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A STUDY ON TRIDIMENSIONAL LUCAS-COBALANCING NUMBERS

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Abstract. In this paper, we aim to introduce and study the tridimensional version of Lucas-cobalancing numbers, with special emphasis on their recurrence relation, key properties, and diverse sum identities.

1. Introduction

Numerical sequences have long been a focus of mathematical research, with numerous studies dedicated to their properties, extensions, and generalizations. In this article, we aim to expand the study of Lucas-balancing numbers, building on the foundational work of Behera and Panda [**1**], who introduced the concept of balancing numbers in 1999 within the framework of solving a Diophantine equation. Their investigation sought to identify a natural number n satisfying the equation

$$1+2+3+\cdots+(n+1)=(n+1)+(n+2)+\cdots+(n+r),$$

for some natural number r.

The resulting sequence of balancing numbers is denoted by $\{B_n\}_{n\geqslant 1}$. Behera and Panda established the following recurrence relation to describe this sequence:

$$B_{n+2}=6B_{n+1}-B_n, \tag{1.1}$$

for $n\geqslant 2$, with the initial conditions $B_0=0$ and $B_1=1$. By solving this recurrence as a second-order linear homogeneous difference equation, they derived a Binet-like formula, the generating function, and various identities for balancing

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numbers. These foundational results have paved the way for further exploration and generalization of balancing sequences, including their extensions to Lucas-balancing numbers.

As demonstrated by Ray [**14**], a fundamental property of balancing numbers is the following: a number b qualifies as a balancing number if and only if $8b^2+1$ is a perfect square. Building on this observation, Lucas-balancing numbers were introduced and are defined as

$$C_n = \sqrt{8(B_n)^2+1},$$

where B_n represents the n-th balancing number.

The sequence of Lucas-balancing numbers $\{C_n\}_{n\geqslant 0}$, corresponding to the balancing numbers $\{B_n\}_{n\geqslant 0}$, satisfies the same recurrence relation as the balancing numbers, namely

$$C_{n+1} = 6C_n - C_{n-1},$$

for $n \geqslant 2$, with initial elements $C_0 = 1$ and $C_1 = 3$. For more details on balancing and Lucas-balancing numbers, see [**1, 6, 7, 8, 9, 10, 11, 12, 14, 16**], among others.

This foundational framework serves as the basis for the study of other sequences and their extensions. One such sequence is the Lucas-cobalancing numbers $\{c_n\}_{n\geqslant 1}$, which are defined by the recurrence relation

$$c_{n+2} = 6c_{n+1} - c_n, \tag{1.2}$$

with initial values $c_1 = 1$ and $c_2 = 7$. This sequence was introduced by Panda and Ray [**13**]. The recurrence relation for the Lucas-cobalancing numbers is similar to that of the balancing numbers but differs in the initial values. For the Lucas-cobalancing sequence, the first element is considered to be c_1. More detailed studies on the Lucas-cobalancing numbers can be found in [**1, 2, 11, 12, 15, 17**].

In [**3, 5**], the bidimensional extensions of the balancing and Lucas-balancing numbers are presented, along with a detailed examination of their properties and characteristics. The bidimensional balancing numbers $\{B_{(n,m)}\}_{n,m\geqslant 0}$ and the bidimensional Lucas-cobalancing numbers $\{c_{(n,m)}\}_{n,m\geqslant 0}$ are defined by the following recurrence relations, respectively:

$$\begin{cases} B_{(n+1,m)} &= 6B_{(n,m)} - B_{(n-1,m)}, \\ B_{(n,m+1)} &= 6B_{(n,m)} - B_{(n,m-1)}, \end{cases}$$

with initial terms $B_{(0,0)} = 0$, $B_{(1,0)} = 1$, $B_{(0,1)} = i$, $B_{(1,1)} = 1+i$ and $i^2 = -1$, and

$$\begin{cases} c_{(n+1,m)} &= 6c_{(n,m)} - c_{(n-1,m)}, \\ c_{(n,m+1)} &= 6c_{(n,m)} - c_{(n,m-1)}, \end{cases}$$

with initial values $c_{(0,0)} = 1$, $c_{(1,0)} = 7$, $c_{(0,1)} = i$, $c_{(1,1)} = 7+i$, where $i^2 = -1$. The recurrence relation for the bidimensional balancing numbers is structured similarly to that of the bidimensional Lucas-cobalancing numbers but differs only in the initial conditions. For additional information on these bidimensional sequences, see [**3, 5, 4**], and other references.

In this paper, we introduce the tridimensional version of the Lucas-cobalancing numbers of integers and derives properties and identities that reveal its connection to its unidimensional counterpart. Our primary goal is to explore and analyze the tridimensional Lucas-cobalancing numbers, investigating their distinctive properties and sum identities to further our understanding of this intriguing numerical sequence.

Besides the Introduction, this work comprises two main sections: Section 2 explores tridimensional Lucas-balancing numbers, examining their properties and sum-related identities. Finally, Section 3 discusses the conclusions and future research directions.

2. Tridimensional Lucas-cobalancing numbers

This section introduces the tridimensional extension of Lucas-cobalancing numbers, exploring their fundamental properties and deriving various sum identities associated with them.

Definition 2.1. The numbers $c_{(n,m,p)}$, where n, m and p are any non-negative integers, represent the tridimensional Lucas-cobalancing number and satisfy the following recurrence relations:

$$\begin{cases} c_{(n+1,m,p)} &= 6c_{(n,m,p)} - c_{(n-1,m,p)}, \\ c_{(n,m+1,p)} &= 6c_{(n,m,p)} - c_{(n,m-1,p)}, \\ c_{(n,m,p+1)} &= 6c_{(n,m,p)} - c_{(n,m,p-1)}, \end{cases}$$

with the initial values $c_{(0,0,0)} = 1$, $c_{(1,0,0)} = 7$, $c_{(0,1,0)} = c_{(0,0,1)} = 1 + i$, $c_{(1,1,0)} = c_{(1,0,1)} = 7 + i$, $c_{(0,1,1)} = 1 + 2i$, $c_{(1,1,1)} = 7 + 2i$ and $i^2 = -1$.

Definition 2.1 holds true because $c_{(n_1,n_2,n_3)}$ is independent of the specific path chosen for its computation. To illustrate this, we will outline three different paths to derive $c_{(2,2,2)}$. Specifically, for calculating $c_{(2,2,2)}$, we observe the following:

$$c_{(2,2,2)} = \begin{cases} 6c_{(1,2,2)} - c_{(0,2,2)}, & \text{path } (a); \\ 6c_{(2,1,2)} - c_{(2,0,2)}, & \text{path } (b); \\ 6c_{(2,2,1)} - c_{(2,2,0)}, & \text{path } (c). \end{cases}$$

To begin, we perform and present the following calculations:

$$\begin{aligned} c_{(0,0,2)} &= 6c_{(0,0,1)} - c_{(0,0,0)} = 6(1+i) - 1 = 5 + 6i; \\ c_{(0,2,0)} &= 6c_{(0,1,0)} - c_{(0,0,0)} = 6(1+i) - 1 = 5 + 6i; \\ c_{(2,0,0)} &= 6c_{(1,0,0)} - c_{(0,0,0)} = 6 \times 7 - 1 = 41; \\ c_{(0,1,2)} &= 6c_{(0,1,1)} - c_{(0,1,0)} = 6(1+2i) - (1+i) = 5 + 11i; \\ c_{(1,0,2)} &= 6c_{(1,0,1)} - c_{(1,0,0)} = 6(7+i) - 7 = 35 + 6i; \\ c_{(0,2,1)} &= 6c_{(0,1,1)} - c_{(0,0,1)} = 6(1+2i) - (1+i) = 5 + 11i; \end{aligned}$$

$$c_{(1,2,0)} = 6c_{(1,1,0)} - c_{(1,0,0)} = 6(7+i) - 7 = 35 + 6i;$$
$$c_{(2,0,1)} = 6c_{(1,0,1)} - c_{(0,0,1)} = 6(7+i) - (1+i) = 41 + 5i;$$
$$c_{(2,1,0)} = 6c_{(1,1,0)} - c_{(0,1,0)} = 6(7+i) - (1+i) = 41 + 5i;$$
$$c_{(1,2,1)} = 6c_{(1,1,1)} - c_{(1,0,1)} = 6(7+2i) - (7+i) = 35 + 11i;$$
$$c_{(2,1,1)} = 6c_{(1,1,1)} - c_{(0,1,1)} = 6(7+2i) - (1+2i) = 41 + 10i.$$

Considering the path (a), we have

$$\begin{aligned} c_{(1,2,2)} &= 6c_{(1,2,1)} - c_{(1,2,0)} \\ &= 6(35+11i) - (35+6i) \\ &= 175 + 60i \end{aligned}$$

and

$$\begin{aligned} c_{(0,2,2)} &= 6c_{(0,2,1)} - c_{(0,2,0)} \\ &= 6(5+11i) - (5+6i) = 25 + 60i. \end{aligned}$$

Hence

$$\begin{aligned} c_{(2,2,2)} &= 6c_{(1,2,2)} - c_{(0,2,2)} \\ &= 6(175+60i) - (25+60i) \\ &= 1025 + 300i. \end{aligned}$$

Considering the path (b), we obtain

$$\begin{aligned} c_{(2,1,2)} &= 6c_{(1,1,2)} - c_{(0,1,2)} \\ &= 6(35+11i) - (5+11i) \\ &= 205 + 55i \end{aligned}$$

and

$$\begin{aligned} c_{(2,0,2)} &= 6c_{(1,0,2)} - c_{(0,0,2)} \\ &= 6(35+6i) - (5+6i) \\ &= 205 + 30i. \end{aligned}$$

So,

$$\begin{aligned} c_{(2,2,2)} &= 6c_{(2,1,2)} - c_{(2,0,2)} \\ &= 6(205+55i) - (205+30i) \\ &= 1025 + 300i. \end{aligned}$$

Considering the path (c), we have the following:

$$\begin{aligned} c_{(2,2,1)} &= 6c_{(2,1,1)} - c_{(2,0,1)} \\ &= 6(41+10i) - (41+5i) = 205 + 55i \end{aligned}$$

and

$$\begin{aligned} c_{(2,2,0)} &= 6c_{(2,1,0)} - c_{(2,0,0)} \\ &= 6(41+5i) - 41 = 205 + 30i. \end{aligned}$$

Thus,

$$\begin{aligned} c_{(2,2,2)} &= 6c_{(2,2,1)} - c_{(2,2,0)} \\ &= 6(205+55i) - (205+30i) = 1025+300i, \end{aligned}$$

which coincides with the value found for $c_{(2,2,2)}$ where we used the path (a) or path (b).

2.1. Some properties. In this subsection, we explore several properties of the tridimensional Lucas-cobalancing numbers, building upon the foundational concepts established by the unidimensional and bidimensional versions of balancing numbers.

LEMMA 2.1. *Assume that c_j and B_j are Lucas-cobalancing and balancing numbers of order j, respectively. Then, the following properties are true for tridimensional Lucas-cobalancing numbers:*

(1) $c_{(n,0,0)} = c_{n+1}$;
(2) $c_{(0,m,0)} = c_{(0,0,m)} = (B_m - B_{m-1}) + B_m i$;
(3) $c_{(n,1,0)} = c_{(n,0,1)} = c_{n+1} + (B_n - B_{n-1})\, i$;
(4) $c_{(n,1,1)} = c_{n+1} + 2\,(B_n - B_{n-1})\, i$;
(5) $c_{(1,m,0)} = c_{(1,0,m)} = 7\,(B_m - B_{m-1}) + B_m i$;
(6) $c_{(0,m,1)} = c_{(0,1,m)} = (B_m - B_{m-1}) + (2B_m - B_{m-1})\, i$;
(7) $c_{(1,m,1)} = c_{(1,1,m)} = 7\,(B_m - B_{m-1}) + (2B_m - B_{m-1})\, i$;
(8) $c_{(n,m,0)} = c_{(n,0,m)} = c_{n+1}\,(B_m - B_{m-1}) + (B_n - B_{n-1})\, B_m i$;
(9) $c_{(0,m,p)} = (B_m - B_{m-1})\,(B_p - B_{p-1}) + \theta$;
(10) $c_{(n,m,1)} = c_{(n,1,m)} = c_{n+1}\,(B_m - B_{m-1}) + (B_n - B_{n-1})\,(2B_m - B_{m-1})\, i$;
(11) $c_{(1,m,p)} = 7\,(B_m - B_{m-1})\,(B_p - B_{p-1}) + \theta$,

where $\theta = \Big(B_m\,(B_p - B_{p-1}) + (B_m - B_{m-1})\, B_p\Big) i$.

PROOF. 1. The proof is carried out by induction on n. For $n = 0$ and taking into account the values of c_1 in (1.2) and $c_{(0,0,0)}$ in Definition 2.1 we have $c_{(0,0,0)} = 1 = c_1$ and the Lemma 2.1 is valid.

For $n = 1$ and given that $c_2 = 7$ in (1.2) and considering the value of $c_{(1,0,0)}$ in Definition 2.1 we have $c_{(1,0,0)} = 7 = c_2$ and the Lemma 2.1 is also valid.

Suppose the proposition is true for any integer $k \leqslant n$, and we will show that it is still valid for $n + 1$.

Thus, applying the first recurrence relation from Definition 2.1, the induction hypothesis and (1.2), we obtain

$$\begin{aligned} c_{(n+1,0,0)} &= 6c_{(n,0,0)} - c_{(n-1,0,0)} \\ &= 6c_{n+1} - c_n \\ &= c_{n+2}. \end{aligned}$$

Hence, the proof of item 1. is complete.

2. The proof is then repeated by induction, but now with m. Firstly, we will prove that $c_{(0,m,0)} = c_{(0,0,m)}$.

For $m = 0$, we have $c_{(0,0,0)} = 1$ and the equality is valid for the first initial value of Definition 2.1.

For $m = 1$, we have $c_{(0,1,0)} = 1 + i = c_{(0,0,1)}$, and the equality is also valid by the third initial condition of Definition 2.1.

Suppose the statement is valid for any integer $k \leqslant m$, and let us show that this continues to be valid for $m + 1$.

Hence, according to the second and third recurrence relations of Definition 2.1 and by induction hypothesis, we obtain

$$\begin{aligned} c_{(0,m+1,0)} &= 6c_{(0,m,0)} - c_{(0,m-1,0)} \\ &= 6c_{(0,0,m)} - c_{(0,0,m-1)} \\ &= c_{(0,0,m+1)}, \end{aligned}$$

what proves that $c_{(0,m,0)} = c_{(0,0,m)}$.

Now, let us prove that $c_{(0,0,m)} = B_{(1,m)}$.

For $m = 0$, we have $c_{(0,0,0)} = 1 = B_{(1,0)}$ and the equality is true by the first initial value of Definition 2.1 and by the second initial condition of (1.3).

For $m = 1$, we have $c_{(0,0,1)} = 1 + i = B_{(1,1)}$ and the equality is also true by the second initial condition of Definition 2.1 and by the third initial value of (1.3).

Suppose the statement is true for any integer less than or equal to m, and we will prove that it continues to be true for $m + 1$.

Thus, applying the third recurrence relation of Definition 2.1, the induction hypothesis and by second recurrence relation of (1.3),

$$\begin{aligned} c_{(0,0,m+1)} &= 6c_{(0,0,m)} - c_{(0,0,m-1)} \\ &= 6B_{(1,m)} - B_{(1,m-1)} \\ &= B_{(1,m+1)}, \end{aligned}$$

which holds.

Now, according Lemma 3.2.4 in [**3**], $B_{(1,m)} = (B_m - B_{m-1}) + B_m i$. Therefore, by the transitivity relation, the item 2. is proved.

3 The proof is also done by induction on n. First, we will prove that $c_{(n,1,0)} = c_{(n,0,1)}$.

For $n = 0$, we have $c_{(0,1,0)} = 1 + i = c_{(0,0,1)}$ and the equality holds again by the third initial condition of Definition 2.1.

For $n = 1$, we have $c_{(1,1,0)} = 7 + i = c_{(1,0,1)}$, and the equality is also true by the fourth initial value of Definition 2.1.

Assume that $c_{(k,1,0)} = c_{(k,0,1)}$ for all integers $k \leqslant n$. Let us now show that it is still true for $n + 1$.

Therefore, using the first recurrence relation from Definition 2.1 and by induction hypothesis, we obtain

$$\begin{aligned} c_{(n+1,1,0)} &= 6c_{(n,1,0)} - c_{(n-1,1,0)} \\ &= 6c_{(n,0,1)} - c_{(n-1,0,1)} \\ &= c_{(n+1,0,1)}, \end{aligned}$$

and the expression $c_{(n,1,0)} = c_{(n,0,1)}$ becomes true.

We will now prove that $c_{(n,0,1)} = c_{n+1} + (B_n - B_{n-1})\, i$.

For $n = 0$, we have $c_{(0,0,1)} = 1 + i = c_1 + (B_0 - B_{-1})\, i$, which is verified taking into account the third initial value of Definition 2.1, the first initial condition of (1.1) and (1.2) and by Proposition 2.1 in [**3**] combined with the second initial value also of (1.1).

For $n = 1$, we have $c_{(1,0,1)} = 7 + i = c_2 + (B_1 - B_0)\, i$, which is also true given the fourth initial condition of Definition 2.1, the first and second initial values of (1.1) and the second initial condition of (1.2).

Suppose that $c_{(k,0,1)} = c_{k+1} + (B_k - B_{k-1})\, i$ is true for all integers $k \leqslant n$. Let us show that it is continuously true for $n + 1$.

Hence, using the first recurrence relation of Definition 2.1, the induction hypothesis and by (1.1) and (1.2), we get

$$\begin{aligned} c_{(n+1,0,1)} &= 6c_{(n,0,1)} - c_{(n-1,0,1)} \\ &= 6\left(c_{n+1} + (B_n - B_{n-1})\, i\right) - \left(c_n + (B_{n-1} - B_{n-2})\, i\right) \\ &= 6c_{n+1} + 6\,(B_n - B_{n-1})\, i - c_n - (B_{n-1} - B_{n-2})\, i \\ &= (6c_{n+1} - c_n) + (6B_n - B_{n-1})\, i - (6B_{n-1} - B_{n-2})\, i \\ &= c_{n+2} + (B_{n+1} - B_n)\, i, \end{aligned}$$

which is true.

Thus, using the transitivity relation, the item 3 is proven.

4. The proof is also carried out by induction on n.

For $n = 0$ we have that $c_{(0,1,1)} = 1 + 2i = c_1 + 2\,(B_0 - B_{-1})\, i$, which is verified according to the value of $c_{(0,1,1)}$ in Definition 2.1, the value of $c_{(1)}$ in (1.2) and the values of $B_{(0)}$ in (1.1) and the value of $B_{(-1)} = -B_{(1)}$ also in (1.1) conjugated with Proposition 2.1 in [**3**].

For $n = 1$, we have $c_{(1,1,1)} = 7 + 2i = c_2 + 2\,(B_1 - B_0)\, i$, which is also true given the values of $c_{(1,1,1)}$, $c_{(2)}$ and $B_{(0)}$ and $B_{(1)}$, in Definition 1, in the recurrence relations (2) and (1), respectively.

Let us assume that the statement is true for all values less than or equal to n and let us prove that it continues to be true for $n + 1$.

Thus, by the first recurrence relation of Definition 2.1, the induction hypothesis and by (1.1) and (1.2), we obtain

$$\begin{aligned} c_{(n+1,1,1)} &= 6c_{(n,1,1)} - c_{(n-1,1,1)} \\ &= 6\left(c_{n+1} + 2\,(B_n - B_{n-1})\, i\right) - \left(c_n + 2\,(B_{n-1} - B_{n-2})\, i\right) \\ &= 6c_{n+1} + 12\,(B_n - B_{n-1})\, i - c_n - 2\,(B_{n-1} - B_{n-2})\, i \\ &= (6c_{n+1} - c_n) + 2\left((6B_n - B_{n-1})\, i - (6B_{n-1} - B_{n-2})\right) i \\ &= c_{n+2} + 2\,(B_{n+1} - B_n)\, i, \end{aligned}$$

which still holds.

Therefore, by the transitivity relation, the item 4. is verified.

We do not present the proof for items 5., 6. and 7. because they are done in a similar way to the previous ones (items 2. and 3.), so we have omitted them, respectively.

8. The proof is first performed by induction on m. Firstly, we will prove that $c_{(n,m,0)} = c_{(n,0,m)}$.

For $m = 0$. we have $c_{(n,0,0)} = c_{n+1}$, and item by 1. this is true.

For $m = 1$, we have $c_{(n,1,0)} = c_{n+1} + (B_n - B_{n-1})\, i = c_{(n,0,1)}$, and by item 2. this is also true.

Let us assume that $c_{(n,k,0)} = c_{n,0,k)}$ for all integers $k \leqslant m$. We want to show that this remains true for $m + 1$.

Thus, applying the second and third recurrence relations of Definition 2.1 and the induction hypothesis, we obtain

$$\begin{aligned} c_{(n,m+1,0)} &= 6c_{(n,m,0)} - c_{(n,m-1,0)} \\ &= 6c_{(n,0,m)} - c_{(n,0,m-1)} \\ &= c_{(n,0,m+1)}, \end{aligned}$$

and the equality $c_{(n,m+1,0)} = c_{(n,0,m+1)}$ is true.

Therefore, item 8. is proven by the law of transitivity.

Now, let us carry out induction on n. We will prove that $c_{(n,m,0)} = c_{(n,0,m)}$.

For $n = 0$, we have $c_{(0,m,0)} = B_{(1,m)} = c_{(0,0,m)}$, and considering item 2., this is true.

For $n = 1$, we have $c_{(1,m,0)} = 7\,(B_n - B_{n-1}) + B_m i = c_{(1,0,m)}$, and considering item 5., this is also true.

Suppose that $c_{(k,m,0)} = c_{(k,0,m)}$ is true for any $k \leqslant n$. Let us prove that it continues true for $n + 1$.

Then, by the first recurrence of Definition 2.1 and by induction hypothesis, we have

$$\begin{aligned} c_{(n+1,m,0)} &= 6c_{(n,m,0)} - c_{(n-1,m,0)} \\ &= 6c_{(n,0,m)} - c_{(n-1,0,m)} \\ &= c_{(n+1,0,m)}, \end{aligned}$$

and equality is also true.

Therefore, item 8. is also proven by the law of transitivity.

We have omitted the proof of items 9., 10. and 11. because they are done in the same way as the previous result. □

The next, and most important, result of this work establishes a connection between the tridimensional Lucas-cobalancing and the unidimensional balancing numbers.

Theorem 2.1. *Given that m, n and p are non-negative integers, the tridimensional Lucas-cobalancing numbers can be expressed in the following form:*

$$\begin{aligned} c_{(n,m,p)} &= c_{n+1}\,(B_m - B_{m-1})\,(B_p - B_{p-1}) \\ &\quad + (B_n - B_{n-1})\,(B_m\,(B_p - B_{p-1}) + (B_m - B_{m-1})\,B_p)\,i\,, \end{aligned}$$

where $\{B_n\}_{n\geqslant 0}$ is the the balancing sequence, and $\{c_n\}_{n\geqslant 1}$ is Lucas-cobalancing sequence.

Proof. We will start by doing the induction on p.

For $p=0$ and, considering the value of B_0 in (1.1) and the value of $B_{-1}=-B_1$ also in (1.1) conjugated with Proposition 2.1 in [**3**], we have

$$\begin{aligned} c_{(n,m,0)} &= c_{n+1}\left(B_m-B_{m-1}\right)\left(B_0-B_{-1}\right) \\ &\quad +\left(B_n-B_{n-1}\right)\left(B_m\left(B_0-B_{-1}\right)+\left(B_m-B_{m-1}\right)B_0\right)i \\ &= c_{n+1}\left(B_m-B_{m-1}\right)+\left(B_n-B_{n-1}\right)B_m i, \end{aligned}$$

which is true by item 8. of Lemma 2.1.

For $p=1$ and, once again, taking into account the values of $B_0=0$ and $B_1=1$ in (1.1), we have

$$\begin{aligned} c_{(n,m,1)} &= c_{n+1}\left(B_m-B_{m-1}\right)\left(B_1-B_0\right) \\ &\quad +\left(B_n-B_{n-1}\right)\left(B_m\left(B_1-B_0\right)+\left(B_m-B_{m-1}\right)B_1\right)i \\ &= c_{n+1}\left(B_m-B_{m-1}\right)+\left(B_n-B_{n-1}\right)\left(2B_m-B_{m-1}\right)i, \end{aligned}$$

which is also true by item 10. of Lemma 2.1.

Suppose the theorem is true for any non-negative integer less than or equal to p. We will show that it continues to be true for $p+1$.

Hence, using the third recurrence relation of Definition 2.1, the induction hypothesis and by (1.1), we obtain

$$\begin{aligned} c_{(n,m,p+1)} &= 6c_{(n,m,p)}-c_{(n,m,p-1)} \\ &= 6c_{n+1}\left(B_m-B_{m-1}\right)\left(B_p-B_{p-1}\right) \\ &\quad +6\left(B_n-B_{n-1}\right)\left(B_m\left(B_p-B_{p-1}\right)+\left(B_m-B_{m-1}\right)B_p\right)i \\ &\quad -c_{n+1}\left(B_m-B_{m-1}\right)\left(B_{p-1}-B_{p-2}\right) \\ &\quad -\left(B_n-B_{n-1}\right)\left(B_m\left(B_{p-1}-B_{p-2}\right)+\left(B_m-B_{m-1}\right)B_{p-1}\right)i \\ &= c_{n+1}\left(B_m-B_{m-1}\right)\left(\left(6B_p-B_{p-1}\right)-\left(6B_{p-1}-B_{p-2}\right)\right) \\ &\quad +\left(B_n-B_{n-1}\right)\Big(B_m\left(\left(6B_p-B_{p-1}\right)-\left(6B_{p-1}-B_{p-2}\right)\right) \\ &\quad +\left(B_m-B_{m-1}\right)\left(6B_p-B_{p-1}\right)\Big)i \\ &= c_{n+1}\left(B_m-B_{m-1}\right)\left(B_{p+1}-B_p\right)+ \\ &\quad \left(B_n-B_{n-1}\right)\left(B_m\left(B_{p+1}-B_p\right)+\left(B_m-B_{m-1}\right)B_{p+1}\right)i, \end{aligned}$$

what we wanted to prove.

Using the same reasoning as before (for induction on p), the result for induction on m and n is also true in the case where n and p, and m and p, respectively, are fixed.

Consequently, the theorem is valid. □

2.2. Some sum identities. This subsection explores several sum identities involving Lucas-cobalancing numbers, which are defined in terms of balancing numbers.

To begin, the partial sum of the third component in the tridimensional Lucas-cobalancing numbers is expressed as follows:

Proposition 2.1. *The sum of the first p numbers $c_{(n,m,l)}$, with index l a non-negative integer, is described as follows:*

$$\sum_{l=1}^{p} c_{(n,m,l)} = (\beta c_{n+1} + \alpha B_m i) B_p + \frac{1}{4}\alpha\beta (B_{p+1} - B_p - 1) i,$$

where $\alpha = B_n - B_{n-1}$ and $\beta = B_m - B_{m-1}$.

Proof. By applying Theorem 2.1, we have

$$\begin{aligned}\sum_{l=1}^{p} c_{(n,m,l)} &= \sum_{l=1}^{p} c_{n+1} (B_m - B_{m-1}) (B_l - B_{l-1}) \\ &\quad + \sum_{l=1}^{p} (B_n - B_{n-1}) (B_m (B_l - B_{l-1}) + (B_m - B_{m-1}) B_l) i.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{l=1}^{p} c_{(n,m,l)} &= c_{n+1} (B_m - B_{m-1}) \sum_{l=1}^{p} (B_l - B_{l-1}) \\ &\quad + (B_n - B_{n-1}) B_m \sum_{l=1}^{p} (B_l - B_{l-1}) i \\ &\quad + (B_n - B_{n-1}) (B_m - B_{m-1}) \sum_{l=1}^{p} B_l i \\ &= c_{n+1} (B_m - B_{m-1}) \left(\sum_{l=1}^{p} B_l - \sum_{l=1}^{p} B_{l-1}\right) \\ &\quad + (B_n - B_{n-1}) B_m \left(\sum_{l=1}^{p} B_l - \sum_{l=1}^{p} B_{l-1}\right) i \\ &\quad + (B_n - B_{n-1}) (B_m - B_{m-1}) \sum_{l=1}^{p} B_l i \\ &= (c_{n+1} (B_m - B_{m-1}) + (B_n - B_{n-1}) B_m i) \left(\sum_{l=1}^{p} B_l - \sum_{l=0}^{p-1} B_l\right) \\ &\quad + (B_n - B_{n-1}) (B_m - B_{m-1}) \sum_{l=0}^{p} B_l i,\end{aligned}$$

and by using item 6 of Proposition 2.6 in [**2**] and taking into account the value of B_0 in (1.1), the result follows. □

The partial sum of tridimensional Lucas-cobalancing numbers, considering only those with odd indices in the third component, can be expressed as:

PROPOSITION 2.2. *The sum of the first p numbers $c_{(n,m,l)}$ of odd index l is obtained as follows:*

$$\sum_{l=1}^{p} c_{(n,m,2l-1)} = (\beta c_{n+1} + \alpha B_m i)\left(B_p^2 + B_{2p} - B_p B_{p+1}\right) + \alpha\beta B_p^2 i,$$

where $\alpha = B_n - B_{n-1}$ and $\beta = B_m - B_{m-1}$.

PROOF. By Theorem 2.1, we have

$$\begin{aligned}\sum_{l=1}^{p} c_{(n,m,2l-1)} &= \sum_{l=1}^{p}\Big(c_{n+1}\left(B_m - B_{m-1}\right)\left(B_{2l-1} - B_{2l-2}\right)\\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m\left(B_{2l-1} - B_{2l-2}\right) + \left(B_m - B_{m-1}\right)B_{2l-1}\right)i\Big).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{l=1}^{p} c_{(n,m,2l-1)} &= c_{n+1}\left(B_m - B_{m-1}\right)\sum_{l=1}^{p}\left(B_{2l-1} - B_{2l-2}\right)\\ &\quad + \left(B_n - B_{n-1}\right)B_m\sum_{l=1}^{p}\left(B_{2l-1} - B_{2l-2}\right)i\\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{l=1}^{p} B_{2l-1} i\\ &= c_{n+1}\left(B_m - B_{m-1}\right)\left(\sum_{l=1}^{p} B_{2l-1} - \sum_{l=1}^{p} B_{2l-2}\right)\\ &\quad + \left(B_n - B_{n-1}\right)B_m\left(\sum_{l=1}^{p} B_{2l-1} - \sum_{l=1}^{p} B_{2l-2}\right)i\\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{l=1}^{p} B_{2l-1} i\\ &= \left(c_{n+1}\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right)B_m i\right)\Big(\sum_{l=1}^{p} B_{2l-1}\\ &\quad - \sum_{l=1}^{p} B_{2l-2}\Big) + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{l=1}^{p} B_{2l-1} i,\end{aligned}$$

and by items (a) and (b) of Corollary 2.3.6 in [**14**] and taking into account item (a) of Corollary 2.3.5 also in [**14**], the result follows. □

The partial sum of tridimensional Lucas-cobalancing numbers with even indices in the third component is given by the following expression:

Proposition 2.3. *The sum of the first p numbers $c_{(n,m,l)}$ of even index l is given by:*

$$\sum_{l=1}^{p} c_{(n,m,2l)} = (\beta c_{n+1} + \alpha B_m i)\left(B_p B_{p+1} - B_p^2\right) + \alpha\beta B_p B_{p+1} i,$$

where $\alpha = B_n - B_{n-1}$ and $\beta = B_m - B_{m-1}$.

Proof. Using Theorem 2.1, we have

$$\begin{aligned}\sum_{l=1}^{p} c_{(n,m,2l)} &= \sum_{l=1}^{p} c_{n+1}\left(B_m - B_{m-1}\right)\left(B_{2l} - B_{2l-1}\right) \\ &\quad + \sum_{l=1}^{p}\left(B_n - B_{n-1}\right)\left(B_m\left(B_{2l} - B_{2l-1}\right) + \left(B_m - B_{m-1}\right)B_{2l}\right)i.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{l=1}^{p} c_{(n,m,2l)} &= c_{n+1}\left(B_m - B_{m-1}\right)\sum_{l=1}^{p}\left(B_{2l} - B_{2l-1}\right) \\ &\quad + \left(B_n - B_{n-1}\right)B_m\sum_{l=1}^{p}\left(B_{2l} - B_{2l-1}\right)i \\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{l=1}^{p} B_{2l} i \\ &= c_{n+1}\left(B_m - B_{m-1}\right)\left(\sum_{l=1}^{p} B_{2l} - \sum_{l=1}^{p} B_{2l-1}\right) + \\ &\quad + \left(B_n - B_{n-1}\right)B_m\left(\sum_{l=1}^{p} B_{2l} - \sum_{l=1}^{p} B_{2l-1}\right)i \\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{l=1}^{p} B_{2l} i \\ &= \left(c_{n+1}\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right)B_m i\right)\left(\sum_{l=1}^{p} B_{2l} - \sum_{l=1}^{p} B_{2l-1}\right) \\ &\quad + \left(B_n - B_{n-1}\right)\left(B_m - B_{m-1}\right)\sum_{l=1}^{p} B_{2l} i,\end{aligned}$$

and the result follows, by using items (a) and (b) of Corollary 2.3.6 in [**14**]. □

The next result follows directly from Propositions 2.2 and 2.3.

Proposition 2.4. *The alternate sum of the first r numbers $c_{(n,m,r)}$ of even index $r = 2p$, with $p > 1$, is given by:*

$$\sum_{l=1}^{r}(-1)^r c_{(n,m,l)} = (\beta c_{n+1} + \alpha B_m i)\left(2B_p B_{p+1} - 2B_p^2 - B_{2p}\right) + \alpha\beta B_p\left(B_{p+1} - B_p\right)i\,,$$

where $\alpha = B_n - B_{n-1}$ *and* $\beta = B_m - B_{m-1}$.

PROOF. Since $r = 2p$ and $p > 1$. So, combining Propositions 2.2 and 2.3, we have

$$\begin{aligned}\sum_{l=1}^{r}(-1)^r c_{(n,m,l)} &= -\sum_{l=1}^{p} c_{(n,m,2l-1)} + \sum_{l=1}^{p} c_{(n,m,2l)} \\ &= -\Big((\beta c_{n+1} + \alpha B_m i)\left(B_p^2 + B_{2p} - B_p B_{p+1}\right) + \alpha\beta B_p^2 i\Big) \\ &\quad +\Big((\beta c_{n+1} + \alpha B_m i)\left(B_p B_{p+1} - B_p^2\right) + \alpha\beta B_p B_{p+1} i\Big) \\ &= (\beta c_{n+1} + \alpha B_m i)\left(2B_p B_{p+1} - 2B_p^2 - B_{2p}\right) + \alpha\beta B_p\left(B_{p+1} - B_p\right) i\,,\end{aligned}$$

and the result follows. □

The following results pertain to Propositions 2.6 through 2.9, which are closely connected to the tridimensional versions of Lucas-cobalancing numbers. The proofs of Propositions 2.6 through 2.5 were carried out using induction on m, while the proofs of Propositions 2.10 through 2.9 were established by induction on n. Specifically, in the latter three propositions, key elements from Proposition 1.1 in [**5**] were utilized: items 1. and 2. were applied in Proposition 2.10 and Proposition 2.11, and item 1. was used in Proposition 2.9.

Since the proofs are analogous to earlier cases, they have been omitted for brevity.

Now, the partial sum of the second component in the tridimensional Lucas-cobalancing numbers is expressed as:

PROPOSITION 2.5. *The sum of the first* m *numbers* $c_{(n,t,p)}$, *with index* t *a non-negative integer, is as follows:*

$$\sum_{t=1}^{m} c_{(n,t,p)} = (\gamma c_{n+1} + \alpha B_p i)\, B_m + \frac{1}{4}\alpha\gamma\left(B_{m+1} - B_m - 1\right) i,$$

where $\alpha = B_n - B_{n-1}$ *and* $\gamma = B_p - B_{p-1}$.

The partial sum of the second component of three-dimensional Lucas-balancing numbers with odd indices is given as follows:

PROPOSITION 2.6. *The sum of the first* m *numbers* $c_{(n,t,p)}$ *of odd index* t *is described as follows:*

$$\sum_{t=1}^{m} c_{(n,2t-1,p)} = (\gamma c_{n+1} + \alpha B_p i)\left(B_m^2 - B_m B_{m+1} - B_{2m}\right) + \alpha\gamma B_m^2 i,$$

where $\alpha = B_n - B_{n-1}$ *and* $\gamma = B_p - B_{p-1}$.

The partial sum of the second component of tridimensional Lucas-cobalancing numbers with even indices is given by:

Proposition 2.7. *The sum of the first m numbers $c_{(n,t,p)}$ of even index t is defined as follows:*

$$\sum_{t=1}^{m} c_{(n,2t,p)} = (\gamma c_{n+1} + \alpha B_p i)\left(B_m B_{m+1} - B_m^2\right) + \alpha\gamma B_m B_{m+1} i,$$

where $\alpha = B_n - B_{n-1}$ and $\gamma = B_p - B_{p-1}$.

The partial and alternate sum of tridimensional Lucas-cobalancing numbers in the second component is present in the next result, and is a direct consequence of the Propositions 2.6 and 2.7.

Proposition 2.8. *The alternate sum of the first r numbers $c_{(n,r,p)}$ of even index $r = 2m$, with $m > 1$, is given by:*

$$\sum_{l=1}^{r} (-1)^r c_{(n,l,p)} = (\gamma c_{n+1} + \alpha B_p i)\left(2B_m B_{m+1} - 2B_m^2 - B_{2m}\right) + \alpha\gamma B_m \left(B_{m+1} - B_m\right) i\,,$$

where $\alpha = B_n - B_{n-1}$ and $\gamma = B_p - B_{p-1}$.

Finally, the partial sum of the first component in the tridimensional Lucas-cobalancing numbers can now be written as:

Proposition 2.9. *The sum of the first n numbers $c_{(r,m,p)}$, where r is a non-negative integer, is defined as follows:*

$$\sum_{r=1}^{n} c_{(r,m,p)} = \frac{1}{2}\beta\gamma\left(c_{n+1} + 3B_n - B_{n-1} - 3\right) + B_n\left(\gamma B_m + \beta B_p\right) i,$$

where $\beta = B_m - B_{m-1}$ and $\gamma = B_p - B_{p-1}$.

The partial sum of the first component of tridimensional Lucas-cobalancing numbers for odd indices is given by:

Proposition 2.10. *The sum of the first n numbers $c_{(r,m,p)}$ of odd index r is given by the following:*

$$\sum_{r=1}^{n} c_{(2r-1,m,p)} = \beta\gamma\left(B_n^2 + B_n B_{n+1}\right) + \left(B_n^2 - B_n B_{n+1} + B_{2n}\right)\left(\gamma B_m + \beta B_p\right) i,$$

where $\beta = B_m - B_{m-1}$ and $\gamma = B_p - B_{p-1}$.

The partial sum of the first component of the tridimensional Lucas-cobalancing numbers, for even indices, is expressed as follows:

Proposition 2.11. *The sum of the first n numbers $c_{(r,m,p)}$ of even index r is described as follows:*

$$\sum_{r=1}^{n} c_{(2r,m,p)} = \frac{1}{2}\beta\gamma\left(B_{n+1}^2 + B_{n+1}B_{n+2}\right) + \left(B_n B_{n+1} + B_n^2\right)\left(\gamma B_m + \beta B_p\right) i,$$

where $\beta = B_m - B_{m-1}$ and $\gamma = B_p - B_{p-1}$.

The partial and alternating sums of the tridimensional Lucas-cobalancing numbers in the first component are presented in the following result, which directly follows from Propositions 2.10 and 2.11.

PROPOSITION 2.12. *The alternate sum of the first r numbers $c_{(n,r,p)}$ of even index $r = 2n$, with $n > 1$, is given by:*

$$\sum_{l=1}^{r}(-1)^r c_{(l,m,p)} = \frac{1}{2}\beta\gamma\left(B_{n+1}^2 - 2B_n^2 + B_{n+1}(B_{n+2} - 2B_n)\right) - B_{2n}\left(\gamma B_m + \beta B_p\right) i\,,$$

where $\beta = B_m - B_{m-1}$ and $\gamma = B_p - B_{p-1}$.

3. Conclusion

In this study, we introduced the tridimensional version of Lucas-cobalancing numbers, exploring their recurrence relations, fundamental properties, and specific sum identities. This work not only extends the concept of Lucas-cobalancing numbers into three dimensions but also lays the groundwork for further investigations. A compelling question emerges from this exploration: could these concepts be expanded to a fourth dimension or beyond the realm of integers? For instance, how might Lucas-balancing and Lucas-cobalancing numbers manifest when defined using Gaussian coefficients? Such inquiries open intriguing possibilities for future research into multidimensional and complex-number extensions of these numerical sequences.

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