First order differential equations with piecewise constant arguments and nonlinear boundary value conditions[☆]

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Abstract

This paper is devoted to the study of differential equations with piecewise constant arguments coupled with nonlinear boundary value conditions. Under suitable assumptions on the data of the equation, by means of the method of (weakly coupled) lower and upper solutions, we derive the existence of extremal solutions and extremal quasi-solutions. Moreover some results are given concerning the uniqueness of solutions. Furthermore, we deduce some maximum principles related to the linear equation which allow us to develop the monotone iterative method. Some illustrative examples are also presented.

Keywords: Nonlinear boundary value conditions, comparison results, piecewise constant arguments, lower and upper solutions.

1. Introduction

The study of differential equations with piecewise constant arguments has been treated widely in the literature. This type of equation, in which techniques of differential and difference equations are combined, models, among others, some biological phenomena (see [1, 6] and references therein), the stabilization of hybrid control systems with feedback discrete controller [8], or damped oscillators [13]. The first studies in this field have been given in [4, 10], after this, some papers related with stability, oscillation properties and existence of periodic solutions have been treated by several authors (see [5] for details).

In [3] the authors construct the Green's function related to the linear operator $x'(t) + mx(t) + Mx([t])$, from where they obtain some maximum principles in the space of periodic solutions depending of the values of the real parameters m and M. Such operators have been also studied in [12] with initial value conditions. The method of lower and upper solutions has been employed in [14] to derive existence of periodic solutions of a first order nonlinear equation with piecewise

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constant argument. This method, as well as the method of weakly coupled lower and upper solutions, is applied in [7] to deduce existence results of a first order nonlinear boundary value problem involving a differential equation with continuous delay. Our arguments combine the techniques used in [7] for equations with continuous delay with the ones developed in [3] for linear first order piecewise equations.

The paper is organized as follows, in section 2 we present the main tools that we will use in the rest of the paper. Section 3 is devoted to obtain the unique solution of the associated linear problem, from which we derive comparison results for operator $x'(t) + m\,x(t) + M\,x([t])$. In section 4, we present results concerning the existence of extremal quasi-solutions and the uniqueness of solution in the presence of weakly coupled lower and upper solutions. We formulate, in section 5, existence results of extremal (unique) solutions lying between a pair of well ordered lower and upper solutions. Some examples are presented to point out the novelty and applicability of the given results.

2. Preliminaries

We study, in this paper, the following boundary value problem

$$x'(t) = f(t, x(t), x([t])) \equiv Fx(t), \quad t \in I = [0, T], \quad 0 = g(x(0), x(T)), \tag{1}$$

where T is a positive real number, $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $[\cdot]$ designates the greatest integer function.

Let N_T be such that

$$N_T := \begin{cases} [T], & \text{if } [T] < T, \\ [T] - 1, & \text{if } [T] = T, \end{cases}$$

then, for each $n \in \{0, 1, \ldots, N_T\}$, we define $I_n := [n, n+1)$ for $n < N_T$ and $I_{N_T} := [N_T, T]$.

We will denote by Λ the set of all functions $y : I \to \mathbb{R}$ that are continuous on I_n for all $n \in \{0, \ldots, N_T\}$ and there exists $y(t^-) \in \mathbb{R}$ for all $t \in \{1, 2, \ldots, N_T\}$. If $y \in \Lambda$, we understand that $y(t) = y(t^+)$ for all $t \in \{0, 1, \ldots, N_T\}$.

We will denote by Ω the set of all functions $x : I \to \mathbb{R}$ that are continuous on I and verify that there exists $x' \in \Lambda$.

A function $x : I \to \mathbb{R}$ is said to be a solution of the boundary value problem (1) if $x \in \Omega$ and satisfies (1), taking $x'(t) = x'(t^+)$ for all $t \in \{0, 1, \ldots, N_T\}$.

Given $u, v \in \Omega$, we will say that $u \le v$ if and only if $u(t) \le v(t)$ for all $t \in I$. In this case, we shall define $[u, v] := \{x \in \Omega \,|\, u \le x \le v\}$.

We will use, in the discussion of the problem, several properties of the function $h_{M,m} : \mathbb{R} \to \mathbb{R}$ defined by (see [3])

$$h(t) := h_{M,m}(t) = \begin{cases} e^{-mt} - \dfrac{M}{m}(1 - e^{-mt}), & \text{if } m \neq 0, \\ 1 - Mt, & \text{if } m = 0. \end{cases} \tag{2}$$

It is easy to verify that $h'_{M,m}(t) = -(m + M)e^{-mt}$ for all t and, as an immediate consequence, $h_{M,m}$ is strictly monotone increasing on $\mathbb{R}$ for $M < -m$ and strictly monotone decreasing on $\mathbb{R}$ if $M > -m$. Obviously, when $M = -m$, $h_{M,m}$ is a constant function equals to 1. Clearly, for all m, $M \in \mathbb{R}$, we have that $h(0) = 1$. Moreover, we denote by $C := h(1)$, which is bigger than 1 or less than 1 whenever $m + M < 0$ or $m + M > 0$ respectively.

3. The linear problem: comparison results.

In this section we analyze the following linear initial value problem

$$x'(t) + m\,x(t) + M\,x([t]) = \sigma(t), \quad t \in I = [0, T],\ x(0) = x_0, \tag{3}$$

where $\sigma \in \Lambda$, and m, M and T are real constants such that $T > 0$.

At a first moment we prove that this problem has a unique solution.

Theorem 3.1. *Problem (3) has a unique solution for any $x_0 \in \mathbb{R}$.*

Proof. For any $x_0 \in \mathbb{R}$, problem (3) can be rewritten as a family of initial value problems of ordinary differential equations on the intervals I_n, $n \in \{0, 1, \ldots, N_T\}$, that is to say,

$$x'(t) + m\,x(t) + M\,x([t]) = \sigma(t), \quad t \in I_n,\ x(n) = x_n. \tag{4}$$

Since $x([t]) = x_n$ for all $t \in I_n$, it is obvious that the unique solution of (4) is given by

$$x(t) = x_n h(t-n) + \int_n^t \sigma(s) e^{-m(t-s)}\, ds. \tag{5}$$

Because of the continuity of x, we arrive at

$$x_{n+1} = x(n+1) = C\,x_n + g_n,\ \forall\, n \in \{0, \ldots, N_T - 1\}, \tag{6}$$

where

$$g_n = \int_n^{n+1} \sigma(s) e^{-m(n+1-s)}\, ds.$$

Solving the linear difference equation (6) we have

$$x_n = C^n\, x_0 + \sum_{j=0}^{n-1} C^{n-1-j} g_j,\ \forall\, n \in \{1, \ldots, N_T\}. \tag{7}$$

Therefore, the unique solution of problem (3) is given by

$$x(t) = \begin{cases} x_0\, h(t) + \displaystyle\int_0^t \sigma(s) e^{-m(t-s)} ds, & \text{if } t \in [0,1], \\ x_1\, h(t-1) + \displaystyle\int_1^t \sigma(s) e^{-m(t-s)} ds, & \text{if } t \in [1,2], \\ \qquad \vdots & \\ x_{N_T}\, h(t - N_T) + \displaystyle\int_{N_T}^t \sigma(s) e^{-m(t-s)} ds, & \text{if } t \in [N_T, T], \end{cases} \tag{8}$$

where x_n is given by expression (7) for all $n \in \{1, \ldots, N_T\}$. □

In order to obtain existence results for problem (1) we will use monotone iterative techniques. It is very well known that a fundamental tool to treat this kind of problems consists in maximum principles of the linear operator studied above. From the form of the solution of problem (3) we can deduce the following.

Lemma 3.2. *Let $\sigma \in \Lambda$ be a non-positive function on I and $x_0 \leq 0$. Then, if the following condition holds:*

$$M \leq b_T(m) := \begin{cases} \max\left\{\dfrac{m}{e^m-1}, \dfrac{m}{e^{mT}-1}\right\}, & \text{if } m \neq 0, \\ \max\left\{1, \dfrac{1}{T}\right\}, & \text{if } m = 0, \end{cases} \tag{9}$$

the unique solution of problem (3) is non-positive on I.

Moreover, this is an optimal condition in the following sense: if $M > b_T(m)$ then, for each $x_0 < 0$, there exists a non-positive function $\sigma \in \Lambda$ for which the unique solution of problem (3) changes its sign on I. Also, whenever $M > b_T(m)$, for each non-positive function $\sigma \in \Lambda$, there exists $x_0 < 0$ for which the unique solution of problem (3) changes its sign on I.

Proof. First, we suppose that $T \geq 1$. In this case we have

$$b_T(m) \equiv b(m) = \begin{cases} \dfrac{m}{e^m-1}, & \text{if } m \neq 0, \\ 1, & \text{if } m = 0. \end{cases}$$

Since $\sigma \leq 0$ on I and $x_0 \leq 0$, from (7) we deduce that, if $C \geq 0$ then $x_n \leq 0$ for all $n \in \{0, 1, \ldots, N_T\}$. Furthermore, for all $n \in \{0, 1, \ldots, N_T\}$, the solution $x(t)$ of problem (3) on I_n is given by the expression (5), which is a sum of non-positive terms whenever $C \geq 0$, so that $x(t) \leq 0$ for all $t \in I$. But it is easy to check that $C \geq 0$ if and only if $M \leq b(m)$, so the proof is finished in this case.

In order to prove the optimal character of the previous condition, let us suppose that $M > b(m)$, and so $C < 0$.

For any $x_0 < 0$ fixed, taking $\sigma \equiv 0$, we obtain that $x_1 = C x_0 > 0$, thus the solution change its sign.

On the other hand, for any fixed non-positive function $\sigma \in \Lambda$ we have that, either $\sigma \equiv 0$ or $k = \min_{t \in I}\{\sigma(t)\} < 0$. In case of $\sigma \equiv 0$, reasoning as above, we obtain that for any $x_0 < 0$ the solution changes its sign. If $\sigma \not\equiv 0$ then, taking $x_0 < k/(M - b(m)) < 0$, we have

$$\begin{aligned} e^m x_1 &= e^m \left(C\, x_0 + \int_0^1 \sigma(s)\, e^{-m(1-s)}\, ds \right) \\ &\geq e^m \left(C\, x_0 + \int_0^1 k\, e^{-m(1-s)} ds \right) = x_0 \frac{b(m) - M}{b(m)} + \frac{k}{b(m)} > -\frac{k}{b(m)} + \frac{k}{b(m)} = 0. \end{aligned}$$

Hence, $x_1 > 0$ and, therefore, the solution also changes its sign.

Let us now consider that $T < 1$, in this case function b_T is given by

$$b_T(m) = \begin{cases} \dfrac{m}{e^{mT}-1}, & \text{if } m \neq 0, \\ \dfrac{1}{T}, & \text{if } m = 0. \end{cases}$$

Moreover, the unique solution $x(t)$ of problem (3) on $I = I_0 = [0, T]$ is given by (5), which is a sum of non-positive terms whenever $h(T) \geq 0$. But $h(T) \geq 0$ if and only if $M \leq b_T(m)$. So that, this case is proved.

Arguing as in the case of $T \geq 1$, one can verify the optimal character of this condition. □

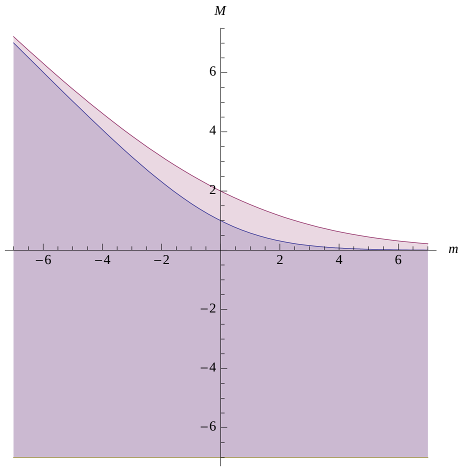

Figure 1: Region, in the plane of parameters m and M, where Lemma 3.2 is satisfied. In dark, the region for the case $T \geq 1$. In pale, the region that has to be added to the previous one in case of $T < 1$ (it is represented the case $T = 1/2$).

Remark 3.1. In Figure 1 we can view the region where the previous maximum principle reaches.

We also obtain next result, that improves Lemma 3.2 in case of $x_0 = 0$.

Lemma 3.3. *Let $\sigma \in \Lambda$ be a non-positive function and $x_0 = 0$. Then the unique solution of problem (3) is non-positive on I if one of the following conditions holds (b_T defined by (9)):*

1. $T \leq 1$.
2. $T > 1$ *and* $M \leq b_{T-1}(m)$.

Moreover, condition 2 is optimal in the following sense: if $T > 1$ and $M > b_{T-1}(m)$, then there exists a non-positive function $\sigma \in \Lambda$ for which the unique solution of problem (3) changes its sign.

Proof. In case of $T \leq 1$, as $x_0 = 0$, from (5) we know that the unique solution of problem (3) on $I = I_0 = [0, T]$ is given by $x(t) = \int_0^t \sigma(s) e^{-m(t-s)}\, ds$. Therefore, if $\sigma \leq 0$ then $x \leq 0$.

In case of $T > 1$, reasoning as above, it is clear that $x_1 \leq 0$. Thus, by applying Lemma 3.2 on the interval $[1, T]$, we arrive at condition 2. □

4. Weakly coupled lower and upper solutions

Now, we present the method of weakly lower and upper solutions applied to problem (1). First, we introduce the following definitions.

Definition 4.1. We say that $\alpha, \beta \in \Omega$ are weakly coupled lower and upper solutions of problem (1) if the following inequalities hold

$$\begin{aligned} \alpha'(t) &\leq F\alpha(t), \quad t \in I, \quad g(\alpha(0), \beta(T)) \leq 0, \\ \beta'(t) &\geq F\beta(t), \quad t \in I, \quad g(\beta(0), \alpha(T)) \geq 0. \end{aligned} \tag{10}$$

Definition 4.2. We say that $x,\ y\in\Omega$ are coupled quasi-solutions of problem (1) if the following equalities hold

$$\begin{aligned} x'(t) &= Fx(t), \quad t\in I, \quad g(x(0),y(T))=0,\\ y'(t) &= Fy(t), \quad t\in I, \quad g(y(0),x(T))=0. \end{aligned} \tag{11}$$

Next, we establish sufficient conditions for problem (1) to have extremal coupled quasi-solutions.

Theorem 4.1. *Let $\alpha,\ \beta\in\Omega$ be weakly coupled lower and upper solutions of problem (1) such that $\alpha\le\beta$. In addition, let us assume that the following assumptions are fulfilled:*

(H_1) *There exist real constants m and M such that $M\le b_T(m)$ and, for all $t\in I$,*

$$f(t,y,z)-f(t,\bar{y},\bar{z})\le m(\bar{y}-y)+M(\bar{z}-z) \tag{12}$$

when $\alpha(t)\le y\le\bar{y}\le\beta(t)$ and $\alpha([t])\le z\le\bar{z}\le\beta([t])$.

(H_2) *For all $y\in[\alpha(0),\beta(0)]$ function $g(y,\cdot)$ is non-decreasing on the interval $[\alpha(T),\beta(T)]$, that is to say,*

$$g(y,z)\le g(y,\bar{z}) \quad \text{if } \alpha(T)\le z\le\bar{z}\le\beta(T) \text{ and } \alpha(0)\le y\le\beta(0). \tag{13}$$

(H_3) *For all $z\in[\alpha(T),\beta(T)]$ function $g(\cdot,z)$ satisfies the following one sided Lipschitz condition: there exists a real constant $K>0$ such that*

$$g(\bar{y},z)-g(y,z)\le K\,(\bar{y}-y) \quad \text{if } \alpha(0)\le y\le\bar{y}\le\beta(0) \text{ and } \alpha(T)\le z\le\beta(T). \tag{14}$$

Then problem (1) has coupled quasi-solutions $\rho\le\gamma$ in $[\alpha,\beta]$.

Moreover, ρ and γ are extremal in the following sense: if μ and η are coupled quasi-solutions of problem (1) such that $\alpha\le\mu\le\eta\le\beta$, then $\rho\le\mu\le\eta\le\gamma$.

Proof. We first consider, for $\xi,\varphi\in[\alpha,\beta]$ given, the following initial value problems

$$x'(t)+m\,x(t)+M\,x([t]) = F\xi(t)+m\,\xi(t)+M\xi([t]), \quad t\in I, \quad x(0)=\xi(0)-\tfrac{1}{K}\,g(\xi(0),\varphi(T)), \tag{15}$$

$$x'(t)+m\,x(t)+M\,x([t]) = F\varphi(t)+m\,\varphi(t)+M\varphi([t]), \quad t\in I, \quad x(0)=\varphi(0)-\tfrac{1}{K}\,g(\varphi(0),\xi(T)). \tag{16}$$

Theorem 3.1 allows us to define the operators $A,B:[\alpha,\beta]\times[\alpha,\beta]\to\Omega$ as $A(\xi,\varphi):=$ unique solution of problem (15), and $B(\xi,\varphi):=$ unique solution of problem (16).

We will first see that, if ξ and φ are weakly coupled lower and upper solutions of problem (1) such that $\alpha\le\xi\le\varphi\le\beta$, then

$$\alpha\le\xi\le u\le v\le\varphi\le\beta, \tag{17}$$

where $u=A(\xi,\varphi)$ and $v=B(\xi,\varphi)$.

Put $r=\xi-u$ and $s=v-\varphi$. Since ξ y φ are weakly coupled lower and upper solutions, we obtain

$$\begin{aligned} r'(t)+m\,r(t)+M\,r([t]) &= \xi'(t)-F\xi(t)\le 0,\\ r(0) &= \tfrac{1}{K}\,g(\xi(0),\varphi(T))\le 0, \end{aligned}$$

and

$$s'(t) + m\, s(t) + M\, s([t]) \; = F\varphi(t) - \varphi'(t) \leq 0,$$
$$s(0) = -\tfrac{1}{K}\, g(\varphi(0), \xi(T)) \leq 0.$$

Consequently, Lemma 3.2 yields $\xi \leq u$ and $v \leq \varphi$.

We now put $w = u - v$. From (H_1), we derive

$$w'(t) + m\, w(t) + M\, w([t]) \; = F\xi(t) - F\varphi(t) + m(\xi(t) - \varphi(t)) + M(\xi([t]) - \varphi([t])) \leq 0,$$

while, from (H_2) and (H_3), we get

$$\begin{aligned} w(0) &= \xi(0) - \varphi(0) + \tfrac{1}{K} \left(g(\varphi(0), \xi(T)) - g(\xi(0), \varphi(T)) \right) \\ &\leq \xi(0) - \varphi(0) + \tfrac{1}{K} \left(g(\varphi(0), \xi(T)) - g(\xi(0), \xi(T)) \right) \leq \xi(0) - \varphi(0) - \xi(0) + \varphi(0) = 0. \end{aligned}$$

By Lemma 3.2 we deduce that $u \leq v$. So that, it is proved that inequality (17) holds.

We will now show that u and v are weakly coupled lower and upper solutions of problem (1). Note that, reasoning as above, (H_1) drives to

$$\begin{aligned} u'(t) - Fu(t) &= F\xi(t) - Fu(t) + \; m(\xi(t) - u(t)) + M(\xi([t]) - u([t])) \leq 0, \\ v'(t) - Fv(t) &= F\varphi(t) - Fv(t) + \; m(\varphi(t) - v(t)) + M(\varphi([t]) - v([t])) \geq 0, \end{aligned}$$

while, making use of (H_2) and (H_3), we deduce

$$0 = K(u(0) - \xi(0)) + g(\xi(0), \varphi(T)) \geq g(u(0), \varphi(T)) \geq g(u(0), v(T))$$

and

$$0 = K(v(0) - \varphi(0)) + g(\varphi(0), \xi(T)) \leq g(v(0), \xi(T)) \leq g(v(0), u(T)).$$

Therefore, it is proved that u and v are weakly coupled lower and upper solutions of problem (1).

Next step is to build up two sequences which converge to the extremal quasi-solutions of problem (1). To this end, take $\alpha_0 = \alpha$ and $\beta_0 = \beta$, and define the sequences $\{\alpha_l\}$ and $\{\beta_l\}$ as $\alpha_{l+1} = A(\alpha_l, \beta_l)$ and $\beta_{l+1} = B(\alpha_l, \beta_l)$. By means of (17), it is immediate to verify that

$$\alpha(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_l(t) \leq \cdots \leq \beta_l(t) \cdots \leq \beta_1(t) \leq \beta(t), \quad \forall t \in I, \quad \forall l \in \mathbb{N},$$

so that the sequences $\{\alpha_l\}$ and $\{\beta_l\}$ are uniformly bounded. From the statement of problems (15) and (16), and due to the properties of f, it is easy to check that the sequences $\{\alpha_l'\}$ and $\{\beta_l'\}$ are also bounded. Thus, sequences $\{\alpha_l\}$ and $\{\beta_l\}$ are equicontinuous on $C(I)$, and the Ascoli-Arzelà theorem guarantees the existence of subsequences $\{\alpha_{l_j}\} \subset \{\alpha_l\}$ and $\{\beta_{l_j}\} \subset \{\beta_l\}$ which converge uniformly to their limits ρ, $\gamma \in C(I)$ respectively. Because of both of sequences $\{\alpha_l\}$ and $\{\beta_l\}$ are monotone, we conclude that they converge uniformly to ρ and γ respectively.

From the definition of operators A and B we know that, for all $l \in \mathbb{N}$, it is verified

$$\alpha_{l+1}(0) = \alpha_l(0) - \tfrac{1}{K} g(\alpha_l(0), \beta_l(T)) \quad \text{and} \quad \beta_{l+1}(0) = \beta_l(0) - \tfrac{1}{K} g(\beta_l(0), \alpha_l(T)).$$

Consequently, due to the continuity of g and passing to the limit, we derive

$$g(\rho(0), \gamma(T)) = 0 = g(\gamma(0), \rho(T)). \tag{18}$$

Moreover, from (8) we know that, on each interval I_n and all $n \in \{0, \ldots, N_T\}$, it is satisfied

$$\alpha_{l+1}(t) = \alpha_{l+1}(n)\, h(t-n) + \int_n^t (f(s, \alpha_l(s), \alpha_l(n)) + m\, \alpha_l(s) + M\alpha_l(n))e^{-m(t-s)}\, ds,$$

$$\beta_{l+1}(t) = \beta_{l+1}(n)\, h(t-n) + \int_n^t (f(s, \beta_l(s), \beta_l(n)) + m \beta_l(s) + M\beta_l(n))e^{-m(t-s)}\, ds.$$

So that, because of the continuity of f and passing to the limit, we deduce that

$$\begin{aligned} \rho(t) &= \rho(n)\, h(t-n) + \int_n^t (f(s, \rho(s), \rho(n)) + m\, \rho(s) + M\rho(n))e^{-m(t-s)}\, ds, \\ \gamma(t) &= \gamma(n)\, h(t-n) + \int_n^t (f(s, \gamma(s), \gamma(n)) + m\, \gamma(s) + M\gamma(n))e^{-m(t-s)}\, ds, \end{aligned} \tag{19}$$

on each interval I_n and all $n \in \{0, \ldots, N_T\}$.

Therefore, ρ and γ are in Ω and so, from (18) and (19), we conclude that ρ and γ are coupled quasi-solutions of problem (1).

Finally, we will show the extremal character of the coupled quasi-solutions ρ and γ. To do this, let $\mu \leq \eta$ be coupled quasi-solutions of problem (1) in $[\alpha, \beta]$. We will prove, by mathematical induction, that $\alpha_l \leq \mu \leq \eta \leq \beta_l$ for all $l \in \mathbb{N}$.

It is obvious that this property holds for $l = 0$. Let us suppose that it is verified for a given $l \in \mathbb{N}$. Put $r = \alpha_{l+1} - \mu$, then

$$r'(t) = F\alpha_l(t) - F\mu(t) + m(\alpha_l(t) - \alpha_{l+1}(t)) + M(\alpha_l([t]) - \alpha_{l+1}([t])) \leq -mr(t) - Mr([t]),$$

and

$$\begin{aligned} r(0) &= \alpha_l(0) - \mu(0) + \tfrac{1}{K}(g(\mu(0), \eta(T)) - g(\alpha_l(0), \beta_l(T))) \\ &\leq \alpha_l(0) - \mu(0) + \tfrac{1}{K}(g(\mu(0), \beta_l(T)) - g(\alpha_l(0), \beta_l(T))) \leq 0, \end{aligned}$$

which implies, by Lemma 3.2, that $\alpha_{l+1} \leq \mu$ on I.

In an analogous way we deduce that $\beta_{l+1} \geq \eta$ on I.

As consequence, passing to the limit, we conclude $\rho \leq \mu \leq \eta \leq \gamma$ on I. This proves the extremal character of the coupled quasi-solutions ρ and γ in $[\alpha, \beta]$. □

In case of weakly coupled lower and upper solutions match at the starting point of the interval I, we can prove the existence of extremal solutions for problem (1). Even more, we can do it under weaker assumptions on the boundary conditions. The result is the following.

Theorem 4.2. *Let α, $\beta \in \Omega$ be weakly coupled lower and upper solutions of problem (1) such that $\alpha \leq \beta$ and $\alpha(0) = \beta(0)$. Let us suppose that assumptions (H_1) and (H_2) hold.*

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. As $\alpha(0) = \beta(0)$, it is immediate to verify that assumption (H_3) holds for any $K > 0$. Then, all of assumptions of theorem 4.1 are fulfilled and, in consequence, there exist extremal coupled quasi-solutions $\rho \leq \gamma$ of problem (1) in $[\alpha, \beta]$. Thus, $\alpha(0) = \rho(0) = \gamma(0) = \beta(0)$, from where we deduce

$$0 = g(\rho(0), \gamma(T)) = g(\gamma(0), \gamma(T)),$$

and

$$0 = g(\gamma(0), \rho(T)) = g(\rho(0), \rho(T)).$$

In consequence we deduce that functions ρ and γ are extremal solutions of problem (1) in $[\alpha, \beta]$. $\square$

Example 4.1. The following problem

$$x'(t) = e^{x([t])} - \frac{1}{t+1}\, x(t), \quad t \in I = [0, \tfrac{\pi}{2}], \quad 0 = x(\tfrac{\pi}{2}) \cos^2(\tfrac{\pi}{2} x(0)), \tag{20}$$

has extremal solutions in the sector $[\,-t-1, \frac{3}{2}\,t-1]$.

Proof. This problem is a particular case of (1) with

$$f(t, y, z) = e^z - \frac{1}{t+1}\, y \quad \text{and} \quad g(y, z) = z \cos^2(\tfrac{\pi}{2}\, y).$$

Put $\alpha(t) = -t-1$ and $\beta(t) = \frac{3}{2}\,t-1$. Then, we get

$$\begin{aligned} \alpha'(t) = -1 \le F\alpha(t) &= \begin{cases} e^{-1} + 1, & \text{if } t \in [0, 1), \\ e^{-2} + 1, & \text{if } t \in [1, \frac{\pi}{2}], \end{cases} \\ \beta'(t) = \frac{3}{2} \ge F\beta(t) &= \begin{cases} e^{-1} - \frac{3t-2}{2(t+1)}, & \text{if } t \in [0, 1), \\ e^{\frac{1}{2}} - \frac{3t-2}{2(t+1)}, & \text{if } t \in [1, \frac{\pi}{2}], \end{cases} \\ g(\alpha(0), \beta(\tfrac{\pi}{2})) &= g(-1, \tfrac{3\pi}{4} - 1) = 0 \le 0, \\ g(\beta(0), \alpha(\tfrac{\pi}{2})) &= g(-1, -\tfrac{\pi}{2} - 1) = 0 \ge 0. \end{aligned}$$

Due to this, α and β are weakly coupled lower and upper solutions or problem (20). Clearly, assumption (H_2) is verified and, with $m = 1$ and $M = 0$, assumption (H_1) holds.

Since $\alpha(0) = -1 = \beta(0)$, we can apply Theorem 4.2. So that, we conclude that problem (20) has extremal solutions in $[\alpha, \beta]$. $\square$

Remark 4.1. In fact, problem (20) has a unique solution in $[\alpha, \beta]$. It is because $\alpha(0) = -1 = \beta(0)$, therefore the solutions of this problem in $[\alpha, \beta]$ are the solutions of the following initial value problem

$$x'(t) + \frac{x(t)}{t+1} = e^{x([t])}, \quad t \in I = [0, \tfrac{\pi}{2}], \quad x(0) = -1. \tag{21}$$

Under analogous arguments to the ones developed in the proof of Theorem 3.1, we rewrite problem (21) as a family of two initial value problems on the intervals $I_0 = [0, 1)$ and $I_1 = [1, \frac{\pi}{2}]$. First, we solve problem

$$x'(t) + \frac{x(t)}{t+1} = e^{-1}, \quad t \in I_0 = [0, 1), \quad x(0) = -1,$$

which has a unique solution given by

$$x(t) = \frac{t^2 + 2t - 2e}{2e(t+1)}.$$

Taking into account that the solution of problem (21) has to be continuous, and that $x(1) = \frac{3-2e}{4e}$, we now solve the problem

$$x'(t) + \frac{x(t)}{t+1} = e^{\frac{3-2e}{4e}}, \quad t \in I_1 = [1, \tfrac{\pi}{2}], \quad x(1) = \frac{3-2e}{4e},$$

which, also, has a unique solution given by

$$x(t) = \frac{e^{\frac{2e+3}{4e}}\left(t^2 + 2t - 3\right) - 2e + 3}{2e(t+1)}.$$

Therefore, the unique solution of problem (21), as well as of problem (20) in $[\alpha, \beta]$, is given by

$$x(t) = \begin{cases} \dfrac{t^2 + 2t - 2e}{2e(t+1)}, & \text{if } 0 \le t < 1, \\[2ex] \dfrac{e^{\frac{2e+3}{4e}}(t^2 + 2t - 3) - 2e + 3}{2e(t+1)}, & \text{if } 1 \le t \le \frac{\pi}{2}. \end{cases}$$

The graphs of the unique solution and the weakly coupled lower an upper solutions of problem (20) are shown in the following figure.

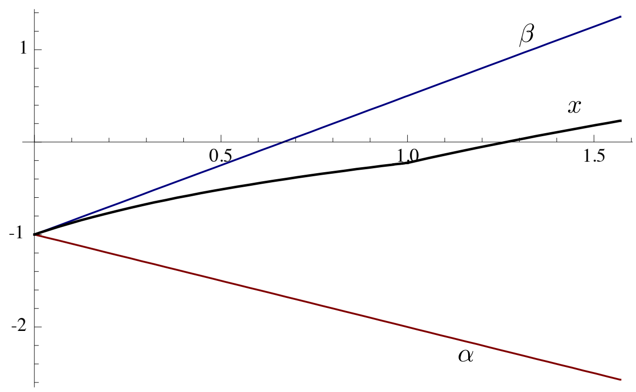

Figure 2: Graphs of α, β and x. Respectively, the weakly coupled lower and upper solutions, and the unique solution of problem (20).

Assuming additional conditions, our next results guarantee that problem (1) is uniquely solvable on the sector formed by a pair of well ordered weakly coupled lower and upper solutions.

Theorem 4.3. *Let $\alpha, \beta \in \Omega$ be weakly coupled lower and upper solutions of problem (1) such that $\alpha \le \beta$. Let us assume that (H_1), (H_2) and (H_3) are satisfied. In addition, let us assume that*

(H_4) *There exist real constants K_1 and K_2 satisfying $K \ge K_1 > 0$ (constant K given in (H_3)), $K_2 \ge 0$ and*

$$g(y, \bar{z}) - g(\bar{y}, z) \le -K_1(\bar{y} - y) + K_2(\bar{z} - z) \tag{22}$$

if $\alpha(0) \le y \le \bar{y} \le \beta(0)$ and $\alpha(T) \le z \le \bar{z} \le \beta(T)$.

(H_5) *For all* $t \in I$ *and* $y \in [\alpha(t), \beta(t)]$*, function* $f(t, y, \cdot)$ *is non–increasing in* $[\alpha([t]), \beta([t])]$*. Moreover there exists a real constant* p *such that* $m + p \geq 0$ *and, for all* $t \in I$ *and* $z \in [\alpha([t]), \beta([t])]$*, the following inequalities hold*

$$f(t, y, z) - f(t, \bar{y}, z) \geq -p(\bar{y} - y) \quad \text{if } \alpha(t) \leq y \leq \bar{y} \leq \beta(t). \tag{23}$$

(H_6) *It is verified that*

$$K_2\, e^{pT} < K_1. \tag{24}$$

Then problem (1) has a unique solution in $[\alpha, \beta]$*.*

Proof. Theorem 4.1 guarantees the existence of $\rho \leq \gamma$ extremal weakly coupled quasi-solutions of problem (1) in $[\alpha, \beta]$. Now, denoting by $q = \rho - \gamma$ and making use of (H_4) we obtain

$$0 = g(\rho(0), \gamma(T)) - g(\gamma(0), \rho(T)) \leq K_1 q(0) - K_2 q(T). \tag{25}$$

On the other hand, for all $t \in I$, (H_5) drives to

$$q'(t) = f(t, \rho(t), \rho([t])) - f(t, \gamma(t), \gamma([t])) \geq f(t, \rho(t), \gamma([t])) - f(t, \gamma(t), \gamma([t])) \geq p\, q(t).$$

Now, since $q' \in \Lambda$, we deduce that

$$q(t) \geq q(n)\, e^{p\,(t-n)}, \quad t \in I_n.$$

From the continuity of function q, by induction in n, we arrive at

$$q(t) \geq q(0)\, e^{p\,t}, \quad t \in I. \tag{26}$$

and, consequently, from (25) we get

$$0 \leq q(0)(K_1 - K_2 e^{pT}).$$

This and (24) imply that $q(0) \geq 0$, so that (26) yields $\rho \geq \gamma$.

Therefore, $\rho \equiv \gamma \in [\alpha, \beta]$ is a solution of problem (1). Because of the extremal character of ρ and γ in $[\alpha, \beta]$, the solution has to be unique in $[\alpha, \beta]$. □

Remark 4.2. We note that assumptions (H_1) and (H_5) can be satisfied simultaneously. If we analyze the behaviour of function f with respect to the second and third variables, then we use these assumptions to show that

$$-m \leq \frac{f(t, \bar{y}, z) - f(t, y, z)}{\bar{y} - y} \leq p, \quad \text{if } \alpha(t) \leq y < \bar{y} \leq \beta(t) \text{ and } \alpha([t]) \leq z \leq \beta([t]),$$

$$-M \leq \frac{f(t, y, \bar{z}) - f(t, y, z)}{\bar{z} - z} \leq 0, \quad \text{if } \alpha([t]) \leq z < \bar{z} \leq \beta([t]) \text{ and } \alpha(t) \leq y \leq \beta(t).$$

In case of f being differentiable, these conditions turn into

$$-m \leq \frac{\partial f}{\partial y}(t, y, z) \leq p \quad \text{if } \alpha(t) \leq y \leq \beta(t) \text{ and } \alpha([t]) \leq z \leq \beta([t]),$$

$$-M \leq \frac{\partial f}{\partial z}(t, y, z) \leq 0 \quad \text{if } \alpha(t) \leq y \leq \beta(t) \text{ and } \alpha([t]) \leq z \leq \beta([t]).$$

It happens the same with assumptions (H_2), (H_3) and (H_4). If we now analyze the behaviour of g, from those assumptions, we obtain the following conditions

$$K_1 \le \frac{g(\bar{y}, z) - g(y, z)}{\bar{y} - y} \le K \quad \text{if } \alpha(0) \le y < \bar{y} \le \beta(0) \text{ and } \alpha(T) \le z \le \beta(T),$$

$$0 \le \frac{g(y, \bar{z}) - g(y, z)}{\bar{z} - z} \le K_2 \quad \text{if } \alpha(T) \le z < \bar{z} \le \beta(T) \text{ and } \alpha(0) \le y \le \beta(0), \tag{27}$$

which, in case of g being differentiable, change into

$$K_1 \le \frac{\partial g}{\partial y}(y, z) \le K \quad \text{if } \alpha(0) \le y \le \beta(0) \text{ and } \alpha(T) \le z \le \beta(T),$$

$$0 \le \frac{\partial g}{\partial z}(y, z) \le K_2 \quad \text{if } \alpha(0) \le y \le \beta(0) \text{ and } \alpha(T) \le z \le \beta(T).$$

Example 4.2. The following problem

$$x'(t) = \tfrac{1}{2} t\, x^2(t) - \tfrac{1}{3} x([t]), \quad t \in I = [0, \tfrac{3}{2}], \quad 0 = \tfrac{11}{2}\, x(0) + e^{x(3/2)}, \tag{28}$$

has a unique solution in the sector $[-\frac{1}{4}, \frac{1}{16}]$.

Proof. This problem is a particular case of problem (1) with

$$f(t, y, z) = \tfrac{1}{2}\, t\, y^2 - \tfrac{1}{3} z \quad \text{and} \quad g(y, z) = \tfrac{11}{2}\, y + e^z.$$

One can verify that $\alpha(t) \equiv -\frac{1}{4}$ and $\beta(t) \equiv \frac{1}{16}$ are weakly coupled quasi-solutions of problem (28). If we take $m = \frac{3}{8}$, $M = \frac{1}{3}$, $K = K_1 = \frac{11}{2}$, $K_2 = e^{1/16}$ and $p = \frac{3}{32}$, then f and g verify the assumptions $(H_1) - (H_5)$.

Therefore, Theorem 4.3 guarantees that problem (28) has a unique solution in $[-\frac{1}{4}, \frac{1}{16}]$. □

Theorem 4.4. *Let $\alpha, \beta \in \Omega$ be weakly coupled lower and upper solutions or problem (1) such that $\alpha \le \beta$ and $\alpha(0) = \beta(0)$. Let us suppose that assumptions (H_1), (H_2) and (H_5) hold.*

Then problem (1) has a unique solution in $[\alpha, \beta]$.

Proof. Theorem 4.2 provides the existence of extremal solutions $\rho \le \gamma$ in $[\alpha, \beta]$. Take $q = \rho - \gamma$, using (H_5) and reasoning as in Theorem 4.3, we derive that (26) holds.

Since $q(0) = \rho(0) - \gamma(0) = 0$, we deduce that $\rho \ge \gamma$. Therefore, $\rho \equiv \gamma$ is the unique solution of the considered problem in $[\alpha, \beta]$. □

Remark 4.3. Notice that the function f in Example 5.1 does not verify the assumption (H_5). Nevertheless, the problem presented in that example has a unique solution. It is clear, therefore, that such an assumption is only a sufficient condition for the uniqueness of solutions.

Remark 4.4. From Remark 4.2 it is immediate to verify that the anti–periodic boundary value conditions $x(0) = -x(T)$, characterized by $g(y, z) = y + z$ fulfill conditions (H_2), (H_3) and (H_4). In consequence all of the previous existence results can be applied to this kind of condition when $p \le 0$.

However, the initial value conditions $x(0) = x_0$ ($g(y, z) = y - x_0$) are covered by (H_2) and (H_3) but not by (H_4). So, we cannot ensure that Theorem 4.3 holds in this case. The same comment is valid for the terminal problem $x(T) = x_T$ ($g(y, z) = z - x_T$), provided that $\alpha(T) = \beta(T) = x_T$ and for the periodic problem $x(0) = x(T)$ ($g(y, z) = z - y$), when $\alpha(0) = \beta(0) = \alpha(T) = \beta(T)$.

5. Lower and upper solutions

In this section, by assuming different monotonicity assumptions on the boundary value conditions, that include, among others, the periodic ones, we deduce some existence results for problem (1) by means of the method of lower and upper solutions.

Definition 5.1. We say that $\alpha \in \Omega$ is a lower solution of problem (1) if the following inequalities hold

$$\alpha'(t) \leq F\alpha(t), \quad t \in I, \quad g(\alpha(0), \alpha(T)) \leq 0. \tag{29}$$

In analogous way, we say that $\beta \in \Omega$ is an upper solution of problem (1) if the following inequalities hold

$$\beta'(t) \geq F\beta(t), \quad t \in I, \quad g(\beta(0), \beta(T)) \geq 0. \tag{30}$$

Theorem 5.1. *Let α, $\beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$. Let us suppose that assumptions (H_1) and (H_3) hold. In addition, let us assume that*

(H_2') *For all $y \in [\alpha(0), \beta(0)]$ function $g(y, \cdot)$ is non–increasing on the interval $[\alpha(T), \beta(T)]$, that is to say,*

$$g(y, z) \geq g(y, \bar{z}) \quad \text{if } \alpha(T) \leq z \leq \bar{z} \leq \beta(T) \text{ and } \alpha(0) \leq y \leq \beta(0). \tag{31}$$

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. Let us consider, for $\xi \in [\alpha, \beta]$ given, the following initial value problem

$$x'(t) + m\, x(t) + M\, x([t]) = F\xi(t) + m\xi(t) + M\xi([t]), \quad t \in I, \quad x(0) = \xi(0) - \tfrac{1}{K}\, g(\xi(0), \xi(T)). \tag{32}$$

Thus, Theorem 3.1 allows us to define operator $L : [\alpha, \beta] \to \Omega$ as $L\xi :=$ unique solution of problem (32).

Then, putting $\alpha_0 = \alpha$ and $\beta_0 = \beta$, we define the following two sequences, $\{\alpha_l\}$ and $\{\beta_l\}$, as $\alpha_{l+1} = L\,\alpha_l$ and $\beta_{l+1} = L\beta_l$, for all $l \in \mathbb{N}$.

Arguing as in the proof of Theorem 4.1, we conclude that $\{\alpha_l\}$ and $\{\beta_l\}$ are two monotone sequences that converge uniformly to the extremal solutions of problem (1) in $[\alpha, \beta]$. □

Corollary 5.2. *Let α, $\beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$ and $\alpha(T) = \beta(T)$. Let us suppose that assumptions (H_1) and (H_3) are satisfied.*

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. Since $\alpha(T) = \beta(T)$, it is obvious that assumption (H_2') holds. Therefore, all of assumptions of Theorem 5.1 are fulfilled. □

Remark 5.1. It is clear that if $\alpha(T) = \beta(T)$ the concept of weakly coupled lower and upper solutions of problem (1) is equivalent to the existence of a pair of well ordered lower and an upper solutions of problem (1). Furthermore, $x, y \in [\alpha, \beta]$ are coupled quasi-solutions of problem (1) if and only if $x, y \in [\alpha, \beta]$ are solutions of problem (1).

On the other hand, it is clear that conditions (H_2) and (H_2') are, in that case, both satisfied.

Because of this, Corollary 5.2 remains valid as a corollary of Theorem 4.1 in case α and β are weakly coupled lower and upper solutions of problem (1) such that $\alpha(T) = \beta(T)$.

Corollary 5.3. *Let α, $\beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$ and $\alpha(0) = \beta(0)$. Let us suppose that assumptions (H_1) and (H_2') are satisfied.*

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. As $\alpha(0) = \beta(0)$, it is immediate to verify that (H_3) holds for any $K > 0$. So that, all of assumptions of Theorem 5.1 hold. □

Corollary 5.4. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$ and $\alpha(0) = \beta(0)$. Let us suppose that assumption (H_2') holds. Additionally, let us assume that $T < 1$ and there exists a real constant m such that, for all $t \in I$,*

$$f(t, y, z) - f(t, \bar{y}, z) \le m(\bar{y} - y) \quad \text{if } \alpha(t) \le y \le \bar{y} \le \beta(t). \tag{33}$$

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. Since $T < 1$ and $\alpha(0) = \beta(0)$, from (33) it is clear that assumption (H_1) holds for any real constant M such that $M \le b_T(m)$. Thus, we can apply Corollary 5.3. □

Corollary 5.5. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$, $\alpha(0) = \beta(0)$ and $\alpha(T) = \beta(T)$. Let us suppose that assumption (H_1) holds.*

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. Since $\alpha(0) = \beta(0)$ and $\alpha(T) = \beta(T)$, it is easy to verify that assumption (H_2') holds, and assumption (H_3) is fulfilled for any $K > 0$. Due to this, Theorem 5.1 guarantees the existence of extremal solutions of problem (1). □

Corollary 5.6. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$, $\alpha(0) = \beta(0)$ and $\alpha(T) = \beta(T)$. Let us assume that $T < 1$ and condition (33) holds.*

Then problem (1) has extremal solutions in $[\alpha, \beta]$.

Proof. From $\alpha(T) = \beta(T)$, it is clear that assumption (H_2') holds, so that, we can apply Corollary 5.4. □

Remark 5.2. Since $\alpha(T) = \beta(T)$ is an assumption both in Corollary 5.5 and in Corollary 5.6, we know (Remark 5.1) that α and β are weakly coupled lower and upper solutions of problem (1). Therefore, these results remain valid as corollaries of Theorem 4.2.

If we add some extra conditions to Theorem 5.1, we can guarantee the uniqueness of solutions of problem (1) under the presence of a pair of well ordered lower and upper solutions.

Theorem 5.7. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$. Let us suppose that assumptions (H_1), (H_2'), (H_3), (H_5) and (H_6) are fulfilled. Additionally, let us assume that*

(H_4') *There exist real constants K_1, K_2 such that $K \ge K_1 > 0$, $K_2 \ge 0$, and*

$$g(y, z) - g(\bar{y}, \bar{z}) \le -K_1(\bar{y} - y) + K_2(\bar{z} - z) \tag{34}$$

if $\alpha(0) \le y \le \bar{y} \le \beta(0)$ and $\alpha(T) \le z \le \bar{z} \le \beta(T)$.

Then problem (1) has a unique solution in $[\alpha, \beta]$.

Proof. We omit the proof, since it is analogous to that of Theorem 4.3. □

Remark 5.3. We note that assumptions (H_2'), (H_3) and (H_4') are congruent. Although assumptions (H_2) and (H_4) have been changed by, respectively, assumptions (H_2') and (H_4'), the analysis made in Remark 4.2 about the behaviour of g with respect to the first variable remains valid. Nevertheless, when we analyze the behaviour or g with respect to the second variable, we now have

$$-K_2 \leq \frac{g(y,\bar{z}) - g(y,z)}{\bar{z} - z} \leq 0 \quad \text{if } \alpha(T) \leq z < \bar{z} \leq \beta(T) \text{ and } \alpha(0) \leq y \leq \beta(0), \tag{35}$$

which, in case of being g differentiable, becomes

$$-K_2 \leq \frac{\partial g}{\partial z}(y,z) \leq 0 \quad \text{if } \alpha(T) \leq z \leq \beta(T) \text{ and } \alpha(0) \leq y \leq \beta(0).$$

On the other hand, whenever $\alpha(T) = \beta(T)$, it is obvious that both conditions (35) and (27) are fulfilled. Therefore, in this case, if the following assumption

(H_4'') There exists a real constant K_1 such that $K \geq K_1 > 0$ satisfying

$$g(y,z) - g(\bar{y},z) \leq -K_1(\bar{y} - y) \quad \text{if } \alpha(0) \leq y \leq \bar{y} \leq \beta(0), \tag{36}$$

holds, then assumptions (H_4) and (H_4') are both satisfied.

Theorem 5.8. *Let α, $\beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$ and $\alpha(T) = \beta(T)$. Let us suppose that assumptions (H_1), (H_3), (H_4'') and (H_5) are satisfied.*

Then problem (1) has a unique solution in $[\alpha,\beta]$.

Proof. Corollary 5.2 guarantees the existence of $\rho \leq \gamma$, extremal solutions of problem (1) in $[\alpha,\beta]$.

Putting $q = \rho - \gamma \leq 0$, and making use of (H_5) as in Theorem 4.3, we obtain inequality (26). If we suppose that $q(0) \neq 0$, then $q(0) < 0$. So that, from (H_4''), we deduce

$$0 = g(\rho(0),\rho(T)) - g(\gamma(0),\rho(T)) \leq K_1\, q(0) < 0,$$

which is a contradiction.

Therefore, $q(0) = 0$ and, hence, $\rho \equiv \gamma$ is the unique solution of problem (1) in $[\alpha,\beta]$. □

Remark 5.4. We notice that, if $\alpha(T) = \beta(T)$ function g obviously satisfies assumption (H_2). In such a case, if assumption (H_4'') holds, assumption (H_4) is also fulfilled. Furthermore, as it is explained in Remark 5.1, the concepts of weakly coupled lower and upper solution and lower and upper solution match in this case, as well as the concepts of quasi-solution and solution.

Because of this, Theorem 5.8 remains valid in case of α and β are weakly coupled lower and upper solutions or problem (1).

Theorem 5.9. *Let α, $\beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$ and $\alpha(0) = \beta(0)$. Let us suppose that assumptions (H_1), (H_2') and (H_5) are satisfied.*

Then problem (1) has a unique solution in $[\alpha,\beta]$.

Proof. This result is analogous to Theorem 4.4, in this case for lower and upper solutions, so we omit the proof. □

Example 5.1. The following problem

$$x'(t) = -\frac{t^2}{4} + x^2(t) - \frac{3}{2}x(t) - \frac{2}{5}x([t]) + \frac{9}{16}, \quad t \in I = [0, \tfrac{3}{2}], \quad 0 = e^{x(3/2)} \cos\left(\tfrac{2\pi}{3} x(0)\right), \tag{37}$$

has extremal solutions in the sector $[-\frac{t}{2} + \frac{3}{4}, \frac{t}{2} + \frac{3}{4}]$.

Proof. This problem is a particular case of (1) with

$$f(t, y, z) = -\frac{t^2}{4} + y^2 - \frac{3y}{2} - \frac{2z}{5} + \frac{9}{16} \quad \text{and} \quad g(y, z) = e^z \cos\left(\tfrac{2\pi}{3} y\right).$$

Functions $\alpha(t) = -\frac{t}{2} + \frac{3}{4}$ and $\beta(t) = \frac{t}{2} + \frac{3}{4}$ are, respectively, lower and upper solutions of problem (37) such that $\alpha \le \beta$ and $\alpha(0) = \beta(0) = \frac{3}{4}$.

Put $m = p = \frac{3}{2}$ and $M = \frac{2}{5}$, thus f and g verify assumptions (H_1), (H_2') and (H_5). Therefore, all of assumptions of Theorem 5.9 are satisfied. □

Corollary 5.10. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$, $\alpha(0) = \beta(0)$ and $\alpha(T) = \beta(T)$. Let us suppose that assumptions (H_1) and (H_5) are satisfied.*

Then problem (1) has a unique solution in $[\alpha, \beta]$.

Proof. Since $\alpha(T) = \beta(T)$, assumption (H_2') is obviously satisfied. Thus, we can apply Theorem 5.9. □

Corollary 5.11. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$ and $\alpha(0) = \beta(0)$. Let us assume that $T < 1$ and condition (33) holds. In addition, let us suppose that assumptions (H_2') and (H_5) are satisfied.*

Then problem (1) has a unique solution in $[\alpha, \beta]$.

Proof. Since $\alpha(0) = \beta(0)$ and $T < 1$, from (33) we derive that assumption (H_1) holds. Then we can apply Theorem 5.9. □

Corollary 5.12. *Let $\alpha, \beta \in \Omega$ be, respectively, lower and upper solutions of problem (1) such that $\alpha \le \beta$, $\alpha(0) = \beta(0)$ and $\alpha(T) = \beta(T)$. Let us assume that $T < 1$ and condition (33) holds. Additionally, let us suppose that assumption (H_5) is satisfied.*

Then problem (1) has a unique solution in $[\alpha, \beta]$.

Proof. Reasoning as in the previous corollary, we know that assumption (H_1) holds. So that, we can apply Corollary 5.10. □

Remark 5.5. In this case, from remarks 4.2 and 5.3 we know that the periodic problem $x(0) = x(T)$ ($g(y, z) = y - z$) maps the conditions (H_2), (H_3) and (H_4'). So we deduce that all the existence results presented in this section are valid for the periodic problem, whenever $p \le 0$. The same assertion, in this case for all $p \in \mathbb{R}$, is valid for the initial value conditions $x(0) = x_0$ ($g(y, z) = y - x_0$).

Since the terminal problem $x(T) = x_T$ ($g(y, z) = -z + x_T$) satisfies conditions (H_2) and (H_3) but it does not satisfy condition (H_4'), we know that Theorem 5.1 holds in this situation. However the uniqueness Theorems 5.7 and 5.8 can be applied only if $\alpha(0) = \beta(0)$ and $\alpha(T) = \beta(T) = x_T$.

The anti–periodic boundary value conditions $x(0) = -x(T)$ ($g(y, z) = y + z$) can be treated under our formulation if and only if $\alpha(0) = \beta(0) = -\beta(T) = -\alpha(T)$.

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