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RELATIVE ROTA-BAXTER OPERATORS, MODULES AND PROJECTIONS

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Abstract. The present article is devoted to introduce, in a braided monoidal setting, the notion of module over a relative Rota-Baxter operator. It is proved that there exists an adjunction between the category of modules associated to an invertible relative Rota-Baxter operator and the category of modules associated to a Hopf brace, which induces an equivalence by assuming certain additional hypothesis. Moreover, the notion of projection between relative Rota-Baxter operators is defined, and it is proved that those which are called "strong" give rise to a module according to the previous definition in the cocommutative setting.

Keywords: braided monoidal category; Hopf algebra; Hopf brace; relative Rota-Baxter operator

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1. Introduction

An important issue in the field of mathematical physics consists in finding solutions of the Quantum Yang-Baxter Equation (QYBE). The QYBE appeared in the 1970s in the field of quantum and statistical mechanics (see [3] and [24]), and a complete classification of its solutions has not yet been obtained at this stage. A solution of such equation is an automorphism $c\colon V\otimes V\to V\otimes V$, where V is a vector space over the field $\mathbb{F}$ and $\otimes$ denotes the tensor product of vector spaces over $\mathbb{F}$, which

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satisfies the following equality:

$$\text{(QYBE)} \qquad (c\otimes id_V)\circ(id_V\otimes c)\circ(c\otimes id_V)=(id_V\otimes c)\circ(c\otimes id_V)\circ(id_V\otimes c).$$

Applications of the QYBE are diverse and cover different areas of mathematics and physics (to mention a few: knot theory, non-commutative geometry or quantum groups, between others), which implies that its solutions have been studied hardly and from different points of view.

The first method to systematically obtain solutions of the QYBE was proposed by Drinfeld in [7]. He introduced the notion of quasitriangular Hopf algebra and proved that their associated modules give rise to solutions of the QYBE. It is an important result that if H is a Hopf algebra, the category of modules over its Drinfeld's double, $D(H)$, which is a quasitriangular Hopf algebra, is equivalent to the category of Yetter-Drinfeld modules over H, see [17], Theorem IX.5.2 and Section XIII.5. So, as a consequence of these facts, every Yetter-Drinfeld module over a Hopf algebra H induce a solution of such equation.

Later, Bespalov, Majid and Radford in [4], [19] and [20], respectively, stated that given a Hopf algebra H such that its antipode is an isomorphism, there exists a categorical equivalence between the category of Hopf algebras in ${}^H_H\mathsf{YD}$ and the category of Hopf algebra projections over H, where ${}^H_H\mathsf{YD}$ denotes the category of Yetter-Drinfeld modules over H, and a Hopf algebra projection over H is a pair of Hopf algebra morphisms $f\colon H\to B$ and $g\colon B\to H$, satisfying that $g\circ f=id_H$. Therefore, taking into account the above mentioned results, to construct a solution of the QYBE it is enough to have a Hopf algebra projection.

On the other hand, in [8] Drinfeld proposed to focus on the task of obtained set-theoretical solutions of the QYBE, which are those, where a solution c is a linear map induced by a mapping $\overline{c}\colon X\times X\to X\times X$, where X is a set (in this situation, V is the $\mathbb{F}$-vector space spanned by X). The study of this kind of solutions was pursued subsequently by several authors, for example, Etingof, Schedler and Soloviev in [9] or Gateva-Ivanova in [12]. With this aim, the notion of brace was introduced by Rump in [21] and then generalized by Guarnieri and Vendramin in [16] for the non-abelian setting obtaining the concept of skew brace. A skew brace is a pair of groups $(G,\star)$ and $(G,\circ)$ which satisfies the compatibility condition

$$g\circ(h\star t)=(g\circ h)\star g^{-1}\star(g\circ t) \tag{1.1}$$

for all $g,h,t\in G$ and where g^{-1} denotes the inverse of g with regard to the group structure $(G,\star)$. In such paper the authors obtain that every skew brace induces a set-theoretical solution of the QYBE not always involutive, i.e., the inverse of such solution c is not necessarily c.

The quantum version of a skew brace is what is known by a Hopf brace, objects introduced by Angiono, Galindo and Vendramin in [2]. Then, a Hopf brace is a pair of Hopf algebras sharing the underlying coalgebra structure, $H_1 = (H, 1, \cdot, \varepsilon, \Delta, \lambda)$ and $H_2 = (H, 1_\circ, \circ, \varepsilon, \Delta, S)$, which satisfy the following compatibility condition between the products:

$$g \circ (h \cdot t) = (g_1 \circ h) \cdot \lambda(g_2) \cdot (g_3 \circ t) \tag{1.2}$$

for all $g, h, t \in H$. As it occurs with skew braces, the subclass of the cocommutative Hopf braces also induces solutions of the QYBE, see [2], Corollary 2.4.

Recently, by combining the two previously mentioned approaches to obtaining solutions for the QYBE, Fernández Vilaboa et al. in [11] studied the theory of projections in the Hopf brace setting, introducing the notion of Hopf brace projection, a suitable definition for the category of Yetter-Drinfeld modules over a Hopf brace and extending the correspondence of Radford-Majid-Bespalov to this framework.

Since the appearence of Hopf braces, another structures related with them have emerged. The first ones we will mention are invertible 1-cocycles, which appeared initially in Angiono's et al. article, see [2]. Mixing Theorem 1.12 of [2] and Theorem 3.2 of [15], it was proved that the categories of invertible 1-cocycles and Hopf braces are equivalent. Also, Brzeziński in [6] generalized the Hopf brace structure by modifying (1.2) using a cocycle, which gave rise to the notion of Hopf truss. Besides, recently in [18], Li, Sheng and Tang introduced the categories of post-Hopf algebras and relative Rota-Baxter operators. Regarding post-Hopf algebras, Fernández Vilaboa, González Rodrguez and Ramos Pérez proved in [10] that the category of Hopf braces and a certain subcategory of post-Hopf algebras are isomorphic in the cocommutative setting. On the other hand, relative Rota-Baxter operators are a generalization of the notion of Rota-Baxter operator given by Goncharov in [13] for cocommutative Hopf algebras. In Theorem 3.3 of [18] it was proved that there exists an adjunction between the category of post-Hopf algebras and the category of relative Rota-Baxter operators in a cocommutative context and, as a consequence of the above mentioned isomorphism between Hopf braces and post-Hopf algebras, this adjunction also holds between the category of Hopf braces and the category of relative Rota-Baxter operators under cocommutativity.

Looking at all the background, it is not unreasonable to tackle the study of projections for the objects mentioned in the previous paragraphs and whose connection with Hopf braces is strong. So, this paper is devoted to study the theory of modules and projections for relative Rota-Baxter operators.

The paper is organized as follows: After the initial preliminary section, where we are going to fix the notation and remember the basic necessary notions for the development of the article, Section 3 is devoted to introducing the concept of module over a relative Rota-Baxter operator, see Definition 3.8. In this section, we will first prove that such category is symmetric monoidal under cocommutativity conditions and assuming that the base category is symmetric, see Theorem 3.10. Then, some functorial results are proved in order to state a correspondence between the category of modules over a certain Hopf brace and the category of modules over a relative Rota-Baxter operator. Using such correspondence, in Theorem 3.18 it is shown that there exists an adjunction between the category of modules over an invertible relative Rota-Baxter operator and the category of modules over such Hopf brace, which induce an equivalence by assuming some additional conditions, see Theorem 3.20. Section 4 is devoted to introducing the category of projections for relative Rota-Baxter operators, see Definition 4.3. As happens between the category of Hopf braces and relative Rota-Baxter operators (see Theorem 3.7), in Theorem 4.13 we will see that there also exists an adjunction between their respective projection categories. To conclude the paper, we will define what a strong projection of relative Rota-Baxter operators is and we will prove that every such projection in the cocommutative setting gives rise to a module in the sense of Definition 3.8, as happens in the classical theory of Hopf algebra projections.

2. PRELIMINARIES

From now, on C denotes a strict braided monoidal category with tensor product $\otimes$, unit object K and braiding c. Considering that it is well known that every non-strict monoidal category is monoidal equivalent to a strict one, we can assume without loss of generality that the category C is strict and then we omit explicitly the associativity and unit constraints. Thus, the results proved in this paper for objects and morphisms in C remain valid for every non-strict braided monoidal category which would include, for example, the category $\mathbb{F}$-Vect of vector spaces over a field $\mathbb{F}$, the category R-Mod of left modules over a commutative ring R, or the category Set of sets. If for all $M, N \in \mathsf{C}$ the braiding satisfies that $c_{N,M} \circ c_{M,N} = id_{M\otimes N}$, where id denotes the identity morphism, we will say that C is symmetric. In what follows, for simplicity of notation, given objects M, N, P in C and a morphism $f\colon M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

Definition 2.1. An algebra in C is a triple $A = (A, \eta_A, \mu_A)$, where A is an object in the category C and $\eta_A\colon K \to A$ (unit), $\mu_A\colon A \otimes A \to A$ (product) are morphisms in C such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f\colon A \to B$ in C is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$.

If A, B are algebras in C, the tensor product $A \otimes B$ is also an algebra in C, where $\eta_{A\otimes B} := \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} := (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in C is a triple $D = (D, \varepsilon_D, \delta_D)$, where D is an object in C and $\varepsilon_D \colon D \to K$ (counit), $\delta_D \colon D \to D \otimes D$ (coproduct) are morphisms in C such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, a morphism $f \colon D \to E$ in C is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$.

Given D, E coalgebras in C, the tensor product $D \otimes E$ is a coalgebra in C, where $\varepsilon_{D\otimes E} := \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} := (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Definition 2.2. Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra and $A = (A, \eta_A, \mu_A)$ an algebra in C. By $\mathrm{Hom}(D, A)$ we denote the set of morphisms $f \colon D \to A$ in C. With the convolution operation $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$, $\mathrm{Hom}(D, A)$ is an algebra, where the unit element is $\eta_A \circ \varepsilon_D = \varepsilon_D \otimes \eta_A$.

Definition 2.3. Let A be an algebra. The pair (M, φ_M) is a left A-module if M is an object in C and $\varphi_M \colon A \otimes M \to M$ is a morphism in C satisfying $\varphi_M \circ (\eta_A \otimes M) = id_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. Given two left A-modules (M, φ_M) and (N, φ_N), $f \colon M \to N$ is a morphism of left A-modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$. We will denote the category of left A-modules by ${}_A\mathsf{Mod}$.

Let D be a coalgebra. The pair (M, ϱ_M) is a left D-comodule if M is an object in C and $\varrho_M \colon M \to D \otimes M$ is a morphism in C satisfying $(\varepsilon_D \otimes M) \circ \varrho_M = id_M$, $(D \otimes \varrho_M) \circ \varrho_M = (\delta_D \otimes M) \circ \varrho_M$. Given two left D-comodules (M, ϱ_M) and (N, ϱ_N), $f \colon M \to N$ is a morphism of left D-comodules if $(D \otimes f) \circ \varrho_M = \varrho_N \circ f$. We will denote the category of left D-comodules by ${}^D\mathsf{Comod}$.

In a similar way we can define the notions of right A-module and right D-comodule.

Definition 2.4. We say that X is a bialgebra in C if (X, η_X, μ_X) is an algebra, $(X, \varepsilon_X, \delta_X)$ is a coalgebra, and ε_X and δ_X are algebra morphisms (equivalently, η_X and μ_X are coalgebra morphisms). Moreover, if there exists a morphism $\lambda_X \colon X \to X$ in C, called the antipode of X, satisfying that λ_X is the inverse of id_X in $\mathrm{Hom}(X, X)$, i.e.,

$$id_X * \lambda_X = \eta_X \circ \varepsilon_X = \lambda_X * id_X, \tag{2.1}$$

we say that X is a Hopf algebra. A morphism of Hopf algebras is an algebra-coalgebra morphism. Note that if $f \colon X \to Y$ is a Hopf algebra morphism, the following equality holds:

$$\lambda_Y \circ f = f \circ \lambda_X. \tag{2.2}$$

With the composition of morphisms in C we can define a category whose objects are Hopf algebras and whose morphisms are morphisms of Hopf algebras. We denote this category by Hopf.

A Hopf algebra is commutative if $\mu_X \circ c_{X,X} = \mu_X$ and cocommutative if $c_{X,X} \circ \delta_X = \delta_X$. In both cases, $\lambda_X \circ \lambda_X = id_X$ and also, by Corollary 5 of [22], the identity

$$c_{X,X} \circ c_{X,X} = id_{X \otimes X} \tag{2.3}$$

holds.

If X is a Hopf algebra, we have the following relevant properties of its antipode λ_X: It is antimultiplicative and anticomultiplicative

$$\lambda_X \circ \mu_X = \mu_X \circ (\lambda_X \otimes \lambda_X) \circ c_{X,X}, \tag{2.4}$$

$$\delta_X \circ \lambda_X = c_{X,X} \circ (\lambda_X \otimes \lambda_X) \circ \delta_X, \tag{2.5}$$

and leaves the unit and counit invariant, i.e.,

$$\lambda_X \circ \eta_X = \eta_X, \tag{2.6}$$

$$\varepsilon_X \circ \lambda_X = \varepsilon_X. \tag{2.7}$$

So, it is a direct consequence of these identities that if X is commutative, then λ_X is an algebra morphism and if X is cocommutative, then λ_X is a coalgebra morphism.

In the following definitions we recall the notion of left (co)module (co)algebra.

Definition 2.5. Let X be a Hopf algebra. An algebra A is said to be a left X-module algebra if (A, φ_A) is a left X-module and η_A, μ_A are morphisms of left X-modules, i.e.,

$$\varphi_A \circ (X \otimes \eta_A) = \varepsilon_X \otimes \eta_A, \tag{2.8}$$

$$\varphi_A \circ (X \otimes \mu_A) = \mu_A \circ \varphi_{A\otimes A}, \tag{2.9}$$

where $\varphi_{A\otimes A} := (\varphi_A \otimes \varphi_A) \circ (X \otimes c_{X,A} \otimes A) \circ (\delta_X \otimes A \otimes A)$ is the left action on $A \otimes A$.

On the other hand, A is said to be a left X-comodule algebra if (A, ϱ_A) is a left X-comodule and η_A and μ_A are morphisms of left X-comodules, i.e.,

$$\varrho_A \circ \eta_A = \eta_X \otimes \eta_A, \tag{2.10}$$

$$\varrho_A \circ \mu_A = (X \otimes \mu_A) \circ \varrho_{A\otimes A}, \tag{2.11}$$

where $\varrho_{A\otimes A} := (\mu_X \otimes A \otimes A) \circ (H \otimes c_{A,X} \otimes A) \circ (\varrho_A \otimes \varrho_A)$ is the coaction on $A \otimes A$. Equivalently, (A, ϱ_A) is a left X-comodule algebra if and only if ϱ_A is an algebra morphism.

Definition 2.6. Let X be a Hopf algebra. A coalgebra D is said to be a left X-module coalgebra if (D, φ_D) is a left X-module and ε_D, δ_D are morphisms of left X-modules, in other words, the following equalities hold:

$$\varepsilon_D \circ \varphi_D = \varepsilon_X \otimes \varepsilon_D, \tag{2.12}$$

$$\delta_D \circ \varphi_D = \varphi_{D\otimes D} \circ (X \otimes \delta_D). \tag{2.13}$$

Equivalently, (D, φ_D) is a left X-module coalgebra if and only if φ_D is a coalgebra morphism.

Finally, a coalgebra D is said to be a left X-comodule coalgebra if (D, ϱ_D) is a left X-comodule and ε_D and δ_D are morphisms of left X-comodules, i.e.,

$$(X \otimes \varepsilon_D) \circ \varrho_D = \eta_X \otimes \varepsilon_D, \tag{2.14}$$

$$(X \otimes \delta_D) \circ \varrho_D = \varrho_{D\otimes D} \circ \delta_D. \tag{2.15}$$

Example 2.7. Every Hopf algebra X in C has a structure of left module algebra over itself with the so called adjoint action

$$\varphi_X^{\mathrm{ad}} := \mu_X \circ (\mu_X \otimes \lambda_X) \circ (X \otimes c_{X,X}) \circ (\delta_X \otimes X).$$

If X is also cocommutative, then $(X, \varphi_X^{\mathrm{ad}})$ is a left X-module algebra-coalgebra. Moreover, X is a left X-comodule coalgebra with the adjoint coaction $\varrho_X^{\mathrm{ad}} := (\mu_X \otimes X) \circ (X \otimes c_{X,X}) \circ (\delta_X \otimes \lambda_X) \circ \delta_X$.

3. Relative Rota-Baxter operators and their modules

The present section of this paper is devoted to introducing what a module over a relative Rota-Baxter operator is. Relative Rota-Baxter operators have been introduced considering the underlying category $\mathsf{C} = \mathbb{F}\text{-}\mathsf{Vect}$ by Li, Sheng and Tang in [18] as a generalization of Rota-Baxter operators defined by Goncharov in [13] for cocommutative Hopf algebras.

First of all we start by remembering the notion and basic properties of relative Rota-Baxter operators, as well as the strong relationship between these structures and Hopf braces, and we show that the category formed by this objects is symmetric monoidal.

After that we will focus on the study of modules over a relative Rota-Baxter operator, giving a definition that allows us to show that there exists an adjunction between the category of modules over a Hopf brace (using the definition of these objects introduced by Gonzlez in [14]) and the category of modules over an invertible relative Rota-Baxter operator assuming cocommutativity. The section will finish by proving that, under certain additional hypothesis, the previous adjunction gives rise to an equivalence of categories.

Definition 3.1. Let $H=(H,\eta_H,\mu_H,\varepsilon_H,\delta_H,\lambda_H)$ and $B=(B,\eta_B,\mu_B,\varepsilon_B,\delta_B,\lambda_B)$ be Hopf algebras in C such that (H,φ_H) is a left B-module algebra-coalgebra. We will say that a coalgebra morphism $T\colon H\to B$ is a relative Rota-Baxter operator if the following condition holds:

$$\mu_B\circ(T\otimes T)=T\circ\mu_H\circ(H\otimes(\varphi_H\circ(T\otimes H)))\circ(\delta_H\otimes H). \tag{3.1}$$

In what follows we will denote relative Rota-Baxter operators by $\left(\begin{smallmatrix} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{smallmatrix}\right)$.

If $\left(\begin{smallmatrix} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & A & \\ L & \downarrow & ,\varphi_A \\ & D & \end{smallmatrix}\right)$ are relative Rota-Baxter operators, a morphism between them is a pair (f,h), where $f\colon H\to A$ and $h\colon B\to D$ are Hopf algebra morphisms and the following conditions hold:

$$L\circ f=h\circ T, \tag{3.2}$$

$$f\circ\varphi_H=\varphi_A\circ(h\otimes f). \tag{3.3}$$

Considering the natural composition of morphisms, relative Rota-Baxter operators give rise to a category that we will denote by rRB. Moreover, we will denote by $\mathsf{rRB}^{\star}$ the full subcategory of rRB whose objects are relative Rota-Baxter operators $\left(\begin{smallmatrix} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{smallmatrix}\right)$ such that H is cocommutative, and by $\mathsf{coc\text{-}rRB}$ to the full subcategory of $\mathsf{rRB}^{\star}$ satisfying that both Hopf algebras involved, H and B, are cocommutative. The objects constituting the latter subcategory will henceforth be called cocommutative relative Rota-Baxter operators.

An important property of relative Rota-Baxter operators is that they preserve the unit. This will be proven in the following result.

Lemma 3.2. *If* $\left(\begin{smallmatrix} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{smallmatrix}\right)$ *is a relative Rota-Baxter operator, then*

$$\eta_B=T\circ\eta_H. \tag{3.4}$$

Proof. By (3.1), the condition of morphism of left B-modules for η_H, the condition of coalgebra morphism for T and the (co)unit property, we obtain that

$$\mu_B\circ((T\circ\eta_H)\otimes(T\circ\eta_H))=T\circ\eta_H. \tag{3.5}$$

Then we have that

$\eta_B = \eta_B \circ \varepsilon_B \circ T \circ \eta_H$ (by the condition of coalgebra morphism for T and (co)unit properties)

$= \mu_B \circ (\lambda_B \otimes B) \circ \delta_B \circ T \circ \eta_H$ (by (2.1) for B)

$= \mu_B \circ (\lambda_B \otimes B) \circ (T \otimes T) \circ (\eta_H \otimes \eta_H)$ (by the condition of coalgebra morphism for T and η_H)

$= \mu_B \circ (\lambda_B \otimes B) \circ ((T \circ \eta_H) \otimes (\mu_B \circ ((T \circ \eta_H) \otimes (T \circ \eta_H))))$ (by (3.5))

$= \mu_B \circ ((\mu_B \circ (\lambda_B \otimes B) \circ \delta_B \circ T \circ \eta_H) \otimes (T \circ \eta_H))$ (by associativity of μ_B and the condition of coalgebra morphism T and η_H)

$= T \circ \eta_H$ (by (2.1) for B, the condition of coalgebra morphism for T and (co)unit properties).

□

If C is symmetric, rRB admits a structure of symmetric monoidal category as can be seen in what follows.

Theorem 3.3. *Let us assume* C *to be symmetric. The category of relative Rota-Baxter operators is a strict symmetric monoidal with tensor functor*

$$\otimes\colon \mathsf{rRB} \times \mathsf{rRB} \to \mathsf{rRB}$$

$$\left(\left(T\ \begin{matrix} H \\ \downarrow \\ B \end{matrix}\ ,\varphi_H\right), \left(L\ \begin{matrix} A \\ \downarrow \\ D \end{matrix}\ ,\varphi_A\right)\right) \mapsto \left(T\otimes L\ \begin{matrix} H\otimes A \\ \downarrow \\ B\otimes D \end{matrix}\ ,\varphi^t_{H\otimes A}\right),$$

where $\varphi^t_{H\otimes A} := (\varphi_H \otimes \varphi_A) \circ (B \otimes c_{D,H} \otimes A)$, *unit object* $\left(id_K\ \begin{matrix} K \\ \downarrow \\ K \end{matrix}\ ,id_K\right)$ *and symmetry given by*

$$\tau_{T,L} := (c_{H,A}, c_{B,D})\colon \left(T\otimes L\ \begin{matrix} H\otimes A \\ \downarrow \\ B\otimes D \end{matrix}\ ,\varphi^t_{H\otimes A}\right) \to \left(L\otimes T\ \begin{matrix} A\otimes H \\ \downarrow \\ D\otimes B \end{matrix}\ ,\varphi^t_{A\otimes H}\right).$$

Proof. When C is symmetric, if $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ and $A = (A, \eta_A, \mu_A, \varepsilon_A, \delta_A, \lambda_A)$ are Hopf algebras in C, then $H \otimes A = (H \otimes A, \eta_H \otimes \eta_A, \mu_{H\otimes A}, \varepsilon_H \otimes \varepsilon_A, \delta_{H\otimes A}, \lambda_H \otimes \lambda_A)$ is also a Hopf algebra in C.

Moreover, $(H\otimes A, \varphi_{H\otimes A})$ is a left $B\otimes D$-module algebra-coalgebra. Indeed, on the one side, the left module axioms are straightforward thanks to naturality of c. On the other side, it is easy to prove that $\varphi^t_{H\otimes A}\circ(B\otimes D\otimes\eta_H\otimes\eta_A)=\varepsilon_B\otimes\varepsilon_D\otimes\eta_H\otimes\eta_A$ and the condition of morphism of left $B\otimes D$-modules for $\mu_{H\otimes A}$ follows by

$$\begin{aligned}
&\mu_{H\otimes A}\circ(\varphi^t_{H\otimes A}\otimes\varphi^t_{H\otimes A})\circ(B\otimes D\otimes c_{B\otimes D,H\otimes A}\otimes H\otimes A)\\
&\qquad\circ(\delta_{B\otimes D}\otimes H\otimes A\otimes H\otimes A)\\
&\quad=((\mu_H\circ(\varphi_H\otimes\varphi_H)\circ(B\otimes c_{B,H}\otimes H)\circ(\delta_B\otimes H\otimes H))\otimes(\mu_A\circ(\varphi_A\otimes\varphi_A)\\
&\qquad\circ(D\otimes c_{D,A}\otimes A)))\circ(B\otimes((H\otimes c_{D,H}\otimes D\otimes A)\circ(c_{D,H}\otimes c_{D,H}\otimes A)\\
&\qquad\circ(D\otimes c_{D,H}\otimes H\otimes A)\circ(\delta_D\otimes H\otimes c_{A,H}))\otimes A)\\
&\qquad\text{(by naturality of c and C symmetric)}\\
&\quad=((\mu_H\circ(\varphi_H\otimes\varphi_H)\circ(B\otimes c_{B,H}\otimes H)\circ(\delta_B\otimes H\otimes H))\otimes(\mu_A\circ(\varphi_A\otimes\varphi_A)\\
&\qquad\circ(D\otimes c_{D,A}\otimes A)\circ(\delta_D\otimes A\otimes A)))\circ(B\otimes((H\otimes c_{D,H}\otimes A)\\
&\qquad\circ(c_{D,H}\otimes c_{A,H}))\otimes A)\ \text{(by naturality of c)}\\
&\quad=((\varphi_H\circ(B\otimes\mu_H))\otimes(\varphi_A\circ(D\otimes\mu_A)))\circ(B\otimes((H\otimes c_{D,H}\otimes A)\\
&\qquad\circ(c_{D,H}\otimes c_{A,H}))\otimes A)\ \text{(by the condition of morphism of B-modules for μ_H}\\
&\qquad\text{and the condition of morphism of D-modules for μ_A)}\\
&\quad=\varphi^t_{H\otimes A}\circ(B\otimes D\otimes\mu_{H\otimes A})\ \text{(by naturality of c).}
\end{aligned}$$

In addition, $(\varepsilon_H\otimes\varepsilon_A)\circ\varphi^t_{H\otimes A}=\varepsilon_B\otimes\varepsilon_D\otimes\varepsilon_H\otimes\varepsilon_A$ is straightforward while

$$\delta_{H\otimes A}\circ\varphi^t_{H\otimes A}=(\varphi^t_{H\otimes A}\otimes\varphi^t_{H\otimes A})\circ(B\otimes D\otimes c_{B\otimes D,H\otimes A}\otimes H\otimes A)\circ(\delta_{B\otimes D}\otimes\delta_{H\otimes A})$$

follows by

$$\begin{aligned}
\delta_{H\otimes A}\circ\varphi^t_{H\otimes A}&=(H\otimes c_{H,A}\otimes A)\circ(((\varphi_H\otimes\varphi_H)\circ(B\otimes c_{B,H}\otimes H)\circ(\delta_B\otimes\delta_H))\\
&\qquad\otimes((\varphi_A\otimes\varphi_A)\circ(D\otimes c_{D,A}\otimes A)\circ(\delta_D\otimes\delta_A)))\circ(B\otimes c_{D,H}\otimes A)\\
&\qquad\text{(by the condition of coalgebra morphism for φ_H and φ_A)}\\
&=(\varphi^t_{H\otimes A}\otimes\varphi^t_{H\otimes A})\circ(B\otimes D\otimes c_{B\otimes D,H\otimes A}\otimes H\otimes A)\circ(\delta_{B\otimes D}\otimes\delta_{H\otimes A})\\
&\qquad\text{(by naturality of c and C symmetric).}
\end{aligned}$$

Also note that, considering the standard structure of tensor coproduct if T and L are coalgebra morphisms, then $T\otimes L$ is also a coalgebra morphism. Therefore, to conclude the monoidal character of rRB we only have to compute that (3.1) holds. Indeed,

$$\begin{aligned}
&(T\otimes L)\circ\mu_{H\otimes A}\circ(H\otimes A\otimes(\varphi^t_{H\otimes A}\circ(T\otimes L\otimes H\otimes A)))\circ(\delta_{H\otimes A}\otimes H\otimes A)\\
&\quad=((T\circ\mu_H\circ(H\otimes(\varphi_H\circ(T\otimes H)))\circ(\delta_H\otimes H))\\
&\qquad\otimes(L\circ\mu_A\circ(A\otimes(\varphi_A\circ(L\otimes A)))\circ(\delta_A\otimes A)))\circ(H\otimes c_{A,H}\otimes A)\\
&\qquad\text{(by naturality of c and C symmetric)}
\end{aligned}$$

$= ((\mu_B \circ (T \otimes T)) \otimes (\mu_D \circ (L \otimes L))) \circ (H \otimes c_{A,H} \otimes A)$ (by (3.1) for T and L)
$= \mu_{B\otimes D} \circ ((T \otimes L) \otimes (T \otimes L))$ (by naturality of c).

On the other hand, $\tau_{T,L}$ is a symmetry for rRB because when the base category C is symmetric, $(c_{H,A}, c_{B,D})$ is a morphism in rRB between $\left(T{\otimes}L \begin{array}{c} H{\otimes}A \\ \downarrow \\ B{\otimes}D \end{array} , \varphi^{t}_{H\otimes A} \right)$ and $\left(L{\otimes}T \begin{array}{c} A{\otimes}H \\ \downarrow \\ D{\otimes}B \end{array} , \varphi^{t}_{A\otimes H} \right)$. ☐

After proving these general properties which relative Rota-Baxter operators satisfy, we will see that under suitable conditions there exists a functorial link between relative Rota-Baxter operators and Hopf braces. First we will remember the definition of Hopf brace and its main properties.

Definition 3.4. Let $H = (H, \varepsilon_H, \delta_H)$ be a coalgebra in C. Let us assume that there are two algebra structures (H, η_H^1, μ_H^1), (H, η_H^2, μ_H^2) defined on H and suppose that there exist two endomorphisms of H denoted by λ_H^1 and λ_H^2. We will say that

$$(H, \eta_H^1, \mu_H^1, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^1, \lambda_H^2)$$

is a Hopf brace in C if:

(i) $H_1 = (H, \eta_H^1, \mu_H^1, \varepsilon_H, \delta_H, \lambda_H^1)$ is a Hopf algebra in C,
(ii) $H_2 = (H, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^2)$ is a Hopf algebra in C,
(iii) the following equality holds:

$$\mu_H^2 \circ (H \otimes \mu_H^1) = \mu_H^1 \circ (\mu_H^2 \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H),$$

where

$$\Gamma_{H_1} := \mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes H).$$

For any Hopf brace $\eta_H^1 = \eta_H^2$ holds and therefore, because of this property, the expression of a Hopf brace is reduced to

$$(H, \eta_H, \mu_H^1, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^1, \lambda_H^2).$$

In the following lines, a Hopf brace will be denoted by $\mathbb{H} = (H_1, H_2)$ or in a simpler way by $\mathbb{H}$.

Definition 3.5. If $\mathbb{H}$ is a Hopf brace in C, we will say that $\mathbb{H}$ is cocommutative if $\delta_H = c_{H,H} \circ \delta_H$, i.e., if H_1 and H_2 are cocommutative Hopf algebras in C.

Definition 3.6. Given two Hopf braces $\mathbb{H}$ and $\mathbb{D}$ in C, a morphism x in C between the two underlying objects is called a morphism of Hopf braces if both $x\colon H_1 \to D_1$ and $x\colon H_2 \to D_2$ are Hopf algebra morphisms.

Hopf braces together with morphisms of Hopf braces form a category which we denote by HBr. Moreover, cocommutative Hopf braces constitute a full subcategory of HBr, which we will denote by $\mathsf{coc\text{-}HBr}$.

Moreover, in our braided context Lemma 1.8 and Remark 1.9 of [2] hold and then we have that the algebra (H, η_H, μ_H^1) is a left H_2-module algebra with action Γ_{H_1} and μ_H^2 admits the following expression:

$$\mu_H^2 = \mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ (\delta_H \otimes H). \tag{3.6}$$

In addition, by Lemma 2.2, of [2], Γ_{H_1} is a coalgebra morphism when $\mathbb{H}$ is cocommutative.

The following result is the braided monoidal version of the result proved by Li et al. in Proposition 3.2 and Theorem 3.3 of [18] for Hopf braces in the category of vector spaces over a field $\mathbb{F}$. We do the proof in detail to clarify certain properties and notations that will be very useful in the rest of the paper.

Theorem 3.7. *There exists a functor* $\mathsf{F}\colon \mathsf{coc\text{-}HBr} \to \mathsf{rRB}^\star$ *defined on objects by*

$$\mathsf{F}(\mathbb{H}) = \left(\begin{array}{ccc} & H_1 & \\ id_H & \downarrow & , \Gamma_{H_1} \\ & H_2 & \end{array} \right).$$

and on morphisms by $\mathsf{F}(x) = (x, x)$.

Moreover, there exists a functor $\mathsf{G}\colon \mathsf{rRB}^\star \to \mathsf{coc\text{-}HBr}$ *defined on objects by*

$$\mathsf{G}\left(\left(\begin{array}{ccc} & H & \\ T & \downarrow & , \varphi_H \\ & B & \end{array} \right) \right) = \overline{H},$$

where $\overline{\mathbb{H}} = (H, \overline{H})$ *is the Hopf brace with* $\overline{H} = (H, \eta_H, \overline{\mu}_H, \varepsilon_H, \delta_H, \overline{\lambda}_H)$ *the Hopf algebra whose product and antipode are given by*

$$\overline{\mu}_H := \mu_H \circ (H \otimes (\varphi_H \circ (T \otimes H))) \circ (\delta_H \otimes H),$$
$$\overline{\lambda}_H := \varphi_H \circ ((\lambda_B \circ T) \otimes \lambda_H) \circ \delta_H,$$

and on morphisms by $\mathsf{G}(f, h) = f$.

In addition, F *is left adjoint to* G *and also* $F(\mathbb{H}) \in \mathsf{coc\text{-}rRB}$ *for all* $\mathbb{H} \in \mathsf{coc\text{-}HBr}$.

Proof. First of all, let us see that F is well-defined. If $\mathbb{H} = (H_1, H_2)$ is a cocommutative Hopf brace in C, then we already know that (H_1, Γ_{H_1}) is a left H_2-module algebra and, thanks to the cocommutativity, Γ_{H_1} is a coalgebra morphism, i.e., (H_1, Γ_{H_1}) is a left H_2-module algebra-coalgebra. Moreover, (3.6) implies that (3.1) holds. Therefore, $\left(\begin{array}{ccc} & H_1 & \\ id_H & \downarrow & ,\Gamma_{H_1} \\ & H_2 & \end{array}\right)$ is a relative Rota-Baxter operator. In addition, if $\mathbb{D}$ is another cocommutative Hopf brace and $x\colon \mathbb{H} \to \mathbb{D}$ is a morphism of Hopf braces, then the pair (x, x) is a morphism of relative Rota-Baxter operators between $\left(\begin{array}{ccc} & H_1 & \\ id_H & \downarrow & ,\Gamma_{H_1} \\ & H_2 & \end{array}\right)$ and $\left(\begin{array}{ccc} & D_1 & \\ id_D & \downarrow & ,\Gamma_{D_1} \\ & D_2 & \end{array}\right)$. Indeed, it is straightforward to compute that (3.2) holds and (3.3) follows by

$$\begin{aligned} x \circ \Gamma_{H_1} &= \mu_D^1 \circ ((x \circ \lambda_H^1) \otimes (x \circ \mu_H^2)) \circ (\delta_H \otimes H) \text{ (by the condition of algebra} \\ &\qquad \text{morphism for } x\colon H_1 \to D_1) \\ &= \mu_D^1 \circ (\lambda_D^1 \otimes \mu_D^2) \circ (((x \otimes x) \circ \delta_H) \otimes x) \text{ (by (2.2) and the condition} \\ &\qquad \text{of algebra morphism for } x\colon H_2 \to D_2) \\ &= \Gamma_{D_1} \circ (x \otimes x) \text{ (by the condition of coalgebra morphism for } x). \end{aligned}$$

Now, let us prove that G is well-defined. On the one hand, consider $\left(\begin{array}{ccc} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{array}\right)$ a relative Rota-Baxter operator such that H is cocommutative. Let us show that $\overline{H}$ is a Hopf algebra. At first, note that it is straightforward to prove that η_H is the unit for $\overline{\mu}_H$ and the associativity of $\overline{\mu}_H$ follows by

$$\begin{aligned} &\overline{\mu}_H \circ (\overline{\mu}_H \otimes H) \\ &= \mu_H \circ (H \otimes (\varphi_H \circ (T \otimes H))) \circ (((\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \\ &\quad \circ (\delta_H \otimes ((\varphi_H \otimes \varphi_H) \circ (B \otimes c_{B,H} \otimes H) \circ (((T \otimes T) \circ \delta_H) \otimes \delta_H)))) \otimes H) \\ &\quad \circ (\delta_H \otimes H \otimes H) \text{ (by the condition of coalgebra morphism for } \mu_H,\ \varphi_H \text{ and } T) \\ &= \mu_H \circ (H \otimes \varphi_H) \circ (\overline{\mu}_H \otimes (T \circ \overline{\mu}_H) \otimes H) \circ (((H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)) \otimes H) \\ &\quad \text{(by the cocommutativity and coassociativity of } \delta_H \text{ and naturality of } c) \\ &= \mu_H \circ (\overline{\mu}_H \otimes (\varphi_H \circ ((\mu_B \circ (T \otimes T)) \otimes H))) \circ (((H \otimes c_{H,H} \otimes H) \\ &\quad \circ (\delta_H \otimes \delta_H)) \otimes H) \text{ (by (3.1))} \\ &= \mu_H \circ (H \otimes (\mu_H \circ (\varphi_H \otimes \varphi_H) \circ (B \otimes c_{B,H} \otimes H) \circ (\delta_B \otimes H \otimes H))) \\ &\quad \circ (((H \otimes T) \circ \delta_H) \otimes ((H \otimes (\varphi_H \circ (T \otimes H))) \circ (\delta_H \otimes H))) \text{ (by module axioms} \\ &\quad \text{for } (H, \varphi_H) \text{, coassociativity of } \delta_H \text{, the condition of coalgebra morphism for } T \\ &\quad \text{and associativity of } \mu_H) \\ &= \overline{\mu}_H \circ (H \otimes \overline{\mu}_H) \text{ (by the condition of morphism of left } B\text{-modules for } \mu_H). \end{aligned}$$

Moreover, note that

$$\begin{aligned}
\delta_H \circ \overline{\mu}_H = {} & (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes ((\varphi_H \otimes \varphi_H) \\
& \circ (B \otimes c_{B,H} \otimes H) \circ (((T \otimes T) \circ \delta_H) \otimes \delta_H))) \circ (\delta_H \otimes H) \text{ (by the condition} \\
& \text{of coalgebra morphism for } \mu_H,\ \varphi_H \text{ and } T) \\
= {} & ((\mu_H \circ (H \otimes (\varphi_H \circ (T \otimes H)))) \otimes (\mu_H \circ (H \otimes (\varphi_H \circ (T \otimes H))))) \\
& \circ (H \otimes ((H \otimes c_{H,H} \otimes H) \circ (c_{H,H} \otimes c_{H,H})) \otimes H) \circ (((\delta_H \otimes \delta_H) \circ \delta_H) \otimes \delta_H) \\
& \text{(by naturality of } c) \\
= {} & (\overline{\mu}_H \otimes \overline{\mu}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by cocommutativity and} \\
& \text{coassociativity of } \delta_H)
\end{aligned}$$

and, by the condition of coalgebra morphism for μ_H, φ_H and T and the counit property, the equality $\varepsilon_H \circ \overline{\mu}_H = \varepsilon_H \otimes \varepsilon_H$ also holds. Thus, to conclude that $\overline{H}$ is a Hopf algebra, it only remains to prove that $\overline{\lambda}_H$ is the inverse of id_H for the convolution in $\mathrm{Hom}(H, \overline{H})$, operation that we will denote by $\overline{*}$. Firstly,

$$\begin{aligned}
id_H \overline{*} \overline{\lambda}_H & = \mu_H \circ (H \otimes (\varphi_H \circ (((id_B * \lambda_B) \circ T) \otimes \lambda_H) \circ \delta_H)) \circ \delta_H \text{ (by the} \\
& \quad \text{coassociativity of } \delta_H \text{, module axioms for } (H, \varphi_H) \text{ and the condition} \\
& \quad \text{of coalgebra morphism for } T) \\
& = id_H * \lambda_H \text{ (by (2.1) for } B \text{, the condition of coalgebra morphism for } T, \\
& \quad \text{the counit property and module axioms for } (H, \varphi_H)) \\
& = \varepsilon_H \otimes \eta_H \text{ (by (2.1) for } H).
\end{aligned}$$

Note also that $\overline{\lambda}_H$ satisfies the following property: $\overline{\lambda}_H$ is a coalgebra morphism because

$$\begin{aligned}
\delta_H \circ \overline{\lambda}_H & = (\varphi_H \otimes \varphi_H) \circ (B \otimes c_{B,H} \otimes H) \circ ((\delta_B \circ \lambda_B \circ T) \otimes (\delta_H \circ \lambda_H)) \circ \delta_H \\
& \quad \text{(by the condition of coalgebra morphism for } \varphi_H) \\
& = (\varphi_H \otimes \varphi_H) \circ (B \otimes c_{B,H} \otimes H) \circ (((\lambda_B \otimes \lambda_B) \circ (T \otimes T) \circ c_{H,H} \circ \delta_H) \\
& \quad \otimes ((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \text{ (by (2.5), the condition of coalgebra} \\
& \quad \text{morphism for } T \text{ and naturality of } c) \\
& = ((\varphi_H \circ ((\lambda_B \circ T) \otimes \lambda_H)) \otimes (\varphi_H \circ ((\lambda_B \circ T) \otimes \lambda_H))) \circ (H \otimes c_{H,H} \otimes H) \\
& \quad \circ (\delta_H \otimes \delta_H) \circ \delta_H \text{ (by cocommutativity of } \delta_H \text{ and naturality of } c) \\
& = (\overline{\lambda}_H \otimes \overline{\lambda}_H) \circ \delta_H \text{ (by coassociativity and cocommutativity of } \delta_H)
\end{aligned}$$

and

$$\varepsilon_H \circ \overline{\lambda}_H = \varepsilon_H,$$

follows by the condition of coalgebra morphism for φ_H and T, (2.7) and the counit property. As a consequence,

$$\begin{aligned}
\overline{\lambda}_H \circ \overline{\lambda}_H &= \overline{\mu}_H \circ ((\eta_H \circ \varepsilon_H) \otimes (\overline{\lambda}_H \circ \overline{\lambda}_H)) \circ \delta_H \text{ (by the (co)unit property)} \\
&= \overline{\mu}_H \circ ((\overline{\mu}_H \circ (H \otimes \overline{\lambda}_H) \circ \delta_H) \otimes (\overline{\lambda}_H \circ \overline{\lambda}_H)) \circ \delta_H \text{ (by } id_H \overline{\ast} \overline{\lambda}_H = \varepsilon_H \otimes \eta_H\text{)} \\
&= \overline{\mu}_H \circ (H \otimes (\overline{\mu}_H \circ (\overline{\lambda}_H \otimes (\overline{\lambda_H} \circ \overline{\lambda}_H)) \circ \delta_H)) \circ \delta_H \text{ (by coassociativity} \\
&\quad \text{of } \delta_H \text{ and associativity of } \overline{\mu}_H\text{)} \\
&= \overline{\mu}_H \circ (H \otimes ((id_H \overline{\ast} \overline{\lambda}_H) \circ \overline{\lambda}_H)) \circ \delta_H \text{ (by the condition of coalgebra} \\
&\quad \text{morphism of } \overline{\lambda}_H\text{)} \\
&= id_H \text{ (by } id_H \overline{\ast} \overline{\lambda}_H = \varepsilon_H \otimes \eta_H\text{, the condition of coalgebra morphism} \\
&\quad \text{of } \overline{\lambda}_H \text{ and the (co)unit property).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{\lambda}_H \overline{\ast}\, id_H &= \overline{\mu}_H \circ (\overline{\lambda}_H \otimes (\overline{\lambda}_H \circ \overline{\lambda}_H)) \circ \delta_H \text{ (by } \overline{\lambda}_H \circ \overline{\lambda}_H = id_H\text{)} \\
&= (id_H \overline{\ast} \overline{\lambda}_H) \circ \overline{\lambda}_H \text{ (by the condition of coalgebra morphism of } \overline{\lambda}_H\text{)} \\
&= \varepsilon_H \otimes \eta_H \text{ (by } id_H \overline{\ast} \overline{\lambda}_H = \varepsilon_H \otimes \eta_H \text{ and the condition of coalgebra} \\
&\quad \text{morphism of } \overline{\lambda}_H\text{).}
\end{aligned}$$

So, to conclude that $\overline{\mathbb{H}}$ is a Hopf brace it is enough to see that (iii) of Definition 3.4 holds. Indeed,

$$\overline{\Gamma}_H = \varphi_H \circ (T \otimes H) \tag{3.7}$$

follows by

$$\begin{aligned}
\overline{\Gamma}_H &= \mu_H \circ ((\lambda_H \ast id_H) \otimes (\varphi_H \circ (T \otimes H))) \circ (\delta_H \otimes H) \text{ (by the coassociativity} \\
&\quad \text{of } \delta_H \text{ and associativity of } \mu_H\text{)} \\
&= \varphi_H \circ (T \otimes H) \text{ (by (2.1) and the (co)unit property).}
\end{aligned}$$

Then,

$$\begin{aligned}
&\mu_H \circ (\overline{\mu}_H \otimes \overline{\Gamma}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H) \\
&= \mu_H \circ (H \otimes (\mu_H \circ (\varphi_H \otimes \varphi_H) \circ (B \otimes c_{B,H} \otimes H) \circ ((\delta_B \circ T) \otimes H \otimes H))) \\
&\quad \circ (\delta_H \otimes H \otimes H) \text{ (by (3.7), coassociativity of } \delta_H\text{, associativity of } \mu_H\text{,} \\
&\quad \text{naturality of } c \text{ and the condition of coalgebra morphism for } T\text{)} \\
&= \overline{\mu}_H \circ (H \otimes \mu_H) \text{ (by the condition of morphism of left } B\text{-modules for } \mu_H\text{).}
\end{aligned}$$

On the other hand, if (f,h) is a morphism in $\mathsf{rRB}^{\star}$ between $\left(T\ \begin{array}{c} H\\ \downarrow\\ B\end{array} ,\varphi_H\right)$ and $\left(L\ \begin{array}{c} A\\ \downarrow\\ D\end{array} ,\varphi_A\right)$, then f is a morphism of Hopf braces between $\overline{\mathbb{H}}$ and $\overline{\mathbb{A}}$. Indeed,

$$\begin{aligned}
f\circ\overline{\mu}_H &= \mu_A\circ(f\otimes(f\circ\varphi_H\circ(T\otimes H)))\circ(\delta_H\otimes H) \text{ (by the condition of algebra}\\
&\qquad \text{morphism for } f)\\
&= \mu_A\circ(f\otimes(\varphi_A\circ((h\circ T)\otimes f)))\circ(\delta_H\otimes H) \text{ (by (3.3))}\\
&= \mu_A\circ(A\otimes(\varphi_A\circ(L\otimes A)))\circ(((f\otimes f)\circ\delta_H)\otimes f) \text{ (by (3.2))}\\
&= \overline{\mu}_A\circ(f\otimes f) \text{ (by the condition of coalgebra morphism for } f).
\end{aligned}$$

To prove that F is left adjoint to G, it is enough to consider the bijection

$$\begin{aligned}
{}^{\mathbb{H}}\Theta_L\colon\ & \mathrm{Hom}_{\mathsf{coc\text{-}HBr}}(\mathbb{H}=(H_1,H_2),\overline{\mathbb{A}}=(A,\overline{A}))\\
&\to \mathrm{Hom}_{\mathsf{rRB}^{\star}}\left(\left(id_H\ \begin{array}{c} H_1\\ \downarrow\\ H_2\end{array}\ ,\Gamma_{H_1}\right),\left(L\ \begin{array}{c} A\\ \downarrow\\ D\end{array}\ ,\varphi_A\right)\right)
\end{aligned}$$

given by ${}^{\mathbb{H}}\Theta_L(y)=(y,L\circ y)$ and $({}^{\mathbb{H}}\Theta_L)^{-1}(f,h)=f$ for all $\mathbb{H}\in\mathsf{coc\text{-}HBr}$ and $\left(L\ \begin{array}{c} A\\ \downarrow\\ D\end{array} ,\varphi_A\right)\in\mathsf{rRB}^{\star}$. $\square$

Taking into account the previous considerations, in the following definition the notion of module over a relative Rota-Baxter operator is introduced.

Definition 3.8. Let $\left(T\ \begin{array}{c} H\\ \downarrow\\ B\end{array} ,\varphi_H\right)$ be a relative Rota-Baxter operator. We will say that a 6-tuple

$$(M,N,\phi_M,\varphi_M,\varphi_N,\gamma)$$

is a left module over $\left(T\ \begin{array}{c} H\\ \downarrow\\ B\end{array} ,\varphi_H\right)$ if the following conditions hold:

(i) (M,ϕ_M) is a left H-module and (M,φ_M) is a left B-module such that the equality

$$\varphi_M\circ(B\otimes\phi_M)=\phi_M\circ(\varphi_H\otimes\varphi_M)\circ(B\otimes c_{B,H}\otimes M)\circ(\delta_B\otimes H\otimes M) \tag{3.8}$$

holds,

(ii) (N,φ_N) is a left B-module,

(iii) $\gamma\colon M\to N$ is a morphism satisfying that

$$\varphi_N\circ(T\otimes\gamma)=\gamma\circ\phi_M\circ(H\otimes(\varphi_M\circ(T\otimes M)))\circ(\delta_H\otimes M). \tag{3.9}$$

Let $(M,N,\phi_M,\varphi_M,\varphi_N,\gamma)$ and $(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)$ be left modules over $\left(T\ \begin{array}{c} H\\ \downarrow\\ B\end{array} ,\varphi_H\right)$.

We will say that a pair (r,s) is a morphism of left modules over $\left(\begin{smallmatrix} & H \\ T & \downarrow \\ & B \end{smallmatrix}, \varphi_H \right)$ between $(M,N,\phi_M,\varphi_M,\varphi_N,\gamma)$ and $(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)$ if the conditions

(i) $r\colon (M,\phi_M) \to (P,\phi_P)$ is a morphism of left H-modules,
(ii) $r\colon (M,\varphi_M) \to (P,\varphi_P)$ is a morphism of left B-modules,
(iii) $s\colon (N,\varphi_N) \to (Q,\varphi_Q)$ is a morphism of left B-modules,
(iv) $s \circ \gamma = \theta \circ r$

hold.

Therefore, with the obvious composition of morphisms, left modules over the relative Rota-Baxter operator $\left(\begin{smallmatrix} & H \\ T & \downarrow \\ & B \end{smallmatrix}, \varphi_H \right)$ constitute a category that we will denote by ${}_{(T,\varphi_H)}\mathsf{Mod}$.

Example 3.9. Note that given $\left(\begin{smallmatrix} & H \\ T & \downarrow \\ & B \end{smallmatrix}, \varphi_H \right)$ a relative Rota-Baxter operator, $(H,B,\mu_H,\varphi_H,\mu_B,T)$ is an object in ${}_{(T,\varphi_H)}\mathsf{Mod}$. Moreover, $(K,K,\varepsilon_H,\varepsilon_B,\varepsilon_B,id_K)$ is called the trivial left module over any relative Rota-Baxter operator $\left(\begin{smallmatrix} & H \\ T & \downarrow \\ & B \end{smallmatrix}, \varphi_H \right)$.

Theorem 3.10. *Let us assume that* C *is symmetric and let* $\left(\begin{smallmatrix} & H \\ T & \downarrow \\ & B \end{smallmatrix}, \varphi_H \right)$ *be an object in* coc-rRB. *Then, the category* ${}_{(T,\varphi_H)}\mathsf{Mod}$ *is monoidal with unit object* $(K,K,\varepsilon_H,\varepsilon_B,\varepsilon_B,id_K)$ *and tensor functor defined by*

$$\begin{aligned} &\otimes\colon {}_{(T,\varphi_H)}\mathsf{Mod} \times {}_{(T,\varphi_H)}\mathsf{Mod} \to {}_{(T,\varphi_H)}\mathsf{Mod} \\ &((M,N,\phi_M,\varphi_M,\varphi_N,\gamma),(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)) \\ &\mapsto (M\otimes P, N\otimes Q, \phi_{M\otimes P}, \varphi_{M\otimes P}, \varphi_{N\otimes Q}, \gamma\otimes\theta). \end{aligned}$$

Moreover, ${}_{(T,\varphi_H)}\mathsf{Mod}$ *is symmetric with symmetry isomorphism given by*

$$\tau_{(M,N,\gamma),(P,Q,\theta)} := (c_{M,P}, c_{N,Q}).$$

Proof. Consider $(M,N,\phi_M,\varphi_M,\varphi_N,\gamma)$ and $(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)$ objects in ${}_{(T,\varphi_H)}\mathsf{Mod}$. Then let us see that $(M\otimes P, N\otimes Q, \phi_{M\otimes P}, \varphi_{M\otimes P}, \varphi_{N\otimes Q}, \gamma\otimes\theta)$ is also an object in ${}_{(T,\varphi_H)}\mathsf{Mod}$. Indeed, note that, due to the monoidal character of the module categories ${}_H\mathsf{Mod}$ and ${}_B\mathsf{Mod}$, the only facts that remain us to compute are that (3.8) and (3.9) hold. At first, we have that

$$\begin{aligned} &\phi_{M\otimes P} \circ (\varphi_H \otimes \varphi_{M\otimes P}) \circ (B\otimes c_{B,H}\otimes M\otimes P) \circ (\delta_B \otimes H\otimes M\otimes P) \\ &\quad = (\phi_M\otimes\phi_P)\circ(H\otimes c_{H,M}\otimes P)\circ(((\varphi_H\otimes\varphi_H)\circ(B\otimes c_{B,H}\otimes H) \\ &\qquad \circ(\delta_B\otimes\delta_H))\otimes((\varphi_M\otimes\varphi_P)\circ(B\otimes c_{B,M}\otimes P)\circ(\delta_B\otimes M\otimes P))) \\ &\qquad \circ(B\otimes c_{B,H}\otimes M\otimes P)\circ(\delta_B\otimes H\otimes M\otimes P) \end{aligned}$$

(by the condition of coalgebra morphism for φ_H)

$$
\begin{aligned}
&= ((\phi_M \circ (\varphi_H \otimes \varphi_M) \circ (B \otimes c_{B,H} \otimes M) \circ (\delta_B \otimes H \otimes M)) \otimes (\phi_P \circ (\varphi_H \otimes \varphi_P) \\
&\quad \circ (B \otimes c_{B,H} \otimes P) \circ (\delta_B \otimes H \otimes P))) \circ (B \otimes ((H \otimes c_{B,M} \otimes H) \\
&\quad \circ (c_{B,H} \otimes c_{H,M})) \otimes P) \circ (\delta_B \otimes \delta_H \otimes M \otimes P) \text{ (by naturality of } c\text{,} \\
&\quad \text{symmetrical character of } \mathsf{C} \text{ and cocommutativity of } \delta_B) \\
&= ((\varphi_M \circ (B \otimes \phi_M)) \otimes (\varphi_P \circ (B \otimes \phi_P))) \circ (B \otimes ((H \otimes c_{B,M} \otimes H) \\
&\quad \circ (c_{B,H} \otimes c_{H,M})) \otimes P) \circ (\delta_B \otimes \delta_H \otimes M \otimes P) \text{ (by (3.8))} \\
&= \varphi_{M\otimes P} \circ (B \otimes \phi_{M\otimes P}) \text{ (by naturality of } c\text{),}
\end{aligned}
$$

so (3.8) holds and, on the other side,

$$
\begin{aligned}
&(\gamma \otimes \theta) \circ \phi_{M\otimes P} \circ (H \otimes (\varphi_{M\otimes P} \circ (T \otimes M \otimes P))) \circ (\delta_H \otimes M \otimes P) \\
&\quad = ((\gamma \circ \phi_M \circ (H \otimes (\varphi_M \circ (T \otimes M)))) \otimes (\theta \circ \phi_P \circ (H \otimes (\varphi_P \circ (T \otimes P))))) \\
&\qquad \circ (H \otimes ((H \otimes c_{H,M} \otimes H) \circ (c_{H,H} \otimes c_{H,M})) \otimes P) \\
&\qquad \circ (((\delta_H \otimes \delta_H) \circ \delta_H) \otimes M \otimes P) \text{ (by naturality of } c \text{ and the condition} \\
&\qquad \text{of coalgebra morphism for } T) \\
&\quad = ((\gamma \circ \phi_M \circ (H \otimes (\varphi_M \circ (T \otimes M))) \circ (\delta_H \otimes M)) \\
&\qquad \otimes (\theta \circ \phi_P \circ (H \otimes (\varphi_P \circ (T \otimes P))) \circ (\delta_H \otimes P))) \circ (H \otimes c_{H,M} \otimes P) \\
&\qquad \circ (\delta_H \otimes M \otimes P) \text{ (by coassociativity and cocommutativity of } \delta_H \\
&\qquad \text{and naturality of } c) \\
&\quad = ((\varphi_N \circ (T \otimes \gamma)) \otimes (\varphi_Q \circ (T \otimes \theta))) \circ (H \otimes c_{H,M} \otimes P) \circ (\delta_H \otimes M \otimes P) \\
&\qquad \text{(by (3.9))} \\
&\quad = \varphi_{N\otimes Q} \circ (T \otimes \gamma \otimes \theta) \text{ (by naturality of } c \text{ and the condition of coalgebra} \\
&\qquad \text{morphism for } T\text{),}
\end{aligned}
$$

which implies that (3.9) holds.

On the other hand, the symmetric character of ${}_{(T,\varphi_H)}\mathsf{Mod}$ follows by the fact that when C is symmetric and H and B are cocommutative Hopf algebras, the pair $(c_{M,P}, c_{N,Q})$ is a morphism in ${}_{(T,\varphi_H)}\mathsf{Mod}$. □

Theorem 3.11. *If* $(f,h)\colon \left(T \overset{H}{\underset{B}{\downarrow}}, \varphi_H \right) \to \left(L \overset{A}{\underset{D}{\downarrow}}, \varphi_A \right)$ *is a morphism of relative Rota-Baxter operators, then there exists a functor*

$$\mathsf{R}_{(f,h)}\colon {}_{(L,\varphi_A)}\mathsf{Mod} \to {}_{(T,\varphi_H)}\mathsf{Mod}$$

acting on objects by

$$\begin{aligned}\mathsf{R}_{(f,h)}((M,N,\phi_M,\varphi_M,\varphi_N,\gamma)) = (M,N,\phi_M^T := \phi_M\circ(f\otimes M),\varphi_M^T \\ := \varphi_M\circ(h\otimes M),\varphi_N^T := \varphi_N\circ(h\otimes N),\gamma)\end{aligned}$$

and on morphisms by the identity.

Proof. Let us show that it is well-defined on objects. First of all, note that it is straightforward to prove that (M,ϕ_M^T) is a left H-module and (M,φ_M^T) and (N,φ_N^T) are left B-modules. Thus, it only remains to see that equalities (3.8) and (3.9) hold. Then

$$\begin{aligned}&\phi_M^T\circ(\varphi_H\otimes\varphi_M^T)\circ(B\otimes c_{B,H}\otimes M)\circ(\delta_B\otimes H\otimes M)\\ &= \phi_M\circ(\varphi_A\otimes\varphi_M)\circ(D\otimes c_{D,A}\otimes M)\circ(\delta_D\otimes A\otimes M)\circ(h\otimes f\otimes M) \text{ (by (3.3)}\\ &\quad \text{for } (f,h)\text{, naturality of } c \text{ and the condition of coalgebra morphism for } h)\\ &= \varphi_M^T\circ(B\otimes\phi_M^T) \text{ (by (3.8)),}\end{aligned}$$

i.e., (3.8) holds, and also

$$\begin{aligned}&\gamma\circ\phi_M^T\circ(H\otimes(\varphi_M^T\circ(T\otimes M)))\circ(\delta_H\otimes M)\\ &= \gamma\circ\phi_M\circ(A\otimes(\varphi_M\circ(L\otimes M)))\circ(((f\otimes f)\circ\delta_H)\otimes M) \text{ (by (3.2) for } (f,h))\\ &= \gamma\circ\phi_M\circ(A\otimes(\varphi_M\circ(L\otimes M)))\circ((\delta_A\circ f)\otimes M) \text{ (by the condition of}\\ &\quad \text{coalgebra morphism for } f)\\ &= \varphi_N\circ((L\circ f)\otimes\gamma) \text{ (by (3.9))}\\ &= \varphi_N^T\circ(T\otimes\gamma) \text{ (by (3.2) for } (f,h)),\end{aligned}$$

which implies that (3.9) holds. □

Let us recall the notion of module over a Hopf brace introduced in [14] and some results related with this concept which can be consulted in [11].

Definition 3.12. Let $\mathbb{H} = (H_1,H_2)$ be a Hopf brace. We will say that a triple (M,ψ_M^1,ψ_M^2) is a left module over $\mathbb{H}$ if (M,ψ_M^1) is a left H_1-module, (M,ψ_M^2) is a left H_2-module and the following compatibility condition holds:

$$\psi_M^2\circ(H\otimes\psi_M^1) = \psi_M^1\circ(\mu_H^2\otimes\Gamma_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes H\otimes M), \tag{3.10}$$

where

$$\Gamma_M := \psi_M^1\circ(\lambda_H^1\otimes\psi_M^2)\circ(\delta_H\otimes M).$$

If (M, ψ_M^1, ψ_M^2) and (N, ψ_N^1, ψ_N^2) are modules over the Hopf brace $\mathbb{H}$ and $f\colon M \to N$ is a morphism between them, we will say that f is a morphism of left $\mathbb{H}$-modules if f is a morphism of left H_1-modules and left H_2-modules.

Then, with the obvious composition of morphisms, modules over $\mathbb{H}$ constitute a category that we will denote by ${}_{\mathbb{H}}\mathsf{Mod}$.

Remark 3.13. Let (M, ψ_M^1, ψ_M^2) be a left module over a Hopf brace $\mathbb{H}$. By composing on the right-hand side of (3.10) with $H \otimes \eta_H \otimes M$, we obtain that the equality

$$\psi_M^2 = \psi_M^1 \circ (H \otimes \Gamma_M) \circ (\delta_H \otimes M) \tag{3.11}$$

holds.

The proof of the following result can be seen in Lemma 2.11 of [11].

Theorem 3.14. *Let $\mathbb{H}$ be a Hopf brace and (M, ψ_M^1, ψ_M^2) a left module over $\mathbb{H}$. The equality*

$$\Gamma_M \circ (H \otimes \psi_M^1) = \psi_M^1 \circ (\Gamma_{H_1} \otimes \Gamma_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M) \tag{3.12}$$

holds and (M, Γ_M) is a left H_2-module.

Taking into account the previous results, in what follows we are going to construct two functors that set a relationship between the category of modules over a Hopf brace and the category of modules over a relative Rota-Baxter operator.

Theorem 3.15. *Let $\mathbb{H}$ be a cocommutative Hopf brace. There exists a functor*

$$\mathsf{W}_{\mathbb{H}}\colon {}_{\mathbb{H}}\mathsf{Mod} \to {}_{(id_H, \Gamma_{H_1})}\mathsf{Mod},$$

where (id_H, Γ_{H_1}) denotes the relative Rota-Baxter operator $\mathsf{F}(\mathbb{H}) = \left(\begin{array}{c} H_1 \\ id_H \ \downarrow \\ H_2 \end{array}, \Gamma_{H_1} \right)$ *introduced in Theorem* 3.7, *which acts on objects by*

$$\mathsf{W}_{\mathbb{H}}((M, \psi_M^1, \psi_M^2)) = (M, M, \psi_M^1, \Gamma_M, \psi_M^2, id_M)$$

and on morphisms by $\mathsf{W}_{\mathbb{H}}(f) = (f, f)$.

Proof. At first we are going to prove that if (M, ψ_M^1, ψ_M^2) is a left module over the Hopf brace $\mathbb{H}$, then the 6-tuple $(M, M, \psi_M^1, \Gamma_M, \psi_M^2, id_M)$ is a left module over the relative Rota-Baxter operator $\mathsf{F}(\mathbb{H})$. Indeed, on the one hand, (3.8) follows by (3.12) and, on the other hand, (3.9) follows by (3.11). Therefore, W is well-defined on objects.

On the other hand, if $f\colon (M,\psi_M^1,\psi_M^2) \to (N,\psi_N^1,\psi_N^2)$ is a morphism of left $\mathbb{H}$-modules, then (f,f) is a morphism in ${}_{(id_H,\Gamma_{H_1})}\mathsf{Mod}$ between the following tuples $(M,M,\psi_M^1,\Gamma_M,\psi_M^2,id_M)$ and $(N,N,\psi_N^1,\Gamma_N,\psi_N^2,id_N)$, which follows by the fact that $f\colon (M,\Gamma_M) \to (N,\Gamma_N)$ is a morphism of left H_2-modules, which is straightforward to show. □

Corollary 3.16. *Let* $\left(T \overset{H}{\underset{B}{\downarrow}} ,\varphi_H \right)$ *be a relative Rota-Baxter operator in* $\mathsf{rRB}^{\star}$. *If* $T\colon H \to B$ *is an isomorphism, then there exists a functor*

$$V\colon {}_{\overline{\mathbb{H}}}\mathsf{Mod} \to {}_{(T,\varphi_H)}\mathsf{Mod},$$

where $\overline{\mathbb{H}} = \mathsf{G}\left(\left(T \overset{H}{\underset{B}{\downarrow}} ,\varphi_H \right)\right)$ *is the Hopf brace introduced in Theorem* 3.7.

Proof. Note that when T is an isomorphism, then (id_H, T^{-1}) is a morphism in the category of relative Rota-Baxter operators between $\left(T \overset{H}{\underset{B}{\downarrow}} ,\varphi_H \right)$ and $\left(id_H \overset{H}{\underset{\overline{H}}{\downarrow}} ,\overline{\Gamma}_H \overset{(3.7)}{=} \varphi_H\circ(T\otimes H) \right)$. Therefore, we can define V as the following composition of functors: If $\mathsf{R}_{(id_H,T^{-1})}$ is the functor introduced in Theorem 3.11, then

$$\mathsf{V} := \mathsf{R}_{(id_H,T^{-1})} \circ \mathsf{W}_{\overline{\mathbb{H}}},$$

which is defined on objects by

$$\mathsf{V}((M,\overline{\psi}_M^1,\overline{\psi}_M^2)) = (M,M,\overline{\psi}_M^1,\overline{\Gamma}_M\circ(T^{-1}\otimes M),\overline{\psi}_M^2\circ(T^{-1}\otimes M),id_M)$$

and on morphisms by $\mathsf{V}(f) = (f,f)$. □

Theorem 3.17. *Let* $\left(T \overset{H}{\underset{B}{\downarrow}} ,\varphi_H \right)$ *be a relative Rota-Baxter operator in* $\mathsf{rRB}^{\star}$. *There exists a functor*

$$\mathsf{U}\colon {}_{(T,\varphi_H)}\mathsf{Mod} \to {}_{\overline{\mathbb{H}}}\mathsf{Mod},$$

where $\overline{\mathbb{H}} = \mathsf{G}\left(\left(T \overset{H}{\underset{B}{\downarrow}} ,\varphi_H \right)\right)$ *is the Hopf brace introduced in Theorem* 3.7, *defined on objects by*

$$\mathsf{U}((M,N,\phi_M,\varphi_M,\varphi_N,\gamma)) = (M,\phi_M,\overline{\varphi}_M),$$

being $\overline{\varphi}_M := \phi_M\circ(H\otimes(\varphi_M\circ(T\otimes M)))\circ(\delta_H\otimes M)$, *and on morphisms by* $\mathsf{U}(r,s) = r$.

Proof. Consider $(M,N,\phi_M,\varphi_M,\varphi_N,\gamma)$ an object in ${}_{(T,\varphi_H)}\mathsf{Mod}$ and let us show that $(M,\phi_M,\overline{\varphi}_M)$ is a module over the Hopf brace $\overline{\mathbb{H}}$. At first, we will see that $(M,\overline{\varphi}_M)$ is a left $\overline{H}$-module. Indeed, it is straightforward to show that $\overline{\varphi}_M\circ(\eta_H\otimes M)=id_M$ and also

$$\begin{aligned}
&\overline{\varphi}_M\circ(H\otimes\overline{\varphi}_M)\\
&\quad=\phi_M\circ(H\otimes(\phi_M\circ((\varphi_H\circ(T\otimes H))\otimes(\varphi_M\circ(T\otimes(\varphi_M\circ(T\otimes M)))))\\
&\qquad\circ(H\otimes c_{H,H}\otimes H\otimes M)\circ(\delta_H\otimes H\otimes H\otimes M)))\circ(\delta_H\otimes\delta_H\otimes M)\\
&\qquad\text{(by (3.8), the condition of coalgebra morphism for } T \text{ and naturality of } c)\\
&\quad=\phi_M\circ(\overline{\mu}_H\otimes(\varphi_M\circ((\mu_B\circ(T\otimes T))\otimes M)))\circ(((H\otimes c_{H,H}\otimes H)\\
&\qquad\circ(\delta_H\otimes\delta_H))\otimes M)\ \text{(by the module axioms for } (M,\phi_M) \text{ and } (M,\varphi_M) \text{ and}\\
&\qquad\text{coassociativity of } \delta_H)\\
&\quad=\phi_M\circ(H\otimes(\varphi_M\circ(T\otimes M)))\circ(((\overline{\mu}_H\otimes\overline{\mu}_H)\circ(H\otimes c_{H,H}\otimes H)\\
&\qquad\circ(\delta_H\otimes\delta_H))\otimes M)\ \text{(by (3.1))}\\
&\quad=\overline{\varphi}_M\circ(\overline{\mu}_H\otimes M)\ \text{(by the condition of coalgebra morphism for } \overline{\mu}_H).
\end{aligned}$$

So, to conclude that U is well-defined on objects we have to see that the triple $(M,\phi_M,\overline{\varphi}_M)$ satisfies (3.10). Note that, in this situation,

$$\overline{\Gamma}_M=\varphi_M\circ(T\otimes M) \tag{3.13}$$

as we will prove in what follows:

$$\begin{aligned}
\overline{\Gamma}_M&=\phi_M\circ((\lambda_H*id_H)\otimes(\varphi_M\circ(T\otimes M)))\circ(\delta_H\otimes M)\ \text{(by coassociativity of } \delta_H\\
&\qquad\text{and module axioms for } (M,\phi_M))\\
&=\varphi_M\circ(T\otimes M)\ \text{(by (2.1) and (co)unit properties).}
\end{aligned}$$

Then, (3.10) follows by

$$\begin{aligned}
&\phi_M\circ(\overline{\mu}_H\otimes\overline{\Gamma}_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes H\otimes M)\\
&\quad=\phi_M\circ(H\otimes(\phi_M\circ((\varphi_H\circ(T\otimes H))\otimes(\varphi_M\circ(T\otimes M)))\circ(H\otimes c_{H,H}\otimes M)\\
&\qquad\circ(\delta_H\otimes H\otimes M)))\circ(\delta_H\otimes H\otimes M)\ \text{(by (3.13), module axioms for } (M,\phi_M)\\
&\qquad\text{and coassociativity of } \delta_H)\\
&\quad=\phi_M\circ(H\otimes(\phi_M\circ(\varphi_H\otimes\varphi_M)\circ(B\otimes c_{B,H}\otimes M)\circ((\delta_B\circ T)\otimes H\otimes M)))\\
&\qquad\circ(\delta_H\otimes H\otimes M)\ \text{(by naturality of } c \text{ and the condition of coalgebra}\\
&\qquad\text{morphism for } T)\\
&\quad=\overline{\varphi}_M\circ(H\otimes\phi_M)\ \text{(by (3.8)).}
\end{aligned}$$

On the other hand, to show that U is well-defined on morphisms it is enough to see that if

$$(r,s)\colon\ (M,N,\phi_M,\varphi_M,\varphi_N,\gamma)\to(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)\in{}_{(T,\varphi_H)}\mathsf{Mod},$$

then $r\colon\ (M,\overline{\varphi}_M)\to(P,\overline{\varphi}_P)$ is a morphism of left $\overline{H}$-modules. Indeed,

$$\begin{aligned}r\circ\overline{\varphi}_M &= \phi_P\circ(H\otimes(r\circ\varphi_M\circ(T\otimes M)))\circ(\delta_H\otimes M)\ \text{(by the condition of}\\ &\qquad \text{morphism of left } H\text{-modules for } f\colon\ (M,\phi_M)\to(P,\phi_P))\\ &= \overline{\varphi}_P\circ(H\otimes f)\ \text{(by the condition of morphism of left } B\text{-modules for}\\ &\qquad r\colon\ (M,\varphi_M)\to(P,\varphi_P)).\end{aligned}$$

□

Theorem 3.18. *Let* $\left(\begin{array}{c} H \\ T\ \downarrow\ ,\varphi_H \\ B\end{array}\right)$ *be a relative Rota-Baxter operator in* $\mathsf{rRB}^{\star}$ *and assume that* $T\colon\ H\to B$ *is an isomorphism. Under these hypothesis, the functor* V *is a left adjoint of* U.

Proof. Let $(M,\overline{\psi}^{1}_{M},\overline{\psi}^{2}_{M})$ be a module over the Hopf brace $\overline{\mathbb{H}}$. Moreover, let $(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)$ be a module over the relative Rota-Baxter operator $\left(\begin{array}{c} H \\ T\ \downarrow\ ,\varphi_H \\ B\end{array}\right)$. We have to set a bijection ${}^{(M,\overline{\psi}^{1}_{M},\overline{\psi}^{2}_{M})}\Lambda_{(P,Q,\theta)}$ between

$$\mathrm{Hom}_{{}_{\overline{\mathbb{H}}}\mathsf{Mod}}((M,\overline{\psi}^{1}_{M},\overline{\psi}^{2}_{M}),(P,\phi_P,\overline{\varphi}_P))$$

and

$$\begin{aligned}\mathrm{Hom}_{{}_{(T,\varphi_H)}\mathsf{Mod}}(&(M,M,\overline{\psi}^{1}_{M},\overline{\Gamma}_M\circ(T^{-1}\otimes M),\overline{\psi}^{2}_{M}\circ(T^{-1}\otimes M),id_M),\\ &(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)).\end{aligned}$$

On the one hand, take $f\colon\ (M,\overline{\psi}^{1}_{M},\overline{\psi}^{2}_{M})\to(P,\phi_P,\overline{\varphi}_P)\in{}_{\overline{\mathbb{H}}}\mathsf{Mod}$ and let us see that $(f,\theta\circ f)$ is a morphism in ${}_{(T,\varphi_H)}\mathsf{Mod}$ between $(M,M,\overline{\psi}^{1}_{M},\overline{\Gamma}_M\circ(T^{-1}\otimes M),\overline{\psi}^{2}_{M}\circ(T^{-1}\otimes M),id_M)$ and $(P,Q,\phi_P,\varphi_P,\varphi_Q,\theta)$. Then we have to show that $f\colon\ (M,\overline{\Gamma}_M\circ(T^{-1}\otimes M))\to(P,\varphi_P)$ and $\theta\circ f\colon\ (M,\overline{\psi}^{2}_{M}\circ(T^{-1}\otimes M))\to(Q,\varphi_Q)$ are morphisms of left B-modules, which follows by

$$\begin{aligned}f\circ\overline{\Gamma}_M\circ(T^{-1}\otimes M) &= \phi_P\circ(\lambda_H\otimes(f\circ\overline{\psi}^{2}_{M}))\circ((\delta_H\circ T^{-1})\otimes M)\ \text{(by the condition}\\ &\qquad \text{of morphism of left } H\text{-modules for } f\colon\ (M,\overline{\psi}^{1}_{M})\to(P,\phi_P))\\ &= \overline{\Gamma}_P\circ(T^{-1}\otimes f)\ \text{(by the condition of morphism of left}\\ &\qquad \overline{H}\text{-modules for } f\colon\ (M,\overline{\psi}^{2}_{M})\to(P,\overline{\varphi}_P))\\ &= \varphi_P\circ((T\circ T^{-1})\otimes f)\ \text{(by (3.13))}\\ &= \varphi_P\circ(B\otimes f),\end{aligned}$$

and

$$\begin{aligned}\theta \circ f \circ \overline{\psi}^2_M \circ (T^{-1} \otimes M) &= \theta \circ \overline{\varphi}_P \circ (T^{-1} \otimes f) \text{ (by the condition of morphism of left}\\ &\qquad \overline{H}\text{-modules for } f\colon\ (M, \overline{\psi}^2_M) \to (P, \overline{\varphi}_P))\\ &= \varphi_Q \circ ((T \circ T^{-1}) \otimes (\theta \circ f)) \text{ (by (3.9))}\\ &= \varphi_Q \circ (B \otimes (\theta \circ f)).\end{aligned}$$

Therefore, let us define

$${}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}(f) = (f, \theta \circ f).$$

On the other hand, consider

$$(r, s)\colon\ (M, M, \overline{\psi}^1_M, \overline{\Gamma}_M \circ (T^{-1} \otimes M), \overline{\psi}^2_M \circ (T^{-1} \otimes M), id_M) \to (P, Q, \phi_P, \varphi_P, \varphi_Q, \theta)$$

a morphism in ${}_{(T,\varphi_H)}\mathsf{Mod}$ and let us prove that r is a morphism in ${}_{\overline{\mathbb{H}}}\mathsf{Mod}$ between $(M, \overline{\psi}^1_M, \overline{\psi}^2_M)$ and $(P, \phi_P, \overline{\varphi}_P)$. To see this fact it is enough to compute that $r\colon\ (M, \overline{\psi}^2_M) \to (P, \overline{\varphi}_P)$ is a morphism of left $\overline{H}$-modules. Indeed,

$$\begin{aligned}\overline{\varphi}_P \circ (H \otimes r) &= \phi_P \circ (H \otimes (r \circ \overline{\Gamma}_M)) \circ (\delta_H \otimes M) \text{ (by the condition of morphism}\\ &\qquad \text{of left } B\text{-modules for } r\colon\ (M, \overline{\Gamma}_M \circ (T^{-1} \otimes M)) \to (P, \varphi_P))\\ &= r \circ \overline{\psi}^1_M \circ (H \otimes \overline{\Gamma}_M) \circ (\delta_H \otimes M) \text{ (by the condition of morphism}\\ &\qquad \text{of left } H\text{-modules for } r\colon\ (M, \overline{\psi}^1_M) \to (P, \phi_P))\\ &= r \circ \overline{\psi}^2_M \text{ (by (3.11))}.\end{aligned}$$

Then we define

$$\left({}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}\right)^{-1}(r, s) = r.$$

So, ${}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}$ is a bijection because

$$\begin{aligned}&\left(\left({}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}\right)^{-1} \circ {}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}\right)(f)\\ &\qquad\qquad = \left({}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}\right)^{-1}(f, \theta \circ f) = f,\end{aligned}$$

and

$$\begin{aligned}&\left({}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)} \circ \left({}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}\right)^{-1}\right)(r, s)\\ &\qquad = {}^{(M,\overline{\psi}^1_M,\overline{\psi}^2_M)}\Lambda_{(P,Q,\theta)}(r) = (r, \theta \circ r)\\ &\qquad = (r, s) \text{ (by condition (iv) of morphism in } {}_{(T,\varphi_H)}\mathsf{Mod}).\end{aligned}$$

□

Definition 3.19. Suppose that $\left(\begin{smallmatrix} & H & \\ T & \downarrow & , \varphi_H \\ & B & \end{smallmatrix} \right)$ is a relative Rota-Baxter operator. We will define ${}_{(T,\varphi_H)}\mathsf{Mod}^{\mathsf{iso}}$ as the full subcategory of ${}_{(T,\varphi_H)}\mathsf{Mod}$ whose objects, $(M, N, \phi_M, \varphi_M, \varphi_N, \gamma)$, satisfy that $\gamma\colon M \to N$ is an isomorphism in C.

Theorem 3.20. *Let* $\left(\begin{smallmatrix} & H & \\ T & \downarrow & , \varphi_H \\ & B & \end{smallmatrix} \right)$ *be a relative Rota-Baxter operator in* $\mathsf{rRB}^{\star}$ *such that* $T\colon H \to B$ *is an isomorphism. The categories* ${}_{(T,\varphi_H)}\mathsf{Mod}^{\mathsf{iso}}$ *and* ${}_{\overline{\mathbb{H}}}\mathsf{Mod}$ *are equivalent.*

Proof. Define the functor U' as the restriction of U to ${}_{(T,\varphi_H)}\mathsf{Mod}^{\mathsf{iso}}$. It is straightforward to show that

$$\mathsf{U}' \circ \mathsf{V} = \mathsf{id}_{{}_{\overline{\mathbb{H}}}\mathsf{Mod}}.$$

On the other hand, consider $(M, N, \phi_M, \varphi_M, \varphi_N, \gamma)$ in ${}_{(T,\varphi_H)}\mathsf{Mod}^{\mathsf{iso}}$. We have that

$$\begin{aligned} &(\mathsf{V} \circ \mathsf{U}')((M, N, \phi_M, \varphi_M, \varphi_N, \gamma)) \\ &\quad = \mathsf{V}((M, \phi_M, \overline{\varphi}_M)) = (M, M, \phi_M, \varphi_M, \overline{\varphi}_M \circ (T^{-1} \otimes M), id_M) \text{ (by (3.13))}, \end{aligned}$$

which is isomorphic to $(M, N, \phi_M, \varphi_M, \varphi_N, \gamma)$ via (id_M, γ^{-1}). Indeed, to show that (id_M, γ^{-1}) is a morphism in ${}_{(T,\varphi_H)}\mathsf{Mod}$ it is enough to see that $\gamma^{-1}\colon (N, \varphi_N) \to (M, \overline{\varphi}_M \circ (T^{-1} \otimes M))$ is a morphism of left B-modules, which follows by

$$\begin{aligned} \gamma^{-1} \circ \varphi_N &= \gamma^{-1} \circ \varphi_N \circ (T \otimes \gamma) \circ (T^{-1} \otimes \gamma^{-1}) \\ &\qquad \text{(by the fact that } T \text{ and } \gamma \text{ are isomorphisms)} \\ &= \overline{\varphi}_M \circ (T^{-1} \otimes \gamma^{-1}) \text{ (by (3.9))}. \end{aligned}$$

As a consequence,

$$\mathsf{V} \circ \mathsf{U}' \simeq \mathsf{id}_{{}_{(T,\varphi_H)}\mathsf{Mod}^{\mathsf{iso}}}$$

which concludes the proof. □

Corollary 3.21. *Suppose that* $\mathbb{H}$ *is a cocommutative Hopf brace. The category* ${}_{(id_H,\Gamma_{H_1})}\mathsf{Mod}^{\mathsf{iso}}$ *is equivalent to* ${}_{\mathbb{H}}\mathsf{Mod}$.

Proof. The proof is a direct consequence of the previous theorem taking into account that $(\mathsf{G} \circ \mathsf{F})(\mathbb{H}) = \mathbb{H}$, where F and G are the functors introduced in Theorem 3.7. □

4. PROJECTIONS OF RELATIVE ROTA-BAXTER OPERATORS

This section is devoted to the study of projections between relative Rota-Baxter operators. A projection between this kind of objects involves projections of Hopf algebras. Then, at the beginning of this section we will make a brief summary of the basic theory linked to projections of Hopf algebras including the Yetter-Drinfeld modules following [1], [19] and [20].

Definition 4.1. Let X be a Hopf algebra in C. We shall denote by ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$ the category of left Yetter-Dinfeld modules over X. More concretely, a triple $M = (M, \varphi_M, \varrho_M)$ is an object in ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$ if (M, φ_M) is a left X-module, (M, ϱ_M) is a left X-comodule and the identity

$$\begin{aligned}(\mu_X \otimes M) \circ (X \otimes c_{M,X}) \circ ((\varrho_M \circ \varphi_M) \otimes X) \circ (X \otimes c_{X,M}) \circ (\delta_X \otimes M)\\ = (\mu_X \otimes \varphi_M) \circ (X \otimes c_{X,X} \otimes M) \circ (\delta_X \otimes \varrho_M)\end{aligned} \tag{4.1}$$

holds. The morphisms in ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$ are morphisms of left X-modules and left X-comodules.

For example, for any Hopf algebra X, $(X, \varphi_X^{\mathrm{ad}}, \varrho_X = \delta_X)$ and $(X, \varphi_X = \mu_X, \varrho_X^{\mathrm{ad}})$ are left Yetter-Drinfeld modules over X. Also, any left X-module (M, φ_M) over a cocommutative Hopf algebra X is a Yetter-Drinfeld module with the trivial left coaction $\varrho_M = \eta_X \otimes M$. Finally, the triple $(M, \varphi_M = \varepsilon_X \otimes M, \varrho_M = \eta_X \otimes M)$ is a left Yetter-Drinfeld module for all Hopf algebra X.

The category ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$ is strict monoidal with the usual tensor product in C, that is to say, for M, N in ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$, $M \otimes N$ is a left Yetter-Drinfeld module over X with the tensor module and comodule structures given by

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (X \otimes c_{X,M} \otimes N) \circ (\delta_X \otimes M \otimes N),$$
$$\varrho_{M\otimes N} = (\mu_X \otimes M \otimes N) \circ (X \otimes c_{M,X} \otimes N) \circ (\varrho_M \otimes \varrho_N).$$

If the antipode of X is an isomorphism, ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$ is a braided monoidal category, where the braiding $t_{M,N} \colon M \otimes N \to N \otimes M$ is given by

$$t_{M,N} = (\varphi_N \otimes M) \circ (X \otimes c_{M,N}) \circ (\varrho_M \otimes N).$$

It is immediate to see that $t_{M,N}$ is natural and it is an isomorphism with inverse

$$t^{-1}_{M,N} = c^{-1}_{M,N} \circ (\varphi_N \otimes M) \circ (\lambda_X^{-1} \otimes N \otimes M) \circ (c^{-1}_{X,N} \otimes M) \circ (N \otimes \varrho_M).$$

Definition 4.2. A projection of Hopf algebras in C is a 4-tuple (X, Y, f, g), where X, Y are Hopf algebras, and $f \colon X \to Y$, $g \colon Y \to X$ are Hopf algebra morphisms such that $g \circ f = id_X$.

A morphism between projections of Hopf algebras (X,Y,f,g) and (X',Y',f',g') is a pair (x,y), where $x\colon\ X\to X'$, $y\colon\ Y\to Y'$ are Hopf algebra morphisms such that

$$y\circ f=f'\circ x,\quad x\circ g=g'\circ y. \tag{4.2}$$

With the obvious composition of morphisms we can define a category whose objects are Hopf algebra projections and whose morphisms are morphisms of Hopf algebra projections. We denote this category by $\mathsf{P}(\mathsf{Hopf})$.

It is obvious that there exists a functor $\mathsf{P}_{\rm triv}\colon\ \mathsf{Hopf}\to\mathsf{P}(\mathsf{Hopf})$ defined on objects by $\mathsf{P}_{\rm triv}(X)=(X,X,id_X,id_X)$ and on morphisms by $\mathsf{P}(f)=(f,f)$.

Let (X,Y,f,g) be an object in $\mathsf{P}(\mathsf{Hopf})$. The morphism $q_Y:=id_Y*(f\circ\lambda_X\circ g)$ is idempotent and, as a consequence, there exists an object $I(q_Y)$, called the object of coinvariants, an epimorphism p_Y and a monomorphism i_Y such that $q_Y=i_Y\circ p_Y$ and $p_Y\circ i_Y=id_{I(q_Y)}$. As a consequence,

$$\begin{tikzcd}
I(q_Y) \arrow[r, "i_Y"] & Y \arrow[r, shift left, "(Y\otimes g)\circ\delta_Y"] \arrow[r, shift right, "Y\otimes\eta_X"'] & Y\otimes X
\end{tikzcd}$$

is an equalizer diagram and $I(q_Y)$ is a left X-module algebra, where the algebra structure is defined by

$$\eta_{I(q_Y)}=p_Y\circ\eta_Y,\quad \mu_{I(q_Y)}=p_Y\circ\mu_Y\circ(i_Y\otimes i_Y), \tag{4.3}$$

i.e., $\eta_{I(q_Y)}$ is the unique morphism such that $i_Y\circ\eta_{I(q_Y)}=\eta_Y$ and $\mu_{I(q_Y)}$ is the unique morphism such that

$$i_Y\circ\mu_{I(q_Y)}=\mu_Y\circ(i_Y\otimes i_Y). \tag{4.4}$$

The action $\psi_{I(q_Y)}\colon\ X\otimes I(q_Y)\to I(q_Y)$ is $\psi_{I(q_Y)}=p_Y\circ\mu_Y\circ(f\otimes i_Y)$, and then $\psi_{I(q_Y)}$ is the unique morphism such that

$$i_Y\circ\psi_{I(q_Y)}=\varphi_Y^{\rm ad}\circ(f\otimes i_Y). \tag{4.5}$$

On the other hand,

$$\begin{tikzcd}
Y\otimes X \arrow[r, shift left, "\mu_Y\circ(Y\otimes f)"] \arrow[r, shift right, "Y\otimes\varepsilon_X"'] & Y \arrow[r, "p_Y"] & I(q_Y)
\end{tikzcd}$$

is a coequalizer diagram and, as a consequence, $I(q_Y)$ is a left X-comodule coalgebra with

$$\varepsilon_{I(q_Y)}=\varepsilon_Y\circ i_Y,\quad \delta_{I(q_Y)}=(p_Y\otimes p_Y)\circ\delta_Y\circ i_Y \tag{4.6}$$

and coaction $\varrho_{I(q_Y)}\colon\ I(q_Y) \to X \otimes I(q_Y)$ defined by

$$\varrho_{I(q_Y)} = (g \otimes p_Y) \circ \delta_Y \circ i_Y.$$

In this case $\varepsilon_{I(q_Y)}$ is the unique morphism such that $\varepsilon_{I(q_Y)} \circ p_Y = \varepsilon_Y$, $\delta_{I(q_Y)}$ is the unique morphism such that

$$\delta_{I(q_Y)} \circ p_Y = (p_Y \otimes p_Y) \circ \delta_Y, \tag{4.7}$$

and the coaction $\varrho_{I(q_Y)}$ is the unique morphism satisfying

$$\varrho_{I(q_Y)} \circ p_Y = (g \otimes p_Y) \circ \varrho_Y^{\mathrm{ad}}. \tag{4.8}$$

The algebra-coalgebra $I(q_Y)$, with the action $\psi_{I(q_Y)}$ and the coaction $\varrho_{I(q_Y)}$, is a Hopf algebra in ${}^{\mathsf{X}}_{\mathsf{X}}\mathsf{YD}$ with antipode

$$\lambda_{I(q_Y)} = \psi_{I(q_Y)} \circ (X \otimes (p_Y \circ \lambda_Y \circ i_Y)) \circ \varrho_{I(q_Y)}.$$

Also, using that i_Y is an equalizer morphism and p_Y is a coequalizer, we obtain the following identities:

$$p_Y \circ \mu_Y \circ (Y \otimes q_Y) = p_Y \circ \mu_Y, \quad (Y \otimes q_Y) \circ \delta_Y \circ i_Y = \delta_Y \circ i_Y. \tag{4.9}$$

Note that i_Y is a coalgebra morphism if and only if

$$(q_Y \otimes Y) \circ \delta_Y \circ i_Y = \delta_Y \circ i_Y. \tag{4.10}$$

If Y is cocommutative, condition (4.10) always holds. This fact was proved by Sweedler in [23] for projections of Hopf algebras in the category of vector spaces. On the other hand, there exist examples where i_Y is not a coalgebra morphism, see [5] for the complete details. In any case, if i_Y is a coalgebra morphism, then we have that $I(q_Y)$ is a Hopf algebra in C because $\varrho_{I(q_Y)}$ is trivial.

Similarly, p_Y is an algebra morphism if and only if

$$p_Y \circ \mu_Y \circ (q_Y \otimes Y) = p_Y \circ \mu_Y. \tag{4.11}$$

Equivalently, p_Y is an algebra morphism if and only if $\psi_{I(q_Y)} = \varepsilon_X \otimes I(q_Y)$, see [1]. Therefore, if p_Y is an algebra morphism, then we have that $I(q_Y)$ is a Hopf algebra in C because $\psi_{I(q_Y)}$ is trivial.

Finally, if (X, Y, f, g) is in $\mathsf{P}(\mathsf{Hopf})$ and Y is cocommutative, then the morphism q_Y is a coalgebra morphism. Also, under these conditions, i_Y is a coalgebra morphism and the following equality

$$i_Y \circ \lambda_{I(q_Y)} = \lambda_Y \circ i_Y \tag{4.12}$$

holds, see [11].

In the following definition the notion of projection between Hopf algebras is extended in order to introduce what a projection between relative Rota-Baxter operators is.

Definition 4.3. Let $\left(T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \; , \varphi_H \right)$ and $\left(L \; \begin{matrix} A \\ \downarrow \\ D \end{matrix} \; , \varphi_A \right)$ be relative Rota-Baxter operators. We will say that a 6-tuple

$$\left(\left(T \quad \begin{matrix} H \\ \downarrow \\ B \end{matrix} \quad , \varphi_H \right), \left(L \quad \begin{matrix} A \\ \downarrow \\ D \end{matrix} \quad , \varphi_A \right), f, h, g, l \right)$$

is a projection of relative Rota-Baxter operators if

(i) the 4-tuple (H, A, f, g) is an object in $\mathsf{P}(\mathsf{Hopf})$,

(ii) the 4-tuple (B, D, h, l) is an object in $\mathsf{P}(\mathsf{Hopf})$,

(iii) the pair $(f, h) \colon \left(T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \; , \varphi_H \right) \to \left(L \; \begin{matrix} A \\ \downarrow \\ D \end{matrix} \; , \varphi_A \right)$ is a morphism of relative Rota-Baxter operators,

(iv) the pair $(g, l) \colon \left(L \; \begin{matrix} A \\ \downarrow \\ D \end{matrix} \; , \varphi_A \right) \to \left(T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \; , \varphi_H \right)$ is a morphism of relative Rota-Baxter operators.

$$\begin{tikzcd}
H \arrow[loop left, "id_H"] \arrow[r, shift left, "f"] \arrow[d, "T"'] & A \arrow[l, shift left, "g"] \arrow[d, "L"] \\
B \arrow[loop left, "id_B"] \arrow[r, shift left, "h"] & D \arrow[l, shift left, "l"]
\end{tikzcd}$$

Figure 1. Projection of relative Rota-Baxter operators.

Projections of relative Rota-Baxter operators give rise to a category whose morphisms are defined as follows: We will say that a 4-tuple (x, y, z, t) is a morphism between the projections of relative Rota-Baxter operators

$$\left(\left(\begin{array}{ccc} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{array}\right), \left(\begin{array}{ccc} & A & \\ L & \downarrow & ,\varphi_A \\ & D & \end{array}\right), f, h, g, l\right)$$
$$\to \left(\left(\begin{array}{ccc} & H' & \\ T' & \downarrow & ,\varphi_{H'} \\ & B' & \end{array}\right), \left(\begin{array}{ccc} & A' & \\ L' & \downarrow & ,\varphi_{A'} \\ & D' & \end{array}\right), f', h', g', l'\right)$$

if the following conditions hold:

(i) The pair $(x, y)\colon (H, A, f, g) \to (H', A', f', g')$ is a morphism in $\mathsf{P(Hopf)}$.

(ii) The pair $(z, t)\colon (B, D, h, l) \to (B', D', h', l')$ is a morphism in $\mathsf{P(Hopf)}$.

(iii) The pair $(x, z)\colon \left(\begin{array}{ccc} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{array}\right) \to \left(\begin{array}{ccc} & H' & \\ T' & \downarrow & ,\varphi_{H'} \\ & B' & \end{array}\right)$ is a morphism in rRB.

(iv) The pair $(y, t)\colon \left(\begin{array}{ccc} & A & \\ L & \downarrow & ,\varphi_A \\ & D & \end{array}\right) \to \left(\begin{array}{ccc} & A' & \\ L' & \downarrow & ,\varphi_{A'} \\ & D' & \end{array}\right)$ is a morphism in rRB.

This category will be denoted by $\mathsf{P(rRB)}$.

$$\begin{tikzcd}
& H \arrow[loop left, "id_H"] \arrow[rr, shift left, "f"] \arrow[d, "T"'] \arrow[ddl, "x"'] & & A \arrow[ll, shift left, "g"] \arrow[d, "L"] \arrow[ddl, "y"] \\
& B \arrow[loop left, "id_B"] \arrow[rr, shift left, "h"] \arrow[ddl, "z"] & & D \arrow[ll, shift left, "l"] \arrow[ddl, "t"] \\
H' \arrow[loop left, "id_{H'}"] \arrow[rr, shift left, "f'"] \arrow[d, "T'"'] & & A' \arrow[ll, shift left, "g'"] \arrow[d, "L'"] & \\
B' \arrow[loop left, "id_B"] \arrow[rr, shift left, "h'"] & & D' \arrow[ll, shift left, "l'"] &
\end{tikzcd}$$

Figure 2. Morphism of projections of relative Rota-Baxter operators.

By $\mathsf{P(rRB^\star)}$ we will denote the full subcategory of $\mathsf{P(rRB)}$, where the involved relative Rota-Baxter operators are objects in $\mathsf{rRB^\star}$. Finally, $\mathsf{P(coc\text{-}rRB)}$ denotes the full subcategory of $\mathsf{P(rRB^\star)}$, where the involved objects are cocommutative relative Rota-Baxter operators.

Let

$$\left(\left(\begin{array}{ccc} & H & \\ T & \downarrow & ,\varphi_H \\ & B & \end{array}\right), \left(\begin{array}{ccc} & A & \\ L & \downarrow & ,\varphi_A \\ & D & \end{array}\right), f, h, g, l\right)$$

be an object in $\mathsf{P}(\mathsf{coc\text{-}rRB})$. Due to being (H, A, f, g) and (B, D, h, l) objects in $\mathsf{P}(\mathsf{Hopf})$ with A and D cocommutative, we know that the respective objects of coinvariants, $I(q_A)$ and $I(q_D)$, are Hopf algebras in C and also the equalizers, i_A and i_D, are coalgebra morphisms. Moreover, the morphism $L \circ i_A$ factors through the equalizer i_D because

$$\begin{aligned}
&(D \otimes l) \circ \delta_D \circ L \circ i_A \\
&\quad = (L \otimes (l \circ L)) \circ \delta_A \circ i_A \text{ (by the condition of coalgebra morphism for } L) \\
&\quad = (L \otimes T) \circ (A \otimes g) \circ \delta_A \circ i_A \text{ (by (3.2) for the morphism } (g, l)) \\
&\quad = ((L \circ i_A) \otimes (T \circ \eta_H)) \text{ (by the equalizer condition for } i_A) \\
&\quad = (T \circ i_A) \otimes \eta_B \text{ (by (3.4) for } T).
\end{aligned}$$

As a conclusion, there exists a unique $L_0 \colon I(q_A) \to I(q_D)$ satisfying that

$$i_D \circ L_0 = L \circ i_A. \tag{4.13}$$

By composing on the left with the epimorphism p_D and taking into account that $p_D \circ i_D = id_{I(q_D)}$, we obtain that

$$L_0 = p_D \circ L \circ i_A. \tag{4.14}$$

Also note that there exists an action

$$\varphi_{I(q_A)} \colon I(q_D) \otimes I(q_A) \to I(q_A),$$

which is obtained by factorization through the equalizer i_A of the morphism $\varphi_A \circ (i_D \otimes i_A)$. Indeed,

$$\begin{aligned}
&(A \otimes g) \circ \delta_A \circ \varphi_A \circ (i_D \otimes i_A) \\
&\quad = (A \otimes g) \circ (\varphi_A \otimes \varphi_A) \circ (D \otimes c_{D,A} \otimes A) \circ ((\delta_D \circ i_D) \otimes (\delta_A \circ i_A)) \\
&\qquad \text{(by the condition of coalgebra morphism for } \varphi_A) \\
&\quad = (\varphi_A \otimes \varphi_H) \circ (D \otimes c_{B,A} \otimes H) \circ (((D \otimes l) \circ \delta_D \circ i_D) \otimes ((A \otimes g) \circ \delta_A \circ i_A)) \\
&\qquad \text{(by (3.3) for } (g, l) \text{ and naturality of } c) \\
&\quad = (\varphi_A \circ (i_D \otimes i_A)) \otimes \eta_H \text{ (by equalizer condition for } i_D \text{ and } i_A, \text{ naturality} \\
&\qquad \text{of } c \text{ and module axioms).}
\end{aligned}$$

Therefore, there exists a unique $\varphi_{I(q_A)} \colon I(q_D) \otimes I(q_A) \to I(q_A)$ satisfying that

$$i_A \circ \varphi_{I(q_A)} = \varphi_A \circ (i_D \otimes i_A). \tag{4.15}$$

Then, by composing on the left with p_A and using that $p_A \circ i_A = id_{I(q_A)}$, the equality

$$\varphi_{I(q_A)} = p_A \circ \varphi_A \circ (i_D \otimes i_A) \tag{4.16}$$

holds.

$$\begin{tikzcd}
I(q_A) \arrow[r, "i_A"] \arrow[d, dashed, "\exists^{\bullet} L_0"] & A \arrow[r, shift left, "(A\otimes g)\circ\delta_A"] \arrow[r, shift right, "A\otimes\eta_H"'] \arrow[d, "L"] & A \otimes H \\
I(q_D) \arrow[r, "i_D"] & D \arrow[r, shift left, "(D\otimes l)\circ\delta_D"] \arrow[r, shift right, "D\otimes\eta_B"'] & D \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
I(q_A) \arrow[r, "i_A"] & A \arrow[r, shift left, "(A\otimes g)\circ\delta_A"] \arrow[r, shift right, "A\otimes\eta_H"'] & A \otimes H \\
 & I(q_D) \otimes I(q_A) \arrow[u, "\varphi_A\circ(i_D\otimes i_A)"'] \arrow[ul, dashed, "\exists^{\bullet}\varphi_{I(q_A)}"] &
\end{tikzcd}$$

Figure 3. The construction of L_0 and $\varphi_{I(q_A)}$.

Theorem 4.4. *Let*

$$\left(\left(T\ \begin{array}{c}H\\ \downarrow\\ B\end{array}\ ,\varphi_H\right),\left(L\ \begin{array}{c}A\\ \downarrow\\ D\end{array}\ ,\varphi_A\right),f,h,g,l\right)$$

be an object in $\mathsf{P}(\mathsf{coc\text{-}rRB})$. *Then if* L_0 *is the morphism defined by* (4.14) *and* $\varphi_{I(q_A)}$ *the action introduced in* (4.16), *we have that*

$$\left(L_0\ \begin{array}{c}I(q_A)\\ \downarrow\\ I(q_D)\end{array}\ ,\varphi_{I(q_A)}\right),$$

is a relative Rota-Baxter operator.

Proof. First of all note that $(I(q_A), \varphi_{I(q_A)})$ is a left $I(q_D)$-module algebra-coalgebra. Indeed, module axioms are straightforward thanks to (4.15) and module axioms for (A, φ_A). Moreover, it is direct to compute that $\eta_{I(q_A)}$ is a morphism of left $I(q_D)$-modules and

$i_A \circ \varphi_{I(q_A)} \circ (I(q_D) \otimes \mu_{I(q_A)})$

$= \varphi_A \circ (D \otimes \mu_A) \circ (i_D \otimes i_A \otimes i_A)$ (by (4.15) and (4.4))

$= \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (D \otimes c_{D,A} \otimes A) \circ ((\delta_D \circ i_D) \otimes i_A \otimes i_A)$ (by the condition of morphism of left D-modules for μ_A)

$$
\begin{aligned}
&= \mu_A \circ ((\varphi_A \circ (i_D \otimes i_A)) \otimes (\varphi_A \circ (i_D \otimes i_A))) \circ (I(q_D) \otimes c_{I(q_D),I(q_A)} \otimes I(q_A)) \\
&\quad \circ (\delta_{I(q_D)} \otimes I(q_A) \otimes I(q_A)) \text{ (by the condition of coalgebra morphism for } i_D \\
&\quad \text{and naturality of } c) \\
&= i_A \circ \mu_{I(q_A)} \circ (\varphi_{I(q_A)} \otimes \varphi_{I(q_A)}) \circ (I(q_D) \otimes c_{I(q_D),I(q_A)} \otimes I(q_A)) \\
&\quad \circ (\delta_{I(q_D)} \otimes I(q_A) \otimes I(q_A)) \text{ (by (4.15) and (4.4)),}
\end{aligned}
$$

so $(I(q_A), \varphi_{I(q_A)})$ is a left $I(q_D)$-module algebra. To finish, $\varphi_{I(q_A)}$ is a coalgebra morphism because

$$
\begin{aligned}
&\delta_{I(q_A)} \circ \varphi_{I(q_A)} \\
&= (p_A \otimes p_A) \circ \delta_A \circ \varphi_A \circ (i_D \otimes i_A) \text{ (by (4.15))} \\
&= (p_A \otimes p_A) \circ (\varphi_A \otimes \varphi_A) \circ (D \otimes c_{D,A} \otimes A) \circ ((\delta_D \circ i_D) \otimes (\delta_A \circ i_A)) \\
&\quad \text{(by the condition of coalgebra morphism for } \varphi_A) \\
&= ((p_A \circ \varphi_A \circ (i_D \otimes i_A)) \otimes (p_A \circ \varphi_A \circ (i_D \otimes i_A))) \\
&\quad \circ (I(q_D) \otimes c_{I(q_D),I(q_A)} \otimes I(q_A)) \circ (\delta_{I(q_D)} \otimes \delta_{I(q_A)}) \\
&\quad \text{(by the condition of coalgebra morphism for } i_D \text{ and } i_A \text{ and naturality of } c) \\
&= (\varphi_{I(q_A)} \otimes \varphi_{I(q_A)}) \circ (I(q_D) \otimes c_{I(q_D),I(q_A)} \otimes I(q_A)) \circ (\delta_{I(q_D)} \otimes \delta_{I(q_A)}) \text{ (by (4.16)),}
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon_{I(q_A)} \circ \varphi_{I(q_A)} \\
&= \varepsilon_A \circ \varphi_A \circ (i_D \otimes i_A) \text{ (by (4.6) and (4.15))} \\
&= ((\varepsilon_D \circ i_D) \otimes (\varepsilon_A \circ i_A)) \text{ (by the condition of coalgebra morphism for } \varphi_A) \\
&= \varepsilon_{I(q_D)} \otimes \varepsilon_{I(q_A)} \text{ (by (4.6)).}
\end{aligned}
$$

On the other hand, let us see that L_0 is a coalgebra morphism. Indeed,

$$
\begin{aligned}
&\delta_{I(q_D)} \circ L_0 \\
&= (p_D \otimes p_D) \circ \delta_D \circ L \circ i_A \text{ (by (4.13))} \\
&= ((p_D \circ L) \otimes (p_D \circ L)) \circ \delta_A \circ i_A \text{ (by the condition of coalgebra morphism} \\
&\quad \text{for } L) \\
&= ((p_D \circ L \circ i_A) \otimes (p_D \circ L \circ i_A)) \circ \delta_{I(q_A)} \text{ (by the condition of coalgebra} \\
&\quad \text{morphism for } i_A) \\
&= (L_0 \otimes L_0) \circ \delta_{I(q_A)} \text{ (by (4.14)).}
\end{aligned}
$$

So, the only condition that remains to compute is (3.1), which follows from:

$$\begin{aligned}
&i_D \circ L_0 \circ \mu_{I(q_A)} \circ (I(q_A) \otimes (\varphi_{I(q_A)} \circ (L_0 \otimes I(q_A)))) \circ (\delta_{I(q_A)} \otimes I(q_A)) \\
&\quad = L \circ i_A \circ \mu_{I(q_A)} \circ (I(q_A) \otimes (\varphi_{I(q_A)} \circ (L_0 \otimes I(q_A)))) \circ (\delta_{I(q_A)} \otimes I(q_A)) \\
&\qquad \text{(by (4.13))} \\
&\quad = L \circ \mu_A \circ (i_A \otimes (i_A \circ \varphi_{I(q_A)} \circ (L_0 \otimes I(q_A)))) \circ (\delta_{I(q_A)} \otimes I(q_A)) \text{ (by (4.4))} \\
&\quad = L \circ \mu_A \circ (A \otimes (\varphi_A \circ (L \otimes A))) \circ (((i_A \otimes i_A) \circ \delta_{I(q_A)}) \otimes i_A) \\
&\qquad \text{(by (4.15) and (4.13))} \\
&\quad = L \circ \mu_A \circ (A \otimes (\varphi_A \circ (L \otimes A))) \circ (\delta_A \otimes A) \circ (i_A \otimes i_A) \text{ (by the condition} \\
&\qquad \text{of coalgebra morphism for } i_A) \\
&\quad = \mu_D \circ ((L \circ i_A) \otimes (L \circ i_A)) \text{ (by (3.1) for } \left(\begin{array}{ccc} & A & \\ L & \downarrow & ,\varphi_A \\ & D & \end{array}\right) \\
&\quad = i_D \circ \mu_{I(q_D)} \circ (L_0 \otimes L_0) \text{ (by (4.13) and (4.4)).}
\end{aligned}$$

□

Corollary 4.5. *Under the conditions of the previous theorem, the pair* (i_A, i_D) *is a morphism of relative Rota-Baxter operators between* $\left(\begin{array}{ccc} & I(q_A) & \\ L_0 & \downarrow & ,\varphi_{I(q_A)} \\ & I(q_D) & \end{array}\right)$ *and* $\left(\begin{array}{ccc} & A & \\ L & \downarrow & ,\varphi_A \\ & D & \end{array}\right)$.

Proof. By (4.13), (3.2) holds and (3.3) follows by (4.15). □

Example 4.6. In [13], Goncharov proved that if X is a cocommutative Hopf algebra, then

$$\left(\begin{array}{ccc} & X & \\ \lambda_X & \downarrow & ,\varphi_X^{\mathrm{ad}} \\ & X & \end{array}\right)$$

is a relative Rota-Baxter operator. So, if H and A are cocommutative Hopf algebras and $(H, A, f, g) \in \mathsf{P}(\mathsf{Hopf})$, then

$$\left(\left(\begin{array}{ccc} & H & \\ \lambda_H & \downarrow & ,\varphi_H^{\mathrm{ad}} \\ & H & \end{array}\right), \left(\begin{array}{ccc} & A & \\ \lambda_A & \downarrow & ,\varphi_A^{\mathrm{ad}} \\ & A & \end{array}\right), f, f, g, g\right)$$

is a projection of relative Rota-Baxter operators. As a consequence, by Theorem 4.4, there exists a relative Rota-Baxter operator, $\left(\begin{array}{ccc} & I(q_A) & \\ L_0 & \downarrow & ,\varphi_{I(q_A)} \\ & I(q_A) & \end{array}\right)$, where L_0 and $\varphi_{I(q_A)}$ satisfy (4.14) and (4.16), respectively. Note that, due to A being cocommutative, by (4.12),

$$L_0 = p_A \circ \lambda_A \circ i_A = \lambda_{I(q_A)}.$$

Then we obtain that $\left(\lambda_{I(q_A)}\ \begin{array}{c} I(q_A) \\ \downarrow \\ I(q_A) \end{array}, \varphi_{I(q_A)}\right)$ is a relative Rota-Baxter operator, where $\varphi_{I(q_A)} = p_A \circ \varphi_A^{\mathrm{ad}} \circ (i_A \otimes i_A)$.

$$\begin{tikzcd}
I(q_A) \arrow[r, "i_A"] \arrow[d, "L_0=\lambda_{I(q_A)}"'] & A \arrow[r, shift left, "g"] \arrow[d, "\lambda_A"] & H \arrow[l, shift left, "f"] \arrow[d, "\lambda_H"] \arrow[loop right] \\
I(q_A) \arrow[r, "i_A"] & A \arrow[r, shift left, "g"] & H \arrow[l, shift left, "f"] \arrow[loop right]
\end{tikzcd}$$

Figure 4. $\lambda_{I(q_A)}$ is a relative Rota-Baxter operator.

Corollary 4.7. *There exists a functor*

$$\mathsf{P}\colon \mathsf{P}(\mathsf{coc\text{-}rRB}) \to \mathsf{coc\text{-}rRB}$$

acting on objects by

$$\mathsf{P}\left(\left(\left(T\ \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H\right), \left(L\ \begin{array}{c} A \\ \downarrow \\ D \end{array}, \varphi_A\right), f, h, g, l\right)\right) = \left(L_0\ \begin{array}{c} I(q_A) \\ \downarrow \\ I(q_D) \end{array}, \varphi_{I(q_A)}\right)$$

and on morphisms by $\mathsf{P}((x, y, z, t)) = (y_0, t_0)$, *where* $y_0\colon I(q_A) \to I(q_{A'})$ *is the unique morphism satisfying that* $i_{A'} \circ y_0 = y \circ i_A$ *and* $t_0\colon I(q_D) \to I(q_{D'})$ *is the unique morphism such that* $i_{D'} \circ t_0 = t \circ i_D$.

Proof. Functor P is well-defined on objects thanks to Theorem 4.4. Consider (x, y, z, t) a morphism in $\mathsf{P}(\mathsf{coc\text{-}rRB})$ between

$$\left(\left(T\ \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H\right), \left(L\ \begin{array}{c} A \\ \downarrow \\ D \end{array}, \varphi_A\right), f, h, g, l\right)$$

and

$$\left(\left(T'\ \begin{array}{c} H' \\ \downarrow \\ B' \end{array}, \varphi_{H'}\right), \left(L'\ \begin{array}{c} A' \\ \downarrow \\ D' \end{array}, \varphi_{A'}\right), f', h', g', l'\right).$$

On the one hand, note that $y \circ i_A$ factors through the equalizer $i_{A'}$. Indeed,

$$\begin{aligned}
&(A' \otimes g') \circ \delta_{A'} \circ y \circ i_A \\
&\qquad = (y \otimes (g' \circ y)) \circ \delta_A \circ i_A \ \text{(by the condition of coalgebra morphism for } y\text{)} \\
&\qquad = (y \otimes (x \circ g)) \circ \delta_A \circ i_A \ \text{(by (4.2) for } (x, y)\text{)} \\
&\qquad = (y \circ i_A) \otimes (x \circ \eta_H) \ \text{(by the equalizer condition for } i_A\text{)} \\
&\qquad = (y \circ i_A) \otimes \eta_{H'} \ \text{(by the condition of algebra morphism for } x\text{)}.
\end{aligned}$$

Therefore, there exists a unique $y_0\colon\ I(q_A) \to I(q_{A'})$ satisfying that

$$i_{A'} \circ y_0 = y \circ i_A. \tag{4.17}$$

As a consequence, if we compose on the left with $p_{A'}$, then the following equality is obtained:

$$y_0 = p_{A'} \circ y \circ i_A. \tag{4.18}$$

Note also that y_0 is a Hopf algebra morphism. On the one side, it is an algebra morphism because

$$\begin{aligned} i_{A'} \circ y_0 \circ \eta_{I(q_A)} &= y \circ i_A \circ \eta_{I(q_A)} \text{ (by (4.17))} \\ &= y \circ \eta_A \text{ (by the equality } i_A \circ \eta_{I(q_A)} = \eta_A) \\ &= \eta_{A'} \text{ (by the condition of algebra morphism for } y) \\ &= i_{A'} \circ \eta_{I(q_{A'})} \text{ (by the equality } i_{A'} \circ \eta_{I(q_{A'})} = \eta_{A'}) \end{aligned}$$

and also

$$\begin{aligned} & i_{A'} \circ y_0 \circ \mu_{I(q_A)} \\ &\quad = y \circ i_A \circ \mu_{I(q_A)} \text{ (by (4.17))} = y \circ \mu_A \circ (i_A \otimes i_A) \text{ (by (4.4))} \\ &\quad = \mu_{A'} \circ ((y \circ i_A) \otimes (y \circ i_A)) \text{ (by the condition of algebra morphism for } y) \\ &\quad = \mu_{A'} \circ (i_{A'} \otimes i_{A'}) \circ (y_0 \otimes y_0) \text{ (by (4.17))} = i_{A'} \circ \mu_{I(q_{A'})} \circ (y_0 \otimes y_0) \text{ (by (4.4))}. \end{aligned}$$

On the other side, $\varepsilon_{I(q_{A'})} \circ y_0 = \varepsilon_{I(q_A)}$ follows by (4.17) and the fact that y preserves the counit and

$$\begin{aligned} & \delta_{I(q_{A'})} \circ y_0 \\ &\quad = (p_{A'} \otimes p_{A'}) \circ \delta_{A'} \circ y \circ i_A \text{ (by (4.17))} \\ &\quad = (p_{A'} \otimes p_{A'}) \circ (y \otimes y) \circ \delta_A \circ i_A \text{ (by the condition of coalgebra morphism for } y) \\ &\quad = (y_0 \otimes y_0) \circ \delta_{I(q_A)} \text{ (by the condition of coalgebra morphism for } i_A \text{ and (4.18))}, \end{aligned}$$

so y_0 is a coalgebra morphism too. Analogously, it is easy to prove that $t \circ i_D$ factors through the equalizer $i_{D'}$, and then there exists a unique $t_0\colon\ I(q_D) \to I(q_{D'})$ such that the equalities

$$i_{D'} \circ t_0 = t \circ i_D, \tag{4.19}$$

$$t_0 = p_{D'} \circ t \circ i_D \tag{4.20}$$

hold and t_0 is a Hopf algebra morphism.

Thus, to conclude the proof it is enough to see that the pair (y_0, t_0) satisfies conditions (3.2) and (3.3). At first, (3.2) follows by

$$\begin{aligned} i_{D'} \circ L'_0 \circ y_0 &= L' \circ i_{A'} \circ y_0 \text{ (by (4.13))} = L' \circ y \circ i_A \text{ (by (4.17))} \\ &= t \circ L \circ i_A \text{ (by (3.2) for } (y,t)) = t \circ i_D \circ L_0 \text{ (by (4.13))} \\ &= i_{D'} \circ t_0 \circ L_0 \text{ (by (4.19)).} \end{aligned}$$

In addition, (3.3) is a consequence of

$$\begin{aligned} i_{A'} \circ y_0 \circ \varphi_{I(q_A)} &= y \circ i_A \circ \varphi_{I(q_A)} \text{ (by (4.17))} = y \circ \varphi_A \circ (i_D \otimes i_A) \text{ (by (4.15))} \\ &= \varphi_{A'} \circ ((t \circ i_D) \otimes (y \circ i_A)) \text{ (by (3.3) for } (y,t)) \\ &= \varphi_{A'} \circ (i_{D'} \otimes i_{A'}) \circ (t_0 \otimes y_0) \text{ (by (4.17) and (4.19))} \\ &= i_{A'} \circ \varphi_{I(q_{A'})} \circ (t_0 \otimes y_0) \text{ (by (4.15)).} \end{aligned}$$

Then P is also well-defined on morphisms.

$$\begin{tikzcd}
I(q_A) \arrow[r, "i_A"] \arrow[d, dashed, "\exists^{\bullet}\, y_0"] & A \arrow[r, shift left, "(A\otimes g)\circ\delta_A"] \arrow[r, shift right, "A\otimes\eta_H"'] \arrow[d, "y"] & A\otimes H \\
I(q_{A'}) \arrow[r, "i_{A'}"] & A' \arrow[r, shift left, "(A'\otimes g')\circ\delta_{A'}"] \arrow[r, shift right, "A'\otimes\eta_{H'}"'] & A'\otimes H'
\end{tikzcd}$$

$$\begin{tikzcd}
I(q_D) \arrow[r, "i_D"] \arrow[d, dashed, "\exists^{\bullet}\, t_0"] & D \arrow[r, shift left, "(D\otimes l)\circ\delta_D"] \arrow[r, shift right, "D\otimes\eta_B"'] \arrow[d, "t"] & D\otimes B \\
I(q_{D'}) \arrow[r, "i_{D'}"] & D' \arrow[r, shift left, "(D'\otimes l')\circ\delta_{D'}"] \arrow[r, shift right, "D'\otimes\eta_{B'}"'] & D'\otimes B'
\end{tikzcd}$$

Figure 5. The construction of y_0 and t_0.

□

In what follows we recall the notion of projection of Hopf braces introduced in [11].

Definition 4.8. A projection of Hopf braces in C is a 4-tuple $(\mathbb{H}, \mathbb{D}, x, y)$, where $\mathbb{H}$, $\mathbb{D}$ are Hopf braces in C, $x\colon \mathbb{H} \to \mathbb{D}$, $y\colon \mathbb{D} \to \mathbb{H}$ are morphisms of Hopf braces in C and the equality $y \circ x = id_{\mathbb{H}}$ holds.

A morphism between two projections of Hopf braces $(\mathbb{H}, \mathbb{D}, x, y)$ and $(\mathbb{H}', \mathbb{D}', x', y')$ is a pair (z, t), where $z\colon \mathbb{H} \to \mathbb{H}'$, $t\colon \mathbb{D} \to \mathbb{D}'$ are morphisms in HBr and the following equalities hold:

$$x' \circ z = t \circ x, \quad y' \circ t = z \circ y. \tag{4.21}$$

With these morphisms and the previous objects we can define the category of projections of Hopf braces. We will denote this category by $\mathsf{P(HBr)}$. With $\mathsf{P(coc\text{-}HBr)}$ we will denote the category of projections between cocommutative Hopf braces.

Remark 4.9. Following [11], if $(\mathbb{H}, \mathbb{D}, x, y)$ is a projection of Hopf braces in C, then we have two projections of Hopf algebras (H_1, D_1, x, y) and (H_2, D_2, x, y). Then with q_D^1 and q_D^2 we will denote the associated idempotent morphisms. Note that if $q_D^1 = i_D^1 \circ p_D^1$ and $q_D^2 = i_D^2 \circ p_D^2$, with $p_D^1 \circ i_D^1 = id_{I(q_D^1)}$ and $p_D^2 \circ i_D^2 = id_{I(q_D^2)}$, we have that

$$\begin{tikzcd} I(q_D^k) \arrow[r, "i_D^k"] & D \arrow[r, shift left, "(D\otimes y)\circ\delta_D"] \arrow[r, shift right, "D\otimes\eta_H"'] & D\otimes H \end{tikzcd}$$

is an equalizer diagram for $k \in \{1, 2\}$ and, as a consequence, we can assume that $i_D^1 = i_D^2 = i_D$ and $I(q_D^1) = I(q_D^2) = I(q_D)$. Then $p_D^1 \circ i_D = id_{I(q_D)} = p_D^2 \circ i_D$ holds.

Just as there exists an adjunction between the category of relative Rota-Baxter operators, $\mathsf{rRB}^\star$, and the category of cocommutative Hopf braces, $\mathsf{coc\text{-}HBr}$ (see Theorem 3.7), it seems reasonable to ask whether this adjunction carries over to their respective projection categories.

Theorem 4.10. *There exists a functor*

$$\mathsf{Q}\colon \mathsf{P(coc\text{-}HBr)} \to \mathsf{P(rRB^\star)}$$

acting on objects by $\mathsf{Q}((\mathbb{H}, \mathbb{D}, x, y)) = (\mathsf{F}(\mathbb{H}), \mathsf{F}(\mathbb{B}), x, x, y, y)$ *and on morphisms by* $\mathsf{Q}((z, t)) = (z, t, z, t)$, *where* F *denotes the functor introduced in Theorem* 3.7.

Proof. It follows by the fact that functor $\mathsf{F}\colon \mathsf{coc\text{-}HBr} \to \mathsf{rRB}^\star$ introduced in Theorem 3.7 is well-defined. □

Remark 4.11. Note that the image of Q is in $\mathsf{P(coc\text{-}rRB)}$. Then if $I^\star\colon \mathsf{P(coc\text{-}rRB)} \to \mathsf{P(rRB^\star)}$ denotes the inclusion functor, we have that $\mathsf{Q} = I^\star \circ \mathsf{Q}'$, where

$$\mathsf{Q}'\colon \mathsf{P(coc\text{-}HBr)} \to \mathsf{P(coc\text{-}rRB)}$$

is the functor defined as Q.

Theorem 4.12. *There exists a functor*

$$\mathsf{R}\colon \mathsf{P(rRB^\star)} \to \mathsf{P(coc\text{-}HBr)}$$

acting on objects by

$$\begin{aligned}
&\mathsf{R}\left(\left(\left(T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H\right), \left(L \begin{array}{c} A \\ \downarrow \\ D \end{array}, \varphi_A\right), f, h, g, l\right)\right) \\
&= \left(\mathsf{G}\left(\left(T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H\right)\right), \mathsf{G}\left(\left(L \begin{array}{c} A \\ \downarrow \\ D \end{array}, \varphi_A\right), f, g\right)\right),
\end{aligned}$$

where G *denotes the functor introduced in Theorem* 3.7*, and on morphisms by* $\mathsf{R}((x,y,z,t)) = (x,y)$.

Proof. The proof is straightforward taking into account that functor $\mathsf{G}\colon \mathsf{rRB}^{\star} \to \mathsf{coc\text{-}HBr}$ is well-defined, as it is possible to consult in Theorem 3.7. $\square$

Theorem 4.13. *The functor* Q *is a left adjoint of functor* R.

Proof. We have to show that there exists a bijection ${}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}$ between

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{P(rRB^{\star})}}\left(\mathsf{Q}((\mathbb{H},\mathbb{D},x,y)), \left(\left(T \begin{array}{c} X \\ \downarrow \\ B \end{array}, \varphi_X\right), \left(L \begin{array}{c} A \\ \downarrow \\ Y \end{array}, \varphi_A\right), f, h, g, l\right)\right) \\
&= \mathrm{Hom}_{\mathsf{P(rRB^{\star})}}\left(\left(\left(id_H \begin{array}{c} H_1 \\ \downarrow \\ H_2 \end{array}, \Gamma_{H_1}\right), \left(id_D \begin{array}{c} D_1 \\ \downarrow \\ D_2 \end{array}, \Gamma_{D_1}\right), x, x, y, y\right),\right. \\
&\qquad \left.\left(\left(T \begin{array}{c} X \\ \downarrow \\ B \end{array}, \varphi_X\right), \left(L \begin{array}{c} A \\ \downarrow \\ Y \end{array}, \varphi_A\right), f, h, g, l\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{P(coc\text{-}HBr)}}\left((\mathbb{H},\mathbb{D},x,y), \mathsf{R}\left(\left(\left(T \begin{array}{c} X \\ \downarrow \\ B \end{array}, \varphi_X\right), \left(L \begin{array}{c} A \\ \downarrow \\ Y \end{array}, \varphi_A\right), f, h, g, l\right)\right)\right) \\
&= \mathrm{Hom}_{\mathsf{P(coc\text{-}HBr)}}((\mathbb{H},\mathbb{D},x,y), (\overline{\mathbb{X}}, \overline{\mathbb{A}}, f, g)).
\end{aligned}$$

At first, consider (a,b,c,d) a morphism of relative Rota-Baxter projections between

$$\left(\left(id_H \begin{array}{c} H_1 \\ \downarrow \\ H_2 \end{array}, \Gamma_{H_1}\right), \left(id_D \begin{array}{c} D_1 \\ \downarrow \\ D_2 \end{array}, \Gamma_{D_1}\right), x, x, y, y\right)$$

and

$$\left(\left(T \quad \begin{array}{c} X \\ \downarrow \\ B \end{array} \quad, \varphi_X\right), \left(L \quad \begin{array}{c} A \\ \downarrow \\ Y \end{array} \quad, \varphi_A\right), f, h, g, l\right)$$

and let us show that (a, b) is a morphism in $\mathsf{P}(\mathsf{coc\text{-}HBr})$ between $(\mathbb{H}, \mathbb{D}, x, y)$ and $(\overline{\mathbb{X}}, \overline{\mathbb{A}}, f, g)$. Indeed, we only need to show that a, b are multiplicative morphisms. Then, on the one hand,

$$\begin{aligned}
&\overline{\mu}_X \circ (a \otimes a) \\
&\quad = \mu_X \circ (a \otimes (\varphi_X \circ ((T \circ a) \otimes a))) \circ (\delta_H \otimes H) \text{ (by the condition of coalgebra} \\
&\qquad \text{morphism for } a) \\
&\quad = \mu_X \circ (a \otimes (\varphi_X \circ (c \otimes a))) \circ (\delta_H \otimes H) \text{ (by (3.2) for } (a, c)) \\
&\quad = \mu_X \circ (a \otimes (a \circ \Gamma_{H_1})) \circ (\delta_H \otimes H) \text{ (by (3.3) for } (a, c)) \\
&\quad = a \circ \mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ (\delta_H \otimes H) \text{ (by the condition of algebra morphism} \\
&\qquad \text{for } a\colon\ H_1 \to X) \\
&\quad = a \circ \mu_H^2 \text{ (by (3.6)),}
\end{aligned}$$

and, on the other hand, $\overline{\mu}_A \circ (b \otimes b) = b \circ \mu_D^2$ following the same arguments as before. Then we define

$${}^{(\mathbb{H}, \mathbb{D}, x, y)}\Sigma_{(T, L, f, g, h, l)}(a, b, c, d) = (a, b).$$

Now, let (z, t) be a morphism of projections of Hopf braces between $(\mathbb{H}, \mathbb{D}, x, y)$ and $(\overline{\mathbb{X}}, \overline{\mathbb{A}}, f, g)$. Then let us prove that $(z, t, T \circ z, L \circ t)$ is a morphism of projections of relative Rota-Baxter operators between

$$\left(\left(id_H \quad \begin{array}{c} H_1 \\ \downarrow \\ H_2 \end{array} \quad, \Gamma_{H_1}\right), \left(id_D \quad \begin{array}{c} D_1 \\ \downarrow \\ D_2 \end{array} \quad, \Gamma_{D_1}\right), x, x, y, y\right)$$

and

$$\left(\left(T \quad \begin{array}{c} X \\ \downarrow \\ B \end{array} \quad, \varphi_X\right), \left(L \quad \begin{array}{c} A \\ \downarrow \\ Y \end{array} \quad, \varphi_A\right), f, h, g, l\right).$$

By hypothesis, $(z, t)\colon\ (H_1, D_1, x, y) \to (X, A, f, g)$ is a morphism of projections of Hopf algebras. On the other hand, $(T \circ z, L \circ t)\colon\ (H_2, D_2, x, y) \to (B, Y, h, l)$ is a morphism of projections of Hopf algebras too. That is due to the fact that $T \circ z$ and $L \circ t$ are Hopf algebra morphisms, which follows by

$$\begin{aligned}
&\mu_B \circ ((T \circ z) \otimes (T \circ z)) \\
&\quad = T \circ \overline{\mu}_X \circ (z \otimes z) \text{ (by (3.1) for } T) \\
&\quad = T \circ z \circ \mu_H^2 \text{ (by the condition of algebra morphism for } z\colon\ H_2 \to \overline{X})
\end{aligned}$$

and

$$\begin{aligned}
&\mu_Y \circ ((L \circ t) \otimes (L \circ t)) \\
&= L \circ \overline{\mu}_A \circ (t \otimes t) \text{ (by (3.1) for } L) \\
&= L \circ t \circ \mu_D^2 \text{ (by the condition of algebra morphism for } t\colon\ D_2 \to \overline{A}),
\end{aligned}$$

and also we have that

$$h \circ T \circ z = L \circ f \circ z \text{ (by (3.2) for } (f,h)) = L \circ t \circ x \text{ (by (4.2) for } (z,t)),$$

and

$$l \circ L \circ t = T \circ g \circ t \text{ (by (3.2) for } (g,l)) = T \circ z \circ y \text{ (by (4.2) for } (z,t)).$$

Note also that $(z, T \circ z)$ is a morphism of relative Rota-Baxter operators between $\left(\begin{array}{c} H_1 \\ id_H \downarrow\ , \Gamma_{H_1} \\ H_2 \end{array}\right)$ and $\left(\begin{array}{c} X \\ T \downarrow\ , \varphi_X \\ B \end{array}\right)$ because (3.2) follows directly and

$$\begin{aligned}
z \circ \Gamma_{H_1} &= \mu_X \circ ((z \circ \lambda_H^1) \otimes (z \circ \mu_H^2)) \circ (\delta_H \otimes H) \text{ (by the condition of algebra} \\
&\quad \text{morphism for } z\colon\ H_1 \to X) \\
&= \mu_X \circ (\lambda_X \otimes \overline{\mu}_X) \circ (((z \otimes z) \circ \delta_H) \otimes z) \text{ (by (2.2) and the condition of} \\
&\quad \text{algebra morphism for } z\colon\ H_2 \to \overline{X}) \\
&= \mu_X \circ ((\lambda_X * id_X) \otimes (\varphi_X \circ (T \otimes X))) \circ (\delta_X \otimes X) \circ (z \otimes z) \text{ (by the} \\
&\quad \text{condition of coalgebra morphism for } z, \text{ the associativity of } \mu_X \text{ and} \\
&\quad \text{coassociativity of } \delta_X) \\
&= \varphi_X \circ ((T \circ z) \otimes z) \text{ (by (2.1) and the (co)unit property).}
\end{aligned}$$

Following the same arguments, we can also show that $(t, L \circ t)$ is a morphism of relative Rota-Baxter operators between $\left(\begin{array}{c} D_1 \\ id_D \downarrow\ , \Gamma_{D_1} \\ D_2 \end{array}\right)$ and $\left(\begin{array}{c} A \\ L \downarrow\ , \varphi_A \\ Y \end{array}\right)$ so we define

$$\left({}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,h,g,l)}\right)^{-1}(z,t) = (z, t, T \circ z, L \circ t).$$

Therefore, it only remains to prove that

$${}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)} \quad \text{and} \quad \left({}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}\right)^{-1}$$

define a bijection. Indeed,

$$\begin{aligned}
&\left(\left({}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}\right)^{-1} \circ {}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}\right)(a,b,c,d) \\
&\qquad\qquad = \left({}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}\right)^{-1}(a,b) = (a, b, T \circ a, L \circ b) \\
&\qquad\qquad = (a,b,c,d)
\end{aligned}$$

because $T \circ a = c$ and $L \circ b = d$ by (3.2) for (a, c) and (b, d), and also

$$\big({}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)} \circ \big({}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}\big)^{-1}\big)(z, t)$$
$$= {}^{(\mathbb{H},\mathbb{D},x,y)}\Sigma_{(T,L,f,g,h,l)}(z, t, T \circ z, L \circ t) = (z, t).$$

□

The following result, proved in [11], asserts that every projection of cocommutative Hopf braces give rise to a new Hopf brace in C.

Theorem 4.14. *If* $(\mathbb{H}, \mathbb{D}, x, y) \in \mathsf{P}(\text{coc-HBr})$, *then*

$$\mathbb{I}(q_D) = (I(q_D), \eta_{I(q_D)}, \mu^1_{I(q_D)}, \mu^2_{I(q_D)}, \varepsilon_{I(q_D)}, \delta_{I(q_D)}, \lambda^1_{I(q_D)}, \lambda^2_{I(q_D)})$$

is a Hopf brace in C, *where*

$$\eta_{I(q_D)} = p^1_D \circ \eta_D = p^2_D \circ \eta_D, \tag{4.22}$$
$$\mu^1_{I(q_D)} = p^1_D \circ \mu^1_D \circ (i_D \otimes i_D) = p^2_D \circ \mu^1_D \circ (i_D \otimes i_D), \tag{4.23}$$
$$\mu^2_{I(q_D)} = p^2_D \circ \mu^2_D \circ (i_D \otimes i_D) = p^1_D \circ \mu^2_D \circ (i_D \otimes i_D), \tag{4.24}$$
$$\varepsilon_{I(q_D)} = \varepsilon_D \circ i_D, \tag{4.25}$$
$$\delta_{I(q_D)} = (p^1_D \otimes p^1_D) \circ \delta_D \circ i_D = (p^2_D \otimes p^2_D) \circ \delta_D \circ i_D, \tag{4.26}$$
$$\lambda^1_{I(q_D)} = p^1_D \circ \lambda^1_D \circ i_D, \tag{4.27}$$
$$\lambda^2_{I(q_D)} = p^2_D \circ \lambda^2_D \circ i_D. \tag{4.28}$$

Corollary 4.15. *There exists a functor*

$$\mathsf{P}' \colon \mathsf{P}(\text{coc-HBr}) \to \text{coc-HBr}$$

acting on objects by $\mathsf{P}'((\mathbb{H}, \mathbb{D}, x, y)) = \mathbb{I}(q_D)$ *and on morphism by* $\mathsf{P}'((z, t)) = t_0$, *where* $t_0 \colon I(q_D) \to I(q_{D'})$ *is the unique morphism verifying that* $i_{D'} \circ t_0 = t \circ i_D$.

Proof. Thanks to the previous theorem, functor P' is well-defined on objects. Let us show that it is well-defined on morphisms too. Take $(z, t) \colon (\mathbb{H}, \mathbb{D}, x, y) \to (\mathbb{H}', \mathbb{D}', x', y')$ a morphism of projections of Hopf braces and note that $t \circ i_D$ factors through the equalizer $i_{D'}$:

$$\begin{aligned}
&(D' \otimes y') \circ \delta_{D'} \circ t \circ i_D \\
&\quad = (t \otimes (y' \circ t)) \circ \delta_D \circ i_D \text{ (by the condition of coalgebra morphism for } t) \\
&\quad = (t \otimes (z \circ y)) \circ \delta_D \circ i_D \text{ (by (4.21))} \\
&\quad = (t \circ i_D) \otimes (z \circ \eta_H) \text{ (by the equalizer condition for } i_D) \\
&\quad = (t \circ i_D) \otimes \eta_{H'} \text{ (by the condition of algebra morphism for } z).
\end{aligned}$$

Therefore, there exists a unique $t_0\colon I(q_D) \to I(q_{D'})$ such that

$$i_{D'} \circ t_0 = t \circ i_D, \tag{4.29}$$

which implies that

$$t_0 = p^1_{D'} \circ t \circ i_D = p^2_{D'} \circ t \circ i_D. \tag{4.30}$$

$$\begin{tikzcd}
I(q_D) \arrow[r, "i_D"] \arrow[d, dashed, "\exists^{\bullet}\, t_0"] & D \arrow[r, shift left, "(D\otimes y)\circ\delta_D"] \arrow[r, shift right, "D\otimes\eta_H"'] \arrow[d, "t"] & D \otimes H \\
I(q_{D'}) \arrow[r, "i_{D'}"] & D' \arrow[r, shift left, "(D'\otimes y')\circ\delta_{D'}"] \arrow[r, shift right, "D'\otimes\eta_{H'}"'] & D' \otimes H'
\end{tikzcd}$$

Figure 6. The construction of t_0.

Then, to conclude, it is enough to show that $t_0\colon \mathbb{I}(q_D) \to \mathbb{I}(q_{D'})$ is a morphism of Hopf braces. The condition of coalgebra morphism for t_0 follows by

$$\begin{aligned}
&\delta_{I(q_{D'})} \circ t_0 \\
&\quad = (p^1_{D'} \otimes p^1_{D'}) \circ \delta_{D'} \circ t \circ i_D \text{ (by (4.29))} \\
&\quad = ((p^1_{D'} \circ t) \otimes (p^1_{D'} \circ t)) \circ \delta_D \circ i_D \text{ (by the condition of coalgebra morphism for } t) \\
&\quad = (t_0 \otimes t_0) \circ \delta_{I(q_D)} \text{ (by the condition of coalgebra morphism for } i_D \text{ and (4.30)),}
\end{aligned}$$

and also t_0 is an algebra morphism between $I(q_D)_k$ and $I(q_{D'})_k$ for all $k = 1, 2$ because

$$\begin{aligned}
&i_{D'} \circ t_0 \circ \mu^k_{I(q_D)} \\
&\quad = t \circ i_D \circ \mu^k_{I(q_D)} \text{ (by (4.29))} = t \circ \mu^k_D \circ (i_D \otimes i_D) \text{ (by (4.4))} \\
&\quad = \mu^k_{D'} \circ ((t \circ i_D) \otimes (t \circ i_D)) \text{ (by the condition of morphism of Hopf braces for } t) \\
&\quad = \mu^k_{D'} \circ (i_{D'} \otimes i_{D'}) \circ (t_0 \otimes t_0) \text{ (by (4.29))} \\
&\quad = i_{D'} \circ \mu^k_{I(q_{D'})} \circ (t_0 \otimes t_0) \text{ (by (4.4)).}
\end{aligned}$$

□

Theorem 4.16. *The diagram of functors*

$$\begin{tikzcd}
\mathsf{P(coc\text{-}rRB)} \arrow[r, "\mathsf{R}'"] \arrow[d, "\mathsf{P}"] & \mathsf{P(coc\text{-}HBr)} \arrow[d, "\mathsf{P}'"] \\
\mathsf{coc\text{-}rRB} \arrow[r, "\mathsf{G}"] & \mathsf{coc\text{-}HBr}
\end{tikzcd}$$

where R' *is the restriction of the functor* $\mathsf{R}\colon \mathsf{P}(\mathsf{rRB}^{\star}) \to \mathsf{P(coc\text{-}HBr)}$, *introduced in Theorem* 4.12, *to the subcategory* $\mathsf{P(coc\text{-}rRB)}$, *is commutative.*

Proof. We only detail that the previous diagram is commutative on objects because the commutativity on morphisms is straightforward. Let

$$\left(\left(\begin{array}{ccc} & H_1 & \\ T & \downarrow & , \varphi_H \\ & B & \end{array}\right), \left(\begin{array}{ccc} & A & \\ L & \downarrow & , \varphi_A \\ & D & \end{array}\right), f, h, g, l\right)$$

be a projection of cocommutative relative Rota-Baxter operators. Therefore, we have to show that the Hopf brace

$$\mathsf{G}\left(\left(\begin{array}{ccc} & I(q_A) & \\ L_0 & \downarrow & , \varphi_{I(q_A)} \\ & I(q_D) & \end{array}\right)\right)$$
$$= \overline{\mathbb{I}(q_A)} = (I(q_A), \eta_{I(q_A)}, \mu_{I(q_A)}, \overline{\mu}_{I(q_A)}, \varepsilon_{I(q_A)}, \delta_{I(q_A)}, \lambda_{I(q_A)}, \overline{\lambda}_{I(q_A)}),$$

where $\eta_{I(q_A)}$, $\mu_{I(q_A)}$, $\varepsilon_{I(q_A)}$, $\delta_{I(q_A)}$ and $\lambda_{I(q_A)}$ are the usual ones, while

$$\overline{\mu}_{I(q_A)} = \mu_{I(q_A)} \circ (I(q_A) \otimes (\varphi_{I(q_A)} \circ (L_0 \otimes I(q_A)))) \circ (\delta_{I(q_A)} \otimes I(q_A)), \tag{4.31}$$
$$\overline{\lambda}_{I(q_A)} = \varphi_{I(q_A)} \circ ((\lambda_{I(q_D)} \circ L_0) \otimes \lambda_{I(q_A)}) \circ \delta_{I(q_A)}, \tag{4.32}$$

is the same as the Hopf brace

$$\mathsf{P}'(\overline{\mathbb{H}}, \overline{\mathbb{A}}, f, g) = \mathbb{I}(q_{\overline{A}}) = (I(q_A), \eta_{I(q_{\overline{A}})}, \mu^1_{I(q_{\overline{A}})}, \mu^2_{I(q_{\overline{A}})}, \varepsilon_{I(q_{\overline{A}})}, \delta_{I(q_{\overline{A}})}, \lambda^1_{I(q_{\overline{A}})}, \lambda^2_{I(q_{\overline{A}})}),$$

where $\eta_{I(q_{\overline{A}})} = \eta_{I(q_A)}$, $\mu^1_{I(q_{\overline{A}})} = \mu_{I(q_A)}$, $\varepsilon_{I(q_{\overline{A}})} = \varepsilon_{I(q_A)}$, $\delta_{I(q_{\overline{A}})} = \delta_{I(q_A)}$ and $\lambda^1_{I(q_{\overline{A}})} = \lambda_{I(q_{\overline{A}})}$, while $\mu^2_{I(q_{\overline{A}})}$ is the unique product satisfying that

$$i_A \circ \mu^2_{I(q_{\overline{A}})} = \overline{\mu}_A \circ (i_A \otimes i_A) \tag{4.33}$$

and $\lambda^2_{I(q_{\overline{A}})}$ is the unique morphism which verifies that

$$i_A \circ \lambda^2_{I(q_{\overline{A}})} = \overline{\lambda}_A \circ i_A.$$

Thus, to conclude the proof it is enough to show that $\overline{\mu}_{I(q_A)} = \mu^2_{I(q_{\overline{A}})}$ which implies that $\overline{\lambda}_{I(q_A)} = \lambda^2_{I(q_{\overline{A}})}$ due to the uniqueness of the antipode for a bialgebra structure. Indeed,

$$\begin{aligned}
& i_A \circ \overline{\mu}_{I(q_A)} \\
&\quad = \mu_A \circ (i_A \otimes (i_A \circ \varphi_{I(q_A)} \circ (L_0 \otimes I(q_A)))) \circ (\delta_{I(q_A)} \otimes I(q_A)) \text{ (by (4.4))} \\
&\quad = \mu_A \circ (i_A \otimes (\varphi_A \circ ((i_D \circ L_0) \otimes i_A))) \circ (\delta_{I(q_A)} \otimes I(q_A)) \text{ (by (4.15))} \\
&\quad = \mu_A \circ (i_A \otimes (\varphi_A \circ (L \otimes A))) \circ (((i_A \otimes i_A) \circ \delta_{I(q_A)}) \otimes i_A) \text{ (by (4.13))} \\
&\quad = \overline{\mu}_A \circ (i_A \otimes i_A) \text{ (by the condition of coalgebra morphism for } i_A) \\
&\quad = i_A \circ \mu^2_{I(q_{\overline{A}})} \text{ (by (4.33))}.
\end{aligned}$$

As a conclusion, due to the fact that i_A is a monomorphism, $\overline{\mu}_{I(q_A)} = \mu^2_{I(q_{\overline{A}})}$. □

Theorem 4.17. *The diagram of functors*

$$\begin{tikzcd}
\mathsf{P}(\mathsf{coc\text{-}HBr}) \arrow[r, "\mathsf{Q}'"] \arrow[d, "\mathsf{P}'"] & \mathsf{P}(\mathsf{coc\text{-}rRB}) \arrow[d, "\mathsf{P}"] \\
\mathsf{coc\text{-}HBr} \arrow[r, "\mathsf{F}"] & \mathsf{coc\text{-}rRB}
\end{tikzcd}$$

is commutative.

Proof. Let $(\mathbb{H}, \mathbb{D}, x, y) \in \mathsf{P}(\mathsf{coc\text{-}HBr})$. We have to show that the relative Rota-Baxter operator

$$\begin{aligned}
&(\mathsf{P} \circ \mathsf{Q}')((\mathbb{H}, \mathbb{D}, x, y)) \\
&\quad = \mathsf{P}\left(\left(\left(\begin{array}{ccc} id_H & \begin{array}{c} H_1 \\ \downarrow \\ H_2 \end{array} & , \Gamma_{H_1} \end{array}\right), \left(\begin{array}{ccc} id_D & \begin{array}{c} D_1 \\ \downarrow \\ D_2 \end{array} & , \Gamma_{D_1} \end{array}\right), x, x, y, y\right)\right) \\
&\quad = \left(\begin{array}{ccc} id_{I(q_D)} & \begin{array}{c} I(q_D) \\ \downarrow \\ I(q_D) \end{array} & , \varphi_{I(q_D)} \end{array}\right),
\end{aligned}$$

where $\varphi_{I(q_D)}$ is the unique action satisfying that

$$i_D \circ \varphi_{I(q_D)} = \Gamma_{D_1} \circ (i_D \otimes i_D), \tag{4.34}$$

is the same as the relative Rota-Baxter operator

$$(\mathsf{F} \circ \mathsf{P}')((\mathbb{H}, \mathbb{D}, x, y)) = \mathsf{F}(\mathbb{I}(q_D)) = \left(\begin{array}{ccc} id_{I(q_D)} & \begin{array}{c} I(q_D)_1 \\ \downarrow \\ I(q_D)_2 \end{array} & , \Gamma_{I(q_D)_1} \end{array}\right).$$

To finish the proof it is enough to see that $\varphi_{I(q_D)} = \Gamma_{I(q_D)_1}$, which follows by (4.34), by the equality $i_D \circ \Gamma_{I(q_D)_1} = \Gamma_{D_1} \circ (i_D \otimes i_D)$ and by the fact that i_D is a monomorphism. □

To finish the article we will introduce the notion of strong projection of relative Rota-Baxter operators in order to prove that any such projection gives rise to a module in the sense of Definition 3.8 in a cocommutative setting.

Definition 4.18. A projection of relative Rota-Baxter operators

$$\left(\left(\begin{array}{ccc} T & \begin{array}{c} H \\ \downarrow \\ B \end{array} & , \varphi_H \end{array}\right), \left(\begin{array}{ccc} L & \begin{array}{c} A \\ \downarrow \\ D \end{array} & , \varphi_A \end{array}\right), f, h, g, l\right)$$

is said to be strong when the following conditions hold:

$$p_A \circ \varphi_A = p_A \circ \varphi_A \circ (D \otimes q_A), \tag{4.35}$$

$$\varphi^{\mathrm{ad}}_{\overline{A}} \circ (f \otimes i_A) = \mu_A \circ (\overline{\mu}_A \otimes \lambda_A) \circ (A \otimes c_{A,A}) \circ ((\delta_A \circ f) \otimes i_A), \tag{4.36}$$

where $\overline{A}$ is the Hopf algebra introduced in Theorem 3.7.

Strong projections of relative Rota-Baxter operators constitute a full subcategory of $\mathsf{P}(\mathsf{rRB})$, which we will denote by $\mathsf{SP}(\mathsf{rRB})$. When the relative Rota-Baxter operators involved in the strong projection are cocommutative, they constitute a full subcategory of $\mathsf{SP}(\mathsf{rRB})$ denoted by $\mathsf{SP}(\mathsf{coc\text{-}rRB})$.

Theorem 4.19. *If*

$$\left(\left(T \quad \begin{matrix} H \\ \downarrow \\ B \end{matrix} \quad , \varphi_H\right), \left(L \quad \begin{matrix} A \\ \downarrow \\ D \end{matrix} \quad , \varphi_A\right), f, h, g, l\right)$$

is an object in $\mathsf{SP}(\mathsf{coc\text{-}rRB})$, *then*

$$(I(q_A), I(q_D), \psi_{I(q_A)}, \varphi_{I(q_A)}, \psi_{I(q_D)}, L_0)$$

is a module over the relative Rota-Baxter operator $\left(T \begin{matrix} H \\ \downarrow \\ B \end{matrix} , \varphi_H\right)$, *where* $L_0\colon I(q_A) \to I(q_D)$ *is the morphism defined by* (4.13) *and the actions are defined as follows:*

$$\psi_{I(q_A)} := p_A \circ \mu_A \circ (f \otimes i_A), \quad \varphi_{I(q_A)} := p_A \circ \varphi_A \circ (h \otimes i_A), \quad \psi_{I(q_D)} := p_D \circ \mu_C \circ (h \otimes i_D).$$

Proof. By the general theory of Hopf algebra projections, it is well-known that $(I(q_A), \psi_{I(q_A)})$ is a left H-module and $(I(q_D), \psi_{I(q_D)})$ is a left B-module. Let us show that $(I(q_A), \varphi_{I(q_A)})$ is also a left B-module. Indeed, on the one hand, it is straightforward to compute that $\varphi_{I(q_A)} \circ (\eta_B \otimes I(q_A)) = id_{I(q_A)}$ and, on the other hand, we have that

$$\begin{aligned}
&\varphi_{I(q_A)} \circ (B \otimes \varphi_{I(q_A)}) \\
&\quad = p_A \circ \varphi_A \circ (h \otimes (q_A \circ \varphi_A \circ (h \otimes i_A))) \text{ (by } q_A = i_A \circ p_A) \\
&\quad = p_A \circ \varphi_A \circ (h \otimes (\varphi_A \circ (h \otimes i_A))) \text{ (by (4.35))} \\
&\quad = p_A \circ \varphi_A \circ ((\mu_D \circ (h \otimes h)) \otimes i_A) \text{ (by module axioms for } (A, \varphi_A)) \\
&\quad = \varphi_{I(q_A)} \circ (\mu_B \otimes I(q_A)) \text{ (by the condition of algebra morphism for } h).
\end{aligned}$$

To conclude the proof it only remains us to show that (3.8) and (3.9) hold. At first, we have that

$$\begin{aligned}
&\psi_{I(q_A)} \circ (\varphi_H \otimes \varphi_{I(q_A)}) \circ (B \otimes c_{B,H} \otimes I(q_A)) \circ (\delta_B \otimes H \otimes I(q_A))\\
&\quad = p_A \circ \mu_A \circ ((f \circ \varphi_H) \otimes (\varphi_A \circ (h \otimes i_A))) \circ (B \otimes c_{B,H} \otimes I(q_A))\\
&\qquad \circ (\delta_B \otimes H \otimes I(q_A)) \ \text{(by (4.9))}\\
&\quad = p_A \circ \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (D \otimes c_{D,A} \otimes A) \circ (((h \otimes h) \circ \delta_B) \otimes f \otimes i_A)\\
&\qquad \text{(by (3.3) for } (f,h))\\
&\quad = p_A \circ \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (D \otimes c_{D,A} \otimes A) \circ (\delta_D \otimes A \otimes A) \circ (h \otimes f \otimes i_A)\\
&\qquad \text{(by the condition of coalgebra morphism for } h)\\
&\quad = p_A \circ \varphi_A \circ (h \otimes (\mu_A \circ (f \otimes i_A)))\\
&\qquad \text{(by the condition of morphism of left } D\text{-modules for } \mu_A)\\
&\quad = \varphi_{I(q_A)} \otimes (B \otimes \psi_{I(q_A)}) \ \text{(by (4.35)),}
\end{aligned}$$

which implies that (3.8) holds. Let us define

$$\kappa := L \circ \varphi^{\mathrm{ad}}_{\overline{A}} \circ (f \otimes i_A).$$

On the one side, it is obtained that

$$\begin{aligned}
&i_D \circ \psi_{I(q_D)} \circ (T \otimes L_0)\\
&\quad = \varphi_D^{\mathrm{ad}} \circ ((h \circ T) \otimes (i_D \circ L_0)) \ \text{(by (4.5))}\\
&\quad = \varphi_D^{\mathrm{ad}} \circ ((L \circ f) \otimes (L \circ i_A)) \ \text{(by (4.13) and (3.2) for } (f,h))\\
&\quad = \kappa \ \text{(by the condition of Hopf algebra morphism for } L\colon \overline{A} \to D\text{),}
\end{aligned}$$

and, on the other side,

$$\begin{aligned}
&i_D \circ L_0 \circ \psi_{I(q_A)} \circ (H \otimes (\varphi_{I(q_A)} \circ (T \otimes I(q_A)))) \circ (\delta_H \otimes I(q_A))\\
&\quad = L \circ q_A \circ \mu_A \circ (A \otimes \varphi_A) \circ (((f \otimes (h \circ T)) \circ \delta_H) \otimes i_A) \ \text{(by (4.9) and (4.13))}\\
&\quad = L \circ \mu_A \circ (\mu_A \otimes (f \circ \lambda_H \circ g \circ \mu_A)) \circ (A \otimes c_{A,A} \otimes A) \circ ((\delta_A \circ f)\\
&\qquad \otimes (((\varphi_A \circ (L \otimes A)) \otimes (\varphi_A \circ (L \otimes A))) \circ (A \otimes c_{A,A} \otimes A) \circ ((\delta_A \circ f) \otimes (\delta_A \circ i_A))))\\
&\qquad \circ (\delta_H \otimes I(q_A)) \ \text{(by (3.2) for } (f,h) \text{ and the condition of coalgebra}\\
&\qquad \text{morphism for } \mu_A,\ \varphi_A \text{ and } L)\\
&\quad = L \circ \mu_A \circ (\mu_A \otimes (f \circ \lambda_H \circ \mu_H)) \circ (A \otimes c_{H,A} \otimes H) \circ (((f \otimes H) \circ \delta_H)\\
&\qquad \otimes (((\varphi_A \circ ((L \circ f) \otimes A)) \otimes (\varphi_H \circ ((l \circ L \circ f) \otimes H)))\\
&\qquad \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes ((A \otimes g) \circ \delta_A \circ i_A)))) \circ (\delta_H \otimes I(q_A))\\
&\qquad \text{(by the condition of algebra morphism for } g\text{, (3.3) for } (g,l)\text{, naturality of } c\text{,}\\
&\qquad \text{the condition of coalgebra morphism for } f \text{ and } g \circ f = id_H)
\end{aligned}$$

$$
\begin{aligned}
&= L\circ\mu_A\circ(\mu_A\otimes(f\circ\lambda_H))\circ(A\otimes c_{H,A})\\
&\quad\circ(((f\otimes H)\circ\delta_H)\otimes(\varphi_A\circ((L\circ f)\otimes i_A)))\circ(\delta_H\otimes I(q_A))\ \text{(by the equalizer}\\
&\quad\text{condition for } i_A\text{, the fact that } \eta_H \text{ is a morphism of left } B\text{-modules and}\\
&\quad\text{(co)unit properties)}\\
&= L\circ\mu_A\circ(\overline{\mu}_A\otimes\lambda_A)\circ(A\otimes c_{A,A})\circ(\delta_A\otimes A)\circ(f\otimes i_A)\ \text{(by (2.2), the}\\
&\quad\text{condition of coalgebra morphism for } f\text{, naturality of } c \text{ and}\\
&\quad\text{cocommutativity of } \delta_A\text{)}\\
&= \kappa\ \text{(by (4.36)),}
\end{aligned}
$$

which implies that (3.9) holds and the proof is concluded due to the fact that i_D is a monomorphism. □

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