New fixed point theorem for discontinuous operators in cones and applications

Jorge Rodríguez–López *

Departamento de Estatística, Análise Matemática e Optimización, Instituto de matemáticas,
Universidade de Santiago de Compostela, 15782, Facultade de Matemáticas, Campus Vida, Santiago,
Spain.
e-mail: jorgerodriguez.lopez@usc.es

Abstract

We provide new fixed point theorems for a class of discontinuous operators by combining a new fixed point theorem of compression-expansion type for these discontinuous operators with monotone iterative methods. As an application we study the existence of positive solutions for a nonlinear fourth-order discontinuous boundary value problem.

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1 Introduction and preliminaries

Recently some authors [3–5,8,15] gave new criteria for the existence of positive fixed points of completely continuous operators in cones. In order to do this they used some well-known techniques in nonlinear analysis, namely, Krasnosel'skiĭ's fixed point theorem or fixed point index theory together to monotone iterative methods. Their new results only need conditions about one boundary instead of on two boundaries as in the fixed point theorems of compression–expansion type, but it is necessary to require some assumption about the monotonicity of the operator.

Here we will generalize the mentioned results in order to allow that they could be applied to a class of possibly discontinuous operators. In this sense we will use the ideas of [10,19] in their generalizations of Schauder's fixed point theorem and [12] where a new degree theory for this class of discontinuous

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operators was built. This topological degree theory lets to define a discontinuous fixed point index theory in cones [13] that it was applied in order to generalize some well-known fixed point theorems: Krasnosel'skiĭ's theorem [11] and Leggett–Williams' three-solution theorem. We will again employ these tools in the proof of our present results.

Notice that the mentioned theory is a consequence of the classical fixed point and topological degree theories for upper semicontinuous multivalued operators (see [14, 20, 22]).

As an application of our fixed point results, in Section 3 we study the existence of positive solutions for a simply supported beam equation. The main aim is to weaken the continuity assumption on the right-hand side of the differential problem. Another interesting point is that we are able to obtain the existence of a positive solution to the fourth order two-points boundary value problem (3.3) by assuming the existence of a strict upper solution for (3.3) and some asymptotic behavior near the origin or infinity. Therefore, our main existence result is new even in the classical case of continuous right-hand sides in (3.3), see Corollary 3.13.

In the sequel we need the following definitions. A closed and convex subset K of a Banach space $(X, \|\cdot\|)$ is a cone if it satisfies the following conditions:

(i) if $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$;

(ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines the partial order in X given by $x \preceq y$ if and only if $y - x \in K$. For $x, y \in X$, with $x \preceq y$, the set $[x, y] = \{z \in X : x \preceq z \preceq y\}$ is said an order interval. The cone K is called normal with a normal constant $c > 0$ if and only if $\|x\| \leq c\,\|y\|$ for all $x, y \in X$ with $0 \preceq x \preceq y$. If K is a solid cone (i.e. with nonempty interior), $x \prec\!\prec y$ means $y - x \in \operatorname{int}(K)$. Moreover, for two subsets A and B of the Banach space X, we denote $A - B := \{a - b \,:\, a \in A,\ b \in B\}$ and $A \setminus B := \{x \in A \,:\, x \notin B\}$.

Let U be a relatively open subset of K and let $T : \overline{U} \subset K \to K$ be an operator, not necessarily continuous.

Definition 1.1 *The closed–convex envelope (cc–envelope, for short) of an operator* $T : \overline{U} \longrightarrow K$ *is the multivalued mapping* $\mathbb{T} : \overline{U} \longrightarrow 2^K$ *given by*

$$\mathbb{T}x = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\, T\left(\overline{B}_\varepsilon(x) \cap \overline{U}\right) \quad \text{for every } x \in \overline{U}, \tag{1.1}$$

where $\overline{B}_\varepsilon(x)$ *denotes the closed ball centered at* x *and radius* ε*, and* $\overline{\mathrm{co}}$ *means closed convex hull.*

In other words, we say that $y \in \mathbb{T}x$ *if for every* $\varepsilon > 0$ *and every* $\rho > 0$ *there exist* $m \in \mathbb{N}$ *and a finite family of vectors* $x_i \in \overline{B}_\varepsilon(x) \cap \overline{U}$ *and coefficients* $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, m)$ *such that* $\sum \lambda_i = 1$ *and*

$$\left\| y - \sum_{i=1}^{m} \lambda_i\, T x_i \right\| < \rho.$$

Using the degree theory of [12], which is based in the topological degree for multivalued mappings, the following fixed point index was defined in [13].

Definition 1.2 *Let* $T : \overline{U} \subset K \longrightarrow K$ *be an operator such that* $T\overline{U}$ *is relatively compact,* T *has no fixed points on* ∂U *and*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for every } x \in \overline{U} \cap \mathbb{T}\overline{U}, \tag{1.2}$$

where $\mathbb{T}$ *is the cc–envelope of* T.

We define the fixed point index of T *in* K *over* U, $i_K(T,U)$, *as*

$$i_K(T,U) = i_K(\mathbb{T},U),$$

where the right–hand fixed point index is that of multivalued operators (see [14, 22]).

Observe that condition (1.2) is simply equivalent to $\operatorname{Fix}\mathbb{T} \subseteq \operatorname{Fix} T$, where $\operatorname{Fix} S$ denotes the set of all fixed points of a given operator S. We will need some useful properties of the fixed point index above which can be obtained from the properties of the degree theory for multivalued mappings (see [12, 13]).

Theorem 1.3 *Let* T *be a mapping in the conditions of Definition 1.2. Then the following properties are satisfied*

i. (Additivity) Let U *the disjoint union of two open sets* U_1 *and* U_2. *If* $0 \notin (I-T)\left(\overline{U} \setminus (U_1 \cup U_2)\right)$, *then*

$$i_K(T,U) = i_K(T,U_1) + i_K(T,U_2).$$

ii. (Excision) Let $A \subset U$ *be a closed set. If* $0 \notin (I-T)(\partial U) \cup (I-T)(A)$, *then*

$$i_K(T,U) = i_K(T,U \setminus A).$$

iii. (Existence) If $i_K(T,U) \neq 0$, *then there exists* $x \in U$ *such that* $Tx = x$.

iv. (Normalization) For every constant map T *such that* $T\overline{U} \subset U$, $i_K(T,U) = 1$.

Proposition 1.4 *Let* $H : \overline{U} \times [0,1] \longrightarrow K$ *be a map satisfying the following conditions:*

(a) *For each* $(x,t) \in \overline{U} \times [0,1]$ *and all* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, x, t) > 0$ *such that*

$$s \in [0,1],\ |t-s| < \delta \implies \|H(z,t) - H(z,s)\| < \varepsilon \quad \forall\, z \in \overline{B}_\delta(x) \cap \overline{U};$$

(b) $H\left(\overline{U} \times [0,1]\right)$ *is relatively compact;*

(c) $\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\}$ *holds for all* $x \in \overline{U} \cap \mathbb{H}_t\overline{U}$ *when* $t = 0$ *and* $t = 1$.

If $x \notin \mathbb{H}(x,t)$ *for all* $(x,t) \in \partial U \times [0,1]$ *then*

$$i_K(H_0,U) = i_K(H_1,U).$$

Now we recall two fixed point theorems for discontinuous operators satisfying condition (1.2). The first one is a generalization of Schauder's fixed point theorem (see [10] or [12]).

Theorem 1.5 *Let* M *be a non-empty, closed and convex subset of* X. *Let* $T : M \to M$ *be a mapping such that* TM *is a relatively compact subset of* X *and fulfills condition (1.2) with* $\overline{U} = M$. *Then* T *has a fixed point in* M.

The second one is a generalization of the Krasnosel'skiĭ's fixed point theorem, see [11]. Let K be a cone, we will denote $K_c = \{x \in K : \|x\| < c\}$ and $\overline{K}_c$ its closure, with $0 < c < \infty$.

Proposition 1.6 *Let $0 < r_i \leq R$ $(i = 1, 2)$ and let $T : \overline{K}_R \to K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition (1.2) in $\overline{K}_R$.*

(a) *If $\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ an all $\lambda \geq 1$, then $i_K(T, K_{r_1}) = 1$.*

(b) *If there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for every $\lambda \geq 0$ and all $x \in K$ with $\|x\| = r_2$, then $i_K(T, K_{r_2}) = 0$.*

Proof.

(a) We define the homotopy $H : \overline{K}_{r_1} \times [0, 1] \longrightarrow K$ given by

$$H(x, t) = t\,Tx.$$

The set $H\left(\overline{K}_{r_1} \times [0, 1]\right)$ is relatively compact because $T\,\overline{K}_R$ is. By assumption, we have

$$\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\} \text{ for } t = 0, 1 \text{ and all } x \in \overline{K}_{r_1}.$$

Therefore, if there exists $(x, t) \in \partial K_{r_1} \times [0, 1]$ such that $x \in \mathbb{H}(t, x)$ then it implies that $\dfrac{1}{t}x \in \mathbb{T}x$ for some $t \in (0, 1]$ and $x \in K$ with $\|x\| = r_1$, a contradiction.

By Proposition 1.4 and normalization property of index, we have

$$i_K(T, K_{r_1}) = i_K(0, K_{r_1}) = 1.$$

(b) Assume on the contrary that $i_K(T, K_{r_2}) \neq 0$. Since $T\,\overline{K}_R$ is relatively compact, we can take $\mu > 0$ such that $\|y\| < \mu$ for every $y \in \mathbb{T}\,x$ and all $x \in \overline{K}_{r_2}$. Choose $\lambda > (r_2 + \mu)/\|w\|$ and consider the homotopy given by

$$H(x, t) = \mathbb{T}\,x + t\lambda w.$$

We have by assumption that $x \notin H(x, t)$ for all $x \in K$ with $\|x\| = r_2$ and every $t \in [0, 1]$. Hence, by virtue of the homotopy invariance property of fixed point index for multivalued mappings we obtain

$$i_K(\mathbb{T}, K_{r_2}) = i_K(\mathbb{T} + \lambda w, K_{r_2}).$$

Now, since $i_K(\mathbb{T}, K_{r_2}) = i_K(T, K_{r_2}) \neq 0$ (see [13]), there exists $x \in K_{r_2}$ such that $x \in \mathbb{T}\,x + \lambda w$. Then,

$$\|x - \lambda w\| \geq \lambda \|w\| - \|x\| > (r_2 + \mu) - \|x\| > \mu > \|y\|,$$

for all $y \in \mathbb{T}x$, which it is not possible. Therefore, $i_K(T, K_{r_2}) = 0$. □

The following corollary is an immediate consequence of the previous proposition because the conditions required in (i) and (ii) are stronger than (a) and (b), respectively.

Corollary 1.7 *Let $0 < r_i \leq R$ $(i = 1, 2)$ and let $T : \overline{K}_R \to K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition (1.2) in $\overline{K}_R$.*

(i) *If* $y \not\succeq x$ *for all* $y \in \mathbb{T}\,x$ *and all* $x \in K$ *with* $\|x\| = r_1$*, then* $i_K(T, K_{r_1}) = 1$*.*

(ii) *If* $y \not\preceq x$ *for all* $y \in \mathbb{T}\,x$ *and all* $x \in K$ *with* $\|x\| = r_2$*, then* $i_K(T, K_{r_2}) = 0$*.*

To finish the preliminaries we enunciate the monotone iterative method for discontinuous mappings given by Heikkilä *et al.* [17, Theorem 1.2.2].

Theorem 1.8 *Let* Y *be a subset of an ordered metric space* X*,* $[a,b]$ *a nonempty order interval in* Y*, and* $G : [a,b] \to [a,b]$ *a nondecreasing mapping. If* $(Gx_n)_{n=0}^{\infty}$ *converges in* Y *whenever* $(x_n)_{n=0}^{\infty}$ *is a monotone sequence in* $[a,b]$*, then the well-ordered chain of* G*-iterations of* a *has the maximum* x_**, and the inversely well-ordered chain of* G*-iterations of* b *has the minimum* x^**, and*

$$x_* = \min\{y \,|\, Gy \le y\}, \quad x^* = \max\{y \,|\, y \le Gy\}.$$

In particular, x_* *and* x^* *are the extremal fixed points of* G*.*

2 Fixed point theorems in cones

Let X be a real Banach space and $K \subset X$ a solid cone. We state our results on the existence of non-trivial fixed points.

Theorem 2.1 *Let* $T : K \to K$ *be an operator such that maps bounded sets into relatively compact sets. Suppose that*

1. *there exists* $r > 0$ *such that* $\mathbb{T}x \subset \{Tx\} - K$ *for all* $x \in K$ *with* $\|x\| = r$*, where* $\mathbb{T}$ *is the multivalued operator associated to* T *defined as in* (1.1)*,*
2. *there exist* $\beta \in K$*, with* $T\,\beta \preceq \beta$*, and* $R > 0$ *such that* $\overline{B}_R(\beta) \subset K$*,*
3. *the mapping* T *is nondecreasing in* $\mathcal{P} = \{x \in K : x \preceq \beta\}$ *and* $T\,\mathcal{P}$ *is relatively compact,*
4. *there exists a bounded open set* $V \subset K$ *such that* $i_K(T, V) = 0$ *and* $\overline{K}_r \subset V$ *or* $\overline{V} \subset K_r$*.*

Moreover suppose that T *fulfills condition (1.2) on* $\mathcal{P} \cup \overline{V}$*. Then* T *has at least a non-trivial fixed point in* K *such that it belongs to* $\mathcal{P}$ *or it belongs to* $\begin{cases} V \setminus \overline{K}_r, & \text{if } \overline{K}_r \subset V, \\ K_r \setminus \overline{V}, & \text{if } \overline{V} \subset K_r. \end{cases}$

Proof. Without loss of generality assume that $r \le R$. Since $\overline{B}_R(\beta) \subset K$, if $x \in K$ with $\|x\| \le R$, then $\beta - x \in K$ and so $x \preceq \beta$. Suppose there exists $\alpha \in K$ with $\|\alpha\| = r$ and $T\alpha \succeq \alpha$. We have $\alpha \preceq \beta$ and if $\alpha \preceq x \preceq \beta$, since T is a nondecreasing mapping and $T\beta \preceq \beta$, we obtain $\alpha \preceq T\alpha \preceq Tx \preceq T\beta \preceq \beta$. Thus T maps the order interval $[\alpha, \beta]$ into itself. Notice that $[\alpha, \beta]$ is a nonempty closed and convex set, $T\left([\alpha, \beta]\right) \subset T\,\mathcal{P}$ is relatively compact and condition (1.2) is fulfilled in $[\alpha, \beta]$, so Theorem 1.5 implies that T has at least a fixed point in $[\alpha, \beta]$.

Otherwise, we have $Tx \not\succeq x$ for all $x \in K$ with $\|x\| = r$, and then $y \not\succeq x$ for all $y \in \mathbb{T}x$ with $x \in K$, $\|x\| = r$. Indeed, for $y \in \mathbb{T}x$ with $x \in K$ and $\|x\| = r$, we have $y - x = Tx - x - k$ for some $k \in K$, by hypothesis (1). Thus $Tx - x \notin K$ implies that $y - x \notin K$. Therefore, by Corollary 1.7, $i_K(T, K_r) = 1$.

Since $i_K(T, V) = 0$, the properties of the fixed point index (see Theorem 1.3) ensure that there exists a non-trivial fixed point of T in $V \setminus \overline{K}_r$ (if $\overline{K}_r \subset V$) or in $K_r \setminus \overline{V}$ (if $\overline{V} \subset K_r$). $\square$

REMARK 2.2 *Observe that hypothesis (1) in Theorem 2.1 is weaker than the continuity of the operator T in a neighborhood of the origin.*

In fact, such a condition is satisfied if the operator T is "upper semicontinuous" for $x \in K$ with $\|x\| = r$, that is,

(1^*) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y - x\| < \delta$, $y \in K$, implies $Ty \in \overline{B}_\varepsilon(Tx) - K$.*

Indeed, if (1^*) holds, then for each $\varepsilon > 0$,

$$\mathbb{T}x \subset \overline{\text{co}}\left(\overline{B}_\varepsilon(Tx) - K\right) = \overline{B}_\varepsilon(Tx) - K,$$

so $\mathbb{T}x \subset \{Tx\} - K$.

REMARK 2.3 *The previous result seems to be new even in the case of a completely continuous operator T. Indeed, in [3–5, 8] it is assumed either that the cone is normal or that it satisfies a suitable condition which involves the partial order induced by the cone and the norm of the Banach space (see [5]). However it can be useful to work in cones which not satisfy those conditions as we will show in the next section.*

The conditions requested in Theorem 2.1 can be too restrictive in some cases: the set $T\mathcal{P}$ must be relatively compact, but $\mathcal{P}$ could not be a bounded set. We note that this assumption can be weakened if the cone is normal. In addition, the hypotheses about the monotonicity of the operator and the condition (1.2) need only to be satisfied in suitable bounded sets of the cone.

THEOREM 2.4 *Let K be a solid and normal cone with normal constant $d \geq 1$ and $T : K \to K$ an operator which maps bounded sets into relatively compact ones. Suppose that*

(*a*) *there exists $r > 0$ such that $\mathbb{T}x \subset \{Tx\} - K$ for all $x \in K$ with $\|x\| = r$,*

(*b*) *there exist $\beta \in K$, with $T\beta \preceq \beta$, and $R > 0$ such that $\overline{B}_R(\beta) \subset K$,*

(*c*) *the mapping T is nondecreasing in $\mathcal{P} = \left\{x \in K : x \preceq \beta \text{ and } \dfrac{r}{d} \leq \|x\|\right\}$,*

(*d*) *there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 0$ and $\overline{K}_r \subset V$ or $\overline{V} \subset K_r$.*

Moreover suppose that T fulfills condition (1.2) for all $x \in \overline{V}$. Then T has at least a non-trivial fixed point in K such that it belongs to $\mathcal{P}$ or it belongs to $\begin{cases} V \setminus \overline{K}_r, & \text{if } \overline{K}_r \subset V, \\ K_r \setminus \overline{V}, & \text{if } \overline{V} \subset K_r. \end{cases}$

Proof. Without loss of generality assume that $r \leq R$ and suppose there exists $\alpha \in K$ with $\|\alpha\| = r$ and $T\alpha \succeq \alpha$. We obtain that T maps the set $[\alpha, \beta]$ into itself. Consider a nondecreasing sequence $\{x_n\}_{n=0}^\infty \subset [\alpha, \beta]$. The sequence $\{Tx_n\}_{n=0}^\infty$ is contained in the bounded order interval $[\alpha, \beta]$, so $\{Tx_n\}_{n=0}^\infty$ is relatively compact and thus it has a convergent subsequence $\{Tx_{n_k}\} \to y$. Since T is nondecreasing, there exists a $N \in \mathbb{N}$ such that for all $n, m \geq N$ is possible to find $k, l \in \mathbb{N}$ such that $Tx_{n_k} \preceq Tx_n \preceq Tx_{n_l}$ and $Tx_{n_k} \preceq Tx_m \preceq Tx_{n_l}$. Therefore, for every $n, m \geq N$ we have $Tx_n - Tx_m \preceq Tx_{n_l} - Tx_{n_k}$, so from the normality of the cone we obtain $\|Tx_n - Tx_m\| \leq d\,\|Tx_{n_l} - Tx_{n_k}\|$. It follows that $\{Tx_n\}_{n=0}^{\infty}$ is a Cauchy sequence and then the whole sequence $\{Tx_n\}_{n=0}^{\infty}$ converges to y. It is similar to see that $\{Tx_n\}_{n=0}^{\infty}$ converges whenever the sequence $\{x_n\}_{n=0}^{\infty}$ is non-increasing. Hence, Theorem 1.8 ensures that T has a fixed point in $[\alpha, \beta]$.

Now suppose that such α does not exist. In this case, $Tx \not\succeq x$ for all $x \in K$ with $\|x\| = r$ which implies $y \not\succeq x$ for all $y \in \mathbb{T}x$ with $x \in K$ and $\|x\| = r$. By Corollary 1.7, $i_K(T, K_r) = 1$. Since $i_K(T, V) = 0$, we conclude from the additivity and existence properties of the fixed point index. $\square$

Following the ideas of [4, 5] it is possible to get a result for non-increasing discontinuous operators.

Theorem 2.5 *Let $T : K \to K$ be an operator which maps bounded sets into relatively compact sets. Suppose that*

(i) *there exists $r > 0$ such that $\mathbb{T}x \subset \{Tx\} + K$ for all $x \in K$ with $\|x\| = r$,*

(ii) *there exist $\alpha \in K$, with $\alpha \preceq T\alpha$, and $R > 0$ such that $\overline{B}_R(\alpha) \subset K$,*

(iii) *the mapping T is non-increasing in $\mathcal{P} = \{x \in K : r \leq \|x\| \leq \|\alpha\|\}$,*

(iv) *there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 1$ and $\overline{K}_r \subset V$ or $\overline{V} \subset K_r$.*

Moreover suppose that T fulfills condition (1.2) for all $x \in \overline{V}$. Then T has at least a non-trivial fixed point in K such that it belongs to $\mathcal{P}$ or it belongs to $\begin{cases} V \setminus \overline{K}_r, & \text{if } \overline{K}_r \subset V, \\ K_r \setminus \overline{V}, & \text{if } \overline{V} \subset K_r. \end{cases}$

Proof. Without loss of generality assume that $r \leq R$. Let $x \in K$ be with $\|x\| = r$. By (ii), $x \preceq \alpha$ and since $x, \alpha \in \mathcal{P}$, it follows from (iii) that $Tx \succeq T\alpha \succeq \alpha \succeq x$.

If for some $x \in K$ with $\|x\| = r$ we have $Tx \preceq x$, then $Tx = x$. Otherwise, by condition (i), $y \npreceq x$ for all $y \in \mathbb{T}x$, with $x \in K$ and $\|x\| = r$. Hence Corollary 1.7 ensures that $i_K(T, K_r) = 0$. Therefore assumption (iv) and the properties of fixed point index imply the existence of the desired fixed point. $\square$

Remark 2.6 *Analogously to Remark 2.2, condition* (i) *holds if the operator T is "lower semicontinuous" at $x \in K$, $\|x\| = r$, i.e.,*

(i^*) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y - x\| < \delta$, $y \in K$, implies $Ty \in \overline{B}_\varepsilon(Tx) + K$.*

Remark 2.7 *The fixed point theorems presented in [3–5, 8] were recently extended to Fréchet spaces and admissibly compact maps in [16]. In addition, some assumptions about the cone (solid and normal) were weakened or removed.*

3 Application to a fourth order problem

In this section we will apply Theorem 2.1 in order to obtain sufficient conditions for the existence of positive solutions for the following fourth order boundary value problem (BVP)

$$\begin{aligned} &u^{(4)}(t) = g(t)f(u(t)), \quad t \in (0,1), \\ &u(0) = u(1) = 0 = u''(0) = u''(1), \end{aligned} \tag{3.3}$$

where $g \geq 0$ a.e. and $g \in L^1(0,1)$ and the function $f : [0,\infty) \to [0,\infty)$ is such that $u \mapsto f(u)$ is measurable for every $u \in \mathcal{C}^2([0,1])$ and $f \in L^\infty_{loc}([0,\infty))$. BVP (3.3) was intensively studied in the literature (see, for example [6–9, 21]) and the results are presented several times as an application to beam problems. However continuity assumptions are usually imposed about f. Our goal is to weaken this hypothesis.

Technical reasons make that we need to work in the Banach space $(\mathcal{C}^2([0,1]), \|\cdot\|)$, where $\|u\| = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$ and $\|\cdot\|_\infty$ is the usual supremum norm.

We shall look for fixed points of the operator $T : \mathcal{C}^2([0,1]) \to \mathcal{C}^2([0,1])$ given by

$$Tu(t) := \int_0^1 G(t,s)g(s)f(u(s))\,ds,$$

where G is the Green's function. It is given by

$$G(t,s) = \begin{cases} \dfrac{1}{6}s(1-t)(2t-s^2-t^2), & s \leq t, \\ \dfrac{1}{6}t(1-s)(2s-t^2-s^2), & s > t, \end{cases}$$

which is non negative and satisfies (see [8, 21])

$$\begin{aligned} G(t,s) &\leq \Phi(s), && \text{for } t,s \in [0,1], \\ c\,\Phi(s) &\leq G(t,s), && \text{for } t \in \left[\tfrac{1}{4},\tfrac{3}{4}\right], s \in [0,1], \end{aligned}$$

where

$$\Phi(s) = \begin{cases} \dfrac{\sqrt{3}}{27}s(1-s^2)^{3/2}, & \text{for } 0 \leq s \leq 1/2, \\ \dfrac{\sqrt{3}}{27}(1-s)s^{3/2}(2-s)^{3/2}, & \text{for } 1/2 \leq s \leq 1, \end{cases}$$

and $c = 45\sqrt{3}/128 \approx 0.608924$.

We shall look for fixed points of T in the cone

$$K = \left\{ u \in \mathcal{C}^2([0,1]) : u \geq 0, \min_{t \in \left[\frac{1}{4},\frac{3}{4}\right]} u(t) \geq \tilde{c}\,\|u\|_\infty \right\},$$

where $0 < \tilde{c} \leq c$ will be fixed later.

Remark 3.1 *K is not a normal cone because there is not information about the derivatives of the functions which belong to it whereas that in the $\mathcal{C}^2$ norm it appears.*

Proposition 3.2 *The operator $T : K \to K$ is well-defined and maps bounded sets into relatively compact sets.*

Proof. The fact that $TK \subset K$ can be verified by using the properties of the Green's function G together with the mapping Φ. In addition, from the hypotheses about f and g and the regularity of the Green's function it is routine to conclude that T maps bounded sets into relatively compact ones by means of the Áscoli–Arzela's theorem. □

Maximum principles for the operator $L_M u := u^{(4)} + Mu$ with the boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ were established in [6]. Here we recall some results in this direction which we will employ in order to give sufficient conditions for the existence of an upper fixed point for the operator T (see condition (2) in Theorem 2.1).

Definition 3.3 *Let $B \subset \mathcal{C}^4([0,1])$ and define the operator $L_M : B \to \mathcal{C}([0,1])$ given by*

$$(L_M u)(t) := u^{(4)}(t) + Mu(t) \quad \text{for all } t \in [0,1].$$

We say that L_M is inverse positive in B if

$$u \in B,\ (L_M u)(t) \geq 0 \text{ for all } t \in [0,1] \text{ implies } u(t) \geq 0 \text{ for all } t \in [0,1],$$

and L_M is strongly inverse positive in B if it is inverse positive in B and

$$u \in B,\ L_M u \gneqq 0 \text{ in } [0,1] \text{ implies } u(t) > 0 \text{ in } (0,1).$$

Proposition 3.4 ([6, Corollary 2.1]) *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$\mathcal{W} = \{u \in \mathcal{C}^4([0,1]) \,:\, u(0) \geq 0,\ u(1) \geq 0,\ u''(0) \leq 0,\ u''(1) \leq 0\}$$

if, and only if, $0 \leq M \leq c_1$, where $c_1 = 4k_1^4 \approx 125.137$ and k_1 is the smallest positive solution of the equation $\tan k = -\tanh k$.

Definition 3.5 *We say that $\beta \in W^{4,1}(I)$ is an upper solution for problem (3.3) if*

$$\begin{array}{l} \beta^{(4)}(t) \geq g(t)f(\beta(t)) \quad \text{for a.a. } t \in [0,1], \\ \beta(0) \geq 0,\ \beta(1) \geq 0,\ \beta''(0) \leq 0,\ \beta''(1) \leq 0. \end{array}$$

Further, $\beta \in W^{4,1}(I)$ is a strict upper solution if it is an upper solution and, moreover, there exists an open subinterval $I_0 \subset [0,1]$ such that

$$\beta^{(4)}(t) > g(t)f(\beta(t)) \quad \text{for a.a. } t \in I_0.$$

Following the notation of [5], we define

$$\gamma_* = \inf_{t \in [1/4,3/4]} \int_0^1 G(t,s)g(s)\,ds, \qquad \gamma^* = \sup_{t \in [0,1]} \int_0^1 G(t,s)g(s)\,ds,$$

and we suppose $\gamma_* > 0$.

We are in a position to present some sufficient conditions for the existence of an upper fixed point for the operator T, that is, $\beta \in K$ such that $T\beta \preceq \beta$.

Lemma 3.6 *Suppose that one of the following two conditions holds:*

(*i*) *there exists $\beta > 0$ such that $\gamma^* f(\beta) < \beta$; or*

(*ii*) *there exists a strict upper solution β for problem (3.3) with $\min_{t \in [0,1]} \beta(t) > 0$.*

Then $T\beta \preceq \beta$ and there exists $R > 0$ such that $\overline{B}_R(\beta) \subset K$.

Proof. First, assume that condition (i) holds. By inequality $\gamma^* f(\beta) < \beta$, we obtain that

$$T\beta = \int_0^1 G(t,s)g(s)f(\beta)\,ds \le \gamma^* f(\beta) < \beta.$$

Moreover, since $\|\beta - T\beta\|_\infty = \beta$ and there exists $0 < \tilde{c} < c$ such that $\beta - \gamma^* f(\beta) > \tilde{c}\beta$ we have for every $t \in [1/4, 3/4]$,

$$\beta(t) - T\beta(t) \ge \beta - \gamma^* f(\beta) > \tilde{c}\beta = \tilde{c}\,\|\beta - T\beta\|_\infty .$$

In addition, β is an interior point of K. Indeed, if $u \in \overline{B}_R(\beta)$ for $0 < R < \beta$ then $\|u - \beta\|_\infty \le R$, that is, $\beta - R \le u(t) \le \beta + R$ for all $t \in [0,1]$, so $u(t) > 0$ for all $t \in [0,1]$ and whenever R is small enough we have

$$\min_{t\in[1/4,3/4]} u(t) \ge \beta - R \ge \tilde{c}(\beta + R) \ge \tilde{c}\,\|u\|_\infty .$$

Now, suppose that condition (ii) is satisfied. It ensures the existence of a nonnegative function $h \in L^1(I)$, $A, B \ge 0$ and $C, D \le 0$ such that

$$\begin{aligned} &\beta^{(4)}(t) - g(t)f(\beta(t)) = h(t) \quad \text{for a.a. } t \in (0,1),\\ &\beta(0) = A,\ \beta(1) = B,\ \beta''(0) = C,\ \beta''(1) = D, \end{aligned}$$

or equivalently,

$$\beta(t) - T\beta(t) = \int_0^1 G(t,s)h(s)\,ds + \vartheta(t),$$

where ϑ is the unique solution of the problem

$$\begin{aligned} &y^{(4)}(t) = 0 \quad \text{for a.a. } t \in (0,1),\\ &y(0) = A,\ y(1) = B,\ y''(0) = C,\ y''(1) = D. \end{aligned}$$

Since $M = 0$, by Proposition 3.4, we deduce that $\beta(t) - T\beta(t) \ge 0$ in $[0,1]$ and $\beta(t) - T\beta(t) > 0$ in $(0,1)$. Hence, there exists $0 < \tilde{c} < c$ small enough such that for $t \in [1/4, 3/4]$,

$$\beta(t) - T\beta(t) > \tilde{c}\,\|\beta - T\beta\|_\infty .$$

As before, we can verify that β is an interior point of K. □

Now we define the points where we allow the function f to be discontinuous. The following definition is an adjustment of the admissible discontinuity curves of [10, 12, 19] in the case of a fourth order problem and an autonomous function f. Moreover it is similar to another admissible discontinuity notions, see [18, Theorem A] and also [1, 2].

Definition 3.7 *An admissible discontinuity point is a nonnegative real number x satisfying one of the following conditions:*

- (a) $f(x) = 0$ *(x is said a viable point),*
- (b) *There exist $\varepsilon > 0$ and $\psi \in L^1(0,1)$, $\psi(t) > 0$ for a.a. $t \in [0,1]$ such that*

$$\psi(t) < g(t)f(y) \quad \text{for a.a. } t \in [0,1] \text{ and all } y \in [x-\varepsilon, x+\varepsilon] \quad (x \text{ is inviable}). \tag{3.4}$$

Now we enunciate three technical results whose proofs will be omitted because they can be found in [19].

LEMMA 3.8 ([19, LEMMA 4.1]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a, b)$, $g \ge 0$ a.e., and $h > 0$ a.e. in (a, b).*

For every measurable set $J \subset (a, b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for every $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \setminus J} g(s)\, ds}{\int_{\tau_0}^{t} h(s)\, ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \setminus J} g(s)\, ds}{\int_{t}^{\tau_0} h(s)\, ds}.$$

COROLLARY 3.9 ([19, COROLLARY 4.2]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a, b)$ be such that $h > 0$ a.e. in (a, b).*

For every measurable set $J \subset (a, b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \cap J} h(s)\, ds}{\int_{\tau_0}^{t} h(s)\, ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \cap J} h(s)\, ds}{\int_{t}^{\tau_0} h(s)\, ds}.$$

COROLLARY 3.10 ([19, COROLLARY 4.3]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $f,\ f_n : [a, b] \longrightarrow \mathbb{R}$ be absolutely continuous functions on $[a, b]$ $(n \in \mathbb{N})$, such that $f_n \to f$ uniformly on $[a, b]$ and for a measurable set $A \subset [a, b]$ with $m(A) > 0$ we have*

$$\lim_{n \to \infty} f_n'(t) = g(t) \quad \textit{for a.a. } t \in A.$$

If there exists $M \in L^1(a, b)$ such that $|f'(t)| \le M(t)$ a.e. in $[a, b]$ and also $|f_n'(t)| \le M(t)$ a.e. in $[a, b]$ $(n \in \mathbb{N})$, then $f'(t) = g(t)$ for a.a. $t \in A$.

We shall also need the following result whose proof is similar to that of Lemma 3.11 in [11].

LEMMA 3.11 *If $M \in L^1(0, 1)$, $M \ge 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^3([0, 1]) : \left|u'''(t) - u'''(s)\right| \le \int_s^t M(r)\, dr \quad \textit{whenever } 0 \le s \le t \le 1 \right\},$$

is closed in $\mathcal{C}^2([0, 1])$.

Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \to u$ in the $\mathcal{C}^2$ norm, then there exists a subsequence $\{u_{n_k}\}$ which tends to u in the $\mathcal{C}^3$ norm.

Now we prove the main result of this section.

THEOREM 3.12 *Assume that the previous hypotheses about f and g hold and*

- (H) *There exist admissible discontinuity points $x_n \ge 0$ such that the function $u \mapsto f(u)$ is continuous in $[0, \infty) \setminus \bigcup_{n \in \mathbb{N}} \{x_n\}$ and $r > 0$ such that f is right-continuous in $[0, r]$.*

Moreover, assume that

- (i) *there exist $\beta \in K$, with $T\,\beta \preceq \beta$, and $R > 0$ such that $\overline{B}_R(\beta) \subset K$;*
- (ii) *f is nondecreasing on $[0, \|\beta\|_\infty]$;*
- (iii) $f_0 := \lim_{u \to 0^+} \dfrac{f(u)}{u} = +\infty$ *or* $f_\infty := \lim_{u \to \infty} \dfrac{f(u)}{u} = +\infty$.

Then BVP (3.3) has at least a positive solution.

Proof. We are going to prove that the conditions of Theorem 2.1 are satisfied. Claims 1 and 3 are similar to those in the proof of [8, Theorem 3.1] and the last one is a technical result which follows the ideas of [19, Theorem 4.4] and [11, Theorem 3.12].

Claim 1: The map is monotone nondecreasing in the set $\mathcal{P} = \{u \in K : u \preceq \beta\}$ *and* $T\mathcal{P}$ *is relatively compact.*

Since f is nondecreasing in $[0, \|\beta\|_\infty]$, it is clear that if we take $u, v \in K$ with $u(t) \le v(t) \le \beta(t)$ for all $t \in [0,1]$ we have $Tv(t) - Tu(t) \ge 0$ and for $t \in [1/4, 3/4]$ and $r \in [0,1]$,

$$\begin{aligned} Tv(t) - Tu(t) &= \int_0^1 G(t,s)g(s)\left[f(v(s)) - f(u(s))\right] ds \ge c\int_0^1 \Phi(s)g(s)\left[f(v(s)) - f(u(s))\right] ds \\ &\ge c\int_0^1 G(r,s)g(s)\left[f(v(s)) - f(u(s))\right] ds \ge \tilde{c}\left[Tv(r) - Tu(r)\right], \end{aligned}$$

so $\min\limits_{t\in[1/4,3/4]} [Tv(t) - Tu(t)] \ge \tilde{c}\,\|Tv - Tu\|_\infty$. Therefore $Tv - Tu \in K$, i.e., $Tu \preceq Tv$. Thus T is nondecreasing.

Notice that $T\mathcal{P}$ is relatively compact because if $u \in \mathcal{P}$ then $0 \le u(t) \le \|\beta\|_\infty$ for all $t \in [0,1]$ so, since $f \in L^\infty_{loc}([0,\infty))$, there exists $N > 0$ such that $f(u) \le N$ for all $u \in \mathcal{P}$. Therefore the conclusion is easily obtained by Áscoli-Arzela theorem.

Claim 2: The operator T *satisfies that* $\mathbb{T}u \subset \{Tu\} - K$ *for all* $u \in K$ *with* $\|u\| = r$.

Without loss of generality, assume that $r \le \|\beta\|_\infty$. Let $u \in K$ with $\|u\| = r$ and $\varepsilon > 0$ given. By the right-continuity and the monotonicity of f in $[0, r]$, for all $x \in [0, r]$ there exists $\delta > 0$ such that $-x \le y - x < \delta$ implies that $f(y) \le f(x) + \varepsilon/\gamma^*$. Therefore, for $\|v - u\| < \delta$, $v \in K$, we have

$$\begin{aligned} Tv(t) &= \int_0^1 G(t,s)g(s)f(v(s))\,ds \le \int_0^1 G(t,s)g(s)\left[f(u(s)) + \frac{\varepsilon}{\gamma^*}\right] ds \\ &= \int_0^1 G(t,s)g(s)f(u(s))\,ds + \varepsilon = Tu(t) + \varepsilon, \end{aligned}$$

so $Tv \preceq Tu + \varepsilon$ and thus $Tv \in \overline{B}_\varepsilon(Tu) - K$. Now, using Remark 2.2, the conclusion is obtained.

Claim 3: There exists a bounded open set $V \subset K$ *such that* $i_K(T, V) = 0$ *and* $\overline{V} \subset K_r$ *or* $\overline{K}_r \subset V$.

Suppose that $f_0 = \infty$ (the case $f_\infty = \infty$ is similar). In this case we shall show that there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 0$ and $\overline{V} \subset K_r$ ($\overline{K}_r \subset V$, if $f_\infty = \infty$). Hypothesis (iii) guarantees that we can choose $L > 0$ large enough such that $\gamma_* L c > 2$ and $C > 0$ satisfying $f(s) \ge Ls$ provided that $0 \le s \le C$. Suppose that $u \in K$ with $\|u\| = \min\{r/2, C/2\} =: \overline{r}$, then for every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \ldots, m)$, with $\sum \lambda_i = 1$ and $\varepsilon = \|u\|_\infty/2$, we have $\|u_i\|_\infty \le 3\overline{r}/2 < C$, so $0 \le u_i(t) \le C$ for all $t \in [1/4, 3/4]$ and

$$\begin{aligned} \sum_{i=1}^m \lambda_i T u_i(t) &\ge \sum_{i=1}^m \lambda_i \int_{1/4}^{3/4} G(t,s)g(s)f(u_i(s))\,ds \\ &\ge \gamma_* L c \sum_{i=1}^m \lambda_i \|u_i\|_\infty \ge \gamma_* L c \left(\|u\|_\infty - \varepsilon\right) > \|u\|_\infty, \end{aligned}$$

which implies that $y \not\preceq u$ for all $y \in \mathbb{T}u$ with $u \in K$ and $\|u\| = \overline{r}$. By Corollary 1.7 we obtain that $i_K(T, K_{\overline{r}}) = 0$, so we can choose $V = K_{\overline{r}}$.

Claim 4: The operator T satisfies the condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ for all $u \in \mathcal{P} \cup \overline{K}_R$.

First, notice that there exists $R_1 > 0$ such that $\|u\|_\infty \le R_1$ for all $u \in \mathcal{P} \cup \overline{K}_R$, so there exists $R_2 > 0$ such that $f(u) \le R_2$ for all $u \in \mathcal{P} \cup \overline{K}_R$. Therefore, there exists $M \in L^1(0,1)$ such that

$$g(t)f(u) \le M(t) \quad \text{for a.a. } t \in [0,1] \text{ and all } u \in \mathcal{P} \cup \overline{K}_R. \tag{3.5}$$

Now we consider the set

$$Q = \left\{ u \in \mathcal{C}^3([0,1]) : \left| u'''(t) - u'''(s) \right| \le \int_s^t M(r)\, dr \quad (s \le t) \right\}, \tag{3.6}$$

which is a closed and convex subset of $\mathcal{C}^2([0,1])$ by Lemma 3.11. It is immediate to see that $T\,K \subset Q$, by the definition of the operator T, and since Q is a closed and convex subset of X we have that $\mathbb{T}K \subset Q$. In particular, $\mathbb{T}(\mathcal{P} \cup \overline{K}_R) \subset Q$. We note that condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ need only be verified for every $u \in (\mathcal{P} \cup \overline{K}_R) \cap \mathbb{T}(\mathcal{P} \cup \overline{K}_R) \subset (\mathcal{P} \cup \overline{K}_R) \cap Q$.

Therefore we fix $u \in (\mathcal{P} \cup \overline{K}_R) \cap Q$ and we consider the following three cases.

Case 1: $m(\{t \in [0,1] : u(t) = x_n\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that then T is continuous at u.

The assumption implies that for a.a. $t \in [0,1]$ the function $f(\cdot)$ is continuous at $u(t)$. Hence, if $u_k \to u$ in Q, then

$$f(u_k(t)) \to f(u(t)) \quad \text{for a.a. } t \in [0,1],$$

which, along with (3.5), yield $Tu_k \to Tu$ in $\mathcal{C}^2([0,1])$.

Case 2: $m(\{t \in [0,1] : u(t) = x_n\}) > 0$ for some $n \in \mathbb{N}$ such that x_n is inviable. In this case, we can prove that $u \notin \mathbb{T}u$.

Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in [0,1] : u(t) = x_n\}) > 0$ and we will simply denote x instead of x_n. There exist $\varepsilon > 0$ and $\psi \in L^1(0,1)$, $\psi(t) > 0$ for a.a. $t \in [0,1]$ such that (3.4) holds.

We denote $J = \{t \in [0,1] : u(t) = x\}$ and we deduce from Lemma 3.8 that there exists a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t] \setminus J} M(s)\, ds}{(1/4)\int_{\tau_0}^t \psi(s)\, ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0] \setminus J} M(s)\, ds}{(1/4)\int_t^{\tau_0} \psi(s)\, ds}. \tag{3.7}$$

By Corollary 3.9 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t] \cap J_0} \psi(s)\, ds}{\int_{\tau_0}^t \psi(s)\, ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0] \cap J_0} \psi(s)\, ds}{\int_t^{\tau_0} \psi(s)\, ds}. \tag{3.8}$$

Let us now fix a point $\tau_0 \in J_1$. From (3.7) and (3.8) we deduce that there exist $t_- < \tilde{t}_- < \tau_0$ and $t_+ > \tilde{t}_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied for all $t \in [\tilde{t}_+, t_+]$:

$$\int_{[\tau_0,t] \setminus J} M(s)\, ds < \frac{1}{4} \int_{\tau_0}^t \psi(s)\, ds, \tag{3.9}$$

$$\int_{[\tau_0,t] \cap J} \psi(s)\, ds \ge \int_{[\tau_0,t] \cap J_0} \psi(s)\, ds > \frac{1}{2} \int_{\tau_0}^t \psi(s)\, ds, \tag{3.10}$$

and for all $t \in [t_-, \tilde{t}_-]$:

$$\int_{[t,\tau_0]\setminus J} M(s)\,ds < \frac{1}{4}\int_t^{\tau_0} \psi(s)\,ds, \tag{3.11}$$

$$\int_{[t,\tau_0]\cap J} \psi(s)\,ds > \frac{1}{2}\int_t^{\tau_0} \psi(s)\,ds. \tag{3.12}$$

Finally, we define a positive number

$$\tilde{\rho} = \min\left\{\frac{1}{4}\int_{\tilde{t}_-}^{\tau_0} \psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{\tilde{t}_+} \psi(s)\,ds\right\}, \tag{3.13}$$

and we are now in a position to prove that $u \notin \mathbb{T}u$. It suffices to prove the following claim:

Claim – Let $\varepsilon > 0$ be given by our assumptions over x and let $\rho = \dfrac{\tilde{\rho}}{2}\min\{\tilde{t}_- - t_-, t_+ - \tilde{t}_+\}$ be where $\tilde{\rho}$ is as in (3.13). For every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$, we have $\|u - \sum \lambda_i T u_i\| \geq \rho$.

Let u_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T u_i$. For a.a. $t \in J = \{t \in [0,1] : u(t) = x\}$ we have

$$y^{(4)}(t) = \sum_{i=1}^m \lambda_i (Tu_i)^{(4)}(t) = \sum_{i=1}^m \lambda_i\, g(t) f(u_i(t)). \tag{3.14}$$

On the other hand, for every $i \in \{1, 2, \dots, m\}$ and every $t \in J$ we have

$$|u_i(t) - x| = |u_i(t) - u(t)| < \varepsilon,$$

and then the assumptions on x ensure that for a.a. $t \in J$ we have

$$y^{(4)}(t) = \sum_{i=1}^m \lambda_i\, g(t) f(u_i(t)) > \sum_{i=1}^m \lambda_i\, \psi(t) = \psi(t) = \psi(t) + u^{(4)}(t). \tag{3.15}$$

Now for $t \in [t_-, \tilde{t}_-]$ we compute

$$\begin{aligned}
y'''(\tau_0) - y'''(t) &= \int_t^{\tau_0} y^{(4)}(s)\,ds = \int_{[t,\tau_0]\cap J} y^{(4)}(s)\,ds + \int_{[t,\tau_0]\setminus J} y^{(4)}(s)\,ds \\
&> \int_{[t,\tau_0]\cap J} \psi(s)\,ds + \int_{[t,\tau_0]\cap J} u^{(4)}(s)\,ds \qquad \text{(by (3.15) and (3.14))} \\
&= \int_{[t,\tau_0]\cap J} \psi(s)\,ds + u'''(\tau_0) - u'''(t) - \int_{[t,\tau_0]\setminus J} u^{(4)}(s)\,ds \\
&\geq \int_{[t,\tau_0]\cap J} \psi(s)\,ds + u'''(\tau_0) - u'''(t) - \int_{[t,\tau_0]\setminus J} M(s)\,ds \\
&> u'''(\tau_0) - u'''(t) + \frac{1}{4}\int_t^{\tau_0} \psi(s)\,ds \quad \text{(by (3.11) and (3.12))},
\end{aligned}$$

hence $u'''(t) - y'''(t) \geq \tilde{\rho}$ provided that $u'''(\tau_0) \geq y'''(\tau_0)$. Therefore, by integration we obtain

$$u''(\tilde{t}_-) - y''(\tilde{t}_-) = u''(t_-) - y''(t_-) + \int_{t_-}^{\tilde{t}_-} (u'''(t) - y'''(t))\,dt \geq u''(t_-) - y''(t_-) + \tilde{\rho}(\tilde{t}_- - t_-).$$

If $u''(t_-) - y''(t_-) \leq -\rho$, then $\|y'' - u''\|_\infty \geq \rho$ and thus $\|y - u\| \geq \rho$ too. Otherwise, that is, if $u''(t_-) - y''(t_-) > -\rho$, then we have $u''(\tilde{t}_-) - y''(\tilde{t}_-) > \rho$ and hence $\|y - u\| \geq \rho$ too.

Similar computations in the interval $[\tilde{t}_+, t_+]$ instead of $[t_-, \tilde{t}_-]$ show that if $u'''(\tau_0) \leq y'''(\tau_0)$ then we have $y'''(t) - u'''(t) \geq \tilde{\rho}$ for all $t \in [\tilde{t}_+, t_+]$ and this also implies $\|y - u\| \geq \rho$. The claim is proven.

Case 3: $m(\{t \in [0,1] \,:\, u(t) = x_n\}) > 0$ *only for some of those* $n \in \mathbb{N}$ *such that* x_n *is viable.* Let us prove that in this case the relation $u \in \mathbb{T}u$ implies $u = Tu$.

Let us consider the subsequence of all viable admissible discontinuity points in the conditions of Case 3, which we denote again by $\{x_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in [0,1] \,:\, u(t) = x_n\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have

$$u^{(4)}(t) = 0 = g(t)f(x_n) = g(t)f(u(t)),$$

and therefore $u^{(4)}(t) = g(t)f(u(t))$ a.e. in $J = \cup_{n\in\mathbb{N}} J_n$.

Now we assume that $u \in \mathbb{T}u$ and we prove that it implies that $u^{(4)}(t) = g(t)f(u(t))$ a.e. in $[0,1] \setminus J$, thus showing that $u = Tu$.

Since $u \in \mathbb{T}u$ then for each $k \in \mathbb{N}$ we can guarantee that we can find functions $u_{k,i} \in \overline{B}_{1/k}(u) \cap K$ and coefficients $\lambda_{k,i} \in [0,1]$ $(i = 1, 2, \ldots, m(k))$ such that $\sum \lambda_{k,i} = 1$ and

$$\left\| u - \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i} \right\| < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i}$, and notice that $y_k \to u$ in the $\mathcal{C}^2$ norm and $\|u_{k,i} - u\| \leq 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, m(k)\}$.

For every $k \in \mathbb{N}$ we have $y_k \in Q$ as defined in (3.6), and therefore Lemma 3.11 guarantees that $u \in Q$ and, up to a subsequence, $y_k \to u$ in the $\mathcal{C}^3$ topology.

For a.a. $t \in [0,1] \setminus J$ we have that $f(\cdot)$ is continuous at $u(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$g(t)|f(u_{k,i}(t)) - f(u(t))| < \varepsilon \quad \text{for all } i \in \{1, 2, \ldots, m(k)\},$$

and therefore

$$|y_k^{(4)}(t) - g(t)f(u(t))| \leq \sum_{i=1}^{m(k)} \lambda_{k,i} g(t) |f(u_{k,i}(t)) - f(u(t))| < \varepsilon.$$

Hence $y_k^{(4)}(t) \to g(t)f(u(t))$ for a.a. $t \in [0,1] \setminus J$, and then Corollary 3.10 guarantees that $u^{(4)}(t) = g(t)f(u(t))$ for a.a. $t \in [0,1] \setminus J$.

Therefore the conditions of Theorem 2.1 are satisfied and we can ensure that BVP (3.3) has at least a positive solution. □

We emphasize that, even in the case of a continuous function f, Theorem 3.12 complements the existence results presented in [8]. As far as we know, the following corollary is new.

Corollary 3.13 *Assume that* f *is continuous and* $g \geq 0$ *a.e. and* $g \in L^1(0,1)$.

Moreover, assume that

(i) there exists a strict upper solution β *for problem* (3.3) *with* $\min_{t\in[0,1]}\beta(t) > 0$*;*

(ii) f *is nondecreasing on* $[0, \|\beta\|_\infty]$*;*

(iii) $f_0 = +\infty$ *or* $f_\infty = +\infty$.

Then BVP (3.3) has at least a positive solution.

To finish we illustrate our theory with an example inspired by [8, Example 2], but that it falls outside the scope of the fixed point theorems presented in [3–5, 8] because the corresponding fixed point operator is not continuous.

Example 3.14 *Consider the BVP*

$$\begin{cases} u^{(4)} = \lfloor h_1(u) \rfloor + h_2(u), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{3.16}$$

where $\lfloor x \rfloor$ *denotes the integer part of* x*.*

Assume that $h_i : [0,\infty) \to [0,\infty)$*,* $i = 1, 2$*, are continuous functions such that* $h_1(0) = 0$*,* $h_2(x) > 0$ *for all* $x \in (0,\infty)$*, both functions* h_1 *and* h_2 *are nondecreasing in* $[0, \beta]$ *for some* $\beta > 0$ *and*

$$\gamma^*(\lfloor h_1(\beta) \rfloor + h_2(\beta)) < \beta,$$

where $\gamma^* = 5/384$ *(see [8]). Moreover,*

$$\lim_{u\to\infty} \frac{h_1(u)}{u} = +\infty \quad \textit{or} \quad \lim_{u\to\infty} \frac{h_2(u)}{u} = +\infty. \tag{3.17}$$

Then Theorem 3.12 guarantees the existence of a positive solution for problem (3.16)*.*

The mapping $f(u) = \lfloor h_1(u) \rfloor + h_2(u)$ *is discontinuous at the points corresponding to the discontinuities of the integer part function. The positivity of the function* h_2 *implies that these points are admissible inviable discontinuity points (see Definition 3.7). In addition, the asymptotic condition* (3.17) *clearly guarantees that* $f_\infty = +\infty$*.*

For instance, we can choose $h_1(u) = 7u^3 - 18u^2 + 12u$ *and* $h_2(u) = \sqrt{u}$*. Then the previous conditions are satisfied by taking* $\beta = 0.69341$*. Notice that, in this case,* f *is not monotone in* $[0,\infty)$*, but it is nondecreasing in* $[0, 0.69341]$*.*

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