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A consistent test of equality of distributions for Hilbert-valued random elements

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ABSTRACT

Two independent random elements taking values in a separable Hilbert space are considered. The aim is to develop a test with bootstrap calibration to check whether they have the same distribution or not. A transformation of both random elements into a new separable Hilbert space is considered so that the equality of expectations of the transformed random elements is equivalent to the equality of distributions. Thus, a bootstrap test procedure to check the equality of means can be used in order to solve the original problem. It will be shown that both the asymptotic and bootstrap approaches proposed are asymptotically correct and consistent. The results can be applied, for example, in functional data analysis. In practice, the test can be solved with simple operations in the original space without applying the mentioned transformation, which is used only to guarantee the theoretical results. Empirical results and comparisons with related methods support and complement the theory.

1. Introduction

The classical problem of testing the equality of distributions of two independent random elements is considered. This topic has been studied extensively for the case of scalar random data (e.g., Gibbons and Chakraborti [1], Thas [2]). The situation involving functional random data or statistical objects in general, has attracted less attention until now. Functional random elements are frequently assumed to take values in a separable Hilbert space. In this paper we will work in this context although, as it is remarked later, this assumption could be relaxed to spaces of negative type. Results developed for Hilbertian random elements are actually applicable for a variety of standard and non-standard data, such as functional or (fuzzy) set-valued data in a unified way (e.g., Boente et al. [3], Lin et al. [4], González-Rodríguez et al. [5,6]). Three main related approaches have been considered for the functional data case in this setting, namely:

1. Comparison of functional means by using, e.g., principal component approaches (e.g., Horváth and Rice [7], Ghale-Joogh and Hosseini-Nasab [8]), adapting the ideas of the $F-$ test to the functional context (e.g., Lee and Follen [9], Zhang and Liang [10], González-Rodríguez et al. [5], Cuevas et al. [11], Górecki and Smaga [12]).
2. Comparison of covariance structures (e.g., Fremdt et al. [13], Boente et al. [14], Guo et al. [15,16,17]).

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3. Comparison of the structure of the distributions in various ways. Tests based on smoothing discrete data observed in potential functional data are developed in Hall and Keilegom [18], and similarly, in Estévez-Pérez and Vilar [19] or Pomann et al. [20]. Empirical processes have been used in Bárcenas et al. [21]. An L_2−type criterion based on empirical distribution functions is used in Jiang et al. [22]. Some Cramér–Von Mises-type statistics adapted to the functional case calibrated with permutation resampling are employed in Bugni and Horowitz [23]. More recently, a new test of comparison of distributions for sparse functional data, using the energy distance ideas of Székely and Rizzo [24], was presented in Zhu and Wang [25].

In this paper, the testing problem of equality of distributions of two independent Hilbert-valued random elements is considered. Using the terminology of Székely and Rizzo [26], given in their review, based on the concept of "energy statistic", the class of derived statistics based on distances is now generalized from the euclidean vectorial case to the Hilbert space context. Under the assumption of separability, the Hilbert space is of strong negative type (see, for instance, Lyons [27]). In this way, the corresponding energy distance can be seen as a L^2 distance between the means of the transformed original random elements, now evaluated in other (embedded) Hilbert space.

Our proposal is based on relating the original distributions to the expected values of the transformation. The equality of expectations of the transformed random elements is equivalent to the equality of distributions of the original ones. Based on this fact, and using the known consistency of the bootstrap procedures for checking the equality of means, a general theory of consistent bootstrap for the calibration of the energy statistic distribution, for the use in the original test problem, is given. In particular, the consistency of two types of resampling: the exchangeable weighted and the wild bootstrap, is proven in the paper. Under our knowledge there are no results of consistency of the bootstrap using this kind of embedding arguments.

The rest of the manuscript is organized as follows. In Section 2, the hypothesis test will be introduced together with the main notation regarding energy statistics. The test will be equivalently rewritten in terms of the energy distance of the pair of variables under analysis. Section 3 will be devoted to the asymptotic analysis of the proposed statistic, including a study of local alternatives. An asymptotic testing procedure will be established. In Section 4, several bootstrap procedures will be proposed proving their consistency and asymptotic correctness. In Section 5, studentized versions of the (bootstrap) statistics are analyzed. Some simulations and comparative studies are presented in Section 6. Section 7 describes some illustrative real data examples. Finally, the conclusions are summarized in Section 8. The proofs are in the appendix.

2. Test of equality of distributions in separable Hilbert spaces

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a separable Hilbert space with associated norm $\|\cdot\|_{\mathcal{H}}$ and consider a probability space (Ω, σ, P). Whenever it is needed, $\mathcal{H}^k$ will denote the direct sum $\mathcal{H} \bigoplus \overset{k}{\cdots} \bigoplus \mathcal{H}$ for any $k \in \mathbb{N}$.

Let $X, Y : (\Omega, \sigma, P) \to \mathcal{H}$ be two independent $\mathcal{H}$-valued random elements and denote by P_X and P_Y their corresponding associated probability measures. These probability measures are usually denoted as μ_X and μ_Y in the context of energy statistics because they will be related to general signed measures μ. However, this notation is not used here to avoid confusion with the standard notation for expected values in functional data analysis.

The aim is to test the equality of distributions of X and Y, that is,

$$\begin{cases} H_0 : P_X = P_Y \\ H_1 : P_X \neq P_Y \end{cases}. \tag{1}$$

This test can be solved using a random projections technique (see Cuesta-Albertos et al. [28,29]), but to achieve this goal, a quite restrictive condition is assumed: the probability measures must be determined by their moments, and all the moments must be finite. In contrast, the approach followed in this paper requires only the first moments to be finite; that is, it is assumed that

$$E\left(\|X\|_{\mathcal{H}}\right) < \infty, \quad E\left(\|Y\|_{\mathcal{H}}\right) < \infty.$$

2.1. Background of embedding with energy distances

Some additional notation and concepts regarding the so-called energy statistics are needed. The notation in Lyons [27], with a slight modification, will be mainly used. Let $M(\mathcal{H})$ be the set of finite signed Borel measures on $\mathcal{H}$ and consider the subset of measures with finite first order moment,

$$M_1(\mathcal{H}) = \left\{ \mu \in M(\mathcal{H})\text{mid} \int \|x\|_{\mathcal{H}} d|\mu|(x) < \infty \right\}.$$

In the same way, let $M_1^P(\mathcal{H}) \subset M_1(\mathcal{H})$ be the set of probability measures with finite first order moment. Finally let the mapping $D : M_1(\mathcal{H}) \to \mathbb{R}$ be given by,

$$D(\mu) = \int \|x - x'\|_{\mathcal{H}} d\mu^2(x, x'), \tag{2}$$

for all $\mu \in M_1(\mathcal{H})$. Clearly, $D\left(P_X - P_Y\right)$ can be computed based on the expectation of pairwise difference of random elements as follows,

$$D\left(P_X - P_Y\right) = E\left(\|X_1 - X_2\|_{\mathcal{H}}\right) + E\left(\|Y_1 - Y_2\|_{\mathcal{H}}\right) - 2E\left(\|X_1 - Y_2\|_{\mathcal{H}}\right), \tag{3}$$

being X_1, X_2 i.i.d. $\mathcal{H}$-valued random elements with the same distribution as X, i.e. with probability measure P_X, and Y_1, Y_2 i.i.d. $\mathcal{H}$-valued random elements with the same distribution as Y. This is the idea underlying the "energy statistics": to rewrite complex comparisons (of distributions, in this particular case) in terms of some appropriate functions of distances between the involved random elements (or "statistical observations", as it is expressed in Székely and Rizzo [26]).

A metric space $(\mathcal{X}, d)$ is of negative type if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{X}$ and all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 0$,

$$\sum_{i,j} \alpha_i \alpha_j d(x_i, x_j) \leq 0.$$

Different examples of spaces of negative type can be found in Meckes [30] including, for instance, L_p spaces with $p \in (0, 2]$. Particularly, if $\mathcal{X}$ is of negative type, then

$$D(P_1 - P_2) \leq 0, \tag{4}$$

for all $P_1, P_2 \in M_1^P(\mathcal{X})$. According to the Schoenberg Embedding Theorem, Schoenberg [31,32], the metric space $(\mathcal{X}, d)$ is of negative type if, and only if, there exists a Hilbert space $\mathcal{G}$ and an embedding $\phi : \mathcal{X} \to \mathcal{G}$ so that

$$d(x_1, x_2) = \|\phi(x_1) - \phi(x_2)\|_{\mathcal{G}}^2,$$

for all $x_1, x_2 \in \mathcal{X}$. If, in addition, the equality in (4) holds if, and only if, $P_1 = P_2$, then it is said that $\mathcal{X}$ is of strong negative type. It is known that every separable Hilbert space is of strong negative type, as it is stated, for instance, in Theorem 3.16. of Lyons [27] (see also Lyons [33]). Consequently, given the separable Hilbert space $\mathcal{H}$, we can consider another separable Hilbert space $\mathcal{G}$ and a mapping $\phi_{\mathcal{H}} : \mathcal{H} \to \mathcal{G}$ so that

$$\|h_1 - h_2\|_{\mathcal{H}} = \|\phi_{\mathcal{H}}(h_1) - \phi_{\mathcal{H}}(h_2)\|_{\mathcal{G}}^2, \tag{5}$$

for all $h_1, h_2 \in \mathcal{H}$. Without loss of generality it will be assumed that $\phi_{\mathcal{H}}(0) = 0$ (otherwise, consider $\phi'(h) = \phi_{\mathcal{H}}(h) - \phi_{\mathcal{H}}(0)$, so (5) also holds for this mapping). This particular property of the "centered" embedding will be involved in the moment assumptions.

Finally, following Székely and Rizzo [26], the energy distance of (X, Y) is defined as

$$\varepsilon(X, Y) = -D(P_X - P_Y). \tag{6}$$

Note that the energy distance can be rewritten in terms of the embedding by combining (3) and some algebra as follows,

$$\varepsilon(X, Y) = 2E\left(\|X_1 - Y_2\|_{\mathcal{H}}\right) - E\left(\|X_1 - X_2\|_{\mathcal{H}}\right) - E\left(\|Y_1 - Y_2\|_{\mathcal{H}}\right) = 2\|E(\phi_{\mathcal{H}}(X)) - E(\phi_{\mathcal{H}}(Y))\|_{\mathcal{G}}^2. \tag{7}$$

Thus, $\varepsilon(X, Y) \geq 0$, and the equality holds if, and only if, X and Y are identically distributed.

2.2. Hypothesis test statistic

According to the previous developments, the Test (1) is equivalent to the following energy-based test,

$$\begin{cases} H_0 : \varepsilon(X, Y) = 0 \\ H_1 : \varepsilon(X, Y) \neq 0 \end{cases},$$

or, equivalently,

$$\begin{cases} H_0 : E(\phi_{\mathcal{H}}(X)) = E(\phi_{\mathcal{H}}(Y)) \\ H_1 : E(\phi_{\mathcal{H}}(X)) \neq E(\phi_{\mathcal{H}}(Y)) \end{cases}.$$

Thus, we can apply a general test for equality of expectations of the transformed random elements, which are independent random elements taking values in a separable Hilbert space. The procedure presented in González-Rodríguez et al. [5] will be considered and extended to accommodate a large variety of bootstrap methods by considering the developments in González-Rodríguez and Colubi [34]. The consistency and asymptotic correctness of the proposed bootstrap methods will be proven. The key point is to express all the developments in terms of the original variables of the initial space, as neither the Hilbert space $\mathcal{G}$, nor the concrete mapping $\phi_{\mathcal{H}}$, needs to be known.

To this aim, let $\{X_i\}_{i=1}^n$ be i.i.d. random elements with the same distribution as X and let $\{Y_j\}_{j=1}^m$ be i.i.d random elements with the same distribution as Y. The testing statistic based on the energy statistics, that is, the empirical estimation of the energy distance, is

$$T_n = n\left(\frac{2}{nm}\sum_{i,j}\|X_i - Y_j\|_{\mathcal{H}} - \frac{1}{nn}\sum_{i,j}\|X_i - X_j\|_{\mathcal{H}} - \frac{1}{mm}\sum_{i,j}\|Y_i - Y_j\|_{\mathcal{H}}\right),$$

that, equivalently, can be expressed in the image space by using the embedding (see (7)) as follows:

$$T_n = 2\left\|\sqrt{n}\left(\overline{\phi_{\mathcal{H}}(X)} - \overline{\phi_{\mathcal{H}}(Y)}\right)\right\|_{\mathcal{G}}^2, \tag{8}$$

where $\overline{\phi_{\mathcal{H}}(X)}$ and $\overline{\phi_{\mathcal{H}}(Y)}$ are the corresponding sample means.

3. Asymptotic analysis

Centering the transformations in the core of (8), a statistics suitable to handle for the asymptotic theory under H_0 is obtained, namely,

$$\tau_n = \sqrt{n}\left(\overline{\phi_{\mathcal{H}}(X) - E(\phi_{\mathcal{H}}(X))}\right) - \sqrt{\frac{n}{m}}\sqrt{m}\left(\overline{\phi_{\mathcal{H}}(Y) - E(\phi_{\mathcal{H}}(Y))}\right).$$

If H_0 holds, then $2\|\tau_n\|_{\mathcal{G}}^2$ and T_n match. Consequently, the following result establishes the asymptotic distribution of the statistic under the null hypothesis. For the sake of simplicity, it will be assumed that $n/m \to 1$. Nevertheless, the results can be straightforwardly extended for the case $n/m \to q \in (0, \infty)$.

Theorem 1. *If $E(\|X\|_{\mathcal{H}}) < \infty$, $E(\|Y\|_{\mathcal{H}}) < \infty$ and $n/m \to 1$ as $n \to \infty$, then*

$$\tau_n \xrightarrow{\mathcal{L}} Z_1 - Z_2,$$

where $Z = (Z_1, Z_2)$ is a $\mathcal{G}^2$-valued Gaussian random element with the same covariance operator as $\left(\phi_{\mathcal{H}}(X), \phi_{\mathcal{H}}(Y)\right)$.

From now on, given G any $\mathcal{G}$-valued random element, being $\mathcal{G}$ an arbitrary separable Hilbert space, Z_G will denote a Gaussian $\mathcal{G}$-valued random element with the same covariance operator as G. In view of the previous theorem, an asymptotic testing procedure can be established. The following result guarantees the asymptotic correctness and consistency of the test and shows its behavior under local alternatives.

Theorem 2. *If $E(\|X\|_{\mathcal{H}}) < \infty$, $E(\|Y\|_{\mathcal{H}}) < \infty$ and $n/m \to 1$ as $n \to \infty$, the test that rejects H_0 in (1) whenever $T_n > c_\alpha$, where c_α is the $(1-\alpha)$-quantile of the distribution $2\|Z_1 - Z_2\|_{\mathcal{G}}^2$ with $Z = (Z_1, Z_2)$ defined as in Theorem 1, is asymptotically correct and consistent at a significance level α.*

In addition, consider as a sequence of Pitman local alternatives (Pitman [35])

$$P_X^{(n)} = \frac{\delta_n}{\sqrt{n}} P_U + \left(1 - \frac{\delta_n}{\sqrt{n}}\right) P_Y,$$

for some $\mathcal{H}$-valued random element U so that $P_U \neq P_Y$ and $0 \le \delta_n/\sqrt{n} \le 1$. If $\delta_n \to \infty$ and $\frac{\delta_n}{\sqrt{n}} \to 0$ as n tends to infinity, then

$$\lim_{n\to\infty} P(T_n > c) = 1,$$

for any $c > 0$.

From an applied viewpoint, the asymptotic test established in Theorem 2 cannot be directly used in practice as the covariance operator of $\left(\phi_{\mathcal{H}}(X), \phi_{\mathcal{H}}(Y)\right)$ is unknown. As a solution, we could pool all the available data of X and Y together, extract large samples of size $n' \gg n$ and $m' \gg m$ at random from the pooled data and compute the value of the statistic $T_{n'}$. By repeating the procedure a large number of times B, the empirical distribution $\hat{F}_B$ of the generated data is an approximation of the distribution of $2\|\hat{Z}_1 - \hat{Z}_2\|_{\mathcal{G}}^2$ in virtue of the Monte Carlo method and Theorem 1, being $\hat{Z} = (\hat{Z}_1, \hat{Z}_2)$ a Gaussian $\mathcal{H}^2$-valued random element with covariance operator equal to the empirical one derived from the pooled data. Under the null hypothesis, $\hat{Z}$ is an approximation of Z. Thus, the corresponding quantile of $\hat{F}_n$ could be used to solve the test. Nevertheless, the asymptotic distribution $2\|Z_1 - Z_2\|_{\mathcal{G}}^2$ typically is a bad approximation of the sampling one for moderate (even large) sample sizes when the random elements take values in high dimensional spaces, whereas bootstrap resampling is a better option.

In the next section, several bootstrap procedures will be proposed, and the consistency of such procedures will be proved.

4. Bootstrap procedures

The aim is to mimic the distribution of the $\mathcal{H}$-valued random element τ_n, as under the null hypothesis $2\|\tau_n\|_{\mathcal{G}}^2 = T_n$. Firstly, the basic notation regarding bootstrap approaches for the sample means of i.i.d. random elements taking values in an arbitrary separable Hilbert space (see González-Rodríguez and Colubi [34]) will be introduced. With respect to the Exchangeable Weighted bootstrap, two arrays of random variables $\{W_{ni}^\star\}_{i=1}^n$ and $\{V_{nj}^\star\}_{j=1}^m$ are considered. These weights must verify the regularity conditions introduced in Praestgaard and Wellner [36], namely:

- A1 $(W_{n1}^\star, \dots, W_{nn}^\star)$ are exchangeable,
- A2 $W_{nj}^\star \ge 0\ \forall j$ and $\sum_{j=1}^n W_{nj}^\star = n\ \forall n \in \mathbb{N}$,
- A3 $\sup_n \int_0^\infty (P(|W_{n1}^\star| > t))^{1/2} dt < \infty$,
- A4 $\lim_{\lambda\to 0} \limsup_{n\to\infty} \sup_{t\ge\lambda} t^2 P(W_{n1}^\star \ge t) = 0$,
- A5 $(1/n)\sum_{j=1}^n (W_{nj}^\star - 1)^2 \to c_W^2 > 0$ in probability,

and analogously for $V^{\star}_{ni}$. The normalizing factors c_W and c_V must match as, in general, the same bootstrap procedure will be considered both for X and Y variables. Let $c = c_W = c_V$ be the common value.

To simplify the notation let us consider the centered weights

$$w^{\star}_{ni} = \left(\frac{W^{\star}_{ni} - 1}{n}\right), \quad v^{\star}_{mj} = \left(\frac{V^{\star}_{mj} - 1}{m}\right),$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.

The Exchangeable Weighted Bootstrap statistic is defined as follows,

$$\mathcal{T}^{E*}_n = 2\frac{n}{c^2}\sum_{i,j} w^{\star}_{ni} v^{\star}_{mj} \|X_i - Y_j\|_{\mathcal{H}} - \frac{n}{c^2}\sum_{i,j} w^{\star}_{ni} w^{\star}_{nj} \|X_i - X_j\|_{\mathcal{H}} - \frac{n}{c^2}\sum_{i,j} v^{\star}_{mi} v^{\star}_{mj} \|Y_i - Y_j\|_{\mathcal{H}}. \tag{9}$$

As mentioned in Section 2, the embedding $\phi_{\mathcal{H}}$ and the image separable Hilbert space $\mathcal{G}$ are not necessarily known. Consequently, to make the procedures applicable in practice, the bootstrap statistic has been expressed involving just operations in the original space $\mathcal{H}$. For the Exchangeable Weighted bootstrap, an equivalent formulation of the proposed statistic in terms of a simple pairwise comparison of the sample means of the transformed random elements can be found. Thus, the asymptotic distribution can be easily derived as it is established in the following theorem.

Theorem 3. *The Exchangeable Weighted Bootstrap statistic can be rewritten in terms of the embedding as follows,*

$$T^{E\star}_n = 2\left\| \frac{\sqrt{n}}{c}\sum_{i=1}^{n} w^{\star}_{ni}\phi_{\mathcal{H}}(X_i) - \frac{\sqrt{n}}{c}\sum_{j=1}^{m} v^{\star}_{mj}\phi_{\mathcal{H}}(Y_j)\right\|^2_{\mathcal{G}}. \tag{10}$$

Therefore, if $E(\|X\|_{\mathcal{H}}) < \infty$, $E(\|Y\|_{\mathcal{H}}) < \infty$ and $n/m \to 1$ as $n \to \infty$ then,

$$T^{E\star}_n \xrightarrow{\mathcal{L}} 2\,\|Z_1 - Z_2\|^2_{\mathcal{G}} \quad P\text{-a.s.}$$

In the same way, concerning the Wild Bootstrap, two sequences of i.i.d. random variables $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ having zero mean, variance 1 and

$$\int_0^{\infty} (P(|\xi_1| > t)^{1/2})dt < \infty,$$

(analogously for η_j) are considered.

As for the Exchangeable Weighted bootstrap, given $n, m \in \mathbb{N}$, consider the centered multipliers defined as

$$\xi^{\star}_i = \frac{\xi_i - \overline{\xi}}{n}, \quad \eta^{\star}_j = \frac{\eta_j - \overline{\eta}}{m},$$

for all $i, j \in \mathbb{N}$. Note that for the centered multipliers the following result holds,

Lemma 1. *If $X_1, X_2, \dots$ is a sequence of i.i.d. random elements in a separable Hilbert space having finite second order moment then*

$$\sqrt{n}\sum_{i=1}^{n} \xi^{\star}_i X_i \xrightarrow{\mathcal{L}} Z_{X_1} \quad P\text{-a.s.}$$

The Wild Bootstrap statistic in the original space $\mathcal{H}$ is defined as follows,

$$T^{W\star}_n = n\left(2\sum_{i,j} \xi^{\star}_i \eta^{\star}_j \|X_i - Y_j\| - \sum_{i,j} \xi^{\star}_i \xi^{\star}_j \|X_i - X_j\| - \sum_{i,j} \eta^{\star}_i \eta^{\star}_j \|Y_i - Y_j\|\right).$$

The following theorem states its weak convergence.

Theorem 4. *The Wild Bootstrap statistic has the following equivalent expansion in terms of the embedding,*

$$T^{W\star}_n = 2\left\| \sqrt{n}\sum_{i=1}^{n} \xi^{\star}_i \phi_{\mathcal{H}}(X_i) - \sqrt{m}\sum_{j=1}^{m} \eta^{\star}_j \phi_{\mathcal{H}}(Y_j)\right\|^2_{\mathcal{G}}. \tag{11}$$

Therefore, if $E(\|X\|_{\mathcal{H}}) < \infty$, $E(\|Y\|_{\mathcal{H}}) < \infty$ and $n/m \to 1$ as $n \to \infty$, then

$$T^{W\star}_n \xrightarrow{\mathcal{L}} 2\,\|Z_1 - Z_2\|^2_{\mathcal{G}} \quad P\text{-a.s.}$$

If $n = m$, then it is possible to select $\eta^{\star}_i = \xi^{\star}_i$ for all $i = 1, \dots, n$ and the same asymptotic result holds.

In order to conclude the analysis, the corresponding bootstrap testing procedures are stated in the following corollary.

Corollary 1. *If $E(\|X\|_{\mathcal{H}}) < \infty$, $E(\|Y\|_{\mathcal{H}}) < \infty$ and $n/m \to 1$ as $n \to \infty$, the test that rejects H_0 in (1) whenever $T_n > c^n_{\alpha}$, where c^n_{α} is the $(1 - \alpha)$-quantile of the distribution of $T^{E\star}_n$ or $T^{W\star}_n$, is asymptotically correct and consistent at a significance level α.*

As usual, the quantile c^n_{α} is approximated by the Monte Carlo method.

5. Studentized version of the test

The considered statistics can be studentized as it is done in the ANOVA test (see González-Rodríguez et al. [5]). To this aim, an appropriate estimate of

$$\mathrm{Var}(\tau_n) = \mathrm{Var}(\phi_{\mathcal{H}}(X)) + \frac{n}{m}\mathrm{Var}(\phi_{\mathcal{H}}(Y)) \tag{12}$$

can be introduced as the denominator of the different statistics. Let us define,

$$\hat{V}_X = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|X_i - X_j\right\|_{\mathcal{H}}. \tag{13}$$

Similarly, for the Exchangeable bootstrap,

$$\hat{V}_X^{E\star} = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} W_{ni}^{\star}W_{nj}^{\star}\left\|X_i - X_j\right\|_{\mathcal{H}}, \tag{14}$$

or, equivalently,

$$\hat{V}_X^{E\star} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(w_{ni}^{\star} + \frac{1}{n}\right)\left(w_{nj}^{\star} + \frac{1}{n}\right)\left\|X_i - X_j\right\|_{\mathcal{H}}.$$

Note that the Wild Bootstrap has been designed as a tool for assessing the weak convergence of a sequence of random elements (see Ledoux and Talagrand [37], Theorem 10.4 – almost sure conditional multiplier CLT) and its "weighting" factors $\xi_i^{\star}$ have a null expectation. Thus, they cannot be used directly to estimate the variability, but as for the second expression of the Exchangeable variance estimator, by adding the term $1/n$ to the centered multipliers, a Wild variance estimator can be obtained as follows:

$$\hat{V}_X^{W\star} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\xi_i^{\star} + \frac{1}{n}\right)\left(\xi_j^{\star} + \frac{1}{n}\right)\left\|X_i - X_j\right\|_{\mathcal{H}}. \tag{15}$$

Analogous definitions are considered for the variable Y. The consistency of the variance estimator, and the bootstrap versions, is analyzed in the following theorem.

Theorem 5. *If $E\|X\|_{\mathcal{H}}^2 < \infty$ then,*

$$\hat{V}_X \xrightarrow{\mathcal{P}} Var\left(\phi_{\mathcal{H}}(X)\right), \quad \hat{V}_X^{E\star} \xrightarrow{\mathcal{P}} Var\left(\phi_{\mathcal{H}}(X)\right) \ P\text{-a.s.} \ \text{and} \ \hat{V}_X^{W\star} \xrightarrow{\mathcal{P}} Var\left(\phi_{\mathcal{H}}(X)\right) \ P\text{-a.s.}$$

The same result also holds for the $\mathcal{H}$-valued random element Y. Therefore, the corresponding studentized testing procedures are also valid, as formalized in the following result.

Theorem 6. *If $E(\|X\|_{\mathcal{H}}^2) < \infty$, $E(\|Y\|_{\mathcal{H}}^2) < \infty$ and $n/m \to 1$ as $n \to \infty$, the test that rejects H_0 in (1) whenever*

$$\frac{T_n}{\hat{V}_X + \frac{n}{m}\hat{V}_Y} > c_\alpha^n,$$

where c_α^n is the $(1-\alpha)$-quantile of the distribution of $T_n^{E\star}/(\hat{V}_X^{E\star} + \frac{n}{m}\hat{V}_Y^{E\star})$ or $T_n^{W\star}/(\hat{V}_X^{W\star} + \frac{n}{m}\hat{V}_Y^{W\star})$, is asymptotically correct and consistent at a significance level α.

6. Simulations

To show the empirical performance of the proposed test procedures, the case of functional data taking values in $(L^2[0,1], \langle\cdot,\cdot\rangle)$ is examined. Two functional processes that may differ on the mean, on the covariance operator or on the distribution have been considered. The difference on mean and covariance depends on two parameters β and γ. In details, two functional sample sets of sizes $n \in \{50, 100\}$ and $m \in \{50, 100\}$, respectively, have been generated for different parameters β and γ as follows:

$$X_i(t) = 10t(1-t)^{1+\beta} + \epsilon_i(t), \quad i \in \{1,\ldots,n\}, \quad t \in [0,1], \tag{16}$$

$$Y_j(t) = 10t^{1+\beta}(1-t) + \varepsilon_j(t), \quad j \in \{1,\ldots,m\}, \quad t \in [0,1], \tag{17}$$

where $\epsilon(t)$, $\varepsilon(t)$ have both zero mean, $\Sigma_\epsilon(t,t') = 0.2\exp(-3|t-t'|)$ and $\Sigma_\varepsilon = \gamma\Sigma_\epsilon$. The curves are observed in a equispaced grid of length 51 in the interval $[0,1]$. For obtaining the trajectories, we represent X (and Y) in a principal components basis (PCs) of length K derived from the spectral decomposition of Σ_ϵ, and so, $X_i(t) \approx \mu_X(t) + \sum_{j=1}^{K} x_{ij}v_j(t)$ where x_{ij}, $v_j(t)$ are, respectively, the scores in the PC basis and the eigenfunctions of Σ_ϵ. Recall that a process X is Gaussian if and only if $x_{\cdot j} \sim N(0, \lambda_j^X)$ where λ_j^X are the eigenvalues of the spectral decomposition. The same can be applied to Y process having the same eigenfunctions as X but scaled eigenvalues due to parameter γ. Using this representation and changing the distribution of x_{ij} and y_{ij}, we can generate non-Gaussian processes. For the different scenarios, β is chosen from $\{0, 0.10\}$, γ from $\{1, 2\}$ and the distribution of the scores is chosen from the following list:

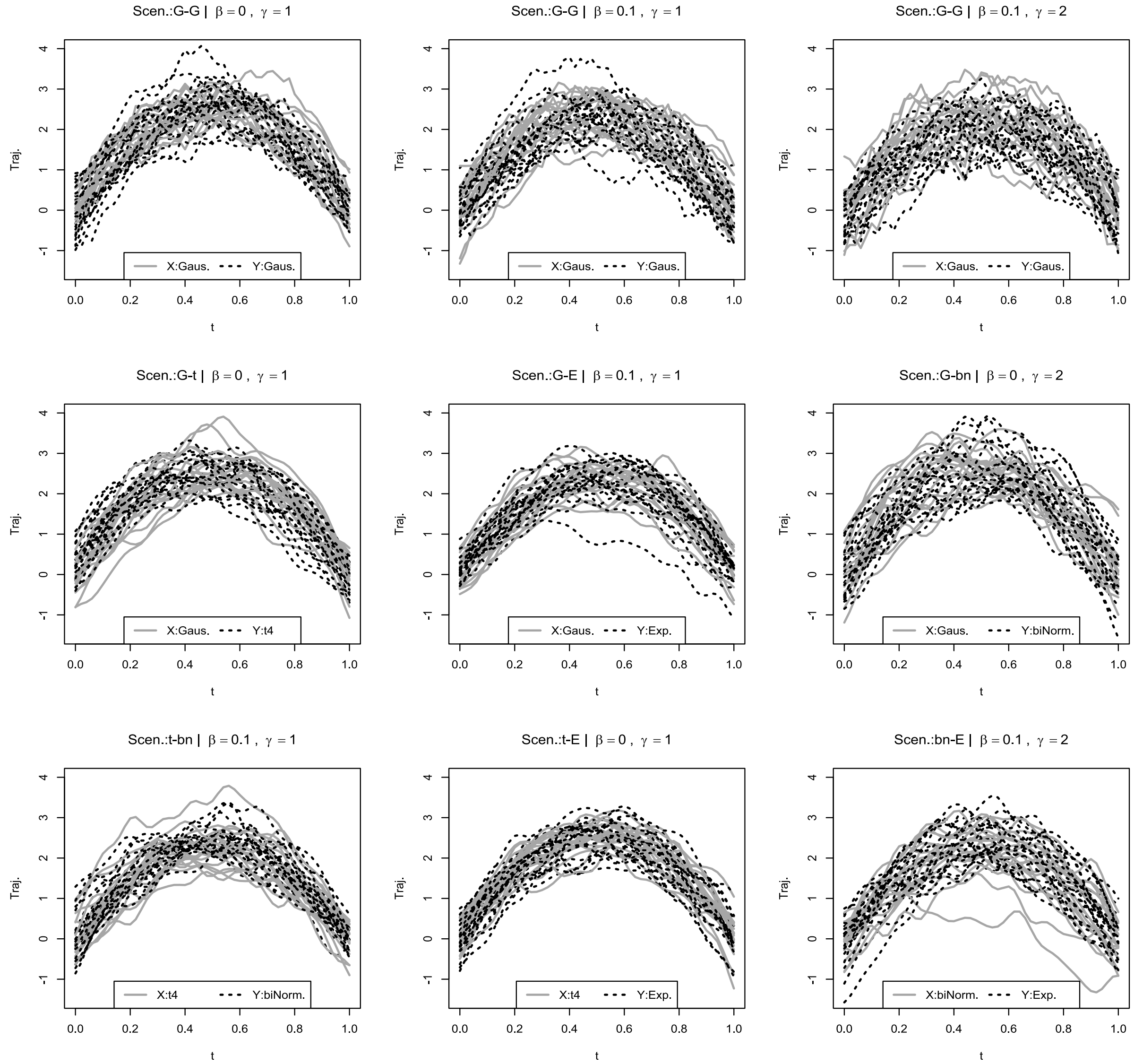

Fig. 1. Each subplot shows twenty generated trajectories under different combinations of β, γ (specified in the main title) and score distributions (specified in the legend).

- (Gaussian): $x_{ij} \sim N(0, \lambda_j^X)$, $y_{ij} \sim N(0, \gamma\lambda_j^X)$;
- (t_4): $x_{ij} \sim \sqrt{\lambda_j^X}\dfrac{r_i}{\sqrt{4/2}}$, $y_{ij} \sim \sqrt{\gamma\lambda_j^X}\dfrac{r_i}{\sqrt{4/2}}$ where $r_i \sim t(4)$;
- (Exp.): $x_{ij} \sim \sqrt{\lambda_i} r_i$, $y_{ij} \sim \sqrt{\gamma\lambda_i} r_i$ where $r_i \sim \text{Exp}(1) - 1$;
- (biNormal): $x_{ij} \sim \sqrt{\lambda_i}\dfrac{r_i}{\sqrt{a^2+s^2}}$, $y_{ij} \sim \sqrt{\gamma\lambda_i}\dfrac{r_i}{\sqrt{a^2+s^2}}$, $r_i \sim 0.5\mathcal{N}(a, s^2) + 0.5\mathcal{N}(-a, s^2)$ with $a = 1$ and $s = 0.5$.

t_4 will produce trajectories similar to Gaussian (G) but with heavier tail scores. Scenario Exp. (E) includes asymmetric scores and scenario biNormal (bn) will produce bimodal scores. In all cases, the scores have zero mean and fixing $\gamma = 1$, variance one, meaning that the spectral decomposition of the covariance operator is different from the Gaussian scenario. So, the particular distribution of X and Y will depend on the way of generating r_i.

A graphical example of twenty generated trajectories is shown in Fig. 1 for several values of β, γ and different scenarios. The first row is devoted to differences in mean and covariance for Gaussian processes. The null hypothesis corresponds to the top-left plot, although the graphical perception of the difference among populations is small, even in the case of maximum deviation from H_0 ($\beta = 0.1$ and $\gamma = 2$, right-top plot). The second row compares a Gaussian process with a Non-Gaussian (also changing mean and covariance), but the overall aspect of the trajectories are similar to the first row. The third row compares the trajectories of several Non-Gaussian processes. In all cases, it is hard to distinguish by eye differences on the mean, covariance or even the score distribution.

For each combination of β, γ and scenario, 500 runs are performed to evaluate the proportion of rejections for the bootstrap procedures (Exchangeable, Wild and their studentized versions) using 5000 bootstrap resamples. The proposed tests are compared with two natural competitors. The first one is a test based on so-called Maximum Mean Discrepancy (MMD) proposed by Gretton et al. [38,39] which, essentially, finds the maximum difference among two populations from all transformations that can be applied maintaining an RKHS structure. The test has two solutions for obtaining the critical point: using the asymptotic spectral decomposition of the kernel (of the RKHS) or using shuffling or a bootstrap approach. Indeed, the kernel can be obtained as a Radial Basis Function (RBF) of the distances or with a similar formula as the energy test. In fact, Sejdinovic et al. [40] proves that there is an equivalence between MMD and the energy distance test. The second competitor is a test for the equality of distributions based on random projections (see Cuesta-Albertos et al. [28,29]). The random projections procedure is as follows:

- Consider a random direction h and compute the projections $X_h = \langle X, h\rangle$ and $Y_h = \langle Y, h\rangle$;
- Perform a test for the equality of distribution in $\mathbb{R}$ for X_h and Y_h (e.g., Kolmogorov–Smirnov or Anderson–Darling) and obtain the p–value;
- Repeat for $h_1, \dots, h_H$ projections and compute the overall p–value of the test using a False Discovery Rate (FDR) approach.

The number of random projections is typically low, e.g., $H \in \{5, 10, 15\}$, and the directions are generated as random linear combinations of the first five principal components of the joint process (X, Y). Finally, the comparison is completed with tests for the equality of means (μ–test, see Horváth and Kokoszka [41]) and variances (Σ–test, see Fremdt et al. [13]) based on the decomposition of the variance–covariance operator. The critical points for both tests can be obtained using asymptotic properties or using a parametric bootstrap (under the assumption that the processes are Gaussian).

The results for the proposed testing procedures for $n = m = 100$ are shown in Table 1. All the methods are implemented in the last version of R–package `fda.usc` (Febrero-Bande and Oviedo de la Fuente [42]). The name of the specific functions for each method are indicated below. The first three columns fix the scenario (*scen*) and difference in mean (β) and in variance (γ). Then, the remaining columns correspond to the empirical size of each competitor:

- *Exch.*, *W.B.*: Energy test using Exchangeable or Wild Bootstrap techniques. (`fda.usc::fEqDistrib.test`);
- *MMDa*, *MMDb*: MMD test using asymptotic or numerical (shuffling) approximation. (`fda.usc::MMDA.test`, `fda.usc::MMD.test`);
- *KS5*, *KS10*, *AD5*, *AD10*: Random projection methods using Kolmogorov–Smirnov or Anderson–Darling univariate tests with 5,10 projections. (`fda.usc::XYRP.test`);
- μ_A, μ_B: μ–tests using asymptotic or parametric bootstrap approaches. (`fda.usc::fmean.test.fdata`);
- Σ_A, Σ_B: Σ–tests using asymptotic or bootstrap approaches. (`fda.usc::cov.test.fdata`).

The studentized versions of the bootstrap proposals show almost exactly the same results as the non-studentized and so, no improvement was obtained using them. In order to save space, its results are not shown in the tables. For the Exchangeable Bootstrap, uniforms weights randomly selected in $[0, 1]$ conveniently normalized and centered have been considered, while for the Wild Bootstrap standard normal variables have been generated to derive the centered multipliers.

When H_0 holds, the test is well calibrated and using 500 repetitions, it is expected to have results in the range $[0.031, 0.069]$. The block of rows with *scen=‘G-G’* in Table 1 is the case where both processes are Gaussian. In this block, the empirical levels in the first row (H_0 holds) are out of the expected interval for MMDa, MMDb, KS10, AD5, AD10 and Σ_B. Curiously, when $n = m = 50$ (see Table 2) only KS5 and KS10 are out of the confidence interval for proportion. The rest of the rows of the same block have empirical powers close to 1 except for random projection tests that have powers between 0.188 and 0.448. The results for the Random Projected Kolmogorov–Smirnov and Anderson–Darling tests are only provided for $H = 5$ and $H = 10$ projections. Other values of H between 1 and 15 provided outputs in the same line. $H = 10$ seems to be a good balance between the empirical size and power for deviations in mean. However, this procedure shows less power when the deviations from H_0 relate to the covariance. In our results, more power can be obtained by increasing the number of projections at the cost of losing the level under the null hypothesis. In this block the tests for equality of mean and variances work as expected.

The rest of the blocks where both processes coincide (*‘bn-bn’, ‘E-E’, ‘t-t’*) and so, the first row is under H_0, our proposals show a similar behavior to the MMD ones in all cases. The tests based on random projections show less power specially in the case where the difference is only due to changes in the variance. μ–tests work quite well although showing a slightly loss of level in certain cases. On the contrary, Σ–tests show an erratic performance not being able to identify the null hypothesis when the processes are not Gaussian. The other blocks (*‘bn-E’, ‘G-bn’, ‘G-E’, ‘G-t’, ‘t-bn’, ‘t-E’*) correspond to H_1 scenarios, showing more power the MMD techniques than our proposals or the RP ones. By comparing Table 1 and Table 2, we conclude that the empirical performance of the Bootstrap procedures in this paper is competitive.

7. Data examples

Two real data examples are studied in this section. The two datasets are well-known in Functional Data Analysis (FDA), and the associated supervised classification problems are considered hard.

Table 1
Proportion of rejections for the equality of distributions in 500 repetitions with $n = m = 100$ at level $\alpha = 0.05$. The distribution of populations are determined by first three columns: scen(X-Y), β and γ. The rows with X = Y, $\beta = 0.00$ and $\gamma = 1.0$ correspond to null hypothesis.

scen	β	γ	Exch.	W.B.	MMDa	MMDb	KS5	KS10	AD5	AD10	μ_A	μ_B	Σ_A	Σ_B
bn-bn	0.00	1.0	0.044	0.044	0.044	0.046	0.038	0.028	0.042	0.038	0.056	0.046	0.010	0.002
bn-bn	0.00	2.0	0.990	0.986	1.000	1.000	0.362	0.352	0.828	0.824	0.044	0.054	1.000	1.000
bn-bn	0.10	1.0	0.978	0.972	0.948	0.946	0.792	0.880	0.884	0.950	0.998	0.978	0.008	0.004
bn-bn	0.10	2.0	1.000	1.000	1.000	1.000	0.912	0.950	0.994	1.000	0.968	0.882	1.000	1.000
bn-E	0.00	1.0	0.606	0.610	0.896	0.898	0.442	0.440	0.460	0.488	0.054	0.044	0.244	0.148
bn-E	0.00	2.0	0.782	0.780	0.976	0.980	0.314	0.364	0.336	0.376	0.074	0.078	1.000	0.940
bn-E	0.10	1.0	0.998	0.998	1.000	1.000	0.962	0.982	0.978	0.988	0.994	0.970	0.218	0.182
bn-E	0.10	2.0	1.000	1.000	1.000	1.000	0.944	0.984	0.958	0.992	0.954	0.862	1.000	0.958
E-E	0.00	1.0	0.030	0.030	0.038	0.040	0.032	0.034	0.036	0.030	0.034	0.028	0.588	0.356
E-E	0.00	2.0	0.894	0.888	0.988	0.988	0.442	0.450	0.656	0.656	0.036	0.030	1.000	0.912
E-E	0.10	1.0	0.998	0.998	1.000	1.000	0.926	0.984	0.938	0.990	0.994	0.970	0.542	0.332
E-E	0.10	2.0	1.000	1.000	1.000	1.000	0.984	1.000	0.994	1.000	0.948	0.864	1.000	0.912
G-bn	0.00	1.0	0.070	0.070	0.088	0.088	0.074	0.062	0.058	0.052	0.048	0.042	0.032	0.018
G-bn	0.00	2.0	0.996	0.996	1.000	1.000	0.556	0.602	0.844	0.848	0.048	0.064	1.000	0.998
G-bn	0.10	1.0	0.986	0.986	0.978	0.982	0.812	0.876	0.908	0.954	0.994	0.980	0.028	0.026
G-bn	0.10	2.0	1.000	1.000	1.000	1.000	0.946	0.972	0.988	0.994	0.946	0.864	1.000	0.990
G-E	0.00	1.0	0.312	0.308	0.754	0.746	0.238	0.246	0.198	0.194	0.032	0.030	0.264	0.194
G-E	0.00	2.0	0.870	0.868	0.994	0.994	0.478	0.454	0.504	0.466	0.056	0.046	1.000	0.936
G-E	0.10	1.0	1.000	1.000	1.000	1.000	0.950	0.990	0.954	0.988	1.000	0.962	0.280	0.210
G-E	0.10	2.0	1.000	1.000	1.000	1.000	0.968	0.990	0.984	0.994	0.952	0.868	1.000	0.924
G-G	0.00	1.0	0.036	0.036	0.030	0.028	0.036	0.028	0.026	0.018	0.042	0.034	0.044	0.022
G-G	0.00	2.0	0.918	0.924	0.994	0.992	0.214	0.188	0.448	0.396	0.050	0.036	1.000	0.976
G-G	0.10	1.0	0.992	0.992	0.994	0.994	0.804	0.904	0.882	0.948	0.998	0.988	0.040	0.044
G-G	0.10	2.0	1.000	1.000	1.000	1.000	0.858	0.914	0.956	0.976	0.972	0.878	1.000	0.972
G-t	0.00	1.0	0.084	0.084	0.188	0.196	0.052	0.046	0.042	0.054	0.058	0.052	0.370	0.238
G-t	0.00	2.0	0.476	0.472	0.784	0.780	0.102	0.072	0.160	0.124	0.062	0.056	1.000	0.928
G-t	0.10	1.0	0.998	0.998	0.998	0.998	0.874	0.946	0.920	0.964	0.998	0.978	0.338	0.192
G-t	0.10	2.0	0.996	0.996	0.998	0.998	0.814	0.894	0.896	0.952	0.968	0.884	1.000	0.936
t-bn	0.00	1.0	0.288	0.280	0.590	0.590	0.264	0.276	0.206	0.180	0.050	0.050	0.290	0.204
t-bn	0.00	2.0	1.000	1.000	1.000	1.000	0.846	0.894	0.978	0.996	0.038	0.040	1.000	0.948
t-bn	0.10	1.0	0.998	0.998	1.000	0.998	0.908	0.960	0.932	0.978	0.994	0.980	0.264	0.178
t-bn	0.10	2.0	1.000	1.000	1.000	1.000	0.992	0.998	0.998	0.998	0.964	0.862	1.000	0.944
t-E	0.00	1.0	0.286	0.278	0.606	0.600	0.300	0.296	0.232	0.238	0.040	0.056	0.628	0.374
t-E	0.00	2.0	0.982	0.984	0.998	0.998	0.656	0.698	0.708	0.764	0.064	0.034	1.000	0.914
t-E	0.10	1.0	1.000	1.000	1.000	1.000	0.974	0.996	0.974	1.000	0.996	0.972	0.594	0.364
t-E	0.10	2.0	1.000	1.000	1.000	1.000	0.980	0.996	0.986	0.998	0.942	0.844	1.000	0.922
t-t	0.00	1.0	0.046	0.048	0.056	0.054	0.044	0.024	0.034	0.028	0.028	0.056	0.598	0.338
t-t	0.00	2.0	0.814	0.808	0.968	0.966	0.168	0.148	0.294	0.258	0.046	0.044	1.000	0.900
t-t	0.10	1.0	1.000	1.000	1.000	1.000	0.906	0.968	0.934	0.982	0.996	0.980	0.596	0.360
t-t	0.10	2.0	1.000	1.000	1.000	1.000	0.906	0.944	0.936	0.972	0.948	0.874	0.996	0.890

The first one is `phoneme` which contains log-periodogram curves (discretized in 150 frequencies) for five phonemes: {"sh","iy","dcl","aa","ao"}. The first three categories are easily distinguishable, but the last two "aa" (open *a*) and "ao" (closed *a*) are more difficult to separate. The dataset employed here is available in the R–package `fda.usc` Febrero-Bande and Oviedo de la Fuente [42]. Focusing on the last two groups, we have 50 curves for both groups in the learn sample and 50 more in the test sample. To verify whether the classification problem is well-posed indeed, we can perform the proposed tests for comparing both groups and comparing the learn and test groups. The results are shown in Table 3, where the equality of groups is clearly rejected but, in both groups, the test and learn groups are homogeneous. Thus, although it is a difficult classification problem, it is well-defined.

The second example is about the longevity and reproduction activity of the Mediterranean fruit flies (*Ceratitis capitata*). The dataset was initially recorded by Carey et al. [43], but it was introduced in the FDA literature as a complex classification problem by Müller and Stadtmüller [44] and Baíllo and Cuevas [45]. The dataset has a total of 534 egg-laying trajectories of flies from day 5 to day 34, and the goal is to predict if the fly will die before or after day 50 using its fertility in the initial period. Therefore, there are two groups: "short-lived"(SL) and "long-lived"(LL) with 256 and 278 cases, respectively. The curves are usually smoothed by representing them in a B-spline basis or applying a kernel smoothing filter. We have applied a Gaussian kernel smoothing filter with bandwidth 1. The best result in Müller and Stadtmüller [44] was a correct classification rate of 65% for SL group and 52% for LL (with a 60% overall). In Baíllo and Cuevas [45], a 54.7% correct classification rate for the non-smoothed trajectories and 57.1% for the smoothed (sm) ones was obtained. These 'poor' results show the difficulty of the classification task. Fig. 2 displays several trajectories of SL and LL groups. Again, the proposed tests will be applied to assess whether the classification problem is well-posed.

Since the trajectories represent the number of laid eggs, it is compelling to consider the log-curves in order to stabilize the variance along days. This is useful because the proposed test statistics are based on distances, and the higher values/variability in the interval $[10, 20]$ of the curves may distort these distances. The results are shown in Table 4. For the original data, both for the

Table 2
Proportion of rejections for the equality of distributions in 500 repetitions with $n = m = 50$ at level $\alpha = 0.05$. The distribution of populations are determined by first three columns: scen(X-Y), β and γ. The rows with X=Y, $\beta = 0.00$ and $\gamma = 1.0$ correspond to null hypothesis.

scen	β	γ	Exch.	W.B.	MMDa	MMDb	KS5	KS10	AD5	AD10	μ_A	μ_B	Σ_A	Σ_B
bn-bn	0.00	1.0	0.058	0.056	0.046	0.048	0.018	0.012	0.046	0.042	0.062	0.060	0.022	0.002
bn-bn	0.00	2.0	0.486	0.486	0.916	0.914	0.114	0.096	0.270	0.232	0.092	0.058	0.984	0.842
bn-bn	0.10	1.0	0.682	0.672	0.622	0.614	0.376	0.424	0.598	0.668	0.886	0.706	0.014	0.002
bn-bn	0.10	2.0	0.936	0.936	0.984	0.986	0.460	0.514	0.660	0.684	0.714	0.466	0.984	0.838
bn-E	0.00	1.0	0.204	0.202	0.502	0.492	0.108	0.128	0.114	0.116	0.056	0.046	0.182	0.180
bn-E	0.00	2.0	0.332	0.316	0.680	0.674	0.132	0.130	0.164	0.134	0.088	0.050	0.956	0.704
bn-E	0.10	1.0	0.936	0.932	0.974	0.974	0.694	0.798	0.768	0.850	0.884	0.716	0.168	0.192
bn-E	0.10	2.0	0.954	0.954	0.996	0.994	0.634	0.736	0.770	0.866	0.714	0.530	0.944	0.690
E-E	0.00	1.0	0.048	0.048	0.052	0.046	0.018	0.016	0.024	0.036	0.034	0.040	0.498	0.310
E-E	0.00	2.0	0.422	0.408	0.822	0.814	0.174	0.166	0.254	0.250	0.068	0.062	0.974	0.768
E-E	0.10	1.0	0.870	0.868	0.930	0.926	0.672	0.800	0.774	0.884	0.894	0.748	0.478	0.368
E-E	0.10	2.0	0.982	0.980	0.996	0.994	0.800	0.908	0.896	0.964	0.716	0.496	0.966	0.720
G-bn	0.00	1.0	0.060	0.056	0.076	0.082	0.042	0.040	0.048	0.046	0.068	0.050	0.026	0.016
G-bn	0.00	2.0	0.722	0.712	0.966	0.966	0.168	0.180	0.298	0.282	0.062	0.050	0.980	0.820
G-bn	0.10	1.0	0.754	0.752	0.734	0.730	0.458	0.526	0.632	0.670	0.860	0.710	0.026	0.016
G-bn	0.10	2.0	0.956	0.954	1.000	1.000	0.598	0.630	0.718	0.734	0.724	0.506	0.994	0.800
G-E	0.00	1.0	0.184	0.178	0.384	0.380	0.098	0.122	0.128	0.114	0.064	0.064	0.264	0.206
G-E	0.00	2.0	0.408	0.388	0.740	0.728	0.184	0.194	0.208	0.224	0.058	0.076	0.948	0.696
G-E	0.10	1.0	0.950	0.952	0.982	0.980	0.722	0.834	0.796	0.876	0.886	0.716	0.280	0.208
G-E	0.10	2.0	0.960	0.958	0.998	0.998	0.728	0.798	0.804	0.856	0.698	0.496	0.954	0.752
G-G	0.00	1.0	0.046	0.042	0.046	0.042	0.032	0.036	0.050	0.048	0.074	0.052	0.030	0.052
G-G	0.00	2.0	0.444	0.432	0.824	0.814	0.058	0.052	0.118	0.106	0.054	0.046	0.980	0.768
G-G	0.10	1.0	0.774	0.776	0.772	0.772	0.466	0.570	0.650	0.720	0.904	0.746	0.044	0.040
G-G	0.10	2.0	0.918	0.914	0.990	0.988	0.488	0.528	0.632	0.666	0.738	0.498	0.988	0.750
G-t	0.00	1.0	0.068	0.064	0.102	0.100	0.048	0.040	0.068	0.052	0.074	0.044	0.244	0.188
G-t	0.00	2.0	0.174	0.156	0.418	0.408	0.040	0.038	0.074	0.056	0.064	0.058	0.930	0.672
G-t	0.10	1.0	0.810	0.800	0.836	0.836	0.564	0.628	0.680	0.736	0.872	0.680	0.250	0.192
G-t	0.10	2.0	0.834	0.834	0.908	0.906	0.414	0.500	0.600	0.664	0.736	0.558	0.942	0.684
t-bn	0.00	1.0	0.120	0.118	0.250	0.248	0.070	0.072	0.074	0.074	0.068	0.054	0.162	0.150
t-bn	0.00	2.0	0.922	0.920	1.000	1.000	0.336	0.364	0.536	0.554	0.050	0.052	0.984	0.748
t-bn	0.10	1.0	0.852	0.844	0.886	0.882	0.562	0.650	0.654	0.740	0.868	0.698	0.170	0.146
t-bn	0.10	2.0	0.996	0.996	1.000	1.000	0.742	0.826	0.826	0.884	0.710	0.500	0.992	0.798
t-E	0.00	1.0	0.158	0.150	0.362	0.356	0.126	0.140	0.122	0.126	0.070	0.052	0.444	0.272
t-E	0.00	2.0	0.652	0.644	0.928	0.924	0.304	0.304	0.334	0.346	0.066	0.048	0.990	0.762
t-E	0.10	1.0	0.956	0.950	0.994	0.990	0.776	0.892	0.834	0.904	0.876	0.718	0.466	0.336
t-E	0.10	2.0	0.986	0.984	0.998	0.998	0.806	0.900	0.860	0.920	0.670	0.510	0.988	0.782
t-t	0.00	1.0	0.056	0.052	0.054	0.052	0.028	0.020	0.044	0.044	0.058	0.052	0.478	0.322
t-t	0.00	2.0	0.394	0.376	0.726	0.724	0.066	0.074	0.092	0.088	0.046	0.048	0.982	0.712
t-t	0.10	1.0	0.870	0.864	0.898	0.896	0.592	0.694	0.706	0.806	0.902	0.728	0.488	0.318
t-t	0.10	2.0	0.936	0.932	0.984	0.980	0.570	0.626	0.692	0.746	0.748	0.546	0.974	0.742

Table 3
p–values for the tests of equality of distributions for `phoneme` example using Exchangeable (Exch.) and Wild Bootstrap (W.Boot.) methods with their studentized versions (Std. Exch. and Std. WB).

Test	Exch.	Std. Exch.	W.Boot.	Std. WB
“aa” vs “ao”	0.0000	0.0000	0.0000	0.0000
“aa” learn vs test	0.1484	0.1570	0.1788	0.2020
“ao” learn vs test	0.9348	0.9388	0.7946	0.8128

non-smoothed and the smoothed versions, the differences among two groups (in terms of distributions) are small but significant. When the curves are transformed by $\ln(\cdot + 1)$, these differences are non-significant at level $\alpha = 0.01$ meaning that this classification problem is quite hard. Clearly, if we can accept that the distribution in both groups is the same, the classification problem is hopeless, or equivalently, harder as p-values move to acceptance region.

8. Conclusions

Powerful tools for testing the equality of distributions among statistical objects in Hilbert spaces have been proposed. Some partial solutions had been considered in the literature, e.g., when the focus is on the mean or the covariance, or those based on

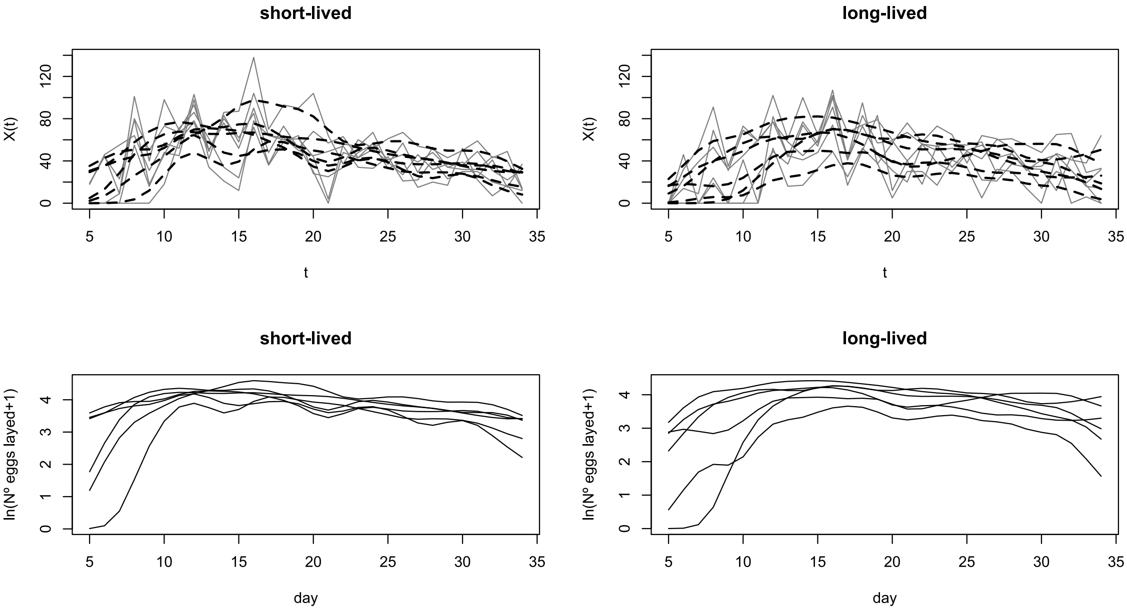

Fig. 2. First row, six example trajectories (non-smoothed and smoothed) for short-lived (left) and long-lived (right) flies. Second row, log-transformed smoothed trajectories of the first row.

Table 4
p–values for the tests of equality of distributions Short-Lived vs Long-lived for the `medflies` dataset.

SL vs LL	Exch.	Std.Exch.	W.Boot.	Std.WB	MMDb	KS10	AD10
medflies	0.0012	0.0012	0.0030	0.0034	0.0015	0.0010	0.0026
sm.medflies	0.0025	0.0025	0.0020	0.0030	0.0005	0.0034	0.0000
ln(medflies+1)	0.0294	0.0304	0.0230	0.0236	0.0235	0.0285	0.0622
ln(sm.mflies+1)	0.0175	0.0170	0.0150	0.0125	0.0100	0.0189	0.0007

random projections for functional data. However, this is the first general approach that has been proved to be asymptotically correct and consistent under mild conditions, thanks to the characterization provided by the so-called energy distance. The proposal is mainly applied for functional data here, but the tool can be used for other types of data such as sets, vectors or circular data, to name but a few, with the only limitation that the associated metric space can be embedded into a separable Hilbert space. The possibility of applying these methods for the (high-dimensional) multivariate case as, e.g., in Modarres [46], is specially relevant. In the machine learning literature there has been also good developments (see, for example, Gretton et al. [39] or Sejdinovic et al. [40]). An alternative proof of the particular case of wild bootstrap consistency is given in Chwialkowski et al. [47].

CRediT authorship contribution statement

Gil González–Rodríguez: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Ana Colubi:** Conceptualization, Methodology, Writing – review & editing. **Wenceslao González–Manteiga:** Conceptualization, Methodology, Writing – review & editing. **Manuel Febrero–Bande:** Conceptualization, Methodology, Formal analysis, Writing – review & editing, Software, Visualization.

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Appendix. Proofs

Proof of Theorem 1. According to (5) and the moment conditions,

$$E\left(\|\phi_{\mathcal{H}}(X)\|\right)^2_{\mathcal{G}} = E\left(\|X\|\right) < \infty,$$

and analogously $E\left(\|\phi_{\mathcal{H}}(Y)\|\right)^2_{\mathcal{G}} < \infty$. Thus, the result is a direct consequence of the CLT for i.i.d. Hilbert-valued random elements (see, e.g., Laha and Rohatgi [48]) and the Continuous Mapping Theorem. □

Proof of Theorem 2. Under the null hypothesis, $T_n = 2\|\tau_n\|^2_{\mathcal{G}}$. Thus, Theorem 1 and the Continuous Mapping Theorem guarantee the asymptotic correctness of the test.

On the other hand, when the null hypothesis does not hold, by applying the Slutsky Theorem and the Continuous Mapping Theorem it is verified that

$$\frac{T_n}{n} = 2\left\|\frac{\tau_n}{\sqrt{n}} + (E(\phi(X)) - E(\phi(Y)))\right\|^2_{\mathcal{G}} \xrightarrow{\mathcal{P}} \|(E(\phi(X)) - E(\phi(Y)))\|^2_{\mathcal{G}} = d > 0.$$

Fix any $c > 0$, $n_0 \in \mathbb{N}$ so that $n_0 d > 2c$, and consider any $n \geq n_0$, then

$$P(T_n > c) \geq P\left(\frac{T_n}{n} > \frac{c}{n_0}\right) \geq P\left(\frac{T_n}{n} - d > -\frac{c}{n_0}\right) \geq P\left(\left|\frac{T_n}{n} - d\right| < \frac{c}{n_0}\right), \tag{A.1}$$

and this last probability converges to 1 as $n \to \infty$, due to the convergence in probability of T_n/n to d. Consequently, the proposed test is asymptotically consistent.

Finally, for the local alternatives study, consider the sequences of random elements $\{Y'_n\}_{n\in\mathbb{N}}$ i.i.d. as Y, $\{U_n\}_{n\in\mathbb{N}}$ i.i.d. as U and $\{q_n\}_{n\in\mathbb{N}}$ i.i.d. as a uniform distribution on $(0,1)$. Define

$$X_i^{(n)} = I_{\{q_i < \frac{\delta_n}{\sqrt{n}}\}} U_i + I_{\{q_i \geq \frac{\delta_n}{\sqrt{n}}\}} Y'_i.$$

Note that $\left\{X_i^{(n)}\right\}_{i=1}^n$ is a sequence of i.i.d. $\mathcal{H}$-valued random elements with associated probability measure $P_X^{(n)}$ for any $n \in \mathbb{N}$. It is straightforward to check that,

$$\frac{T_n}{\delta_n^2} = 2|\left|S_n + \left(E\left(\phi_{\mathcal{H}}(U)\right) - E\left(\phi_{\mathcal{H}}(Y)\right)\right)\right|^2_{\mathcal{G}},$$

being

$$S_n = \frac{\sqrt{n}}{\delta_n}\overline{\left(I_{\{q<\frac{\delta_n}{\sqrt{n}}\}}\phi_{\mathcal{H}}(U)\right)^c} + \frac{\sqrt{n}}{\delta_n}\overline{\left(I_{\{q\geq\frac{\delta_n}{\sqrt{n}}\}}\phi_{\mathcal{H}}(Y')\right)^c} + \frac{\sqrt{n}}{\delta_n}\overline{\left(\phi_{\mathcal{H}}(Y)\right)^c},$$

where the superscript 'c' indicates that the corresponding variables have been centered. In order to prove the result, it is enough to check that T_n/δ_n^2 converges in probability to 0 and then to use the same reasoning as in (A.1).

To simplify the notation let us rewrite $S_n = S_{n,1} + S_{n,2} + S_{n,3}$. By applying the Central Limit Theorem and the Slutsky Theorem it is straightforward to prove that $S_{n,3} \xrightarrow{\mathcal{P}} 0$. Concerning $S_{n,2}$ we have that

$$\begin{aligned}
E\left\|S_{n,2}\right\|^2_{\mathcal{G}} &= \frac{1}{\delta_n^2}\left\|I_{\{q\geq\frac{\delta_n}{\sqrt{n}}\}}\phi_{\mathcal{H}}(Y') - \left(1 - \frac{\delta_n}{\sqrt{n}}\right)\phi_{\mathcal{H}}(Y')\right\|^2_{\mathcal{G}} \\
&= \frac{1}{\delta_n^2}\left(\left(1 - \frac{\delta_n}{\sqrt{n}}\right)E\left\|\phi_{\mathcal{H}}(U)\right\|^2_{\mathcal{G}} - \left(1 - \frac{\delta_n}{\sqrt{n}}\right)^2\left\|E(\phi_{\mathcal{H}}(U))\right\|^2_{\mathcal{G}}\right),
\end{aligned}$$

that converges to 0 as $n \to \infty$. By following a similar reasoning it can be shown that $E\left\|S_{n,2}\right\|^2_{\mathcal{G}} \xrightarrow{n\to\infty} 0$, which concludes the proof. □

Proof of Theorem 3. Regarding the identity just note that condition A2 guarantees that

$$\sum_{i=1}^n w^\star_{ni} = \sum_{j=1}^m v^\star_{mj} = 0.$$

Hence, by using the embedding (5), it holds that

$$\sum_{i,j} w^\star_{ni} v^\star_{mj} \langle\phi_{\mathcal{H}}(X_i), \phi_{\mathcal{H}}(Y_j)\rangle_{\mathcal{G}} = \sum_{i,j} w^\star_{ni} v^\star_{mj} \|X_i - Y_j\|_{\mathcal{H}}.$$

Similarly for the terms $\langle\phi_{\mathcal{H}}(X_i), \phi_{\mathcal{H}}(X_j)\rangle_{\mathcal{G}}$ and $\langle\phi_{\mathcal{H}}(Y_i), \phi_{\mathcal{H}}(Y_j)\rangle_{\mathcal{G}}$). Thus, a simple expansion of the squared norm leads to the result.

On the other hand, Corollary 1 in González-Rodríguez and Colubi [34] guarantees that,

$$\frac{\sqrt{n}}{c}\sum_{i=1}^{n} w^{\star}_{ni}\phi_{\mathcal{H}}(X_i) \xrightarrow{\mathcal{L}} Z_{\phi_{\mathcal{H}}(X)} \quad P\text{-a.s.}, \qquad \frac{\sqrt{m}}{c}\sum_{i=1}^{n} v^{\star}_{nj}\phi_{\mathcal{H}}(Y_j) \xrightarrow{\mathcal{L}} Z_{\phi_{\mathcal{H}}(Y)} \quad P\text{-a.s.}$$

The application of the Slutsky Theorem and the Continuous Mapping Theorem concludes the proof. □

Proof of Lemma 1. The result is a direct consequence of the CTL for i.i.d. random elements in a separable Hilbert space (see, e.g., Laha and Rohatgi [48]), Theorem 2.1. of Wellner [49] and Slutsky Theorem. □

Proof of Theorem 4. For the centered multipliers, it holds that $\sum_{i=1}^{n}\xi_i^{\star} = \sum_{j=1}^{m}\eta_j^{\star} = 0$. Thus, by reasoning as in the proof of Theorem 3, the identity is proven.

The asymptotic result is a direct consequence of Theorem 2 in González-Rodríguez and Colubi [34], the Continuous Mapping Theorem and the Slutsky Theorem. □

Proof of Theorem 5. The property of the embedding shown in (5) and a simple expansion of the squared norm yields to an equivalent expression of the estimators in the image space $\mathcal{G}$. In this way, it is easy to check that

$$\hat{V}_X = \frac{1}{n}\sum_{i=1}^{n}\|\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}} - \left\|\frac{1}{n}\sum_{i=1}^{n}\phi_{\mathcal{H}}(X_i)\right\|^2_{\mathcal{G}}.$$

As a consequence, the strong law of large numbers in separable Hilbert spaces guarantees that $\hat{V}_X$ converges P-a.s. to $\mathrm{Var}(\phi_{\mathcal{H}}(X)) = E\|\phi_{\mathcal{H}}(X)\|^2_{\mathcal{G}} - \|E\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}}$ as $n \to \infty$.

In the same way, regarding the Exchangeable Weighted Bootstrap estimator, it follows that

$$\hat{V}_X^{E\star} = \sum_{i=1}^{n}\left(w^{\star}_{ni} + \frac{1}{n}\right)\|\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}} - \left\|\sum_{i=1}^{n}\left(w^{\star}_{ni} + \frac{1}{n}\right)\phi_{\mathcal{H}}(X_i)\right\|^2_{\mathcal{G}} =$$

$$= \sum_{i=1}^{n} w^{\star}_{ni}\|\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}} + \frac{1}{n}\sum_{i=1}^{n}\|\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}} - \left\|\sum_{i=1}^{n} w^{\star}_{ni}\phi_{\mathcal{H}}(X_i) + \frac{1}{n}\sum_{i=1}^{n}\phi_{\mathcal{H}}(X_i)\right\|^2_{\mathcal{G}}. \tag{A.2}$$

By the moment assumption, applying the embedding (5), we have that

$$E\|\phi_{\mathcal{H}}(X)\|^4_{\mathcal{G}} = E\|X\|^2_{\mathcal{H}} < \infty.$$

The Corollary 1 in González-Rodríguez and Colubi [34] applied to the sequences $\{\|\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}}\}_{i=1}^{n}$ and $\{\phi_{\mathcal{H}}(X_i)\}_{i=1}^{n}$ together with the Slutsky Theorem guarantee that

$$\sum_{i=1}^{n} w^{\star}_{ni}\|\phi_{\mathcal{H}}(X_i)\|^2_{\mathcal{G}} \xrightarrow{\mathcal{P}} 0 \quad P-\text{a.s.}, \qquad \sum_{i=1}^{n} w^{\star}_{ni}\phi_{\mathcal{H}}(X_i) \xrightarrow{\mathcal{P}} 0 \quad P-\text{a.s.}$$

The Strong Law of Large Numbers and the Continuous Mapping Theorem completes the proof. □

Concerning the Wild Bootstrap statistic, the result follows by applying similar arguments as before.

References

[1] J.D. Gibbons, S. Chakraborti, Nonparametric Statistical Inference: Revised and Expanded, CRC Press, New York, 2014.
[2] O. Thas, Comparing distributions, in: Springer Series in Statistics, Springer, New York, 2010, p. XVIII, 353.
[3] G. Boente, M. Salibian Barrera, D.E. Tyler, A characterization of elliptical distributions and some optimality properties of principal components for functional data, J. Multivariate Anal. 131 (2014) 254–264.
[4] Z. Lin, H.-G. Muller, F. Yao, Mixture inner product spaces and their application to functional data analysis, Ann. Statist. 46 (1) (2018) 370–400.
[5] G. González-Rodríguez, A. Colubi, M. Gil, Fuzzy data treated as functional data: A one-way ANOVA test approach, Comput. Statist. Data Anal. 56 (2012) 943–955.
[6] G. González-Rodríguez, A. Ramos-Guajardo, A. Colubi, A. Blanco-Fernández, A new framework for the statistical analysis of set-valued random elements, Internat. J. Approx. Reason. 92 (2018) 279–294.
[7] L. Horváth, G. Rice, An introduction to functional data analysis and a principal component approach for testing the equality of mean curves, Rev. Mat. Complut. 28 (2015) 505–548.
[8] H.S. Ghale-Joogh, S.M.E. Hosseini-Nasab, A two-sample test for mean functions with increasing number of projections, Statistics 52 (4) (2018) 852–873.
[9] J. Lee, M. Follen, A two sample test for functional data, Commun. Stat. Appl. Methods 22 (2015) 121–135.
[10] J.-T. Zhang, X. Liang, One-way ANOVA for functional data via globalizing the pointwise F-test, Scand. J. Stat. 41 (1) (2014) 51–71.
[11] A. Cuevas, M. Febrero, R. Fraiman, An ANOVA test for functional data, Comput. Statist. Data Anal. 47 (1) (2004) 111–122.
[12] T. Górecki, L. Smaga, fdANOVA: An R software package for analysis of variance for univariate and multivariate functional data, Comput. Statist. 34 (2) (2019) 571–597.
[13] S. Fremdt, J.G. Steinebach, L. Horvath, P. Kokoska, Testing the equality of covariance operators in functional samples, Scand. J. Stat. 40 (1) (2013) 138–152.
[14] G. Boente, D. Rodríguez, M. Sued, Testing equality between several populations covariance operators, Ann. Inst. Statist. Math. 70 (2018) 919–950.
[15] J. Guo, B. Zhou, J.-T. Zhang, Testing the equality of several covariance functions for functional data: A supremum-norm based test, Comput. Statist. Data Anal. 124 (2018) 15–26.

[16] J. Guo, B. Zhou, J.-T. Zhang, New tests for equality of several covariance functions for functional data, J. Amer. Statist. Assoc. 114 (527) (2019) 1251–1263.
[17] J. Guo, B. Zhou, J. Chen, J.-T. Zhang, An L2-norm-based test for equality of several covariance functions: A further study, TEST 28 (2019) 1092–1112.
[18] P. Hall, I.V. Keilegom, Two-sample tests in funcional data analysis starting from discrete data, Statist. Sinica 17 (4) (2007) 1511–1531.
[19] G. Estévez-Pérez, J. Vilar, Functional ANOVA starting from discrete data: an application to air quality data, Environ. Ecol. Stat. 20 (2013) 495–517.
[20] G.-M. Pomann, A.-M. Staicu, S. Ghosh, A two-sample distribution-free test for functional data with application to a diffusion tensor imaging study of multiple sclerosis, J. R. Stat. Soc. Ser. C. Appl. Stat. 65 (3) (2016) 395–414.
[21] R. Bárcenas, J. Ortega, A.J. Quiroz, Quadratic forms of the empirical processes for the two-sample problem for functional data, TEST 26 (2017) 503–526.
[22] Q. Jiang, M. Huskova, S.G. Meintanis, L. Zhu, Asymptotics, finite-sample comparisons and applications for two-sample tests with functional data, J. Multivariate Anal. 170 (2019) 202–220.
[23] F.A. Bugni, J.L. Horowitz, Permutation tests for equality of distributions of functional data, J. Appl. Econometrics (2021) 1–17.
[24] G. Székely, M. Rizzo, Testing for equal distributions in high dimension, InterStat 5 (2004).
[25] C. Zhu, J.-L. Wang, Testing homogeneity: The trouble with sparse functional data, J. R. Stat. Soc. Ser. B Stat. Methodol. 85 (2023) 705–731.
[26] G.J. Székely, M.L. Rizzo, Energy statistics: Statistics based on distances, J. Statist. Plann. Inference 143 (2013) 1249–1272.
[27] R. Lyons, Distance covariance in metric spaces, Ann. Probab. 41 (2013) 3284–3305.
[28] J. Cuesta-Albertos, R. Fraiman, T. Ransford, Random projections and goodness-of-fit tests in infinite-dimensional spaces, Bull. Braz. Math. Soc. 37 (2006) 477–501.
[29] J.A. Cuesta-Albertos, R. Fraiman, T. Ransford, A sharp form of the Cramér–Wold theorem, J. Theoret. Probab. 20 (2007) 201–209.
[30] M. Meckes, Positive definite metric spaces, Positivity 3 (2013) 733–757.
[31] I. Schoenberg, On certain metric spaces arising from euclidean spaces by a change of metric and their imbedding in Hilbert space, Ann. of Math. 38 (1937) 787–793.
[32] I. Schoenberg, Metric spaces and positive definite functions, Trans. Am. Math. Soc. 44 (1938) 522–536.
[33] R. Lyons, Errata to 'distance covariance in metric spaces', Ann. Probab. 46 (2018) 2400–2405.
[34] G. González-Rodríguez, A. Colubi, On the consistency of bootstrap methods in separable Hilbert spaces, Econometr. Stat. 1 (2017) 118–127.
[35] E.J. Pitman, Some Basic Theory for Statistical Inference: Monographs on Applied Probability and Statistics, CRC Press, Boca Raton, 2018.
[36] J. Praestgaard, J. Wellner, Exchangeably weighted bootstraps of the general empirical process, Ann. Probab. 21 (1993) 2053–2086.
[37] M. Ledoux, M. Talagrand, Probability in Banach Spaces: Isoperimetry and Processes, Springer, Berlin, 2006.
[38] A. Gretton, K. Fukumizu, Z. Harchaoui, B.K. Sriperumbudur, A fast, consistent kernel two-sample test, Adv. Neural Inf. Process. Syst. 22 (2009).
[39] A. Gretton, K.M. Borgwardt, M.J. Rasch, B. Schölkopf, A. Smola, A kernel two-sample test, J. Mach. Learn. Res. 13 (1) (2012) 723–773.
[40] D. Sejdinovic, B. Sriperumbudur, A. Gretton, K. Fukumizu, Equivalence of distance-based and RKHS-based statistics in hypothesis testing, Ann. Statist. (2013) 2263–2291.
[41] L. Horváth, P. Kokoszka, Inference for Functional Data with Applications, Vol. 200, Springer, New York, 2012.
[42] M. Febrero-Bande, M. Oviedo de la Fuente, fda.usc: Functional data analysis. Utilities for statistical computing, 2022, R package version 2.1.0.
[43] J.R. Carey, P. Liedo, H.-G. Muller, J.-L. Wang, J.-M. Chiou, Relationship of age patterns of fecundity to mortality, longevity, and lifetime reproduction in a large cohort of mediterranean fruit fly females, J. Gerontol. A: Biol. Sci. Med. Sci. 53A (4) (1998) B245–B251.
[44] H. Müller, U. Stadtmüller, Generalized functional linear models, Ann. Statist. 33 (2) (2005) 774–805.
[45] A. Baíllo, A. Cuevas, Supervised functional classification: A theoretical remark and some comparisons, 2008, arXiv:0806.2831.
[46] R. Modarres, Graphical comparison of high-dimensional distributions, Internat. Statist. Rev. 88 (2020) 698–714.
[47] K. Chwialkowski, D. Sejdinovic, A. Gretton, A wild bootstrap for degenerate kernel tests, 2016, arXiv:1408.5404.
[48] R. Laha, V. Rohatgi, Probability Theory, Wiley, New York, 1979.
[49] J. Wellner, Bootstrap limit theorems: A partial survey, in: A.K.M.E. Saleh (Ed.), Nonparametric Statistics and Related Topics, North-Holland, Amsterdam, 1992, pp. 313–329.