Cahiers du CERO, volume **37** (1995) 25–41

Bargaining with graph-restricted communication[1]

B. Casas, I. García-Jurado and J. M. Prada-Sánchez

Department of Statistics and OR, University of Santiago de Compostela, E-15771 Santiago de Compostela, Spain.

Abstract

In this paper we study n-person bargaining problems where not all players can freely communicate, but there are some restrictions on the communication embodied in an undirected connected graph, in such a way that two players can directly communicate if and only if there is a link of the graph joining them. These restrictions can produce asymmetries in the negotiation process which might affect its final result. In this setup we propose some solutions and study some of their properties.

1 Introduction

A cooperative game is a conflict situation where the players are allowed to make enforceable agreements. In classical cooperative game theory, it was assumed that free communication between players was permitted. However, in the last years, cooperative models with communication restrictions were widely studied since the Myerson value ([4] Myerson, 1977) was introduced for TU communication situations. One of these situations is determined by a TU game and an undirected graph embodying the communication possibilities of the players (in such a way

[1]We thank University of Santiago de Compostela and Xunta de Galicia for financial support through projects 60902.25064(5060), XUGA20701B91 and XUGA20702B93.

that two players can directly communicate if and only if there is a link of the graph joining them). Another important value for TU communication situations is the position value ([1] Borm, Owen and Tijs, 1992). In [6] van den Nouweland (1993) a complete survey on TU and NTU cooperative games with restricted communication is included.

An interesting class of cooperative games where the effect of communication restrictions had not been explored yet is the class of bargaining games. Of course, in a bargaing game the consensus of all players is needed in order for an agreement to be reached, so the communication graph should be connected. However, it is possible that certain pairs of players are unable to communicate directly and, then, some can have a predominant position in the negotiation process (because they are better "located" in the communication graph). This non-symmetric aspect of the situation must be considered by the final solution if we want it to propose a sensible allocation for the players.

This paper is devoted to study bargaining games with graph-restricted communication and to introduce and characterize axiomatically a solution for them (which results in a generalization of the Nash solution for bargaining games). It is organized as follows. In the next section, we set up the notation and recall some definitions and results. Section 3 is devoted to present and study an allocation rule for the so-called *bargaining communication situations.* Finally, in Section 4 we make some remarks about how to define other allocation rules.

2 Preliminaries

In this section we present some basic notation and definitions. Since the position value for TU games will be a useful tool for the present study, we start by providing its definition and some other concepts concerning TU games.

A TU game is a pair (N, v) where $N := \{1, \dots, n\}$ is the set of players and v is the characteristic function (defined from $2^N := \{S \,/\, S \subset N\}$ to $\mathbb{R}$ and satisfying that $v(\emptyset) = 0$). We often identify a TU game (N, v) with its characteristic function v and denote by $G(N)$ the set of TU games with set of players N. We say that $v \in G(N)$ is zero-normalized if $v(i) = 0 \; \forall i \in N$. For every $v \in G(N)$, we write $\Phi(v)$ for its Shapley value (see [8] Shapley, 1953) and v^0 for the zero-normalized TU-game given by $v^0(T) = v(T) - \sum_{i \in T} v(i) \; \forall T \subset N$. For each $L \subset 2^N \backslash \{\emptyset\}$, the unanimity game $u^L \in G(N)$ is defined by $u^L(T) = 1$ if $L \subset T$ and $u^L(T) = 0$ otherwise (for all $T \subset N$).

An undirected graph on N without loops is a set B of unordered pairs of distinct elements of N. The elements $i : j \in B$ are called links. We denote by $g(N)$ the set of all undirected graphs on N without loops. For any $T \subset N$ and any $B \in g(N)$ we say that $i, j \in T$ are connected in T by B if B induces a path in T connecting i and j, and we denote by T/B te set of connected components in T (maximal subsets of connected elements in T) associated with the graph B. Note that T/B is a partition of T. We say that $B \in g(N)$ is connected if N/B has a unique class. We denote by $c(N)$ the set of all connected elements of $g(N)$

and by $w(N)$ the set of all graphs in $c(N)$ without cycles (paths connecting an element of N with itself).

A TU communication situation with set of players N is a pair (v, B) such that $v \in G(N)$ and $B \in g(N)$. The graph B embodies the communication channels existing between players, in the sense that $i, j \in N$ can directly communicate if and only if $i : j \in B$. We denote by $\overline{G}(N)$ the set of all TU communiation situations with set of players N. Given $(v, B) \in \overline{G}(N)$ such that v is zero-normalized, its link game v_B is defined by:

$$v_B(R) = \sum_{T \in N/R} v(T) \ \forall R \subset B.$$

Note that the players of v_B are the links of B. Now, the position value is a map $\pi : \overline{G}(N) \longrightarrow \mathbb{R}^n$ defined by:

$$\pi_i(v, B) = \sum_{b \in B_i} \frac{1}{2} \Phi_b(v_B^0) + v(i)$$

for all $i \in N$, where $B_i = \{r : s \in B \,/\, r = i \text{ or } s = i\}$ and v_B^0 is the link game associated with (v^0, B).

Now, let us set up the notation related to bargaining problems. A bargaining game with set of players N is a pair (S, d) satisfying:

[1] $S \subset \mathbb{R}^n$ is a convex, closed and comprehensive set, and

[2] $S_d := \{x \in S \,/\, x > d\}$ is a nonempty and bounded set.

S is the set of feasible allocations and d the disagreement point. $\mathcal{B}(N)$ is the set of all bargaining games with set of players N. We denote by $\mathcal{N}(S,d)$ and $\mathcal{N}^p(S,d)$ the prescription of the Nash solution (see [5] Nash, 1950) and the prescription of the nonsymetric Nash solution with vector of weights p (see [3] Kalai, 1977) for the bargaining game (S,d) (respectively). For more details about bargaining solutions see, for instance, [7] Peters (1992).

3 An allocation rule for bargaining communication situations

A bargaining communication situation with set of players N is a triplet (S,d,B) where $(S,d) \in \mathcal{B}(N)$ and $B \in c(N)$. In one of such situations, the players in N have to choose jointly an allocation in S (if an agreement of all players is not reached, then d, the disagreement point, is the outcome of the game) through a negotiation process which is limited by the communication restrictions given by B (i and j can directly negotiate if and only if $i:j \in B$). Note that B is supposed to be connected; otherwise a general consensus would not be possible and d would be the result of the game. We denote by $\overline{\mathcal{B}}(N)$ the set of all bargaining communication situations with player set N.

For these situations we wish to define an allocation rule, i.e. a map from $\overline{\mathcal{B}}(N)$ to $\mathbb{R}^n$ which allocates a vector of payoffs (to the players) to any bargaining communication situation. Let us see some properties we would like a map f :

$A \subset \overline{\mathcal{B}}(N) \longrightarrow \mathbb{R}^n$ to satisfy. First, some standard ones (in axiomatic bargaining theory):

[1] *Feasibility (F)*: for all $(S, d, B) \in A$, $f(S, d, B) \in S_d$.

[2] *Independence of Affine Transformations (IAT)*: for all $(S, d, B) \in A$ and all positive affine transformation F, $F(f(S, d, B)) = f(F(S), F(d), B)$ (a positive affine transformation F maps every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ in an $F(x) \in \mathbb{R}^n$ whose i-th component is $a_i x_i + b_i$, where $a_1, \ldots, a_n$ are positive real numbers and $b_1, \ldots, b_n \in \mathbb{R}$).

[3] *Independence of Irrelevant Alternatives (IIA)*: for all $(S, d, B) \in A$ and $(T, d, B) \in A$, if $S \subset T$ and $f(T, d, B) \in S$, then $f(S, d, B) = f(T, d, B)$.

Another standard property in bargaining literature is the symmetry property. However, when the communication is limited, we can only ask for a kind of graph-weighted symmetry in order to incorporate in the solution all the non-symmetric aspects of the situation embodied in the communication graph. Before going into the details of our new property, let us give some definitions.

We say that a graph $B \in g(N)$ is levelled if there exists $k \in \mathbb{N}$ such that, for any $R \subset B$, $|N/R| = 1$ if and only if $|R| \geq k$. Note that in a levelled graph all links are equally important, in the sense that, in order to know if a certain subset of B connects all players in N is sufficient to know how many elements (links) it has. We define S^N and θ by:

$$S^N = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,/\, \sum_{i=1}^{n} x_i \leq 1\} \text{ and } \theta = (0, \ldots, 0) \in \mathbb{R}^n.$$

[4] *Graph-Weighted Symmetry (GWS)*: for every $(S^N, \theta, B) \in A$ such that B is levelled and for every $i \in N$, $f_i(S^N, \theta, B) = \alpha |B_i|$, with $\alpha \in \mathbb{R}$ (recall that B_i is the subset of B given by the links in which i is involved; $|B_i|$ is the cardinality of B_i).

This property means that, if players have to divide one unit and all the communication links are equally important, f proposes for each player a quantity which is proportional to his (or her) communication possibilities.

With the four axioms above in mind and denoting by $\overline{\mathcal{B}}^w(N)$ the set of all bargaining communication situations in $\overline{\mathcal{B}}(N)$ whose communication graphs are in $w(N)$ (i.e. do not contain cycles), we can state the following theorem.

Theorem 1. *There exists a unique map* $\varphi : \overline{\mathcal{B}}^w(N) \longrightarrow \mathbb{R}^n$ *satisfying* (F), (IAT), (IIA) *and* (GWS). *Such a map assigns, to each* $(S, d, B) \in \overline{\mathcal{B}}^w(N)$, *the value* $\mathcal{N}^p(S, d)$ *where* $p = (p_1, \ldots, p_n)$ *is such that, for every* $i \in N$, $p_i = \dfrac{|B_i|}{2|B|}$.

Proof. (a) Existence: The nonsymmetric Nash solutions verify (F), (IAT) and (IIA) (see [3] Kalai, 1977, for more details on nonsymmetric Nash solutions). Moreover, since $\mathcal{N}^p(S^N, \theta) = p$ if $\sum_{i=1}^n p_i = 1$, it is straightforward that φ satisfies (GWS).

(b) Uniqueness: Take f a map from $\overline{\mathcal{B}}^w(N)$ to $\mathbb{R}^n$ satisfying (F), (IAT), (IIA) and

(GWS). We will show that f must be equal to φ. First, consider an $(S^N, \theta, B) \in \overline{\mathcal{B}}^w(N)$. As f is feasible, $f_i(S^N, \theta, B) > 0$ for all $i \in N$ and $\sum_{i=1}^n f_i(S^N, \theta, B) \leq 1$. Let F be the positive affine transformation given by:

$$F(x) = (\sum_{i=1}^n f_i(S^N, \theta, B))x$$

for all $x \in \mathbb{R}^n$. Now, take $F(S^N) = \{x \in \mathbb{R}^n \,/\, \sum_{i=1}^n x_i \leq \sum_{i=1}^n f_i(S^N, \theta, B)\}$. Clearly, $F(S^N) \subset S^N$ and $f(S^N, \theta, B) \in F(S^N)$. As f satisfies (IIA), then

$$f(F(S^N), \theta, B) = f(S^N, \theta, B)) \tag{1}$$

and, since f verifies (IAT)

$$f(F(S^N), \theta, B) = F(f(S^N, \theta, B)) = (\sum_{i=1}^n f_i(S^N, \theta, B))f(S^N, \theta, B). \tag{2}$$

From Equations (1) and (2),

$$\sum_{i=1}^n f_i(S^N, \theta, B) = 1. \tag{3}$$

Further, as B does not contain cycles, it is levelled. Then, taking into account that f satisfies (GWS) and Equation (3),

$$f_i(S^N, \theta, B) = \frac{|B_i|}{2|B|} = \varphi_i(S^N, \theta, B)$$

for all $i \in N$. Finally, consider $(S, d, B) \in \overline{\mathcal{B}}^w(N)$ and let F be the affine transformation such that $F(\varphi(S, d, B)) = f(S^N, \theta, B)$ and $F(d) = \theta$. Then, it can bee seen that $F(S) \subset S^N$ (see, for instance, [3] Kalai, 1977, for details about the proof) and, as φ is feasible, $f(S^N, \theta, B) \in F(S)$. Since f satisfies (IIA), $f(F(S), \theta, B) = f(S^N, \theta, B)$. Now, taking into account that f satisfies (IAT), $f(S, d, B) = \varphi(S, d, B)$. $\square$

It is easy to check that, if B does not contain cycles p, defined as above, is equal to $\pi(u^N, B)$. From this, we define our allocation rule $\varphi : \overline{\mathcal{B}}(N) \longrightarrow \mathbb{R}^n$ in the following way:

$$\varphi(S, d, B) = \mathcal{N}^p(S, d),$$

for all $(S, d, B) \in \overline{\mathcal{B}}(N)$ (with $p = \pi(u^N, B)$). Note that φ, defined as above, obviously satisfies (F), (IAT) and (IIA) in $\overline{\mathcal{B}}(N)$. Moreover, as π satisfies component-efficiency and arc anonimity (see [1] Borm et al., 1992), it is clear that, if $B \in g(N)$ is levelled, then $\pi_i(u^N, B) = \frac{|B_i|}{2|B|}$ for all $i \in N$. Then since $\mathcal{N}^p(S^N, \theta) = p$ if $\sum_{i=1}^n p_i = 1$, it is straightforward that φ satisfies (GWS) in $\overline{\mathcal{B}}(N)$.

We have seen that φ satisfies a number of interesting properties. Let us see now how it works in an example.

Example 1. *Consider the bargaining communication situation* (S, d, B) *where* $S = \{(x, y, z) \in \mathbb{R}^3 \,/\, 3x + y + z \leq 1\}$, $d = (0, 0, 0)$ *and* $B = \{1 : 2, 2 : 3\}$. *Then,* $\mathcal{N}(S, d) = (4/36, 12/36, 12/36)$ *and* $\varphi(S, d, B) = (3/36, 18/36, 9/36)$. *In* *this example, our solution rewards the second player for his (or her) better position in the negotiation process, and punishes the other two players proportionaly to their possibilities (in the Nash sense) in the complete communication game.*

In addition to the axiomatic characterization of φ on a subset of $\overline{\mathcal{B}}(N)$, the following results and comments show other interesting properties of φ. Fist, we present a proposition which states that our allocation rule is a generalization of the Nash bargaining solution.

Proposition 1. *If $B^N \in g(N)$ denotes the complete graph on N (i.e. $B^N = \{i : j \,/\, i \in N,\, j \in N,\, i \neq j\}$), then, for every $(S,d) \in \mathcal{B}(N)$, $\varphi(S,d,B^N) = \mathcal{N}(S,d)$.*

Proof. It is very easy to check that $\Phi_a(u^N_{B^N}) = \Phi_b(u^N_{B^N})$ for all $a, b \in B^N$. Hence, $\pi_i(u^N, B^N) = \pi_j(u^N, B^N)$ for all $i, j \in N$ and $\varphi(S,d,B^N) = \mathcal{N}(S,d)$. $\square$

Now, observe that a bargaining communication situation with player set N can be viewed as an NTU communication situation with the same player set (an NTU communication situation with player set N is a pair (V, B) such that V is an NTU game with player set N and $B \in g(N)$; recall that an NTU game with player set N is characterized by a map V that assigns to each coalition $T \in 2^N \backslash \{\emptyset\}$ a nonempty, closed and comprehensive set $V(T) \subset \mathbb{R}^t$, t being the cardinality of T; we denote by $\Gamma(N)$ the set of NTU games with player set N and we assume that, for every $V \in \Gamma(N)$ and every $i \in N$, there exists an upper bound $\underline{\mathrm{x}}_i = \sup\{x_i \,/\, x_i \in V(i)\}$; in these conditions, each bargaining communication situation $(S,d,B) \in \overline{\mathcal{B}}(N)$ can be identified with the NTU communication situation (V, B) where $V \in \Gamma(N)$

is given by: $V(N) = S$, $V(T) = \{x \in \mathbb{R}^t \,/\, x_i \leq d_i \ \forall i \in T\}$, $\forall T \subset 2^N \backslash \{\emptyset, N\}$). In [6] van den Nouweland (1993) the *position set* is introduced as a solution concept for NTU communication situations, following the λ-transfer philosophy proposed in [9] Shapley (1969). We will recall soon such a concept but, first, let us make some useful comments. We say that an NTU game $V \in \Gamma(N)$ is zero-normalized if $V(i) = (-\infty, 0]$ for all $i \in N$. For any $V \in \Gamma(N)$, we write V^0 for the zero-normalized NTU game given by

$$V^0(T) = \{x \in \mathbb{R}^t \,/\, \text{there exists } y \in V(T) \text{ with } x_i = y_i - \underline{\mathrm{x}}_i \, \forall i \in T\}.$$

Now, let (V, B) be an NTU communication situation with player set N such that V is zero-normalized. A weight vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with $\lambda_i > 0$ $\forall i \in N$ and $\sum_{i \in N} \lambda_i = 1$ is link-admissible if, for all $R \subset B$, the supremum

$$v_B^\lambda(R) := \sup\{\sum_{i \in N} \lambda_i x_i \,/\, x \in \Pi_{T \in N/R} V(T)\}$$

exists. Observe that the players of the TU game v_B^λ generated by the link-admissible vector λ are the links in B; hence, v_B^λ is a link game. Now, the position set for any (V, B) with V zero-normalized is given by:

$$P(V, B) := \{x \in \Pi_{T \in N/B} V(T) \ / \ \exists \text{ a link} - \text{admissible } \lambda$$

$$\text{such that } \lambda_i x_i = \sum_{b \in B_i} \frac{1}{2} \Phi_b(v_B^\lambda) \ \forall i \in N\}.$$

For any NTU communication situation (V, B) with player set N, its position set $P(V, B)$ is given by:

$P(V, B) = \{x \in \mathbb{R}^n \,/\, \text{there exists } y \in P(V^0, B) \text{ with } x_i = y_i + \underline{\mathrm{x}}_i \ \forall i \in N\}.$

We write $P(S, d, B)$ for the position set of the bargaining communication situation (S, d, B), when viewed as an NTU communication situation, and $P(v, B)$ for the position set of the TU communication situation (v, B) when viewed as an NTU communication situation. In [6] van den Nouweland (1993), it is proved that $P(v, B) = \{\pi(v, B)\}$. The next result shows a similar property for our allocation rule.

Theorem 2. *The position set of a bargaining communication situation contains a unique point. This point is the allocation proposed by* φ *for the bargaining communication situation. This means that* $P(S, d, B) = \{\mathcal{N}^p(S, d)\}$ *with* $p = \pi(u^N, B)$.

Proof. Note first that the bargaining communication situation (S^N, θ, B) can be identified with the TU-game (u^N, B) (for any $B \in c(N)$). Then, $P(S^N, \theta, B) = P(u^N, B) = \{\pi(u^N, B)\}$. But, clearly, $\pi(u^N, B) = \mathcal{N}^p(S^N, \theta)$ for any $B \in c(N)$ (remind that, for simplicity, we denote $p = \pi(u^N, B)$). Hence,

$$P(S^N, \theta, B) = \{\mathcal{N}^p(S^N, \theta)\}. \tag{4}$$

Now, it is not difficult to check that, if we restrict to bargaining communication situations, P satisfies *Independence of Affine Transformations (IAT)* (i.e. $P(F(S), F(d), B) = F(P(S, d, B))$ for all $(S, d, B) \in \overline{\mathcal{B}}(N)$ and all positive affine transformation F) and *Independence of Irrelevant Alternatives (IIA)* (i.e. $P(T, d, B) \cap S \subset P(S, d, B)$ for all $(S, d, B), (T, d, B) \in \overline{\mathcal{B}}(N)$ with $S \subset T$). We will use this fact to prove that, for any $(S, d, B) \in \overline{\mathcal{B}}(N)$, $P(S, d, B) = \{\mathcal{N}^p(S, d)\}$. Let us see first that $P(S, d, B)$ is nonempty (in fact, we will check that $\mathcal{N}^p(S, d) \in P(S, d, B)$). Take F to be the positive affine transformation such that $F(d) = \theta$ and $F(\mathcal{N}^p(S, d)) = p$. It can be proved that, in these conditions, $F(S) \subset S^N$. But then, in view of Equation 4 and the fact that P satisfies (IIA) and $\mathcal{N}^p(S^N, \theta) = p \in F(S)$,

$$\mathcal{N}^p(S^N, \theta) \in P(F(S), F(d), B). \tag{5}$$

Note now that $\mathcal{N}^p(S^N, \theta) = p = F(\mathcal{N}^p(S, d))$. In view of this, Equation 5 and the fact that P satisfies (IAT), we have that $\mathcal{N}^p(S, d) \in P(S, d, B)$. Next, let us demonstrate that, if $\overline{x} \in P(S, d, B)$, then $\overline{x} = \mathcal{N}^p(S, d)$. Since P and $\mathcal{N}^p$ satisfy (IAT), it is enough to prove it when $d = \theta$. Now, if we consider (for every link-admissible λ) the TU-game $v_\lambda \in G(N)$ defined by:

$$v_\lambda(T) = \begin{cases} \sup\{\sum_{i=1}^n \lambda_i x_i \,/\, x \in S\} & if \quad T = N \\ 0 & \text{in other case} \end{cases}$$

it is easy to see that $\overline{x} \in P(S, \theta, B)$ if and only if there exists a link-admissible λ such that $\lambda_i \overline{x}_i = \pi_i(v_\lambda, B)$ for all $i \in N$. Now, if we denote by F_λ the positive affine transformation which maps every $(x_1, \ldots, x_n) \in \mathbb{R}^n$ in $(\lambda_1 x_1, \ldots, \lambda_n x_n)$, we can identify the TU-game v_λ with the bargaining problem $(F_\lambda(S_\lambda), \theta)$ where

$$S_\lambda = \{y \in \mathbb{R}^n \,/\, \sum_{i=1}^{n} \lambda_i y_i \leq \sum_{i=1}^{n} \lambda_i \overline{x}_i\}.$$

Moreover, in view of Equation 4, and taking into account that P and $\mathcal{N}^p$ satisfy (IAT),

$$P(F_\lambda(S_\lambda), \theta, B) = \{\mathcal{N}^p(F_\lambda(S_\lambda), \theta)\}.$$

Then, combining all the facts above,

$$\{F_\lambda(\overline{x})\} = \{\pi(v_\lambda, B)\} = P(v_\lambda, B) = P(F_\lambda(S_\lambda), \theta, B) = \{\mathcal{N}^p(F_\lambda(S_\lambda), \theta)\}.$$

Now, as $\mathcal{N}^p$ satisfies (IAT) and (IIA),

$$\overline{x} = \mathcal{N}^p(S_\lambda, \theta) = \mathcal{N}^p(S, \theta). \ \Box$$

4 Concluding remarks

In Section 3 we introduced an allocation rule for bargaining communication situations based on the Nash solution. Analogously, we could have defined alternative allocation rules related to other bargaining solutions. For instance, an allocation rule inspired on the Kalai-Smorodinsky solution could be introduced in the following way. Take

$$\overline{\mathcal{B}}^{\partial}(N) = \{(S, d, B) \in \overline{\mathcal{B}}(N) \,/\, PA(S_d) = \partial(S_d)\}$$

where $\partial(S_d)$ denotes the frontier of S_d and

$$PA(S_d) = \{x \in S_d \,/\ \nexists y \in S_d \text{ with } y \neq x\}$$

i.e. $PA(S_d)$ is the Pareto-optimal subset of S_d. Then, if $(S, d, B) \in \overline{\mathcal{B}}^{\partial}(N)$, we can define φ^K as $\varphi^K(S, d, B) = \tilde{F}^{-1}(k)$ where $\tilde{F}$ is the positive affine transformation with $\tilde{F}(d) = \theta$ and $\tilde{F}(U(S, d)) = (1, \ldots, 1)$ ($U(S, d) = (U_1(S, d), \ldots, U_n(S, d))$ denotes the utopia point associated to (S, d), i.e. for all $i \in N$ $U_i(S, d) = \max\{x_i \,/\, x \in S_d\}$), and k is the unique point in $PA(\tilde{F}(S))$ lying on the line joining θ and $\pi(u^N, B)$. Defined in this way, it can be proved that φ^K satisfies:

- *Pareto Efficiency (PE)*: for all $(S, d, B) \in \overline{\mathcal{B}}^{\partial}(N)$, $\varphi^K(S, d, B) \in PA(S)$.
- *Utopia Proportional Monotonicity (UPM)*: for all $(S, d, B), (T, d, B) \in \overline{\mathcal{B}}^{\partial}(N)$ with $\varphi^K(S, d, B) \in T$ and

$$\frac{U_i(S, d) - d_i}{U_j(S, d) - d_j} = \frac{U_i(T, d) - d_i}{U_j(T, d) - d_j}$$

$\forall i, j \in N$, then

$$\varphi^K(T, d, B) \geq \varphi^K(S, d, B).$$

(The (UPM) axiom has been introduced in [2] Gutiérrez, 1993). Moreover, using the proofs about weighted Kalai-Smorodinsky solutions in [2] Gutiérrez (1993), it can be proved the following theorem.

Theorem 3. *φ^K is the unique map from $\overline{B}^{\partial}(N) \cap \overline{B}^{w}(N)$ to $\mathbb{R}^n$ satisfying (F), (PE), (IAT), (UPM) and (GWS).*

One of the most interesting features related to the allocation rule we defined in Section 3 is that it results to be the position set (which contains, in this case, only one point) of the communication bargaining problem considered as an NTU communication situation. Obviously, this is not true for φ^K. One interesting question is of what NTU solution φ^K is a restriction.

References

[1] BORM, P., OWEN, G. and TIJS, S. (1992). On the position value for communication situations. *SIAM Journal on Discrete Mathematics* **5**, 305–320.

[2] GUTIÉRREZ, E. (1993). *Three essays on the bargaining problem.* Ph.D. Dissertation. University of the Basque Country, Bilbao, Spain.

[3] KALAI, E. (1977). Nonsymmetric Nash solutions and replications of 2-person bargaining. *International Journal of Game Theory* **6**, 129–133.

[4] MYERSON, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research* **2**, 225–229.

[5] NASH, J. F. (1950). The bargaining problem. *Econometrica* **18**, 155–162.

[6] NOUWELAND, A. van den. (1993). *Games and graph in economic situations.* Ph.D. Dissertation. Tilburg University, Tilburg, The Netherlands.

[7] PETERS, H. J. M. (1992). *Axiomatic bargaining game theory.* Kluwer Academic Publishers, Amsterdam, The Netherlands.

[8] SHAPLEY, L. S. (1953). A value for n-person games, in: H. Kuhn and A. W. Tucker, eds., *Contributions to the Theory of Games II.* Princeton University Press, Princeton, USA.

[9] SHAPLEY, L. S. (1969). Utility comparison and the theory of games, in: G. T. Guilbaud, ed., *La Décision.* Edition du CNRS, Paris, France.