Characterization of non-disconjugacy for a one parameter family of $n^{\rm th}-$order linear differential equations.*

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Abstract

The aim of this paper is to obtain different criteria which allow us to affirm that the one parameter family of $n^{\rm th}-$order linear differential equations, given by the following expression

$$T_n[M]\,u(t) \equiv u^{(n)}(t) + a_1(t)\,u^{(n-1)}(t) + \cdots + a_{n-1}(t)\,u'(t) + (a_n(t)+M)\,u(t) = 0\,, \quad t \in I \equiv [a,b]\,, \qquad (1)$$

is not disconjugate for every $M \in \mathbb{R}$.

Three different sufficient criteria, which ensure that such property holds, are presented. Moreover, a characterization of this property is given.

To finish the paper, three examples, where the different criteria are applied, are shown.

Key Words: Disconjugacy, Green's functions, spectral theory,

1 Introduction

In this paper we are going to study the disconjugacy of the $n^{\rm th}-$order linear differential equation (1) for different values of $M \in \mathbb{R}$. More concisely, we are going to obtain a characterization to ensure that equation (1) is not disconjugate for every $M \in \mathbb{R}$. The main difference with previous results given in the literature in this direction is that we are not considering a specific equation. Our goal consist in to warrant that the equation is not disconjugate with independence of the value of the real parameter M.

Taking into account that the coefficient of u can be uniquely decomposed as

$$a_n(t) = \tilde{a}_n(t) + \frac{1}{b-a}\int_a^b a_n(s)\,ds, \quad t \in I,$$

it is obvious that the study of the equation $u^{(n)}(t) + a_1(t)\,u^{(n-1)}(t) + \cdots + a_{n-1}(t)\,u'(t) + a_n(t)\,u(t) = 0$ on I is equivalent to the study of the linear differential equation (1) in I for $M \in \mathbb{R}$.

Disconjugacy has been studied along the time by many authors, see [4]–[9]. Different sufficient criteria to ensure the disconjugacy character of the linear operator $T_n[0]$ has been developed in classical references, see for instance [8, 9]. The interest of studying disconjugacy is due to the fact that this property implies many others on the linear differential equation (1) and on its related operator $T_n[M]$. For instance, in many cases we can know the sign of the Green's function related to suitable boundary value conditions.

Disconjugacy has been studied for particular cases, in many papers, see for example [7], where the even order equation $y^{(2\,n)}(t) - (-1)^n\,p(t)\,y(t) = 0$, with $p \geq 0$ is studied, or [5], where sufficient conditions for the disconjugacy of $y^{(n)}(t) + p(t)\,y(t) = 0$ in $[a,+\infty)$, with p of constant sign are obtained.

Moreover, in [2] it is obtained a characterization of the parameter set where the general equation (1) is disconjugate on a given interval under the hypothesis that the parameter set of disconjugacy is not empty. As consequence, the main difficulty to apply those results consists on verify the non empty character of the disconjugacy set. This paper, is devoted to obtain a characterization to ensure that such parameter set is empty (or, equivalently, not empty).

Once we have obtained our characterization, we will reformulate it into an equivalent form to express it in terms of spectral theory.

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Now, in order to make the paper more readable, we introduce some previous concepts and results.

First, we introduce the so called $(k, n-k)$ boundary conditions by

$$u(a) = \cdots = u^{(k-1)}(a) = u(b) = \cdots = u^{(n-k-1)}(b) = 0\,, \quad 1 \leq k \leq n-1\,, \tag{2}$$

and the related definition set

$$X_{k\,[a,b]} = \left\{ u \in C^n(I) \ \mid u(a) = \cdots = u^{(k-1)}(a) = u(b) = \cdots = u^{(n-k-1)}(b) = 0 \right\}.$$

Definition 1.1. *Let $a_k \in C^{n-k}(I)$ for $k = 1, \ldots, n$. The $n^{\rm th}-$order linear differential equation (1) is said to be disconjugate on an interval $J \subset I$ if every non trivial solution has, at most, $n-1$ zeros on J, multiple zeros being counted according to their multiplicity.*

Definition 1.2. *Let $a \in \mathbb{R}$, denote the first right point conjugate of a for the linear differential equation (1) by*

$$\eta_M(a) = \sup\{b > a \quad | \quad \text{equation (1) is disconjugate on } [a,b]\} \in (a, \infty]\,.$$

We consider a fundamental system of solutions $y_1[M](t), \ldots, y_n[M](t)$ of equation (1) where every $y_k[M](t)$ satisfies the following initial conditions:

$$y_k^{(n-k)}[M](a) = 1\,, \quad y_k^{(n-j)}[M](a) = 0\,,\ j = 1, \ldots, n\,,\ j \neq k\,. \tag{3}$$

Then, we denote the $n-1$ Wronskians as

$$W_k^n[M](t) := \begin{vmatrix} y_1[M](t) & \cdots & y_k[M](t) \\ \vdots & \cdots & \vdots \\ y_1[M]^{(k-1)}(t) & \cdots & y_k[M]^{(k-1)}(t) \end{vmatrix}\,, \quad t \in \mathbb{R}\,,\ k = 1, \ldots, n-1\,. \tag{4}$$

Now, we enunciate some results which are collected on [4, Chapter 3].

Proposition 1.3. *There exists a solution of equation (1) which verifies the boundary conditions $(k, n-k)$ on I if, and only if, $W_{n-k}^n[M](b) = 0$.*

Definition 1.4. *Denote $\omega_M(a)$ as the least $b > a$, if one exists, at which one of the Wronskians $W_k^n[M](b)$ vanishes for $k = 1, \ldots, n-1$.*

The next result gives us a relation between this concept and the one introduced on Definition 1.2.

Proposition 1.5. *$\eta_M(a) = \omega_M(a)$.*

The following result, which appears on [7, Theorem 3.2], shows a property of the eigenvalues of a disconjugate operator.

Theorem 1.6. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]\,u(t) = 0$ is disconjugate on I. Then*

- *If $n-k$ is even, there is not any eigenvalue of $T_n[\bar{M}]$ on $X_{k\,[a,b]}$ such that $\lambda < 0$.*
- *If $n-k$ is odd, there is not any eigenvalue of $T_n[\bar{M}]$ on $X_{k\,[a,b]}$ such that $\lambda > 0$.*

Moreover, we have the following result, obtained in [3].

Lemma 1.7. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]\,u(t) = 0$ is disconjugate on I. Then the following properties are fulfilled:*

- *If $n-k$ is even, then there exists $\lambda_1 > 0$, the least eigenvalue in absolute value of operator $T_n[\bar{M}]$ in $X_{k\,[a,b]}$.*
- *If $n-k$ is odd, then there exists $\lambda_2 < 0$, the least eigenvalue in absolute value of operator $T_n[\bar{M}]$ in $X_{k\,[a,b]}$.*

2 Non disconjugacy criteria

In this section, we are going to give some results about disconjugacy and we will obtain sufficient criteria to affirm that the linear differential equation (1) is not disconjugate for every $M \in \mathbb{R}$. Finally, we will prove that one of these sufficient criteria gives also a necessary condition to obtain the characterization of non-disconjugacy.

Lemma 2.1. *If the linear differential equation* (1) *is disconjugate on I for $M = \bar{M}$, then, if $n-k$ is even, $W_{n-k}^n[M](t) \neq 0$ for every $t \in (a,b]$ and $M \geq \bar{M}$.*

Proof. Trivially, for $M = \bar{M}$ the assertion is true, see Proposition 1.5.

If there exist $c \in (a,b]$ and $M^* > \bar{M}$, such that $W_{n-k}^n[M^*](c) = 0$, with $n-k$ even, we have, using Proposition 1.3, that $\bar{M} - M^* < 0$ is an eigenvalue of $T_n[\bar{M}]$ in $X_{k[a,c]}$, with $n-k$ even.

Hence, using Theorem 1.6, we can affirm that the linear differential equation (1) is not disconjugate on $[a,c]$ for $M = \bar{M}$. But in such a case, since $[a,c] \subset I$, the linear differential equation (1) cannot be disconjugate for $M = \bar{M}$ on I, which contradicts our assumption. □

As a consequence, we deduce the following corollary.

Corollary 2.2. *If $W_{n-k}^n[M^*](c) = 0$ for any $c \in (a,b]$, with $n-k$ even, then the linear differential equation* (1) *is not disconjugate for every $M \leq M^*$.*

Now, if $n-k$ is odd, in an analogous way we can prove the following results.

Lemma 2.3. *If the linear differential equation* (1) *is disconjugate on I for $M = \bar{M}$, then, if $n-k$ is odd, $W_{n-k}^n[M](t) \neq 0$ for every $t \in (a,b]$ and $M \leq \bar{M}$.*

Corollary 2.4. *If $W_{n-k}^n[M^*](c) = 0$ for any $c \in (a,b]$ and $n-k$ odd, then the linear differential equation* (1) *is not disconjugate for every $M \geq M^*$.*

Now, combining Corollaries 2.2 and 2.4, we obtain the following result which gives a sufficient criteria to ensure that the linear differential equation (1) is not disconjugate for every $M \in \mathbb{R}$.

Corollary 2.5. *If there exist $c_1, c_2 \in (a,b]$ and $M_1 \geq M_2$ such that*

- *There exists $n-k^*$ even such that $W_{n-k^*}[M_1](c_1) = 0$.*
- *There exits $n-k^{**}$ odd such that $W_{n-k^{**}}[M_2](c_2) = 0$.*

Then, the linear differential equation (1) *is not disconjugate for every $M \in \mathbb{R}$.*

And, as a particular case, we have the following result.

Corollary 2.6. *Suppose that there exists $\bar{M} \in \mathbb{R}$ such that there are two different Wronskians satisfying $W_{k^*}^n[\bar{M}](c_1) = W_{\tilde{k}}^n[\bar{M}](c_2) = 0$ for some $c_1, c_2 \in (a,b]$, with $k^* - \tilde{k}$ an odd number.*

Then there is not any $M \in \mathbb{R}$ such that $T_n[M]\, u(t) = 0$ is a disconjugate equation on I.

Now, using Corollary 2.5, we introduce another sufficient criteria to ensure that the equation (1) is not disconjugate for every $M \in \mathbb{R}$.

Corollary 2.7. *If there exists $\bar{M} \in \mathbb{R}$ such that $W_{n-k}^n[\bar{M}]$ has a double zero at $c \in (a,b]$, for any $n-k \in \{1,\dots,n-1\}$. Then, there is not any $M \in \mathbb{R}$ such that the linear differential equation* (1) *is disconjugate on I.*

Proof. We have that $W_{n-k}^n[\bar{M}](c) = 0$.

If $W_{n-k-1}[\bar{M}](c) = 0$, we can apply Corollary 2.5 with $M_1 = M_2 = \bar{M}$ and $c_1 = c_2 = c$ and we conclude that the linear differential equation (1) is not disconjugate for every $M \in \mathbb{R}$.

If $W_{n-k-1}[\bar{M}](c) \neq 0$, we consider the following non trivial solution of equation (1):

$$y[\bar{M}](t) := \begin{vmatrix} y_1[\bar{M}](c) & \dots & y_{n-k}[\bar{M}](c) \\ \vdots & \ddots & \vdots \\ y_1^{(n-k-2)}[\bar{M}](c) & \dots & y_{n-k}^{(n-k-2)}[\bar{M}](c) \\ y_1[\bar{M}](t) & \dots & y_{n-k}[\bar{M}](t) \end{vmatrix}.$$

By construction, this solution satisfies the boundary conditions $(k, n-k)$ on $[a,c]$ and, from the expression of a derivative of a Wronskian, it also satisfies the boundary conditions $(k-1, n-k+1)$. Hence, using Proposition 1.3, we can affirm that $W_{n-k+1}^n[\bar{M}](c) = 0$, and then we can apply again Corollary 2.5 to affirm that the linear differential equation (1) is not disconjugate for every $M \in \mathbb{R}$. □

Now, let us see that the reciprocal of Corollary 2.5 is true, i.e., we can state a sufficient and necessary criteria to ensure that the parameter set where the linear differential equation (1) is disconjugate on I, is, or not, empty.

Theorem 2.8. *The linear differential equation* (1) *is not disconjugate on I for every $M \in \mathbb{R}$ if, and only if, there exist $c_1, c_2 \in (a,b]$ and $M_1 \geq M_2$ such that*

- *There exists $n-k^*$ even such that $W_{n-k^*}[M_1](c_1) = 0$.*
- *There exits $n-k^{**}$ odd such that $W_{n-k^{**}}[M_2](c_2) = 0$.*

Proof. We only need to prove the first implication, since the other one is given in Corollary 2.5.

Let us denote the following parameters

$$\begin{aligned}\widehat{M_1} &= \sup\{M \in \mathbb{R} \quad | \quad \exists\, c \in (a,b], W_{n-k}[M](c) = 0\,,\ \text{with } n-k \text{ even}\}\ ,\\ \widehat{M_2} &= \inf\{M \in \mathbb{R} \quad | \quad \exists\, c \in (a,b], W_{n-k}[M](c) = 0\,,\ \text{with } n-k \text{ odd}\}\ .\end{aligned}$$

We can affirm that both previous sets are not empty because, otherwise, there is not any eigenvalue of $T_n[M]\,u(t)$ in $X_{k[a,c]}$ for some $k \in \{1,\dots,n-1\}$, every $c \in (a,b]$ and every $M \in \mathbb{R}$. But, we know that for every $M \in \mathbb{R}$ there exists an interval, which we denote by $[a, \alpha_M(a)] \subset [a,b]$, where the linear differential equation (1) is disconjugate, see [4, Chapter 3, Proposition 1]. By Lemma 1.7, it must exist an eigenvalue of $T[M]$ in $X_{k[a,\alpha_M(a)]}$ Hence, by Proposition 1.3, there exist $\bar{M} \in \mathbb{R}$ such that $W_{n-k}[\bar{M}](\alpha_M(a)) = 0$.

Suppose now that the assertion of the Theorem is not verified, then necessarily $\widehat{M_1} < \widehat{M_2}$, and we have that for $M \in (\widehat{M_1}, \widehat{M_2})$ all Wronskians do not vanish for $t \in (a,b]$, so we can affirm that for those M the linear differential equation (1) is disconjugate on $[a,b]$, by means of Proposition 1.5. Then we have that the disconjugacy set is not empty and our result is proved. □

Now, by using Proposition 1.3, we can reformulate previous result in terms of spectral theory.

Theorem 2.9. *The linear differential equation* (1) *is not disconjugate on I for every $M \in \mathbb{R}$ if, and only if, there exist $c_1, c_2 \in (a,b]$ and $\bar{M} \in \mathbb{R}$ such that*

- *There exist $k^* \in \{1,\dots,n-1\}$, such that $n-k^*$ is even, and $\lambda_1 \leq 0$, an eigenvalue of $T[\bar{M}]$ in $X_{k^*[a,c_1]}$.*
- *There exist $k^{**} \in \{1,\dots,n-1\}$, such that $n-k^{**}$ is odd, and $\lambda_2 \geq 0$, an eigenvalue of $T[\bar{M}]$ in $X_{k^{**}[a,c_2]}$.*

We can rewrite previous result to characterize when the disconjugacy set is nonempty as follows.

Theorem 2.10. *There exists an interval (M_1, M_2), such that the linear differential equation* (1) *is disconjugate on I for every $M \in (M_1, M_2)$ if, and only if, for all $M \in \mathbb{R}$ one of the two following assertions is verified*

- *If $n-k$ is even, there is not any eigenvalue of $T_n[M]$ on $X_{k[a,c]}$ such that $\lambda \leq 0$ for all $c \in (a,b]$.*
- *If $n-k$ is odd, there is not any eigenvalue of $T_n[M]$ on $X_{k[a,c]}$ such that $\lambda \geq 0$ for all $c \in (a,b]$.*

In fact, by Theorem 2.9 we can only ensure the existence of at least one $\bar{M}$ for which equation (1) is disconjugate on I. The open character of the disconjugacy set shown in [4], allows us to warrant the existence of such interval.

Realize that, to ensure that the disconjugacy set is not empty, from this result, we obtain that for all $M \in \mathbb{R}$ one of the assertions of Theorem 1.6 must be verified. Moreover, using Theorem 1.6, we deduce that if for $M = \bar{M}$ the linear differential equation (1) is disconjugate on I, then, since the disconjugacy on I, implies the disconjugacy on every interval $[a,c] \subset I$, both assertions of Theorem 2.10 are satisfied for such $\bar{M}$.

3 Examples

In this section we present three examples, where we the different criteria given in the previous section are applied:

Example 1, is a fourth order example, where we apply Corollary 2.6 to see that two consecutive Wronskians oscillate for an interval of $M \in \mathbb{R}$.

Example 2, is an eight order example, where we find two non consecutive Wronskians which oscillate simultaneously for an interval of $M \in \mathbb{R}$.

Example 3, is a fourth order example, where, in order to apply Corollary 2.7, we find a Wronskian with a double zero for a $\widehat{M}$. Furthermore, we can prove that this $\widehat{M}$ is the unique $M \in \mathbb{R}$ for which the sufficient and necessary condition given in Theorem 2.8 is satisfied. Since the calculus are very hard, we show our conclusions without putting the explicit calculus.

3.1 Example 1

Firstly, we show an example where the non-disconjugacy criteria holds as an straight consequence of Corollary 2.6.

We consider the operator $T_4[M]\,u(t) = u^{(4)}(t) + 200\,u''(t) + M\,u(t)$ for $t \in [0,1]$.

Let's see that for $\bar{M} = 8^4$, both $W_1^4[\bar{M}](t)$ and $W_2^4[\bar{M}](t)$ oscillate on $[0,1]$.

We can show that they are given by the following expressions

$$\begin{aligned} W_1^4[\bar{M}](t) &= \frac{\left(3+\sqrt{41}\right)\sin\left(\sqrt{2}\left(\sqrt{41}-3\right)t\right) - \left(\sqrt{41}-3\right)\sin\left(\sqrt{2}\left(3+\sqrt{41}\right)t\right)}{768\sqrt{82}}\,, \\ W_2^4[\bar{M}](t) &= \frac{41\cos\left(6\sqrt{2}t\right) - 9\cos\left(2\sqrt{82}t\right) - 32}{377856}\,, \end{aligned}$$

and, in particular, they satisfy $W_1^4[\bar{M}](1/3) > 0 > W_1^4[\bar{M}](1)$, $\quad W_2^4[\bar{M}](4/5) > 0 > W_2^4[\bar{M}](1)$.

So, both of them oscillate on $[0,1]$ and Corollary 2.6 allows us to affirm that it does not exist $M \in \mathbb{R}$ such that $u^{(4)}(t) + 200\,u''(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$.

Realize that in this case, by means of the continuity of the Wronskians as functions of M, we can affirm that there exists $\varepsilon > 0$, such that the non-disconjugacy criteria is satisfied for $M \in (8^4 - \varepsilon, 8^4 + \varepsilon)$.

In order to apply the spectral characterization given in Theorem 2.9, we obtain numerically, for this case, that $7.03^4 > 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{3[0,1]}$ and $-5.075^4 < 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{2[0,0.9]}$.

3.2 Example 2

In this example we consider the eight-order linear equation

$$T_8[M]\,u(t) = u^{(8)}(t) + 11^4\,u^{(4)}(t) + M\,u(t) = 0, \quad t \in [0,1]\,. \tag{5}$$

We are going to see that for $\bar{M} = -40917842$, $W_1[\bar{M}]$ and $W_4[\bar{M}]$ oscillate simultaneously on $[0,1]$.

In order to obtain the expression of $W_1[\bar{M}]$ and $W_4[\bar{M}]$, since the coefficients are constant, we are able to calculate the solutions $y_k[\bar{M}]$, $k = 1,\ldots,4$, defined in (3) by means of Mathematica Program.

As a consequence, we can verify that $W_1[\bar{M}]\left(\frac{4}{5}\right) > 0 > W_1[\bar{M}](1)$ and, therefore, $W_1[\bar{M}]$ oscillates on $(0,1]$.

Moreover, one can verify that $W_4[\bar{M}](t) = \dfrac{w_4(t)}{467795255437017340368147360}$, where $w_4(t)$ is a function whose explicit expression is obtained by means of Mathematica program, but its expression is to complicated to show here. However, it is not difficult to verify that $w_4\left(\frac{4}{5}\right) > 0 > w_4(1)$, hence it also oscillates on $[0,1]$.

So, by means of Corollary 2.6 again, we can affirm that $T_8[M]\,u(t) = 0$ is not disconjugate on $[0,1]$ for all $M \in \mathbb{R}$. As in Example 1 we obtain an interval of $M \in \mathbb{R}$ for which the non-disconjugacy hypothesis is satisfied.

Remark 3.1. *In previous example we have applied the non-disconjugacy criteria to two non consecutive Wronskians. In order to have information of the rest of the Wronskians, we realize that, since the problem is self-adjoint, we only need to study the Wronskians* $W_1[M]$, $W_2[M]$, $W_3[M]$ *and* $W_4[M]$.

By numerical studies, we observe that

- $W_1[M]$ *does not have any zero on* $(0,1]$ *for every* $M \le M_1 = -\dfrac{235165923}{4}$,
- $W_2[M]$ *does not have any zero on* $(0,1]$ *for every* $M \ge M_1$,
- $W_3[M]$ *does not have any zero on* $(0,1]$ *for every* $M \le 0$,
- $W_4[M]$ *does not have any zero on* $(0,1]$ *for every* $M \le 0$.

With analytical calculus, we are able to prove first and fourth items. Moreover, we also can prove:

- $W_2[M]$ *does not have any zero on* $(0,1]$ *for* $M = M_1$,
- $W_3[M]$ *does not have any zero on* $(0,1]$ *for* $M = 0$.

However, because of the hardness of the calculus, we have not been able to completely prove analytically that $W_2[M]$ *does not have any zero on* $(0,1]$ *for every* $M \ge M_1$ *and* $W_3[M]$ *does not have any zero on* $(0,1]$ *for every* $M \le 0$.

If this assertions are true, we have that two consecutive Wronskian do not oscillate simultaneously for any $M \in \mathbb{R}$ *and allow us to prove that it does not exist any* $M \in \mathbb{R}$ *such that the disconjugacy criteria of Corollary 2.7 is satisfied. Since a double zero on a Wronskian implies that a consecutive Wronskian vanishes at the same value.*

So, we would have proved that Corollary 2.7 and 2.5 are not equivalent. In particular, Corollary 2.7 would be a sufficient but not necessary condition. In any case, an analytical proof remains as an open problem.

In this case, to apply Theorem 2.9, we obtain numerically that $7.6^8 > 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{7[0,1]}$ and $-7.7^8 < 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{4[0,1]}$.

3.3 Example 3

Next example follows by means of Corollary 2.7. We consider the fourth order equation

$$T_4[M]\,u(t) = u^{(4)}(t) + 1000\,u'(t) + M\,u(t) = 0\,, \quad t \in [0,1]\,. \tag{6}$$

To prove the non-disconjugacy property for this equation is really complicated. This is due, in part, from the fact that the parameter $\bar{M}$, for which the non-disconjugacy criteria given in Theorem 2.8 follows, is unique. The calculus made on this case are very hard, so they will be avoided.

Next, we summarize the steps that we have followed in order to obtain the unique $\bar{M} \in \mathbb{R}$ for which the hypotheses of Theorem 2.8 are fulfilled:

- First, we prove that the first Wronskian, $W_1[M]$ has at least a zero on $(0,1]$ for $M = 0$.
- Then, we obtain the expression of such Wronskian for negative values of M and we prove that there exists $M^* < 0$ such that $W_1[M^*](t) \neq 0$ for all $t \in (0,1]$.
- After that, we verify that $W_1[M](1) \neq 0$ for all $M \in [M^*, 0]$. Since $W_1[M]$ is a continuous function on M, we can affirm that there exist $\widehat{M} \in (M^*, 0)$ and $\widehat{t} \in (0,1)$ such that $W_1[\widehat{M}]$ has a double zero at $\widehat{t}$ and that $W_1[M]$ oscillates on $[0,1]$ for all $0 \geq M > \widehat{M}$.

Hence, as a consequence of the previous assertions, using Corollary 2.7, we can conclude that the linear differential equation (6) is not disconjugate for every $M \in \mathbb{R}$.

To see that there is not any $M \in \mathbb{R}$, $M \neq \widehat{M}$ such that the non-disconjugacy criteria given in Theorem 2.8 could be applied, we argue as follows:

- First, we see that if $M < \widehat{M}$, then $W_1[M] > 0$ on $(0,1]$.
- After that, we verify that $W_3[M]$ is of constant sign for every $M \leq 0$.
- Finally, we prove that $W_2[M] > 0$ is of constant sign on $(0,1]$ for $M \geq \widehat{M}$.

This three last items allow us to affirm that the hypothesis of Corollaries 2.5, 2.6 and 2.7 and Theorem 2.8 are only satisfied for $\widehat{M}$.

References

[1] A. Cabada, *Green's Functions in the Theory of Ordinary Differential Equations*, Springer Briefs in Mathematics, 2014.

[2] A. Cabada, L. Saavedra *Disconjugacy characterization by means of spectral* $(k, n-k)$ *problems*, Appl. Math. Lett. 52 (2016), 21-29.

[3] A. Cabada, L. Saavedra,*The eigenvalue Characterization for the constant Sign Green's functions of* $(k, n-k)$ *problems*, Boundary Value Problems (2016:44) 35pp.

[4] W. A. Coppel, *Disconjugacy. Lecture Notes in Mathematics*, Vol. 220. Springer-Verlag, Berlin-New York, 1971.

[5] U. Elias, Necessary conditions and sufficient conditions for disfocality and disconjugacy of a differential equation. Pacific J. Math. 81 (1979), 379–397.

[6] Z. Nehari, A disconjugacy criterion for self-adjoint linear differential equations. J. Math. Anal. Appl. 35 (1971), 591–599.

[7] Z. Nehari, Disconjugate linear differential operators. Trans. Amer. Math. Soc. 129 (1967), 500–516.

[8] A. Zettl, *A constructive characterization of disconjugacy*, Bull. Amer. Math. Soc. 81 (1975), 145–147.

[9] A. Zettl, *A characterization of the factors of ordinary linear differential operators*, Bull. Amer. Math. Soc. 80 (1974), 498–499.