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Non-convolutional general fractional operators and some of their properties

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ABSTRACT

In this work, we propose a general framework for fractional integrals without following a convolution-kernel approach, and consider the corresponding notions for fractional derivatives under different perspectives (Riemann–Liouville and Caputo-type), analyzing their main mathematical properties such as semi-group condition, and the linearity of the integral, as well as the first theorem of calculus. We study some connections between the different notions, and provide a generalized Sonin condition.

1. Introduction

After his work on the generalization of an Abel formula [1], Sonin [2] in 1884 was the first to propose a contemporary notion of the so-called General Fractional Integrals (GFIs) and General Fractional Derivatives (GFDs). For the validity of many of the properties, it was needed to establish a restriction on the type of acceptable kernels, this restriction is known as Sonin condition. The work by Sonin included examples of kernels satisfying this restriction based on products of power functions and functions represented as series, in analytical way. The link between these notions and Fractional Calculus (FC) was not provided at that moment, as it is explained in [3], and some details were not specified in the presentation of results.

According to [3], it was the work by Kochubei [4] in 2011 the first comprehensive treatment of GFIs and GFDs including a proper connection with the theory of FC, including Sonin kernels in relation with the Laplace transforms, and proposed some regularized fractional derivatives based on the kernels proposed, which now take his name. It is also thanks to Kochubei the starting of the study of ODE and PDE including these generalized derivation operators, studying problems as Cauchy problems for diffusion equations. We can see the works [5,6] also for a study of the GFDs and related fractional differential equations. In particular, in [6], the estimates obtained allowed the authors to deduce a weak maximum principle for generalized time-fractional diffusion equations subject to initial–boundary-conditions by considering kernels independent of those proposed by Kochubei. New classes of kernels were later proposed in other research works by Luchko [7–9]. Some of these works were focused on the case of the domain as the positive real semi-axis with singular kernels at the origin with integrable singularities.

The application of this type of GFIs and GFDs has been also discussed by other authors in the literature. For instance, in [10], a nonlocal generalization of classical statistical mechanics is proposed in terms of generalized FC. Also, recently, in [11], some problems on classical mechanics are analyzed from the perspective of generalized FC, studying the action principle or Euler–Lagrange equations. The study of the mathematical properties of the operators is also being developed in detail in order to support the

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applicability to differential equations. As example, in [12], the author includes the first and the second fundamental theorems of FC for the GFD of convolution type, deducing generalized convolution Taylor formulas and considering some related Cauchy problems. We also mention the following generalizations: [13], where an extended formulation for the fractional calculus is achieved by using different types of pairs of operator kernels, or [14], which is based on a generalization of the Riesz fractional calculus, or even [15], based on a multi-kernel approach, in which the Laplace convolutions of different Sonin kernels are used. See also [16] for GFC and applications.

The general framework provided by Sonin [2] allows different theories concerning generalized FC depending on several classes of kernels and base spaces selected, so the specific formalisms have to be developed in detail. In this particular context of convolutional kernels, the left-sided and right-sided fractional operators have been studied in the recent paper [17], were the authors relate one notion to each other by a conjugation relation using negation operators.

The selection of the domain of work is also an important issue, as specified in [3], where it is explained how the mathematical properties and definitions appearing in the classical theory of FC of Riemann–Liouville type depend strongly on the type of domain considered, such as bounded intervals, semi-axes, or the real line. For instance, in the Ref. [18], they build a Mikusiński-type convolution algebra C_α, including functions with power-type singularities at the origin as well as all the functions that are continuous on $[0, \infty)$, and they apply it to fractional operators and problems. In this work, we follow the lines in [3], in what respects the consideration of bounded intervals, and we give new concepts of generalized fractional integrals and derivatives based on two-variable kernels, which are not necessarily of convolution type. Concerning the derivative, since several different notions are given, we study the connection between them. Moreover, we study some of the main mathematical properties such as the linearity and semi-group properties for the integral, and well as the first fundamental theorem of generalized FC. We provide a generalized concept for Sonin condition, establishing the relation between kernels in order to deduce some properties coherent to the development made for the convolution case.

We refer to [19–21] for the main monographs on FC theory. We also mention Ref. [22] for the analysis of fractional differential equations using operators of Caputo type.

The main section of the paper, Section 2, includes the establishment of the proper spaces of work, the proposal of the new concepts of generalized fractional integrals and derivatives on finite intervals, the deduction of their main mathematical properties, the establishment of a generalized Sonin condition, and the connection between the different concepts. We explain how the results obtained are a natural extension of the corresponding properties for convolution-type operators. Finally, some additional examples and conclusions are included in Sections 3 and 4, respectively.

2. The general non-convolution fractional integrals on a finite interval

In this section, we present the main concepts of generalized fractional integrals and derivatives on finite intervals based on two-variable kernels that are not necessarily of convolution type, we study some of their properties, we establish a generalized Sonin condition, and we analyze the connection between the different concepts introduced. We include, as an example, the case of convolution-type operators.

We start by recalling some spaces of functions that will be very helpful for the good-definition of the generalized fractional integrals (GFIs) and the generalized fractional derivatives (GFDs) we propose on a finite interval. The type of spaces considered are inspired by those used in [3], as introduced by Dimovski [23].

We consider, initially, the space of functions $\mathcal{F}_a^b$ of functions defined on the interval $[a, b]$, where $a < b$. We take this as a basis for the subsequent construction. However, for the validity of most of the properties of GFIs and GFDs, we will need to work in a more restrictive setting. In different subsections, we explore the different aspects (definitions and properties) in relation with the operators defined on proper spaces.

2.1. Definition of function spaces

We consider some spaces of interest in the rest of the paper.

Definition 1 (*[23]*). For $\alpha \geq -1$ and $n \in \mathbb{N}$, we define the spaces of functions

$$C_\alpha^n(a, b] = \left\{ f \in C_\alpha(a, b] \,:\, f^{(n)} \in C_\alpha(a, b] \right\}, \tag{1}$$

$$C_\alpha^n[a, b) = \left\{ f \in C_\alpha[a, b) \,:\, f^{(n)} \in C_\alpha[a, b) \right\}, \tag{2}$$

where

$$C_\alpha(a, b] = \left\{ f \,:\, (a, b] \to \mathbb{R} \,:\, \text{there exists } p > \alpha \text{ with } f(t) = (t - a)^p f_1(t),\ f_1 \in C[a, b] \right\}, \tag{3}$$

$$C_\alpha[a, b) = \left\{ f \,:\, [a, b) \to \mathbb{R} \,:\, \text{there exists } p > \alpha \text{ with } f(t) = (b - t)^p f_1(t),\ f_1 \in C[a, b] \right\}, \tag{4}$$

and the spaces $C_a(a, b]$ and $C_a(a, b]$ are interpreted as $C_\alpha^0(a, b]$ and $C_\alpha^0(a, b]$, respectively.

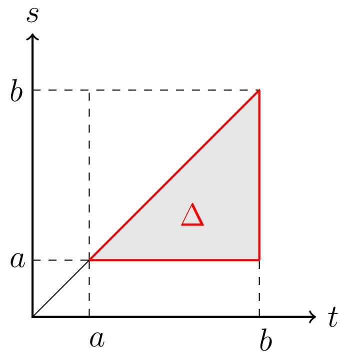

Fig. 1. Domain Δ of a kernel κ.

This type of spaces is very useful and suitable for research in FC, due to the singularity that the kernels usually present at one of the endpoints of the interval. Moreover, some interesting properties can be derived for the collection of spaces $C^n_\alpha(a,b]$ and $C^n_\alpha[a,b)$, for $n = 0, 1, 2, \ldots$, since they present a particular ordering with respect to the parameter α, in the sense that, for $\alpha_1 > \alpha_2 \geq -1$, we have

$$C^n_{\alpha_1}(a,b] \subseteq C^n_{\alpha_2}(a,b] \quad \text{and} \quad C^n_{\alpha_1}[a,b) \subseteq C^n_{\alpha_2}[a,b). \tag{5}$$

Hence, the spaces obtained taking $\alpha = -1$, that is, $C^n_{-1}(a,b]$ and $C^n_{-1}[a,b)$, are, respectively, the biggest ones, including all the other spaces $C^n_\alpha(a,b]$ or $C^n_\alpha[a,b)$, respectively. This is the main reason to consider in the literature, and also in what follows, the base spaces $C^n_{-1}(a,b]$ and $C^n_{-1}[a,b)$. However, the properties normally mentioned for these spaces are also satisfied by $C^n_\alpha(a,b]$ and $C^n_\alpha[a,b)$ for every $\alpha \geq -1$.

In this paper, we introduce and investigate the General Fractional Integrals (GFIs) and the General Fractional Derivatives (GFDs) on a finite interval (a,b) involving a two-variable kernel defined on a certain set Δ belonging to a proper space that will be defined later, with

$$\Delta := \{(t,s) \in [a,b] \times [a,b] \mid s \leq t\}. \tag{6}$$

(see Fig. 1).

When using a convolution-based generalized notions of integrals and derivatives (of Riemann–Liouville and Caputo type), see [7]:

$$\begin{aligned} (\mathbb{I}_{(\kappa)} f)(t) &:= (\kappa * f)(t) = \int_0^t \kappa(t-s) f(s)\, ds, \\ (\mathbb{D}_{(k)} f)(t) &:= \frac{d}{dt}(k * f)(t), \\ ({}_*\mathbb{D}_{(k)} f)(t) &:= (\mathbb{D}_{(k)} f)(t) - f(0)k(t), \end{aligned} \tag{7}$$

it is assumed that the kernels are in the space $C_{-1}(0, b-a]$, and also that the kernel $\kappa \in C_{-1}(0, b-a]$ of the GFIs and the kernel $k \in C_{-1}(0, b-a]$ of the GFDs are Sonin kernels in the sense that they satisfy the so-called Sonin condition [2], written as $\kappa * k \equiv 1$, where 1 stands for the function identically equal to one for $t \in (0, b-a]$, or, equivalently:

$$(\kappa * k)(t) = \int_0^t \kappa(t-s) k(s)\, ds = 1, \text{ for every } t \in (0, b-a]. \tag{8}$$

The set of such kernels will be denoted by $\mathcal{S}$.

As mentioned in [3], different pairs of kernels satisfying the Sonin condition, that is, pairs belonging to $\mathcal{S}$, have been obtained using elementary and special functions. In the Ref. [2], we find one example very relevant, the one given by power functions

$$\kappa(t) = h_\alpha(t), \qquad k(t) = h_{1-\alpha}(t), \qquad 0 < \alpha < 1, \tag{9}$$

$$h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad h_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \tag{10}$$

which was already known by Abel. From [24], we have another interesting example:

$$\kappa(t) = h_{1-\beta+\alpha}(t) + h_{1-\beta}(t), \quad k(t) = t^{\beta-1} E_{\alpha,\beta}(-t^\alpha), \ 0 < \alpha < \beta < 1, \tag{11}$$

where h_α comes from (10), and the definition of the two-parameter Mittag-Leffler function is as follows

$$E_{\alpha,\beta}(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha > 0,\ \beta,\ z \in \mathbb{C}. \tag{12}$$

In the rest of the section, generalized notions of fractional integrals and derivatives will be given for two-variable non-convolution kernels, and an appropriate extension of the Sonin condition will be provided in order to prove some basic properties for the notions presented. The properties proved in [3,7] for convolution-type operators will serve as inspiration in the process. Some examples will be given as illustration.

2.2. Definition of general fractional integrals

Consider $\alpha \geq -1$, and the space

$$C_\alpha(\Delta) = \left\{\kappa : \Delta \to \mathbb{R} : \kappa(t,s) = (t-s)^q \kappa_1(t,s),\ q > \alpha,\ \kappa_1 \in C(\Delta)\right\}. \tag{13}$$

Definition 2. Let a kernel κ belong to the space $C_{-1}(\Delta)$. The left-sided General Fractional Integral (LGFI) of a function $f \in C_{-1}(a,b]$ is defined by:

$$({}_l I^a_{(\kappa)} f)(t) = \int_a^t \kappa(t,s) f(s) ds, \qquad a < t \leq b. \tag{14}$$

Also, for a kernel κ belonging to the space $C_{-1}(\Delta)$, the right-sided General Fractional Integral (RGFI) of a function $f \in C_{-1}[a,b)$ is defined by the following formula:

$$({}_r I^b_{(\kappa)} f)(t) = \int_t^b \kappa(s,t) f(s) ds, \quad a \leq t < b. \tag{15}$$

Here, $\kappa(t,s)$ represents the kernel function belonging to the space $C_{-1}(\Delta)$. The LGFI and RGFI operators are defined on the intervals $(a,b]$ and $[a,b)$, respectively, for suitable functions $f(t)$.

In the rest of this section, we discuss some of the properties of the LGFIs (14) and the RGFIs (15) that are valid for any kernel κ from the space $C_{-1}(\Delta)$. However, in the next subsection, where the General Fractional Derivatives (GFDs) are introduced and investigated, we will assume that the kernel k satisfies a generalized condition that extends the property of being a Sonin kernel (in the set $\mathcal{S}$) associated to the kernel κ of the notion of the generalized integral.

Example 1. In particular, the power law kernel $\kappa(t,s) = h_\alpha(t-s)$, for $\alpha \in (0,1)$, belongs to $C_{-1}(\Delta)$, since

$$\kappa(t,s) = h_\alpha(t-s) = (t-s)^{\alpha-1}\frac{1}{\Gamma(\alpha)} \tag{16}$$

with $q := \alpha - 1 > -1$, and $\kappa_1(t,s) := \frac{1}{\Gamma(\alpha)} \in C(\Delta)$. This kernel generates the well-known left- and right-sided Riemann–Liouville fractional integrals that have been extensively studied in the Fractional Calculus (FC) literature (see, e.g., Ref. [20] for their properties).

Besides, for the previous reasoning, the kernel

$$\kappa(t,s) = h_{1-\beta+\alpha}(t-s) + h_{1-\beta}(t-s), \qquad 0 < \alpha < \beta < 1, \tag{17}$$

from (9) also belongs to $C_{-1}(\Delta)$, and leads to a sum of two left- and right-sided Riemann–Liouville fractional integrals of the orders $1-\beta+\alpha$ and $1-\beta$, respectively.

Moreover, the kernel $\kappa(t,s) = (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}\left(2\sqrt{t-s}\right)$ [2], where

$$J_\delta(z) = \sum_{j=0}^{\infty} \frac{(-1)^j (z/2)^{2j+\delta}}{j!\Gamma(j+\delta+1)} \tag{18}$$

is the Bessel function, belongs to $C_{-1}(\Delta)$, since

$$\begin{aligned}\kappa(t,s) &= (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}\left(2\sqrt{t-s}\right) = (t-s)^{\frac{\alpha-1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j (t-s)^{j+\frac{\alpha-1}{2}}}{j!\Gamma(j+\alpha)} \\ &= (t-s)^{\alpha-1} \sum_{j=0}^{\infty} \frac{(-1)^j (t-s)^j}{j!\Gamma(j+\alpha)} =: (t-s)^{\alpha-1}\kappa_1(t,s),\end{aligned} \tag{19}$$

with $q := \alpha - 1 > -1$, and $\kappa_1(t,s) \in C(\Delta)$. As we know, this kernel generates the following pair of convolution kernel left- and right-sided General Fractional Integrals:

- Left-sided General Fractional Integral (LGFIs) of a function f:

$$({}_l I^a_{(\kappa)} f)(t) = \int_a^t \left(\sqrt{t-s}\right)^{\alpha-1} J_{\alpha-1}\left(2\sqrt{t-s}\right) f(s) ds, \quad a < t \leq b. \tag{20}$$

- Right-sided General Fractional Integral (RGFIs) of a function f:

$$({}_r I^b_{(\kappa)} f)(t) = \int_t^b \left(\sqrt{s-t}\right)^{\alpha-1} J_{\alpha-1}\left(2\sqrt{s-t}\right) f(s) ds, \quad a \leq t < b. \tag{21}$$

We can also check that, if $\kappa_0 \in C_{-1}(0, b-a]$, then the two-variable kernel

$$\kappa(t,s) := \kappa_0(t-s) \tag{22}$$

belongs to $C_{-1}(\Delta)$. Indeed, since $\kappa_0 \in C_{-1}(0, b-a]$, then

$$\kappa_0(t) = t^p \kappa_{0,1}(t), \tag{23}$$

where $p > -1$, and $\kappa_{0,1} \in C[0, b-a]$, so that

$$\kappa(t,s) = \kappa_0(t-s) = (t-s)^p \kappa_{0,1}(t-s), \tag{24}$$

with $q := p > -1$, and $\kappa_1(t,s) := \kappa_{0,1}(t-s) \in C(\Delta)$.

Finally, an example of non-convolution two-variable kernel in $C_{-1}(\Delta)$ would be, for instance,

$$\kappa(t,s) = \frac{\sin(t)\cos(s)}{\sqrt{t-s}}, \tag{25}$$

where $q := -\frac{1}{2} > -1$, and $\kappa_1(t,s) = \sin(t)\cos(s) \in C([0,\pi]\times[0,\pi])$. This kernel cannot be written as $\kappa(t,s) = \kappa_0(t-s)$ for a certain $\kappa_0 \in C_{-1}(0,\pi]$.

Example 2. We can also include in our general framework the notions of fractional integrals of variable order $\alpha(t)$, since the function $\alpha(t)$ can be easily included in the expression of the kernel $\kappa(t,s)$. For instance, following the proposals in [25], we can consider the examples

$$({}_l I^0_{(\kappa)} f)(t) = \int_0^t \frac{1}{\Gamma(\alpha(t))}(t-s)^{\alpha(t)-1} f(s)ds, \quad 0 < t \le b,$$

$$({}_l I^0_{(\kappa)} f)(t) = \int_0^t \frac{1}{\Gamma(\alpha(t,s))}(t-s)^{\alpha(t,s)-1} f(s)ds, \quad 0 < t \le b, \tag{26}$$

or even

$$({}_l I^0_{(\kappa)} f)(t) = \int_0^t \frac{1}{\Gamma(\alpha^*(t,s))} s^{\alpha^*(t,s)-1} f(t-s)ds, \quad 0 < t \le b,$$

where $\alpha^*(t,s) = \alpha(t, t-s)$.

In particular, if we take the second expression (26), if there exists $q > -1$ such that

$$\frac{1}{\Gamma(\alpha(t,s))}(t-s)^{\alpha(t,s)-q-1} \text{ is continuous on } \Delta, \tag{27}$$

then the kernel defined by

$$\kappa(t,s) = \frac{1}{\Gamma(\alpha(t,s))}(t-s)^{\alpha(t,s)-1} \tag{28}$$

belongs to $C_{-1}(\Delta)$.

Remark 1. We note that, in Definition 2, we had two options for the concept of right-sided General Fractional Integral: to change the order of the variables in the kernel, as we did, keeping the same domain as the lower triangle Δ, or keeping $k(t,s)$ with the same order of variables, which would lead to a change in the domain of the kernel for the right-sided integral to the upper triangle

$$\widetilde{\Delta} := \{(t,s) \in [a,b]\times[a,b] \mid t \le s\}. \tag{29}$$

(see Fig. 2).

We chose a definition that has the domains for left- and right- cases as lower triangles in the plane (with the second argument less than or equal to the first argument), which forces to swap between $\kappa(t,s)$ in the operator (14) to $\kappa(s,t)$ in the operator (15). Both approaches are possible and analogous, with the corresponding adaptations of properties in the right-sided case. We have chosen to keep the same domain for simplicity, and change the order of variables. One of the reasons for choosing this option was the idea of the generalization of the Riemann–Liouville notions, where left- and right- cases are consistent with the use of the same convolution kernel function of one variable

$$\kappa(t,s) = h_\alpha(t-s) = (t-s)^{\alpha-1}\frac{1}{\Gamma(\alpha)}, \tag{30}$$

which easily gives

$$\kappa(s,t) = h_\alpha(s-t) = (s-t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}, \tag{31}$$

and the function h_α can be used directly in the definition of both operators with a common domain. We have also chosen this option because we are interested in the connections of these properties with Volterra equations. However, with the aim of completeness,

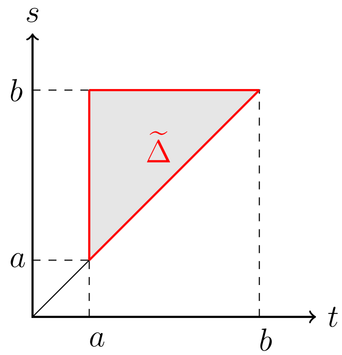

Fig. 2. Domain $\widetilde{\Delta}$ of a kernel κ for an alternative notion of right-sided General Fractional Integral.

we introduce some comments trying to illustrate how some of the conditions imposed would be transformed if we consider the alternative notion of right-sided integral with $k(t,s)$, as follows:

$$({}_rI^b_{(\kappa)}f)(t) = \int_t^b \kappa(t,s)f(s)ds, \quad a \le t < b. \tag{32}$$

We will especially focus on those restrictions different from the obvious change of the domain (see Remarks 2–5, and 7). We note that, in the development of the theory that includes this alternative notion, all the kernels need to be defined in the whole rectangle $[a,b]\times[a,b]$.

In the rest of the paper, we consider that the corresponding kernel κ satisfies that $\kappa \in C_{-1}(\Delta)$.

Proposition 1. *The left-sided general fractional integral and the right-sided general fractional integral are both self-mappings on the spaces* $C_{-1}(a,b]$ *and* $C_{-1}[a,b)$*, respectively.*

Proof. We prove it for the left-sided General Fractional Integral. Let $f \in C_{-1}(a,b]$. Then, we prove that the left-sided fractional integral ${}_lI^a_{(\kappa)}f$ of f, defined by:

$$({}_lI^a_{(\kappa)}f)(t) = \int_a^t \kappa(t,s)f(s)\,ds, \qquad a < t \le b, \tag{33}$$

is also in the space $C_{-1}(a,b]$, that is, we aim to show that ${}_lI^a_{(\kappa)}f \in C_{-1}(a,b]$. Indeed, since $f \in C_{-1}(a,b]$, then $f(t) = (t-a)^p f_1(t)$, with $p > -1$ and $f_1 \in C[a,b]$, so that, using also that $\kappa(t,s) = (t-s)^q\kappa_1(t,s)$, for some $q > -1$, and $\kappa_1 \in C([a,b]\times[a,b])$, we get

$$\begin{aligned}({}_lI^a_{(\kappa)}f)(t) &= \int_a^t \kappa(t,s)(s-a)^p f_1(s)\,ds,\\ &= \int_a^b (t-s)^q\kappa_1(t,s)(s-a)^p f_1(s)\,ds,\\ &= \int_a^t (t-s)^q(s-a)^p\kappa_1(t,s)f_1(s)\,ds.\end{aligned} \tag{34}$$

With the change of variable $z = \frac{t-s}{t-a}$, we have

$$({}_lI^a_{(\kappa)}f)(t) = (t-a)^{q+p+1}\int_0^1 z^q(1-z)^p\kappa_1(t,t-z(t-a))f_1(t-z(t-a))\,dz, \tag{35}$$

and the conclusion follows since $q+p+1 > -1$, κ_1 and f_1 are continuous on a compact set and $\int_0^1 z^q(1-z)^p\,dz = B(q+1,p+1)$, being B the Beta function. □

In the next proposition, we use the following notation, for $\kappa, \hat{\kappa} \in C_{-1}(\Delta)$:

$$\langle \kappa, \hat{\kappa}\rangle(x,u) := \int_u^x \kappa(x,t)\hat{\kappa}(t,u)\,dt, \qquad (x,u) \in \mathring{\Delta}. \tag{36}$$

Proposition 2 (*Semi-Group Property*)**.** *We consider two kernels* κ *and* $\hat{\kappa}$ *in the space* $C_{-1}(\Delta)$*. Then the left-sided general fractional integral satisfies a non-commutative couple of semi-group properties of the form:*

$$\left({}_lI^a_{(\kappa)}\left({}_lI^a_{(\hat{\kappa})}f\right)\right)(x) = {}_lI^a_{(\tilde{\kappa})}f(x), \qquad a < x \le b, \tag{37}$$

$$\left({}_lI^a_{(\hat{\kappa})}\left({}_lI^a_{(\kappa)}f\right)\right)(x) = {}_lI^a_{(\bar{\kappa})}f(x), \qquad a < x \le b, \tag{38}$$

where the kernels $\widetilde{\kappa}(x,u) = \langle \kappa, \widehat{\kappa}\rangle(x,u)$ *and* $\bar{\kappa}(x,u) = \langle \widehat{\kappa}, \kappa\rangle(x,u)$ *belong to* $C_{-1}(\Delta)$*. Analogously, the right-sided general fractional integral also satisfies a non-commutative couple of semi-group properties of the form:*

$$\left({}_rI^b_{(\kappa)}\left({}_rI^b_{(\widehat{\kappa})}f\right)\right)(x) = {}_rI^b_{(\bar{\kappa})}f(x), \qquad a \le x < b, \tag{39}$$

$$\left({}_rI^b_{(\widehat{\kappa})}\left({}_rI^b_{(\kappa)}f\right)\right)(x) = {}_rI^b_{(\widetilde{\kappa})}f(x), \qquad a \le x < b, \tag{40}$$

where the kernels $\widetilde{\kappa}$ *and* $\bar{\kappa}$ *belong to* $C_{-1}(\Delta)$*.*

Proof. Consider κ, $\widehat{\kappa} \in C_{-1}(\Delta)$, $f \in C_{-1}(a,b]$, and the left-sided general fractional integral of f, then, by interchanging the order of the variables, we get:

$$\begin{aligned}\left({}_lI^a_{(\kappa)}\left({}_lI^a_{(\widehat{\kappa})}f\right)\right)(x) &= \int_a^x \kappa(x,t)\left(\int_a^t \widehat{\kappa}(t,u)f(u)\,du\right)dt\\ &= \int_a^x\int_a^t \kappa(x,t)\widehat{\kappa}(t,u)f(u)\,du\,dt\\ &= \int_a^x\int_u^x \kappa(x,t)\widehat{\kappa}(t,u)f(u)\,dt\,du\\ &= \int_a^x f(u)\left(\int_u^x \kappa(x,t)\widehat{\kappa}(t,u)\,dt\right)du\end{aligned} \tag{41}$$

Now, the inner integral represents the expression of the kernel $\widetilde{\kappa}$, which belongs to $C_{-1}(\Delta)$. Indeed, since κ, $\widehat{\kappa} \in C_{-1}(\Delta)$, then

$$\kappa(x,t) = (x-t)^q\kappa_1(x,t), \qquad \widehat{\kappa}(t,u) = (t-u)^{\widehat{q}}\widehat{\kappa}_1(t,u), \tag{42}$$

with q, $\widehat{q} > -1$, and κ_1, $\widetilde{\kappa}_1 \in C(\Delta)$. Then

$$\begin{aligned}\widetilde{\kappa}(x,u) &= \int_u^x \kappa(x,t)\widehat{\kappa}(t,u)\,dt\\ &= \int_u^x (x-t)^q\kappa_1(x,t)(t-u)^{\widehat{q}}\widehat{\kappa}_1(t,u)\,dt.\end{aligned} \tag{43}$$

By making the change of variable $y = \frac{t-u}{x-u}$, we have

$$\begin{aligned}\widetilde{\kappa}(x,u) &= \int_u^x \kappa(x,t)\widehat{\kappa}(t,u)\,dt\\ &= \int_0^1 (x-u)^{q+\widehat{q}}(1-y)^q y^{\widehat{q}}\kappa_1\left(x, u+(x-u)y\right)\widehat{\kappa}_1\left(u+(x-u)y, u\right)(x-u)\,dy\\ &= (x-u)^{q+\widehat{q}+1}\int_0^1 (1-y)^q y^{\widehat{q}}\kappa_1(x,u+(x-u)y)\widehat{\kappa}_1(u+(x-u)y,u)\,dy.\end{aligned} \tag{44}$$

Therefore, $\widetilde{\kappa} \in C_{-1}(\Delta)$, since $\widetilde{q} := q+\widehat{q}+1 > -1$, the functions in the integral are continuous on a compact set and $\int_0^1 y^{\widehat{q}}(1-y)^q\,dz = B(\widehat{q}+1, q+1)$.

Moreover, the outer integral in the last line of (41) represents the left-sided general fractional integral of f with respect to the kernel $\tilde{\kappa}$, proving that

$$\left({}_lI^a_{(\kappa)}\left({}_lI^a_{(\widehat{\kappa})}f\right)\right)(x) = {}_lI^a_{(\widetilde{\kappa})}f(x), \tag{45}$$

and the fact that ${}_lI^a_{(\kappa)}\left({}_lI^a_{(\widehat{\kappa})}f\right) \in C_{-1}(a,b]$, by Proposition 1. On the other hand,

$$\left({}_lI^a_{(\widehat{\kappa})}\left({}_lI^a_{(\kappa)}f\right)\right)(x) = \int_a^x f(u)\left(\int_u^x \widehat{\kappa}(x,t)\kappa(t,u)\,dt\right)du = {}_lI^a_{(\bar{\kappa})}f(x). \tag{46}$$

If we consider κ, $\widehat{\kappa} \in C_{-1}(\Delta)$, $f \in C^{-1}[a,b)$, and the right-sided general fractional integral of f, we get:

$$\begin{aligned}\left({}_rI^b_{(\kappa)}\left({}_rI^b_{(\widehat{\kappa})}f\right)\right)(x) &= \int_x^b \kappa(t,x)\left(\int_t^b \widehat{\kappa}(u,t)f(u)\,du\right)dt\\ &= \int_x^b\int_t^b \kappa(t,x)\widehat{\kappa}(u,t)f(u)\,du\,dt\\ &= \int_x^b\int_x^u \kappa(t,x)\widehat{\kappa}(u,t)f(u)\,dt\,du\\ &= \int_x^b f(u)\left(\int_x^u \kappa(t,x)\widehat{\kappa}(u,t)\,dt\right)du\\ &= \int_x^b f(u)\bar{\kappa}(u,x)\,du.\end{aligned} \tag{47}$$

Similarly to the left-sided case, the outer integral represents the right-sided general fractional integral of f with respect to the kernel $\bar{\kappa}$, proving that

$$\left({}_rI^b_{(\kappa)}\left({}_rI^b_{(\hat{\kappa})}f\right)\right)(x) = {}_rI^b_{(\bar{\kappa})}f(x), \tag{48}$$

and the fact that ${}_rI^b_{(\kappa)}\left({}_rI^b_{(\hat{\kappa})}f\right) \in C_{-1}(a,b]$, by Proposition 1. The rest of the proof is completed accordingly. □

Corollary 1. *If the two kernels κ and $\hat{\kappa}$ in the space $C_{-1}(\Delta)$ satisfy that $\widetilde{\kappa}(x,u) = \bar{\kappa}(x,u)$, for every $(x,u) \in \mathring{\Delta}$, that is, if*

$$\langle \kappa, \hat{\kappa}\rangle(x,u) = \int_u^x \kappa(x,t)\hat{\kappa}(t,u)\,dt = \int_u^x \hat{\kappa}(x,t)\kappa(t,u)\,dt = \langle \hat{\kappa}, \kappa\rangle(x,u),\ (x,u) \in \mathring{\Delta}, \tag{49}$$

then the left-sided general fractional integral satisfies the semi-group property of the form:

$$\left({}_lI^a_{(\kappa)}\left({}_lI^a_{(\hat{\kappa})}f\right)\right)(x) = {}_lI^a_{(\widetilde{\kappa})}f(x) = \left({}_lI^a_{(\hat{\kappa})}\left({}_lI^a_{(\kappa)}f\right)\right)(x), \qquad a < x \le b, \tag{50}$$

where the kernel $\widetilde{\kappa}(x,u) = \bar{\kappa}(x,u)$ belongs to $C_{-1}(\Delta)$. Analogously, under the same restrictions, the right-sided general fractional integral also satisfies the semi-group property of the form:

$$\left({}_rI^b_{(\kappa)}\left({}_rI^b_{(\hat{\kappa})}f\right)\right)(x) = {}_rI^b_{(\bar{\kappa})}f(x) = \left({}_rI^b_{(\hat{\kappa})}\left({}_rI^b_{(\kappa)}f\right)\right)(x), \qquad a \le x < b, \tag{51}$$

where the kernel $\widetilde{\kappa} = \bar{\kappa}$ belongs to $C_{-1}(\Delta)$.

Example 3. In the particular case where $\kappa(t,s) := \kappa_0(t-s)$, and $\hat{\kappa}(t,s) := \hat{\kappa}_0(t-s)$ with κ_0, $\hat{\kappa}_0 \in C_{-1}(0, b-a]$, then the kernel

$$\begin{aligned}\widetilde{\kappa}(x,u) &= \int_u^x \kappa(x,t)\hat{\kappa}(t,u)\,dt \\ &= \int_u^x \kappa_0(x-t)\hat{\kappa}_0(t-u)\,dt.\end{aligned} \tag{52}$$

Using the change of variable $z = t - u$, then we have

$$\begin{aligned}\widetilde{\kappa}(x,u) &= \int_0^{x-u} \kappa_0(x-u-z)\hat{\kappa}_0(z)\,dz \\ &= \left(\kappa_0 * \hat{\kappa}_0\right)(x-u),\end{aligned} \tag{53}$$

where $*$ denotes the convolution operator. Besides,

$$\begin{aligned}\bar{\kappa}(x,u) &= \int_u^x \hat{\kappa}(x,t)\kappa(t,u)\,dt \\ &= \int_u^x \hat{\kappa}_0(x-t)\kappa_0(t-u)\,dt.\end{aligned} \tag{54}$$

Using the change of variable $s = x + u - t$, then we have

$$\begin{aligned}\bar{\kappa}(x,u) &= \int_x^u \hat{\kappa}_0(s-u)\kappa_0(x-s)(-1)\,ds \\ &= \int_u^x \hat{\kappa}_0(s-u)\kappa_0(x-s)\,ds = \widetilde{\kappa}(x,u).\end{aligned} \tag{55}$$

This shows that Proposition 2 (and also Corollary 1) extends Proposition 3 [3].

Example 4. Concerning the expression (26), if we take

$$\kappa(t,s) = \hat{\kappa}(t,s) = \frac{1}{\Gamma(\alpha(t,s))}\,(t-s)^{\alpha(t,s)-1}\,, \tag{56}$$

with α satisfying condition (27), then, it is obvious that $\widetilde{\kappa}(x,u) = \bar{\kappa}(x,u)$, for every $(x,u) \in \mathring{\Delta}$, so that the semi-group property is derived for variable variable-order integral operators.

Example 5. If the kernels κ and $\hat{\kappa}$ in the space $C_{-1}(\Delta)$ are equal, then it is obvious that $\widetilde{\kappa}(x,u) = \bar{\kappa}(x,u)$ for every $(x,u) \in \mathring{\Delta}$, so that the semi-group property follows. Analogously, if $\kappa \equiv \mu\hat{\kappa}$, for some $\mu \in \mathbb{R}$.

Remark 2. In Proposition 2, with the alternative notion of right-sided General Fractional Integral given by (32) (See Remark 1), if we consider $\kappa, \widehat{\kappa} \in C_{-1}(\widetilde{\Delta})$, $f \in C^{-1}[a,b)$, we get, for this alternative right-sided general fractional integral of f:

$$\begin{aligned}\left({}_rI^b_{(\kappa)}\left({}_rI^b_{(\widehat{\kappa})}f\right)\right)(x) &= \int_x^b \kappa(x,t)\left(\int_t^b \widehat{\kappa}(t,u)f(u)\,du\right)dt\\ &= \int_x^b\int_t^b \kappa(x,t)\widehat{\kappa}(t,u)f(u)\,du\,dt\\ &= \int_x^b\int_x^u \kappa(x,t)\widehat{\kappa}(t,u)f(u)\,dt\,du\\ &= \int_x^b f(u)\left(\int_x^u \kappa(x,t)\widehat{\kappa}(t,u)\,dt\right)du\\ &= \int_x^b f(u)\bar{\kappa}^*(x,u)\,du,\end{aligned} \tag{57}$$

where

$$\bar{\kappa}^*(x,u) := \int_x^u \kappa(x,t)\widehat{\kappa}(t,u)\,dt = -\int_u^x \kappa(x,t)\widehat{\kappa}(t,u)\,dt, \text{ for } (x,u)\in\widetilde{\Delta},$$

that is,

$$\bar{\kappa}^*(x,u) = -\langle\kappa,\widehat{\kappa}\rangle(x,u), \text{ for } (x,u)\in\widetilde{\Delta},$$

where we have considered the extension of the definition of $\langle\cdot,\cdot\rangle$ to the rectangle $[a,b]\times[a,b]$, in order to apply it to pairs in $\widetilde{\Delta}$.

This represents the alternative notion of right-sided general fractional integral of f with respect to the kernel $\bar{\kappa}^*$, proving also the property

$$\left({}_rI^b_{(\kappa)}\left({}_rI^b_{(\widehat{\kappa})}f\right)\right)(x) = {}_rI^b_{(\bar{\kappa}^*)}f(x), \tag{58}$$

and the fact that ${}_rI^b_{(\kappa)}\left({}_rI^b_{(\widehat{\kappa})}f\right) \in C_{-1}(a,b]$. Similarly, we get

$$\left({}_rI^b_{(\widehat{\kappa})}\left({}_rI^b_{(\kappa)}f\right)\right)(x) = {}_rI^b_{(\widetilde{\kappa}^*)}f(x), \qquad a\le x<b, \tag{59}$$

where

$$\widetilde{\kappa}^*(x,u) := -\langle\widehat{\kappa},\kappa\rangle(x,u), \text{ for } (x,u)\in\widetilde{\Delta},$$

Concerning Corollary 1 with the alternative notion of right-sided integral, if we impose that $\widetilde{\kappa}(x,u)=\bar{\kappa}(x,u)$, for every $(x,u)\in\mathring{\Delta}$, and the analogous identity with the extended definition

$$\bar{\kappa}^*(x,u) = \widetilde{\kappa}^*(x,u), \text{ for every } (x,u)\in\left(\mathring{\widetilde{\Delta}}\right),$$

then the semigroup property follows for left- and right-sided integral operators (according to (32)).

Proposition 3. *If κ is a two variable kernel in the space $C_{-1}(\Delta)$, then the left-sided generalized fractional integral with kernel κ is a linear operator on the space $C_{-1}(a,b]$. Similarly, if κ is a two variable kernel in the space $C_{-1}(\Delta)$, then the right-sided generalized fractional integral with kernel κ is a linear operator on the space $C_{-1}[a,b)$.*

Proof. Let us assume that f and g are functions in the space $C_{-1}(a,b]$. Then, by Proposition 1, ${}_lI^a_{(\kappa)}f$ and ${}_lI^a_{(\kappa)}g$ are also functions in this space. Let us also assume that μ and ν are constants. Then:

$$\begin{aligned}\left({}_lI^a_{(\kappa)}[\mu f+\nu g]\right)(t) &= \int_a^t k(t,s)(\mu f(s)+\nu g(s))\,ds\\ &= \mu\int_a^t k(t,s)f(s)\,ds + \nu\int_a^t k(t,s)g(s)\,ds\\ &= \mu\left({}_lI^a_{(\kappa)}f\right)(t) + \nu\left({}_lI^a_{(\kappa)}g\right)(t).\end{aligned} \tag{60}$$

Thus, ${}_lI^a_{(\kappa)}$ is a linear operator. Analogous proof can be written for the right-sided integral. □

2.3. The general fractional derivatives on a finite interval

In this section, we introduce several different kinds of the GFDs on a finite interval and study their basic properties including the 1st and the 2nd fundamental theorems of FC. As in the case of the Riemann–Liouville and the Caputo fractional derivatives with the power law kernels, we define the GFD (of the Riemann–Liouville type) and the regularized GFD (of the Caputo type). Moreover, both the left- and the right-sided GFDs will be introduced and studied.

In what follows, we suppose that the kernels of the GFIs and the GFDs are in the space $C_{-1}(\Delta)$ for the left- and right-sided case. Besides, we will need to provide a connection between the kernels of the integral and the derivative in each of the cases that extends the well-known family of Sonin kernels in the set $\mathcal{S}$. We will represent the kernels for the integrals with the letter κ and its variants, and those for the derivatives with the letter k (and its variants). We also represent by $\mathring{\Delta}$ the interior of Δ.

Definition 3 (*Generalized Non-Convolution Sonin Condition*)**.** We say that a pair of kernels satisfy the generalized non-convolution Sonin condition if the following two conditions hold:

$$\langle k,\kappa\rangle(t,z) := \int_z^t k(t,s)\kappa(s,z)\,ds = 1, \qquad (t,z)\in\mathring{\Delta},$$

and

$$\langle \kappa,k\rangle(t,z) := \int_z^t \kappa(t,s)k(s,z)\,ds = 1, \qquad (t,z)\in\mathring{\Delta}. \tag{61}$$

We denote the pair of kernels satisfying this property by $\mathcal{S}_G$.

Example 6. In the particular case where $\kappa(t,s) := \kappa_0(t-s)$, and $k(t,s) := k_0(t-s)$ with $\kappa_0,\ k_0 \in C_{-1}(0,b-a]$, then

$$\begin{aligned}\langle k,\kappa\rangle(t,z) &= \int_z^t k(t,s)\kappa(s,z)\,ds\\ &= \int_z^t k_0(t-s)\kappa_0(s-z)\,ds\\ &= \int_0^{t-z} k_0(t-z-u)\kappa_0(u)\,ds = \left(k_0 * \kappa_0\right)(t-z).\end{aligned} \tag{62}$$

Hence, since convolution is commutative, the generalized Sonin condition (61) is reduced to (8).

Definition 4. Let (κ,k) be a pair of kernels belonging to the set $\mathcal{S}_G$, and such that $k(t,s)$ is a two-variable function with first-order partial derivative with respect to the variable t belonging to $C_{-1}(\Delta)$ for the left-sided case, and with first-order partial derivative with respect to the variable s belonging to $C_{-1}(\Delta)$, for the right-sided case.

The left-sided general fractional derivative (LGFD) of a function $f \in C_{-1}(a,b]$ is defined by:

$$({}_l D^a_{(k)} f)(t) = \frac{d}{dt}\int_a^t k(t,s)f(s)\,ds, \quad a<t\le b. \tag{63}$$

Similarly, the right-sided general fractional derivative (RGFD) of a function $f \in C_{-1}[a,b)$ is defined by the following formula:

$$({}_r D^b_{(k)} f)(t) = -\frac{d}{dt}\int_t^b k(s,t)f(s)\,ds, \quad a\le t< b. \tag{64}$$

In the notion of derivative of Riemann–Liouville type, the parameter α represents the order of the fractional derivative. In this case, the ‘order’ of derivative is given by two elements, the particular expression of the kernel k and the fact that the outer classical derivative is of order 1, so that Definition 4 extends the notion of Riemann–Liouville fractional derivative for $\alpha \in (0,1)$.

Remark 3. In Definition 4, with the alternative notion of right-sided General Fractional Integral given by (32) (See Remark 1), we can define the right-sided general fractional derivative of f as

$$({}_r D^b_{(k)} f)(t) = -\frac{d}{dt}\int_t^b k(t,s)f(s)\,ds, \quad a\le t< b. \tag{65}$$

Proposition 4 (*First Fundamental Theorem*)**.** *Let (κ,k) be a pair of kernels belonging to the set $\mathcal{S}_G$, with k in the conditions in Definition 4. Then, the LGFD (63) is a left inverse operator to the LGFI (14) on the space $C_{-1}(a,b]$, that is,*

$$({}_l D^a_{(k)}\, {}_l I^a_{(\kappa)} f)(t) = f(t), \quad f\in C_{-1}(a,b]. \tag{66}$$

Similarly, the RGFD (64) is a left inverse operator to the RGFI (15) on the space $C_{-1}[a,b)$, that is,

$$({}_r D^b_{(k)}\, {}_r I^b_{(\kappa)} f)(t) = f(t), \quad f\in C_{-1}[a,b). \tag{67}$$

Proof. Indeed, by the properties of $\mathcal{S}_G$,

$$\begin{aligned}({}_l D^a_{(k)}\, {}_l I^a_{(\kappa)} f)(t) &= \frac{d}{dt}\int_a^t k(t,s)\left(\int_a^s \kappa(s,z)f(z)dz\right)ds\\ &= \frac{d}{dt}\int_a^t\int_a^s k(t,s)\kappa(s,z)f(z)dz\,ds\\ &= \frac{d}{dt}\int_a^t\left(\int_z^t k(t,s)\kappa(s,z)\,ds\right)f(z)dz\\ &= \frac{d}{dt}\int_a^t f(z)dz = f(t),\end{aligned} \tag{68}$$

where we have used that $\left(\int_z^t k(t,s)\kappa(s,z)\,ds\right) = \langle k,\kappa\rangle(t,z)$ is identically one. Similarly,

$$\begin{aligned}({}_rD^b_{(k)}\,{}_rI^b_{(\kappa)}f)(t) &= -\frac{d}{dt}\int_t^b k(s,t)\left(\int_s^b \kappa(u,s)f(u)du\right)ds\\ &= -\frac{d}{dt}\int_t^b\left(\int_t^u \kappa(u,s)k(s,t)\,ds\right)f(u)du\\ &= f(t),\end{aligned} \tag{69}$$

where, again, $\left(\int_t^u \kappa(u,s)k(s,t)\,ds\right) = \langle\kappa,k\rangle(u,t)$ is identically one. □

Remark 4. In Proposition 4, with the alternative notion of right-sided General Fractional Integral given by (32) (See Remark 1), we also have the left inverse property of the derivative with respect to the integral, both for left- and right-sided operators, just by considering the alternative Generalized non-convolution Sonin condition

$$\langle k,\kappa\rangle(t,z) := \int_z^t k(t,s)\kappa(s,z)\,ds = 1, \qquad (t,z)\in\mathring{\Delta},$$

and

$$\langle k,\kappa\rangle(t,z) := \int_z^t k(t,s)\kappa(s,z)\,ds = -1, \qquad (t,z)\in\left(\mathring{\tilde{\Delta}}\right). \tag{70}$$

Indeed, for $f\in C_{-1}[a,b)$, we deduce that

$$\begin{aligned}({}_rD^b_{(k)}\,{}_rI^b_{(\kappa)}f)(t) &= -\frac{d}{dt}\int_t^b k(t,s)\left(\int_s^b \kappa(s,u)f(u)du\right)ds\\ &= -\frac{d}{dt}\int_t^b\left(\int_t^u k(t,s)\kappa(s,u)\,ds\right)f(u)du\\ &= f(t),\end{aligned} \tag{71}$$

where, again, $\int_t^u k(t,s)\kappa(s,u)\,ds = -\int_u^t k(t,s)\kappa(s,u)\,ds = -\langle k,\kappa\rangle(t,u)$ is identically one for $t\le u$.

For a general regularized notion of derivative, in the sense of Caputo derivative, several notions arise. This is mainly due to the existence of a two-variable kernel, so that some properties are not exactly corresponding to the properties derived for a convolution definition. Hence, we propose different notions for regularized derivatives. We note that, in what follows, we will consider the derivative operators for functions defined on the spaces $C^1_{-1}(a,b]$ and $C^1_{-1}[a,b)$, respectively:

$$C^1_{-1}(a,b] := \left\{f\in C_{-1}(a,b] \mid f'\in C_{-1}(a,b]\right\}, \tag{72}$$

$$C^1_{-1}[a,b) := \left\{f\in C_{-1}[a,b) \mid f'\in C_{-1}[a,b)\right\}. \tag{73}$$

The above-mentioned spaces have already been introduced in the literature (see Definition 1 in [26] based on Definition 2.2 in the classical paper [27]). Note that Property (3) in [27, page 211] shows that any f in the space given by (72) or (73) is actually in $C[a,b]$, that is, it is continuous on the closed interval. This fact will be needed in the sequel (see Definition 5 and Proposition 6, where $f(a)$ is used for $f\in C^1_{-1}(a,b]$ and $f(b)$ for $f\in C^1_{-1}[a,b)$). In these cases, Property (3) in [27, page 211] provides that $f\in C[a,b]$, so that $f(a)$, $f(b)$, and, thus, the regularized derivatives are well defined.

Definition 5. Let (κ,k) be a pair of kernels belonging to the set $\mathcal{S}_G$, with k in the conditions in Definition 4.

The regularized left-sided general fractional derivative RLGFD of a differentiable function $f\in C^1_{-1}(a,b]$ is defined by:

$$({}^*_lD^a_{(k)}f)(t) = ({}_lD^a_{(k)}f)(t) - f(a)\frac{d}{dt}\int_a^t k(t,s)\,ds, \quad a<t\le b. \tag{74}$$

Analogously, the regularized right-sided general fractional derivative RRGFD of a differentiable function $f\in C^1_{-1}[a,b)$ is defined as follows:

$$({}^*_rD^b_{(k)}f)(t) = ({}_rD^b_{(k)}f)(t) + f(b)\frac{d}{dt}\int_t^b k(s,t)\,ds, \quad a\le t<b. \tag{75}$$

According to Definition 5, the RLGFD and RRGFD of Caputo-type can be written as regularized LGFD and RGFD, respectively, in the Riemann–Liouville sense, as follows:

$$({}^*_lD^a_{(k)}f)(t) = ({}_lD^a_{(k)}[f(\cdot)-f(a)])(t), \quad a<t\le b, \tag{76}$$

$$({}^*_rD^b_{(k)}f)(t) = ({}_rD^b_{(k)}[f(\cdot)-f(b)])(t), \quad a\le t<b, \tag{77}$$

for the functions in the specified respective spaces. Indeed,

$$\begin{aligned}({}_l^*D^a_{(k)}f)(t) &= ({}_lD^a_{(k)}[f(\cdot)-f(a)])(t) = \frac{d}{dt}\int_a^t k(t,s)[f(s)-f(a)]\,ds\\ &= ({}_lD^a_{(k)}f)(t) - f(a)\frac{d}{dt}\int_a^t k(t,s)\,ds,\end{aligned} \tag{78}$$

and, analogously for the right case:

$$\begin{aligned}({}_r^*D^b_{(k)}f)(t) &= ({}_rD^b_{(k)}[f(\cdot)-f(b)])(t) = -\frac{d}{dt}\int_t^b k(s,t)[f(s)-f(b)]\,ds\\ &= ({}_rD^b_{(k)}f)(t) + f(b)\frac{d}{dt}\int_t^b k(s,t)\,ds.\end{aligned} \tag{79}$$

Example 7. For the case where $k(t,s) := k_0(t-s)$ with $k_0 \in C_{-1}(0, b-a]$, then

$$\begin{aligned}\frac{d}{dt}\int_a^t k(t,s)\,ds &= \frac{d}{dt}\int_a^t k_0(t-s)\,ds\\ &= \frac{d}{dt}\int_0^{t-a} k_0(u)\,du = k_0(t-a),\end{aligned} \tag{80}$$

and

$$\frac{d}{dt}\int_t^b k(s,t)\,ds = \frac{d}{dt}\int_t^b k_0(s-t)\,ds = \frac{d}{dt}\int_0^{b-t} k_0(u)\,du = -k_0(b-t), \tag{81}$$

so that

$$({}_l^*D^a_{(k)}f)(t) = ({}_lD^a_{(k)}f)(t) - f(a)k_0(t-a) = ({}_lD^a_{(k)}f)(t) - f(a)k(t,a), \tag{82}$$

and

$$({}_r^*D^b_{(k)}f)(t) = ({}_rD^b_{(k)}f)(t) - f(b)k_0(b-t) = ({}_rD^b_{(k)}f)(t) - f(b)k(b,t), \quad a \le t < b. \tag{83}$$

This proves that the left case in Definition 5 is coherent with the notion given in [3] (see Proposition 4 [3]).

Remark 5. In Definition 5, with the alternative notion of right-sided General Fractional Integral given by (32) (See Remark 1), we could define

$$({}_r^*D^b_{(k)}f)(t) = ({}_rD^b_{(k)}f)(t) + f(b)\frac{d}{dt}\int_t^b k(t,s)\,ds, \quad a \le t < b. \tag{84}$$

in order to respect the property

$$({}_r^*D^b_{(k)}f)(t) = ({}_rD^b_{(k)}[f(\cdot)-f(b)])(t), \quad a \le t < b. \tag{85}$$

Next, we prove the first fundamental theorem of fractional calculus for general regularized fractional derivatives of Caputo-type given in Definition 5.

Proposition 5. *Let (κ, k) be a pair of kernels belonging to the set S_G, with k in the conditions in Definition 4.*
Then, the LRGFD (74) is a left inverse operator to the LGFI (14) on the space $C^1_{-1,k}(a,b]$, that is,

$$({}_l^*D^a_{(k)}\,{}_lI^a_{(\kappa)}f)(t) = f(t), \quad f \in C^1_{-1,k}(a,b], \tag{86}$$

where

$$C^1_{-1,k}(a,b] := \left\{ f : (a,b] \to \mathbb{R} \mid f(t) = ({}_lI^a_{(k)}\phi)(t),\ \phi \in C_{-1}(a,b]. \right\} \tag{87}$$

Similarly, the RRGFD (75) is a left inverse operator to the RGFI (15) on the space $C^{1,*}_{-1,k}[a,b)$, that is,

$$({}_r^*D^a_{(k)}\,{}_rI^a_{(\kappa)}f)(t) = f(t), \quad f \in C^{1,*}_{-1,k}[a,b), \tag{88}$$

where

$$C^{1,*}_{-1,k}[a,b) := \left\{ f : [a,b) \to \mathbb{R} \mid f(t) = ({}_rI^b_{(k)}\phi)(t),\ \phi \in C_{-1}[a,b). \right\} \tag{89}$$

Proof. We prove it just for the left case. For every $f \in C^1_{-1,k}(a,b]$, we have $f(t) = ({}_lI^a_{(k)}\phi)(t)$, and the following chain of identities:

$$({}_lI^a_{(\kappa)}f)(t) = ({}_lI^a_{(\kappa)}\,{}_lI^a_{(k)}\phi)(t) = ({}_lI^a_{(\widetilde{\kappa})}\phi)(t), \tag{90}$$

where $\widetilde{\kappa} = \langle \kappa, k\rangle \equiv 1$.

Then,

$$({}_lI^a_{(\kappa)}f)(t) = \int_a^t \phi(s)\,ds \in C^1_{-1}(a,b], \tag{91}$$

and, also,

$$\frac{d}{dt}({}_lI^a_{(\kappa)}f)(t) = \phi(t) \in C_{-1}(a,b]. \tag{92}$$

Moreover, since $\phi \in C_{-1}(a,b]$, there exist $q > -1$ and $\phi_1 \in C[a,b]$ such that

$\phi(t) = (t-a)^q\phi_1(t)$, so that

$$\left|({}_lI^a_{(\kappa)}f)(t)\right| = \left|\int_a^t \phi(s)\,ds\right| \tag{93}$$

$$\leq \int_a^t |\phi(s)|\;ds = \int_a^t (s-a)^q\left|\phi_1(s)\right|\;ds \tag{94}$$

$$\leq C\frac{(t-a)^{q+1}}{q+1},\; t\in(a,b]. \tag{95}$$

This proves that $\lim_{t\to a}({}_lI^a_{(\kappa)}f)(t) = 0$, so that$({}_lI^a_{(\kappa)}f)(a) = 0$. Therefore,

$$\begin{aligned}({}^*_lD^a_{(k)}\,{}_lI^a_{(\kappa)}f)(t) &= ({}_lD^a_{(k)}\,{}_lI^a_{(\kappa)}f)(t) - \left({}_lI^a_{(\kappa)}f\right)(a)\frac{d}{dt}\int_a^t k(t,s)\,ds\\ &= ({}_lD^a_{(k)}\,{}_lI^a_{(\kappa)}f)(t) = f(t). \quad\square\end{aligned} \tag{96}$$

According to the convolution-type notions, we find in the literature another equivalent definition for generalized Caputo fractional derivatives. They are based on the integral operator of fractional type applied to the derivative of the function f. However, in this context of two-variable kernels, we find that this concept is not always equivalent to the notion given in Definition 5.

Definition 6. Let (κ, k) be a pair of kernels belonging to the set $\mathcal{S}_G$. The second-type regularized left-sided general fractional derivative (RLGFD) of a differentiable function $f \in C^1_{-1}(a,b]$ is defined by:

$$({}^\star_lD^a_{(k)}f)(t) = \int_a^t k(t,s)f'(s)\,ds, \quad a < t \leq b. \tag{97}$$

Analogously, the second-type regularized right-sided general fractional derivative (RRGFD) of a differentiable function $f \in C^1_{-1}[a,b)$ is defined as follows:

$$({}^\star_rD^b_{(k)}f)(t) = -\int_t^b k(s,t)f'(s)\,ds, \quad a \leq t < b. \tag{98}$$

Example 8. Next, we give some examples of RLGFD and RRGFD with kernels satisfying the generalized Sonin condition. They are based on the fact that, for one-variable convolution kernels satisfying the original Sonin condition (8), the generalized Sonin condition (61) is also satisfied.

One well-known example of the left- and right-sided GFDs introduced above are the Riemann–Liouville and the Caputo left- and right-sided fractional derivatives with the power law kernel $k(t,s) = \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}$, for $0 < \alpha < 1$.

Another important example [3] is generated by the kernel

$$k(t,s) = (t-s)^{\beta-1}E_{\alpha,\beta}(-(t-s)^\alpha), \qquad 0 < \alpha \leq \beta < 1. \tag{99}$$

For this kernel, the left- and right-sided regularized GFD on the positive real semi-axes have already been defined and investigated, and the convolution-generalized versions are given on a finite interval in [3], as follows:

$$({}_rD^b_{(k)}f)(t) = -\frac{d}{dt}\int_t^b (s-t)^{\beta-1}E_{\alpha,\beta}(-(s-t)^\alpha)f(s)\,ds, \quad a \leq t < b, \tag{100}$$

$$({}^\star_rD^b_{(k)}f)(t) = -\int_t^b (s-t)^{\beta-1}E_{\alpha,\beta}(-(s-t)^\alpha)f'(s)\,ds, \quad a \leq t < b. \tag{101}$$

Finally, we mention the right-sided GFDs on a finite interval with the Sonin kernel $k(t,s) = (\sqrt{t-s})^{-\alpha}I_{1-\alpha}(2\sqrt{t-s})$, for $0 < \alpha < 1$:

$$({}_rD^b_{(k)}f)(t) = -\frac{d}{dt}\int_t^b (\sqrt{s-t})^{1-\alpha}I_{1-\alpha}(2\sqrt{s-t})f(s)\,ds, \quad a \leq t < b, \tag{102}$$

$$({}^{\star}_{r}D^{b}_{(k)}f)(t) = -\int_t^b (\sqrt{s-t})^{1-\alpha} I_{1-\alpha}(2\sqrt{s-t}) f'(s)\,ds, \quad a \le t < b. \tag{103}$$

Note that Definition 6 is equivalent to Definition 5 in the convolution case for f in the corresponding suitable space, since:

$$\begin{aligned}({}_{l}D^{a}_{(k)}f)(t) &= \frac{d}{dt}\int_a^t k(t,s)f(s)\,ds \\ &= \frac{d}{dt}\int_a^t k_0(t-s)f(s)\,ds = \frac{d}{dt}\int_0^{t-a} k_0(u)f(t-u)\,du \\ &= \int_0^{t-a} k_0(u)f'(t-u)\,du + k_0(t-a)f(a) \\ &= \int_a^t k_0(t-s)f'(s)\,ds + k_0(t-a)f(a) = k_0 * f' + k_0(t-a)f(a),\end{aligned} \tag{104}$$

for $f \in C^1_{-1}(a,b]$, and

$$\begin{aligned}({}_{r}D^{b}_{(k)}f)(t) &= -\frac{d}{dt}\int_t^b k(s,t)f(s)\,ds \\ &= -\frac{d}{dt}\int_t^b k_0(s-t)f(s)\,ds = -\frac{d}{dt}\int_0^{b-t} k_0(u)f(u+t)\,du \\ &= -\int_0^{b-t} k_0(u)f'(u+t)\,du + k_0(b-t)f(b) \\ &= -\int_t^b k_0(s-t)f'(s)\,ds + k_0(b-t)f(b),\end{aligned} \tag{105}$$

for $f \in C^1_{-1}[a,b)$.

2.4. Relationship between different notions of GFDs

In the following proposition, we establish the relationship between the two different notions of regularized GFDs for functions in the particular space $C^1_{-1}(a,b]$ (respectively, $C^1_{-1}[a,b)$) given by Definitions 5 and 6. In Example 8, dealing with the convolution case, we observe that the two concepts of regularized GFD $({}^{*}_{r}D^{b}_{(k)}f)(t)$ and $({}^{\star}_{r}D^{b}_{(k)}f)(t)$ are coincident, and, thus, that the regularized GFD $({}^{\star}_{r}D^{b}_{(k)}f)(t)$ can be obtained from the GFD with a translation by a function that is related to the kernel and the value of the function at the corresponding endpoint. In this context, the regularized GFD show a better behavior in comparison with the GFD, and this fact has been thus recognized as the possibility of adapting better to the solvability of different problems in the field of applications. However, as we have already mentioned, in the context of two-variable kernels, the connection between the two concepts of regularized GFDs is not always straightforward. In this section, we try to motivate the essence of their different expressions, by putting into correspondence the GFD and the notion given by Definition 6.

To establish the relation between these notions, we have to introduce, for the remaining part of this section, the following spaces. Consider $\alpha \ge -1$, and define

$$C^1_{\alpha}(\Delta) = \left\{ k \in C_{\alpha}(\Delta) \;:\; \frac{\partial k}{\partial t}, \frac{\partial k}{\partial s} \in C_{\alpha}(\Delta) \right\}. \tag{106}$$

In the following proposition and remark, we suppose that the kernels k for the derivatives belongs to the space $C^1_{-1}(\Delta)$ for the left- and the right-sided cases. For simplicity, we note the partial derivatives of k with respect to t and s, respectively, by k_1 and k_2.

Proposition 6 (*Connection Between GFD and the Regularized Derivatives Given By Definition 6*)**.** *Let k be a kernel satisfying the conditions in Definition 4, and lying in the space $C^1_{-1}(\Delta)$ for the left- and the right-sided cases.*

Then, for any function $f \in C^1_{-1}(a,b]$, the following relation holds:

$$({}_{l}D^{a}_{(k)}f)(t) = f(a)k(t,a) + \int_a^t \left[k_1(t,s) + k_2(t,s)\right] f(s)\,ds + ({}^{\star}_{l}D^{a}_{(k)}f)(t), \; a < t \le b. \tag{107}$$

Also, for any function $f \in C^1_{-1}[a,b)$, the following relation holds:

$$({}_{r}D^{b}_{(k)}f)(t) = -f(b)k(b,t) - \int_t^b \left[k_1(s,t) + k_2(s,t)\right] f(s)\,ds + ({}^{\star}_{r}D^{b}_{(k)}f)(t), \; a \le t < b. \tag{108}$$

Proof. First, for $f \in C^1_{-1}(a,b]$, we have, for $a < t \le b$:

$$\begin{aligned}
({}_l D^a_{(k)} f)(t) &= \frac{d}{dt}\int_a^t k(t,s)f(s)\,ds \\
&\overset{u=t-s}{=} \frac{d}{dt}\int_0^{t-a} k(t,t-u)f(t-u)\,du \\
&= k(t,a)f(a) + \int_0^{t-a} \frac{d}{dt}\Big(k(t,t-u)f(t-u)\Big)\,du \\
&= k(t,a)f(a) + \int_0^{t-a} \Big(\frac{d}{dt}k(t,t-u)f(t-u) + k(t,t-u)f'(t-u)\Big)\,du \\
&= k(t,a)f(a) + \int_0^{t-a} \left[k_1(t,t-u) + k_2(t,t-u)\right] f(t-u)\,du \\
&\quad + \int_0^{t-a} k(t,t-u)f'(t-u)\,du \\
&\overset{s=t-u}{=} k(t,a)f(a) + \int_a^t \left[k_1(t,s)+k_2(t,s)\right] f(s)\,ds + \int_a^t k(t,s)f'(s)\,ds \\
&= k(t,a)f(a) + \int_a^t \left[k_1(t,s)+k_2(t,s)\right] f(s)\,ds + ({}^{\star}_l D^a_{(k)} f)(t).
\end{aligned} \tag{109}$$

On the other hand, for $f \in C^1_{-1}[a,b)$, we have, for $a \le t < b$:

$$\begin{aligned}
({}_r D^b_{(k)} f)(t) &= -\frac{d}{dt}\int_t^b k(s,t)f(s)\,ds \\
&\overset{u=s-t}{=} -\frac{d}{dt}\int_0^{b-t} k(u+t,t)f(u+t)\,du \\
&= -k(b,t)f(b) - \int_0^{b-t} \frac{d}{dt}\Big(k(u+t,t)f(u+t)\Big)\,du \\
&= -k(b,t)f(b) - \int_0^{b-t} \Big(\frac{d}{dt}k(u+t,t)f(u+t) + k(u+t,t)f'(u+t)\Big)\,du \\
&= -k(b,t)f(b) - \int_0^{b-t} \left[k_1(u+t,t)+k_2(u+t,t)\right] f(u+t)\,du \\
&\quad - \int_0^{b-t} k(u+t,t)f'(u+t)\,du \\
&\overset{s=u+t}{=} -k(b,t)f(b) - \int_t^b \left[k_1(s,t)+k_2(s,t)\right] f(s)\,ds \\
&\quad - \int_t^b k(s,t)f'(s)\,du \\
&= -k(b,t)f(b) - \int_t^b \left[k_1(s,t)+k_2(s,t)\right] f(s)\,ds + ({}^{\star}_r D^b_{(k)} f)(t). \qquad \square
\end{aligned} \tag{110}$$

Remark 6. Suppose that k is a kernel satisfying the conditions in Definition 4, and lying in the space $C^1_{-1}(\Delta)$ for the left- and the right-sided cases. Take $f \in C^1_{-1}(a,b]$ and $g \in C^1_{-1}[a,b)$. Then, from Definition 5, we know that

$$({}^{*}_l D^a_{(k)} f)(t) = ({}_l D^a_{(k)} f)(t) - f(a)\frac{d}{dt}\int_a^t k(t,s)\,ds, \quad a < t \le b. \tag{111}$$

$$({}^{*}_r D^b_{(k)} g)(t) = ({}_r D^b_{(k)} g)(t) + g(b)\frac{d}{dt}\int_t^b k(s,t)\,ds, \quad a \le t < b, \tag{112}$$

and, by Proposition 6, we have deduced that

$$({}^{\star}_l D^a_{(k)} f)(t) = ({}_l D^a_{(k)} f)(t) - f(a)k(t,a) - \int_a^t \left[k_1(t,s)+k_2(t,s)\right] f(s)\,ds, \; a < t \le b. \tag{113}$$

$$({}^{\star}_r D^b_{(k)} g)(t) = ({}_r D^b_{(k)} g)(t) + g(b)k(b,t) + \int_t^b \left[k_1(s,t)+k_2(s,t)\right] g(s)\,ds, \; a \le t < b. \tag{114}$$

This illustrates that the notions in Definitions 5 and 6 are different in general. Indeed, for the left case, we observe that

$$\begin{aligned} \frac{d}{dt}\int_a^t k(t,s)\,ds &\overset{u=t-s}{=} \frac{d}{dt}\int_0^{t-a} k(t,t-u)\,du \\ &= k(t,a) + \int_0^{t-a} \frac{d}{dt}k(t,t-u)\,du \\ &= k(t,a) + \int_0^{t-a} \left[k_1(t,t-u) + k_2(t,t-u)\right]\,du \\ &\overset{s=t-u}{=} k(t,a) + \int_a^t \left[k_1(t,s) + k_2(t,s)\right]\,ds. \end{aligned} \tag{115}$$

3. Some additional examples and considerations

As we have seen, one of the core contributions of this work is the concept of a generalized non-convolution Sonin condition relating the kernels that allow to define the integral operator and the derivative operator. One interesting question is if the new concept provided allows the introduction of some new pairs of fractional operators. We will show, in this section, how the new concept of integral includes some other interesting examples of non-convolution two-variable kernels. In particular, we some interesting integral kernels can be obtained, for instance, as follows:

$$\kappa(t,s) = h_1(t)h_2(s), \tag{116}$$

$$\kappa(t,s) = h_1(t) + h_2(s), \tag{117}$$

and

$$\kappa(t,s) = \int_s^t w(u)\,du, \tag{118}$$

where the involved functions h_1, h_2, and w are positive. We note that, if W is a primitive of w in the last expression, then

$$\kappa(t,s) = \int_s^t w(u)\,ds = W(t) - W(s), \tag{119}$$

different from $W(t-s)$ in general. Therefore, these different expressions provide interesting different non-convolution kernels. As particular $h_i(u)$ functions, for $i = 1, 2$, we can explore the use of some power-law type functions that appear usually in the study of general fractional calculus, as generators for several convolution two-variable kernels with application to model diverse phenomena, such as

$$h(t) = \frac{\lambda(\lambda t)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{120}$$

which is used to model memory effects in dynamical systems, or the Gamma distribution type kernel, represented as

$$h(t) = \lambda\frac{\gamma(1-\alpha,\lambda t)}{\Gamma(1-\alpha)}, \tag{121}$$

which is often applied in systems exhibiting exponential decay, or even Mittag-Leffler type kernels, also employed in fractional models to describe processes with anomalous diffusion. However, when combining these functions properly in the integral kernel, it would be necessary to analyze the expression of the corresponding kernel for the derivative in terms of the Sonin condition, and to explore the relation, if any, with those kernels associated with the functions separately, that is:

$$k(t) = \frac{(\lambda t)^{\alpha-1}}{\Gamma(\alpha)}, \tag{122}$$

$$k(t) = \frac{(\lambda t)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t}, \tag{123}$$

and

$$k(t) = (\lambda t)^{\beta-1}E_{\alpha,\beta}[-(\lambda t)^\alpha], \tag{124}$$

respectively.

In relation with this, once the two-variable kernel for the integral operator has been fixed and, independently of its type, an interesting question to be analyzed is the calculation of the corresponding pair of fractional operators by providing also the associated kernel for the derivative operator. As mentioned before, both kernels for integral and derivative operators should satisfy the generalized Sonin condition. As we will see, the explicit expression for the kernel for the derivative is not always easy to calculate, since it requires to solve a problem in the field of integral equations, but its existence can be graphically justified easily.

On the other hand, the concept of variable-order fractional calculus is quite old, but due to the lack of some fundamental properties, it is quite difficult to work with the corresponding operators analytically. We also discuss in this section to what extent the generalized notion of the Sonin kernel can also be applied to variable-order fractional calculus. In fact, as we mention in Example 2, some types of variable-order fractional integral operators can be included directly in the framework proposed, providing new examples of application of the definitions provided. It will be our objective now to give a first attempt, in the lines of this paper, of a generalized Sonin condition for variable-order fractional calculus. We study this problem in the next part, since we will provide some new definitions which are also useful to provide additional examples for non-convolutional operators afterwards.

3.1. Extension of notions for variable-order integral operators

Concerning variable-order fractional operators, we recall that the left-sided integral operator is given by

$$({}_lI^a_{(\kappa)}f)(t) = \int_a^t \frac{1}{\Gamma(\alpha(t))}(t-s)^{\alpha(t)-1} f(s)ds, \quad a < t \le b, \tag{125}$$

with $0 < \alpha(t) < 1$ for simplicity. Then, the kernel is given by

$$\kappa(t,s) = \frac{1}{\Gamma(\alpha(t))}(t-s)^{\alpha(t)-1}. \tag{126}$$

Our aim is to extend the Sonin condition for this type of operators, by proposing a notion of definition of derivative that represents an extension of Definition 4 and can be related to variable-order fractional derivative operator.

Definition 7 (*An Even More Generalized Non-Convolution Sonin Condition for Left-Sided Operators*)**.** We say that a pair of kernels satisfy the generalized non-convolution Sonin condition for left-sided operators if there exists a function $g_l(t)$ such that:

$$g_l(t)\frac{d}{dt}\int_a^t \langle k,\kappa\rangle(t,z)f(z)\,dz = f(t), \qquad t\in(a,b],\ f\in C_{-1}(a,b], \tag{127}$$

where

$$\langle k,\kappa\rangle(t,z) := \int_z^t k(t,s)\kappa(s,z)\,ds, \tag{128}$$

We denote the pair of kernels satisfying this property by $\mathcal{SC}^L$.

Definition 8. Let (κ, k) be a pair of kernels belonging to the set $\mathcal{SC}^L$, and such that $k(t,s)$ is a two-variable function with first-order partial derivative with respect to the variable t belonging to $C_{-1}(\Delta)$.

The left-sided generalized fractional derivative of a function $f\in C_{-1}(a,b]$ is defined by:

$$({}^G_lD^a_{(k)}f)(t) = g_l(t)\frac{d}{dt}\int_a^t k(t,s)f(s)\,ds, \quad a < t \le b, \tag{129}$$

where function g is given by Eq. (127).

In fact, with these notions, we can immediately prove that the derivative operator is left inverse to the integral on the space $C_{-1}(a,b]$, that is,

$$({}^G_lD^a_{(k)}\,{}_lI^a_{(\kappa)}f)(t) = f(t), \quad f\in C_{-1}(a,b]. \tag{130}$$

Indeed, by the definition of $\mathcal{SC}^L$, we have, similarly to (68) ,

$$\begin{aligned}({}^G_lD^a_{(k)}\,{}_lI^a_{(\kappa)}f)(t) &= g_l(t)\frac{d}{dt}\int_a^t k(t,s)\left(\int_a^s \kappa(s,z)f(z)dz\right)ds\\ &= g_l(t)\frac{d}{dt}\int_a^t\left(\int_z^t k(t,s)\kappa(s,z)\,ds\right)f(z)dz = f(t),\end{aligned} \tag{131}$$

where we have used that (127) holds.

Definition 9 (*The Corresponding Generalized Non-Convolution Sonin Condition for Right-Sided Operators*)**.** We say that a pair of kernels satisfy the generalized non-convolution Sonin condition for right-sided operators if there exists a function $g_r(t)$ such that:

$$g_r(t)\frac{d}{dt}\int_t^b\left(\int_t^u \kappa(u,s)k(s,t)\,ds\right)f(u)du = f(t), \quad t\in(a,b],\ f\in C_{-1}[a,b). \tag{132}$$

We denote the pair of kernels satisfying this property by $\mathcal{SC}^R$.

Definition 10. Let (κ, k) be a pair of kernels belonging to the set $\mathcal{SC}^R$, and such that $k(t,s)$ is a two-variable function with first-order partial derivative with respect to the variable s belonging to $C_{-1}(\Delta)$.

The right-sided generalized fractional derivative of a function $f\in C_{-1}[a,b)$ is defined by the following formula:

$$({}^G_rD^b_{(k)}f)(t) = g_r(t)\frac{d}{dt}\int_t^b k(s,t)f(s)\,ds, \quad a\le t < b, \tag{133}$$

where function g_r is given by (132).

To prove that the right-sided derivative operator is left inverse of the integral on the space $C_{-1}[a,b)$, we have to check that

$$({}^G_rD^b_{(k)}\,{}_rI^b_{(\kappa)}f)(t) = f(t), \quad f\in C_{-1}[a,b). \tag{134}$$

Indeed, by the definition of $\mathcal{SC}^R$,

$$\begin{aligned}({}_r^G D^b_{(k)}\, {}_r I^b_{(\kappa)} f)(t) &= g_r(t)\frac{d}{dt}\int_t^b k(s,t)\left(\int_s^b \kappa(u,s)f(u)du\right)ds \\ &= g_r(t)\frac{d}{dt}\int_t^b \left(\int_t^u \kappa(u,s)k(s,t)\,ds\right) f(u)du \\ &= f(t),\end{aligned} \tag{135}$$

where we have used condition (132).

In the literature, it is suggested that the corresponding fractional derivative is possibly

$$D_a^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t (t-s)^{-\alpha(t)} f(s)\,ds, \tag{136}$$

hence the suggested values are

$$g_l(t) = \frac{1}{\Gamma(1-\alpha(t))} \tag{137}$$

and

$$k(t,s) = (t-s)^{-\alpha(t)}. \tag{138}$$

However, checking the generalized Sonin condition is equivalent to prove that

$$\frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t\int_z^t \frac{1}{\Gamma(\alpha(t))}(t-s)^{\alpha(t)-1}(s-z)^{-\alpha(s)}\,ds\,f(z)\,dz = f(t), \tag{139}$$

for every $t\in(a,b]$, and $f\in C_{-1}(a,b]$.

According to the new Definition 7, the kernel corresponding to the variable-order left-sided integral

$$({}_l I^a_{(\kappa)} f)(t) = \int_a^t \frac{1}{\Gamma(\alpha(t))}(t-s)^{\alpha(t)-1} f(s)ds, \quad a<t\le b, \tag{140}$$

would be the one satisfying

$$g_l(t)\frac{d}{dt}\int_a^t \frac{k(t,s)}{\Gamma(\alpha(s))}\left(\int_a^s (s-z)^{\alpha(s)-1} f(z)dz\right)ds = f(t), \tag{141}$$

for every $t\in(a,b]$, and $f\in C_{-1}(a,b]$, for a certain function g_l, and the notion of corresponding left-sided derivative would be given by (129) with these k, g_l. This expression can be considered as an extended Sonin condition for left-sided variable-order operators. An analogous expression can be derived for the right-sided case.

3.2. Additional examples for non-convolutional operators and existence of the associated kernel for the derivative

Some interesting examples of non-convolutional integral operators can be those generated by the product, addition or integration kernels of the type:

$$\kappa(t,s) = h_1(t)h_2(s), \tag{142}$$

$$\kappa(t,s) = h_1(t)+h_2(s), \tag{143}$$

and

$$\kappa(t,s) = \int_s^t w(u)\,du, \tag{144}$$

where h_1, h_2 and w are appropriately chosen positive functions.

Consider a kernel κ in $C_{-1}(\Delta)$. Attending to the Sonin condition given by $\mathcal{S}_G$ (see Eq. (61)), we would have to calculate the kernel k for the derivative operator fulfilling that

$$\int_z^t k(t,s)\kappa(s,z)\,ds = 1, \qquad (t,z)\in\mathring{\Delta},$$

and

$$\int_z^t \kappa(t,s)k(s,z)\,ds = 1, \qquad (t,z)\in\mathring{\Delta}. \tag{145}$$

The explicit expression of k is difficult to calculate, since it requires the resolution of an integral equation.

For illustrating the existence of such a kernel k, we will proceed with a geometrical reasoning, starting with the explanation of the procedure for the traditional Sonin condition for convolution operators (see (8)). Indeed, by the change of variable $u=\frac{s}{t}$, the Sonin condition is rewritten as:

$$\int_0^t \kappa(t-s)k(s)\,ds = t\int_0^1 \kappa(t-tu)k(tu)\,du = 1, \text{ for every } t\in(0,b-a]. \tag{146}$$

This means that the calculation of k consists of finding, for every $t\in(0,b-a]$ fixed, and given the function $\varphi_t(u)=\kappa(t-tu)$ defined on $(0,1)$, a corresponding function $\psi_t(u)$ such that the area under the curve defined by the graph of the product function $\varphi_t\psi_t$ on

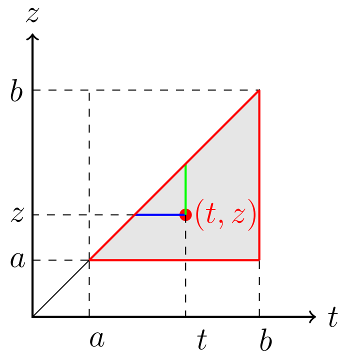

Fig. 3. Construction of kernel k.

the interval $(0,1)$ is equal to $\frac{1}{t}$. This part, the existence of ψ_t, is easily achieved. However, the selection has to be made in such a way that the choice of $\psi_t(u)$ satisfies that $\psi_t(u) = k(tu)$ for a certain k in the proper space, for every $u \in (0,1)$ and $t \in (0, b-a]$, that is, in such a way that the function $k(s) := \psi_t\left(\frac{s}{t}\right)$ is independent of $t \in (0, b-a]$, which is an additional restriction.

If we revise this procedure with Eq. (61), for every $(t,z) \in \mathring{\Delta}$ fixed, we can write, by the change $s = t + (u-1)(t-z)$, the first identity as

$$\begin{aligned} 1 &= \int_z^t k(t,s)\kappa(s,z)\,ds \\ &= (t-z)\int_0^1 k\left(t, t+(u-1)(t-z)\right)\kappa\left(t+(u-1)(t-z), z\right)\,du, \end{aligned} \tag{147}$$

and the second identity as

$$\begin{aligned} 1 &= \int_z^t \kappa(t,s)k(s,z)\,ds \\ &= (t-z)\int_0^1 \kappa\left(t, t+(u-1)(t-z)\right)k\left(t+(u-1)(t-z), z\right)\,du. \end{aligned} \tag{148}$$

We can start a constructive procedure for defining (at least theoretically) the sought function k as follows. For every $(t,z) \in \mathring{\Delta}$ fixed (see Fig. 3), we consider the one-variable function $\varphi_{t,z}(u) = \kappa\left(t+(u-1)(t-z), z\right)$ defined on $(0,1)$, which is the restriction of κ to the blue segment line, and seek a function $\psi_{t,z}(u)$ such that the area under the curve defined by the graph of the product function $\varphi_{t,z}\psi_{t,z}$ on the interval $(0,1)$ is equal to $\frac{1}{t-z}$. Again, the existence of $\psi_{t,z}$, is straightforward. Besides, for every $(t,z) \in \mathring{\Delta}$ fixed, we consider the one-variable function $\phi_{t,z}(u) = \kappa\left(t, t+(u-1)(t-z)\right)$ defined on $(0,1)$, which is the restriction of κ to the green segment line, and seek a function $\Psi_{t,z}(u)$ such that the area under the curve defined by the graph of the product function $\phi_{t,z}\Psi_{t,z}$ on the interval $(0,1)$ is equal to $\frac{1}{t-z}$. The existence of $\Psi_{t,z}$, is also straightforward.

For the proper calculation of k, the selection made has to be consistent, in the sense that the definition through the segments

$$\begin{aligned} &k\left(t, t+(u-1)(t-z)\right) = \psi_{t,z}(u),\ u \in (0,1), \\ &k\left(t+(u-1)(t-z), z\right) = \Psi_{t,z}(u),\ u \in (0,1), \end{aligned} \tag{149}$$

has to be a consistent, for $(t,z) \in \mathring{\Delta}$. Then, the selection has to be made in such a way that the choices of $\psi_{t,z}(u)$ and $\Psi_{t,z}(u)$ satisfy that the following expressions are consistent

$$k\left(t,s\right) = \psi_{t,z}\left(1+\frac{1}{t-z}(s-t)\right), \quad k\left(s,z\right) = \Psi_{t,z}\left(1+\frac{1}{t-z}(s-t)\right), \tag{150}$$

that is, the first one has to be independent of $z \in (0,t)$, and the second one independent of $t \in (a,b)$, both expressions being consistent.

If we consider, for instance, $\kappa(t,s) = \frac{(t^2-s^2)^{\alpha-1}}{\Gamma(\alpha)}$, with $\alpha \in (0,1)$, the geometrical procedure can illustrate the attempt to obtain the corresponding kernel k.

For the integral kernel

$$\kappa(t,s) = \int_s^t w(u)\,du, \tag{151}$$

for an appropriate function w, conditions (61) are written as

$$\begin{aligned} 1 &= \int_z^t k(t,s)\kappa(s,z)\,ds = \int_z^t k(t,s)\int_z^s w(u)\,du\,ds \\ &= \int_z^t w(u)\int_u^t k(t,s)\,ds\,du \end{aligned} \tag{152}$$

and

$$1 = \int_z^t \kappa(t,s)k(s,z)\,ds = \int_z^t \int_s^t w(u)\,du\,k(s,z)\,ds = \int_z^t w(u) \int_z^u k(s,z)\,ds\;du. \tag{153}$$

In this case, for every $(t,z) \in \mathring{\Delta}$ fixed, we seek a function $\psi_{t,z}(u)$ such that the area under the curve defined by the graph of the product function $w\psi_{t,z}$ on the interval (z,t) is equal to 1. Then, we should select k in the proper space such that $\int_u^t k(t,s)\,ds = \psi_{t,z}(u)$. Moreover, for every $(t,z) \in \mathring{\Delta}$ fixed, we seek a function $\Psi_{t,z}(u)$ such that the area under the curve defined by the graph of the product function $w\Psi_{t,z}$ on the interval (z,t) is equal to 1. Then, we should select k in the proper space such that $\int_z^u k(s,z)\,ds = \Psi_{t,z}(u)$. The two conditions to be fulfilled are, therefore,

$$\int_u^t k(t,s)\,ds = \psi_{t,z}(u), \qquad \int_z^u k(s,z)\,ds = \Psi_{t,z}(u), \quad u \in (z,t), \tag{154}$$

and $\psi_{t,z}$, $\Psi_{t,z}$ should be such that the definition of k is consistent for $(t,z) \in \mathring{\Delta}$.

We can also provide some details for the case of the two-variable kernel

$$\kappa(t,s) = h_1(t)h_2(s), \tag{155}$$

where h_1 and h_2 are functions such that the generated κ belongs to $C_{-1}(\Delta)$. More generally than stated before, according to the properties $\mathcal{SC}^L$, and $\mathcal{SC}^R$, if we seek the kernel satisfying the more generalized non-convolution Sonin condition, we have to guarantee that

$$g_l(t)\frac{d}{dt}\int_a^t h_2(z)\left(\int_z^t k(t,s)h_1(s)\,ds\right) f(z)dz = f(t), \tag{156}$$

for every $t \in (a,b]$, and $f \in C_{-1}(a,b]$, and

$$g_r(t)\frac{d}{dt}\int_t^b h_1(u)\left(\int_t^u h_2(s)k(s,t)\,ds\right) f(u)du = f(t), \tag{157}$$

for every $t \in [a,b)$, and $f \in C_{-1}[a,b)$.

If the Sonin condition is not valid for two integral and derivative kernels, then many interesting properties are not valid, this is the case, for instance, for the left-inverse character of the integral with respect to the derivative. Hence, an interesting exercise is, for instance, fixing the integral and derivative kernels, and trying to obtain the subspace of functions for which the left-inverse property is satisfied.

In this particular case of the product, if we choose, for example, $h_1(t) = \sin(\theta t)$, $h_2(t) = \cos(\theta t)$, thus $\kappa(t,s) = h_1(t)h_2(s)$, and we also select $k \equiv \kappa$, we can try to calculate the subspace of specific functions f for which the properties (156) and (157) hold, for every $t \in [a,b)$. Indeed, we are interested in the following fractional operator $I[f](t)$:

$$I[f](t) = \int_a^t h_2(z)\left(\int_z^t \kappa(t,s)h_1(s)\,ds\right) f(z)\,dz, \tag{158}$$

where, as mentioned before, $h_1(s) = \sin(\theta s)$ and $h_2(z) = \cos(\theta z)$.

The inner integral $I_1(z,t)$ is given by:

$$I_1(z,t) = \int_z^t \kappa(t,s)h_1(s)\,ds \tag{159}$$

$$= \int_z^t \sin(\theta t)\cos(\theta s)\sin(\theta s)\,ds. \tag{160}$$

Using the identity:

$$\cos(\theta s)\sin(\theta s) = \frac{1}{2}\sin(2\theta s), \tag{161}$$

we have:

$$I_1(z,t) = \frac{\sin(\theta t)}{2}\int_z^t \sin(2\theta s)\,ds. \tag{162}$$

Since a primitive of $\sin(2\theta s)$ is given by $\int \sin(2\theta s)\,ds = -\frac{1}{2\theta}\cos(2\theta s)$, thus:

$$I_1(z,t) = -\frac{\sin(\theta t)}{4\theta}\left[\cos(2\theta t) - \cos(2\theta z)\right]. \tag{163}$$

By substituting $I_1(z,t)$ into the expression for $I[f](t)$, we have:

$$I[f](t) = -\frac{\sin(\theta t)}{4\theta}\int_a^t h_2(z)\left[\cos(2\theta t) - \cos(2\theta z)\right] f(z)\,dz. \tag{164}$$

We recall that $h_2(z) = \cos(\theta z)$, so:

$$I[f](t) = -\frac{\sin(\theta t)}{4\theta}\left[\cos(2\theta t)\int_a^t \cos(\theta z)f(z)\,dz - \int_a^t \cos(\theta z)\cos(2\theta z)f(z)\,dz\right] \tag{165}$$

$$= -\frac{\sin(\theta t)}{4\theta}\left[\cos(2\theta t)\cdot A(t) - B(t)\right], \tag{166}$$

where:

$$A(t) = \int_a^t \cos(\theta z) f(z)\, dz, \tag{167}$$

$$B(t) = \int_a^t \cos(\theta z)\cos(2\theta z) f(z)\, dz. \tag{168}$$

Now, we differentiate $I[f](t)$ with respect to t:

$$\frac{dI[f]}{dt} = -\frac{d}{dt}\left(\frac{\sin(\theta t)}{4\theta}\left[\cos(2\theta t)A(t) - B(t)\right]\right) \tag{169}$$

$$= -\left(\frac{d}{dt}\left[\frac{\sin(\theta t)}{4\theta}\right](\cos(2\theta t)A(t) - B(t)) + \frac{\sin(\theta t)}{4\theta}\frac{d}{dt}\left[\cos(2\theta t)A(t) - B(t)\right]\right). \tag{170}$$

First, we compute the derivatives:

- Derivative of $\frac{\sin(\theta t)}{4\theta}$:

$$\frac{d}{dt}\left(\frac{\sin(\theta t)}{4\theta}\right) = \frac{\cos(\theta t)}{4}. \tag{171}$$

- Derivative of $\cos(2\theta t)A(t)$:

$$\frac{d}{dt}\left[\cos(2\theta t)A(t)\right] = -2\theta\sin(2\theta t)A(t) + \cos(2\theta t)A'(t). \tag{172}$$

- Derivative of $B(t)$:

$$B'(t) = \cos(\theta t)\cos(2\theta t) f(t). \tag{173}$$

- Derivative of $A(t)$:

$$A'(t) = \cos(\theta t) f(t). \tag{174}$$

Therefore:

$$\frac{dI[f]}{dt} = -\frac{\cos(\theta t)}{4}(\cos(2\theta t)A(t) - B(t)) \tag{175}$$

$$- \frac{\sin(\theta t)}{4\theta}\left(-2\theta\sin(2\theta t)A(t) + \cos(2\theta t)A'(t) - B'(t)\right), \tag{176}$$

and, after simplification:

$$\frac{dI[f]}{dt} = -\frac{\cos(\theta t)}{4}(\cos(2\theta t)A(t) - B(t)) \tag{177}$$

$$+ \frac{\sin(\theta t)\sin(2\theta t)}{2}A(t) - \frac{\sin(\theta t)}{4\theta}\left(\cos(2\theta t)A'(t) - B'(t)\right). \tag{178}$$

Given the complexity of the expression $\frac{dI[f]}{dt}$, it is challenging to simplify in order to reduce it to the expression depending on $f(t)$ needed for the validity of (156) without imposing specific conditions or choosing a particular form of $f(t)$. In this case, we seek for specific types of functions f satisfying (156). We can assume, for instance, that $f(t)$ is a trigonometric function, such as $f(t) = \cos(\theta t)$ or $f(t) = \sin(\theta t)$. In these cases, it is possible to obtain $g_I(t) = \frac{4\theta}{\cos(\theta t)}$.

3.3. Some connections with other approaches

On the other hand, as we have mentioned in the Introduction, in the particular context of convolutional kernels, the left-sided and right-sided fractional operators have been studied in the recent paper [17], where the authors have proved that the operators are related to each other by a conjugation relation using negation operators. In this part, we try to prove if a similar conjugation relation applies in our case to connect the left-sided and right-sided operators with each other.

To explore this, we first recall that our definitions for left-sided and right-sided fractional operators for a given kernel $\kappa(t,s)$ were given as follows (with a simplified notation):

- **Left-sided operator**: $I_{\kappa,L} f(t) = \int_a^t \kappa(t,s) f(s)\, ds$.
- **Right-sided operator**: $I_{\kappa,R} f(t) = \int_t^b \kappa(s,t) f(s)\, ds$.

We aim to investigate if a conjugation relation, similar to the ones proven in [17] for convolutional fractional operators, exists between I_L and I_R. Specifically, we will check if there is a relation of the form:

$$I_{\kappa,L} f(t) = \mathcal{N}\left(I_{\kappa,R}(\mathcal{N}^{-1} f)\right)(t), \tag{179}$$

where $\mathcal{N}$ is either a negation operator or some transformation specifically designed for connecting the left-sided operator with the right-sided one.

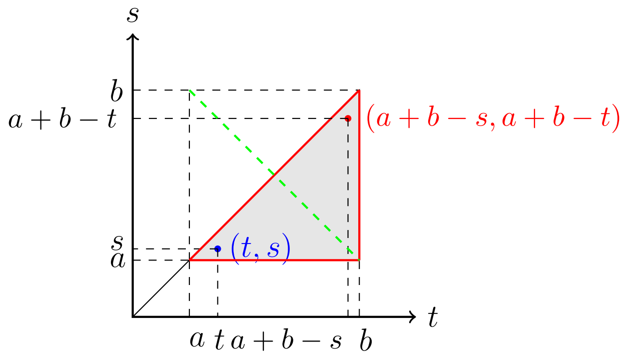

Fig. 4. Relation between the values of the kernels $\hat{\kappa}$ and κ.

To this purpose, we need to analyze how the action of $I_{\kappa,L}$ and $I_{\kappa,R}$ works on test functions and determine if we can express one of the operator in terms of the other and the operator $\mathcal{N}$.

Parting from the definition of both operators, we choose, for instance,

$$(\mathcal{N} f)(t) := f(a+b-t), \tag{180}$$

whose inverse is the same mapping, then

$$I_{\kappa,R}(\mathcal{N}^{-1} f)(t) = \int_t^b \kappa(s,t)(\mathcal{N}^{-1} f)(s)\,ds = \int_t^b \kappa(s,t) f(a+b-s)\,ds. \tag{181}$$

By the change of variable $u = a+b-s$, we get:

$$I_{\kappa,R}(\mathcal{N}^{-1} f)(t) = \int_a^{a+b-t} \kappa(a+b-u,t) f(u)\,du. \tag{182}$$

Applying $\mathcal{N}$, we have:

$$\mathcal{N}\left(I_{\kappa,R}(\mathcal{N}^{-1} f)\right)(t) = \int_a^t \kappa(a+b-u,a+b-t) f(u)\,du. \tag{183}$$

Now, we need to compare this result with $I_{\kappa,L} f(t) = \int_a^t \kappa(t,s) f(s)\,ds$, so that, we obtain the left-sided integral operator for the 'transposed' kernel

$$\hat{\kappa}(t,s) := \kappa(a+b-s, a+b-t), \tag{184}$$

that is, $I_{\hat{\kappa},L}$. In Fig. 4, we illustrate how the value of $\hat{\kappa}$ at the blue point coincides with the value of κ at the red point, so it is a kind of 'transposition' of values with respect to the triangle. In fact, the transposition consists of the composition of the three following planar movements: symmetry with respect to the line $t = \frac{a+b}{2}$, symmetry with respect to the line $s = \frac{a+b}{2}$, and, finally, symmetry with respect to the line $s = t$, in order to go back to the original triangle Δ. This composition of movements coincides with the symmetry with respect to the line $s = a+b-t$ that divides the triangle Δ into two halves by crossing perpendicularly the hypotenuse and passing through the opposite vertex (b,a) (see the green line in Fig. 4 passing through the points (a,b) and (b,a)). For instance, for $a=0$ and $b=2$, $\hat{\kappa}\left(\frac{7}{4},\frac{3}{2}\right) = \kappa\left(\frac{1}{2},\frac{1}{4}\right)$.

If we want to establish the conjugation relation (179) and recover exactly the left-sided integral operator for the same kernel κ, then we have to assume the symmetry condition

$$\kappa(t,s) = \kappa(a+b-s, a+b-t), \quad \forall (t,s). \tag{185}$$

Thus, if the kernel κ satisfies this symmetry condition, then the left-sided and right-sided operators for the same kernel κ are conjugated through the operator $\mathcal{N}$.

In conclusion, if condition (185) is satisfied, then a conjugation relation similar to the one discussed in [17] does indeed hold for our nonconvolutional integral kernel, and this represents an interesting connection between the left-sided and right-sided operators. In other cases, new integral one-sided operators are obtained. In the future, further investigation on properties and examples will serve as illustration of the practical implications of this relationship and how it can be applied to specific kernels and functions.

Remark 7. If we have considered, alternatively, the right-sided integral operator given by (32):

$$I_{\kappa,R} f(t) = \int_t^b \kappa(t,s) f(s)\,ds,$$

and the same $\mathcal{N}$, then

$$I_{\kappa,R}(\mathcal{N}^{-1} f)(t) = \int_t^b \kappa(t,s)(\mathcal{N}^{-1} f)(s)\,ds = \int_t^b \kappa(t,s) f(a+b-s)\,ds. \tag{186}$$

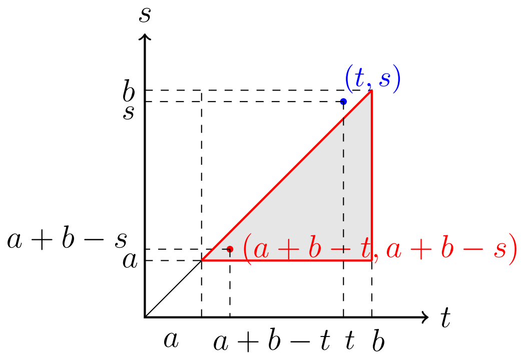

Fig. 5. Relation between the values of the kernels $\hat{\kappa}$ and κ.

By the change of variable $u = a + b - s$, we get:

$$I_{\kappa,R}(\mathcal{N}^{-1} f)(t) = \int_a^{a+b-t} \kappa(t, a + b - u) f(u)\, du. \tag{187}$$

Applying $\mathcal{N}$, we have:

$$\mathcal{N}\left(I_{\kappa,R}(\mathcal{N}^{-1} f)\right)(t) = \int_a^t \kappa(a + b - t, a + b - u) f(u)\, du. \tag{188}$$

Comparing this result with $I_{\kappa,L} f(t) = \int_a^t \kappa(t,s) f(s)\, ds$, we have obtained the left-sided integral operator for the 'translated' kernel

$$\hat{\kappa}(t, s) := \kappa(a + b - t, a + b - s), \tag{189}$$

that is, $I_{\hat{\kappa},L}$. In Fig. 5, we illustrate how the value of $\hat{\kappa}$ at the blue point coincides with the value of κ at the red point, so it is a kind of 'symmetrization' of values based on the horizontal and vertical distances to the hypotenuse of the triangle. Note that, with the alternative definition of right-sided integral, the domain of $\hat{\kappa}$ is the triangle

$$\widetilde{\Delta} := \{(t, s) \in [a, b] \times [a, b] \ \mid \ t \le s\}. \tag{190}$$

This way, the 'symmetrization' consists of the composition of the two following planar movements: symmetry with respect to the line $t = \frac{a+b}{2}$, and symmetry with respect to the line $s = \frac{a+b}{2}$. For instance, for $a = 0$ and $b = 2$, $\hat{\kappa}\left(\frac{3}{2}, \frac{7}{4}\right) = \kappa\left(\frac{1}{2}, \frac{1}{4}\right)$.

To establish the conjugation relation (179) and recover exactly the left-sided integral operator for the same kernel κ, then we have to impose the symmetry condition

$$\kappa(t, s) = \kappa(a + b - t, a + b - s), \quad \forall (t, s). \tag{191}$$

Thus, if the kernel κ satisfies the symmetry condition (191), then the left-sided and right-sided operators for the same kernel κ are conjugated through the operator $\mathcal{N}$, for the alternative definition of right-sided integral operator.

4. Conclusion

In this work, we have proposed a general framework for fractional integrals without following a convolution-kernel approach, and have considered the corresponding notions for fractional derivatives under different perspectives. In particular, we have introduced a Riemann–Liouville-type and two notions of Caputo-type derivatives all of them extending the notions proposed in the literature for the convolution case.

We have analyzed some relevant mathematical properties such as linearity and semi-group properties for the integral, and the first fundamental theorem to prove that the derivative in the sense of Riemann–Liouville is a left inverse operator of the integral.

We have also given an extended notion for Sonin condition that connects the kernels used for integration and derivation processes, and deduced some connections between the different notions proposed.

In future works, we will include some other useful properties for general fractional order operators, extending some relevant examples of non-convolution fractional integrals and derivatives.

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Data availability

No data was used for the research described in the article.

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