + \frac{1-\beta}{p_2} = \frac{1}{p} = \frac{\beta}{p_3} + \frac{1-\beta}{p_4},$$

and interpolate as in [5]

$$\begin{aligned}
\int_a^T \int_{\mathbb{R}^n} v^p &= \int_a^T \int_{\mathbb{R}^d} v^{p\beta} v^{p(1-\beta)} \\
&\leq \int_a^T \left(\int_{\mathbb{R}^d} v^{p_1} \right)^{\frac{p\beta}{p_1}} \left(\int_{\mathbb{R}^d} v^{p_2} \right)^{\frac{p(1-\beta)}{p_2}} \\
&\leq \left(\int_a^T \left(\int_{\mathbb{R}^d} v^{p_1} \right)^{\frac{p_3}{p_1}} \right)^{\frac{\beta p}{p_3}} \left(\int_a^T \left(\int_{\mathbb{R}^d} v^{p_2} \right)^{\frac{p_4}{p_2}} \right)^{\frac{p(1-\beta)}{p_4}},
\end{aligned}$$

so that

$$\left(\int_a^T \int_{\mathbb{R}^d} v^p \right)^{\frac{2}{p}} \leq \beta \left(\int_a^T \left(\int_{\mathbb{R}^d} v^{p_1} \right)^{\frac{p_3}{p_1}} \right)^{\frac{2}{p_3}} + (1-\beta) \left(\int_a^T \left(\int_{\mathbb{R}^d} v^{p_2} \right)^{\frac{p_4}{p_2}} \right)^{\frac{2}{p_4}}.$$

Set

$$p_1 = 2, \qquad p_2 = \frac{2d}{d-2}, \qquad p_3 = \frac{2}{1-\alpha}, \qquad p_4 = 2$$

so that

$$p = 2\left(\frac{2+\alpha d}{\alpha d + 2(1-\alpha)} \right) \text{ and } \beta = \frac{2}{2+\alpha d}. \tag{5.4.13}$$

By Minkowski's inequality

$$\left(\int_a^T \left(\int_{\mathbb{R}^d} v^2 \right)^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \leq \int_{\mathbb{R}^d} \left(\int_a^T v^{\frac{2}{1-\alpha}} \right)^{1-\alpha}.$$

Then

$$\|v\|^2_{L^p(\mathbb{R}^d \times [a,T])} \leq C_{\alpha,d,\Lambda_1,\Lambda_2} \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \int_a^T \int_{\mathbb{R}^d} v^2 + v + \chi_{\{v>0\}} \, dt \, dx, \tag{5.4.14}$$

where $p > 2$ is defined in (5.4.13).

Second step: Nonlinear recurrence. From the short form of the energy inequality (5.4.14), we establish a nonlinear recurrence relation to the following sequence of truncated energy:

$$U_k := \int_a^t \int_{\mathbb{R}^d} (u - \psi_{L_k})_+^2 + (u - \psi_{L_k})_+ + \chi_{\{u - \psi_{L_k} > 0\}}.$$

Now, since $L_k = L_{k-1} + 2^{-k-1}$, we have that $u_k > 0$ implies $u_{k-1} > 2^{-k-1}$, which in turn gives the Chebyshev type inequality

$$\int_{Q_k} v_k^p \leq 2^{(k+1)(p-q)} \int_{Q_k} v_{k-1}^p$$

for every $p > q$. Thus, for some $p > 2$ to be chosen, from (5.4.14), we have

$$\begin{aligned}
\int_{Q_k} (u - \psi_{L_k})_+ &\leq \int_{Q_k} (u - \psi_{L_{k-1}})_+ \chi_{\{u - \psi_{L_{k-1}} > \frac{1}{2^{k+1}}\}} \\
&\leq (2^{k+1})^{p-1} \int_{Q_k} (u - \psi_{L_{k-1}})_+^p \leq (2^{k+1})^{p-1} C_d^p U_{k-1}^{p/2}, \\
\int_{Q_k} \chi_{\{u - \psi_{L_k} > 0\}} &\leq (2^{k+1})^p \int_{Q_k} (u - \psi_{L_{k-1}})_+^p \leq (2^{k+1})^p C_d^p U_{k-1}^{p/2}, \\
\int_{Q_k} (u - \psi_{L_k})_+^2 &\leq \int_{Q_k} (u - \psi_{L_{k-1}})_+^2 \chi_{\{u - \psi_{L_{k-1}} > \frac{1}{2^{k+1}}\}} \\
&\leq (2^{k+1})^{p-2} \int_{Q_k} (u - \psi_{L_{k-1}})_+^p \leq (2^{k+1})^{p-2} C_d^p U_{k-1}^{p/2}.
\end{aligned}$$

From the above three inequalities we conclude that

$$U_k \leq \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \overline{C}^k U_{k-1}^{p/2}, \qquad \text{for every } k \geq 0, \tag{5.4.15}$$

for some universal constant $\overline{C}$ that depends only on $\alpha, d, \Lambda_1, \Lambda_2$. Since $p > 2$, it follows from the nonlinear recurrence relation (5.4.15) that there exists some sufficiently small constant κ_0

$$\kappa_0 = \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) U_0^{p/2},$$

depending only on $\alpha, d, \Lambda_1, \Lambda_2$ (but not ε or a) such that if $U_1 \leq \kappa_0$, it follows that

$$\lim_{k \to \infty} U_k \to 0.$$

From (5.4.12) we have

$$\int_a^T \int_{\mathbb{R}^d} |u - \psi|^2 \, dx \, dt < \varepsilon \left(\frac{\Lambda_1}{\Lambda_2 - \Lambda_1 + 1} \right)^{2/p}.$$

Furthermore, $U_k \to 0$ implies that

$$u \leq \frac{1}{2} + (|t|^{\alpha/2} - 1)_+ \text{ for } (x,t) \in [a, T] \times \mathbb{R}^d.$$

□

For this next corollary we define

$$\overline{\psi}(t) := (|t|^{\alpha/4} - 1)_+ \text{ if } t \leq 0.$$

Corollary 5.4.3. *There exists a constant* $\kappa_0 \in (0,1)$

$$\kappa_0 = R^{-d-\frac{2}{\alpha}}(1+\overline{\psi}(-2,2))^{-2}\delta(\vartheta)^{\left(1-\frac{2\alpha}{2+\alpha d}\right)}, \tag{5.4.16}$$

and $\varepsilon_0 > 0$, *both depending only on* d, α *such that for any solution* $u : [a,0] \times \mathbb{R}^d \hookrightarrow \mathbb{R}$ *to* (5.4.7) *with* $\varepsilon < \varepsilon_0$, $a \leq -2$, *and* $\|f\|_{L^\infty((a,T)\times\mathbb{R}^d)} \leq 1$ *satisfying*

$$u(t,x) \leq 1 + \overline{\psi}(t) \text{ on } [a,0] \times \mathbb{R}^d$$

and

$$|\{u > 0\} \cap ([-2,0] \times B_2)| \leq \kappa_0.$$

Then we have,

$$u(t,x) \leq \frac{1}{2} \text{ for } (t,x) \in [-1,0] \times B_1.$$

Proof. If $R \geq 2^{1/2}$ and $R^{2/\alpha} \geq 2^{1/\alpha}$, then

$$\begin{aligned} 1 + ((|y|+1)^{1/2} - 1)_+ &\leq (|y|) \quad \text{if } |y| \geq R, \\ 1 + ((|t|+1)^{\alpha/4} - 1)_+ &\leq (|t|^{\alpha/2}) \quad \text{if } |t| \geq R^{2/\alpha}. \end{aligned}$$

Thus, we may choose R even larger such that

$$2 + \overline{\psi}(|t|+1) \leq \psi(|t|) \text{ if } (y,t) \notin [-R^{\frac{2}{\alpha}}, 0] \times B_R(0). \tag{5.4.17}$$

R is now fixed and is dependent on d, α.

For any $(t_0, x_0) \in [-1,0] \times B_1$ with $t_0 \in a + \varepsilon\mathbb{Z}_+$, we introduce the rescaled function u_R defined on $[R^{2/\alpha}(a-t_0), R^{2/\alpha} - t_0] \times \mathbb{R}^d$ by

$$u_R(\tau, y) := u\left(t_0 + \frac{\tau}{R^{2/\alpha}}, x_0 + \frac{y}{R}\right).$$

The function u_R satisfies equation (5.4.7) with initial time $R^{2/\alpha}(a-t_0)$, with discrete time increment $\varepsilon R^{2/\alpha}$ and with the right hand side of the equation

$$f_R := f\left(t_0 + \frac{\tau}{R^{2/\alpha}}, x_0 + \frac{y}{R}\right),$$

for $|f_R| \leq 1$. We then choose $\varepsilon_0 R^{2/\alpha} = 1$, so that $\varepsilon R^{2/\alpha} < 1$. We can apply Lemma 5.4.2 to u_R. In [5], it is shown that for $t_0 \in [-1,0]$,

$$\left(\left|t_0 + tR^{-2/\alpha}\right|^{\alpha/4} - 1\right)_+ \leq \left(|t_0 + t|^{\alpha/4} - 1\right)_+,$$

and we conclude that $\overline{\psi}(t_0 + t/R^{2/\alpha}) \leq \overline{\psi}(t_0 + t)$. Now since $\overline{\psi}(t)$ increases with respect to $|t|$ for $|t| > 1$ we have

$$u_R(\tau, y) \leq 2 + \overline{\psi}(t_0 + \frac{\tau}{R^{2/\alpha}}) \leq 2 + \overline{\psi}(t_0 + \tau) \leq 2 + \overline{\psi}(|\tau| + 1).$$

Then utilizing (5.4.17), we have chosen R large enough so that $u_R(\tau,y) \leq \psi(\tau)$ for any $(\tau,y) \notin [-R^{2/\alpha},0] \times B_R$.

$$\int_{R^{\frac{2}{\alpha}}(a-t_0)}^{0} \int_{\mathbb{R}^d} [u_R(\tau,y) - (|\tau|^{\alpha/2}-1)_+]_+^2 = \int_{-R^{\frac{2}{\alpha}}}^{0} \int_{|y|\leq R} [u_R(\tau,y) - (|\tau|^{\alpha/2}-1)_+]_+^2$$
$$\leq \int_{-R^{\frac{2}{\alpha}}}^{0} \int_{|y|\leq R} [u_R(\tau,y)]_+^2 = R^{d+\frac{2}{\alpha}} \int_{t_0-1}^{t_0} \int_{\{x_0\}+B_1} [u_R(\tau,y)]_+^2$$
$$\leq R^{d+\frac{2}{\alpha}} \int_{-2}^{0} \int_{B_2} [u_R(\tau,y)]_+^2 \leq R^{d+\frac{2}{\alpha}} (1+\overline{\psi}(-2))^2 \kappa_0.$$

Choosing $\kappa_0 = R^{-d-\frac{2}{\alpha}}(1+\overline{\psi}(-2,2))^{-2}\delta(\vartheta)^{\left(1-\frac{2\alpha}{2+\alpha d}\right)}$ gives that $u(t_0,x_0) \leq 1/2$ for $(t_0,x_0) \in (-1,0) \times B_1$ with $t_0 \in a + \varepsilon\mathbb{Z}_+$, and therefore for all $(t,x) \in (-1,0) \times B_1$. $\square$

5.5 The Second De Giorgi Lemma

In order to proceed with the regularity proof, we need to analyse what happens when the solution is neither mostly negative nor mostly positive, in the sense of Lemma 5.4.2, in space-time measure [5, 38, 61]. To this aim, we will use De Giorgi's idea of loss of mass at intermediate levels, obtaining a quantitative version of the fact that a function with a jump discontinuity cannot be in the energy space.

The key idea is to impose conditions on the nonlinearity guaranteeing that the equation is neither degenerate nor singular at the intermediate values. Hence, we are in the linear setting studied in [38, 61]. As in [5], the result is written in terms of the functions

$$F_1(t) := \sup(-1, \inf(0, |t|^2 - 16)).$$

We note that F_i are both Lipschitz. F_1 is compactly supported in $[-4,4]$ and is equal to -1 in $[-3,3]$. We will only use $F(t)$ for $t \leq 0$. This function is used to localize the function in $[-4,4]$.

For $\lambda < 1/3$, we also define

$$\psi_\lambda(t) := ((|t| - \lambda^{-4/\alpha})^{\alpha/4} - 1)_+ \chi_{\{|t|\geq \lambda^{-4/\alpha}\}}.$$

Our lemma will involve the following sequence of five cutoffs:

$$\phi_i := 2 + \psi_{\lambda^3}(t,x) + \lambda^i F_1(t),$$

for $0 \leq i \leq 4$.

Lemma 5.5.1. *Let κ_0 be the constant defined in Corollary 5.4.3 and assume $C_1(\Lambda_1) \leq \vartheta'(\tau) \leq C_2(\Lambda_2)$ for every $1/2 \leq \tau \leq 2$. For $0 < \mu < 1/8$ fixed, there exists $\lambda \in (0,1)$, depending only on $d, \Lambda_1, \Lambda_2, \alpha$ such that for any solution $u : [a,0] \times \mathbb{R}^d \to \mathbb{R}$ to* (5.4.7) *with $0 < \varepsilon < \varepsilon_0$, $a \leq -4$, and $|f| \leq 1$ satisfying*

$$u(t,x) \leq 2 + \psi_{\lambda^3}(t) \quad on \quad [a,0] \times \mathbb{R}^d, \qquad |\{u < \phi_0\} \cap ((-3,-2) \times B_1)| \geq \mu,$$

then we have

$$|\{u > \phi_4\} \cap ((-2,0) \times B_2)| \leq \kappa_0.$$

The main idea of the proof is as follows: We use the fact that u satisfies the equation as well as the local nature of the spatial operator to make use of the assumption that the set where u is small on $(-3,-2)\times B_1$ is significant enough, and show there is a large set of points in $(-3,-2)\times B_2$ in which u is small. After that we then use the nonlocal nature of the time derivative to show that this implies that the set in $(-2,0)\times B_2$ on which u is large is a very small set, under the assumptions on the nonlinearity of $\partial_t^\alpha \vartheta$ that

$$C_1(\Lambda_1)\|u-\bar{\psi}\|^2_{H^{\alpha/2}} \leq \int_a^T (u-\phi_{i_+})\partial_\varepsilon^\alpha \vartheta \, dt \leq C_2(\Lambda_2)\|u-\bar{\psi}\|^2_{H^{\alpha/2}}.$$

The proof in [5, Lemma 5.1] works verbatim, using the above equivalence whenever required.

5.6 Proof of the Hölder regularity

In this section, we are now in position to prove our main result. Since De Giorgi's lemmas were proven independent of ε, the conclusions hold in the limit. Therefore, we may prove the results for weak solutions to (5.4.1); however, the proofs can be given for analogous results of the discretized solutions to (5.4.7). Though the general argument follows closely [38, Section 5], [43, Section 10], [5, Section 6], [61, Section 4], an interesting modification is needed to accommodate the nonlinearity of the equation with the fractional derivative: in doing the scaling of the solution in each iteration we will now find solutions of a family of related equations. For λ as in the previous section, we define, for any ς

$$\psi_{\varsigma,\lambda}(t,x) := \Big((|x|-\lambda^{-2})^{\varsigma}-1\Big)_+ \chi_{\{|x|\geq \frac{1}{\lambda^2}\}} + \Big((|t|-\lambda^{-4/\alpha})^{\varsigma}-1\Big)_+ \chi_{\{|t|\geq \frac{1}{\lambda^{4/\alpha}}\}}.$$

Lemma 5.6.1. *Let ϑ be such that $\delta(\vartheta)>0$ and $\delta(\widetilde{\vartheta})>0$, where $\widetilde{\vartheta}(\tau)=-\vartheta(-\tau)$. Assume in addition that $C_1\leq \vartheta'(\tau)\leq C_2$ for $\tau\in[1/2,2]$ or $\tau\in[-2,-1/2]$. There exist constants $\varsigma_0>0$ and $\lambda^*\in(0,1)$ such that if for any u, solution to (5.4.1) in $[a,0]\times\mathbb{R}^d$ with $|f|\leq\lambda^4$ and $a\leq -4$, and if*

$$-2-\psi_{\varsigma,\lambda}\leq u\leq 2+\psi_{\varsigma,\lambda},$$

we have

$$\sup_{[-1,0]\times B_1} u - \inf_{[-1,0]\times B_1} u \leq 4-\lambda^*.$$

Proof. We fix $\varsigma>0$ depending on $\Lambda_1,\Lambda_2,\alpha$ such that

$$\frac{(|x|^{\varsigma}-1)_+}{\lambda^4}\leq (|x|^{1/2}-1)_+ \quad \frac{(|t|^{\varsigma}-1)_+}{\lambda^4}\leq (|t|^{\alpha/4}-1)_+.$$

If u (or $-u$) is subcritical at the level 0, i.e., if $|\{u>\phi_4\}\cap((-2,0)\times B_2)|\leq\kappa_0$, see (5.4.16), we are done thanks to Lemma 5.4.2. Notice that $-u$ solves (5.4.1) with ϑ replaced by $\widetilde{\vartheta}$. Otherwise, thanks to the hypotheses on ϑ', either u or $-u$ satisfies the hypotheses of Lemma 5.5.1. We assume for definiteness that it is u.

We consider the sequence of rescaled functions

$$u_{j+1}=\frac{1}{\lambda^4}(u_j-2(1-\lambda^4)),$$

starting from $u_0(x,t) = u(x,t)$, the solution of the time porous medium equation under consideration. We have that u_j is a weak solution of problem (5.4.1) with a nonlinearity ϑ_{j+1} given iteratively by

$$\vartheta_{j+1}(\tau) = \frac{1}{\lambda^4}\vartheta_j(\lambda^4\tau + 2 - 2\lambda^4), \qquad \vartheta_0 = \vartheta.$$

We will prove that for each j we can apply either Lemma 5.4.2 or Lemma 5.5.1. Repeated application of Lemma 5.5.1 will give that in fact. Lemma 5.4.2 can be applied after a finite number of steps.

The key point is that $\vartheta'_{j+1}(\tau) = \vartheta'_j(\lambda^4\tau + 2 - 2\lambda^4)$. On one hand, since $\lambda^4\tau + 2 - 2\lambda^4 \in [1/2, 2]$ whenever $\tau \in [1/2, 2]$, we have $C_1 \leq \vartheta'(\tau) \leq C_2$. Hence, the fractional derivative $\partial_\tau^\alpha \vartheta(\tau)$ is bounded below and above. On the other hand, since $[1-\lambda^4, 1+\lambda^4] \subset [1/2, 2]$, we get $\delta(\vartheta_j) \geq \bar{\delta} > 0$ for all j.

Let $\nu = R^{-d-\frac{2}{\alpha}}(1+\overline{\psi}(-2,2))^{-2}\bar{\delta}(\vartheta)^{\left(1-\frac{2\alpha}{2+\alpha d}\right)}$. Assume by contradiction that no u_j is subcritical, that is $|\{u > \phi_4\} \cap ((-2,0) \times B_2)| > \nu$ for all j, so that we could never apply Lemma 5.4.2. Let $\mu > 0$ be such that $|\{u < \phi_0\} \cap ((-3,-2) \times B_1)| \geq \mu$. By construction,

$$|\{u_{j+1} < \phi_0\} \cap ((-3,-2) \times B_1)| \geq |\{u_j < \phi_0\} \cap ((-3,-2) \times B_1)| \geq \mu.$$

Furthermore, since λ was chosen so that

$$\phi_4(t) = 2 - 2\lambda^4 \text{ for } t \in [-2,0] \times B_2,$$

we have

$$|u_{j+1} > 0\} \cap ([-2,0] \times B_2)| \leq |\{u_j > \phi_4\} \cap ((-2,0) \times B_2)| \leq \kappa_0.$$

Also,

$$u_{j+1}(x,t) \leq u_j(x,t) \leq 2 + \frac{\psi_{\varsigma,\lambda^3}(x,t)}{\lambda^4} \leq 2 + \psi_{\lambda^3} \leq 2\psi_1 \leq 2 + \overline{\psi}.$$

Furthermore, u_{j+1} satisfies (5.2.2) with right hand side $|f| \leq 1$. Then we may apply Corollary 5.4.3 to u_{j+1}, and conclude $\tilde{u} \leq 1/2$ on $(-1,0) \times B_1$. Hence

$$u(t,x) \leq 2 - 3\lambda^* \text{ for } (t,x) \in [-1,0] \times B_1,$$

with $\lambda^* = \frac{1}{2}\lambda^4$. □

This result shows in particular that the oscillation of u in $(-2,0) \times B_2$ is reduced in $(-1,0) \times B_1$ by a factor $\varpi^* = 2 - \lambda^*/2$. From this, we are in a condition to prove the Hölder regularity stated in Theorem 5.6.2 by mean of scaling arguments. As in [43, 61], we have to consider separately the degenerate and nondegenerate cases.

Theorem 5.6.2. *Let φ satisfy* (5.1.1)*, and let u be a bounded weak solution to the Cauchy problem* (TPME)*. Then u is continuous in Q. If φ satisfies in addition* (5.1.2)*, then, there is some $\beta \in (0,1)$ such that $u \in C^\beta(\mathbb{R}^d \times (-\infty, T))$.*

Proof. We present the two parts (degenerate and nondegenerate cases) of the proof separately. But before that, we state the normalization and the modulus of continuity of the solution.

• **Normalization.** Let $(t_0, x_0) \in (a, \infty) \times \mathbb{R}^d$. We assume that $(t_0 - a) > 4$, otherwise we may rescale and have a new norm depending on the rescaling. We translate to the origin by considering

$$u_0(t,x) := u(t_0 + t, x_0 + x),$$

and dilate by considering $\gamma_0 = \inf(1, t_0/4)^{\alpha/2}$. We define

$$v_0(x,t) = \frac{u(t_0 + \gamma_0^{2/\alpha} t^{2/\alpha}, x_0 + \gamma_0 x)}{\|u(\cdot,0)\|_\infty + \|f(\cdot,0)\|_\infty}.$$

v_0 will satisfy an equation of the type

$$\partial_t^\alpha \vartheta_0(v_0) - \Delta v_0 = f.$$

Furthermore, the initial time for u_0 will be $(a - t_0)\gamma_0^{-2/\alpha} < -4$ and the function ϑ_0 satisfies the hypotheses of Lemma 5.6.1.

• **Modulus of continuity.**

In the following we prove that v_0 is continuous at $(0,0)$. Given $R > 1$, we define by induction the sequence of functions, for $k \geq 1$:

$$v_k(t,x) = \frac{v_0(R^{-2(k+1)/\alpha} t, R^{-(k+1)} x) - \mu_k}{\omega_k}, \quad (t,x) \in (a,0) \times \mathbb{R}^d,$$

where ω_k and μ_k are respectively the semi-oscillation and a certain mean of v_0 in the parabolic cylinder $Q_k = \Gamma_{R^{-k}}$,

$$\omega_k = \frac{\sup_{Q_k} v_0 - \inf_{Q_k} v_0}{2}, \quad \mu_k = \frac{\sup_{Q_k} v_0 + \inf_{Q_k} v_0}{2}.$$

They satisfy the equation

$$\partial_t^\alpha \vartheta_k(v_k) - \Delta v_k = f, \qquad \vartheta_k(\tau) = \frac{\vartheta_0\left(\omega_k \tau + \mu_k\right)}{\omega_k}.$$

Assuming by contradiction that $\omega_k \geq \theta > 0$, we have that the function ϑ_k satisfies the hypotheses of Lemma 5.6.1, since $\omega_k \tau + \mu_k \geq \theta/2$ for $\tau \geq 1/2$ if $\mu_k \geq 0$ and $\omega_k \tau + \mu_k \leq -\theta/2$ for $\tau \leq -1/2$ if $\mu_k \leq 0$.

Also we notice that, $|v_k| \leq 1 \leq 2 + \psi_{\varsigma,\lambda}(x)$ for $|x| \leq R$, if we apply Lemma 5.6.1 by induction to v_{k-1}, since it can be applied to v_0. Also, if we take $R > 1$ large enough so that $\psi_{\varsigma,\lambda}(R) \geq \frac{2-\theta}{\theta}$, we get $|v_k(x,t)| \leq 2 + \psi_{\varsigma,\lambda}(x)$ if $|x| \geq R$. Hence, applying one more Lemma 5.6.1, we conclude that $\omega_k \leq (2 - \lambda^*/2)^k$, which is a contradiction. Therefore we have a modulus of continuity.

• **Hölder regularity at nondegeneracy points.** We assume $u(x_0,t_0) > 0$, the case $u(x_0,t_0) < 0$ being similar. We consider $\gamma < 1$ such that

$$\frac{1}{1-(\lambda^*/2)}\psi_{\varsigma,\lambda}(x) \leq \psi_{\varsigma,\lambda}(x), \text{ for } |x| \geq R := 1/\gamma,$$

γ only depends on λ, λ^*, and ς. We define by induction the sequence of functions, for $k \geq 1$:

$$v_1(t,x) = \frac{u_0(t,x)}{\|u_0\|_{L^\infty} + \lambda^4\|f\|_{L^\infty}}, \qquad (t,x) \in (a,0)\times\mathbb{R}^d,$$
$$v_{k+1}(t,x) = \frac{v_k(\gamma^{2/\alpha}t,\gamma x) - \mu_k}{\varpi^*}, \qquad (t,x) \in (a\gamma^{-2k},0)\times\mathbb{R}^d,$$

where

$$\varpi^* = 2-\lambda^*/2, \qquad \mu_k^* = \frac{\sup_{Q_1} v_k + \inf_{Q_1} v_k}{2}.$$

Observe that the recurrence relation can be written explicitly,

$$v_k(t,x) = \frac{v_0(\gamma^{2(k+1)/\alpha}t,\gamma^{(k+1)}x) - \widetilde{v}_k}{(\varpi^*)^k}, \qquad (t,x) \in (a,0)\times\mathbb{R}^d,$$

where $\widetilde{v}_k = \sum_{j=1}^{k}\mu_j^*(\varpi^*)^k$. Also,

$$\mu_k = (\varpi^*)^k\mu_{k+1}^* + \widetilde{v}_k,$$

so that, since

$$\mu_k \to \frac{u(x_0,t_0)}{\|u(\cdot,0)\|_\infty + \|f(\cdot,0)\|_\infty} > 0,$$

then $\widetilde{v}_k \to \varepsilon > 0$.

The functions v_k satisfy $|v_k| \leq 1$ in Γ_R, and the equation

$$\partial_t^\alpha \vartheta_k(v_k) - \Delta v_k = f,$$

where the new nonlinearity is

$$\vartheta_k(\tau) = \frac{\vartheta_0\left((\varpi^*)^k\tau + \widetilde{v}_k\right)}{(\varpi^*)^k}.$$

By construction, the functions u_k, ϑ_k satisfy the hypotheses of Lemma 5.6.1 for any k. Finally, we have

$$\sup_{t_0+(-\gamma^{2/\alpha},0)\times(x_0+B_{\gamma^k})} - \inf_{t_0+(-\gamma^{2/\alpha},0)\times(x_0+B_{\gamma^k})} u \leq C(1-\lambda^*/4)^k.$$

We then conclude that u is C^β with

$$\beta = \frac{\ln(1-\lambda^*/4)}{\ln \gamma^{2/\alpha}}. \tag{5.6.1}$$

• **Hölder regularity at degeneracy points.** Now, let $u(x_0,t_0)=0$. Here we consider the sequence of functions defined by means of a recurrence that takes into account the nonlinearity, and the possible singularity of β' at zero:

$$v_{k+1}(x,t) = \frac{v_k(R^{-1}x, \gamma R^{-\sigma}t) - \mu_k^*}{\varpi^*}, \qquad \gamma = \frac{\vartheta_0(\varpi^*)}{\varpi^*},$$

with μ_k^* and ϖ^* as before. The rescaled nonlinearity turns out to be

$$\vartheta_k(\tau) = \frac{\vartheta_0\left((\varpi^*)^k\tau + \widetilde{\nu}_k\right)}{\vartheta_0\left((\varpi^*)^k\right)}.$$

We observe that

$$\frac{|\widetilde{\nu}_k|}{(\varpi^*)^k} \leq \frac{|\mu_k|}{(\varpi^*)^k} + |\mu_{k+1}^*| \leq C. \tag{5.6.2}$$

The conditions of Lemma 5.6.1 are fulfilled as long as, for every $k \geq 1$,

$$0 < C_1 \leq \frac{(\varpi^*)^k \vartheta_0'((\varpi^*)^k\tau + \widetilde{\nu}_k)}{\vartheta_0((\varpi^*)^k)} \leq C_2 \quad \text{for every } \tau \in (1/2, 2),$$

which hold from condition (5.1.2) using (5.6.2). See also [61] for similar analysis. Thus we conclude as before. □

5.7 Conclusion

Regularity theorems are an important result in the theory of PDEs, and their fractional counterparts also play a significance role in the study of problems involving nonlocal behaviour. As already observed in various papers [5,6,40,61–63,82] in the theory of fractional nonlocal PDEs, it is possible to find the qualitative behaviour of a solution. In this chapter, we prove the A De Giorgi-Nash type theorem for the time porous medium equation which provide Hölder estimate of weak solutions. The current work fits in with some results obtained in the case of local porous medium equation in the limit when $\alpha \to 1$. This opens the door to a possibly approach to treat nonlinear nonlocal problems with fractional derivative, say of the porous medium type. This will be done in the next chapter.

Chapter 6
Regularity theory for the nonlocal time-porous medium equation with fractional laplacian

We consider nonlinear nonlocal diffusive evolution equations, governed by a Lévy-type nonlocal operator, fractional time derivative and involving porous medium type nonlinearities. As a main result, we prove that the solution is bounded and Hölder continuous for all positive time. As a key ingredient, we use the De Giorgi-Nash-Moser technique.

The results presented in this chapter have been published in [73].

6.1 Introduction

In this chapter, we analyze nonlocal nonlinear equations with fractional time derivative. The basic operators involved are the so-called Lévy-type nonlocal operator with a kernel having a singularity at the origin like the fractional Laplacian in $\mathbb{R}^d$ [125, 154], and the so-called fractional derivative in the sense of extended Caputo or Marchaud in $(0,T)$, for $T > 0$ (see Chapter 1 or [84, 116, 134] for more details on the definitions and properties of those integro-differentials operators).

Nonlinear diffusion problems of the parabolic type involving the Lévy-type nonlocal operator and fractional time derivative operators have recently attracted the interest of many authors (see e.g. [5, 31, 154] and references therein). In this chapter, we focus our attention on a problem posed on the whole space domain in the case of regularity purpose. More precisely, we consider the fractional nonlocal diffusion equation with fractional time derivative known as the fractional time-porous medium equation as described in

$$\partial_t^\alpha u + \mathcal{L}^s\left(u^m\right) = f, \quad \text{for all } (t,x) \in Q, \quad u(x,0) = u_0(x), \text{ for all } x \in \mathbb{R}^d, \quad \text{(TPMEFL)}$$

where $m > 1$, $s, \alpha \in (0,1)$, and $d \geq 1$. The nonlocal operator $\mathcal{L}^s$ is a Lévy-type nonlocal operator. The initial value data $u_0 \in L^1(\mathbb{R}^d)$ has compact support and the right hand side $f \in L^\infty(Q)$ is a smooth bounded forcing function.

The nonlocal operator $\mathcal{L}^s$ with measurable kernel J (see Chapter 1 or [37, 45, 61, 63, 85, 125] for more details) is defined as

$$\mathcal{L}^s f(x) = \text{P.V.} \int_{\mathbb{R}^d} \left(f(x) - f(y)\right) J(x,y)\, dy, \tag{6.1.1}$$

and is assumed to satisfy

$$\begin{cases} J(x,y) \geq 0, \quad J(x,y) = J(y,x), \\ \dfrac{\mathbb{1}_{\{|x-y|\leq 3\}}}{\Lambda |x-y|^{d+s}} \leq J(x,y) \leq \dfrac{\Lambda}{|x-y|^{d+s}}, \end{cases} \quad x,\, y \in \mathbb{R}^d,\, x \neq y, \tag{$\mathrm{H_J}$}$$

for some constants $s \in (0,2)$ and $\Lambda > 0$. As seen in [61–63], when $J(x,y) = |x-y|^{-(d+s)}$, $\mathcal{L}^s$ is a multiple of the fractional Laplacian $(-\Delta)^{s/2}$, whose action on smooth functions is well defined and has a pointwise meaning. However, the pointwise expression (6.1.1) may not have sense for more general kernels in the class that we are considering here, even if f is very smooth. Hence, we have to deal with weak solutions to give sense both to the fractional time derivative and to the nonlocal operator. The precise definition of a weak solution, in terms of a bilinear form associated to the kernel J, is given in Section 6.2, which is devoted to some preliminaries.

The upper bound in $(\mathrm{H_J})$ implies in particular that the operator is of Lévy type,

$$\int_{\mathbb{R}^d} \min(1, |x-y|^2) J(x,y)\, dy < \infty,$$

for almost every $x \in \mathbb{R}^d$. Moreover, the singularity on the diagonal $x = y$ is that of the fractional Laplacian.

Thus, $\mathcal{L}^s$ can be seen as an integro-differential operator of order s with bounded measurable coefficients. The bounds in (6.1.3) allow the kernels J to be very oscillating and irregular. That is why they are referred to as *rough* kernels [61]. Observe that rapidly decreasing or even compactly supported kernels are permitted.

When the permeability of the medium changes over time such as in porous medium equation, it might be interesting to use a fractional time derivative. Among the different fractional derivatives existing in the literature, we consider the extended Caputo or Marchaud derivative in this chapter, since the problem we study is in the divergence from.

Using an integration by parts and proceeding as defined in [3, 5, 31, 32, 84], defining $v(t) \equiv v(a)$ for $t < a$, the extended form or the Marchaud derivative (see also Chapter 5) is defined as

$$\partial_t^\alpha v(t,\cdot) = \alpha \int_{-\infty}^{t} [v(t,\cdot) - v(\tau,\cdot)]\, \mathscr{K}(t,\tau)\, d\tau. \tag{6.1.2}$$

The kernel $\mathscr{K}$ also satisfies the conditions

$$\mathscr{K}(t,t-\tau) = \mathscr{K}(t+\tau,t) \quad \text{and} \quad \frac{\Lambda_1}{(t-\tau)^{1+\alpha}} \leq \mathscr{K}(t,\tau) \leq \frac{\Lambda_2}{(t-\tau)^{1+\alpha}}. \tag{6.1.3}$$

The formulation (6.1.2) is also known as the Marchaud derivative [16, 116, 134]. The reason for working with formulation (6.1.2) is that it allows us to easily utilize the nonlocal nature of the fractional time derivative for regularity purposes. This was succesfully accomplished for divergence problems in [5] as well as for nondivergence problems in [3, 4]. The existence and uniqueness results for (TPMEFL) have been studied in Chapter 5.

The evolutionary nonlinear equation (TPMEFL) with the fractional time derivative is analogous to the abstract evolutionary equations with usual time derivative

$$\partial_t u + \mathcal{L}^s \varphi(u) = f, \tag{6.1.4}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. This type of equation arises in flow in porous medium or plasma physics, depending on the choice of φ. Solution to (6.1.4) has been studied [57] for the regularizing effect of (6.1.4). The case (6.1.4) involves the Lévy type operator $\mathcal{L}^s$ and $\varphi(u) := u^m = |u|^{m-1}u$, with $m > 1$, and $f = 0$, the problem becomes the nonlocal porous medium equation. Regularity and existence results for this specific case have been obtained in [23, 61–63], by using De Giorgi methods in order to show that the solutions of (6.1.4) are s-Hölder continuous, which implies Hölder continuity. The control of the oscillation in the nonlocal setting followed the procedure developed in [38] for a linear problem with a rough kernel, combined with some ideas which deal with the nonlinearity borrowed from [12].

Recently another variant of the problem (6.1.4) was extended in [3–5], with $\mathcal{L}^s$ and $\varphi(u) := u$, with fractional derivative ∂_t^α in the sense of extended Caputo, so that the problem becomes a nonlocal linear problem

$$\partial_t^\alpha u - \int [u(t,y) - u(t,x)]\, J(t,x,y) = f(t,x). \tag{6.1.5}$$

Again, the De Giorgi techniques were used to prove Hölder continuity for solutions to (6.1.5) of divergence form which is a linear analogue of our problem (TPMEFL).

The authors in [3] utilized solutions that are weak-in-time to prove Hölder continuity. The same authors also adapted the methods in [6] to prove Hölder continuity for a nonlocal porous medium equation with inverse potential pressure. As already mentioned in [2], considering weak-in-time solutions is advantageous for existence and regularity results. As far as we know, the case of fully nonlinear and nonlocal variant of (6.1.4) studied in [73] is still open. More precisely, with $\mathcal{L}^s$ and $\varphi(u) := u^m$, with $m > 1$ and the fractional derivative ∂_t^α in the sense of extended Caputo or Marchaud. This is the main aim of this chapter, i.e., to analyze regularity properties of the problem (TPMEFL).

Presentation of the main results. The purpose of this work is to completely analyze the regularity result for the problem (TPMEFL).

In order to study the Hölder regularity, we rewrite the problem (TPMEFL) in the form

$$\partial_t^\alpha \vartheta(u) + \mathcal{K}u = f, \tag{P_ϑ}$$

for different functions ϑ and operator ∂_t^α related, respectively, to our original function $\beta = \varphi^{-1}$ and our original nonlocal operator ∂_t^α. To be more precise, ϑ will have the form $\vartheta(s) = a\beta(bs + c)$ for some $a,\ b > 0$ [61]. Hence, in the sequel, we always assume without further mention that

$$\vartheta \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{s_0\}), \quad \vartheta'(s) > 0 \quad \text{for } s \neq s_0, \qquad \text{for some } s_0 \in \mathbb{R}.$$

We state the main theorem on Hölder regularity that will be proved in Section 5.6. The key ingredient of the proof is the De Giorgi's method, (see [3, 5, 38, 61, 89]). We control the oscillation following the procedure developed in [3,5,61], combined with some ideas to deal with the nonlinearity that will appear in the fractional derivative. More precisely, we will prove that the oscillation of the solution in space-time β-cylinders of radius R, is reduced in a fraction of the cylinder $\mathcal{C}_{\alpha R}$, $\zeta < 1$, at least by a constant factor κ_*. This implies β–Hölder continuity.

Theorem 6.1.1. *Let u be a bounded weak solution which satisfies* (TPMEFL) *with $f \in L^\infty(Q)$. Furthermore, let the kernels $\mathcal{K}$ and J satisfy respectively* (6.1.3) *and*($\mathrm{H_J}$). *Then u is Hölder continuous in Q for some exponent $\beta \in (0,1)$ such that $u \in C^\beta(Q)$, with $\beta = \log \kappa_* / \log \zeta$.*

The chapter is organized as follows. In Section 6.2, we give weak formulation of the problem. Next we provide some functional inequalities and some useful lemmas. Next, Sections 6.3 and 6.4 respectively, are devoted to the proofs of some intermediate results on the energy inequalities, first and second De Giorgi lemmas which are oscillations estimates commonly known as the " L^2, L^∞ " estimates. Finally in Section 6.5 we provide the proof of our main result stated in Theorem 6.1.1 and Conclusion appears Section 6.6.

We recall the notation that will be intensively used throughout the chapter:

- α denotes the order of the extended Caputo derivative or Marchaud derivative;
- s denotes the order of the spatial fractional operator associated to the Lévy type operator;
- d refers to the space dimension;
- Λ denotes the elliptic positive constant which gives the bound of the kernel of the Lévy type operator;
- ε refers to the time length of the discrete approximation;
- t, τ denote time variables;
- ψ stands for a cut-off function;
- $Q = (0,T) \times \mathbb{R}^d$ denotes the domain;
- B_R denotes the bounded domain with radius R;
- $\Gamma_R = \{x \in B_R, t \in (0,T)\}$ denotes the space whose center is the ball with radius B_R.

6.2 Preliminaries

In this section, we establish the notion of weak solution to problem (TPMEFL) and fix the required functional framework borrowed from [61].

The validity of (6.1.1) is guaranteed only for $s < 1$ and functions f in $C^{s+\varepsilon}$ that do not grow too much at infinity. However, we notice that if we assume the additional condition

$$J(x, x+y) = J(x, x-y), \tag{6.2.1}$$

then the operator has a pointwise expression, in terms of second differences, even for $1 \leq s < 2$, for regular enough, $C^{1,s-1+\varepsilon}$, functions,

$$\mathcal{L}^s f(x) = -\frac{1}{2}\int_{\mathbb{R}^d} (f(x+y) + f(x-y) - 2f(x))\, J(x, x-y)\, dy. \tag{6.2.2}$$

In the case $J(x,y) = \widetilde{J}(x-y)$, condition (6.2.1) follows from the symmetry of the kernel.

In order to define the action of the operator $\mathcal{L}^s$ in a weak sense, we consider the bilinear form (nonlocal interaction energy)

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) J(x,y)\, dx\, dy,$$

and the quadratic form $\overline{\mathcal{E}}(f) = \mathcal{E}(f,f)$. For kernels satisfying the symmetry condition (6.2.1) and functions f, $g \in C_0^2(\mathbb{R}^d)$ we have

$$\langle \mathcal{L}^s f, g\rangle = \mathcal{E}(f,g);$$

see [61]. The bilinear form $\mathcal{E}$ is well defined for more general kernels, not necessarily satisfying (6.2.1), and for functions in the space $\dot{\mathcal{H}}_{\mathcal{L}^s}(\mathbb{R}^d)$, which is the closure of $C_0^\infty(\mathbb{R}^d)$ with the seminorm associated to the quadratic form $\overline{\mathcal{E}}$. We also define

$$\mathcal{H}_{\mathcal{L}^s}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \,:\, \overline{\mathcal{E}}(f) < \infty\}.$$

When $J(x,y) = |x-y|^{-d-s}$ for some $0 < s < 2$, the operator reduces to a multiple of the fractional Laplacian $(-\Delta)^{s/2}$. It is clear then from $(\mathrm{H_J})$ that the space $\mathcal{H}_{\mathcal{L}^s}(\mathbb{R}^d)$ coincides with the fractional Sobolev space

$$H^{s/2}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \,:\, (-\Delta)^{s/4} \in L^2(\mathbb{R}^d)\}.$$

Actually, $(\mathrm{H_J})$ implies

$$c_1\, \overline{\mathcal{E}}(f) \leq \|(-\Delta)^{s/4} f\|_2^2 \leq c_2 \left(\|f\|_2^2 + \overline{\mathcal{E}}(f)\right), \tag{6.2.3}$$

where c_1, c_2 depend only on d, s, Λ. We get the stronger result $\overline{\mathcal{E}}(f) \sim \|(-\Delta)^{s/4} f\|_2^2$.

We recall also the inclusions $H^{s/2}(\mathbb{R}^d) \subset L^{\frac{2d}{d-s}}(\mathbb{R}^d)$ if $d > s$, and $H^{s/2}(\mathbb{R}) \subset L^q(\mathbb{R})$ for every $q \geq 2$ if $1 \leq s < 2$. More precisely, we have the Hardy-Littlewood-Sobolev inequality [92, 143],

$$\|(-\Delta)^{s/4} f\|_2 \geq c\|f\|_{\frac{2d}{d-s}}, \qquad d > s, \tag{6.2.4}$$

and the Nash-Gagliardo-Nirenberg inequality [63],

$$\|(-\Delta)^{s/4} f\|_2^2 \|f\|_p^p \geq c\|f\|_{\frac{d(p+2)}{2d-s}}^{p+2}, \qquad d \geq 1,\ 0 < s < 2,\ p \geq 1. \tag{6.2.5}$$

These two inequalities combined with the upper estimate in (6.2.3) yield useful inclusions for functions in $\mathcal{H}_{\mathcal{L}}$.

When dealing with bounded domains $\Omega \subset \mathbb{R}^d$, we consider the operator acting on functions vanishing outside Ω. The corresponding Sobolev type space is $\mathcal{H}_{\mathcal{L},0}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with the norm given by $\overline{\mathcal{E}}^{1/2}$. Functions in this space satisfy a Poincaré inequality [7],

$$\overline{\mathcal{E}}(f) \geq c(\Omega)\|f\|_2^2. \tag{6.2.6}$$

We now define the concept of *weak solution* to the Cauchy problem (TPMEFL), a function $u \in C([0,\infty) : L^1(\mathbb{R}^d))$ with $\varphi(u) \in L^2_{\text{loc}}((0,\infty) : \dot{\mathcal{H}}_{\mathcal{L}}(\mathbb{R}^d))$ such that

$$\int_0^\infty \int_{\mathbb{R}^d} u\partial_t\zeta - \int_0^\infty \mathcal{E}(\varphi(u),\zeta) = 0 \tag{6.2.7}$$

for every $\zeta \in C_c^\infty(Q)$, and taking the initial datum $u(\cdot,0) = u_0$ almost everywhere.

One of the tools needed in the following sections is a generalized Stroock-Varopoulos inequality; see [28].

6.3 The First De Giorgi Lemma

In this section we prove Theorem 6.1.1. The main ingredient of the proof are the De Giorgi lemmas, commonly known as the " L^2, L^∞ " estimates. These techniques have been previously used e.g. in [5, 38, 61]. The nonlinearity part being confined into the fractional time derivative operators, we will need to establish some energy estimates for the weak solutions of (P_ϑ).

Indeed the first step is to derive a priori estimates for weak solutions. We need integral inequalities that measure the behaviour of the weak solutions for (P_ϑ) near its infimum and its supremum in the interior of an appropriate cylinder. At this point, we can ignore the equation and it remains to show that functions that satisfy these inequalities are Hölder continuous.

6.3.1 Energy inequalities estimate

We start by developing the necessary energy inequalities associated to the structure of equation (P_ϑ). The first inequality that needs to be proven is, somehow, a generalization of the famous Caccioppoli inequality [87] that is often used for elliptic equations. We use this inequality to control, in some sense, the norm of the fractional derivative $\partial_t^\alpha \vartheta(u)$, the norm of the bilinear form $\mathcal{E}$ with the norm of the function u itself.

We first introduce the following Lipschitz function ψ:

$$\psi(t,x) := (|t|^{\alpha/2} - 1)_+ + (|x|^{s/2} - 1)_+,$$

so that for $L \geq 0$, we define

$$\psi_L(t,x) = L + \psi(t,x). \tag{6.3.1}$$

Let us introduce (see e.g. [61]) $\ell = \inf_{\{u-\psi_L \geq 0\}} u \geq 0$ and $M = \sup_{\{u-\psi_L \geq 0\}} u < \infty$.

Lemma 6.3.1 (Energy estimates)**.** *Let u be a weak solution of the nonlocal time-porous medium equation* (P_ϑ) *in some finite time interval I including (t_1,t_2). Then,*

$$\begin{aligned}
\Lambda_1 \int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\,dx &+ \int_{t_1}^{t_2} \frac{1}{\Lambda}\|v\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)}\,dt \\
&- (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\,dx\Big|_{t=t_1} \\
&\leq C_{\alpha,s,\Lambda,\Lambda_1,\Lambda_2}\int_{\mathbb{R}^d}\int_{t_1}^{t_2}\left[(u-\psi_L)_+^2 + (u-\psi_L)_+ + \chi_{\{(u>\psi_L\}}\right],
\end{aligned} \tag{6.3.2}$$

where $\Lambda_1 = \inf_{\ell \leq \tau \leq M} \vartheta'(\tau)$ and $\Lambda_2 = \vartheta(M) - \vartheta(\ell)$.

Proof. If we multiply (P_ϑ) by the function $(u-\psi_L)_+$ and integrate over some finite time interval $(t_1,t_2) \subset I$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{t_1}^{t_2} \partial_t\left[(u-\psi_L)_+(\partial_t^{\alpha-1}\vartheta)\right]dx\,dt &- \int_{\mathbb{R}^d}\int_{t_1}^{t_2} \partial_t(u-\psi_L)_+(\partial_t^{\alpha-1}\vartheta)\,dx\,dt \\
&+ \int_{t_1}^{t_2} \mathcal{E}\left(u,(u-\psi_L)_+\right)(t)\,dt = \int_{\mathbb{R}^d}\int_{t_1}^{t_2}(u-\psi_L)_+ f(x,t)\,dx\,dt,
\end{aligned} \tag{6.3.3}$$

where the bilinear form $\mathcal{E}$ is defined as in 6.2.

Using the properties of ϑ, $\partial_t\vartheta$ and $\partial_t^\alpha\vartheta$, if

$$\ell = \inf_{\{u \geq \psi\}} u \geq 0 \quad \text{and} \quad M = \sup_{\{u\geq\psi\}} u < \infty,$$

then the above weak formulation (6.3.3) takes the form

$$\begin{aligned}
\Lambda_1 \int_{\mathbb{R}^d}\int_{t_1}^{t_2}(u-\psi_L)_+(\partial_t^\alpha u)\,dx\,dt &+ \int_{t_1}^{t_2}\mathcal{B}\left[u,(u-\psi_L)_+\right]dt \\
&\leq (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d}(u-\psi_L)_+(\partial_t^{\alpha-1}u)\,dx\Big|_{t=t_1} + \int_{\mathbb{R}^d}\int_{t_1}^{t_2}(u-\psi_L)_+ f(x,t)\,dx\,dt,
\end{aligned} \tag{6.3.4}$$

where

$$\Lambda_1 = \inf_{\ell\leq\tau\leq M}\vartheta'(\tau), \qquad \Lambda_2 = \vartheta(M)-\vartheta(\ell). \tag{6.3.5}$$

Since we do not know a priori if the cut-offs of u were valid test functions, we prove the lemmas for the sequence of approximating functions u_ε and obtain the results of the lemmas for the solution u. In the next two sections we will abuse notation for convenience and also to make the proofs more transparent. We will write u to mean a solution of (P_ϑ) and assume that ε is understood.

We subdivide the interval (a,T) into k intervals and let $\varepsilon = T/k$. We extend $u(t,x) = u(a+\varepsilon j, x)$, $\vartheta(t,x) = \vartheta(a+\varepsilon j,x)$, and $\psi(t,x) = \psi(a+\varepsilon j,x)$ for $a+\varepsilon(j-1) < t \leq a+\varepsilon j$. In this setting, for each fixed k we may solve via recursion

$$\alpha\varepsilon \sum_{i<j} \frac{\vartheta(a+\varepsilon j,x) - \vartheta(a+\varepsilon i,x)}{(\varepsilon(j-i))^{1+\alpha}} - \mathcal{K}u(a+\varepsilon j,x) = f(a+\varepsilon j,x), \tag{6.3.6}$$

for each $-\infty < j \leq k$. Here $f(t,x) = f(a,x)$ for $t < a$, so that $u(a+\varepsilon j,x) = u(a,x)$ for $j < 0$.

If u is a solution to (6.3.6) we consider the truncated function $\varepsilon[u-\psi_L]_+$ as a test function, where ψ_L is defined as in (6.3.1). We will only consider $L \leq 1/2$, so that the assumption on the initial condition will apply. We add in j and integrate over $\mathbb{R}^d$ to obtain

$$\begin{aligned}
\Lambda_1 \int_{\mathbb{R}^d} \sum_{0<j_1<j_2\leq k} \varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx \\
+ \sum_{0<j_1<j_2\leq k} \varepsilon\mathcal{E}\left[u,(u-\psi_L)_+\right](a+\varepsilon j,x) \\
\leq (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d} \varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx\Big|_{t=t_1} \\
+ \int_{\mathbb{R}^d} \sum_{0<j\leq k} \varepsilon(u-\psi_L)_+(a+\varepsilon j,x) f(a+\varepsilon j,x)\,dx.
\end{aligned} \tag{6.3.7}$$

We start by writing $u = (u-\psi_L)_+ + (u-\psi_L)_- + \psi_L$.

• **Control of the elliptic portion**. The elliptic portion $\mathcal{E}[u,(u-\psi_L)_+]$ can be controlled as in [38]. For the brevity of the reader we recall it here.

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{E}[u,(u-\psi_L)_+] &= \mathcal{E}[(u-\psi_L)_+,(u-\psi_L)_+] \\
&\quad + \mathcal{E}[(u-\psi_L)_-,(u-\psi_L)_+] + \mathcal{E}[\psi_L,(u-\psi_L)_+],
\end{aligned}$$

Now, due to the observation that $(u-\psi_L)_+\cdot(u-\psi_L)_- = 0$ and the symmetry of J in x,y, we have

$$\mathcal{E}[(u-\psi_L)_-,(u-\psi_L)_+] = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(t,x,y)(u-\psi_L)_+(x)(u-\psi_L)_{\mathrm{neg}}(y)dxdy,$$

where we denote $(u-\psi_L)_{\mathrm{neg}} = -(u-\psi_L)_- \geq 0$. In particular

$$\mathcal{E}[(u-\psi_L)_-,(u-\psi_L)_+] \geq 0.$$

This "good term" is not fully exploited in this section. It will be used in a crucial way in the next section. The remainder can be written as:

$$\begin{aligned}
&\mathcal{E}[\psi_L,(u-\psi_L)_+] \\
&=\frac{1}{2}\int\int_{|x-y|\geq 1} J(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}dxdy \\
&+\frac{1}{2}\int\int_{|x-y|<1} J(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}dxdy.
\end{aligned} \tag{6.3.8}$$

Using the inequality $|\psi(x)-\psi(y)|\leq 2|y-x|^{\frac{s}{2}}$, for any x and y with $|y-x|\geq 1$, we get the following estimate of the "far-away" contribution.

$$\begin{aligned}
&\left|\int\int_{|x-y|\geq 1} J(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot[u-\psi_L]_+(x)dxdy\right| \\
&=\left|\int\int_{|x-y|\geq 1} J(t,x,y)[\psi(x)-\psi(y)]\cdot[u-\psi_L]_+(x)\,dxdy\right| \\
&\leq\int_{\mathbb{R}^d}\int_{|y-x|\geq 1}\frac{2\Lambda}{|x-y|^{d+s}}2|y-x|^{\frac{s}{2}}dy\cdot(u-\psi_L)_+(x)dx \\
&=4\Lambda|S^{d-1}|\int_1^\infty r^{-\frac{s}{2}}dr\int_{\mathbb{R}^d}(u-\psi_L)_+(x)dx\leq C\int_{\mathbb{R}^d}(u-\psi_L)_+(x)dx.
\end{aligned}$$

By symmetry we end up to

$$\begin{aligned}
&\left|\int\int_{|x-y|\geq 1} J(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}dxdy\right| \\
&\leq C\int_{\mathbb{R}^d}(u-\psi_L)_+(x)dx.
\end{aligned} \tag{6.3.9}$$

The other part of the remainder ca be controlled in the following way:

$$\begin{aligned}
&\left|\int\int_{|x-y|<1} J(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}dxdy\right| \\
&\qquad\leq 2\int\int_{|x-y|<1} J(t,x,y)\chi_{\{[u-\psi_L](x)>0\}}|\psi_L(x)-\psi_L(y)T(u,\psi_L)\,dx\,dy,
\end{aligned} \tag{6.3.10}$$

where, in the above inequality, we have used the fact that

$$\begin{aligned}
T(u,\psi_L)&=|(u-\psi_L)_+(x)-(u-\psi_L)_+(y)| \\
&\leq\{\chi_{\{[u-\psi_L](x)>0\}}+\chi_{\{[u-\psi_L](y)>0\}}\}|(u-\psi_L)_+(x)-(u-\psi_L)_+(y)|,
\end{aligned}$$

and the symmetry in x and y.

Now, by Hölder's inequality, and using the inequality $|\psi(y)-\psi(x)|<|y-x|$, for any

x, y in $\mathbb{R}^d$, we can have the following estimation.

$$\begin{aligned}
&\int\int_{|x-y|<1} J(t,x,y)\chi_{\{[u-\psi_L](x)>0\}}|\psi_L(x)-\psi_L(y)||(u-\psi_L)_+(x)-(u-\psi_L)_+(y)|dxdy \\
&\leq \frac{a}{2}\cdot\int\int_{|x-y|<1} J(t,x,y)\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}^2 dydx \\
&+\frac{1}{2a}\cdot\int\int_{|x-y|<1} J(t,x,y)|\psi(x)-\psi(y)|^2\cdot\chi_{\{[u-\psi_L](x)>0\}}dydx,
\end{aligned} \tag{6.3.11}$$

in which the arbitrary $a>0$ will be chosen later. Finally

$$\begin{aligned}
&\int\int_{|x-y|<1} J(t,x,y)|\psi(x)-\psi(y)|^2 dy\cdot\chi_{\{[u-\psi_L](x)>0\}}dx \\
&\leq \int_{\mathbb{R}^d}\int_{|x-y|<1}\frac{2\Lambda}{|x-y|^{d+s}}|y-x|^2 dy\cdot\chi_{\{[u-\psi_L](x)>0\}}dx = C_s\int_{\mathbb{R}^d}\chi_{\{[u-\psi_L](x)>0\}}dx.
\end{aligned}$$

Pulling this inequality in (6.3.10) with $a=1/2$, and gathering it together with (6.3.8), (6.3.9), (6.3.10), we can rewrite the energy inequality as

$$\begin{aligned}
\Lambda_1\int_{\mathbb{R}^d}\sum_{0<j_1<j_2\leq k} &\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx \\
&+\sum_{0<j_1<j_2\leq k}\varepsilon\mathcal{E}\left[u,(u-\psi_L)_+\right](a+\varepsilon j,x) \\
&\leq(\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx\Big|_{t=t_1} \\
&+C_{d,\Lambda,s}\int_{\mathbb{R}^d}\sum_{0<j\leq k}\varepsilon\left((u-\psi_L)_+(x)+\chi_{\{[u-\psi_L](x)>0\}}dx\right) \\
&+\int_{\mathbb{R}^d}\sum_{0<j\leq k}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)f(a+\varepsilon j,x)\,dx,
\end{aligned} \tag{6.3.12}$$

where $C_{d,\Lambda,s}$ is some universal constant depending only on d, Λ and s. Next, in order to employ the Sobolev embedding theorem, we need to compare $\mathcal{E}[(u-\psi_L)_+,(u-\psi_L)_+]$ with $\|(u-\psi_L)_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)}$ as follow.

$$\begin{aligned}
\|(u-\psi_L)_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)} &= \int\int_{|x-y|\leq 2}\frac{\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}^2}{|x-y|^{d+s}} \\
+\int\int_{|x-y|>2}&\frac{\{(u-\psi_L)_+(x)-(u-\psi_L)_+(y)\}^2}{|x-y|^{d+s}} - \Lambda\cdot\mathcal{E}[(u-\psi_L)_+,(u-\psi_L)_+] \\
&\leq +2\int\int_{|x-y|>2}\frac{1}{|x-y|^{d+s}}\{(u-\psi_L)_+^2(x)+(u-\psi_L)_+^2(y)\}dx\,dy \\
&\qquad\leq \Lambda\cdot\mathcal{E}[(u-\psi_L)_+,(u-\psi_L)_+]+C\int_{\mathbb{R}^d}(u-\psi_L)_+^2 dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Lambda_1 \int_{\mathbb{R}^d} \sum_{0<j_1<j_2\le k} \varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx \\
&\quad + \sum_{0<j_1<j_2\le k} \varepsilon\mathcal{E}\left[u,(u-\psi_L)_+\right](a+\varepsilon j,x) + \frac{1}{\Lambda}\sum_{0<j\le k}\varepsilon\|(u-\psi_L)_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)} \\
&\qquad \le (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d} \varepsilon(u-\psi_L)_+(a+\varepsilon j,x)\partial_\varepsilon^\alpha u(a+\varepsilon j,x)\,dx\Big|_{t=t_1} \\
&\qquad + C_{d,\Lambda,s}\int_{\mathbb{R}^d}\sum_{0<j\le k}\varepsilon\left((u-\psi_L)_+^2+(u-\psi_L)_+ + \chi_{\{u>\psi_L\}}\right)dx \\
&\qquad\qquad + \int_{\mathbb{R}^d}\sum_{0<j\le k}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)f(a+\varepsilon j,x)\,dx. \qquad (6.3.13)
\end{aligned}$$

• **Control of the piece in time**. We now control the piece in time.

We write $v=(u-\psi_L)_+$. We also define v and f for non-integer values as we did for u and ψ. Then

$$\int_{\mathbb{R}^d}\sum_{0<j\le k}\varepsilon(u-\psi_L)_+(a+\varepsilon j,x)f(a+\varepsilon j,x) = \int_{\mathbb{R}^d}\int_a^T fv.$$

The term

$$(u-\psi_L)_+(a+\varepsilon j)\partial_\varepsilon^\alpha(u-\psi_L)_-(a+\varepsilon j)\ge 0,$$

so we may ignore this term on the left hand side of the equation. We will however utilize it in a crucial way in the second De Giorgi Lemma. We also recognize that $(u-\psi_L)_+(x,a)=0$ for all $x\in\mathbb{R}^d$ by assumption, so that

$$\sum_{0<j\le k}\varepsilon v(a+\varepsilon j,x)\partial_\varepsilon^\alpha v(a+\varepsilon j,x) = \sum_{-\infty<j\le k}\varepsilon v(a+\varepsilon j,x)\partial_\varepsilon^\alpha v(a+\varepsilon j,x).$$

We move the term involving $\partial_\varepsilon^\alpha\psi_L$ to the right hand side of the equation and use [5, Lemma 2.3] to control this term by the L^1 norm of v.

For the last term in the right hand side, we have that $|f|v$ is controlled by v, since $|f|\le 1$. Adding all the terms together and utilizing Lemma 5.2.4 and Lemma 5.2.7 respectively, we have that

$$\begin{aligned}
&\Lambda_1\int_{\mathbb{R}^d}\left(\int_a^T\frac{v^2(t,x)}{(t-a)^\alpha}\,dt + \int_a^t\int_a^t\frac{(v(t,x)-v(\tau,x))^2}{(t-\tau)^{1+\alpha}}\,d\tau\,dt\right)dx + \int_{t_1}^{t_2}\frac{1}{\Lambda}\|v\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)}\,dt \\
&\le (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d}\left(\int_a^T\frac{v^2(t,x)}{(t-a)^\alpha}\,dt + \int_a^t\int_a^t\frac{(v(t,x)-v(\tau,x))^2}{(t-\tau)^{1+\alpha}}\,d\tau\,dt\right)dx\Bigg|_{t=t_1} \\
&+ C_{d,\Lambda,\alpha}\left[\int_a^t\int_{\mathbb{R}^d} v^2+v+\chi_{\{v>0\}}\right].
\end{aligned}$$

(6.3.14)

Next, using the fractional Sobolev norm coming from the fractional derivative, we get the desired energy inequality

$$\Lambda_1 \int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\,dx + \int_{t_1}^{t_2} \frac{1}{\Lambda}\|v\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)}\,dt$$
$$- (\Lambda_2-\Lambda_1)\int_{\mathbb{R}^d} \|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})}\,dx\Big|_{t=t_1}$$
$$\leq C_{\alpha,s,\Lambda,\Lambda_1,\Lambda_2}\int_{\mathbb{R}^d}\int_{t_1}^{t_2}\left[(u-\psi_L)^2_+ + (u-\psi_L)_+ + \chi_{\{(u>\psi_L\}}\right].$$

□

6.3.2 The oscillation reduction lemmas

The next step is to obtain a first De Giorgi type oscillation reduction lemma. These technical results need only be proved for bounded nonnegative weak solutions defined in a strip $\Gamma_R = B_R \times [-R,0]$. As done in the previous Chapter 5, if u is mostly negative in space-time measure in a certain parabolic cylinder, then the supremum goes down if we restrict to a smaller nested cylinder (in order keep the natural homogeneity between the space and time variables, since the problem involves the nonlocal time variables). Due to the nonlocal character of the space-time operators involved, it is necessary to have some control of the far away behaviour of the solution. This is done, as in [5,38,43,61], via a barrier function. In order to simplify our approach we work with normalised cylinders. The general case is treated by scaling. Indeed, one of the lemmas controls the decrease of the supremum of the solution once we restrict the size of the parabolic neighbourhood of $(0,0)$, the other one implies that under suitable assumptions the solution separates from zero. A third one improves the first result so as to obtain a real alternative between going a bit down and a bit up, which leads to the proof of regularity [43]. Here is the first basic lemma.

Lemma 6.3.2 (De Giorgi's first lemma)**.** *There exists a positive constant* $\delta(\vartheta) \in (0,1)$

$$\delta(\vartheta) := \frac{\inf\limits_{0\leq\tau\leq 2}\vartheta'(\tau)}{1+\vartheta(2)-\vartheta(0)-\inf\limits_{0\leq\tau\leq 2}\vartheta'(\tau)}, \tag{6.3.15}$$

depending only on d and α – but independent of ε and a– such that for any solution u : $[a,0]\times\mathbb{R}^d \hookrightarrow \mathbb{R}$ to (6.3.6) *with $\|f\|_{L^\infty(Q)} \leq 1$ and $a \leq -1$, the following implication for u holds true.*

If it is verified that

$$\int_a^T \int_{\mathbb{R}^d} [u(t,x)-\psi]^2_+\,dx\,dt \leq \delta(\vartheta)^{2/p},$$

and

$$u(a,x) \leq \psi(a,x) + \frac{1}{2} \qquad \textit{for all } x\in\mathbb{R}^d,$$

then we have

$$u(t,x) \le \frac{1}{2} + \psi(x,t) \qquad \textit{for } (x,t) \in [a,T] \times \mathbb{R}^d.$$

Hence, we have in particular that $u(t,x) \le \frac{1}{2}$ *on* $[-1,0] \times B_1(0)$.

Proof. Let $L_k = \frac{1}{2}(1 - \frac{1}{2^k})$ and $t_k = -1 - \frac{1}{2^k}$. Moreover, we will use the abbreviation $Q_k = [a,0] \times \mathbb{R}^d$, for $t_k \in [a,0]$. We take

$$\psi_{L_k}(t) = L_k + (|t|^{\alpha/2} - 1)_+,$$

in (6.3.1) and $u_k(t) = (u - \psi_{L_k})_+(\cdot,t)$. Since $\psi_k \ge 0$ we take $\ell = 0$ in (6.3.15). Observe that if we start the iteration from $k = 1$ we may take $\ell = 1/4$ and $M = 2$ in (6.3.5) to simplify.

We split the proof in two parts.

First step: Short form of the energy inequality. Let us consider two variables σ, t that satisfy $a \le t_{k-1} \le \sigma \le t_k \le 0$. By taking the time integral over $[\sigma, t]$ in inequality (6.3.2), we obtain

$$\begin{aligned}\Lambda_1 \int_{\mathbb{R}^d} \|(u - \psi_{L_k})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})} dx + \frac{1}{\Lambda} \int_\sigma^t \|(u - \psi_{L_k})_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)} d\tau \\ \le (\Lambda_2 - \Lambda_1) \int_{\mathbb{R}^d} \|(u - \psi_{L_k})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})} dx \Big|_{t=t_1} \\ + C \int_{\mathbb{R}^d} \int_2^t \Big[(u - \psi_{L_k})_+^2 + (u - \psi_{L_k})_+ + \chi_{\{(u > \psi_{L_k}\}} \Big],\end{aligned}$$

where $C \equiv C(\alpha, s, d, \Lambda, \Lambda_1, \Lambda_2)$.

Next, by first taking the average over $\sigma \in [t_{k-1}, t_k]$, and then taking the sup over $t \in [a,0]$ in the above inequality, we deduce from the above inequality that

$$\begin{aligned}\int_{\mathbb{R}^d} \|(u - \psi_{L_k})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R})} + \int_a^T \|(u - \psi_{L_k})_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^d)} \\ \le C_{\alpha,s,d,\Lambda,\Lambda_1,\Lambda_2} \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \left[\int_{\mathbb{R}^d} \int_a^t (u - \psi_{L_k})_+^2 + (u - \psi_{L_k})_+ + \chi_{\{(u > \psi_{L_k}\}} \right].\end{aligned} \tag{6.3.16}$$

We now use the Sobolev embedding $H^{\frac{\alpha}{2}}(\mathbb{R}) \subset L^{\frac{2}{1-\alpha}}(\mathbb{R})$ to obtain from (6.3.16)

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\int_a^T v^{\frac{2}{1-\alpha}} \right)^{1-\alpha} + \int_a^T \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}} \\ \le C_{\alpha,s,d,\Lambda,\Lambda_1,\Lambda_2} \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \int_a^T \int_{\mathbb{R}^d} v^2 + v + \chi_{\{v>0\}}\, dt\, dx,\end{aligned} \tag{6.3.17}$$

where we have settled $v = (u - \psi_{L_k})_+$ for simplicity purposes.

In the following computation we use Holder's inequality twice with

$$\frac{\beta}{p_1}+\frac{1-\beta}{p_2}=\frac{1}{p}=\frac{\beta}{p_3}+\frac{1-\beta}{p_4},$$

and interpolate as in [5] and as in Chapter 5,

$$\begin{aligned}\int_a^T\int_{\mathbb{R}^d} v^p &= \int_a^T\int_{\mathbb{R}^d} v^{p\beta}v^{p(1-\beta)}\\ &\le \int_a^T\left(\int_{\mathbb{R}^d} v^{p_1}\right)^{\frac{p\beta}{p_1}}\left(\int_{\mathbb{R}^d} v^{p_2}\right)^{\frac{p(1-\beta)}{p_2}}\\ &\le \left(\int_a^T\left(\int_{\mathbb{R}^d} v^{p_1}\right)^{\frac{p_3}{p_1}}\right)^{\frac{\beta p}{p_3}}\left(\int_a^T\left(\int_{\mathbb{R}^d} v^{p_2}\right)^{\frac{p_4}{p_2}}\right)^{\frac{p(1-\beta)}{p_4}}\end{aligned}$$

so that

$$\left(\int_a^T\int_{\mathbb{R}^d} v^p\right)^{\frac{2}{p}} \le \beta\left(\int_a^T\left(\int_{\mathbb{R}^d} v^{p_1}\right)^{\frac{p_3}{p_1}}\right)^{\frac{2}{p_3}}+(1-\beta)\left(\int_a^T\left(\int_{\mathbb{R}^d} v^{p_2}\right)^{\frac{p_4}{p_2}}\right)^{\frac{2}{p_4}}.$$

Set

$$p_1=2,\qquad p_2=\frac{2d}{d-2s},\qquad p_3=\frac{2}{1-\alpha},\qquad p_4=2$$

so that

$$p=2\left(\frac{s+\alpha d}{\alpha d+s(1-\alpha)}\right)\text{ and }\beta=\frac{s}{s+\alpha d}. \tag{6.3.18}$$

By Minkowski's inequality

$$\left(\int_a^T\left(\int_{\mathbb{R}^d} v^2\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} \le \int_{\mathbb{R}^d}\left(\int_a^T v^{\frac{2}{1-\alpha}}\right)^{1-\alpha}.$$

Then

$$\|v\|^2_{L^p(\mathbb{R}^d\times[a,T])}\le C_{\alpha,s,d,\Lambda,\Lambda_1,\Lambda_2}\left(\frac{\Lambda_2-\Lambda_1+1}{\Lambda_1}\right)\int_a^T\int_{\mathbb{R}^d} v^2+v+\chi_{\{v>0\}}\,dt\,dx, \tag{6.3.19}$$

where $p>2$ is defined in (6.3.18).

Second step: Nonlinear recurrence. From the short form of the energy inequality (6.3.19), we establish a nonlinear recurrence relation to the following sequence of truncated energy:

$$U_k:=\int_a^t\int_{\mathbb{R}^d}(u-\psi_{L_k})_+^2+(u-\psi_{L_k})_+ +\chi_{\{u-\psi_{L_k}>0\}}.$$

Now, since $L_k=L_{k-1}+2^{-k-1}$, we have that $u_k>0$ implies $u_{k-1}>2^{-k-1}$, which in turn gives the Chebyshev type inequality

$$\int_{Q_k} v_k^p\le 2^{(k+1)(p-q)}\int_{Q_k} v_{k-1}^p$$

for every $p > q$. Thus, for some $p > 2$ to be chosen then (6.3.19), we have

$$\int_{Q_k} (u - \psi_{L_k})_+ \leq \int_{Q_k} (u - \psi_{L_{k-1}})_+ \chi_{\{u-\psi_{L_{k-1}} > \frac{1}{2^{k+1}}\}}$$
$$\leq (2^{k+1})^{p-1} \int_{Q_k} (u - \psi_{L_{k-1}})_+^p \leq (2^{k+1})^{p-1} C_d^p U_{k-1}^{p/2},$$
$$\int_{Q_k} \chi_{\{u-\psi_{L_k}>0\}} \leq (2^{k+1})^p \int_{Q_k} (u - \psi_{L_{k-1}})_+^p \leq (2^{k+1})^p C_d^p U_{k-1}^{p/2}$$
$$\int_{Q_k} (u - \psi_{L_k})_+^2 \leq \int_{Q_k} (u - \psi_{L_{k-1}})_+^2 \chi_{\{u-\psi_{L_{k-1}} > \frac{1}{2^{k+1}}\}}$$
$$\leq (2^{k+1})^{p-2} \int_{Q_k} (u - \psi_{L_{k-1}})_+^p \leq (2^{k+1})^{p-2} C_d^p U_{k-1}^{p/2}.$$

From the above three inequalities we conclude

$$U_k \leq \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) \overline{C}^k U_{k-1}^{p/2}, \qquad \text{for every } k \geq 0, \tag{6.3.20}$$

for some universal constant $\overline{C}$ that depends only on $\alpha, s, \Lambda, d, \Lambda_1, \Lambda_2$. Since $p > 2$ it follows from the nonlinear recurrence relation (6.3.20) that there exists some sufficiently small constant κ_0

$$\kappa_0 = \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1} \right) U_0^{p/2}$$

depending only on $\alpha, s, \Lambda, d, \Lambda_1, \Lambda_2$ (but not ε or a) such that if $U_1 \leq \kappa_0$, it follows that

$$\lim_{k\to\infty} U_k \to 0.$$

From (6.3.17), we have

$$\int_a^T \int_{\mathbb{R}^d} |u - \psi|^2 \, dx \, dt < \varepsilon \left(\frac{\Lambda_1}{\Lambda_2 - \Lambda_1 + 1} \right)^{2/p},$$

Furthermore, $U_k \to 0$ implies that

$$u \leq \frac{1}{2} + (|t|^{\alpha/2} - 1)_+ + (|x|^{s/2} - 1)_+ \text{ for } (x,t) \in [a,T] \times \mathbb{R}^d.$$

□

For this next corollary we define

$$\overline{\psi}(t) := (|t|^{\alpha/4} - 1)_+ + (|x|^{s/4} - 1)_+ \text{ if } t \leq 0.$$

Corollary 6.3.3. *There exists a constant* $\kappa_0 \in (0,1)$

$$\kappa_0 = R^{-d-\frac{s}{\alpha}} (1 + \overline{\psi}(-2,2))^{-2} \delta(\vartheta)^{2/p}, \tag{6.3.21}$$

and $\varepsilon_0 > 0$*, both depending only on* d, s, α *such that for any solution* $u : [a, 0] \times \mathbb{R}^d \hookrightarrow \mathbb{R}$ *to* (6.3.6) *with* $\varepsilon < \varepsilon_0$*,* $a \le -2$*, and* $\|f\|_{L^\infty((a,T)\times\mathbb{R}^d)} \le 1$ *satisfying*

$$u(t,x) \le 1 + \overline{\psi}(t) \text{ on } [a,0] \times \mathbb{R}^d$$

and

$$|\{u > 0\} \cap ([-2,0] \times B_2)| \le \kappa_0.$$

Then we have,

$$u(t,x) \le \frac{1}{2} \text{ for } (t,x) \in [-1,0] \times B_1.$$

Proof. If $R \ge 2^{1/s}$ and $R^{s/\alpha} \ge 2^{1/\alpha}$, then

$$\begin{aligned} 1 + ((|y|+1)^{s/4} - 1)_+ &\le (|y|^{s/2}) \quad \text{if } |y| \ge R, \\ 1 + ((|t|+1)^{\alpha/4} - 1)_+ &\le (|t|^{\alpha/2}) \quad \text{if } |t| \ge R^{s/\alpha}. \end{aligned}$$

Thus, we may choose R even larger such that

$$2 + \overline{\psi}(|t|+1) \le \psi(|t|) \text{ if } (y,t) \notin [-R^{\frac{s}{\alpha}}, 0] \times B_R(0). \tag{6.3.22}$$

R is now fixed and is dependent on d, s, α.

For any $(t_0, x_0) \in [-1,0] \times B_1$ with $t_0 \in a + \varepsilon\mathbb{Z}_+$, we introduce the rescaled function u_R defined on $[R^{s/\alpha}(a - t_0), R^{2/\alpha} - t_0] \times \mathbb{R}^d$ by

$$u_R(\tau, y) := u\left(t_0 + \frac{\tau}{R^{s/\alpha}}, x_0 + \frac{y}{R}\right).$$

The function u_R satisfies equation (6.3.6) with initial time $R^{2/\alpha}(a - t_0)$, with discrete time increment $\varepsilon R^{s/\alpha}$ and with a rescaled kernel

$$J_R(x,y) = R^{-(d+s)} J(x_0 + R^{-1}x, x_0 + R^{-1}y).$$

Observe that this kernel satisfies again hypothesis $(\mathrm{H_J})$ with the same constant provided $R \ge 1$. The right hand side of the equation is

$$f_R := f\left(t_0 + \frac{\tau}{R^{s/\alpha}}, x_0 + \frac{y}{R}\right),$$

for $|f_R| \le 1$. We then choose $\varepsilon_0 R^{s/\alpha} = 1$, so that $\varepsilon R^{s/\alpha} < 1$. We can apply Lemma 6.3.2 to u_R. In [5] it is shown that for $t_0 \in [-1,0]$, and for $x_0 \in B_1$,

$$\left(\left|x_0 + xR^{-1}\right|^{s/4} - 1\right)_+ \le \left(|x_0 + x|^{s/4} - 1\right)_+,$$

and

$$(\left|t_0 + tR^{-s/\alpha}\right|^{\alpha/4} - 1)_+ \le \left(|t_0 + t|^{\alpha/4} - 1\right)_+,$$

and we conclude that $\overline{\psi}(t_0 + t/R^{s/\alpha}) \leq \overline{\psi}(t_0 + t, x_0 + x)$. Now since $\overline{\psi}(t,x)$ increases with respect to $|x|$ and $|t|$ for $|x|,|t| > 1$ we have

$$u_R(\tau,y) \leq 2 + \overline{\psi}(t_0 + \frac{\tau}{R^{2/\alpha}}, x_0 + \frac{y}{R}) \leq 2 + \overline{\psi}(t_0 + \tau, x_0 + y) \leq 2 + \overline{\psi}(|\tau| + 1, |y| + 1).$$

Then utilizing (6.3.22) we have chosen R large enough so that $u_R(\tau,y) \leq \psi(\tau,y)$ for any $(\tau,y) \notin [-R^{s/\alpha}, 0] \times B_R$, and:

$$\int_{R^{\frac{s}{\alpha}}(a-t_0)}^{0} \int_{\mathbb{R}^d} [u_R(\tau,y) - \psi(\tau,y)]_+^2 = \int_{-R^{\frac{s}{\alpha}}}^{0} \int_{|y|\leq R} [u_R(\tau,y) - \psi(\tau,y)]_+^2$$
$$\leq \int_{-R^{\frac{s}{\alpha}}}^{0} \int_{|y|\leq R} [u_R(\tau,y)]_+^2 = R^{d+\frac{s}{\alpha}} \int_{t_0-1}^{t_0} \int_{\{x_0\}+B_1} [u_R(\tau,y)]_+^2$$
$$\leq R^{d+\frac{s}{\alpha}} \int_{-2}^{0} \int_{B_2} [u_R(\tau,y)]_+^2 \leq R^{d+\frac{s}{\alpha}} (1 + \overline{\psi}(-2,2))^2 \kappa_0.$$

Choosing $\kappa_0 = R^{-d-\frac{s}{\alpha}}(1 + \overline{\psi}(-2,2))^{-2}\delta(\vartheta)^{2/p}$ gives that $u(t_0,x_0) \leq 1/2$ for $(t_0,x_0) \in (-1,0) \times B_1$ with $t_0 \in a + \varepsilon\mathbb{Z}_+$, and therefore for all $(t,x) \in (-1,0) \times B_1$. □

6.4 The Second De Giorgi Lemma

The proof of this part follows the one in Chapter 5. We need to analyse what happens when the solution is neither mostly negative nor mostly positive, in the sense of Lemma 6.3.2, in space-time measure [5, 38, 61]. To this aim we will use De Giorgi's idea of loss of mass at intermediate levels, obtaining a quantitative version of the fact that a function with a jump discontinuity cannot be in the energy space.

The key idea is to impose conditions on the nonlinearity guaranteeing that the equation is neither degenerate nor singular at the intermediate values. Hence, we are in the linear setting studied in [38, 61]. As in [5], the result is written in terms of the functions

$$F_1(t) := \sup(-1, \inf(0, |t|^2 - 16)), \quad F_2(t) := \sup(-1, \inf(0, |x|^2 - 16)).$$

We note that F_i are both Lipschitz. F_1 is compactly supported in B_3 and equal to -1 in B_2. Similarly, F_2 is compactly supported in $[-4,4]$ and is equal to -1 in $[-3,3]$. We shall only use $F(t)$ for $t \leq 0$. This function is used to localize the function in $[-4,4]$.

For $\lambda < 1/3$, we also define

$$\psi_\lambda(t,x) := ((|x| - \lambda^{-4/s})^{s/4} - 1)_+ \chi_{\{|x| \geq \lambda^{-4/s}\}} + ((|t| - \lambda^{-4/\alpha})^{\alpha/4} - 1)_+ \chi_{\{|t| \geq \lambda^{-4/\alpha}\}}.$$

Our Lemma will involve the following sequence of five cutoffs:

$$\phi_i := 2 + \psi_{\lambda^3}(t,x) + \lambda^i F_1(x) + \lambda^i F_2(t)$$

for $0 \leq i \leq 4$.

Lemma 6.4.1. *Let κ_0 be the constant defined in Corollary 6.3.3 and assume $C_1(\Lambda_1) \leq \vartheta'(\tau) \leq C_2(\Lambda_2)$ for every $1/2 \leq \tau \leq 2$. For $0 < \mu < 1/8$ fixed, there exists $\lambda \in (0,1)$, depending only on $d, \Lambda_1, \Lambda_2, \alpha, s$ such that for any solution $u : [a,0] \times \mathbb{R}^d \to \mathbb{R}$ to* (6.3.6) *with $0 < \varepsilon < \varepsilon_0$, $a \leq -4$, and $|f| \leq 1$ satisfying*

$$u(t,x) \leq 2 + \psi_{\lambda^3}(t) \quad \text{on } [a,0] \times \mathbb{R}^d, \qquad |\{u < \phi_0\} \cap ((-3,-2) \times B_1)| \geq \mu,$$

then we have

$$|\{u > \phi_4\} \cap ((-2,0) \times B_2)| \leq \kappa_0.$$

The main idea of the proof is as follows: We exploit the fact that u satisfies the equation as well as the local nature of the spatial operator to make use of the assumption that the set where u is small on $(-3,-2) \times B_1$ is significant enough, and show there is a large set of points in $(-3,-2) \times B_2$ in which u is small. After that, we then use the nonlocal nature of the time derivative to show that this implies that the set in $(-2,0) \times B_2$ on which u is large is a very small set, under the assumptions on the nonlinearity of $\partial_t^\alpha \vartheta$ that

$$C_1(\Lambda_1)\|u - \bar{\psi}\|^2_{H^{\alpha/2}} \leq \int_a^T (u - \phi_{i_+})\partial_\varepsilon^\alpha \vartheta \, dt \leq C_2(\Lambda_2)\|u - \bar{\psi}\|^2_{H^{\alpha/2}}.$$

The proof in [5, Lemma 5.1] works verbatim, using the above equivalence whenever required. We present it here for the clarity purpose.

Proof. Throughout this proof, the constants c and C will denote constants that only depend on the parameters $d, s, \alpha, \Lambda, \Lambda_1, \Lambda_2$. They can change from line to line in the proof. We first consider $0 < \lambda < 1/3$ and small enough such that

$$\psi_\lambda(t,x) = 0 \quad \text{if } (t,x) \in [-2,0] \times B_2.$$

We split the proof into three steps.

First Step: The energy inequality. We return to the energy inequality (6.3.4), but this time we utilize the two positive terms that were previously ignored. We have for $v = (u - \phi_1)_+$

$$\begin{aligned}
&\left(\frac{\Lambda_1}{\Lambda_2 - \Lambda_1 + 1}\right) \int_{\mathbb{R}^d} \int_a^T (u - \phi_1)_+ \partial_\varepsilon^\alpha [(u - \phi_1)_+ + (u - \phi_1)_- + \phi_1] dt dx \\
&+ \int_a^T \mathcal{E}((u - \phi_1)_+, (u - \phi_1)_+) + \mathcal{E}((u - \phi_1)_+, \phi_1) + \mathcal{E}((u - \phi_1)_+, (u - \phi_1)) dt \\
&\leq \int_a^T \int_{\mathbb{R}^d} (u - \phi_1)_+ |f|.
\end{aligned}$$

Since $(u - \phi_1)_+$ is compactly supported in $[-4,0] \times B_3$,

$$\int_Q |f|(u - \phi_1)_+ \leq C\lambda$$

The spatial pieces involving $\mathcal{E}$ are controlled as follows as in [38]:

$$|\mathcal{E}((u-\phi_1)_+,\phi_1)| \leq \frac{1}{2}\mathcal{E}((u-\phi_1)_+,(u-\phi_1)_+) \\ +2\int\int[\phi_1(x)-\phi_1(y)]^2 J(x,y)[\chi_{B_3(x)}].$$

The first term is absorbed on the left hand side and

$$2\int\int[\phi_1(x)-\phi_1(y)]^2 J(x,y)[\chi_{B_3(x)}] \leq C\lambda^2.$$

Then our inequality becomes

$$C\lambda^2\left(\frac{\Lambda_2-\Lambda_1+1}{\Lambda_1}\right) \geq \int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_+ dt\,dx \\ +\int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_- dt\,dx + \int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha\phi_1 dt\,dx \\ +\int_a^T \frac{1}{2}\mathcal{E}(v,v)+\mathcal{E}(v,(u-\phi_1)_-)dt.$$

The time piece is controlled as follows: As shown in the first De Giorgi Lemma 6.3.2 we may use the Sobolev embedding to obtain

$$\int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_+ \geq c\int_{\mathbb{R}^d}\left(\int_a^T (u-\phi_1)_+^p\right)^{2/p}.$$

By utilizing that $(u-\phi_1)_+$ is compactly supported in the time interval $[-4,0]$, we have the bound

$$\int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_+ \geq c\int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+^2. \tag{6.4.1}$$

We now control the other time piece by moving it to the right hand side of the equation and showing:

$$0 \leq -\int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha\phi_1 \leq \int_{\mathbb{R}^d}\int_a^T c_1(u-\phi_1)_+^2 + \frac{1}{4c_1}(\partial_\varepsilon^\alpha\phi_1)^2 dtdx. \tag{6.4.2}$$

By choosing c_1 appropriately, we absorb the first term on the left hand side of the equation by using (6.4.1).

Now we also have for $t\in[a,0]$ with $t\in a+\varepsilon\mathbb{Z}_+$,

$$0 \leq -\partial_\varepsilon^\alpha\tilde{\psi}_{\lambda^3} \leq C\lambda^3, \\ 0 \leq -\partial_\varepsilon^\alpha\lambda^i F_2(t) \leq C\lambda^i.$$

Thus,

$$\int_{\mathbb{R}^d}\int_a^T\left(\frac{1}{2}(u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_+ + (u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_-\right)dtdx \\ +\int_a^T \frac{1}{2}\mathcal{E}((u-\phi_1)_+,(u-\phi_1)_+)+\mathcal{E}((u-\phi_1)_+,(u-\phi_1)_-) \leq C\lambda^2\left(\frac{\Lambda_2-\Lambda_1+1}{\Lambda_1}\right)$$

and so

$$\int_{\mathbb{R}^d}\int_a^T (u-\phi_1)_+\partial_\varepsilon^\alpha(u-\phi_1)_-\,dt\,dx+\int_a^T \mathcal{E}((u-\phi_1)_+,(u-\phi_1)_+)dt \leq C\lambda^2\left(\frac{\Lambda_2-\Lambda_1+1}{\Lambda_1}\right). \tag{6.4.3}$$

Second step: An estimate on those time slices where the "good" spacial term helps. We recall that $\mu<1/8$ is fixed from the beginning of the proof. From our hypothesis,

$$|\{u<\phi_0\}\cap((-3,-2)\times B_1)|\geq\mu,$$

the set of times Σ in $(-3,-2)$ for which $|u(\cdot,T)<\phi_0|\geq\mu/4$ has at least measure $\mu/(2|B_1|)$. As in [5, 38] we obtain

$$\int_\Sigma\int_{\mathbb{R}^d}(u-\phi_1)_+dxdt\leq C\Lambda\lambda^2.$$

Now

$$\{u-\phi_2>0\}\cap(\Sigma\times B_2)\subset\{u-\phi_1>\lambda/2\}\cap(\Sigma\times B_2),$$

and so from Tchebychev inequality, we have

$$|\{u>\phi_2\}\cap(\Sigma\times B_2)|\leq C\lambda,$$

which we rewrite as

$$|\{u\leq\phi_2\}\cap(\Sigma\times B_2)|\geq|\Sigma\times B_2|-C\lambda=\mu/2-C\lambda. \tag{6.4.4}$$

This will be positive for λ chosen small enough. This will only depend on μ,d,σ,α.

Third Step: Utilizing the extra good piece in time. We now utilize the second "good" extra term in time. Since we chose ψ_{λ^3} in place of ψ_λ in the definition of ϕ, then substituting ϕ_3 in the place of ϕ_1 in (6.4.3) we obtain

$$\int_{\mathbb{R}^d}\int_a^T (u-\phi_3)_+\partial_\varepsilon^\alpha(u-\phi_3)_-\,dt\,dx+\int_a^T \mathcal{E}((u-\phi_3)_+,(u-\phi_3)_+)dt \leq C\lambda^6\left(\frac{\Lambda_2-\Lambda_1+1}{\Lambda_1}\right). \tag{6.4.5}$$

Next we define the set

$$A:=\{x\in B_2:|(x_0\times\Sigma)\cap\{u\leq\phi_2\}|\geq|\Sigma|/2\}.$$

Then from (6.4.4) we obtain

$$|A|\geq|B_2|-\frac{C\lambda}{2|\Sigma|}\geq|B_2|(1-C\lambda\mu^{-1}).$$

We choose λ small enough such that $|B_2|C\lambda\mu^{-1} \leq \kappa_0/4$, so that

$$|B_2 \setminus A| \leq \kappa_0/4, \tag{6.4.6}$$

where κ_0 is the constant in the statement of the theorem. Recalling that for $x_0 \in A$

$$|\{u(x_0,t) \leq \phi_2\} \cap (x_0 \times [-3,-2])| \geq \frac{\mu}{2|B_2|},$$

then for $x_0 \in A$ and $t_0 \in [-2,0]$ and $t_0 \in a + \varepsilon\mathbb{Z}_+$ with $(u-\phi_3)(x_0,t_0) > 0$

$$\begin{aligned}
\partial_\varepsilon^\alpha (u-\phi_3)_-(t_0) &\geq \alpha\varepsilon \sum_{0\leq i<j} \frac{-(\phi_3-u)(\varepsilon i)}{(\varepsilon(k-i))^{1+\alpha}} \\
&\geq \alpha \int_{[-3,-2]\cap\{u>\phi_3\}} \frac{\lambda^2}{|3|^{1+\alpha}} d\tau \geq \alpha\mu/(2|B_1|)\frac{\lambda^2}{|3|^{1+\alpha}} \\
&\geq c\mu\lambda^2.
\end{aligned}$$

Putting the above inequality together with (6.4.5):

$$\begin{aligned}
C\lambda^6 \left(\frac{\Lambda_2 - \Lambda_1 + 1}{\Lambda_1}\right) &\geq \int_{\mathbb{R}^d}\int_a^T (u-\phi_3)_+ \partial_\varepsilon^\alpha (u-\phi_3)_- \\
&\geq \int_A \int_{-2}^0 (u-\phi_3)_+ \partial_\varepsilon^\alpha (u-\phi_3)_- \\
&\geq \int_A \int_{-2}^0 (u-\phi_3)_- c\mu\lambda^2
\end{aligned}$$

and thus

$$\int_A \int_{-2}^0 (u-\phi_3)_+ \leq C\frac{\lambda^4}{\mu}.$$

We utilize Tchebyschev inequality one more time to get

$$\begin{aligned}
|\{u-\phi_4 > 0\} \cap (A\times[-2,0])| &\leq |\{u-\phi_3 > \lambda^3\} \cap (B_2\times[-2,0])| \\
&\leq \lambda^{-3}\int_{A\times[-2,0]} (u-\phi_3)_+ \leq C\frac{\lambda}{\mu},
\end{aligned}$$

so we finally obtain

$$|\{u-\phi_4 > 0\} \cap (A\times[-2,0])| \leq C\frac{\lambda}{\mu}.$$

We choose λ small enough so that $C\frac{\lambda}{\mu} < \kappa_0/2$. Combining this estimate with (6.4.6) we have the desired inequality:

$$|\{u > \phi_4\} \cap (B_2 \times [-2,0])| \leq \kappa_0. \tag{6.4.7}$$

□

6.5 Proof of the Hölder regularity

In this section we are now in position to prove our main result. Since De Giorgi's lemmas were proven independent of ε, the conclusions hold in the limit. Therefore we may prove the results for weak solutions to (P_ϑ); however, the proofs can be given for analogous results of the discretized solutions to (6.3.6). Though the general argument follows closely [38, Section 5], [43, Section 10], [5, Section 6], [61, Section 4], an interesting modification is needed to accommodate the nonlinearity of the equation with the fractional derivative: in doing the scaling of the solution in each iteration we will now find solutions of a family of related equations. For λ as in the previous section, we define, for any ς

$$\psi_{\varsigma,\lambda}(t,x) := \Big((|x|-\lambda^{-4/s})^{\varsigma}-1\Big)_+\chi_{\{|x|\geq \frac{1}{\lambda^{4/s}}\}} + \Big((|t|-\lambda^{-4/\alpha})^{\varsigma}-1\Big)_+\chi_{\{|t|\geq \frac{1}{\lambda^{4/\alpha}}\}}.$$

Lemma 6.5.1. *Let ϑ be such that $\delta(\vartheta)>0$ and $\delta(\widetilde{\vartheta})>0$, where $\widetilde{\vartheta}(\tau)=-\vartheta(-\tau)$. Assume in addition that $C_1\leq\vartheta'(\tau)\leq C_2$ for $\tau\in[1/2,2]$ or $\tau\in[-2,-1/2]$. There exist constants $\varsigma_0>0$ and $\lambda^*\in(0,1)$ such that if for any solution to (P_ϑ) in $[a,0]\times\mathbb{R}^d$ with $|f|\leq\lambda^4$ and $a\leq -4$ such that if*

$$-2-\psi_{\varsigma,\lambda}\leq u\leq 2+\psi_{\varsigma,\lambda},$$

we have

$$\sup_{[-1,0]\times B_1} u - \inf_{[-1,0]\times B_1} u \leq 4-\lambda^*.$$

Proof. We fix $\varsigma>0$ depending on $\Lambda_1,\Lambda_2,\alpha,s$ such that

$$\frac{(|x|^{\varsigma}-1)_+}{\lambda^4}\leq(|x|^{s/4}-1)_+\quad \frac{(|t|^{\varsigma}-1)_+}{\lambda^4}\leq(|t|^{\alpha/4}-1)_+.$$

If u (or $-u$) is subcritical at the level 0, i.e., if $|\{u>\phi_4\}\cap((-2,0)\times B_2)|\leq\kappa_0$, see (6.3.21), we are done thanks to Lemma 6.3.2. Notice that $-u$ solves (P_ϑ) with ϑ replaced by $\widetilde{\vartheta}$. Otherwise, thanks to the hypotheses on ϑ', either u or $-u$ satisfies the hypotheses of Lemma 6.4.1. We assume for definiteness that it is u.

We consider the sequence of rescaled functions

$$u_{j+1}=\frac{1}{\lambda^4}(u_j-2(1-\lambda^4)),$$

starting from $u_0(x,t)=u(x,t)$, the solution of the time-porous medium equation under consideration. We have that u_j is a weak solution of problem (P_ϑ) with a nonlinearity ϑ_{j+1} given iteratively by

$$\vartheta_{j+1}(\tau)=\frac{1}{\lambda^4}\vartheta_j(\lambda^4\tau+2-2\lambda^4),\qquad \vartheta_0=\vartheta.$$

We will prove that for each j we can apply either Lemma 6.3.2 or Lemma 6.4.1. Repeated application of Lemma 6.4.1 will give that in fact Lemma 6.3.2 can be applied after a finite number of steps.

The key point is that $\vartheta'_{j+1}(\tau) = \vartheta'_j(\lambda^4\tau + 2 - 2\lambda^4)$. Hence, on one hand, since $\lambda^4\tau + 2 - 2\lambda^4 \in [1/2,2]$ whenever $\tau \in [1/2,2]$, we have $C_1 \leq \vartheta'(\tau) \leq C_2$, and consequently, the fractional derivative $\partial_\tau^\alpha \vartheta(\tau)$ is bounded below and above. On the other hand, since $[1-\lambda^4, 1+\lambda^4] \subset [1/2,2]$, we get $\delta(\vartheta_j) \geq \bar{\delta} > 0$ for all j.

Let $\nu = R^{-d-\frac{s}{\alpha}}(1+\overline{\psi}(-2,2))^{-2}\bar{\delta}(\vartheta)^{2/p}$. Assume by contradiction that no u_j is subcritical, that is $|\{u > \phi_4\} \cap ((-2,0)\times B_2)| > \nu$ for all j, so that we could never apply Lemma 6.3.2. Let $\mu > 0$ be such that $|\{u < \phi_0\} \cap ((-3,-2)\times B_1)| \geq \mu$. By construction,

$$|\{u_{j+1} < \phi_0\} \cap ((-3,-2)\times B_1)| \geq |\{u_j < \phi_0\} \cap ((-3,-2)\times B_1)| \geq \mu.$$

Furthermore, since λ was chosen so that

$$\phi_4(t) = 2 - 2\lambda^4 \text{ for } t \in [-2,0]\times B_2,$$

we have

$$|u_{j+1} > 0\} \cap ([-2,0]\times B_2)| \leq |\{u_j > \phi_4\} \cap ((-2,0)\times B_2)| \leq \kappa_0.$$

Also,

$$u_{j+1}(x,t) \leq u_j(x,t) \leq 2 + \frac{\psi_{\varsigma,\lambda^3}(x,t)}{\lambda^4} \leq 2 + \psi_{\lambda^3} \leq 2\psi_1 \leq 2 + \overline{\psi}.$$

Furthermore, u_{j+1} satisfies (5.2.2) with right hand side $|f| \leq 1$. Then we may apply Corollary 6.3.3 to u_{j+1}, and conclude $\tilde{u} \leq 1/2$ on $(-1,0)\times B_1$. Hence,

$$u(t,x) \leq 2 - 3\lambda^* \text{ for } (t,x) \in [-1,0]\times B_1,$$

with $\lambda^* = \frac{1}{2}\lambda^4$. □

This result shows in particular that the oscillation of u in $(-2,0)\times B_2$ is reduced in $(-1,0)\times B_1$ by a factor $\varpi^* = 2 - \lambda^*/2$. From this we are in position to prove the Hölder regularity stated in Theorem 6.1.1 by means of scaling arguments. As in [43, 61], we have to consider separately the degenerate and nondegenerate cases.

Proof of Theorem 6.1.1. We present the two parts (degenerate and nondegenerate cases) of the proof separately. But before that, we state the normalization and the modulus of continuity of the solution.

• **Normalization.** Let $(t_0,x_0) \in (a,\infty)\times\mathbb{R}^d$. We assume that $(t_0 - a) > 4$, otherwise we may rescale and have a new norm depending on the rescaling. We translate to the origin by considering

$$u_0(t,x) := u(t_0+t, x_0+x).$$

and dilate by considering $\gamma_0 = \inf(1, t_0/4)^{\alpha/s}$. We define

$$v_0(x,t) = \frac{u(t_0 + \gamma_0^{s/\alpha} t^{s/\alpha}, x_0 + \gamma_0 x)}{\|u(\cdot,0)\|_\infty + \|f(\cdot,0)\|_\infty}.$$

v_0 will satisfy an equation of the type

$$\partial_t^\alpha \vartheta_0(v_0) + \mathcal{K} v_0 = f.$$

Furthermore, the initial time for u_0 will be $(a-t_0)\gamma_0^{-s/\alpha} < -4$ and the function ϑ_0 satisfies the hypotheses of Lemma 6.5.1.

• Modulus of continuity.

In the following we prove that v_0 is continuous at $(0,0)$. Given $R > 1$, we define by induction the sequence of functions, for $k \geq 1$:

$$v_k(t,x) = \frac{v_0(R^{-s(k+1)/\alpha}t, R^{-(k+1)}x) - \mu_k}{\varpi_k}, \qquad (t,x) \in (a,0) \times \mathbb{R}^d,$$

where ϖ_k and μ_k are respectively the semi-oscillation and a certain mean of v_0 in the parabolic cylinder $Q_k = \Gamma_{R^{-k}}$,

$$\varpi_k = \frac{\sup_{Q_k} v_0 - \inf_{Q_k} v_0}{2}, \qquad \mu_k = \frac{\sup_{Q_k} v_0 + \inf_{Q_k} v_0}{2}.$$

They satisfy the equation

$$\partial_t^\alpha \vartheta_k(v_k) + \mathcal{K}_k v_k = f, \qquad \vartheta_k(\tau) = \frac{\vartheta_0\left(\varpi_k \tau + \mu_k\right)}{\varpi_k}.$$

Assuming by contradiction that $\varpi_k \geq \theta > 0$, we have that the function ϑ_k satisfies the hypotheses of Lemma 6.5.1, since $\varpi_k\tau + \mu_k \geq \theta/2$ for $\tau \geq 1/2$ if $\mu_k \geq 0$ and $\varpi_k\tau + \mu_k \leq -\theta/2$ for $\tau \leq -1/2$ if $\mu_k \leq 0$.

Also we notice that, $|v_k| \leq 1 \leq 2 + \psi_{\varsigma,\lambda}(x)$ for $|x| \leq R$, applying Lemma 6.5.1 by induction to v_{k-1}, since it can be applied to v_0. Also, if we take $R > 1$ large enough so that $\psi_{\varsigma,\lambda}(R) \geq \frac{2-\theta}{\theta}$, we get $|v_k(x,t)| \leq 2 + \psi_{\varsigma,\lambda}(x)$ if $|x| \geq R$. Hence, applying once more Lemma 6.5.1 we conclude that $\varpi_k \leq (2 - \lambda^*/2)^k$, which is a contradiction. Therefore we have a modulus of continuity.

• Hölder regularity at nondegeneracy points. We assume $u(x_0,t_0) > 0$, the case $u(x_0,t_0) < 0$ being similar. We consider $\gamma < 1$ such that

$$\frac{1}{1-(\lambda^*/2)}\psi_{\varsigma,\lambda}(x) \leq \psi_{\varsigma,\lambda}(x), \text{ for } |x| \geq R := 1/\gamma,$$

γ only depends on λ, λ^*, and ς. We define by induction the sequence of functions, for $k \geq 1$:

$$v_1(t,x) = \frac{u_0(t,x)}{\|u_0\|_{L^\infty} + \lambda^4 \|f\|_{L^\infty}}, \qquad (t,x) \in (a,0) \times \mathbb{R}^d,$$

$$v_{k+1}(t,x) = \frac{v_k(\gamma^{s/\alpha}t, \gamma x) - \mu_k}{\varpi^*}, \qquad (t,x) \in (a\gamma^{-sk}, 0) \times \mathbb{R}^d,$$

where

$$\varpi^* = 2 - \lambda^*/2, \qquad \mu_k^* = \frac{\sup_{Q_1} v_k + \inf_{Q_1} v_k}{2}.$$

Observe that the recurrence relation can be written explicitly,

$$v_k(t,x) = \frac{v_0(\gamma^{2(k+1)/\alpha}t, \gamma^{(k+1)}x) - \widetilde{\nu}_k}{(\varpi^*)^k}, \qquad (t,x) \in (a,0) \times \mathbb{R}^d,$$

where $\widetilde{\nu}_k = \sum_{j=1}^{k} \mu_j^* (\varpi^*)^k$. Also, $\mu_k = (\varpi^*)^k \mu_{k+1}^* + \widetilde{\nu}_k$, so that, since

$$\mu_k \to \frac{u(x_0,t_0)}{\|u(\cdot,0)\|_\infty + \|f(\cdot,0)\|_\infty} > 0,$$

then $\widetilde{\nu}_k \to \varepsilon > 0$.

The functions v_k satisfy $|v_k| \leq 1$ in Γ_R, and the equation

$$\partial_t^\alpha \vartheta_k(v_k) + \mathcal{K}_k v_k = f,$$

where the new nonlinearity is

$$\vartheta_k(\tau) = \frac{\vartheta_0\left((\varpi^*)^k \tau + \widetilde{\nu}_k\right)}{(\varpi^*)^k}.$$

By construction, the function u_k, ϑ_k satisfy the hypotheses of Lemma 6.5.1 for any k. So we have finally

$$\sup_{t_0+(-\gamma^{s/\alpha},0)\times(x_0+B_{\gamma^k})} - \inf_{t_0+(-\gamma^{s/\alpha},0)\times(x_0+B_{\gamma^k})} u \leq C(1-\lambda^*/4)^k,$$

We then conclude that u is C^β with

$$\beta = \frac{\ln(1-\lambda^*/4)}{\ln \gamma^{s/\alpha}}. \tag{6.5.1}$$

• **Hölder regularity at degeneracy points.** Now, let $u(x_0,t_0) = 0$. Here, we consider the sequence of functions defined by means of a recurrence that takes into account the nonlinearity, and the possible singularity of β' at zero:

$$v_{k+1}(x,t) = \frac{v_k(R^{-1}x, \gamma R^{-\sigma}t) - \mu_k^*}{\varpi^*}, \qquad \gamma = \frac{\vartheta_0(\varpi^*)}{\varpi^*},$$

with μ_k^* and ϖ^* as before. The rescaled nonlinearity turns out to be

$$\vartheta_k(\tau) = \frac{\vartheta_0\left((\varpi^*)^k \tau + \widetilde{\nu}_k\right)}{\vartheta_0\left((\varpi^*)^k\right)}$$

We observe that

$$\frac{|\widetilde{\nu}_k|}{(\varpi^*)^k} \leq \frac{|\mu_k|}{(\varpi^*)^k} + |\mu_{k+1}^*| \leq C. \tag{6.5.2}$$

The conditions of Lemma 6.5.1 are fulfilled as long as, for every $k \geq 1$,

$$0 < C_1 \leq \frac{(\varpi^*)^k \vartheta_0'((\varpi^*)^k \tau + \tilde{\nu}_k)}{\vartheta_0((\varpi^*)^k)} \leq C_2 \quad \text{for every } \tau \in (1/2, 2),$$

which hold from condition (5.1.2) using (6.5.2). See also [61] for similar analysis. Thus we conclude as before.

□

6.6 Conclusion and comment on boundary regularity

In this chapter we have proved the A De Giorgi–Nash-type theorem for the time-porous medium equation with nonlocal operator with a kernel having a singularity at the origin as that of the fractional Laplacian. The current work fits in with some results obtained in the case of time-porous medium equation studied in Chapter 5 in the limit when $s \to 2$. This opens the door to a possible approach to treat boundary regularity for nonlocal porous medium equation with fractional derivative with nonlocal operaters of Lévy-type.

Bibliography

[1] N. Abatangelo and E. Valdinoci. Getting acquainted with the fractional Laplacian. *Springer-INdAM Series (Springer, chan, to appear)*, 2017.

[2] M. Allen. Uniqueness for weak solutions of parabolic equations with a fractional time derivative. Technical report, Arxiv: 1705.03959, 2017.

[3] M. Allen. Hölder regularity for nondivergence nonlocal parabolic equations. *Calc. Var. Partial Dif.*, 57, 2018.

[4] M. Allen. A nondivergence parabolic problem with a fractional time derivative. *Differ. Integral Equ.*, 31(3/4):215–230, 2018.

[5] M. Allen, L. Caffarelli, and A. Vasseur. A parabolic problem with a fractional time derivative. *Arch. Ration. Mech. Anal.*, 221(2):603–630, 2016.

[6] M. Allen, L. Caffarelli, and A. Vasseur. Porous medium flow with both a fractional potential pressure and fractional time derivative. *Chin. Ann. Math. Ser. B*, 38(1):45–82, 2017.

[7] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions. *SIAM J. Math. Anal.*, 40(5):1815–1851, 2008/09.

[8] I. Area, J. D. Djida, J. Losada, and J. J. Nieto. On fractional orthonormal polynomials of a discrete variable. *Discrete Dyn. Nat. Soc.*, (141325), 2015.

[9] I. Area, J. Losada, and J. J. Nieto. A note on the fractional logistic equation. *Phys. A*, 444(15):182–187, 2016.

[10] A. Atangana. *Fractional Operators with Constant and Variable Order with Application to Geo-hydrology*. Academic Press, New York, 2017.

[11] A. Atangana and D. Baleanu. New Fractional Derivatives with Non-Local and Non-Singular Kernel: Theory and Application to Heat Transfer Model. *Therm. Sci.*, 20:763–769, 2016.

[12] I. Athanasopoulos and L. A. Caffarelli. Continuity of the temperature in boundary heat control problems. *Adv. Math.*, 224(1):293–315, 2010.

[13] D. Baleanu and A. Fernandez. A generalisation of the Malgrange-Ehrenpreis theorem to find fundamental solutions to fractional PDEs. *Electron. J. Qual. Theory Differ. Equ.*, 15:1–12, 2017.

[14] D. Baleanu and A. Fernandez. On some new properties of fractional derivatives with Mittag-Leffler kernel. *Commun. Nonlinear Sci. Numer. Simul.*, 59:444–462, 2018.

[15] E. G. Bazhlekova. *Fractional evolution equations in Banach spaces*. PhD thesis, TUE: Department of Mathematics and Computer Science, 2001.

[16] A. Bernardis, F. J. Martín-Reyes, P. R. Stinga, and J. L. Torrea. Maximum principles, extension problem and inversion for nonlocal one-sided equations. *J. Differential Equations*, 260(7):6333–6362, 2016.

[17] P. Biler, C. Imbert, and G. Karch. The nonlocal porous medium equation: Barenblatt profiles and other weak solutions. *Arch. Ration. Mech. Anal.*, 215(2):497–529, 2014.

[18] P. Biler, G. Karch, and R. Monneau. Nonlinear Diffusion of Dislocation Density and Self-Similar Solutions. *Commun. Math. Phys.*, 294(1):145, Jun 2009.

[19] G. M. Bisci and D. V. Rădulescu. Multiplicity results for elliptic fractional equations with subcritical term. *Nonlinear Differ. Equ. Appl.*, 22(4):721–739, 2015.

[20] G. M. Bisci, V. D. Radulescu, and R. Servadei. *Variational Methods for Nonlocal Fractional Problems*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1st edition, 2016.

[21] R. M. Blumenthal and R. K. Getoor. The asymptotic distribution of the eigenvalues for a class of Markov operators. *Pacific J. Math.*, 9:399–408, 1959.

[22] K. Bogdan, T. Grzywny, and M. Ryznar. Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.*, 38:1901–1923, 2010.

[23] M. Bonforte, A. Figalli, and X. Ros-Oton. Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains. *Comm. Pure Appl. Math.*, 70(8):1472–1508, 2017.

[24] M. Bonforte, Y. Sire, and J. L. Vázquez. Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete Continuous Dyn. Syst. Ser. A*, 35(12):5725–5767, 2015.

[25] M. Bonforte, Y. Sire, and J. L. Vázquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete Continuous. Dyn. Syst. Ser. A*, 35(12):5725–5767, 2015.

[26] M. Bonforte and J. L. Vázquez. Quantitative local and global a priori estimates for fractional nonlinear diffusion equations. *Adv. Math.*, 250(3):242–284, 2014.

[27] M. Bonforte and J. L. Vázquez. A Priori Estimates for Fractional N onlinear Degenerate Diffusion Equations on bounded domains. *Arch. Ration. Mech. Anal.*, 218:317–362, 2015.

[28] C. Brändle and A. de Pablo. Nonlocal heat equations: Regularizing effect, decay estimates and Nash inequalities. *Commun. Pure Appl. Math.*, 17(3):1161–1178, 2018.

[29] H. Brezis. Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In Eduardo H. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, volume 339, pages 101–156. Academic Press, 1971.

[30] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland, 1973.

[31] C. Bucur. *Some nonlocal operators and effects due to nonlocality*. PhD thesis, Dipartimento di Matematica "Federigo Enriques", 2017.

[32] C. Bucur and F. Ferrari. An extension problem for the fractional derivative defined by Marchaud. *Fract. Calc. Appl. Anal.*, 19(4):867–887, 2016.

[33] H. Q. Bui. Weighted Besov and Triebel spaces: interpolation by the real method. *Hiroshima Math. J.*, 12(3):581–605, 1982.

[34] X. Cabré and Y. Sire. Regularity at infinity for a mixed problem for degenerate elliptic operators in a half-cynlider. *Math. Scand.*, 81:101–126, 1997.

[35] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians I Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré - AN*, 31:23–53, 2014.

[36] L. Caffarelli and X. Cabré. *Fully Nonlinear Elliptic Equations*, volume 43 of *Colloquium Publications*. American Mathematical Society, Providence, 1995.

[37] L. Caffarelli and X. Cabré. Fully Nonlinear Elliptic Equations. *Amer. Math. Soc. Colloq. Publ.*, 43(6):754–767, 1997.

[38] L. Caffarelli, C. H. Chan, and A. Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Am. Math. Soc.*, 24(3):849–869, 2011.

[39] L. Caffarelli and A. Figalli. Regularity of solutions to the parabolic fractional obstacle problem. *Reine Angew. Math.*, pages 191–233, 2013.

[40] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Part. Diff. Eq.*, 32:1245–1260, 2007.

[41] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Commun. Pure Appl. Anal.*, 62(5):597—638, 2009.

[42] L. Caffarelli and L. Silvestre. The Evans-Krylov theorem for nonlocal fully nonlinear equations. *Ann. of Math.*, 174(2):1163–1187, 2011.

[43] L. Caffarelli, F. Soria, and J. L. Vázquez. Regularity of solutions of the fractional porous medium flow. *J. Eur. Math. Soc.*, 15(5):1701–1746, 2013.

[44] L. Caffarelli and J. L. Vázquez. Asymptotic behaviour of a porous medium equation with fractional diffusion. *Discrete Continuous. Dyn. Syst.*, 29(8):1393–1404, 2011.

[45] L. Caffarelli and J. L. Vázquez. Nonlinear porous medium flow with fractional potential pressure. *Arch. Ration. Mech. Anal.*, 202(2):537–565, 2011.

[46] L. A. Caffarelli, S. Salsa, and L. Silvestre. *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[47] L. A. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[48] M. Caputo. Linear model of dissipation whose q is almost frequency independent-II. *Geophys. J. R. Astron. Soc.*, 13(5):529–539, 1967.

[49] M. Caputo. Diffusion of fluids in porous media with memory. *Geothermics*, 28(1):113–130, 1999.

[50] M. Caputo. Linear models of dissipation whose Q is almost frequency independent. II. *Fract. Calc. Appl. Anal*, 11(1):4–14, 2008.

[51] M. Caputo and M. Fabrizio. A new Definition of Fractional Derivative without Singular Kernel. *Progr. Fract. Differ. Appl.*, 1(2):73–85, 2015.

[52] A. R. M. Carvalho, C. M. A. Pinto, and D. Baleanu. HIV/HCV coinfection model: a fractional-order perspective for the effect of the HIV viral load. *Adv. Differ. Equ.*, 2:1–22, 2018.

[53] C. H. Chan. *The De Giorgi's Method as Applied to The Regularity Theory for Incompressible Navier Stokes Equations*. PhD thesis, Faculty of the Graduate School of Texas at Austin, Department of mathematics, 2008.

[54] Ph. G. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications*. Philadelphia. SIAM, 2013.

[55] R. R. Coifman and C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Stud. Math.*, 51:241–250, 1974.

[56] M. Crandall and M. Pierre. Regularizing effects for $u_t + A\varphi(u) = 0$ in L^1. *J. Funct. Anal.*, 45(2):194–212, 1982.

[57] M. G. Crandall, H. Ishii, and P. L. Lions. Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited. *J. Math. Soc. Japan*, 39(4):581–596, 1987.

[58] M. G. Crandall and T. M. Liggett. Generation of semi-groups of nonlinear transformations on general Banach spaces. *Am. J. Math.*, 93(2):265–298, 1971.

[59] R. Dautray and J-L. Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques*. Masson, Paris, 1988.

[60] A. de Pablo and L. Vázquez J.˙ Regularity of solutions and interfaces of a generalized porous medium equation in $\mathbb{R}^N$. *Ann. Mat. Pura Appl.*, 158(4):51–74, 1991.

[61] A. de Pablo, F. Quirós, and A. Rodríguez. Nonlocal filtration equations with rough kernels. *Nonlinear Analysis*, 137:402–425, 2016.

[62] A. de Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez. A fractional porous medium equation. *Adv. Math.*, 226(2):1378–1409, 2011.

[63] A. de Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.

[64] E. DiBenedetto. *Degenerate Parabolic Equations*. Springer, New York, 1993.

[65] E. DiBenedetto, U.P. Gianazza, and V. Vespri. *Harnack's Inequality for Degenerate and Singular Parabolic Equations*. Springer Monographs in Mathematics. Springer, New York, 2011.

[66] E. Dipierro, E. Valdinoci, and V. Vespri. Decay estimates for evolutionary equations with fractional time diffusion. *J. Evol. Equ.*, pages –, 2019.

[67] S. Dipierro, N. Soave, and E. Valdinoci. On fractional elliptic equations in Lipschitz sets and epigraphs: regularity, monotonicity and rigidity results. *Math. Ann.*, 369(1):1283–1326, 2017.

[68] J-D. Djida. Wellposedness and boundary regularity for nonlocal parabolic problem with fractional derivative. Technical report, Submitted, 2019.

[69] J-D. Djida, I. Area, and J. J. Nieto. A De Giorgi-Nash type theorem for nonlocal time porous medium equations. Technical report, Submitted, 2018.

[70] J-D. Djida, A. Atangana, and I. Area. Numerical Computation of a Fractional Derivative with Non-Local and Non-Singular Kernel. *Math. Model. Nat. Phenom.*, 12(3):4–13, 2017.

[71] J-D. Djida and A. Fernandez. Interior Regularity Estimates for a Degenerate Elliptic Equation with Mixed Boundary Conditions. *Axioms*, 7(65):1–16, 2018.

[72] J-D. Djida, J. J. Nieto, and I. Area. Nonlocal time-porous medium equation: weak solutions and finite speed of propagation. *Discrete Continuous Dyn. Syst. Ser. B*, 22:1, 2019.

[73] J-D. Djida, J. J. Nieto, and I. Area. Nonlocal time porous medium equation with fractional time derivative. *Rev. Mat. Complut, in press*, pages 1–32, 2019.

[74] J-D. Djida, J. J. Nieto, and I. Area. Parabolic problem with fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel. *Discrete Continuous Dyn. Syst. Ser. S, in press*, 2019.

[75] A. Dokoumetzidis and P. Macheras. Fractional kinetics in drug absorption and disposition processes. *J. Pharmacokinet Pharmacodyn.*, 2(36):165–178, 2009.

[76] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. Providence, RI, 2nd edition, 2010.

[77] E. Fabes, D. Jerison, and C. Kenig. The Wiener test for degenerate elliptic equations. *Ann. Inst. Fourier (Grenoble):vi*, 32(3):151–182, 1982.

[78] E. Fabes, C. Kenig, and R. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1982):77–116, 1991.

[79] M. M. Fall. Regularity estimates for nonlocal Schrödinger equations. *Discrete Continuous Dyn. Syst. Ser. A*, 39:1405, 2019.

[80] M. M. Fall and T. Weth. Monotonicity and nonexistence results for some fractional elliptic problems in the half-space. *Commun. Contemp. Math*, 18(01):1550012, 2016.

[81] M. Felsinger and M. Kassmann. Local Regularity for Parabolic Nonlocal Operators. *Comm. Part. Diff. Eq.*, 38(9):1539–1573, 2013.

[82] A. Fernandez. An elliptic regularity theorem for fractional partial differential operators. *Comp. Appl. Math.*, 37:5542–5553, 2018.

[83] X. Fernández-Real and X. Ros-Oton. Boundary regularity for the fractional heat equation. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 110:49–64, 3 2016.

[84] F. Ferrari. Weyl and Marchaud derivatives: a forgotten history. *Mathematics*, 6(1), 2018.

[85] F. Ferrari and I. E. Verbitsky. Radial fractional Laplace operators and Hessian inequalities. *J. Differential Equations*, 253(1):244–272, 2012.

[86] L. Geisinger. A short proof of weyl's law for fractional differential operators. *Journal of Mathematical Physics*, 55:011504–, 2014.

[87] M. Giaquinta and L. Martinazzi. L^2-regularity: The Caccioppoli inequality. In: An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs. Technical report, Publications of the Scuola Normale Superiore. Edizioni della Normale, Pisa, 2012.

[88] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag Berlin Heidelberg, 2001.

[89] E. De Giorgi. Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.*, 3(3):25–43, 1957.

[90] W. G. Glöckle and T. F. Nonnenmacher. A fractional calculus approach to self-similar protein dynamics. *Biophys. J.*, 68:46–53, 1995.

[91] P. Glowacki and W. Hebisch. Pointwise estimates for densities of stable semigroups of measures. *Studia Math.*, 104:243–258, 1992.

[92] G. H. Hardy and J. E. Littlewood. Some properties of fractional integrals. I. *Math. Z.*, 34(1):565–606, 1928.

[93] T. Jin, Y. Y. Li, and J. Xiong. On a fractional nirenberg problem part I: blow up analysis and compactness solutions. *J. Eur. Math. Soc.*, 16:1111–1171, 2014.

[94] M. Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var.*, 34(1):1–21, 2009.

[95] M. Kassmann and W. R. Madych. Difference quotients and elliptic mixed boundary value problems of second order. *Indiana Univ. Math. J.*, 56(3):1047–1082, 2007.

[96] M. Kassmann and R. W. Schwab. Regularity results for nonlocal parabolic equations. *Riv. Mat. Univ. Parma*, 5:183–212, 2014.

[97] I. Kazufumi. Fractional Evolution Equations and Applications. Technical report, 2013. `https://kito.wordpress.ncsu.edu/files/2018/04/fract0.pdf`.

[98] J. Kemppainen, J. Siljander, and R. Zacher. Representation of solutions and large-time behavior for fully nonlocal diffusion equations. *J. Differ. Equ.*, 263:149–201, 2017.

[99] V. Keyantuo, C. Lizama, and M. Warma. Existence, regularity and representation of solutions of time fractional diffusion equations. *Adv. Differential Equations*, 21:837–886, 2016.

[100] V. Keyantuo and M. Warma. On the interior approximate controllability for fractional wave equations. *Discrete Continuous. Dyn. Syst.*, 36:3719–3739, 2016.

[101] A. A. Kilbas, H. .M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[102] P. E. Kloeden and J. Simsen. Pullback attractors for non-autonomous evolution equations with spatially variable exponents. *Commun. Pure Appl. Anal.*, 13:2543–2557, 2014.

[103] A. N. Kochubei. Fractional-order diffusion. *Differ. Equations*, 26(4):485–492, 1990.

[104] R. C. Koeller. Applications of Fractional Calculus to the Theory of Viscoelasticity. *J. App. Mech.*, 51:299–307, 1984.

[105] M. Köhne, J. Prüss, and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p-spaces. *J. Evol. Equ.*, 10(2):443–463, 2010.

[106] A. Kufner. *Weighted Sobolev spaces*. Wiley, New York, 1985.

[107] O. A. Ladyzhenskaya. On uniqueness and smoothness of generalized solutions to the Navier-Stokes equations. *Sem. Math. V.A. Steklov Math. Inst. Leningard*, 5:60–66, 1969.

[108] H. Chang Lara and G. Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var.*, 49(1):139–172, 2014.

[109] N. N. Lebedev. *Special Functions and Their Applications*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[110] S. Levendorskii. *Degenerate Elliptic Equations*, volume 258 of *Mathematics and Its Applications*. Springer Netherlands, 1993.

[111] L. Liu, T. Caraballo, and P. E. Kloeden. Long time behavior of stochastic parabolic problems with white noise in materials with thermal memory. *Rev. Mat. Complut.*, 30(3):687–717, 2017.

[112] R. L. Magin. *Fractional Calculus in Bioengineering*. Begell House Publishers, Connecticut, 2006.

[113] F. Mainardi. *Fractional calculus and waves in linear viscoelasticity. An introduction to mathematical models*. Imperial College Press, London, 2010.

[114] F. Mainardi, Y. Luchko, and G. Pagnini. The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.*, 4(2):153–192, 2001.

[115] F. Mainardi and G. Pagnini. The Wright functions as solutions of the time-fractional diffusion equation. *Appl. Math. Comput.*, 141(1):51–62, 2003.

[116] A. Marchaud. *Sur les dérivées et sur les différences des fonctions de variables réelles*. PhD thesis, Faculté des Sciences de Paris, 1927. `http://www.numdam.org/issue/THESE_1927__78__1_0.pdf`.

[117] V. G. Maz'ya. *Sobolev Spaces: with Applications to Elliptic Partial Differential Equations*, volume 342 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin Heidelberg, 2nd edition, 2011.

[118] M. Meyries and M. Veraar. Sharp embedding results for spaces of smooth functions with power weights. *Studia Mathematica*, 208:257–293, 2012.

[119] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.

[120] G. Mophou and G. M. N'Guérékata. On a class of fractional differential equations in a Sobolev space. *Appl. Anal.*, 91(1):15–34, 2012.

[121] J. Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964.

[122] J. Moser. On a pointwise estimate for parabolic differential equations. *Comm. Pure Appl. Math.*, 24:727–740, 1971.

[123] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[124] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Am. J. Math.*, 80(3):931–954, 1958.

[125] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136:521–573, 2012.

[126] A. Niang. Boundary regularity for a degenerate elliptic equation with mixed boundary conditions. *Commun. Pure Appl. Anal*, 18:107, 2019.

[127] B. Opic and A. Kufner. *Hardy-type inequalities*. Longman Scientific & Technical, 1990.

[128] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[129] X. Ros-Oton and J. Serra. Boundary regularity for fully nonlinear integro-differential equations. *Duke Math. J.*, 165:2079–2154, 2014.

[130] X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.*, 101:275–302, 2014.

[131] X. Ros-Oton and J. Serra. Regularity theory for general stable operators. *J. Differential Equations*, 260(2016):8675–8715, 2014.

[132] I. Petráš and R. L. Magin. Simulation of Drug Uptake in a Two Compartmental Fractional Model for a Biological System. *Commun. Nonlinear Sci. Numer. Simul.*, 16(12):4588–4595, 2011.

[133] M. K. Saad, A. Atangana, and D. Baleanu. New fractional derivatives with non-singular kernel applied to the Burgers equation. *Chaos*, 28(6):1–7, 2018.

[134] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. CRC, 1993.

[135] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance (Stochastic Modeling Series)*. Chapman and Hall/CRC, 1st edition, 1994.

[136] G. Savaré. Regularity and perturbation results for mixed second order elliptic problems. *Comm. Part. Diff. Eq.*, 22(5–6):869–899, 1997.

[137] J. Serra. Regularity for fully nonlinear nonlocal parabolic equations with rough kernels. *Calc. Var.*, 54(1):615–629, 2015.

[138] J. Serrin. The initial value problem for the Navier-Stokes equations. In R. E. Langer, editor, *Nonlinear Problems*, pages 69–98, Madison, 1963. Univ. of Wisconsin Press.

[139] R. Servadei and E. Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

[140] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[141] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60:67–112, 2007.

[142] J. Simon. Compact sets in the space $L^p(0,t;b)$. *Ann. Mat. Pura Appl.*, 146(1):65–96, 1986.

[143] S. L. Sobolev. On a theorem of functional analysis. *Transl. Amer. Math. Soc.*, 34(2):471–497, 1963.

[144] P. Sopasakis, H. Sarimveis, P. Macheras, and A. Dokoumetzidis. Fractional calculus in pharmacokinetics. *J Pharmacokinet Pharmacodyn*, pages 1–10, 2017.

[145] D. Stan. *Nonlinear and Nonlocal Diffusion Equations*. PhD thesis, Facultad de Ciencias, Departamento de Matemáticas, 2014.

[146] D. Stan, F. D. Teso, and J. L. Vázquez. Finite and infinite speed of propagation for porous medium equations with fractional pressure. *C. R. M*, 352(2):123–128, 2014.

[147] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, 1970.

[148] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series, Monographs in Harmonic Analysis, III*. Princeton University Press, Princeton, NJ, 1993.

[149] P. R. Stinga and J. L. Torrea. Extension Problem and Harnack's Inequality for Some Fractional Operators. *Comm. Part. Diff. Eq.*, 35(11):2092–2122, 2010.

[150] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[151] H. Triebel. *Interpolation theory, function spaces, differential operators*. second edition. Johann Ambrosius Barth, Heidelberg, 1995.

[152] A. Vasseur. A new proof of partial regularity of solutions to Navier-Stokes equations. *Nonlinear differ. equ. appl.*, 14(5-6):753–785, 2007.

[153] J. L. Vázquez. *The Porous Medium Equation*. Oxford University Press, Oxford, 2007.

[154] J. L. Vázquez. Nonlinear Diffusion with Fractional Laplacian Operators. In Holden Helge, Karlsen, and Kenneth H., editors, *Nonlinear Partial Differential Equations*, pages 271–298, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.

[155] J. L. Vázquez. *The Mathematical Theories of Diffusion: Nonlinear and Fractional Diffusion*, pages 205–278. Springer International Publishing, Cham, 2017.

[156] J. L. Vázquez, J. J. Trujillo, and M. P. Velasco. Fractional heat equation and the second law of thermodynamics. *Frac. Calc. App. Anal.*, 14:334–342, 2011.

[157] V. Zaburdaev, S. Denisov, and J. Klafter. Lévy walks. *Rev. Mod. Phys.*, 87:483–530, 2015.

[158] R. Zacher. A De Giorgi-Nash type theorem for time fractional diffusion equations. *Math. Ann.*, 356(1):99–146, 2013.

[159] S. Zaremba. *Sur un problème mixte relatif à l'équation de Laplace*, volume 41. Bulletin international de l'Académie des Sciences de Cracovie. Classe des Sciences Mathématiques et Naturelles, Serie A: Sciences mathématiques (in French), JFM, 1910.

[160] G. M. Zaslavsky. Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.*, 371:461–580, 2012.

[161] Z. Q. Zhen, P. Kim, and R. Song. Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc*, 12:1307–1329, 2010.