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Existence and multiplicity of solutions of Stieltjes differential equations via topological methods

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Abstract. In this work, we use techniques from Stieltjes calculus and fixed point index theory to show the existence and multiplicity of solution of a first order non-linear boundary value problem with linear boundary conditions that extend the periodic case. We also provide the Green's function associated to the problem as well as an example of application.

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1. Introduction

In recent years, there has been a surge regarding the interest in *Stieltjes Calculus*, that is, the study of differentiability, integration and differential and integral equations where a derivator is involved, that is, a nondecreasing left continuous function $g : [0, T] \to \mathbb{R}$. The reason behind this interest is the fact that Stieltjes differential equations include ordinary, difference, impulsive and time scale equations, providing a context where stationary and impulsive behavior is allowed with great generality and the powerful tools of measure theory can be applied.

Many previous works in this area deal with the basic properties of the Stieltjes derivative (which sometimes behaves in unexpected ways), see, for instance, [5–7,16,23], whereas others focus their attention on Stieltjes differential equations and systems [1,6,8,11,17,18]. Despite all of these articles, there is much less work concerning the application of topological methods to Stieltjes differential equations, the exceptions being [20,22,24,25].

Topological methods provide a series of powerful tools to study the existence, nonexistence and multiplicity of solutions of differential problems of all types. In regards to their use for Stieltjes differential equations, in [25] the authors applied Schaeffer's fixed point theorem to study Stieltjes differential problems, whereas in [24] they used Bohnenblust-Karlin set-valued fixed-point theorem for differential inclusions. In [20] they use the upper and lower solution method. Some of the methods yet to be explored are those that rely on the fixed point index theory for cones in Banach spaces—see [4,9,10,13–15,19,26,27] for the case or ordinary differential equations—which provide simple criteria to study the qualitative properties of a given problem.

As said before, in [25] the authors applied Schaeffer's fixed point theorem to study a Stieltjes differential problem. In particular, they considered the problem

$$\begin{cases} u_g'(t) + b(t)u(t) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T). \end{cases}$$

In this paper we will consider the more general case

$$\begin{cases} u_g'(t) + b(t)u(t) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T) + k\, B(u), \end{cases} \tag{1.1}$$

where k is a constant and B is a linear functional.

To this end, we will use fixed point index methods on cones and, in particular, we will adapt the results given in [3] for the n-th order non-local linear boundary value problem with parameter dependence:

$$\begin{cases} T_n\left[M\right] u\left(t\right) = \sigma\left(t\right), & t \in \left[a, b\right], \\ \quad B_i\left(u\right) = \delta_i\, C_i\left(u\right), & i = 1, \ldots, n, \end{cases} \tag{1.2}$$

where

$$T_n\left[M\right] u\left(t\right) := L_n u\left(t\right) + M\, u(t), \quad t \in [a, b],$$

with

$$L_n u\left(t\right) := u^{(n)}\left(t\right) + a_1\left(t\right) u^{(n-1)}\left(t\right) + \cdots + a_n\left(t\right) u\left(t\right), \quad t \in [a, b].$$

Here, σ and a_k are continuous functions for all $k = 0, \ldots, n-1$, $M \in \mathbb{R}$ and $\delta_i \in \mathbb{R}$ for all $i = 1, \ldots, n$. The $C_i : C([a, b]) \to \mathbb{R}$ are linear continuous operators and the B_i are such that they cover the general two-point linear boundary conditions, i.e.,:

$$B_i\left(u\right) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}\left(a\right) + \beta_j^i u^{(j)}\left(b\right)\right), \quad i = 1, \ldots, n,$$

where α_j^i, β_j^i are real constants for all $i = 1, \ldots, n$, $j = 0, \ldots, n-1$.

In [3], the authors described the spectrum of the non-local problem (1.2) by assuming that the local problem ($\delta_i = 0$, $i = 1, \ldots, n$) is uniquely solvable and they obtained the expression of its related Green's function in terms of the one associated to the local problem.

Furthermore, in the particular case of the problem

$$\begin{cases} u'(t) + M u(t) = \sigma(t), & t \in [0, 1], \\ u(0) - u(1) = k \displaystyle\int_0^1 u(s)\, \mathrm{d}\, s, \end{cases}$$

the authors characterize the values of M and k that ensure the constant sign of its Green's function G, and, as a direct consequence of [3, Theorem 2], they prove that it is given by the expression

$$G(t,s,k,M) = g_M(t,s) + \frac{k g_M(t,0)}{1 - k\int_0^1 g_M(t,0)\,\mathrm{d}\,t}\int_0^1 g_M(t,s)\,\mathrm{d}\,t,$$

where g_M is the Green's function of the homogeneous periodic problem

$$\begin{cases} u'(t) + Mu(t) = \sigma(t), \quad t \in [0,1], \\ u(0) - u(1) = 0. \end{cases}$$

Here we will take the same approach to study problem (1.1).

2. Preliminaries

Through this work we will consider a nondecreasing left continuous function $g : [0,T] \to \mathbb{R}$ called a *derivator*, and we will denote by μ_g the Lebesgue–Stieltjes measure associated to g, which is given by

$$\mu_g([a,b)) = g(b) - g(a) \quad a,b \in [0,T],\ a < b,$$

for intervals of the form $[a,b)$ and extended to a complete measure through Carathéodory's extension theorem.

We will consider $\mathscr{L}^1_g([a,b),\mathbb{R})$, the set of functions $f : [a,b) \to \mathbb{R}$ which are μ_g–measurable and $\int_{[a,b)} |f|\,\mathrm{d}\,\mu_g < \infty$.

We will also take into account some special sets in $[0,T]$ related to g such as

$$\begin{aligned} C_g &:= \{t \in [0,T] : g \text{is constant on} (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\} \\ &= \bigcup\nolimits_{n\in\Lambda} (a_n, b_n), \end{aligned}$$

where $(a_l, b_l) \cap (a_j, b_j) = \emptyset$ for $l \neq j$ and Λ is countable; and

$$D_g = \{t \in [0,T] : \Delta^+ g(t) > 0\},$$

where $\Delta^+ g(t) := g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right hand side limit of g at t.

Now, we introduce the concept of g-continuity—see e.g., [6,21].

Definition 2.1. A function $f : [a,b] \to \mathbb{R}$ is *g–continuous* at a point $t \in [a,b]$, or *continuous with respect to g at t*, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \epsilon, \text{ for all } s \in [a,b] \text{ such that } |g(t) - g(s)| < \delta.$$

If f is g–continuous at every point $t \in [a,b]$, we say that f is *g-continuous* on $[a,b]$. We denote by $\mathscr{C}_g([a,b],\mathbb{R})$ the set of g-continuous functions. We denote by $\mathscr{BC}_g([a,b],\mathbb{R})$ the set of g-continuous functions which are also bounded.

$\mathscr{BC}_g([a,b],\mathbb{R})$ is Banach space with the sup-norm

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

Let us now define the *Stieltjes derivative*, or *g-derivative* (we follow here [6] for this form of the definition), of a function $f : [a,b] \to \mathbb{R}$ at a point $t \in [a,b]$ as

$$f'_g(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup C_g, \\ \lim\limits_{s \to t^+} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g, \\ \lim\limits_{s \to b_n^+} \dfrac{f(s)-f(b_n)}{g(s)-g(b_n)}, & t \in (a_n, b_n) \subset C_g, \end{cases}$$

provided the corresponding limits exist (when considering the function inside the limit to be defined wherever the expression makes sense). In that case, we say that f is *g-differentiable at t*.

For Stieltjes derivatives we have the following result—see [6, 21].

Proposition 2.2. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be two g-differentiable functions at a point t.*

Then:

1. *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and*
$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$
2. *The product $f_1 f_2$ is g-differentiable at t and*
$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + (f_2)'_g(t) f_1(t) + (f_1)'_g(t)(f_2)'_g(t)\Delta^+ g(t).$$
3. *If $f_2(t)(f_2(t)+(f_2)'_g(t)\Delta^+ g(t)) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*
$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t) - (f_2)'_g(t) f_1(t)}{f_2(t)(f_2(t) + (f_2)'_g(t)\Delta^+ g(t))}.$$

The next result connects the definition of *g-absolutely continuous* function and the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral.

Theorem 2.3. [21, Theorem 2.7 - Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral] *Let $F : [a,b] \to \mathbb{R}$. The following conditions are equivalent:*

1. *The function F is g-absolutely continuous on $[a,b]$, that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ satisfying*
$$\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta,$$
we have that
$$\sum_{n=1}^m |F(b_n) - F(a_n)| < \epsilon.$$
2. *(i) there exists $F'_g(t)$ for μ_g–a.a. $t \in [a,b)$;*
(ii) $F'_g(t) \in \mathscr{L}^1_g([a,b),\mathbb{R})$;

(iii) for each $t \in [a, b]$,

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, \mathrm{d}\,\mu_g(s).$$

Theorem 2.4. [11, Theorem 3.26 - Fundamental Theorem of Calculus for the Stieltjes derivative] *Let* $f \in \mathscr{L}^1_g([a,b), \mathbb{R})$. *Then the function* $F : [a, b] \to \mathbb{R}$,

$$F(t) = \int_{[a,t)} f(s)\, \mathrm{d}\,\mu_g(s)$$

is well-defined, g-absolutely continuous on $[a, b]$ *and*

$$F'_g(t) = f(t) \text{ for } \mu_g - a.a.\ t \in [a, b).$$

Finally we present the definitions and results regarding fixed point index in cones that we will use.

Definition 2.5. [9, Definition 2.1] Let $(N, \|\cdot\|)$ be a real normed space. A *cone* K in N is a closed set such that

1. $u + v \in K$ for all $u, v \in K$;
2. $\lambda u \in K$ for all $u \in K, \lambda \in [0, \infty)$;
3. $K \cap (-K) = \{0\}$.

Consider a cone P in the Banach space $\mathscr{BC}_g([0,T], \mathbb{R})$. Let Ω be a bounded open subset of $\mathscr{BC}_g([0,T], \mathbb{R})$. We denote by $\overline{\Omega}$ its closure and by $\partial\Omega$ its boundary. We will denote $\Omega_P = \Omega \cap P$.

Lemma 2.6. [2, Theorem 11.1] *Let* Ω *be an open bounded set with* $0 \in \Omega_P$. *Assume that* $F : \overline{\Omega_P} \to P$ *is a continuous compact map such that* $x \neq Fx$ *for all* $x \in \partial\Omega_P$. *Then the fixed point index* $i_P(F, \Omega_P)$ *has the following properties.*

1. *If there exists* $e \in P \setminus \{0\}$ *such that* $x \neq Fx + \lambda e$ *for all* $x \in \partial\Omega_P$ *and all* $\lambda > 0$, *then* $i_P(F, \Omega_P) = 0$.
2. *If* $\lambda x \neq Fx$ *for all* $x \in \partial\Omega_P$ *and for every* $\lambda \geq 1$, *then* $i_P(F, \Omega_P) = 1$.
3. *If* $i_P(F, \Omega_P) \neq 0$, *then* F *has a fixed point in* Ω_P.
4. *Let* Ω^1 *be open in* $\mathscr{BC}_g([0,T], \mathbb{R})$ *with* $\overline{\Omega^1_P} \subset \Omega_P$. *If* $i_P(F, \Omega_P) = 1$ *and* $i_P\left(F, \Omega^1_P\right) = 0$, *then* F *has a fixed point in* $\Omega_P \setminus \overline{\Omega^1_P}$. *The same result holds if* $i_P(F, \Omega_P) = 0$ *and* $i_P\left(F, \Omega^1_P\right) = 1$.

3. The Green's function

In this section, we obtain a Green's function of the linear boundary value problem for the Stieltjes differential equation

$$\begin{cases} u'_g(t) + b(t)u(t) = f(t), & \text{for } \mu_g - \text{a.a. } t \in [0,T], \\ u(0) - u(T) = kB(u), \end{cases} \tag{3.1}$$

where

$$f \in \mathscr{L}^1_g([0,T), \mathbb{R}), \tag{3.2}$$

$$b \in \mathscr{L}^1_g([0,T), \mathbb{R}), \text{ such that } 1 - b(t)\Delta^+ g(t) \neq 0 \quad \forall t \in [0,T), \tag{3.3}$$

and $B : \mathscr{BC}_g([0,T],\mathbb{R}) \to \mathbb{R}$ is a linear bounded functional.

In order to obtain this function, we will make use of the results in [21]. First, we introduce the g-exponential function.

Theorem 3.1. [21, Theorem 3.4] *Let $c \in \mathscr{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that $1 + c(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0+T) \cap D_g$. Then the set*

$$T_c^- = \{t \in [t_0, t_0+T) \cap D_g \ \ 1 + c(t)\Delta^+ g(t) < 0\}$$

has a finite cardinality. Furthermore, if $T_c^- = \{t_1, \dots, t_k\}$, $t_0 \le t_1 < t_2 < \dots < t_k < t_{k+1} = t_0 + T$, then the map $\widehat{c} : [t_0, t_0+T) \to \mathbb{R}$, defined as

$$\widehat{c}(t) = \begin{cases} c(t) & \text{if } t \in [t_0, t_0+T) \setminus D_g, \\ \dfrac{\log|1 + c(t)\Delta^+ g(t)|}{\Delta^+ g(t)} & \text{if } t \in [t_0, t_0+T) \cap D_g, \end{cases}$$

belongs to $\mathscr{L}^1_g([t_0, t_0+T), \mathbb{R})$; the map $\widehat{e}_c(\cdot, t_0) : [t_0, t_0+T] \to \mathbb{R}\backslash\{0\}$, given by

$$\widehat{e}_c(t, t_0) = \begin{cases} \exp\left(\displaystyle\int_{[t_0,t)} \widehat{c}(s)\,\mathrm{d}\,\mu_g(s)\right) & \text{if } t_0 \le t \le t_1, \\ (-1)^j \exp\left(\displaystyle\int_{[t_0,t)} \widehat{c}(s)\,\mathrm{d}\,\mu_g(s)\right) & \text{if } t_j < t \le t_{j+1},\ j = 1, \dots, k, \end{cases}$$

is well-defined and g-absolutely continuous on $[t_0, t_0+T]$; and the map $x : [t_0, t_0+T] \to \mathbb{R}$, given by $x(t) = x_0 \widehat{e}_c(t, t_0)$, $t \in [t_0, t_0+T]$, solves the initial value problem

$$x'_g(t) = c(t)x(t), \quad \mu_g\text{-a.e. } t \in [t_0, t_0+T), \qquad x(t_0) = x_0.$$

We will call the map $\widehat{e}_c$ the *g-exponential function associated to c.*

Now, from Theorem 3.1, the corresponding result for the nonlinear case follows.

Theorem 3.2. [21, Theorem 4.7] *Let $d, h \in \mathscr{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that $1 - d(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0+T) \cap D_g$. Then the unique g-absolutely continuous solution of an initial value problem*

$$x'_g(t) + d(t)x(t) = h(t), \qquad \mu_g\text{-a.e. } t \in [t_0, t_0+T), \qquad x(t_0) = x_0,$$

is given by the map

$$x(t) = \widehat{e}_{-d}(t, t_0)\left(x_0 + \int_{[t_0,t)} \frac{h(s)}{\widehat{e}_{-d}(s, t_0)(1 - d(s)\Delta^+ g(s))}\,\mathrm{d}\,\mu_g(s)\right),\ t \in [t_0, t_0+T].$$

Remark 3.3. In this paper we will use the map $\widehat{e}_c$ with $t_0 = 0$, $c = -b$ and b satisfying (3.3). The function $\widehat{e}_{-b}(\cdot, 0)$ will be denoted as $\widehat{e}_{-b}$.

Let us return our attention to the original boundary value problem (3.1). First, we investigate the periodic problem

$$\begin{cases} u'_g(t) + b(t)u(t) = f(t), & \mu_g\text{-a.e. } t \in [0,T], \\ u(0) - u(T) = 0, \end{cases} \tag{3.4}$$

The thesis of next Lemma is essentially that of [25, Theorem 17], but with simpler assumptions. A similar result can be found in [24, Section 3].

Lemma 3.4. *Let (3.2), (3.3) and*

$$1-\widehat{e}_{-b}(T)\neq 0 \tag{3.5}$$

hold. Then there exists a unique solution of the problem (3.4), which is in the form

$$v(t)=\int_{[0,T)} h_b(t,s)\frac{f(s)}{1-b(s)\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s),$$

where

$$h_b(t,s)=\frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)\,(1-\widehat{e}_{-b}(T))}\cdot\begin{cases}1, & 0\le s<t\le T,\\ \widehat{e}_{-b}(T), & 0\le t\le s\le T.\end{cases} \tag{3.6}$$

Proof. According to Theorem 3.2 and Remark 3.3, each solution of the Stieltjes differential equation in problem (3.4) can be written in the form

$$v(t)=\widehat{e}_{-b}(t)\left(v(0)+\int_{[0,t)}\frac{f(s)}{\widehat{e}_{-b}(s)\,(1-b(s)\Delta^+g(s))}\,\mathrm{d}\,\mu_g(s)\right). \tag{3.7}$$

Taking $t=T$ we get

$$v(T)=\widehat{e}_{-b}(T)\left(v(0)+\int_{[0,T)}\frac{f(s)}{\widehat{e}_{-b}(s)\,(1-b(s)\Delta^+g(s))}\,\mathrm{d}\,\mu_g(s)\right).$$

Therefore, v satisfies the boundary condition in (3.4) if and only if

$$(1-\widehat{e}_{-b}(T))\,v(0)=\widehat{e}_{-b}(T)\int_{[0,T)}\frac{f(s)}{\widehat{e}_{-b}(s)\,(1-b(s)\Delta^+g(s))}\,\mathrm{d}\,\mu_g(s).$$

By (3.5) we arrive at the identity

$$v(0)=\frac{\widehat{e}_{-b}(T)}{1-\widehat{e}_{-b}(T)}\int_{[0,T)}\frac{f(s)}{\widehat{e}_{-b}(s)\,(1-b(s)\Delta^+g(s))}\,\mathrm{d}\,\mu_g(s).$$

Using this expression to substitute $v(0)$ in (3.7) we get (after some calculation) the desired formula for the solution of (3.4). □

Lemma 3.5. *Let (3.2), (3.3) and (3.5) hold and h_b be the Green's function of problem (3.4) given by expression (3.6). Then the function $h_b(\cdot,0)$ is a solution of the problem*

$$\begin{cases}u_g'(t)+b(t)u(t)=0, & \mu_g\text{-a.e. } t\in[0,T],\\ u(0)-u(T)=1.\end{cases}$$

Proof. Since

$$h_b(t,0)=\frac{\widehat{e}_{-b}(t)}{1-\widehat{e}_{-b}(T)},\quad t\in[0,T),$$

it is direct to check that the equation and the boundary conditions hold. □

Now we are ready to write and prove the main theorem of this section. Its proof was inspired by a similar result [3, Theorem 2].

Theorem 3.6. *Let (3.2), (3.3), (3.5) hold, $k \in \mathbb{R}$, and $B : \mathscr{BC}_g([0,T],\mathbb{R}) \to \mathbb{R}$ be a linear bounded operator such that*

$$1 \neq kB(h_b(\cdot,0)). \tag{3.8}$$

Then there exists a unique solution of the problem (3.1) given by

$$u(t) = \int_{[0,T)} H(t,s,k,b)\frac{f(s)}{1-b(s)\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) \tag{3.9}$$

where

$$H(t,s,k,b) = h_b(t,s) + \frac{kh_b(t,0)}{1-kB(h_b(\cdot,0))}B(h_b(\cdot,s)), \quad (t,s)\in[0,T)^2. \tag{3.10}$$

Proof. *Existence:* Let us write

$$u(t) = v(t) + h_b(t,0)\cdot\frac{kB(v)}{1-kB(h_b(\cdot,0))}, \quad t\in[0,T), \tag{3.11}$$

where v is the unique solution of (3.4) and h_b is given by (3.6). Then, according to Lemmas 3.4 and 3.5 we have, for $t \in [0,T)$,

$$\begin{aligned} u'_g(t) &= v'_g(t) + (h_b(\cdot,0))'_g(t)\cdot\frac{kB(v)}{1-kB(h_b(\cdot,0))} \\ &= -b(t)v(t) + f(t) - b(t)h_b(t,0)\cdot\frac{kB(v)}{1-kB(h_b(\cdot,0))} \\ &= -b(t)\left(v(t) + h_b(t,0)\frac{kB(v)}{1-kB(h_b(\cdot,0))}\right) + f(t) \\ &= -b(t)u(t) + f(t). \end{aligned}$$

Since

$$\begin{aligned} &u(0)-u(T) \\ &\quad= v(0) + h_b(0,0)\cdot\frac{kB(v)}{1-kB(h_b(\cdot,0))} - v(T) - h_b(T,0)\cdot\frac{kB(v)}{1-kB(h_b(\cdot,0))} \\ &\quad= (h_b(0,0)-h_b(T,0))\frac{kB(v)}{1-kB(h_b(\cdot,0))} = \frac{kB(v)}{1-kB(h_b(\cdot,0))}, \end{aligned}$$

and

$$\begin{aligned} kB(u) &= kB(v) + kB(h_b(\cdot,0))\cdot\frac{kB(v)}{1-kB(h_b(\cdot,0))} \\ &= \frac{kB(v)\left(1-kB(h_b(\cdot,0)) + kB(h_b(\cdot,0))\right)}{1-kB(h_b(\cdot,0))} = \frac{kB(v)}{1-kB(h_b(\cdot,0))}, \end{aligned}$$

u also satisfies the boundary condition in (3.1). Substituting v given by (3.6) into (3.11), we arrive at formula (3.9).

Uniqueness: Let u, v be two solutions of (3.1). Let us write $w = u - v$. Then w is a solution of the boundary value problem

$$\begin{cases} w'_g(t) + b(t)w(t) = 0, & t\in[0,T], \\ w(0)-w(T) = kB(w). \end{cases} \tag{3.12}$$

Let us prove that problem (3.12) has only a trivial solution. From (3.12) and Theorem 3.2 we get that there exists $w_0 \in \mathbb{R}$ such that

$$w(t) = w_0 \cdot \widehat{e}_{-b}(t), \quad t \in [0, T).$$

This function satisfies the boundary condition if and only if

$$w_0 - w_0 \cdot \widehat{e}_{-b}(T) = kB(w_0 \cdot \widehat{e}_{-b}).$$

The linearity of B implies that this equation is equivalent to

$$w_0 \left(1 - \widehat{e}_{-b}(T) - kB(\widehat{e}_{-b})\right) = 0.$$

From the Assumption (3.5) we get

$$w_0 \left(1 - kB\left(\frac{\widehat{e}_{-b}}{1 - \widehat{e}_{-b}(T)}\right)\right) = 0,$$

which is

$$w_0(1 - kB(h_b(\cdot, 0))) = 0.$$

From (3.8) we get that $w_0 = 0$. Therefore, the problem (3.1) has only one solution. □

4. The integral operator

We will consider the operator

$$\mathscr{T}x(t) = \int_{[0,T)} H(t, s, k, b)\frac{f(s, x(s))}{1 - b(s)\Delta^+ g(s)} \,\mathrm{d}\,\mu_g(s), \tag{4.1}$$

where H is defined in (3.10). Let us recall that $B : \mathscr{BC}_g\left([0, T], \mathbb{R}\right) \to \mathbb{R}$ is a linear bounded functional, i.e., there exists $c \in \mathbb{R}^+$ such that

$$|B(x)| \le c \cdot \|x\|_\infty \tag{4.2}$$

for every $x \in \mathscr{BC}_g\left([0, T], \mathbb{R}\right)$.

Now, we aim to prove that the operator $\mathscr{T}$ maps $\mathscr{BC}_g\left([0, T], \mathbb{R}\right)$ into $\mathscr{BC}_g\left([0, T], \mathbb{R}\right)$ and that it is continuous and compact.

Let us recall some theorems and definitions.

Definition 4.1. [11, Definition 7.1] Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function, $J \subset \mathbb{R}$ be an interval and $X \subset \mathbb{R}^m$. A function $f : J \times X \to \mathbb{R}^n$ is said to be *g–Carathéodory* if the following properties are satisfied:

1. $f(\cdot, x)$ is μ_g–measurable for all $x \in X$,
2. $f(t, \cdot)$ is continuous for μ_g – a.a. $t \in J$,
3. for all $r > 0$ there exists $p_r \in \mathscr{L}^1_g\left(J, [0, \infty)\right)$ such that

$$\|f(t, x)\| \le p_r(t),\ \mu_g - \text{a.a. } t \in J,\ x \in X,\ \|x\| \le r.$$

Lemma 4.2. [12, Lemma 8.2] *Let X be a nonempty subset of $\mathbb{R}^n$ and $f : [0, T] \times X \to \mathbb{R}^n$ a g–Carathéodory function. Then, for every $x \in \mathscr{BC}_g([0, T], X)$, the map $f(\cdot, x(\cdot))$ is in $\mathscr{L}^1_g\left([0, T), \mathbb{R}^n\right)$.*

Now we are ready to prove next theorem.

Theorem 4.3. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a g–Carathéodory function. Then $\mathscr{T}$ defined in (4.1) maps $\mathscr{BC}_g([0,T],\mathbb{R})$ into $\mathscr{BC}_g([0,T],\mathbb{R})$.*

Proof. By Theorem 3.6, for each $x \in \mathscr{BC}_g([0,T],\mathbb{R})$ we have that $f(\cdot,x(\cdot)) \in \mathscr{L}_g^1([0,T),\mathbb{R})$ and $\mathscr{T}x$ is a solution of a differential equation

$$u_g'(t) + b(t)u(t) = f(t,x(t)), \quad \mu_g - \text{a.a. } t \in [0,T).$$

From Theorem 3.2, it follows that $\mathscr{T}x \in \mathscr{AC}_g([0,T],\mathbb{R}) \subset \mathscr{BC}_g([0,T],\mathbb{R})$. □

Remark 4.4. From the proof of Theorem 4.3, we get that $\mathscr{T} : \mathscr{BC}_g([0,T],\mathbb{R}) \to \mathscr{AC}_g([0,T],\mathbb{R})$.

Lemma 4.5. ([21, Lemma 3.1], [24, Section 3]) *Let (3.3) hold. Then*

$$\sum_{t\in D_g} \left|\log\left|1 - b(t)\Delta^+ g(t)\right|\right| < +\infty. \tag{4.3}$$

Lemma 4.6. *Let (3.3) hold. Then H, as defined in (3.10), is bounded.*

Proof. First, we prove that the function h_b defined in (3.6) is bounded. The Lebesgue-Stieltjes integrability of the function

$$\widehat{-b}(t) = \begin{cases} -b(t) & \text{if } t \in [0,T) \setminus D_g, \\ \dfrac{\log|1-b(t)\Delta^+ g(t)|}{\Delta^+ g(t)} & \text{if } t \in [0,T) \cap D_g, \end{cases}$$

follows from Theorem 3.1. Now, we denote

$$M := \sup_{(s,t)\in[0,1]\times[0,1]} \exp\left(\int_{[s,t)} \widehat{-b}(r)\,\mathrm{d}\,\mu_g(r)\right)$$

and obtain the bound

$$\left|\frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)}\right| = \exp\left(\int_{[s,t)} \widehat{-b}(r)\,\mathrm{d}\,\mu_g(r)\right) \le M.$$

Let us denote

$$C_1 := \frac{M}{1-\widehat{e}_{-b}(T)}, \qquad C_2 := \frac{M}{1-\widehat{e}_{-b}(T)} \cdot \widehat{e}_{-b}(T).$$

We have $|h_b(t,s)| \le \max\{C_1, C_2\}$ for every $t \in [0,T]$. Furthermore, given $t, s \in [0,T]$,

$$\begin{aligned} |H(t,s,k,b)| &\le \max\{C_1,C_2\}\left(1 + \frac{|k|}{|1-kB(h_b(\cdot,0))|}\,|B(h_b(\cdot,s))|\right) \\ &\le \max\{C_1,C_2\}\left(1 + \frac{|k|}{|1-kB(h_b(\cdot,0))|} c \cdot \|h_b(\cdot,s)\|_\infty\right) \\ &\le \max\{C_1,C_2\}\left(1 + \frac{|k|}{|1-kB(h_b(\cdot,0))|} c \cdot \max\{C_1,C_2\}\right), \end{aligned}$$

where c is the constant in (4.2) (for instance, the operator norm). Denoting

$$C := \max\{C_1,C_2\}\left(1 + \frac{|k|}{|1-kB(h_b(\cdot,0))|} c \cdot \max\{C_1,C_2\}\right), \tag{4.4}$$

we have that $|H(t,s,k,b)| \leq C$. □

Theorem 4.7. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a g–Carathéodory function. Let (3.3) holds. Then $\mathscr{T}$, defined in (4.1), is continuous.*

Proof. From the convergence of the series (4.3) we conclude that there exists $\delta > 0$ such that

$$\left|1 - b(t)\Delta^+ g(t)\right| > \delta, \quad \forall t \in D_g.$$

Furthermore

$$\left|1 - b(t)\Delta^+ g(t)\right| = 1, \quad \forall t \in [0,T] \setminus D_g.$$

Thus, we can write, for $t \in [0,T]$,

$$\left|1 - b(t)\Delta^+ g(t)\right| \geq \min\{\delta, 1\}. \tag{4.5}$$

Let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence convergent to x in $\mathscr{BC}_g([0,T],\mathbb{R})$. The sequence $\{x_m\}_{m\in\mathbb{N}}$ is then bounded, that is, there exists $r > 0$ such that $\|x_m\|_\infty \leq r$ for all $m \in \mathbb{N}$.
For every $t \in [0,T]$,

$$\begin{aligned}
|\mathscr{T}x_m(t) - \mathscr{T}x(t)| &= \left| \int_{[0,T)} H(t,s,k,b) \frac{f(s,x_m(s)) - f(s,x(s))}{1 - b(s)\Delta^+ g(s)} \, \mathrm{d}\,\mu_g(s) \right| \\
&\leq \int_{[0,T)} |H(t,s,k,b)| \frac{|f(s,x_m(s)) - f(s,x(s))|}{|1 - b(s)\Delta^+ g(s)|} \, \mathrm{d}\,\mu_g(s) \\
&\leq \frac{C}{\min\{\delta,1\}} \cdot \int_{[0,T)} |f(s,x_m(s)) - f(s,x(s))| \, \mathrm{d}\,\mu_g(s).
\end{aligned}$$

Then, since f is a g-Carathéodory function, we have $f(s,x_m(s)) \to f(s,x(s))$ for μ_g-a.a. $s \in [0,T]$ and, moreover, $|f(s,x_m(s))| \leq p_r(s)$ for μ_g-a.a. $s \in [0,T)$ and all $m \in \mathbb{N}$, where $p_r \in \mathscr{L}^1_g([0,T],[0,\infty))$. Due to Lebesgue dominated convergence theorem we have

$$\int_{[0,T)} |f(s,x_m(s)) - f(s,x(s))| \, \mathrm{d}\,\mu_g(s) \to 0 \text{ as } m \to \infty.$$

Therefore, $\mathscr{T}$ is continuous. □

Our task is now to prove the compactness of the operator $\mathscr{T}$. We recall the following useful theorem.

Theorem 4.8. [11, Proposition 5.6] *Let g be a nondecreasing left-continuous function and $S \subset \mathscr{AC}_g([a,b],\mathbb{R}^n)$. Assume that the following conditions are satisfied:*

1. *The set $\{F(a) : F \in S\}$ is bounded.*
2. *There exists $h \in \mathscr{L}^1_g([a,b),[0,+\infty))$ such that*

$$\left\|F'_g(t)\right\| \leq h(t), \ \mu_g - a.a. \ t \in [a,b), \ \textit{for all } F \in S.$$

Then S is a relatively compact subset of $\mathscr{BC}_g([a,b],\mathbb{R}^n)$.

Now we are ready to prove the compactness of $\mathscr{T}$.

Theorem 4.9. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a g-Carathéodory function. Let (3.3) hold. Then the operator $\mathscr{T}$, defined in (4.1), is compact.*

Proof. Let $A\subset\mathscr{BC}_g([0,T],\mathbb{R})$ be such that there exists $r\in\mathbb{R}^+$ such that $\|x\|_\infty\le r, \forall x\in A$. Let us denote by $\mathscr{T}(A)$ the image of A under $\mathscr{T}$. According to Remark 4.4, $\mathscr{T}(A)$ is a subset of $\mathscr{AC}_g([0,T],\mathbb{R})$. We are going to verify the conditions of Theorem 4.8.

1. We consider the set

$$\{\mathscr{T}x(0)\ \ \mathscr{T}x\in\mathscr{T}(A)\}.$$

We have to show that this set is bounded. We can write

$$\begin{aligned}|\mathscr{T}x(0)|&=\left|\int_{[0,T)}H(0,s,k,b)\frac{f(s,x(s))}{1-b(s)\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)\right|\\&\le\int_{[0,T)}\frac{C}{\min\{\delta,1\}}p_r(s)\,\mathrm{d}\,\mu_g(s),\end{aligned}$$

where C is the constant occurring in Lemma 4.6, δ occurs in (4.5) and $p_r\in\mathscr{L}^1_g([0,T),[0,\infty))$ is the function in Definition 4.1.

2. Since $\mathscr{T}x$ is a solution of the differential equation

$$u_g'(t)+b(t)u(t)=f(t,x(t)),$$

we have

$$(\mathscr{T}x)_g'(t)=-b(t)\mathscr{T}x(t)+f(t,x(t))\qquad \mu_g-\text{a.e. on }[0,T).$$

Hence we get

$$\begin{aligned}\left|(\mathscr{T}x)_g'(t)\right|&\le|b(t)|\,|\mathscr{T}x(t)|+|f(t,x(t))|\\&\le|b(t)|\frac{C}{\min\{\delta,1\}}\int_{[0,T)}p_r(s)\,\mathrm{d}\,\mu_g(s)+p_r(t)\end{aligned}$$

for μ_g − a.a. $t\in[0,T)$ and all $x\in A$.

Due to (3.3) and condition (3) of Definition 4.1, the set $\mathscr{T}(A)$ is relatively compact, so the operator $\mathscr{T}$ is compact. □

5. Fixed point index results

In this section we obtain some existence results for the problem (1.1). To do this, we use the operator $\mathscr{T}$ defined in (4.1) together with following assumptions. Let us note, that assumption used in previous sections (i.e. (3.2), (3.3) and (4.2)) are included in following assumptions.

(i) $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is g–Carathéodory function such that

$$f(t,x)\ge 0\quad\text{ for } t\in[0,T],\ x\ge 0. \tag{5.1}$$

(ii) $b\in\mathscr{L}^1_g([0,T),\mathbb{R})$ is such that

$$b(t)>0\quad \mu_g\text{-a.a. } t\in[0,T], \tag{5.2}$$

and

$$1-b(t)\Delta^+g(t)>0\quad\forall t\in[0,T]. \tag{5.3}$$

(iii) $B : \mathscr{BC}_g([0,T],\mathbb{R}) \to \mathbb{R}$ is linear bounded functional (i.e. (4.2)) such that

$$u(t) \geq 0, \forall t \in [0,T] \Rightarrow B(u) \geq 0 \tag{5.4}$$

and we suppose a real number $k \geq 0$,

$$1 - kB(h_b(\cdot,0)) > 0. \tag{5.5}$$

First, we prove some properties of the Green's function $H(t,s,k,b)$.

Lemma 5.1. *Let $b \in \mathscr{L}_g^1([0,T),\mathbb{R})$ be such that (5.2), (5.3) hold. Then $h_b(t,s)$ defined in (3.6) is positive for all $t,s \in [0,T]$.*

Proof. Under Assumptions (5.2) and (5.3), one can see that

$$T_{-b}^- = \{t \in [0,T) \cap D_g \ \ 1 - b(t)\Delta^+ g(t) < 0\}$$

is empty so, according to Theorem 3.1, we have that

$$\widehat{e}_{-b}(t) = \exp\left(\int_{[0,t)} \widehat{-b}(s)\,\mathrm{d}\,\mu_g(s)\right), \quad t \in [0,T]. \tag{5.6}$$

Assumptions (5.2) and (5.3) also guarantee that

$$\widehat{-b}(t) < 0 \quad \forall t \in [0,T],$$

so we can conclude that

$$0 < \widehat{e}_{-b}(t) \leq 1 \quad \forall t \in [0,T] \tag{5.7}$$

and, from (3.6), $h_b(t,s)$ is positive for $t,s \in [0,T]$. □

The next theorem is a direct consequence of Lemma 5.1 and Assumptions (5.4), (5.5).

Theorem 5.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left-continuous function. Let $b \in \mathscr{L}_g^1([0,T),\mathbb{R})$ and (5.2), (5.3), (5.4), (5.5) hold. Then the Green's function H defined in (3.10) is positive on $[0,T] \times [0,T]$.*

Lemma 5.3. *Let the assumptions of Lemma 5.1 be fulfilled. Then, for $t,s \in [0,T]$, it is satisfied that*

$$\frac{\widehat{e}_{-b}(T)}{1 - \widehat{e}_{-b}(T)} \leq h_b(t,s) \leq \frac{1}{1 - \widehat{e}_{-b}(T)}.$$

Proof. We divide the proof into two parts. Let $0 \leq s < t \leq T$. According to (5.6) we get

$$\frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)} = \exp\left(\int_{[s,t)} \widehat{-b}(r)\,\mathrm{d}\,\mu_g(r)\right).$$

Similarly to the proof of Lemma 5.1, we have that

$$\widehat{e}_{-b}(T) \leq \frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)} < 1.$$

From (3.6) it follows that

$$h_b(t,s) = \frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)\left(1 - \widehat{e}_{-b}(T)\right)},$$

so

$$\frac{\widehat{e}_{-b}(T)}{1-\widehat{e}_{-b}(T)} \leq h_b(t,s) < \frac{1}{1-\widehat{e}_{-b}(T)}.$$

Let $0 \leq t \leq s \leq T$. Then,

$$h_b(t,s) = \frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)\left(1-\widehat{e}_{-b}(T)\right)} \cdot \widehat{e}_{-b}(T),$$

and

$$\frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)} = \frac{1}{\exp\left(\int_{[t,s)} \widehat{-b}(r)\,\mathrm{d}\,\mu_g(r)\right)}.$$

This expression can be bounded by

$$1 \leq \frac{\widehat{e}_{-b}(t)}{\widehat{e}_{-b}(s)} \leq \frac{1}{\widehat{e}_{-b}(T)}.$$

Hence, we get

$$\frac{\widehat{e}_{-b}(T)}{1-\widehat{e}_{-b}(T)} \leq h_b(t,s) \leq \frac{\widehat{e}_{-b}(T)}{\widehat{e}_{-b}(T)\left(1-\widehat{e}_{-b}(T)\right)} = \frac{1}{1-\widehat{e}_{-b}(T)}.$$

Thus,

$$\frac{\widehat{e}_{-b}(T)}{1-\widehat{e}_{-b}(T)} \leq h_b(t,s) \leq \frac{1}{1-\widehat{e}_{-b}(T)}$$

for $t, s \in [0,T]$. □

The following theorem follows directly from Lemma 5.3.

Theorem 5.4. *Let the assumptions of Theorem 5.2 be fulfilled. Then, for* $t, s \in [0,T]$*, we have that*

$$\frac{\widehat{e}_{-b}(T)}{1-\widehat{e}_{-b}(T)}\left(1+\frac{k}{1-kB\left(h_b(\cdot,0)\right)}B\left(h_b(\cdot,s)\right)\right) \leq H(t,s,k,b),$$

and

$$H(t,s,k,b) \leq \frac{1}{1-\widehat{e}_{-b}(T)}\left(1+\frac{k}{1-kB\left(h_b(\cdot,0)\right)}B\left(h_b(\cdot,s)\right)\right).$$

Now we consider the cone

$$P := \{x \in \mathscr{BC}_g\left([0,T],\mathbb{R}\right) : x(t) \geq \widehat{e}_{-b}(T)\,\|x\|_\infty,\ t \in [0,T]\}. \tag{5.8}$$

Note that, according to (5.7), we have $\widehat{e}_{-b}(T) \in (0,1)$. The same type of cone can be found in many works, such as [4,9,10,13–15,19,26,27]. We will show that the operator $\mathscr{T}$ maps P into itself.

Theorem 5.5. *Let the assumptions of Theorem 5.2 be fulfilled and consider* P *to be the cone defined in (5.8). If (5.1) holds, then* $\mathscr{T}(P) \subset P$.

Proof. Let $x \in P$. According to the upper estimate of $H(t,s,k,b)$, from Theorem 5.4 and Theorem 5.2, we can write

$$\begin{aligned}\|\mathscr{T}x\|_\infty &= \sup_{t\in[0,T]}\left|\int_{[0,T)} H(t,s,k,b)\,\frac{f(s,x(s))}{1-b(s)\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)\right|\\ &\le \int_{[0,T)} \frac{1}{1-\widehat{e}_{-b}(T)}\left(1+\frac{k}{1-kB(h_b(\cdot,0))}B(h_b(\cdot,s))\right)\\ &\quad\times \frac{f(s,x(s))}{1-b(s)\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s).\end{aligned}$$

Now we can use the lower estimate of $H(t,s,k,b)$ from Theorem 5.4, so

$$\begin{aligned}\mathscr{T}x(t) &= \int_{[0,T)} H(t,s,k,b)\,\frac{f(s,x(s))}{1-b(s)\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)\\ &\ge \int_{[0,T)} \frac{\widehat{e}_{-b}(T)}{1-\widehat{e}_{-b}(T)}\left(1+\frac{k}{1-kB(h_b(\cdot,0))}B(h_b(\cdot,s))\right)\\ &\quad\times \frac{f(s,x(s))}{1-b(s)\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s).\end{aligned}$$

Therefore, for $t \in [0,T]$,

$$\mathscr{T}x(t) \ge \widehat{e}_{-b}(T)\,\|\mathscr{T}x\|_\infty.$$

□

Now we will take same approach as in [9]. We will find assumptions under which the operator $\mathscr{T}$ has one or two fixed points in the cone P. Each fixed point guarantees the existence of the solution of the problem (1.1). This follows from Lemma 4.2 and Theorem 3.6. Indeed: According to Theorem 3.6, the function u in (3.9), i.e.

$$u(t) = \int_{[0,T)} H(t,s,k,b)\frac{f(s)}{1-b(s)\triangle^+g(s)}\,\mathrm{d}\,\mu_g(s),$$

is a solution of the boundary value problem

$$u_g'(t) + b(t)u(t) = f(t), \qquad u(0) - u(T) = kB(u).$$

Let $f : [0,T] \times \mathbb{R} \times \mathbb{R}$ be a g-Carathéodory function. Then for each $x \in \mathscr{BC}_g([0,T))$, we have $f(\cdot,x(\cdot)) \in \mathscr{L}_g^1([0,T))$. And, therefore, the function

$$\mathscr{T}x(t) = \int_{[0,T)} H(t,s,k,b)\frac{f(s,x(s))}{1-b(s)\triangle^+g(s)}\,\mathrm{d}\,\mu_g(s)$$

is a solution of the (again linear) boundary value problem

$$u_g'(t) + b(t)u(t) = f(t,x(t)), \qquad u(0) - u(T) = kB(u).$$

Again, this statement follows from Theorem 3.6. Finally, if moreover $x = \mathscr{T}x$, this function x satisfies

$$x_g'(t) + b(t)x(t) = f(t,x(t)), \qquad x(0) - x(T) = kB(x),$$

i.e. x is a solution of problem (1.1).

Now, according to Lemma 2.6, we give conditions that ensure the fixed point index is 1 or 0 in a certain open subset of P. First, for some $\rho > 0$, we define the sets

$$M^\rho := \left\{ u \in \mathscr{BC}_g\left([0,T],\mathbb{R}\right) : \inf_{t\in[0,T)} u(t) < \rho \right\} \cap P,$$

$$N^\rho := \left\{ u \in \mathscr{BC}_g\left([0,T],\mathbb{R}\right) : \sup_{t\in[0,T)} u(t) < \rho \right\} \cap P.$$

We also define a new function $n : \mathbb{R} \to \mathbb{R}$ by

$$n(\rho) := \frac{\rho}{\widehat{e}_{-b}(T)}, \quad \rho > 0.$$

We are ready to deduce the following inclusions.

Lemma 5.6. *For every $\rho > 0$ we have*

$$N^\rho \subset M^\rho \subset N^{n(\rho)}. \tag{5.9}$$

Proof. Let $u \in N^\rho$ i.e., $\sup_{t\in[0,T)} u(t) < \rho$. Therefore, $\inf_{t\in[0,T)} u(t) < \rho$ and first inclusion is valid.

Now we take $u \in M^\rho$ i.e.,

$$\inf_{t\in[0,T)} u(t) \geq \widehat{e}_{-b}(T) \sup_{t\in[0,T)} u(t)$$

and $\inf_{t\in[0,T)} u(t) < \rho$. Thus, $\sup_{t\in[0,T)} u(t) < \frac{\rho}{\widehat{e}_{-b}(T)} = n(\rho)$ and (5.9) holds. □

Remark 5.7. Notice that, from previous result, we deduce that for any $\rho > 0$ given, the sets M^ρ and N^ρ are bounded. Moreover, from the definition of M^ρ and N^ρ it directly follows that, for every $\rho_1 < \rho_2$, we get

$$M^{\rho_1} \subset M^{\rho_2}, \tag{5.10}$$

$$N^{\rho_1} \subset N^{\rho_2}. \tag{5.11}$$

Lemma 5.8. *Assume that there exists $\rho > 0$ such that*

$$f^\rho \cdot \int_{[0,T)} \frac{1}{1-\widehat{e}_{-b}(T)} \left(1 + \frac{k}{1-kB\left(h_b(\cdot,0)\right)} B\left(h_b(\cdot,s)\right)\right) \mathrm{d}\,\mu_g(s) < 1, \tag{5.12}$$

where

$$f^\rho = \frac{1}{\rho} \sup\left\{ \frac{f\left(t,u(t)\right)}{1-b(t)\Delta^+ g(t)}; t \in [0,T), u \in P, \inf_{t\in[0,T)} u(t) = \rho \right\}.$$

Then $i_P\left(\mathscr{T}, M^\rho\right) = 1$.

Proof. In the proof we will use Lemma 2.6. Assume that there exist $u \in \partial M^\rho$ and $\lambda \geq 1$ such that $\lambda u = \mathscr{T} u$, that is,

$$\lambda u(t) = \int_{[0,T)} H(t,s,k,b) \frac{f(s,u(s))}{1-b(s)\Delta^+ g(s)} \,\mathrm{d}\,\mu_g(s).$$

Taking the infimum on $[0,T)$ on both sides of previous equality, and using upper estimate of H, from Theorem 5.4, we deduce that

$$\begin{aligned}\lambda\rho &= \lambda \inf_{t\in[0,T)} u(t)\\ &\leq \int_{[0,T)} \frac{1}{1-\widehat{e}_{-b}\,(T)}\\ &\quad\times\left(1+\frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)}B\left(h_b\left(\cdot,s\right)\right)\right)\frac{f(s,u(s))}{1-b(s)\Delta^+g(s)}\,\mathrm{d}\,\mu_g\left(s\right)\\ &\leq \rho f^\rho \int_{[0,T)} \frac{1}{1-\widehat{e}_{-b}\,(T)}\\ &\quad\times\left(1+\frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)}B\left(h_b\left(\cdot,s\right)\right)\right)\mathrm{d}\,\mu_g\left(s\right)<\rho.\end{aligned}$$

This is a contradiction. □

Lemma 5.9. *Assume that there exists $\rho>0$ such that*

$$f_\rho\cdot\int_{[0,T)}\frac{\widehat{e}_{-b}\,(T)}{1-\widehat{e}_{-b}\,(T)}\left(1+\frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)}B\left(h_b\left(\cdot,s\right)\right)\right)\mathrm{d}\,\mu_g\left(s\right)>1, \tag{5.13}$$

where

$$f_\rho=\frac{1}{\rho}\inf\left\{\frac{f\left(t,u(t)\right)}{1-b(t)\Delta^+g(t)};t\in[0,T),u\in P,\sup_{t\in[0,T)}u(t)=\rho\right\}.$$

Then $i_P\left(\mathscr{T},N^\rho\right)=0$.

Proof. In the proof we will again use Lemma 2.6. Let $p\in P$, such that $p\neq 0$. Assume that there exist $u\in\partial N^\rho$ and $\lambda>0$ such that $u=\mathscr{T}u+\lambda p$, that is,

$$u(t)=\int_{[0,T)}H(t,s,k,b)\frac{f(s,u(s))}{1-b(s)\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)+\lambda p(t).$$

Taking the supremum on $[0,T)$ on both sides of previous equality, and using the lower estimate of H, from Theorem 5.4 we arrive at the following contradiction

$$\begin{aligned}\rho &= \sup_{t\in[0,T)} u(t)\\ &\geq \int_{[0,T)}\frac{\widehat{e}_{-b}\,(T)}{1-\widehat{e}_{-b}\,(T)}\left(1+\frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)}B\left(h_b\left(\cdot,s\right)\right)\right)\\ &\quad\times\frac{f(s,u(s))}{1-b(s)\Delta^+g(s)}\,\mathrm{d}\,\mu_g\left(s\right)\\ &\geq \rho f^\rho\int_{[0,T)}\frac{\widehat{e}_{-b}\,(T)}{1-\widehat{e}_{-b}\,(T)}\left(1+\frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)}B\left(h_b\left(\cdot,s\right)\right)\right)\mathrm{d}\,\mu_g\left(s\right)\\ &>\rho.\end{aligned}$$

□

Due to the properties of the fixed point index from Lemma 2.6 and the above results we get the following theorem which is the main result of the paper.

Theorem 5.10. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left-continuous function. Let $b \in \mathscr{L}_g^1([0,T), \mathbb{R})$ and (5.2), (5.3), (5.4), (5.5) hold. Then, problem (1.1) has at least one nontrivial solution in P if one of the following conditions hold:*

(S_1) *there exist $\rho_1, \rho_2 > 0$ with $\rho_2 > \rho_1$ such that (5.13) holds for ρ_1 and (5.12) holds for ρ_2,*

(S_2) *there exist $\rho_1, \rho_2 > 0$ with $\rho_2 > n(\rho_1)$ such that (5.12) holds for ρ_1 and (5.13) holds for ρ_2.*

Problem (1.1) has at least two nontrivial solutions in P if one of the following conditions hold:

(S_3) *there exist $\rho_1, \rho_2, \rho_3 > 0$ with $\rho_2 > \rho_1$ and $\rho_3 > n(\rho_2)$ such that (5.13) holds for ρ_1, (5.12) holds for ρ_2 and (5.13) holds for ρ_3,*

(S_4) *there exist $\rho_1, \rho_2, \rho_3 > 0$ with $\rho_2 > n(\rho_1)$ and $\rho_3 > \rho_2$ such that (5.12) holds for ρ_1, (5.13) holds for ρ_2 and (5.12) holds for ρ_3.*

Proof. First we point out that u is a solution of the problem (1.1) if and only if it is a fixed point of the operator $\mathscr{T}$. We give the proof for case (S_3). Proofs of the other cases are similar.

Let (S_3) hold. From (5.9) and (5.11) we get

$$M^{\rho_2} \subset N^{n(\rho_2)} \subset N^{\rho_3}.$$

From (5.9) and (5.10) it follows

$$N^{\rho_1} \subset M^{\rho_1} \subset M^{\rho_2}.$$

So

$$N^{\rho_1} \subset M^{\rho_2} \subset N^{\rho_3}.$$

Therefore, from Lemma 5.9, it follows that $i_P(\mathscr{T}, N^{\rho_1}) = 0$ and $i_P(\mathscr{T}, N^{\rho_3}) = 0$ and, from Lemma 5.8, it follows that $i_P(\mathscr{T}, M^{\rho_2}) = 1$.

We obtain the existence of solution by applying Lemma 2.6. □

6. Example

In this section we apply the theory developed in the previous ones to study whether a given problem has a nontrivial solution. We will be considering the following problem

$$\begin{cases} u_g'(t) + bu(t) = |u(t)|^\alpha, & t \in [0,2], \\ u(0) = u(2) + k \int_{[0,2)} u(t) \, \mathrm{d}\mu_g(t), \end{cases} \tag{6.1}$$

where $b \in (0,1)$, $\alpha > 1$,

$$g(t) = \begin{cases} t, & t \in [0,1], \\ t+1, & t \in (1,2] \end{cases}$$

and $k \in [0, k_0)$, where

$$k_0 = \frac{1-(1-b)\mathrm{e}^{-2b}}{\int_{[0,2)} \widehat{e}_{-b}(s)\,\mathrm{d}\,\mu_g(s)}.$$

Note that

$$\widehat{e}_{-b}(t) = \begin{cases} \mathrm{e}^{-bt}, & t \in [0,1], \\ (1-b)\mathrm{e}^{-bt}, & t \in (1,2]. \end{cases}$$

It is clear that g is a nondecreasing and left continuous function such that

$$D_g = \{1\} \quad \text{and} \quad \Delta^+ g(1) = 1.$$

We will prove that there exists at least one nontrivial solution of (6.1). We start by verifying the Assumptions (5.1)–(5.5). Evidently, $|u|^\alpha \geq 0$ whenever $u \geq 0$, so (5.1) holds. Note that $b(t) = b$ is a constant function with $b \in (0,1)$, so (5.2) holds as well. Also, we have that

$$1 - b(t)\Delta^+ g(t) = \begin{cases} 1, & t \in [0,2] \setminus \{1\}, \\ 1-b, & t \in \{1\}. \end{cases}$$

Therefore, (5.3) is satisfied. Operator B is defined by

$$B(u) = \int_{[0,2)} u(t)\,\mathrm{d}\,\mu_g(t), \quad u \in \mathscr{BC}_g([0,2], \mathbb{R}).$$

It is obvious that (5.4) holds. We also have that $k \in [0, k_0)$, so

$$1 > k\frac{\int_{[0,2)} \widehat{e}_{-b}(s)\,\mathrm{d}\,\mu_g(s)}{1-(1-b)\mathrm{e}^{-2b}}.$$

Therefore (5.5) is satisfied.

Now we work with the cone

$$P := \{u \in \mathscr{BC}_g([0,2], \mathbb{R}) : u(t) \geq (1-b)\mathrm{e}^{-2b}\,\|u\|_\infty, \ t \in [0,2]\}.$$

We take a constant ρ_1 such that

$$0 < \rho_1 < \left(\frac{(1-b)^{\alpha+1}\,\mathrm{e}^{-2\alpha b}}{I}\right)^{\frac{1}{\alpha-1}},$$

where

$$I = \int_{[0,2)} \frac{1}{1-(1-b)\,\mathrm{e}^{-2b}} \times \left(1 + \frac{k}{1-k\int_{[0,2)} h_b(t,0)\,\mathrm{d}\,\mu_g(t)} \int_{[0,2)} h_b(t,s)\,\mathrm{d}\,\mu_g(t)\right) \mathrm{d}\,\mu_g(s).$$

Observe that $I > 0$. Such ρ_1 meets Assumption (5.12). Observe as well that, due to the definition of the cone P, we get

$$u(t) \leq \frac{\inf_{s\in[0,2]} u(s)}{(1-b)\mathrm{e}^{-2b}}.$$

As

$$\frac{f(t,u(t))}{1-b(t\Delta^+g(t))}=\begin{cases}|u(t)|^\alpha & t\in[0,2]\setminus\{1\},\\ \frac{|u(t)|^\alpha}{1-b}, & t\in\{1\},\end{cases}$$

we have that

$$f^{\rho_1}\leq\max\left\{\frac{\rho_1^{\alpha-1}}{((1-b)\mathrm{e}^{-2b})^\alpha},\frac{\rho_1^{\alpha-1}}{((1-b)\mathrm{e}^{-2b})^\alpha}\cdot\frac{1}{1-b}\right\}=\frac{\rho_1^{\alpha-1}}{((1-b)\mathrm{e}^{-2b})^\alpha}\cdot\frac{1}{1-b}.$$

Therefore, Assumption (5.12) is valid for ρ_1.

Now we take a constant ρ_2 such that

$$\rho_2>\left(\frac{1}{(1-b)^{\alpha+1}\,\mathrm{e}^{-2b(\alpha+1)}\cdot I}\right)^{\frac{1}{\alpha-1}}.$$

The constant ρ_2 then meets Assumption (5.13). Indeed, due to the definition of the cone P, we have that

$$u(t)\geq(1-b)\mathrm{e}^{-2b}\sup_{s\in[0,2]}u(s).$$

In this case we get

$$f_{\rho_2}\geq\min\left\{\left((1-b)\mathrm{e}^{-2b}\right)^\alpha\rho_2^{\alpha-1},\frac{\left((1-b)\mathrm{e}^{-2b}\right)^\alpha\rho_2^{\alpha-1}}{1-b}\right\}=\left((1-b)\mathrm{e}^{-2b}\right)^\alpha\rho_2^{\alpha-1},$$

so (5.13) holds for ρ_2. Besides,

$$n(\rho_1)=\frac{\rho_1}{(1-b)\mathrm{e}^{-2b}}<\left(\frac{(1-b)^{\alpha+1}\,\mathrm{e}^{-2\alpha b}}{I}\right)^{\frac{1}{\alpha-1}}.$$

$$\frac{1}{(1-b)\mathrm{e}^{-2b}}<\left(\frac{1}{(1-b)^{\alpha+1}\,\mathrm{e}^{-2b(\alpha+1)}\cdot I}\right)^{\frac{1}{\alpha-1}}<\rho_2.$$

Thus, all assumptions in (S_2) in Theorem 5.10 hold and problem (6.1) has at least one nontrivial solution in P.

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Declarations

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