A generalization of an extensible beam equation with critical growth in $\mathbb{R}^N$

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Abstract

In this work we obtain an existence result for a generalized extensible beam equation with critical growth in $\mathbb{R}^N$ of the type

$$\Delta^2 u - M\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)\Delta u + u = \lambda f(u) + |u|^{2^{**}-2}u \text{ in } \mathbb{R}^N,$$

where $N \geq 5$ and $\lambda > 0$. The functions $M : [0, +\infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous. Since there is a competition between the function M and the critical exponent given by $2^{**} = \frac{2N}{N-4}$, we need to make a truncation on function M. Using the size of λ, we show that each solution of auxiliary problem is a solution of original problem. Our approach is variational and uses minimax point critical theorems.

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1 Introduction

In this work we deal with questions of existence of solutions for a generalized extensible beam equation of the type

$$\begin{cases} \Delta^2 u - M\left(\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)\Delta u + u = \lambda f(u) + |u|^{2^{**}-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \qquad (P_\lambda)$$

where λ is a real positive parameter and $M : [0,+\infty) \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy conditions which will be stated later. Here $2^{**} = \frac{2N}{N-4}$ with $N \geq 5$ and Δ^2 is the biharmonic operator, that is,

$$\Delta^2 u = \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} u + \sum_{i \neq j}^{N} \frac{\partial^4}{\partial x_i^2 x_j^2} u.$$

Before stating our main result, we need the following hypotheses on the function $M : \mathbb{R}^+ \to \mathbb{R}^+$:

(M_1) The function M is increasing.

There exists $0 < m_0$ such that

(M_2) $$M(t) \geq m_0 = M(0), \text{ for all } t \in \mathbb{R}^+.$$

(M_3) The map $t \in (0,\infty) \mapsto \dfrac{M(t)}{t}$ is nonincreasing.

A typical example of a function satisfying the conditions $(M_1)-(M_3)$ is given by

$$M(t) = m_0 + bt$$

with $b \geq 0$ and for all $t \geq 0$, which is the one considered in the equation

$$(WK) \qquad \frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L |\frac{\partial u}{\partial x}|^2\, dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

that was studied in 1950, by Woinowsky-Krieger [19] in a bounded domain.

The parameters in equation (WK) have the following meanings: L is the length of the beam in the rest position, E is the Young modulus of the material, I is the cross-sectional moment of inertia, ρ is the mass density, H is the tension in the rest position and A is the cross-sectional area. This model was proposed to modify the theory of the dynamic Euler-Bernoulli beam, assuming a nonlinear dependence of the axial strain on the deformation of the gradient.

However, our hypotheses about the function M include other functions, such as $M(t) = 1 + bt + \sum_{i=1}^{k} b_i t^{d_i}$ with $b_i \geq 0$ and $d_i \in (0,1)$ for all $i \in \{1, 2, \ldots, k\}$.

We assume the following growth conditions on function f at the origin and at infinity:

$$(f_1) \qquad \lim_{|t|\to 0} \frac{|f(t)|}{|t|} = 0$$

and there exists $q \in (2, 2^{**})$ verifying;

$$(f_2) \qquad \lim_{|t|\to \infty} \frac{|f(t)|}{|t|^{q-1}} = 0.$$

In this article, we use the Ambrosetti-Rabinowitz superlinear condition. There is $\theta \in (2, 2^{**})$ such that

$$(f_3). \qquad 0 < \theta F(t) = \theta \int_0^t f(\xi) d\xi \leq t f(t) \text{ for all } |t| \neq 0,$$

$$(f_4) \qquad \text{The map } t \in \mathbb{R} \mapsto \frac{f(t)}{t} \text{ is increasing for } t \neq 0.$$

The main result is given by

Theorem 1.1. *If $(M_1) - (M_3)$ and $(f_1) - (f_4)$ hold, then there is a positive constant λ^* such that problem (P_λ) has a nontrivial solution, for all $\lambda \in (\lambda^*, +\infty)$. Moreover, if u_λ is a solution of problem (P_λ), then $u_\lambda \in C^{4,\alpha}_{loc}(\mathbb{R}^N)$ and*

$$\lim_{\lambda\to+\infty} \|u_\lambda\| = 0.$$

Owing to its importance in engineering, physics and material mechanics, since that the model for the beam equation was proposed, this class of problems there have been many researches focused on the properties of its solutions, as can be seen in [3], [4], [6] and [15] and references therein. More recent references with important details about the physical motivation of the (WK) problem can be seen in [2], [8], [11], [12], [13], [14] and [16].

Only recently the generalized version of the beam equation began to be studied in bounded domain. The case $N \geq 5$ in bounded domain was studied by [17] and [18]. The existence of few results for the case $N \geq 5$ is, possibly, because there is a competition between the operator M and the nonlinearity in the case bounded domain. In [18] the author overcome this difficult making M uniformly bounded. The difficult caused by growth of function M was overcome in [18] making $M(t) = m_0 + bt$ with m_0 and b small.

In the $\mathbb{R}^N$ case, besides the previously mentioned difficulties, there is the lack of compactness. When $M = 1$, this difficult is overcome showing that the weak limit of a Palais-Smale sequence (u_n) is a weak solution of our problem. When M is a general function, the weak limit is a weak solution of the problem

$$\begin{cases} \Delta^2 u - \alpha\Delta u + u = \lambda f(u) + |u|^{2^{**}-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\alpha = \lim_{n\to\infty} M\left(\int_{\mathbb{R}^N} \nabla u_n|^2\, dx\right)$. To overcome these difficulties we adapt some arguments that can be found in [1] and [7].

This work is, possibly, the first that treat on the generalized version of the beam equation in $\mathbb{R}^N$.

The plan of this paper is as follows. In section 2 it is presented an auxiliary truncated problem, for which the existence of solution is deduced by means of a variational approach. Section 3 is devoted to prove the existence result.

2 The auxiliary problem and variational framework

Since we are intending to work with $N \geq 5$, we use a truncation argument. Here we are assuming, without loss of generality, that M is unbounded. Otherwise, the truncation of the function M is not necessary. We make a truncation on function M case as follows:

From (M_1), there is $t_0 > 0$ such that

$$m_0 < M(t_0) < \frac{\theta}{2}m_0, \tag{2.1}$$

where θ is the positive constant that appear in the hypothesis (f_3).

Now we set

$$M_0(t) := \begin{cases} M(t), \text{ if } 0 \leq t \leq t_0, \\ M(t_0) \text{ if } t_0 \leq t. \end{cases}$$

For all $t \in \mathbb{R}^+$, from (M_1) and (M_2) we get

$$m_0 \leq M_0(t) \leq \frac{\theta}{2}m_0. \tag{2.2}$$

The proof of the Theorem 1.1 is based on a careful study of solutions of the following auxiliary problem

$$(T_\lambda) \qquad \begin{cases} \Delta^2 u - M_0\left(\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)\Delta u + u = \lambda f(u) + |u|^{2^{**}-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where N and λ are as in the introduction.

We say that $u \in H^2(\mathbb{R}^N)$ is a weak solution of the problem (T_λ) if it verifies

$$\begin{aligned} & \int_{\mathbb{R}^N} \Delta u \Delta \phi \, dx + M_0\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)\int_{\mathbb{R}^N} \nabla u \nabla \phi \, dx + \int_{\mathbb{R}^N} u\phi \, dx \\ = \ & \lambda \int_{\mathbb{R}^N} f(u)\phi \, dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2} u\phi \, dx, \end{aligned}$$

for all $\phi \in H^2(\mathbb{R}^N)$.

We will look for solutions of (T_λ) by finding critical points of the C^1-functional $I_\lambda : H^2(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2}\widehat{M_0}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \, dx \\ &- \lambda \int_{\mathbb{R}^N} F(u) \, dx - \frac{1}{2^{**}}\int_{\mathbb{R}^N} |u|^{2^{**}} \, dx, \end{aligned}$$

where $\widehat{M_0}(t) = \displaystyle\int_0^t M_0(s) \, ds$ and $F(t) = \displaystyle\int_0^t f(s) \, ds$.

Note that

$$\begin{aligned} I'_\lambda(u)\phi &= \int_{\mathbb{R}^N} \Delta u \Delta \phi \, dx + M_0\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \nabla \phi dx + \int_{\mathbb{R}^N} u\phi \, dx \\ &- \lambda \int_{\mathbb{R}^N} f(u)\phi \, dx - \int_{\mathbb{R}^N} |u|^{2^{**}-2} u\phi \, dx, \end{aligned}$$

for all $\phi \in H^2(\mathbb{R}^N)$. Hence critical points of I_λ are weak solutions for (T_λ).

Firstly one proves that functional I_λ has the geometry of Mountain Pass Theorem.

Lemma 2.1. *For each $\lambda > 0$, the functional I_λ satisfies that $I_\lambda(0) = 0$ together with the following conditions:*
(i) There exist $r \equiv r(\lambda)$, $\rho > 0$ such that:

$$I_\lambda(u) \geq \rho \ \text{ with } \ \|u\| = r.$$

(ii) There exists $e \in B_r^c(0)$ with $I_\lambda(e) < 0$.

Proof: i) By (f_1) and (f_2), given $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$|f(t)| \leq \epsilon|t| + C_\epsilon |t|^{q-1}. \tag{2.3}$$

Using (M_2) and (2.3) with $0 < \epsilon < 1$, we obtain

$$I_\lambda(u) \geq \frac{k_0}{2}\|u\|^2 - \lambda \frac{C_\epsilon}{q}\int_{\mathbb{R}^N} F(u)\, dx - \frac{1}{2^{**}}\int_{\mathbb{R}^N} |u|^{2^{**}}\, dx,$$

where $k_0 = \min\{1-\lambda\,\epsilon, m_0\}$ and $\|u\|^2 = \int_{\mathbb{R}^N} |\Delta u|^2\, dx + \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} |u|^2\, dx.$

By Sobolev's embedding, there exist positive constants C_1, C_2 and C_3 such that

$$I_\lambda(u) \geq C_1\|u\|^2 - \lambda C_2 \|u\|^q - C_3\|u\|^{2^{**}}.$$

Since that $2 < q < 2^{**}$, the item (i) is proved.

ii) From (f_3), there exist $C_4, C_5 > 0$ such that

$$F(t) \geq C_4 t^\theta - C_5, \quad \forall t > 1.$$

Thus, fixing $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi > 0$ on $\mathbb{R}^N$, $\|\phi\| = 1$ and using (2.2), we get

$$\begin{aligned} I_\lambda(t\phi) \quad &\leq \quad \frac{k_1 t^2}{2} - \lambda\, C_4 t^\theta \int_{\mathbb{R}^N} \phi^\theta\, dx \\ &\quad + \lambda\, C_5 |supp\, \phi| - \frac{t^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |\phi|^{2^{**}}\, dx, \end{aligned}$$

where $k_1 = \max\{1, \frac{\theta}{2} m_0\}$. Since $2 < \theta < 2^{**}$, there exists $\bar{t} = \bar{t}(\lambda) > 1$ such that $e = \bar{t}\phi$ satisfies $I_\lambda(e) < 0$ and $\|e\| > \rho$.

■

From Lemma 2.1, we can conclude that there exists a sequence $(u_n) \subset H^2(\mathbb{R}^N)$ such that

$$I_\lambda(u_n) \to c_\lambda \text{ and } \|I'_\lambda(u_n)\| \to 0 \text{ in } H^2(\mathbb{R}^N)', \tag{2.4}$$

where

$$c_\lambda = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I_\lambda(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], H) : \eta(0) = 0,\ I_\lambda(\eta(1)) < 0\}.$$

Moreover, arguing as in [1, Lemma 2.3], we obtain $c_\lambda = \overline{d} = \widehat{d}$, where

$$\overline{d} = \inf_{u \in H^2(\mathbb{R}^N)\setminus\{0\}} \left\{ \max_{t \geq 0} I_\lambda(tu) \right\}$$

and

$$\widehat{d} = \inf_{u \in \mathcal{M}} I_\lambda.$$

Here

$$\mathcal{M} = \{u \in H^2(\mathbb{R}^N) \setminus \{0\} : I'_\lambda(u)u = 0\}. \tag{2.5}$$

Such sequence is called Palais-Smale sequence for the functional I_λ and c_λ is its mountain pass level. From now on, we shall obtain an estimate for c_λ involving the best constant of the Sobolev embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$ given by

$$S := \inf\left\{ \int_{\mathbb{R}^N} |\Delta u|^2 \ dx : u \in H^2(\mathbb{R}^N), \ \int |u|^{2^{**}} \ dx = 1 \right\}.$$

This estimate is important to prove that each sequence that satisfies (2.4), also converges in $L^{2^{**}}_{loc}(\mathbb{R}^N)$.

Lemma 2.2. *If the conditions $(M_1)-(M_2)$ and $(f_1)-(f_3)$ hold, then there exists $\lambda_* > 0$ such that $c_\lambda \in (0, \frac{2}{N} S^{N/4})$ for all $\lambda \geq \lambda^*$.*

Proof: From Lemma 2.1, there exists $t_\lambda > 0$ verifying $I_\lambda(t_\lambda\, \phi) = \max_{t\geq 0} I_\lambda(t\, \phi)$, where ϕ is the function given in the second part of the Lemma 2.1 again.

Hence, from inequality (2.2) we get

$$t_\lambda^2 k_1 \geq \lambda \int_{\mathbb{R}^N} f(t_\lambda \phi) t_\lambda \phi \ dx + t_\lambda^{2^{**}} \int_\Omega |\phi|^{2^{**}} \ dx \geq t_\lambda^{2^{**}} \int_\Omega |\phi|^{2^{**}} \ dx,$$

which implies that (t_λ) is bounded. Thus, there exists a sequence $\lambda_n \to +\infty$ and $\beta_0 \geq 0$ such that $t_{\lambda_n} \to \beta_0$ as $n \to +\infty$. Consequently, there is $D > 0$ such that

$$t_{\lambda_n}^2 k_1 \leq D \ \ \forall n \in \mathbb{N},$$

and so

$$\lambda_n \int_{\mathbb{R}^N} f(t_{\lambda_n}\phi) t_{\lambda_n} \phi \ dx \leq \lambda_n \int_{\mathbb{R}^N} f(t_{\lambda_n}\phi) t_{\lambda_n} \phi \ dx + t_{\lambda_n}^{2^{**}} \int_\Omega |\phi|^{2^{**}} \ dx \leq D \ \ \forall n \in \mathbb{N}.$$

If $\beta_0 > 0$, the last inequality leads to

$$\lambda_n \int_{\mathbb{R}^N} f(t_{\lambda_n}\phi) t_{\lambda_n} \phi \ dx = +\infty,$$

which is an absurd. Thus, we conclude that $\beta_0 = 0$. Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0,1]$, where e is the function given in the end of prove of the Lemma 2.1. From direct calculation we have that γ_* belongs to Γ and

$$0 < c_\lambda \leq \max_{t\in[0,1]} I_\lambda(\gamma_*(t)) = I(t_\lambda \phi) \leq \frac{1}{2} k_1 t_\lambda^2.$$

In this way, there exists $\lambda^* > 0$ such that $c_\lambda \in (0, \frac{2}{N} S^{N/4})$ for all $\lambda \geq \lambda^*$. ■

Remark 2.3. *Note that, from lemma above, if $\lambda \to \infty$, then $c_\lambda \to 0$.*

Lemma 2.4. *Let $(u_n) \subset H^2(\mathbb{R}^N)$ be the sequence given in (2.4). Then, for $n \in \mathbb{N}$, we have*

$$\|u_n\|^2 \leq t_0.$$

Proof: Assuming, by contradiction, that, for some $n \in \mathbb{N}$ we have $\|u_n\|^2 > t_0$. Thus, from (f_3) and since $M_0(t) \leq M(t_0)$, we get

$$\begin{aligned} c_\lambda = I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)u_n \quad &+ \quad o_n(1) \geq (\frac{1}{2} - \frac{1}{\theta}) \Big(\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \int_{\mathbb{R}^N} |u_n|^2 dx \Big) \\ &+ \quad \frac{1}{2} \widehat{M}_0 (\int_{\mathbb{R}^N} |\nabla u_n|^2 dx) - \frac{1}{\theta} M(t_0) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o_n(1). \end{aligned}$$

Thus, from (M_2) we obtain

$$\begin{aligned} c_\lambda \quad &\geq \quad (\frac{1}{2} - \frac{1}{\theta}) \Big(\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \int_{\mathbb{R}^N} |u_n|^2 dx \Big) \\ &+ \quad (\frac{1}{2} m_0 - \frac{1}{\theta} M(t_0)) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o_n(1) \\ &\geq \quad \frac{\overline{k_0}}{2\theta} \|u_n\|^2 + o_n(1) \geq \frac{\overline{k_0}}{2\theta} t_0 + o_n(1), \end{aligned} \tag{2.6}$$

where $\overline{k_0} = \min\{\theta - 2, (\theta m_0 - 2M(t_0))\}$. But by Remark 2.3, this last inequality is an absurd. Hence (u_n) is bounded in $H^2(\mathbb{R}^N)$ by constant $\sqrt{t_0}$. ■

Lemma 2.5. *Let $(u_n) \subset H^2(\mathbb{R}^N)$ be the sequence given in (2.4). Then,*

$$u_n \to u \quad in \quad L^{2^{**}}_{loc}(\mathbb{R}^N),$$

for some $u \in H^2(\mathbb{R}^N)$.

Proof: By Lemma 2.4, there exists $u \in H^2(\mathbb{R}^N)$ such, up to a subsequence,

$$u_n \rightharpoonup u \ \text{ in } \ H^2(\mathbb{R}^N).$$

We may suppose that

$$\begin{aligned} |\Delta u_n|^2 \rightharpoonup |\Delta u|^2 + \mu, \quad |\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \sigma \\ \text{and} \quad |u_n|^{2^*} \rightharpoonup |u|^{2^*} + \nu \qquad \text{(weak}^*\text{-sense of measures).} \end{aligned}$$

Using the concentration compactness-principle due to Lions (cf. [10, Lemma 2.1]), we obtain an at most countable index set Λ, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\sigma_i), (\nu_i), \subset [0, \infty)$, such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \quad \sigma \geq \sum_{i\in\Lambda} \sigma_i \delta_{x_i} \quad \text{and} \quad S\nu_i^{2/2^{**}} \leq \mu_i, \tag{2.7}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega$.

Now we claim that $\Lambda = \emptyset$. Arguing by contradiction, assume that $\Lambda \neq \emptyset$ and fix $i \in \Lambda$. Consider $\psi \in C_0^\infty(\Omega, [0, 1])$ such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$ and $|\nabla\psi|_\infty \leq 2$. Defining $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\varrho > 0$, we have that $(\psi_\varrho u_n)$ is bounded. Thus $I'_\lambda(u_n)(\psi_\varrho u_n) \to 0$, that is,

$$\begin{aligned} & \int_{\mathbb{R}^N} u_n \Delta u_n \Delta\psi_\varrho \, dx + \int_{\mathbb{R}^N} \psi_\varrho |\Delta u_n|^2 \, dx + \int_{\mathbb{R}^N} |u_n|^2 \psi_\varrho \, dx \\ & + M_0\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx\right) \int_{\mathbb{R}^N} u_n \nabla u_n \nabla\psi_\varrho \, dx \\ = \; & \lambda \int_{\mathbb{R}^N} f(u_n) u_n \psi_\varrho \, dx + \int_{\mathbb{R}^N} \psi_\varrho |u_n|^{2^{**}} \, dx + o_n(1). \end{aligned}$$

Since the support of ψ_ϱ is $B_{2\varrho}(x_i)$, we obtain

$$\left| \int_{\mathbb{R}^N} u_n \Delta u_n \Delta\psi_\varrho \, dx \right| \leq \int_{B_{2\rho}(x_i)} |\Delta u_n| |u_n \Delta\psi_\varrho| \, dx.$$

By Hölder inequality and the fact that the sequence (u_n) is bounded in $H^2(\mathbb{R}^N)$ we have

$$\left| \int_{\mathbb{R}^N} u_n \Delta u_n \Delta\psi_\varrho \, dx \right| \leq C\left(\int_{B_{2\varrho}(x_i)} |u_n \Delta\psi_\varrho|^2 \, dx\right)^{1/2}.$$

By the Dominated Convergence Theorem $\int_{B_{2\varrho}(x_i)} |u_n \Delta\psi_\varrho|^2 \, dx \to 0$ as $n \to +\infty$ and $\varrho \to 0$. Thus, we obtain

$$\lim_{\varrho\to 0} [\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n \Delta u_n \Delta\psi_\varrho \, dx] = 0.$$

Using the same reasoning we obtain

$$\lim_{\varrho\to 0} [\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n \nabla u_n \nabla\psi_\varrho \, dx] = 0.$$

Since $0 < m_0 \leq M_0(t) \leq M(t_0)$, for all $t \in \mathbb{R}$, we get

$$\lim_{\varrho\to 0} \lim_{n\to\infty} [M_0(\|u_n\|^2) \int_{\mathbb{R}^N} u_n \nabla u_n \nabla\psi_\varrho \, dx] = 0.$$

Moreover, with a similar reasoning we conclude

$$\lim_{\varrho\to 0}\lim_{n\to\infty}[\int_{\mathbb{R}^N}\psi_\varrho f(u_n)u_n\ dx]=0$$

and

$$\lim_{\varrho\to 0}\lim_{n\to\infty}[\int_{\mathbb{R}^N}\psi_\varrho|u_n|^2\ dx]=0.$$

Thus, we have

$$\int_{\mathbb{R}^N}\psi_\varrho \mathrm{d}\mu\le\int_{\mathbb{R}^N}\psi_\varrho \mathrm{d}\mu+m_0\int_{\mathbb{R}^N}\psi_\varrho \mathrm{d}\sigma\le\int_{\mathbb{R}^N}\psi_\varrho \mathrm{d}\nu+o_\varrho(1).$$

Letting $\varrho\to 0$, using standard theory of Radon measures and from (2.7) we obtain

$$\mu_i\ge\nu_i\ge S^{N/4}. \tag{2.8}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that $\mu_i\ge S^{N/4}$, for some $i\in\Lambda$. Thus,

$$c_\lambda\ =\ I_\lambda(u_n)-\frac{1}{2^{**}}I'_\lambda(u_n)u_n+o_n(1).$$

Since $M_0(t)\le\dfrac{2^{**}}{2}m_0$ for all $t\in\mathbb{R}$ and (f_3), we have that

$$c_\lambda\ge\frac{2}{N}\int_{\mathbb{R}^N}|\Delta u_n|^2\ dx.$$

Letting $n\to\infty$, we get

$$c_\lambda\ge\frac{2}{N}\mu_i+\frac{2}{N}\int_{\mathbb{R}^N}|\Delta u|^2\ dx\ge\frac{2}{N}S^{N/4}$$

for all $\lambda>0$, which is a contradiction with the Lemma 2.2. Thus Λ is empty and it follows that $u_n\to u$ in $L^{2^{**}}_{loc}(\mathbb{R}^N)$. ■

Lemma 2.6. *Let $(u_n)\subset H^2(\mathbb{R}^N)$ be a sequence that satisfies (2.4) and $\lambda\ge\lambda^*$. Then, there exist a sequence $(y_n)\subset\mathbb{R}^N$ and constants $R,\ \eta>0$ such that*

$$\liminf_{n\to\infty}\int_{B_R(y_n)}|u_n|^2\ dx\ge\eta>0.$$

Proof: Suppose that the lemma does not hold. Then, it follows from [9, Lemma I.1] that $u_n \to 0$ in $L^q(\mathbb{R}^N)$, and thus, $\displaystyle\int_{\mathbb{R}^N} f(u_n)u_n = o_n(1)$. Recalling that $I'_\lambda(u_n)u_n = o_n(1)$, we have, up to a subsequence that,

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx + M_0(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx) \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} |u_n|^2 \, dx = L_\lambda + o_n(1)$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2^{**}} \, dx = L_\lambda + o_n(1).$$

for some $L_\lambda \geq 0$. Since $c_\lambda > 0$, we have that $L_\lambda > 0$. Using the best Sobolev constant S, we get

$$L_\lambda \geq S^{\frac{N}{4}} \quad \forall \lambda > 0.$$

On the other hand, using (f_3) and (2.2), we obtain

$$c_\lambda + o_n(1) \geq \frac{2}{N} \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \geq \frac{2}{N} L_\lambda \geq \frac{2}{N} S^{N/4},$$

which is a contradiction with the Lemma 2.2. ■

Lemma 2.7. *If condition (f_4) and (M_3) hold, then*

$$sf(s) - 2F(s)$$

and

$$\widehat{M}(s) - \frac{1}{2} M(s)s$$

are increasing for $s \neq 0$.

Proof: Supposing $s < t$, from (f_4) we obtain

$$\begin{aligned} sf(s) - 2F(s) &= \frac{f(s)}{s} s^2 - 2F(t) + 2 \int_s^t f(\tau) d\tau \\ &< \frac{f(t)}{t} s^2 - 2F(t) + \frac{f(t)}{t}(t^2 - s^2) \\ &= tf(t) - 2F(t). \end{aligned}$$

Using condition (M_3), this reasoning also proves the result involving the function M and this proves the lemma. ■

3 Proof of the Theorem 1.1

By Lemmas 2.1, 2.4 and 2.5, for all $\lambda \geq \lambda^*$, there is a sequence $(u_n) \subset H^2(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$u_n \rightharpoonup u \in H^2(\mathbb{R}^N), \tag{3.1}$$

$$u_n \to u \in L^s_{loc}(\mathbb{R}^N), \quad 1 \leq s \leq 2^{**} \tag{3.2}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to \varrho_0, \text{ in } \mathbb{R}, \tag{3.3}$$

for some $\varrho_0 \geq 0$.

From Lemma 2.4, we have that $\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq t_0$. Thus,

$$M_0(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx) = M(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx).$$

From continuity of function M and (3.3), we get

$$M(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx) \to M(\varrho_0) \text{ in } \mathbb{R}. \tag{3.4}$$

Without loss of generality, we can assume that $u \neq 0$, because by Lemma 2.6, there exist $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \, dx \geq \eta > 0. \tag{3.5}$$

Considering $v_n(x) = u_n(x - y_n)$, we have that (v_n) is also bounded in $H^2(\mathbb{R}^N)$ and its weak limit denoted by v is nontrivial, because the last inequality together Sobolev's embedding implies that

$$\int_{B_R(0)} |v|^2 \, dx \geq \eta > 0.$$

Furthermore, a routine calculus leads to

$$I_\lambda(v_n) \to c_\lambda \text{ and } I'_\lambda(v_n) = o_n(1).$$

Now, for each $\psi \in C_0^\infty(\mathbb{R}^N)$ and using (3.1), (3.2) and (3.4) we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} \Delta u \Delta \psi \, dx + M(\varrho_0) \int_{\mathbb{R}^N} \nabla u \nabla \psi \, dx + \int_{\mathbb{R}^N} u\psi \, dx \\ = \; & \lambda \int_{\mathbb{R}^N} f(u)\psi \, dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2} u\psi \, dx. \end{aligned}$$

Since $\overline{C_0^\infty(\mathbb{R}^N)} = H^2(\mathbb{R}^N)$, we conclude

$$\begin{aligned} & \int_{\mathbb{R}^N} \Delta u \Delta \phi \ dx + M(\varrho_0) \int_{\mathbb{R}^N} \nabla u \nabla \phi \ dx + \int_{\mathbb{R}^N} u\phi \ dx \\ = \ & \lambda \int_{\mathbb{R}^N} f(u)\phi \ dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2} u\phi \ dx, \end{aligned}$$

for all $\phi \in H^2(\mathbb{R}^N)$.

We claim that

$$M(\varrho_0) = M(\int_{\mathbb{R}^N} |\nabla u|^2 \ dx). \tag{3.6}$$

Considering this claim, (M_2) and arguing as [5, Theorem 2.1], we have that $u_\lambda \in C^{4,\alpha}_{loc}(\mathbb{R}^N)$ with $\alpha \in (0,1)$ is a nontrivial solution of the problem (P_λ), for all $\lambda \geq \lambda^*$.

Finally, by (2.6) and Fatou's Lemma we conclude

$$c_\lambda \geq \frac{k_0}{2\theta} \|u_\lambda\|^2.$$

By Remark 2.3 we get $\lim_{\lambda \to \infty} \|u_\lambda\| = 0$ and the Theorem 1.1 is proved.

To conclude our this section, we need to prove (3.6). We recall that from the weak convergence

$$M(\varrho_0) \geq M(\int_{\mathbb{R}^N} |\nabla u|^2 \ dx).$$

Supposing by contradiction that

$$M(\varrho_0) > M(\int_{\mathbb{R}^N} |\nabla u|^2 \ dx),$$

it follows that $I'_\lambda(u)u < 0$. Therefore, there exists $\bar{t} \in (0,1)$ such that $\bar{t}u \in \mathcal{M}$, with $\mathcal{M}$ defined in (2.5).

Combining this information with the characterization of mountain pass level, we derive

$$c_\lambda \leq I_\lambda(\bar{t}u) = I_\lambda(\bar{t}u) - \frac{1}{2} I'_\lambda(\bar{t}u)\bar{t}u.$$

From Lemma 2.7,

$$c_\lambda < I_\lambda(u) - \frac{1}{2} I'_\lambda(u)u.$$

On the other hand, by Fatou's Lemma,

$$I_\lambda(u) - \frac{1}{2} I'_\lambda(u)u \leq I_\lambda(u_n) - \frac{1}{2} I'_\lambda(u_n)u_n + o_n(1).$$

Thus,

$$c_\lambda < \liminf_{n\to\infty}\left[I_\lambda(u_n) - \frac{1}{2}I'_\lambda(u_n)u_n\right] = c_\lambda,$$

which is an absurd. This way, $M(\varrho) = M(\int_{\mathbb{R}^N} |\nabla u|^2\, dx)$ and the proof of claim (3.6) is finished. ■

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