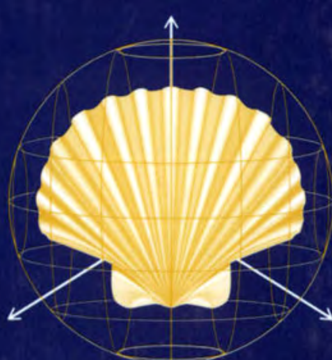
SHELLS

Mathematical modelling and scientific computing

Edited by

M. BERNADOU, P. G. CIARLET
and J. M. VIAÑO



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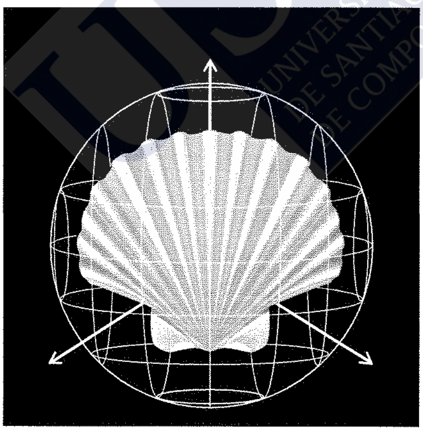

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CONTENTS

One of the fundamental traits of nature seems to be that fundamental physic laws are described in terms of mathematical theories of the utmost power and beauty, the comprehension of which demands a high level of mathematical knowledge.

We could wonder why nature is built in that way. The only answer we can come up with is that our present knowledge seems to show that nature is indeed built that way. We simply have to accept this fact.

Perhaps the situation could be described by asserting that God is a high-level mathematician, and that on creating the Universe he utilised high-level mathematics. Our feeble attempts in the field of mathematics allow us to understand a tiny bit of the Universe; the more complex mathematics we proceed to develop, the better we must expect to understand the Universe ...

P. A. M. Dirac

PREFACE

An "International Conference on Shells" was held on July 14-18, 1997, at the University of Santiago de Compostela, Spain, as part of, and supported by, the Human Capital and Mobility Programme: *Shells, Mathematical Modelling, Analysis and Scientific Computing* of the Commission of the European Communities (Contract No. ERBCHRXCT 940536), in which nine European laboratories and universities were involved. Thirty-five invited lectures by speakers from ten different countries were delivered in this Conference. They covered: asymptotic techniques, numerical methods, convergence analysis for linear and nonlinear lower-dimensional models, eversion problems, junctions, control, and other mathematical and mechanical aspects for shells, plates and rods. These communications are gathered in this Volume.

We express our sincere thanks to all invited speakers and participants in the poster sessions, whose inspired communications made this Conference a success. The support of the following contributing organizations is also deeply appreciated: Universidade de Santiago de Compostela, Dirección General de Enseñanza Superior (M.E.C.), Sociedad Española de Matemática Aplicada, Caixa Galicia and the Laboratories participating in the aforementioned Human Capital and Mobility Programme: Laboratoire d'Analyse Numérique - Université Pierre et Marie Curie (Paris, France), Institut National de Recherche en Informatique et Automatique-INRIA (Rocquencourt, France), Mathematisches Institut A (Stuttgart, Germany), Institut für Statik und Dynamik (Bochum, Germany), Istituto di Analisi Numerica - CNR (Pavia, Italy), Departamento de Matemática - Facultade de Ciencias e Tecnologia (Coimbra, Portugal), Centro de Matematica e Aplicaçoes Fundamentais (Lisbon, Portugal), Departamento de Análisis Matemático - Facultad de Ciencias (Málaga, Spain), Departamento de Matemática Aplicada (Santiago de Compostela, Spain).

Our warmest thanks are also due to Professor Vicente Moreno and to Professor Elena Vázquez who made it possible to locate this Conference within the building of School of Optics, and to the members of Local Organizing Committee: Professors J.A. Álvarez-Dios, J.A. Álvarez-Vázquez, M. Burguera, P. Mato and J.M. Rodríguez-Seijo. They all took a very active share in the general organization and in the editing of the manuscripts.

Santiago de Compostela, July 1997

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AN ASYMPTOTIC BENDING MODEL FOR A GENERAL PLANAR CURVED ROD

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ABSTRACT

Using asymptotic techniques on the re-scaled tridimensional linear elasticity problem for a general planar curved rod in curvilinear coordinates, we prove that the solution of that problem converges to the solution of a one-dimensional bending model generalising the usual Bernoulli-Navier bending model for straight rods.

1. The three-dimensional linear elasticity problem for the curved rod in Cartesian coordinates

We shall henceforth set the notations concerning the curved rod, mainly drawn from Trabucho-Viaño[6], Ciarlet-Lods-Miara[4] and Alvarez-Dios—Viaño[1,2]. Einstenian indices will be routinely used, Latin indices as i, j, k taking the values 1, 2, 3 and Greek ones as α, β the values 1, 2. We fix $\omega \subset \mathbb{R}^2$ an open bounded connected set of constant area equal to A (with no loss of generality we can suppose $A = 1$ as done in early works on asymptotic straight rod studies), and Lipschitz-continuous boundary. We define

$$\omega^\varepsilon = \varepsilon\omega,\ \Omega^\varepsilon = \omega^\varepsilon \times (0, L) \tag{1}$$

where $L > 0$ is of the order of the length of the rod, and $\varepsilon \leq 1$ is taken small when compared to L, for instance in a ratio of $1:10$ or smaller. Note that Ω^ε is naturally the volume enclosed by a general straight rod. We shall also set up the following related notations:

$$\gamma^\varepsilon = \partial\omega^\varepsilon,\ \Gamma_0^\varepsilon = \omega^\varepsilon \times \{0\},\ \Gamma_L^\varepsilon = \omega^\varepsilon \times \{L\},\ \Gamma^\varepsilon = \omega^\varepsilon \times (0, L). \tag{2}$$

A generic point in $\bar\Omega^\varepsilon$ will be noted by $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3)$. We shall represent the outward unit normal vector to $\partial\Omega^\varepsilon$ by $n^\varepsilon = (n_i^\varepsilon)$ and the differential operators $\partial/\partial x_i^\varepsilon$ by ∂_i^ε, even when dealing with functions of x_3 only, and also $\partial_{ij}^\varepsilon = \partial_i^\varepsilon \partial_j^\varepsilon$, etc. Superscripts ε will be dropped when equal to one, so that $\Gamma_L = \Gamma_L^1$, $\Omega = \Omega^1$, etc.

Obviously no generality is lost if we suppose that system $Ox_1^\varepsilon x_2^\varepsilon$ is principal of inertia to the body Ω^ε and therefore $\displaystyle\int_{\omega^\varepsilon} x_\alpha^\varepsilon \, d\omega^\varepsilon = \int_{\omega^\varepsilon} x_1^\varepsilon x_2^\varepsilon \, d\omega^\varepsilon = 0$. We also define the inertia moments by

$$I_\alpha^\varepsilon = \int_{\omega^\varepsilon} (x_\alpha^\varepsilon)^2 \tag{3}$$

We shall consider the following parametrised space curve:

$$C = \{\theta(x_3) = (\theta_i(x_3)) = (\theta_1(x_3), \theta_2(x_3), x_3) \in \mathbb{R}^3 : \; x_3 \in [0, L]\} \tag{4}$$

where $\theta_\alpha(x_3)$ are supposed to be differentiable functions of x_3 of class C^3 at least. It is a known fact that we can define a local orthonormal basis formed by the tangent, normal and binormal vectors constituting the Frenet trihedron to curve C. The curvature of C will be denoted by $\kappa(x_3)$ and the torsion by $\tau(x_3)$. For simplicity we shall suppose that $\tau = 0$, although this is not strictly necessary by any means. We also denote:

$$\mathcal{A} = |\partial_3\theta|^2 = \partial_3\theta_\alpha\partial_3\theta_\alpha + 1. \tag{5}$$

Next, for each $\varepsilon > 0$, we shall define map $\varphi : \; \bar{\Omega}^\varepsilon \longrightarrow \varphi(\bar{\Omega}^\varepsilon) \subset \mathbb{R}^3$ by the following expression

$$\varphi(x^\varepsilon) = (\theta_1(x_3),\, \theta_2(x_3),\, x_3) + x_1^\varepsilon n(x_3) + x_2^\varepsilon b(x_3) \tag{6}$$

We assume that the three vectors $g_i^\varepsilon = \partial_i^\varepsilon\varphi$ are linearly independent at all points of $\bar{\Omega}^\varepsilon$, thus constituting a local basis of $\mathbb{R}^3$ called the *covariant basis*. Consequently we define the *contravariant basis* $(g^{j,\varepsilon})$ in the usual manner:

$$g_i^\varepsilon \cdot g^{j,\varepsilon} = \delta_{ij}. \tag{7}$$

The covariant and contravariant components of the metric tensor are then given by

$$g_{ij}^\varepsilon = g_i^\varepsilon \cdot g_j^\varepsilon, \; g^{ij,\varepsilon} = g^{i,\varepsilon} \cdot g^{j,\varepsilon}. \tag{8}$$

We also define

$$g^\varepsilon = \det(g_{ij}^\varepsilon) \tag{9}$$

and we note by $\Gamma_{ij}^{p,\varepsilon}$ the Christoffel symbols measuring the curvature of the curvilinear coordinate axes, given by

$$\Gamma_{ij}^{p,\varepsilon} = g^{p,\varepsilon} \cdot \partial_j^\varepsilon g_i^\varepsilon. \tag{10}$$

The elastic body occupying volume $\{\hat{\Omega}^\varepsilon\}^-$ is called a *curved rod of axis C*. In a similar way as with Ω^ε, a generic point of $\{\hat{\Omega}^\varepsilon\}^-$ will be denoted by $\hat{x}^\varepsilon = (\hat{x}_i^\varepsilon)$, also $\hat{\partial}_i^\varepsilon$ will stand for the differential operators $\partial/\partial\hat{x}_i^\varepsilon$, and we define

$$\hat{\Gamma}^\varepsilon = \varphi(\Gamma^\varepsilon), \hat{\Gamma}_0^\varepsilon = \varphi(\Gamma_0^\varepsilon) \text{ and } \hat{\Gamma}_L^\varepsilon = \varphi(\Gamma_L^\varepsilon). \tag{11}$$

We shall endeavour to write down a weak formulation of the linear elasticity problem for the solid $\{\hat{\Omega}^\varepsilon\}^-$. Let therefore $\lambda^\varepsilon > 0$, $\mu^\varepsilon > 0$ be the Lamé constants for the material the rod is made of. We denote the displacement field in $\hat{\Omega}^\varepsilon$ by $\hat{u}^\varepsilon = (\hat{u}_i^\varepsilon)$. The Green-Saint Venant linear strain tensor then takes the form:

$$\hat{e}_{ij}^\varepsilon(\hat{u}^\varepsilon) = \frac{1}{2}\left(\hat{\partial}_i^\varepsilon\hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon\hat{u}_i^\varepsilon\right). \tag{12}$$

As boundary conditions we shall suppose both ends of the rod to be clamped (other conditions could of course have been considered). Consequently the natural space for the displacement field can be written as:

$$V(\hat{\Omega}^\varepsilon) \;=\; \{\hat{v}^\varepsilon = (\hat{v}_i^\varepsilon) \in [H^1(\hat{\Omega}^\varepsilon)]^3 : \; \hat{v}^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon \cup \hat{\Gamma}_L^\varepsilon\} \tag{13}$$

endowed with the usual norm, denoted by $\| \ \|_{1,\hat{\Omega}^\varepsilon}$. Supposing the rod is acted on by volume forces $\hat{f}^\varepsilon = (\hat{f}_i^\varepsilon) : \hat{\Omega}^\varepsilon \longrightarrow \mathbb{R}^3$ and surface forces $\hat{h}^\varepsilon = (\hat{h}_i^\varepsilon) : \hat{\Gamma}^\varepsilon \longrightarrow \mathbb{R}^3$, we have the following formulation for the tridimensional linear elasticity problem for the curved rod:

$$\begin{aligned}
&\text{Find } \hat{u}^\varepsilon \in V(\hat{\Omega}^\varepsilon) \text{ such that} \\
&\int_{\hat{\Omega}^\varepsilon} \{\lambda^\varepsilon \hat{e}_{pp}^\varepsilon(\hat{u}^\varepsilon)\hat{e}_{pp}^\varepsilon(\hat{v}^\varepsilon) + 2\mu^\varepsilon \hat{e}_{ij}^\varepsilon(\hat{u}^\varepsilon)\hat{e}_{ij}^\varepsilon(\hat{v}^\varepsilon)\}\, d\hat{x}^\varepsilon = \\
&\qquad \int_{\hat{\Omega}^\varepsilon} \hat{f}_i^\varepsilon \hat{v}_i^\varepsilon \, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon} \hat{h}_i^\varepsilon \hat{v}_i^\varepsilon \, d\hat{a}^\varepsilon, \ \forall \hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon).
\end{aligned} \tag{14}$$

2. Passing to curvilinear coordinates in the straight rod Ω^ε

A necessary step to the asymptotic study is to write a problem equivalent to (14) but posed in the domain Ω independent of ε (see Ciarlet-Lods-Miara[4], Trabucho-Viaño[6],...). Therefore we shall pose problem (14) in curvilinear coordinates on domain Ω^ε as a previous step. With this aim in mind, we express any field $\hat{v}^\varepsilon = (\hat{v}_i^\varepsilon) \in H^1(\hat{\Omega}^\varepsilon)$ in the contravariant basis $\hat{v}^\varepsilon = v_i^\varepsilon g^{i,\varepsilon}$ where $v_i^\varepsilon \in H^1(\Omega^\varepsilon)$ are called the *contravariant components* to $\hat{v}^\varepsilon$. On the contrary, applied forces $\hat{f}^\varepsilon$, $\hat{h}^\varepsilon$ will be expressed by their *covariant components* $f^{i,\varepsilon}$, $h^{i,\varepsilon}$, and we have

$$\hat{f}^\varepsilon(\hat{x}^\varepsilon) = f^{i,\varepsilon}(x^\varepsilon) g_i^\varepsilon(x^\varepsilon), \hat{h}^\varepsilon(\hat{x}^\varepsilon) = h^{i,\varepsilon}(x^\varepsilon) g_i^\varepsilon(x^\varepsilon). \tag{15}$$

We define the *covariant derivative* of a field $v^\varepsilon = v_i^\varepsilon g^{i,\varepsilon}$ in the usual fashion:

$$v_{k||l}^\varepsilon = \partial_l^\varepsilon v_k^\varepsilon - \Gamma_{lk}^q v_q^\varepsilon \tag{16}$$

The quantity $e_{i||j}^\varepsilon(u^\varepsilon) = \frac{1}{2}(u_{i||j}^\varepsilon + u_{j||i}^\varepsilon)$ will be called the *curvilinear strain tensor*. Taking stock of (16) and the symmetry of the Christoffel symbols we obtain

$$e_{i||j}^\varepsilon(u^\varepsilon) = \frac{1}{2}(\partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_j^\varepsilon) - \Gamma_{ij}^p u_p^\varepsilon. \tag{17}$$

We have the following identities relating the area and surface elements before and after the change to curvilinear coordinates:

$$d\hat{x}^\varepsilon = \sqrt{g^\varepsilon}\, dx^\varepsilon \tag{18}$$
$$d\hat{a}^\varepsilon = \sqrt{g^\varepsilon}\, |g^{\alpha,\varepsilon} n_\alpha^\varepsilon| \, da^\varepsilon = \tilde{g}^\varepsilon \, da^\varepsilon \text{ on } \Gamma^\varepsilon. \tag{19}$$

We can state the following result:
Theorem 1 *In the above conditions, we have that the contravariant components of the displacement $(u_i^\varepsilon) \in V(\Omega^\varepsilon)$ solve the following problem:*

$$\begin{aligned}
&\textit{Find } u^\varepsilon \in V(\Omega^\varepsilon) \textit{ such that} \\
&\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{i||j}^\varepsilon(u^\varepsilon) e_{k||l}^\varepsilon(v^\varepsilon)\sqrt{g^\varepsilon}\, dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\, dx^\varepsilon + \int_{\Gamma^\varepsilon} h^{i,\varepsilon} v_i^\varepsilon \tilde{g}^\varepsilon \, d\hat{a}^\varepsilon, \\
&\qquad \forall v^\varepsilon \in V(\Omega^\varepsilon).
\end{aligned} \tag{20}$$

where

$$A^{ijkl,\varepsilon} = \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{kj,\varepsilon}). \tag{21}$$

Problem (20) possesses one and only one solution as a consequence of the following generalised version of Korn's inequality (Ciarlet-Lods-Miara[4]):

$$\|v^\varepsilon\|_{1,\Omega^\varepsilon} \le c(\Omega^\varepsilon)\left\{\sum_{i,j}\left|e^\varepsilon_{i\|j}(v^\varepsilon)\right|_{0,\Omega^\varepsilon} + |v^\varepsilon|^2_{0,\Omega^\varepsilon}\right\}^{1/2}, \forall v^\varepsilon \in H^1(\Omega^\varepsilon) \tag{22}$$

where $c(\Omega^\varepsilon)$ is a constant depending on Ω^ε only.

3. The obtained model

We suppose the volume forces to be of order $O(\varepsilon^2)$ and the surface forces of order $O(\varepsilon^3)$. Scaling the points of Ω^ε so that they are defined over the fixed domain Ω, we apply the techniques in Ciarlet-Lods-Miara[4] to study the limit of the solution to problem (20). We shall give an overview of the obtained results.

We introduce the following preliminary notations: for any $v = (v_i) \in [H^1(\Omega)]^3$ let functions $\rho_\alpha(v) \in H^{-1}(\Omega)$, $\gamma_{33}(v) \in L^2(\Omega)$ be defined by:

$$\rho_1(v) = \partial_{33}v_1 - \frac{\partial_3\mathcal{A}}{2\mathcal{A}}\partial_3 v_1 + \partial_3\kappa v_3 + \kappa\gamma_{33}(v) \tag{23}$$

$$\rho_2(v) = \partial_{33}v_2 - \frac{\partial_3\mathcal{A}}{2\mathcal{A}}\partial_3 v_2 \tag{24}$$

$$\gamma_{33}(v) = \partial_3 v_3 - \mathcal{A}\kappa v_1 - \frac{\partial_3\mathcal{A}}{2\mathcal{A}}v_3 \tag{25}$$

and let space $V_F(0, L)$ be defined by:

$$V_F(0,L) = \{\eta = (\eta_i) \in [H^2(0,L)]^3 : \gamma_{33}(v) = 0, \tag{26}$$

$$\eta_i(0) = \eta_i(L) = \partial_3\eta_\alpha(0) = \partial_3\eta_\alpha(L) = 0\}. \tag{27}$$

Next we state our main result:

Theorem 2 *For $0 < \varepsilon < \varepsilon_0$, let us denote by $u(\varepsilon) = (u_i(\varepsilon)) \in V(\Omega)$ the solution of the scaled variational problem (20). Then there exists $u = (u_i) \in V(\Omega)$ such that*

$$u(\varepsilon) \longrightarrow u \text{ in } H^1(\Omega) \text{ as } \varepsilon \to 0 \tag{28}$$

$$u \text{ is independent of planar variables } x_\alpha \tag{29}$$

$$\bar{u} = (\bar{u}_i) = \frac{1}{A}\int_\omega u\, d\omega \in V_F(0,L) \tag{30}$$

and function $\bar{u}$ solves the following one-dimensional variational problem:

$$EI_\alpha \int_0^L \frac{1}{\mathcal{A}^{3/2}}\rho_\alpha(\bar{u})\rho_\alpha(\eta)\, dx_3 = \int_0^L F^i\eta_i\, dx_3,\ \forall \eta \in V_F(0,L) \tag{31}$$

where

$$F^i = \int_\omega f^i\sqrt{\mathcal{A}}\, d\omega + \int_\gamma g^i\sqrt{\mathcal{A}}\, d\gamma. \tag{32}$$

and

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{33}$$

4. Acknowledgements

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5. References

1. J. A. Alvarez-Dios and J. M. Viaño, Une théorie asymptotique de flexion-extension pour les poutres élastiques faiblement courbées. *C. R. Acad. Sci. Paris* **Sér. I Math.** 321 (1995), no. 10, 1395–1400.
2. J. A. Alvarez-Dios and J. M. Viaño, Mathematical justification of a one-dimensional model for general elastic shallow arches, *Math. Meth. Appl. Sc.* To appear.
3. P. G. Ciarlet and V. Lods, Asymptotic analysis of linearly elastic shells. I. Justification of membrane shell equations, *Arch. Rational Mech. Anal.*, **136** (1996),119-161.
4. P. G. Ciarlet, V. Lods, and B. Miara, Asymptotic analysis of linearly elastic shells, II. Flexural shells, *Arch. Rational Mech. Anal.*, **136** (1996),163-190.
5. Figueiredo, I. de and L. Trabucho, A Galerking approximation for curved beams, *Comp. Meth. Appl. Mech. Eng.*, **102** (1993), 235-253.
6. Trabucho, L. and J. M. Viaño, Existence and characterization of higher order terms in an asymptotic expansion method for linearized elastic beams, *Asymptotic Analysis* **2** (1989), 223–255.

ASYMPTOTIC MODELING OF GENUINELY CLAMPED BEAMS

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ABSTRACT

We present several limit models for genuinely clamped elastic beams obtained from linear three-dimensional elasticity using an asymptotic method and we prove existence and uniqueness of solution for these models. We also obtain the convergence to the usual clamped beam model.

1. The Three-Dimensional Problem

The principal aim of this paper is the obtention of models for the displacement of a linearly elastic beam with non-classical boundary conditions. Usual beam models are obtained for strongly or weakly clamped ends (a complete analysis of rod models with exhaustive bibliographic references may be found in the extensive work of Trabucho-Viaño [5]), but we introduce here, following the notation of Blanchard-Xiang [3] for plates, a more realistic condition: the genuine clamping, where the beam is not only fixed at both extremities but also at thin neighbourhoods in the lateral surface. Let ε be a positive real parameter and let ω^ε be an open bounded connected set of the plane $0x_1^\varepsilon x_2^\varepsilon$ with boundary γ^ε smooth enough and with area $A(\omega^\varepsilon) = \varepsilon^2$. Given the constants $L > 0,\ \delta > 0$ we define:

$$\Omega^\varepsilon = \omega^\varepsilon \times (0, L), \qquad \Gamma_0^\varepsilon = \overline{\omega^\varepsilon} \times \{0\}, \qquad \Gamma_L^\varepsilon = \overline{\omega^\varepsilon} \times \{L\}, \tag{1}$$

$$\Gamma_0^{\varepsilon\delta} = \gamma^\varepsilon \times (0, \delta], \qquad \Gamma_L^{\varepsilon\delta} = \gamma^\varepsilon \times [L-\delta, L), \qquad \Gamma^{\varepsilon\delta} = \gamma^\varepsilon \times (\delta, L-\delta). \tag{2}$$

We consider the elastic beam of length L occupying the reference configuration $\overline{\Omega^\varepsilon}$. We denote the generic points of Ω^ε by $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$, the partial derivatives $\partial/\partial x_i^\varepsilon$ by ∂_i and, for functions only depending on x_3^ε, the derivative by a prime. We use, as it is customary in elasticity theory, the summation convention on repeated indices and we assume that Latin indices range over $\{1,2,3\}$ and Greek ones over $\{1,2\}$. We also assume that $0x_1^\varepsilon x_2^\varepsilon x_3^\varepsilon$ is a principal system of inertia.

We study here, in the linearized elasticity framework, the physical problem corresponding to the mechanical behavior of an elastic beam, supposed to be genuinely clamped at both ends (i.e., the beam is not only fixed at both extremities Γ_0^ε and Γ_L^ε but also at the neighbourhoods $\Gamma_0^{\varepsilon\delta}$ and $\Gamma_L^{\varepsilon\delta}$ in the lateral surface of width δ) and submitted to body forces of density $f^\varepsilon \equiv (f_i^\varepsilon)$ in Ω^ε and surface forces of density $g^{\varepsilon\delta} \equiv (g_i^{\varepsilon\delta})$ on $\Gamma^{\varepsilon\delta}$. We assume the constitutive material of the beam to be an homogeneous isotropic elastic material of Saint Venant-Kirchhoff's type with Young's modulus E and Poisson's ratio ν. Then, the displacement field $u^{\varepsilon\delta} \in V^{\varepsilon\delta}(\Omega^\varepsilon)$ and the Piola-Kirchhoff stress tensor $\sigma^{\varepsilon\delta} \in \Sigma^\varepsilon(\Omega^\varepsilon)$ are the solution of the following mixed variational problem:

$$\int_{\Omega^\varepsilon} \left(\frac{1+\nu}{E}\sigma_{ij}^{\varepsilon\delta} - \frac{\nu}{E}\sigma_{kk}^{\varepsilon\delta}\delta_{ij}\right)\tau_{ij}^\varepsilon dx^\varepsilon - \int_{\Omega^\varepsilon} \gamma_{ij}(u^{\varepsilon\delta})\tau_{ij}^\varepsilon dx^\varepsilon = 0, \quad \forall\, \tau^\varepsilon \in \Sigma^\varepsilon(\Omega^\varepsilon), \tag{3}$$

$$\int_{\Omega^\varepsilon} \sigma_{ij}^{\varepsilon\delta}\gamma_{ij}(v^\varepsilon)dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma^{\varepsilon\delta}} g_i^{\varepsilon\delta} v_i^\varepsilon da^\varepsilon, \quad \forall\, v^\varepsilon \in V^{\varepsilon\delta}(\Omega^\varepsilon), \tag{4}$$

where $\gamma_{ij}(u^{\varepsilon\delta}) = \frac{1}{2}(\partial_i u_j^{\varepsilon\delta} + \partial_j u_i^{\varepsilon\delta})$ denotes the linearized strain tensor, $\Sigma^\varepsilon(\Omega^\varepsilon) = \{\tau \equiv (\tau_{ij}) \in [L^2(\Omega^\varepsilon)]^9 : \tau_{ij} = \tau_{ji}\}$ and $V^{\varepsilon\delta}(\Omega^\varepsilon)$ is a subspace of $[H^1(\Omega^\varepsilon)]^3$ appropiate for clamping conditions. As it is well known (Ciarlet [4]), this problem has an unique solution.

In order to study the behaviour of $(u^{\varepsilon\delta}, \sigma^{\varepsilon\delta})$ as the area of the section goes to zero, we follow the classical procedure ([2], [5]): change to a fixed domain independent of ε, scaling of the fields, assumptions on the applied forces and passing to the limit.

So, we consider the domain ω with unitary area given by $\omega = \varepsilon^{-1}\omega^\varepsilon$ with boundary γ and the reference beam of section ω occupying the volume $\overline{\Omega}$, where:

$$\Omega = \omega \times (0, L), \qquad \Gamma_0 = \overline{\omega} \times \{0\}, \qquad \Gamma_L = \overline{\omega} \times \{L\}, \tag{5}$$

$$\Gamma_0^\delta = \gamma \times (0, \delta], \qquad \Gamma_L^\delta = \gamma \times [L-\delta, L), \qquad \Gamma^\delta = \gamma \times (\delta, L-\delta). \tag{6}$$

Thus, we can define the change of variable from Ω^ε to the fixed domain Ω:

$$\pi^\varepsilon : x \equiv (x_1, x_2, x_3) \in \overline{\Omega} \longrightarrow \pi^\varepsilon(x_1, x_2, x_3) = (\varepsilon x_1, \varepsilon x_2, x_3) \equiv (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \equiv x^\varepsilon \in \overline{\Omega^\varepsilon} \tag{7}$$

and we define the rescaled fields $u^\delta(\varepsilon)$ and $\sigma^\delta(\varepsilon)$ by:

$$u_\alpha^\delta(\varepsilon)(x) = \varepsilon u_\alpha^{\varepsilon\delta}(x^\varepsilon), \quad u_3^\delta(\varepsilon)(x) = u_3^{\varepsilon\delta}(x^\varepsilon), \tag{8}$$

$$\sigma_{\alpha\beta}^\delta(\varepsilon)(x) = \varepsilon^{-2}\sigma_{\alpha\beta}^{\varepsilon\delta}(x^\varepsilon), \quad \sigma_{\alpha 3}^\delta(\varepsilon)(x) = \varepsilon^{-1}\sigma_{\alpha 3}^{\varepsilon\delta}(x^\varepsilon), \quad \sigma_{33}^\delta(\varepsilon)(x) = \sigma_{33}^{\varepsilon\delta}(x^\varepsilon). \tag{9}$$

We also assume that the applied forces are such that:

$$f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon f_\alpha(x), \quad f_3^\varepsilon(x^\varepsilon) = f_3(x), \quad g_\alpha^{\varepsilon\delta}(x^\varepsilon) = \varepsilon^2 g_\alpha^\delta(x), \quad g_3^{\varepsilon\delta}(x^\varepsilon) = \varepsilon g_3^\delta(x), \tag{10}$$

where $f_i \in L^2(\Omega)$ and $g_i^\delta \in L^2(\Gamma^\delta)$ are independent on ε.

Then, in a classical way, we obtain that $(u^\delta(\varepsilon), \sigma^\delta(\varepsilon))$ is the unique solution of a rescaled variational problem posed in Ω (see [1]), equivalent to (3)–(4).

2. The One-Dimensional Models

The main result of this paper is the following theorem, whose proof is made in an analogous way to the case of classical beams ([5]), where we study the convergence as $\varepsilon \to 0$:

Theorem 1.- We assume that $f_i \in L^2(\Omega)$ and $g_i^\delta \in L^2(\Gamma^\delta)$. We introduce the notations:

$$I_\alpha = \int_\omega {x_\alpha}^2, \qquad F_i^\delta = \int_\omega f_i + \int_\gamma \tilde{g}_i^\delta, \qquad M_\alpha^\delta = \int_\omega x_\alpha f_3 + \int_\gamma x_\alpha \tilde{g}_3^\delta, \tag{11}$$

where $\tilde{g}_i^\delta$ is the extension by zero of g_i^δ to $\Gamma_0^\delta \cup \Gamma_L^\delta$. Then:

(i) As ε goes to zero the sequences $\{u^\delta(\varepsilon)\}$ and $\{\sigma^\delta(\varepsilon)\}$ verify:

$$u^\delta(\varepsilon) \to u^{0\delta} \quad \text{in } H^1(\Omega), \tag{12}$$
$$\sigma^\delta_{33}(\varepsilon) \to \sigma^{0\delta}_{33}, \quad \varepsilon\sigma^\delta_{\alpha 3}(\varepsilon) \to 0, \quad \varepsilon^2\sigma^\delta_{\alpha\beta}(\varepsilon) \to 0 \quad \text{in } L^2(\Omega), \tag{13}$$
$$q^\delta_\alpha(\varepsilon) = \int_\omega \sigma^\delta_{\alpha 3}(\varepsilon) \to q^{0\delta}_\alpha \quad \text{in } H^{-1}(0,L). \tag{14}$$

(ii) The element $u^{0\delta}$ is a Bernoulli-Navier displacement, that is:

$$u^{0\delta}_\alpha(x_1,x_2,x_3) = \xi^\delta_\alpha(x_3), \qquad \xi^\delta_\alpha \in H^2_0(0,L), \tag{15}$$
$$u^{0\delta}_3(x_1,x_2,x_3) = \xi^\delta_3(x_3) - x_\alpha(\xi^\delta_\alpha)'(x_3), \qquad \xi^\delta_3 \in H^1_0(0,L). \tag{16}$$

(iii) The stress component $\sigma^{0\delta}_{33} \in L^2(\Omega)$ is given by $\sigma^{0\delta}_{33} = E\{(\xi^\delta_3)' - x_\alpha(\xi^\delta_\alpha)''\}$.

(iv) The shear forces $q^{0\delta}_\alpha \in H^{-1}(0,L)$ are given by $q^{0\delta}_\alpha = M^\delta_\alpha - EI_\alpha(\xi^\delta_\alpha)'''$. □

The characterization of the components (ξ^δ_i) depends on the type of genuine clamping condition. We introduce the following functional spaces:

$$H^1_\delta(0,L) = \{v \in H^1_0(0,L) \ : \ v = 0 \quad \text{in } [0,\delta] \cup [L-\delta, L]\}, \tag{17}$$
$$H^2_\delta(0,L) = \{v \in H^2_0(0,L) \ : \ v = v' = 0 \quad \text{in } [0,\delta] \cup [L-\delta, L]\}, \tag{18}$$
$$W^1_\delta(0,L) = \{v \in H^1_0(0,L) \ : \ \int_0^\delta v = \int_{L-\delta}^L v = 0\}, \tag{19}$$
$$\begin{aligned} W^2_\delta(0,L) = \{v \in H^2_0(0,L) \ &: \ v(\delta) = v(L-\delta) = 0, \\ \int_0^\delta v = \int_{L-\delta}^L v = 0,& \ \int_0^\delta x_3 v = \int_{L-\delta}^L x_3 v = 0\}. \end{aligned} \tag{20}$$

We considere now four different examples of genuine clamping:

Model 1: Strong clamping.- The beam is strongly genuinely clamped, that is:

$$u^\delta(\varepsilon) = 0 \quad \text{on } \Gamma_0 \cup \Gamma_L \cup \Gamma^\delta_0 \cup \Gamma^\delta_L. \tag{21}$$

Then, $(\xi^\delta_i) \in [H^2_\delta(0,L)]^2 \times H^1_\delta(0,L)$ is the unique solution of the variational problem:

$$\int_\delta^{L-\delta} EI_\alpha(\xi^\delta_\alpha)''v''_\alpha = \int_\delta^{L-\delta} F^\delta_\alpha v_\alpha - \int_\delta^{L-\delta} M^\delta_\alpha v'_\alpha, \quad \forall\ (v_\alpha) \in [H^2_\delta(0,L)]^2, \tag{22}$$
$$\int_\delta^{L-\delta} E(\xi^\delta_3)'v'_3 = \int_\delta^{L-\delta} F^\delta_3 v_3, \quad \forall\ v_3 \in H^1_\delta(0,L). \tag{23}$$

Model 2: Weak clamping.- The beam is weakly genuinely clamped, that is:

$$\int_{\Gamma_0} u^\delta(\varepsilon) = \int_{\Gamma_L} u^\delta(\varepsilon) = \int_{\Gamma^\delta_0} u^\delta(\varepsilon) = \int_{\Gamma^\delta_L} u^\delta(\varepsilon) = 0, \tag{24}$$
$$\int_{\Gamma_0} x \times u^\delta(\varepsilon) = \int_{\Gamma_L} x \times u^\delta(\varepsilon) = \int_{\Gamma^\delta_0} x \times u^\delta(\varepsilon) = \int_{\Gamma^\delta_L} x \times u^\delta(\varepsilon) = 0. \tag{25}$$

Then, under a simple geometric hypothesis, $(\xi^\delta_i) \in [W^2_\delta(0,L)]^2 \times W^1_\delta(0,L)$ is the unique solution of the variational problem:

$$\int_0^L EI_\alpha(\xi^\delta_\alpha)''v''_\alpha = \int_0^L F^\delta_\alpha v_\alpha - \int_0^L M^\delta_\alpha v'_\alpha, \quad \forall\ (v_\alpha) \in [W^2_\delta(0,L)]^2, \tag{26}$$
$$\int_0^L E(\xi^\delta_3)'v'_3 = \int_0^L F^\delta_3 v_3, \quad \forall\ v_3 \in W^1_\delta(0,L). \tag{27}$$

Model 3: Mixed (strong-weak) clamping.- The beam is strongly genuinely clamped in the directions x_α and weakly in x_3, that is:

$$u_\alpha^\delta(\varepsilon) = 0 \quad \text{on } \Gamma_0 \cup \Gamma_L \cup \Gamma_0^\delta \cup \Gamma_L^\delta, \tag{28}$$

$$\int_{\Gamma_0} u_3^\delta(\varepsilon) = \int_{\Gamma_L} u_3^\delta(\varepsilon) = \int_{\Gamma_0^\delta} u_3^\delta(\varepsilon) = \int_{\Gamma_L^\delta} u_3^\delta(\varepsilon) = 0. \tag{29}$$

Then, $(\xi_i^\delta) \in [H_\delta^2(0,L)]^2 \times W_\delta^1(0,L)$ is the unique solution of the variational problem (22), (27).

Model 4: Mixed (weak-strong) clamping.- The beam is weakly genuinely clamped in the directions x_α and strongly in x_3, that is:

$$\int_{\Gamma_0} u_\alpha^\delta(\varepsilon) = \int_{\Gamma_L} u_\alpha^\delta(\varepsilon) = \int_{\Gamma_0^\delta} u_\alpha^\delta(\varepsilon) = \int_{\Gamma_L^\delta} u_\alpha^\delta(\varepsilon) = 0, \tag{30}$$

$$\begin{aligned} \int_{\Gamma_0} (x_1 u_2^\delta(\varepsilon) - x_2 u_1^\delta(\varepsilon)) &= \int_{\Gamma_L} (x_1 u_2^\delta(\varepsilon) - x_2 u_1^\delta(\varepsilon)) \\ &= \int_{\Gamma_0^\delta} (x_1 u_2^\delta(\varepsilon) - x_2 u_1^\delta(\varepsilon)) = \int_{\Gamma_L^\delta} (x_1 u_2^\delta(\varepsilon) - x_2 u_1^\delta(\varepsilon)) = 0, \end{aligned} \tag{31}$$

$$u_3^\delta(\varepsilon) = 0 \quad \text{on } \Gamma_0 \cup \Gamma_L \cup \Gamma_0^\delta \cup \Gamma_L^\delta. \tag{32}$$

Then, $(\xi_i^\delta) \in [H_\delta^2(0,L)]^2 \times H_\delta^1(0,L)$ is the unique solution of the variational problem (22), (23), corresponding to model 1.

3. Conclusions

We have obtained, as the limit of the three-dimensional elasticity model and without any *a priori* hipothesis, new one-dimensional models for the genuinely clamped beams. In order to achieve a strongly genuinely clamped model for the linear beam it is indispensable to impose a strong genuine clamping condition in direction x_3, not being necessary to impose it in the other directions. The convergence of the genuinely clamped model to the classical clamped model as δ goes to zero is also obtained, as can be seen in [1], where we deal too with the nonlinear case.

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4. References

1. L. J. Alvarez-Vázquez, A. R. Rodríguez and J. M. Viaño, (to appear).
2. A. Bermúdez and J. M. Viaño, *RAIRO Anal. Num.* **18** (1984) 347–376.
3. D. Blanchard and Y. Xiang, *Asymptotic Analysis* **5** (1992) 495–516.
4. P. G. Ciarlet, *Mathematical Elasticity, I. Three-Dimensional Elasticity* (North-Holland, Amsterdam, 1988).
5. L. Trabucho and J. M. Viaño, in *Handbook of numerical analysis, IV*, eds. P. G. Ciarlet and J. L. Lions (North-Holland, Amsterdam, 1996) 487–974.

THE EVERSION OF NONLINEARLY ELASTIC SHELLS

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ABSTRACT

This note describes the eversion of axisymmetric, nonlinearly elastic shells within a general, geometrically exact theory in which the shell can suffer flexure, extension, and shear. We give conditions on a thickness parameter ε and on the data ensuring that there is an everted state under zero applied load, and we show how to approximate it effectively with an asymptotic series in ε whose error we can estimate.

1. The Governing Equations

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a fixed right-handed orthonormal basis for Euclidean 3-space. For each angle ϕ we set

$$\mathbf{e}_1(\phi) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}, \quad \mathbf{e}_2(\phi) = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}. \tag{1}$$

The *axisymmetric configuration of a shell* that can suffer flexure, mid-surface extension, and shear is determined by a pair of vector-valued functions $\mathbf{r}$ and $\mathbf{b}$ of s and ϕ of the form

$$\mathbf{r}(s,\phi) = r(s)\mathbf{e}_1(\phi) + z(s)\mathbf{k}, \quad \mathbf{b}(s,\phi) = -\sin\theta(s)\mathbf{e}_1(\phi) + \cos\theta(s)\mathbf{k}. \tag{2}$$

We define $\mathbf{a}(s,\phi) = \mathbf{e}_2(\phi) \times \mathbf{b}(s,\phi) = \cos\theta(s)\mathbf{e}_1(\phi) + \sin\theta(s)\mathbf{k}$. The reference configuration is defined by $\theta = -\psi$ and $\mathbf{r}_s = \cos\psi\mathbf{e}_1 - \sin\psi\mathbf{k}$, where subscripted letters denote partial derivatives. The latter equation is interpreted as defining undeformed material base surface (e.g., a mid-surface) in a shell-like three-dimensional body. The variable s, which is the arc-length parameter of each meridian of the base surface, lies in an interval of the form $[0, l]$, and ϕ, which is the longitude for the base surface, lies in $[0, 2\pi]$. The function $\mathbf{r}$ in (2) gives the deformed image of the base surface, and the vector $\mathbf{b}(s,\phi)$ is interpreted as characterizing the deformed configuration of the material fiber whose reference configuration is on the normal to the base surface through the point with coordinates (s,ϕ).

We assume that the axisymmetric surface $\mathbf{r}$ is simply-connected and smooth so that $r(0) = 0$, $\theta(0) = 0$. Let ρ be the value of r in the undeformed configuration. For the particular problems we treat, we assume that $\rho(s) > 0$ for $0 < s \leq l$.

Our asymptotic analysis is carried out in terms of a small positive dimensionless thickness parameter ε, with ε^2 regarded as the ratio of a typical thickness to a typical length. The disposition of this parameter in both the kinematic relations and the constitutive equations given below is inspired by three-dimensional interpretations of our problem, which we do not pause to describe.

The strains for our problem are

$$\tau \equiv \frac{r}{\rho}, \quad \nu \equiv \mathbf{r}_s \cdot \mathbf{a}, \quad \eta \equiv \mathbf{r}_s \cdot \mathbf{b}(s,\phi), \quad \omega \equiv \varepsilon^2 \frac{\sin\theta}{\rho}, \quad \mu \equiv \varepsilon^2 \theta_s, \tag{3}$$

from which we derive

$$(\rho\tau)_s = \nu\cos\theta - \eta\sin\theta. \tag{4}$$

Let $\varepsilon^2 N(s)\mathbf{a}(s,\phi) + \varepsilon^2 H(s)\mathbf{b}(s,\phi)$ and $-\varepsilon^4 M(s)\mathbf{e}_2(\phi)$ denote the resultant contact force and contact couple per unit reference length of the base circle of latitude s that are exerted across this section at (s,ϕ). Let $\varepsilon^2 T(s)\mathbf{e}_2(\phi)$ and $\varepsilon^4 \Omega(s)\mathbf{a}(s,\phi)$ denote the resultant contact force and contact couple per unit reference length of the meridian ϕ that are exerted across this section at (s,ϕ). These representations are consistent with axisymmetry.

We assume that the shell is subjected to no applied force and no applied couple. Then by summing forces and moments on a typical region we obtain the classical form of the equilibrium equations:

$$[\rho(N\cos\theta - H\sin\theta)]_s - T = 0, \tag{5}$$

$$N\sin\theta + H\cos\theta = 0, \tag{6}$$

$$\varepsilon^2(\rho M)_s - \varepsilon^2 \Omega\cos\theta + \rho(\nu H - \eta N) = 0. \tag{7}$$

We introduce a family of *nonlinearly elastic* shells parametrized by ε^2 by assuming that there are constitutive functions $(\tau,\nu,\eta;\omega,\mu;s,\delta) \mapsto \hat{T}(\tau,\nu,\eta;\omega,\mu;s,\delta)$, etc., such that

$$T(s) = \hat{T}(\tau(s),\nu(s),\eta(s);\varepsilon^2\rho(s)^{-1}\sin\theta(s),\varepsilon^2\theta_s(s);s,\varepsilon^2), \quad \text{etc.} \tag{8}$$

We assume that the functions $\hat{T}$, etc., are $O(1)$ in δ as $\delta \to 0$. The common domain of definition of these functions corresponds to those strains that preserve orientation and to nonnegative ε's. We assume that the reference configuration is natural (i.e., stress-free), so that

$$\hat{T}(1,1,0;-\varepsilon^2\rho^{-1}\sin\psi,-\varepsilon^2\psi_s;s,\varepsilon^2) = 0, \quad \text{etc.}, \tag{9}$$

for all $\varepsilon \geq 0$. We adopt a standard *positive-definiteness* assumption of linear elasticity for shells: The matrix of partial derivatives of $\hat{T},\hat{N},\hat{H},\hat{\Omega},\hat{M}$ with respect to τ,ν,η,ω,μ at the natural state $(1,1,0;-\varepsilon^2\rho^{-1}\sin\psi,-\varepsilon^2\psi_s;s,\varepsilon^2;s,\varepsilon^2)$ is positive-definite for all $\varepsilon \geq 0$.

We require that the response to shear be symmetric: $\hat{H}$ is odd in η, and $\hat{T},\hat{N},\hat{\Omega},\hat{M}$ are even in η. We assume that the constitutive functions satisfy isotropy conditions at the north pole $s = 0$, which we do not spell out.

We limit our attention here to edges $s = l$ that are completely free, so that

$$N(l) = 0, \quad H(l) = 0, \quad M(l) = 0. \tag{10}$$

Our methods also handle hinged edges.

Our boundary-value problem for the basic unknowns τ,ν,η,θ is obtained by substituting the constitutive equations (8) into the the equilibrium equations (5)–(7) and supplementing the resulting equations with (4). This quasilinear system is subject to the boundary condition (10) and to regularity conditions at the pole. See [1] for a detailed discussion of this shell theory.

2. Asymptotic Representation of the Solution

If we set $\varepsilon = 0$ in (5)–(8), we obtain the *reduced* problem. Of its many solutions, we limit our attention to $\tau = 1 = \nu$, $\eta = 0$, $\theta = \psi$, which represents the eversion of the reference configuration of $\mathbf{r}$ for a membrane.

To study configurations close to that of this solution of the reduced problem, we now obtain a simpler formulation of our boundary-value problem: We introduce the horizontal component $\varepsilon^2\rho^{-1}P$ of the contact force $N\mathbf{a} + H\mathbf{b}$ by

$$\varepsilon^2\rho^{-1}P \equiv (N\mathbf{a} + H\mathbf{b}) \cdot \mathbf{e}_1 = N\cos\theta - H\sin\theta, \tag{11}$$

so that (5), (6) imply that

$$T = \varepsilon^2 P_s, \quad N = \varepsilon^2\rho^{-1}P\cos\theta, \quad H = -\varepsilon^2\rho^{-1}P\sin\theta. \tag{12}$$

These scalings with ε^2 do not come from the three-dimensional interpretation of our constitutive functions, but rather from our wish to simplify the asymptotics of the specific problems we treat.

For ε sufficiently small, the positive-definiteness assumption allows us to replace our constitutive equations with an equivalent set giving $\tau, \nu, \eta, \Omega, M$ as functions of $T, N, H, \omega, \mu, s, \delta$:

$$\tau = \tau^\sharp(T, N, H; \omega, \mu; s, \delta), \quad \text{etc.} \tag{13}$$

We can therefore convert system (5)–(7), (4), (8) to the following equivalent system for (θ, P):

$$(\rho M^\sharp)_s - \Omega^\sharp\cos\theta - P(\nu^\sharp\sin\theta + \eta^\sharp\cos\theta) = 0, \tag{14}$$

$$(\rho\tau^\sharp)_s - \nu^\sharp\cos\theta + \eta^\sharp\sin\theta = 0, \tag{15}$$

where the arguments of the constitutive functions bearing sharp signs are

$$\varepsilon^2 P_s, \varepsilon^2\rho^{-1}P\cos\theta, -\varepsilon^2\rho^{-1}P\sin\theta; \varepsilon^2\rho^{-1}\sin\theta, \varepsilon^2\theta_s; s, \varepsilon^2. \tag{16}$$

The boundary conditions corresponding to (10) are

$$P(l) = 0, \quad M^\sharp|_{s=l} = 0. \tag{17}$$

We introduce the stretched variable $t = (s - l)/\varepsilon$. Assuming that the data have sufficient smoothness, we seek solutions of (14)–(17) in which the unknown functions θ, P have the form

$$\theta(s,\varepsilon) = \theta^n(s,\varepsilon) + o(\varepsilon^n) \equiv \psi(s) + \sum_{k=1}^{n}\frac{\varepsilon^k}{k!}\theta_k(s) + \sum_{k=1}^{n}\frac{\varepsilon^k}{k!}\tilde{\theta}_k(t) + \sum_{k=1}^{n}\frac{\varepsilon^k}{k!}\theta_k^*(s) + o(\varepsilon^n), \quad \text{etc.,} \tag{18}$$

where the θ_k^* are given smooth functions equal to $-\tilde{\theta}_k(-l/\varepsilon)$ for s near 0, and equal to 0 for s near l.

A straightforward but complicated analysis delivers explicit expressions for the lower-order terms θ_k, P_k of the regular expansion and $\tilde{\theta}_k, \tilde{P}_k$ of the boundary-layer expansion. For the regular expansion, the first nontrivial terms occur for $k = 2$, the terms for $k = 3$ are trivial, and the effects of nonlinear constitutive equations are manifested for $k = 4$. For the boundary-layer expansion, the first nontrivial terms occur for $k = 1$ and the effects of nonlinear constitutive equations are manifested for $k = 3$.

The boundary-layer expansion shows the presence of a lip near the edge, just like the one that appears on an everted hemispherical orange peel.

3. Justification of the Expansion

We use a refinement by Srubshchik and Yudovich [4] of Kantorovich's [3] operator-theoretic version of Newton's method to prove

Theorem. *Let all the hypotheses described above hold. Let n be a given positive integer. Let all the data have enough smoothness for the expansions* (18) *up to order $n+1$ to make sense. Then there is a positive number ε_0 such that if $\varepsilon \leq \varepsilon_0$, then the boundary-value problem has a nontrivial classical solution (θ, P), which is approximated by (θ^n, P^n) in the sense that there is a number $C > 0$, independent of ε, for which*

$$\max_{s\in[0,l]} |\partial_s^j(\theta - \theta^n)| \leq C\varepsilon^{n+1-j}, \quad \max_{s\in[0,l]} |\partial_s^j(P - P^n)| \leq C\varepsilon^{n+1-j} \quad \textit{for} \quad j = 0, 1, 2. \tag{19}$$

To set up the proof of this theorem, we define $F[\theta, P; \varepsilon]$ to consist of the left-hand sides of (14), of (15), and of the second equation of (17). Clearly F maps the Banach space

$$\mathcal{X} \equiv \{(\theta, P) : \theta, P \in C^2[0, l],\ \theta(0) = 0, P(0) = 0, P(l) = 0\} \tag{20}$$

endowed with the usual norm, to the Banach space

$$\mathcal{Y} \equiv \{(f, g, : \rho^{-1} f, \rho^{-1} g \in C^0[0, l],\ \alpha \in \mathbf{R}\} \tag{21}$$

endowed with the weighted norm

$$\max_{[0,l]} |\rho^{-1} f| + \max_{[0,l]} |\rho^{-1} g| + |\alpha|. \tag{22}$$

The heart of the proof of our theorem is to get sharp estimates for the inverse of the Fréchet derivative of F with respect to (θ, P) at $(\theta^n(\cdot, \varepsilon), P^n(\cdot, \varepsilon))$. This involves a tricky analysis requiring very sharp control of the polar singularity and negative powers of ε.

The work reported here is based on Antman and Srubshchik [2].

4. References

1. S. S. Antman, *Nonlinear Problems of Elasticity* (Springer, 1995).
2. S. S. Antman and L. S. Srubshchik, Asymptotic analysis of the eversion of nonlinearly elastic shells, to appear.
3. L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, 2nd edition (Pergamon, 1982).
4. L. S. Srubshchik and V. I. Yudovich, Asymptotic integration of the system of equations for the large deflections of symmetrically loaded shells of revolution, *J. Appl. Math. Mech.* **26** (1962) 1378–1391.

DIMENSIONAL REDUCTION FOR PLATES BASED ON MIXED VARIATIONAL PRINCIPLES*

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ABSTRACT

We consider the derivation and rigorous justification of models for thin linearly elastic plates using mixed variational principles.

We consider an isotropic, homogeneous, linearly elastic plate occupying the region $P_t = \Omega \times (-t/2, t/2)$, with Ω a smoothly bounded domain in $\mathbb{R}^2$ and $t \in (0,1]$. We denote the union of the top and bottom surfaces of the plate by $\partial P_t^{\pm} = \Omega \times \{-t/2, t/2\}$ and the lateral boundary by $\partial P_t^{\mathrm{L}} = \partial\Omega \times (-t/2, t/2)$. We suppose that the plate is loaded by a surface force density $\underline{g} : \partial P_t^{\pm} \to \mathbb{R}^3$ and a volume force density $\underline{f} : P_t \to \mathbb{R}^3$, and is clamped along its lateral boundary. The resulting stress $\underline{\underline{\sigma}}^* : P_t \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and displacement $\underline{u}^* : P_t \to \mathbb{R}^3$ then satisfy the boundary-value problem

$$\underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\sigma}}^* = \underline{\underline{\varepsilon}}(\underline{u}^*), \quad -\underline{\operatorname{div}}\,\underline{\underline{\sigma}}^* = \underline{f} \quad \text{in } P_t, \qquad \underline{\underline{\sigma}}^*\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm}, \qquad \underline{u}^* = 0 \text{ on } \partial P_t^{\mathrm{L}}, \tag{1}$$

where $\underline{\underline{\varepsilon}}(\underline{u}^*)$ denotes the infinitesimal strain tensor and $\underline{\underline{\underline{\underline{A}}}}$ is the usual isotropic compliance tensor.

We discuss systematic procedures of dimensional reduction of the three-dimensional problem to two-dimensional plate models which proceed from variational formulations of the three-dimensional problem (1). Besides the derivation of models, we also consider their rigorous justification. Namely, we study the convergence to zero of the relative error in energy norm on the three-dimensional plate domain P_t of an approximation of the three-dimensional solution

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determined from the solution of the dimensionally reduced model. For some of the models we show that this error tends to zero, and establish the rate of convergence. A fuller development of the ideas discussed here may be found in [1].

The variational approach to dimensional reduction is systematic and is tied naturally to rigorous convergence theory. These characteristics are shared with another important approach to dimensional reduction of shells, the asymptotic approach developed, for example, in the book [3] of Ciarlet. However the variational approach also differs from this asymptotic analysis in a number of significant ways. The asymptotic approach essentially identifies one particular canonical plate model, the limiting solution of the three-dimensional elastic problem. This is the Kirchhoff–Love model, the superposition of the generalized plane stess model of plate stretching and the biharmonic model of plate bending. By contrast, the variational approach naturally generates a hierarchical family of plate models by using polynomial approximations of increasing degree. In fact, we consider two different mixed variational principles, and each leads to several different hierarchical families of models. It is interesting to note that the simplest plate stretching model arising from a variational approach to dimensional reduction is again generalized plane stress, but the biharmonic plate bending model does *not* arise naturally from this approach. Instead, the simplest bending model arising is (a form of) the Reissner–Mindlin model.

Before proceeding, we summarize some notational conventions. We write first-order tensors (or 3-vectors) with one underbar, second-order tensors (or 3×3 matrices) with two underbars, etc. For tensors in two variables we use undertildes in the same way. Any 3-vector may be expressed in terms of a 2-vector giving its in-plane components and a scalar giving its transverse component, and any 3×3 symmetric matrix may be expressed in terms of a 2×2 symmetric matrix, a 2-vector, and a scalar thus:

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 \end{pmatrix}, \quad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau} & \underset{\sim}{\tau} \\ \underset{\sim}{\tau}^T & \tau_{33} \end{pmatrix}.$$

The starting point for the variational approach to dimensional reduction is the Hellinger–Reissner variational principle. We consider two variant forms of this principle. The first, which we refer to as HR, characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as the unique critical point (namely a saddle point) of the HR functional

$$J(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P_t} \underline{\underline{\underline{\underline{A}}}}\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x} - \int_{P_t} \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x} + \int_{\partial P_t^{\pm}} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}$$

over $\underline{\underline{\Sigma}}^\bullet \times \underline{V}^\bullet := \underline{\underline{L}}^2(P_t) \times \{\, \underline{v} \in \underline{H}^1(P_t) \,:\, \underline{v} = 0 \text{ on } \partial P_t^{\mathrm{L}} \,\}$. The second form of the Hellinger–Reissner principle, HR′ characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as the unique critical point (again a saddle point) of the HR′ functional

$$J'(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P_t} \underline{\underline{\underline{\underline{A}}}}\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x} + \int_{P_t} \underline{\operatorname{div}}\, \underline{\underline{\tau}} \cdot \underline{v}\, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x}$$

on $\underline{\underline{\Sigma}}_g^* \times \underline{V}^* := \{\, \underline{\underline{\sigma}} \in \underline{\underline{H}}(\underline{\operatorname{div}}, P_t) \,|\, \underline{\underline{\sigma}}\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm} \,\} \times \underline{L}^2(P)$.

Plate models may be derived by replacing $\underline{\underline{\Sigma}}^\bullet \times \underline{V}^\bullet$ in HR with subspaces which admit only a specified polynomial dependence on x_3. If the subspaces $\underline{\underline{\Sigma}}$ and $\underline{V}$ are chosen carelessly, there may not exist any such critical point or it may not be unique. We insure a unique solution by insisting that $\underline{\underline{\varepsilon}}(\underline{V}) \subset \underline{\underline{\Sigma}}$. In the simplest example, we seek a saddle-point of the HR functional over $\underline{\underline{\sigma}} \in \underline{\underline{\Sigma}}^{\bullet}$ with $\underset{\approx}{\sigma}$ linear in x_3, $\underset{\sim}{\sigma}$ constant in x_3, and σ_{33} zero, and $\underline{u} \in \underline{V}^{\bullet}$ with $\underset{\sim}{u}$ linear and u_3 constant in x_3. This leads to the lowest order model in the family we refer to as HR_1. This family of models and two other families are described in Table 1.

Table 1. The principle plate models based on the HR principle. The degree p is a positive integer.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}_1(p)$	p	$p-1$	$p-2$	p	$p-1$
$\mathrm{HR}_2(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}_3(p)$	p	$p+1$	p	p	$p+1$

The $\mathrm{HR}_1(1)$ turns out to yield the classical generalized plane stress model for stretching and a form of the Reissner–Mindlin model (with shear correction factor 1) for bending. It is perhaps the simplest way to derive these models.

The model families $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ are *minimum energy* or *energy projection models*. Namely, because they satisfy the condition $\underline{\underline{\underline{\underline{A}}}}^{-1}\,\underline{\underline{\varepsilon}}(\underline{V}) \subset \underline{\underline{\Sigma}}$, the three-dimensional constitutive equation is satisfied exactly and $\underline{u}$ is determined as the minimizer in $\underline{V}$ of the potential energy. In the literature there has been a great deal of attention paid to the minimum energy models (cf., e.g., [2], [6], [7]), and much less to other models arising mixed variational principles. However, the restriction to minimum energy principles eliminates many of the best features of the variational approach.

A striking failure of the minimum energy approach occurs with the simplest minimum energy model, $\mathrm{HR}_2(1)$. It turns out that this is an incorrect plate model, one which is not even consistent with the Kirchhoff–Love reduced problem in the limit of vanishing thickness. More precisely, the $\mathrm{HR}_2(1)$ model equations are of same form as generalized plane stress and Reissner–Mindlin, but these equations contain spurious terms, causing divergence as t tends to 0 (for both stretching and bending). This phenomenon is well-known and has been studied in some generality recently in [6], where it is shown that for a minimum energy method to be consistent in the $t = 0$ limit, the polynomial spaces must be of higher degree. By contrast, the $\mathrm{HR}_1(1)$ method, which uses spaces of *lower* degree than $\mathrm{HR}_2(1)$, is a consistent method.

While the $\mathrm{HR}_2(1)$ model is incorrect, for $p \geq 3$ it can be shown that the $\mathrm{HR}_2(p)$ model is convergent. (For $p = 3$, it can be shown to be identical to a method of Lo, Christensen, and Wu [4].) The $\mathrm{HR}_3(p)$ is also convergent for all $p \geq 1$. However, even in the simplest case ($p = 1$), the models in this family are more complex (involve more dependent variables) than $\mathrm{HR}_1(1)$.

Table 2. The principle plate models based on the HR′ principle. The degree p is a positive integer.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}'_1(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}'_2(p)$	p	$p+1$	p	p	$p-1$
$\mathrm{HR}'_3(p)$	p	$p+1$	p	p	$p+1$
$\mathrm{HR}'_4(p)$	p	$p+1$	$p+2$	p	$p+1$

Table 2 shows the principle model families for the HR′ principle. Here we wish to particularly emphasize the model $\mathrm{HR}'_4(1)$, which is the simplest *complementary energy model.* Namely, $\underline{\underline{\sigma}}$ minimizes the complementary energy $E_c(\underline{\underline{\tau}}) = (1/2)\int_{P_t} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}\,d\underline{x}$ over all $\underline{\underline{\tau}} \in \underline{\underline{\Sigma}}_g$ satisfying the equilibrium condition $\underline{\operatorname{div}}\,\underline{\underline{\tau}} = -P_V \underline{f}$ (P_V is the L^2-projection onto $\underline{V}$). This model agains gives rise to the classical generalized plane stress stretching equations, but with the load constructed in a more sophisticated way from the three-dimensional loading. The bending model is again Reissner–Mindlin, but with a shear correction factor of 5/6 and, again, more complicated loads. As we discuss below, this method is not only consistent with the Kirchhoff–Love solution in the limit of vanishing thickness, but, more importantly, it is convergent in relative energy norm. This is a strong property not shared by all methods which are consistent in the thin plate limit. In fact, the Kirchhoff–Love solution is itself not convergent in relative energy norm. Morgenstern, in his pioneering work on the energy convergence of the biharmonic plate model [5], showed that three-dimensional displacement and stress fields could be constructed from the biharmonic solution which converge to the full three-dimensional solution in relative energy norm. However the construction of these fields is rather ad-hoc, and not suggested by the biharmonic model itself. By contrast, the approximation delivered by the $\mathrm{HR}_4'(1)$ model is, without any post-processing, convergent.

The key to the error analysis in [5] is the two energies principle or Prager–Synge theorem, and we follow that approach. This approach requires a stress field which is in equilibrium with the imposed volume and surface loads. Generally such a field is not trivial to construct (especially if volume loading is present—this case was not treated in [5]). However the $\mathrm{HR}_4'(1)$ method, being a complementary energy method, automatically generates such a stress field. Combining the two energies principle with careful *a priori* estimates of the plate model solutions, we are able to obtain precise bounds on the energy error $\|\underline{\underline{\varepsilon}}(\underline{u}^* - \underline{u})\|_{0,P_t} + \|\underline{\underline{\sigma}}^* - \underline{\underline{\sigma}}\|_{0,P_t}$ in terms of the thickness t and various norms of the loading functions $\underline{f}$ and $\underline{g}$. For example, if the surface load is purely in-plane and is even in x_3 and the volume load vanishes, we find that $\|\underline{\underline{\varepsilon}}(\underline{u}^* - \underline{u})\|_{0,P_t} + \|\underline{\underline{\sigma}}^* - \underline{\underline{\sigma}}\|_{0,P_t} \le$ const., while $\|\underline{\underline{\varepsilon}}(\underline{u})\|_{0,P_t}$ and $\|\underline{\underline{\sigma}}\|_{0,P_t}$ behave as $O(t^{-1/2})$. Thus

$$\frac{\|\underline{\underline{\varepsilon}}(\underline{u}^* - \underline{u})\|_{0,P_t}}{\|\underline{\underline{\varepsilon}}(\underline{u})\|_{0,P_t}} + \frac{\|\underline{\underline{\sigma}}^* - \underline{\underline{\sigma}}\|_{0,P_t}}{\|\underline{\underline{\sigma}}\|_{0,P_t}} \le C t^{1/2},$$

so the $\mathrm{HR}_1'(4)$ plate model converges with order $t^{1/2}$ measured in relative energy norm in this stretching situation. The same result holds for many other loading cases.

References

1. S. M. Alessandrini, D. N. Arnold, R. S. Falk, and A. L. Madureira, *Proceedings of the Summer Seminar on Plates and Shells* (Québec, 1996).
2. I. Babuška and L. Li, *Comput. Methods Appl. Mech. Engrg.* **100** (1992) 249-273.
3. P. G. Ciarlet, *Plates and junctions in elastic multi-structures, an asymptotic analysis* (Masson, Paris, 1990).
4. K. H. Lo, R. M. Christensen, and E. M. Wu, *J. Appl. Mech.* **46** (1977) 663-676.
5. D. Morgenstern, *Arch. Rational Mech. Anal.* **4** (1959) 145-152.
6. J.-C. Paumier and A. Raoult (IMAG RT 164, 1996).
7. C. Schwab, *Hierarchic models of elliptic boundary-value problems on thin domains* (preprint 1996).

MULTI-DIRECTOR AND MULTI-LAYER FINITE SHELL ELEMENTS

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ABSTRACT

The objective of this contribution is the development of multi-layer, multi-director finite shell elements able to deal with finite rotations, large strains and applicable both to compressible and incompressible materials the last case being relevant for rubber-like structures. Incompressible materials are simulated by MOONEY–RIVLIN as well as OGDEN model. The principal stretches appearing in the last model are evaluated by an incremental-iterative procedure starting from the definition of the strain invariants. By using C^0-displacement continuity condition on all interfaces the multi-director shell kinematics is extended to multi-layer models capable to predict arbitrarily complex through-thickness stress distributions at any desired accuracy level. The ability of the finite elements concerning the prediction of finite rotations, large strains and stress singularities are demonstrated by adequate examples.

1. Introduction

Multi-director shell kinematics is understood to be a formulation, where the displacement field is described by a higher order polynomial in thickness coordinate, a quadratic approximation in the present case. If the given shell structure is subdivided into a number of sub-elements and if each of them is described by multi-director kinematics considering the C^0-continuity of the displacement field along all interfaces the resulting model is referred to as *multi-director, multi-layer shell* model. This paper is devoted to the unified derivation of the finite shell elements on the basis of this general applicable shell kinematics simulating large elastic strains by various constitutive laws. The applicability of the adopted shell kinematics to incompressible materials is accomplished simply by eliminating the stretching parameters at the element level through 2D incompressibility constraints.

A large deal of works has been performed in the last decade to develop finite rotation elements on the basis of classical single director kinematics ([2],[4],[6]). Finite shell elements capable to deal with large strains of rubber-like structures are, however, still scarce in literature and mostly dedicated to the simulation of membrane deformations ([5],[9]). Models proposed for the general bending analysis are mainly connected with MOONEY–RIVLIN model ([3],[7]). Only in few contributions ([8]) attention is paid to the determination of principal stretches in order to consider OGDEN material model.

2. Basic Concepts

Shell equations will be present in symbolic notation. Let $\mathbf{x}^* = \mathbf{x}^*(\theta^\alpha)$ be the position vector of any point of the deformed shell continuum, where θ^α are curvilinear coordinates. The starting point of the present development is the approximation of the vector $\mathbf{x}^*$ by a quadratic polynomial in thickness coordinate θ^3:

$$\mathbf{x}^* = \mathbf{x} + \theta^3\lambda\mathbf{d} + \left(\theta^3\right)^2 y\mathbf{d}, \qquad \mathbf{d}\cdot\mathbf{d} = 1 \tag{1}$$

with a director $\mathbf{d}$ supposed to be a unit vector. The multiplicative decomposition $\lambda\mathbf{d}$ used in (1) provides the advantage that numerically sensitive stretching parameter λ is decoupled from the director $\mathbf{d}$ subjected to a pure rotation. The second order term y is of significance to ensure numerical stability in the case of inclusion of transverse strains E_{33}. It will be however, suppressed, if incompressible materials are considered.

For the definition of 2D strains we transform the GREEN strain tensor $\mathbf{E}$ by means of the shell shifter $\mathbf{Z}$ into the surface tensor $\hat{\mathbf{E}}$

$$\mathbf{E} = \mathbf{Z}^{-T}\hat{\mathbf{E}}\mathbf{Z}^{-1} = \mathbf{Z}^{-T}\left(\hat{E}_{ij}\mathbf{A}^i\otimes\mathbf{A}^j\right)\mathbf{Z}^{-1}, \quad \hat{E}_{ij} = \frac{1}{2}\left(\mathbf{g}_i\cdot\mathbf{g}_j - \mathbf{G}_i\cdot\mathbf{G}_j\right) \tag{2}$$

referring in contrast to $\mathbf{E}$ to the basis $\mathbf{A}^i\otimes\mathbf{A}^j$ of the midsurface. By using (1) for $\mathbf{x}^*$ and the expression $\mathbf{X}^* = \mathbf{X} + \theta^3\mathbf{N}$ for its undeformed counterpart $\mathbf{X}^*$ we obtain from (2) within the accuracy level (1)

$$\hat{\mathbf{E}} = \overset{0}{\mathbf{E}} + \theta^3\,\overset{1}{\mathbf{E}} + \left(\theta^3\right)^2\overset{2}{\mathbf{E}}, \tag{3}$$

where 2D strains $\overset{n}{\mathbf{E}}$ (n=1,2,3) are expressible in terms of the kinematic variables of the approximation (1), the corresponding relation being e.g. for the constant term $\overset{0}{\mathbf{E}}$ of the form

$$\overset{0}{\mathbf{E}} = \overset{0}{E}_{ij}\,\mathbf{A}^i\otimes\mathbf{A}^j,$$

$$\overset{0}{E}_{\alpha\beta} = \frac{1}{2}\left(\mathbf{a}_\alpha\cdot\mathbf{a}_\beta - \mathbf{A}_\alpha\cdot\mathbf{A}_\beta\right), \quad \overset{0}{E}_{\alpha 3} = \frac{1}{2}\lambda\mathbf{d}\cdot\mathbf{a}_\alpha, \quad \overset{0}{E}_{33} = \frac{1}{2}\left(\lambda^2 - 1\right). \tag{4}$$

Higher order strains $\overset{1}{\mathbf{E}}$ and $\overset{2}{\mathbf{E}}$ are described by similar kinematic equations given in [2].

When dealing with finite rotations an essential problem is a suitable parametrization of the director $\mathbf{d}$ so that the constraint $\mathbf{d}\cdot\mathbf{d} = 1$ is satisfied a priori during the incremental-iterative procedure. This is achieved in the present study by two different approaches:

- the use of EULER *angles* ψ_α determining the director $\mathbf{d}$ with respect to a fixed reference frame [1],
- the *up-dated formulation* transferring the first two variations of $\mathbf{d}$, $\delta\mathbf{d}$ and $\delta^2\mathbf{d}$, into a rotation vector $\boldsymbol{\omega}$, which is to be used with two or three independent components according to the requirements of the problem under consideration; three rotation parameters are needed only in the case of compound shells (see [6]).

Now we shall show how the above kinematic model can be specialized for incompressible materials. An essential problem in this case is a suitable implementation of the incompressibility condition described by

$$I_3(\mathbf{C}) = 1, \qquad I_3(\mathbf{C}) = \det\mathbf{C} = \det C_{i\cdot}^{\cdot i} = \left(\lambda_1\lambda_2\lambda_3\right)^2 \tag{5}$$

in terms of the third invariant I_3 of the right CAUCHY–GREEN tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F} = 2\mathbf{E} + \mathbf{G}$, $\mathbf{F}$ denoting the deformation gradient. In the present formulation the constraint (5) is first used to determinate the transversal strains E_{33} in the internal potential energy. But it also considered as 2D constraint:

$$\sqrt{g/G} = 1 \qquad \rightarrow \qquad \lambda = \sqrt{A}/[\mathbf{a}_1 \mathbf{a}_2\, \mathbf{d}] \tag{6}$$

for the elimination of the stretching parameter λ at the element level.

Rubber-like materials are simulated by two different procedures: by MOONEY–RIVLIN model with the strain energy density (per unit volume of the undeformed state)

$$\rho_0 e = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{7}$$

depending on the first two invariants of $\mathbf{C}$:

$$I_1(\mathbf{C}) = \mathrm{tr}\mathbf{C} = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\,, \tag{8}$$

$$I_2(\mathbf{C}) = \frac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2\right] = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \tag{9}$$

and by OGDEN model where $\rho_0 e$ is expressed in terms of the principal stretches λ_i $(i = 1, 2, 3)$

$$\rho_0 e = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n}\left[(\lambda_1)^{\alpha_n} + (\lambda_2)^{\alpha_n} + (\lambda_3)^{\alpha_n} - 3\right], \tag{10}$$

C_1, C_2 and μ_n, α_n being the corresponding material constants. The first model expressed in terms of $\mathbf{C}$ can be directly incorporated in the FE–model. On the contrary, the OGDEN model requires the numerical evaluation of the principal stretches λ_i $(i = 1, 2, 3)$. This is achieved by means of the relations (5),(8) and (9) forming a complete equation set for the determination of λ_i in terms of $\mathbf{C}$. To this, the cited relations are transformed into variational equations of first and second order and then treated by an iterative solution technique. This procedure is essentially the same as is employed for the treatment of nonlinear shel equations.

3. Numerical example and conclusion

The theoretical fundamentals presented above are transformed into isoparametric 4-node shell elements where primary kinematic variables are approximated by bilinear interpolation polynomials. In the following a single example, a conical shell with prescribed vertical displacement Δ along upper edge (Fig.1-5), is presented demonstrating the capability of the finite element concerning the prediction of large strains as well as finite rotations.

4. References

1. Y.Başar , Y.Ding and W.B.Krätzig, *Computational Mechanics* **10** (1992) 289-306.
2. Y.Başar and Y.Ding, *Composites Engineering* **5** (1995) 485-499.
3. Y.Başar and Y.Ding, to appear in *Int. J. Solids Structures.* (1997)
4. N.Büchter and E.Ramm, *Int. J. Num. Meth. Eng.* **34** (1992) 39-59.
5. F.Gruttmann and R.L.Taylor, *Int. J. Num. Meth. Eng.* **35** (1992) 1111-1126.
6. W.Menzel, *Gemischt-hybride Elementformulierungen für komplexe Schalenstrukturen unter endlichen Rotationen* (Dissertationsschrift, Bochum,1996)
7. B.Schieck, W.Pietraszkiewicz and H.Stumpf, *Int.J.Solids Struct.* **29** (1992) 689-709.
8. J.C.Simo and R.L.Taylor, *Comput. Meth. Appl. Mech. Eng.* **85** (1991) 273-310.
9. P.Wriggers and R.L.Taylor, *Eng. Comput.* **7** (1990) 303-310.

Geometry:
R = 2.0 ; r = 1.0 ; H = 1.0

OGDEN - material :
$m_1 = 6.300$; $a_1 = 1.3$
$m_2 = 0.012$; $a_2 = 5.0$
$m_3 = -0.100$; $a_3 = -2.0$

Boundary conditions:
$X^3 = 0$: hinged edges : $\Delta X^i = 0$
$X^3 = H$: r = constant (edge movable only in X^3-direction)
r = variable (free edge)

Load:
prescribed vertical displacement Δ

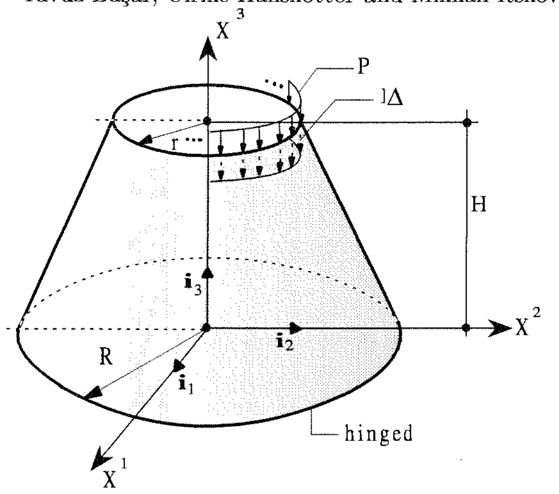

Figure 1.: Conical shell turned inside out

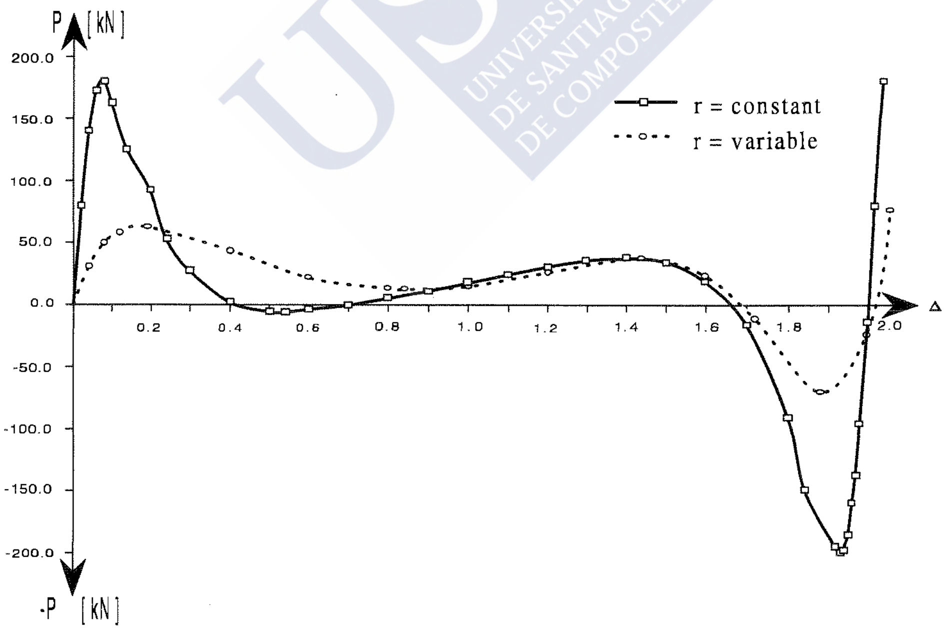

Figure 2: Conical shell: Load - displacement diagram

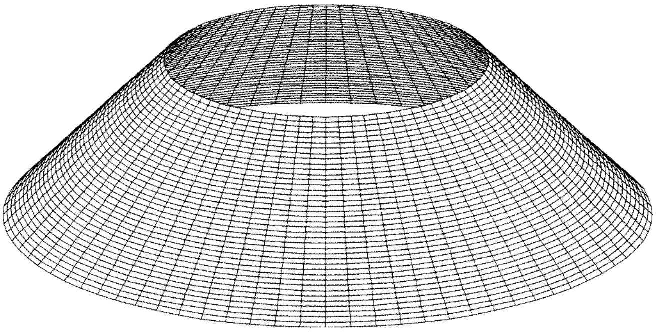

Figure 3: Conical shell: Deformed configuration (Δ=0)

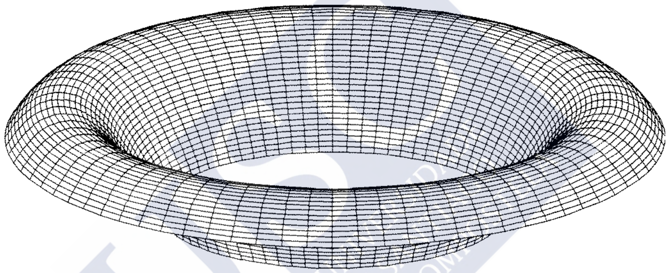

Figure 4: Conical shell: Deformed configuration (Δ=1.5)

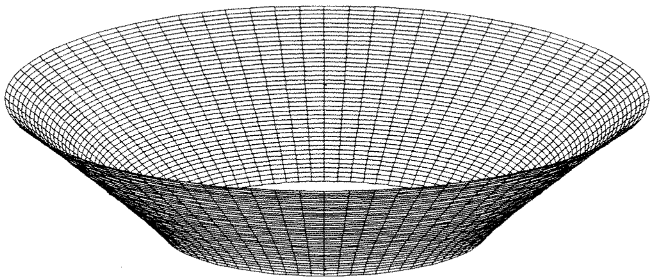

Figure 4: Conical shell: Deformed configuration (Δ=2.0, turned inside out)

A FINITE ELEMENT METHOD FOR 3D ELASTOACOUSTIC VIBRATIONS

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ABSTRACT

We propose a finite element approximation of the vibration modes of a fluid-structure interaction problem in a 3D-domain. Displacement variables are used for both the fluid and the solid. To avoid the typical spurious modes of this formulation we introduce a nonconforming discretization, following the 2D case analyzed in [1]. Optimal error estimates for the approximation of eigenvalues and eigenfunctions are given and numerical results for a cylindrical vessel filled with a fluid are shown.

1. Introduction

This paper deals with a typical problem of fluid-structure vibrations: the elastoacustics one, i.e., the interaction between a compressible fluid and an elastic structure (see for instance [7]). This is a very important problem with application in physics and engineering (e.g. for treatment of noise in cars or planes).

Under the usual assumptions leading to linear problems, the motion is governed by a linear second order in time equation. Its solution can be written in terms of the vibration modes of the coupled system which are solutions of a linear eigenvalue problem.

Continuous piecewise linear and bilinear finite elements, for both the fluid and the solid, have been used for the numerical solution of this problem in the two dimensional case, but spurious modes arise (see [6] and [3]). To avoid this drawback a non conforming discretization is introduced in [1] and analyzed for the 2D case. We adapt these results to the 3D case and improve the estimates given there by following [8].

Numerical results for a test problem consisting of a cylindrical vessel containing a liquid are presented.

2. Statement of the problem

We consider the problem of determining the small amplitude vibrations of a coupled system consisting of an inviscid barotropic fluid contained in a linear elastic structure.

Let Ω_F and Ω_S be the domains occupied by the fluid and the solid, respectively. Let us denote by Γ_I the interface between the solid and the fluid and ν its unit normal vector pointing outwards Ω_F. Let us assume that the exterior boundary of the solid is the union of two parts Γ_D and Γ_N and that the structure is fixed on Γ_D and free on Γ_N.

The classical acoustic approximation for the small amplitude motions yields the following eigenvalue problem for the vibration modes of the coupled system:

$$\left\{\begin{aligned} \lambda\rho_S\vec{v} - \mathcal{L}(\vec{v}) &= \vec{0}, && \text{in } \Omega_S \\ \lambda\rho_F\vec{u} - c^2\rho_F\vec{\nabla}(\operatorname{div}\vec{u}) &= \vec{0}, && \text{in } \Omega_F \\ \vec{u}\cdot\vec{\nu} - \vec{v}\cdot\vec{\nu} &= 0, && \text{in } \Gamma_I \\ \sigma(\vec{v})\vec{\nu} - \rho_F c^2 \operatorname{div}\vec{u}\,\vec{\nu} &= \vec{0}, && \text{in } \Gamma_I \\ \vec{v} &= \vec{0}, && \text{in } \Gamma_D \\ \sigma(\vec{v})\vec{\eta} &= \vec{0}, && \text{in } \Gamma_N \end{aligned}\right.$$

where $\lambda = \omega^2$, with ω being the vibration frequency; $\vec{u}$ and $\vec{v}$ are the amplitudes of the displacements of the fluid and the solid, respectively; $\mathcal{L}(\vec{v}) := -(\lambda_S + \mu_S)\vec{\nabla}(\operatorname{div}\vec{v}) - \mu_S\Delta\vec{v}$, with λ_S and μ_S being the Lamé's coefficients for the solid; ρ_F and ρ_S are the densities of the fluid and the solid, respectively, and c is the acoustic speed in the fluid.

The above problem is equivalent to the variational problem:

VP. *Find $\lambda \in \mathbf{R}$ and $(\vec{u},\vec{v}) \in \mathcal{V}$, $(\vec{u},\vec{v}) \neq (\vec{0},\vec{0})$, such that:*

$$a\left((\vec{u},\vec{v}),(\vec{\phi},\vec{\psi})\right) = \lambda b\left((\vec{u},\vec{v}),(\vec{\phi},\vec{\psi})\right), \quad \forall(\vec{\phi},\vec{\psi}) \in \mathcal{V},$$

where $\mathcal{V} := \left\{(\vec{u},\vec{v}) \in H(\operatorname{div},\Omega_F) \times \left[H^1(\Omega_S)\right]^3 :\ \vec{u}\cdot\vec{\nu} = \vec{v}\cdot\vec{\nu} \text{ on } \Gamma_I,\ \vec{v} = \vec{0} \text{ on } \Gamma_D\right\}$ is the space of admissible displacements and

$$a\left((\vec{u},\vec{v}),(\vec{\phi},\vec{\psi})\right) := \int_{\Omega_F} \rho_F c^2(\operatorname{div}\vec{u})(\operatorname{div}\vec{\phi}) + \int_{\Omega_S} \sigma(\vec{v}) : \epsilon(\vec{\psi}),$$

$$b\left((\vec{u},\vec{v}),(\vec{\phi},\vec{\psi})\right) := \int_{\Omega_F} \rho_F\vec{u}\cdot\vec{\phi} + \int_{\Omega_S} \rho_S\vec{v}\cdot\vec{\psi}.$$

We use spectral theory together with a regularity hypothesis and an orthogonal decomposition of $\mathcal{V}$ to characterize the eigenvalues of **VP**. We have the following:

Theorem 2.1 *The spectrum of* **VP** *consists of the eigenvalue $\lambda = 0$ and a sequence of finite multiplicity eigenvalues $\{\lambda_n :\ n \in \mathbf{N}\}$ converging to ∞. Furthermore, the infinite-dimensional subspace $\mathbf{K} := \left\{(\vec{u},\vec{0}) \in \mathcal{V} :\ \operatorname{div}\vec{u} = 0 \text{ in } \Omega_F\right\}$ is the eigenspace of $\lambda = 0$.*

3. Discretization

Let $\{\mathcal{T}_h\}$ be a family of triangulations. To discretize the displacements in the fluid we use the Raviart-Thomas finite element space given by $R_h(\Omega_F) := \{\vec{u} \in H(\operatorname{div},\Omega_F) :\ \vec{u}\,|_T = (a+dx, b+dy, c+dz),\ \forall T \in \mathcal{T}_h,\ T \subset \Omega_F\}$. The degrees of freedom in $R_h(\Omega_F)$ are the (constant) values of the normal components of $\vec{u}$ along the faces of the tetrahedra. In addition, for each component of the displacements in the solid we use the space $L_h(\Omega_S)$ of classical piecewise linear finite elements.

If we take a conforming approximation of $\mathcal{V}$, only functions with constant normal components along each face of Γ_I will be well approximated. Hence we use a weaker condition to define the discrete spaces:

$$\mathcal{V}_h := \left\{(\vec{u}_h,\vec{v}_h) \in R_h(\Omega_F) \times [L_h(\Omega_S)]^3 :\ \vec{v}_h|_{\Gamma_D} = \vec{0},\ \int_C (\vec{u}_h - \vec{v}_h)\cdot\vec{\nu} = 0,\ \forall C \subset \Gamma_I\right\} \not\subset \mathcal{V}.$$

We consider $\mathcal{V}_h$ as a conforming discretization of the space $H(\text{div}, \Omega_F) \times \left[H^1(\Omega_S)\right]^3$ and use the theory developed in [4] for non compact operators to prove the non existence of spurious modes. Finally, by adapting the results in [8] to three dimensional domains we prove that the eigenfunctions of **VP** are approximated with optimal order. This order γ depends on the strength of the singularities of the exact eigenfunctions (it would be 1 in absence of such singularities). Actually $\gamma := \min\{\alpha, \beta\}$ where α (resp. β) denotes the order of convergence of the finite element methods given above for the uncoupled source problem in the fluid (resp. in the solid). The order of convergence for the eigenvalues doubles that for the eigenfunctions.

More precisely, let λ be an eigenvalue of **VP** of multiplicity m and $\mathbf{E}$ its associated eigenspace. For h small enough there are exactly m eigenvalues of $\mathbf{VP}_h$, $\lambda_{h1}, \ldots, \lambda_{hm}$, (not necessarily different) converging to λ. Let $\mathbf{E}_h$ be the direct sum of their associated eigenspaces. Then we have (see [2] and [8]):

Theorem 3.2 *There exist strictly positive constants C and h_0 (both depending on λ) such that if $h \leq h_0$ then*

i) for each $(\vec{u}_h, \vec{v}_h) \in \mathbf{E}_h$, $\operatorname{dist}((\vec{u}_h, \vec{v}_h), \mathbf{E}) \leq Ch^\gamma \|(\vec{u}_h, \vec{v}_h)\|$,

ii) for each $(\vec{u}, \vec{v}) \in \mathbf{E}$, $\operatorname{dist}((\vec{u}, \vec{v}), \mathbf{E}_h) \leq Ch^\gamma \|(\vec{u}, \vec{v})\|$,

iii) $\max_{1 \leq i \leq m} |\lambda - \lambda_{hi}| \leq Ch^{2\gamma}$,

where dist *denotes the distance in the norm of the space $\mathcal{V}$.*

4. Numerical results

In this section numerical results are given for a test case consisting of a thin steel cylindrical vessel (height 3.5 m, inner diameter 2.0 m, thickness 0.1 m) clamped at both ends and filled with water. To take advantage of the simmetry we have considered just one eight of the cylinder and the fluid inside. We have used three succesively refined meshes.

In the table below we show the lowest eigenfrequencies of the coupled problem computed with each of these meshes. We denote ω_i^S a coupled mode which is a perturbation of the solid in vacuo (resp., ω_i^F a perturbation of the fluid in a rigid cavity). The last column of the table, labelled "Uncoupled", shows, for the "solid" modes ω_i^S, the corresponding frequency of the solid in vacuo for the finest mesh, whereas for the "fluid" modes ω_i^F it shows the exact frequency of the water in a rigid cavity, which is analytically known. Finally, Figures 1 and 2 below, show the deformed eight of cylinder and the pressure in the fluid for the first "solid" eigenmode.

Mode	Mesh 1	Mesh 2	Mesh 3	Uncoupled
ω_1^S	1701.500	1153.671	1009.377	1232.094
ω_1^F	1188.443	1166.649	1158.687	1283.565
ω_2^S	1311.843	1237.666	1219.264	1624.648
ω_3^S	4003.714	2291.144	1707.784	2001.229
ω_2^F	2348.552	2281.999	2255.354	2567.130
ω_3^F	2948.532	2775.609	2695.288	2632.916

Numerical results show a quite poor convergence of the "solid" modes which makes neccessary to use a shell model for the cylindrical vessel. This is being investigated now and some first results have been obtained in the case of the solid being a plate ([5]).

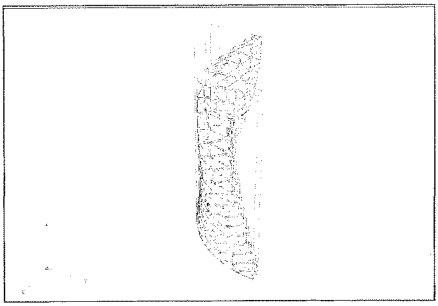

Figure 1: Mode ω_1^S: deformed structure.

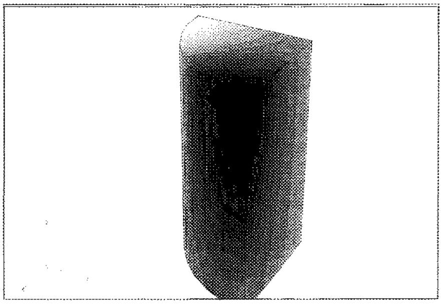

Figure 2: Mode ω_1^S: pressure in the fluid.

5. Acknowledgements

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6. References

1. A. Bermúdez, R. Durán, M.A. Muschietti, R. Rodríguez and J. Solomin, Finite element vibration analysis of fluid-solid systems without spurious modes, *SIAM J. Numer. Anal.*, **32** (1995) 1280-1295.
2. A. Bermúdez, L. Hervella-Nieto and R. Rodríguez, Numerical solution of three-dimensional elastoacoustic problems, in *Numerical Methods in Engineering '96*, J.A. Désidéri *et al.* eds., Wiley, New York, 1996, 874-880.
3. A. Bermúdez and R. Rodríguez, Finite element computation of the vibration modes of a fluid-solid system, *Comp. Methods in Appl. Mech. Eng.*, **119** (1994) 355-370.
4. J. Descloux, N. Nassif and J. Rappaz, On spectral approximation. Part I: The problem of convergence. Part 2: Error estimates for the Galerkin methods, *R.A.I.R.O. Anal. Numér.*, **12** (1978) 97-119.
5. R. Durán, L.M. Hervella-Nieto, E. Liberman, R. Rodríguez and J. Solomin, Finite element analysis of the vibration problem of a plate coupled with a fluid (in preparation).
6. L. Kiefling and G.C. Feng, Fluid-structure finite element vibration analysis, *AIAA J.*, **14** (1976), 199-203.
7. H.J-P. Morand and R. Ohayon, *Interactions Fluides-Structures*, Recherches en Mathématiques Appliquées **23**, Masson, Paris, 1992.
8. R. Rodríguez and J. Solomin, The order of convergence of eigenfrequencies in finite element approximations of fluid-structure interaction problems, *Math. Comp.*, **65** (1996) 1463-1475.

ON THE NUMERICAL MODELIZATION OF LAMINATED SHALLOW SHELLS

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1. Introduction

The construction of many recent large naval structures uses thin laminated shallow shells. Thus there is a need for *good numerical modelizations* of such structures: they must be able to give a good approximation of the state of local stresses in order to prevent damages.

The objective is to develop a *two-dimensional modelization* which gives a good account of the distribution of three-dimensional *displacement* and *stresses* over *a laminated thin shell.* In this way, we use the *asymptotic development technique* with the half-thickness of the laminate as small parameter. Such a method has been used by several authors to obtain various thin plate or thin shell models. We can cite i) the pioneering classification by [13]; ii) for *thin shallow shells*, the works by [7] and [12] which also include the proof of convergence between the two-dimensional and three-dimensional models; iii) for *anisotropic and nonhomogeneous thin shells*, the work by [8]. Besides these mathematical oriented works, we can cite some computational oriented works on thin laminated shells: [2], [14] and [20].

The model we develop here has been motivated by Ifremer and has been initially studied in [16]. It extends a model for plates by [17]. The main achievements of this study are

i) no a priori hypotheses are made on the variations of displacement and stresses through the thickness;

ii) the laminate is assumed to be made of several layers of homogeneous materials having a monoclinic behaviour through the thickness;

iii) by the asymptotic development of the three-dimensional problem, the three-dimensional displacement and stress fields can be approximated by a series of solutions of two-dimensional problems. These problems are recurrent: they are of thin shallow shell type for homogeneous anisotropic materials and they only differ by the second hand members;

iv) some experimental and numerical tests give a first validation of such an approach (due to lack of place, they will be included in [6]).

2. The Three-dimensional Problem

2.1. The Geometry of the Laminated Shallow Shell

The thin shell is the "product" of its middle surface ${}^{\varepsilon}\omega$ by its thickness 2ε ([3], [4], [15]).

The middle surface of the thin shell

Let $\mathcal{E}^3$ be the usual euclidean space referred to a fixed orthonormal reference system $(0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ and let ω be a bounded open subset of the plane. Then, the *middle surface* ${}^{\varepsilon}\omega$ of the *shallow shell* is the image in $\mathcal{E}^3$ of the set $\bar{\omega}$ by the mapping

$$ {}^{\varepsilon}\vec{\phi} \; : (x_1, x_2) \in \bar{\omega} \subset \mathcal{E}^2 \longrightarrow {}^{\varepsilon}\vec{\phi}(x_1, x_2) = (x_1, x_2, \varepsilon\phi(x_1, x_2)), $$

where $\phi \; : (x_1, x_2) \in \bar{\omega} \longrightarrow \phi(x_1, x_2)$ is a function independent of ε.

At any point of the middle surface, we define the *surface covariant basis*

$$ {}^{\varepsilon}\vec{a}_\alpha = {}^{\varepsilon}\vec{\phi}_{,\alpha} = \vec{e}_\alpha + \varepsilon\phi_{,\alpha}\vec{e}_3, \quad \alpha = 1, 2; $$
$$ {}^{\varepsilon}\vec{a}_3 = (\vec{e}_3 - \varepsilon\phi_{,1}\vec{e}_1 - \varepsilon\phi_{,2}\vec{e}_2)/{}^{\varepsilon}d, \text{ with } {}^{\varepsilon}d = [1 + \varepsilon^2(\phi_{,1})^2 + \varepsilon^2(\phi_{,2})^2]^{1/2}. $$

Then, we could define, as usual, first and second fundamental forms ${}^{\varepsilon}a_{\alpha\beta}$, ${}^{\varepsilon}b_{\alpha\beta}$, surface contravariant basis ${}^{\varepsilon}\vec{a}^{i}$, surface Christoffel symbols ${}^{\varepsilon}\Gamma^{\gamma}_{\alpha\beta}$. Here and subsequently, the greek (resp. latin) exponents take their values in the set $\{1, 2\}$ (resp. $\{1, 2, 3\}$) and the summation convention is applied for the upper and lower repeated indices.

The three-dimensional thin shell

The generic point and the volume covariant basis ${}^{\varepsilon}\vec{g}_i$ of the three-dimensional thin shell are obtained as follows :

$$ {}^{\varepsilon}\vec{\Phi}({}^{\varepsilon}x) = {}^{\varepsilon}\phi(x_1, x_2) + {}^{\varepsilon}x_3 \, {}^{\varepsilon}\vec{a}_3(x_1, x_2), \quad \forall \, {}^{\varepsilon}x = (x_1, x_2, x_3) \in {}^{\varepsilon}\bar{\Omega} = \bar{\omega} \times [-\varepsilon, \varepsilon]. \tag{1} $$

$$ {}^{\varepsilon}\vec{g}_\alpha = {}^{\varepsilon}\vec{\Phi}_{,\alpha} = {}^{\varepsilon}\vec{a}_\alpha - {}^{\varepsilon}x_3 \, {}^{\varepsilon}b^{\lambda}_{\alpha} \, {}^{\varepsilon}\vec{a}_\lambda, \quad {}^{\varepsilon}\vec{g}_3 = {}^{\varepsilon}\vec{\Phi}_{,3} = {}^{\varepsilon}\vec{a}_3. $$

The three-dimensional elasticity equations

Equation (1) gives the representation of the 3D thin shell by using curvilinear coordinates. The 3D elasticity equations referred to the basis $({}^{\varepsilon}\vec{g}_i)$ can be found in [15, Chap. 2]

$$ {}^{\varepsilon}\underline{\sigma}^{ij}||_j + {}^{\varepsilon}\underline{f}^{i} = 0 \quad \text{in } {}^{\varepsilon}\Omega, \tag{2} $$

$$ {}^{\varepsilon}\underline{\sigma}^{ij} = {}^{\varepsilon}\underline{a}^{ijk\ell} \, {}^{\varepsilon}\underline{\varepsilon}_{k\ell} \quad \text{in } {}^{\varepsilon}\Omega, \tag{3} $$

$$ \left. \begin{aligned} {}^{\varepsilon}\underline{u}_i = 0 \quad & \text{on } {}^{\varepsilon}\Gamma_0 = \partial\omega_0 \times [-\varepsilon, \varepsilon], \\ {}^{\varepsilon}\underline{\sigma}^{i\alpha}\tilde{n}_\alpha = {}^{\varepsilon}\underline{F}^{i} \quad & \text{on } {}^{\varepsilon}\Gamma_1 = \partial\omega_1 \times [-\varepsilon, \varepsilon], \\ {}^{\varepsilon}\underline{\sigma}^{i3} = \pm \, {}^{\varepsilon}\underline{G}^{i}_{\pm} \quad & \text{on } {}^{\varepsilon}\Gamma_{\pm} = \omega \times \{\pm\varepsilon\}. \end{aligned} \right\} \tag{4} $$

These equations are written on the basis ${}^{\varepsilon}\vec{g}_i$ and ${}^{\varepsilon}\vec{g}^{j}$; by using mapping ${}^{\varepsilon}\vec{\Phi}$, they are expressed on the reference domain ${}^{\varepsilon}\Omega$. Moreover, ${}^{\varepsilon}\underline{\sigma}^{ij}$, ${}^{\varepsilon}\underline{\varepsilon}_{k\ell}$, ${}^{\varepsilon}\underline{a}^{ijk\ell}$ denote the components of the stress, strain and elasticity tensors. In particular, the assumption of multilayered shell made of monoclinic homogeneous layers involves [15, p.155] ${}^{\varepsilon}\underline{a}^{3\alpha\beta\gamma} = {}^{\varepsilon}\underline{a}^{333\alpha} = 0$.

Finally, equations (4) mean that the shell is i) clamped along the part $\vec{\Phi}(\Gamma_0)$ of its lateral boundary; ii) loaded by volume forces whose density is ${}^\varepsilon\vec{\hat f} = {}^\varepsilon\vec{\Phi}({}^\varepsilon\vec{f})$ in ${}^\varepsilon\vec{\Phi}({}^\varepsilon\Omega)$; iii) loaded by a distribution of surface forces upon the upper and lower surfaces of the shell, i.e. ${}^\varepsilon\vec{\hat G}_\pm = {}^\varepsilon\vec{\Phi}({}^\varepsilon\vec{G}_\pm)$ upon $\hat\Gamma_\pm = {}^\varepsilon\vec{\Phi}({}^\varepsilon\Gamma_\pm)$; iv) loaded by surface forces upon the complementary part $\partial\omega_1 \times [-\varepsilon, +\varepsilon]$ of its side boundary, where $\partial\omega_1 = \partial\omega - \partial\omega_0$, i.e., ${}^\varepsilon\vec{\hat F} = {}^\varepsilon\vec{\Phi}({}^\varepsilon\vec{F})$.

Furthermore, we assume that the different layers which constitute the composite thin shell are perfectly sticked: the displacement and the normal stress components are continuous at the interfaces between adjacent layers, i.e., $[\![{}^\varepsilon\vec{u}]\!] = \vec{0}$ and $[\![{}^\varepsilon\tilde\sigma \cdot \vec{n}]\!] = \vec{0}$ at the layer interfaces, where $[\![\cdot]\!]$ denotes the jump at a possible discontinuity area.

3. The "Scaled" 3D Problem over a Domain Independent of ε

3.1. The Unidirectional Zoom

We use the transformation ${}^\varepsilon\theta\ :\ {}^\varepsilon x = (x_1, x_2, {}^\varepsilon x_3) \in I\!R^3 \longrightarrow x = (x_1, x_2, x_3 = \frac{1}{\varepsilon}\,{}^\varepsilon x_3) \in I\!R^3$ to formulate the problem over the reference domain $\Omega = \omega \times]-1, 1[$. In particular

$$\partial\Omega = \Gamma_+ \cup \Gamma \cup \Gamma_-\ ;\ \Gamma_+ = \omega \times \{1\}\ ;\ \Gamma_- = \omega \times \{-1\}\ ;\ \Gamma = \partial\omega \times [-1, 1].$$

3.2. Dependence of the Data on ε

For the asymptotic study

i) it is convenient to rewrite equations (2) to (4) over the basis ${}^\varepsilon\vec{a}_i$ and ${}^\varepsilon\vec{a}^i$; corresponding components are noted ${}^\varepsilon\sigma^{ij}$, u_i, ..., instead of ${}^\varepsilon\underline{\sigma}^{ij}$, $\underline{u}_i$,...;

ii) we make the following *basic assumptions*:

$$ {}^\varepsilon f^i = \varepsilon^{p_i} f^i\ ;\ {}^\varepsilon F^i = \varepsilon^{q_i} F^i\ ;\ {}^\varepsilon G^i_\pm = \varepsilon^{r_i} G^i_\pm \tag{5}$$

where p_i, q_i and $r_i \in I\!N$. Likewise, we introduce the developments

$$ {}^\varepsilon u^i \circ {}^\varepsilon\theta = u^i(\varepsilon) \simeq \sum_{n=P}^{N} \varepsilon^n u^{i(n)}, \quad {}^\varepsilon\sigma^{ij} \circ {}^\varepsilon\theta = \sigma^{ij}(\varepsilon) \simeq \sum_{n=P}^{N} \varepsilon^n \sigma^{ij(n)} \tag{6}$$

$$ {}^\varepsilon a^{ijk\ell} \circ {}^\varepsilon\theta = a^{ijk\ell}(\varepsilon) \simeq a^{ijk\ell} + \sum_{m=1}^{M} \varepsilon^{2m} a^{ijk\ell(2m)} \tag{7}$$

where $P \in Z\!\!\!Z$ and $M, N \in I\!N$ are integers. Relations (5) can be rewritten

$$ {}^\varepsilon f^i \circ {}^\varepsilon\theta = {}^i f(\varepsilon) = \sum_{n=P}^{N} \varepsilon^n f^{i(n)} \text{ with } f^{i(n)} = 0 \text{ if } n \neq p_i \tag{8}$$

and we have similar relations for ${}^\varepsilon F^i$ and ${}^\varepsilon G^i_\pm$. Finally, the geometrical parameters ${}^\varepsilon a_{\alpha\beta}$, ${}^\varepsilon b_{\alpha\beta}$, ${}^\varepsilon\Gamma^\gamma_{\alpha\beta}$, ${}^\varepsilon m^\alpha_\beta$, ..., can be developed in the same spirit. For instance

$$ {}^\varepsilon\Gamma^\gamma_{\alpha\beta} \simeq \sum_{m=1}^{M} \varepsilon^{2m} \Gamma^{\gamma(2m)}_{\alpha\beta}, \quad {}^\varepsilon b_{\alpha\beta} \simeq \sum_{m=1}^{M} \varepsilon^{2m-1} b^{(2m-1)}_{\alpha\beta}. \tag{9}$$

3.3. Recurrent Relations

Substitution of relations (5) to (9) into relations (2) to (4) gives the recurrent relations

$$\left.\begin{aligned} \sigma^{3\alpha(n)}_{,3} + \sigma^{\alpha\beta(n-1)}_{,\beta} + f^{\alpha(n-1)} + \chi^{\alpha(n-2)} = 0 \\ \sigma^{33(n)}_{,3} + \sigma^{3\alpha(n-1)}_{,\alpha} + b^{(1)}_{\alpha\beta}\sigma^{\alpha\beta(n-2)} + f^{3(n-1)} + \chi^{3(n-2)} = 0 \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \sigma^{\alpha\beta(n)} = a^{\alpha\beta\gamma\delta}(u^{(n)}_{\gamma,\delta} - b^{(1)}_{\gamma\delta}u_3^{(n-1)}) + a^{\alpha\beta33}u^{(n+1)}_{3,3} + \theta^{\alpha\beta(n-2)} \\ \sigma^{3\alpha(n)} = a^{\alpha3\beta3}(u^{(n+1)}_{\beta,3} + u^{n}_{3,\beta} + b^{\eta(1)}_{\beta}u^{(n-1)}_{\eta}) + \theta^{3\alpha(n-2)} \\ \sigma^{33(n)} = a^{33\gamma\delta}(u^{(n)}_{\gamma,\delta} - b^{(1)}_{\gamma\delta}u_3^{(n-1)}) + a^{3333}u^{(n+1)}_{3,3} + \theta^{33(n-2)} \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} u^{(n+1)}_{\alpha} = u^{(n+1)}_{3} = 0 \\ \sigma^{\alpha\beta(n)}n_\beta = F^{\alpha(n)}\ ;\ \sigma^{3\beta(n)}n_\beta = F^{3(n)}\ ;\ \sigma^{\alpha3(n)} = \pm G^{\alpha(n)}_{\pm}\ ;\ \sigma^{33(n)} = \pm G^{3(n)}_{\pm} \end{aligned}\right\} \tag{12}$$

where $\vec{\chi}^{(n-2)}$ and $\theta^{(n-2)}$ denote the recurrence terms of lower orders determined by previous iterations. Then, solution of problem (10) to (12) amounts to solve a sequence of 2D problems.

4. The Associated Two-dimensional Problems

4.1. The Displacement Vector $\vec{u}^{(i)}$ as a Function of x_3

Equations (10) to (12) show that the knowledge of vectors $\vec{u}^{(i)}$, $i \leq n$, involves the knowledge of tensors $\sigma^{(i-1)}$. Thus the dependence on x_3 of displacement $\vec{u}^{(n+1)}$ is governed by ($S^{i(n)}$, $s^{i(n)}$ are given by previous iterations, $D^{\alpha\beta} = a^{\alpha3\beta3}$, $D^{\alpha3} = 0$, $D^{33} = a^{3333}$)

$$\left.\begin{aligned} \sigma^{i3(n)}_{,3} = -S^{i(n)}, \quad \sigma^{i3(n)} = D^{ij}u^{(n+1)}_{j,3} + s^{i(n)} \quad \text{in } \Omega \\ \sigma^{i3(n)} = G^{i(n)}_{+} \quad \text{for } x_3 = +1, \quad = -G^{i(n)}_{-} \quad \text{for } x_3 = -1 \end{aligned}\right\}.$$

Theorem 4.1.1: The displacement vector $\vec{u}^{(n)}$ can be written as

$$u^{(n)}_\alpha(x_1,x_2,x_3) = \zeta^{(n)}_\alpha(x_1,x_2) - x_3\zeta^{(n-1)}_{3,\alpha}(x_1,x_2) + I^{(n)}_\alpha(x_1,x_2,x_3) \tag{13}$$

$$u^{(n)}_3(x_1,x_2,x_3) = \zeta^{(n)}_3(x_1,x_2) + I^{(n)}_3(x_1,x_2,x_3) \tag{14}$$

where $I^{(n)}_j(x_1,x_2,x_3)$ are recurrence terms entirely determined by previous iterations. ■

4.2. The Two-dimensional Problems

Relations (13) (14) lead to solve 2D problem on the middle surface of the shell.

Theorem 4.2.1: The displacement vector $(\zeta^{(n+1)}_\alpha, \zeta^{(n)}_3)$ of the middle surface is solution of the thin shallow shell problem

$$\left.\begin{aligned} X^{\alpha\beta(n+1)}_{,\beta} + \mathcal{F}^{\alpha(n+1)} = 0 \quad \text{upon } \omega \\ Y^{\alpha\beta(n+1)}_{,\alpha\beta} + b^{(1)}_{\alpha\beta}X^{\alpha\beta(n+1)} + \mathcal{F}^{3(n+1)} = 0 \quad \text{upon } \omega \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} X^{\alpha\beta(n+1)} = A^{\alpha\beta\gamma\delta}\varphi^{(n+1)}_{\gamma\delta} - B^{\alpha\beta\gamma\delta}\rho^{(n+1)}_{\gamma\delta} \quad \text{upon } \omega \\ Y^{\alpha\beta(n+1)} = B^{\alpha\beta\gamma\delta}\varphi^{(n+1)}_{\gamma\delta} - C^{\alpha\beta\gamma\delta}\rho^{(n+1)}_{\gamma\delta} \quad \text{upon } \omega \end{aligned}\right\} \tag{16}$$

$$\left.\begin{array}{rl} \zeta_\gamma^{(n+1)} = \zeta_3^{(n)} = \zeta_{3,\gamma}^{(n)} = 0 & \textit{upon } \partial\omega_0 \\ X^{\alpha\beta(n+1)} n_\beta = g^{\alpha(n+1)} & \textit{upon } \partial\omega_1 \\ Y^{\alpha\beta(n+1)} n_\beta = h^{\alpha(n+1)} \ ; \ Y_{,\beta}^{\alpha\beta(n+1)} n_\alpha = g^{3(n+1)} & \textit{upon } \partial\omega_1 \end{array}\right\} \quad (17)$$

where $\varphi_{\gamma\delta}^{(n+1)} = \zeta_{\gamma,\delta}^{(n+1)} - b_{\gamma\delta}^{(1)} \zeta_3^{(n)}$; $\rho_{\gamma\delta}^{(n+1)} = \zeta_{3,\gamma\delta}^{(n)}$; $A^{\alpha\beta\gamma\delta} = \sum_{\ell=1}^{L} (z_{\ell+1} - z_\ell) Q_{(\ell)}^{\alpha\beta\gamma\delta}$; $B^{\alpha\beta\gamma\delta} = \frac{1}{2} \sum_{\ell=1}^{L} (z_{\ell+1}^2 - z_\ell^2) Q_{(\ell)}^{\alpha\beta\gamma\delta}$; $C^{\alpha\beta\gamma\delta} = \frac{1}{3} \sum_{\ell=1}^{L} (z_{\ell+1}^3 - z_\ell^3) Q_{(\ell)}^{\alpha\beta\gamma\delta}$; $Q_{(\ell)}^{\alpha\beta\gamma\delta} = a_{(\ell)}^{\alpha\beta\gamma\delta} - \frac{a_{(\ell)}^{\alpha\beta 33} a_{(\ell)}^{33\gamma\delta}}{a_{(\ell)}^{3333}}$, $\mathcal{F}^{i(n+1)}$, $g^{i(n+1)}$ *and* $h^{\alpha(n+1)}$ *are the associated loading and recurrence term resultants over the middle surface, and,* L *= number of layers.* ■

Then, it is possible to give a variational formulation of problem (15) to (17) which has a unique solution $(\zeta_\alpha^{(n+1)}, \zeta_3^{(n)})$ in a suitable space when the data are sufficiently regular.

5. Return to the Three-dimensional problem

The sequence of two-dimensional solutions $\vec{\zeta}^{(n)}$ and the recurrence process give the 3D displacement vector and 3D stress stensor as follows:

Theorem 5.1.1: Let n_0 *be the order of the first non-zero solution* $(\zeta_\alpha^{(n_0)}, \zeta_3^{(n_0-1)})$. *Then, the three dimensional displacement of the reference domain* Ω *is given by*

$$u_\gamma = \varepsilon^{n_0} u_\gamma^{(n_0)} + \varepsilon^{n_0+1} u_\gamma^{(n_0+1)} + \ldots \quad ; \quad u_3 = \varepsilon^{n_0-1} u_3^{(n_0-1)} + \varepsilon^{n_0} u_3^{(n_0)} + \ldots$$

where $u_3^{(n_0-1)} = \zeta_3^{(n_0-1)}$, $u_\gamma^{(n_0)} = \zeta_\gamma^{(n_0)} - x_3 \zeta_{3,\gamma}^{(n_0-1)}$ *and where the next terms are defined by* (13) (14) *while* $(\zeta_\gamma^{(n)}, \zeta_3^{(n-1)})$ *solve equations* (15)-(17) *written at the appriopriate order.* ■

Theorem 5.2.1: Let n_0 *be the order of the first non-zero solution* $(\zeta_\alpha^{(n_0)}, \zeta_3^{(n_0-1)})$. *Then, the three dimensional stress components for the multilayered shell are given by*

$$\sigma^{\alpha\beta} = \varepsilon^{n_0} \sigma^{\alpha\beta(n_0)} + \ldots \quad ; \quad \sigma^{3\alpha} = \varepsilon^{n_0+1} \sigma^{3\alpha(n_0+1)} + \ldots \quad ; \quad \sigma^{33} = \varepsilon^{n_0+2} \sigma^{33(n_0+2)} + \ldots$$

where $\sigma^{\alpha\beta(n)}$, $\sigma^{3\alpha(n+1)}$, $\sigma^{33(n+2)}$ *are calculated from the value of* $(\zeta_\gamma^{(n)}, \zeta_3^{(n-1)})$. ■

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7. References

1. J. H. Argyris, I. Fried and D. W. Scharpf, *The tuba family of plate elements for the matrix displacement method*, 14 (1968) 701–709.
2. Y. Basar, Y. Ding and R. Schultz, *Refined shear-deformation models for composite laminates with finite rotations*, Intern. J. Sol. Struct. ( ).
3. M. Bernadou, *Méthodes d'Eléments Finis pour les Problèmes de Coques Minces*, (Masson, Paris, 1994).

4. M. Bernadou, *Finite Element Methods for Thin Shell Problems*, (J. Wiley, Chichester, 1995).
5. M. Bernadou, P. L. George, A. Hassim, P. Joly, P. Laug, A. Perronnet, E. Saltel, D. Steer, G. Vanderborck, M. Vidrascu, *Modulef: A Modular Library of Finite Elements*, (INRIA, Rocquencourt, 1986).
6. M. Bernadou, R. Kail, F. Léné and Y. H. De Roeck, *Modelization and approximation of laminated shallow shells*, (to appear).
7. S. Busse, P. G. Ciarlet and B. Miara, *Justification d'un modèle linéaire de coques "faiblement courbées" en coordonnées curvilignes* (to appear).
8. D. Caillerie and E. Sanchez-Palencia, *Elastic thin shells: asymptotic theory in the anisotropic and heterogeneous cases*, Math. Models Methods Appl. Sci. **5** (1995), 473–476.
9. P. G. Ciarlet and V. Lods, *Asymptotic analysis of linearly elastic shells. I. Justification of membrane shell equations*, Arch. Rational Mech. Anal. **136** (1996) 119–161.
10. P. G. Ciarlet and V. Lods, *Asymptotic analysis of linearly elastic shells. II. Justification of Koiter's shell equations*, Arch. Rational Mech. Anal. **136** (1996) 191–200.
11. P. G. Ciarlet, V. Lods and B. Miara, *Asymptotic analysis of linearly elastic shells. II. Justification of flexural shell equations*, Arch. Rational Mech. Anal. **136** (1996) 163–190.
12. P. G. Ciarlet and B. Miara, *Justification of the two-dimensional equations of a linearly elastic shallow shell*, Comm. Pures & Appl. Math **XLV** (1992) 327–360.
13. P. Destuynder, *A classification of thin shell theories*, Acta Aplicandæ Mathematicæ **4** (1985) 15–63.
14. S. Di and E. Ramm, *Hybrid stress formulation for higher-order theory of laminated shell analysis*, Comput. Meth. Appl. Mech. Engng. **109** (1993) 359–376.
15. Chap.2 A. E. Green and W. Zerna, *Theoretical Elasticity*, Second Edition (Oxford University Press, 1988).
16. R. Kail, *Modélisation asymptotique et numérique de plaques et coques stratifiées*, (Thèse d'Université, Paris VI, 1994).
17. R. Kail, F. Léné and Y. H. De Roeck, *An asymptotic model of laminated plates*, Revue Européenne des Eléments Finis **6** (1997) 23–42.
18. W. T. Koiter, *On the nonlinear theory of thin elastic shells*, *Proc. Kon. Nederl. Akad. Wetensch.* **B69** (1966) 1–54.
19. B. Miara and E. Sanchez-Palencia, *Asymptotic analysis of linearly elastic shells*, Asymptotic Analysis **12** (1966) 41–54.
20. J. N. Reddy and K. Chandrashekhara, *Nonlinear analysis of laminated shells including transverve shear strains*, AIAA J. **23** (1985) 440–441.

TOWARDS SHELL ELEMENTS AVOIDING LOCKING IN THE GENERAL CASE

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ABSTRACT

We consider a general strategy for building shell elements that avoid membrane locking for Koiter shell formulation if the shell is bending dominated, but retain good approximation properties in the membrane dominated case. A similar strategy could also be applied to Naghdi shell models. So far, the strategy is vague, here and there, and the elements produced by it are quite ugly. However, there is hope that the strategy could be improved in order to provide more elegant and easy-to-use elements. Moreover it seems important, to start with (see for instance [3]), to ascertain that elements capable of provably work in both bending and membrane dominated cases do actually exist.

1. The continuous problems

In this Section we present the formulation and the assumptions on the continuous shell problem. Discretization problems will be tackled in the next Section.

1.1. The Koiter model

In the following we are going, for the sake of simplicity, to deal with the following setting. Ω will be a convex polygone in $\mathbf{R}^2$; we assume that the equations of the midsurface of the shell are given in Cartesian form $x_3 = g(x_1, x_2)$ so that, in particular, the second fundamental form is given by

$$b_{\alpha\beta} = \frac{\partial^2 g}{\partial x_\alpha \partial x_\beta} \qquad \alpha, \beta = 1, 2. \tag{1}$$

In this setting the problem can be formulated as follows. Let $V \subset (H^1(\Omega))^2$ and $W \subset H^2(\Omega)$ be closed subspaces containing the boundary conditions to be imposed on the shell. We want to find $u \in V$ and $w \in W$ minimizing the potential energy

$$2\Pi_o = a_o^b(w, w) + a_o^m(\gamma, \gamma) - 2F(u, w) \tag{2}$$

where: $a_o^b(w, w)$ is a quadratic coercive form on W representing the bending energy in the Koiter assumptions, $a_o^m(\gamma, \gamma)$ is a quadratic coercive form on the space of symmetric tensors $\Sigma = (L^2(\Omega))_s^4$, representing the membrane energy when $\gamma = \gamma(u, w)$ is given by

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - \Gamma^\delta_{\alpha\beta} u_\delta - b_{\alpha\beta} w \qquad \alpha, \beta = 1, 2, \tag{3}$$

and finally $F(u, w)$ represents the action of the external forces. In Eq. (3) $\Gamma^\delta_{\alpha\beta}$ are the usual Christoffel symbols. There and in the following the summation convention of repeated indices is employed. More generally, by default the notation of [1] is used.

Taking into account the particular description of the shell in Cartesian form, it is easy to check that

$$\Gamma^{\delta}_{\alpha\beta} = b_{\alpha\beta}\chi^{\delta} \qquad \alpha, \beta = 1, 2 \tag{4}$$

for a suitable choice of the (smooth) functions (χ^1, χ^2). Setting moreover, as usual,

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \qquad \alpha, \beta = 1, 2 \tag{5}$$

the expression for γ becomes

$$\gamma_{\alpha\beta} = \varepsilon_{\alpha\beta} - b_{\alpha\beta}(\chi^{\delta}u_{\delta} + w) \qquad \alpha, \beta = 1, 2. \tag{6}$$

1.2. Bending dominated and membrane dominated shells

In order to study the locking phenomenon, it is convenient to renormalize the terms appearing in the expression of the energy, considering a sequence of problems with variable thickness t and making explicit the dependence on t of the various terms. Doing that we reach the final form

$$2\Pi_t = a^b(w, w) + t^{-2}a^m(\gamma, \gamma) - 2t^c F(u, w) \tag{7}$$

where Π_t represents the scaled energy, and the exponent c can change according to the character of the shell (bending dominated or membrane dominated) in the following way. We say that the shell is *membrane dominated* if the only solution of the system of equations

$$\gamma_{\alpha\beta}(u, w) = 0 \qquad u \in V, \;\; w \in W, \quad \alpha, \beta = 1, 2 \tag{8}$$

is given by $u = 0, \; w = 0$. This will obviously depend on the goemetric nature of the shell (here expressed by the mapping $x_3 = g(x_1, x_2)$) and on the boundary conditions (here hidden in the definition of the spaces V and W.) With the terminology of Sanchez Palencia a membrane dominated shell is said to be *well inhibited.* In this case we shall make the further assumption that

$$a^m(\gamma, \gamma) \geq constant(||u||^2_V + ||w||^2_{L^2(\Omega)}). \tag{9}$$

This might not always be the case, but we are not ready to face the shell teratology in the full power of its complete generality. In the other case, that is, when Eq. (8) has nonzero solutions, we say that the shell is *bending dominated.* In order to have a sequence of problems where the solutions (for t going to 0) stay bounded without converging to 0, we have to scale the external forces differently in the two cases. More precisely, we take $c = 0$ in the bending dominated case and $c = -2$ in the membrane dominated one. With this choice, the limit problem for bending dominated shells amounts to minimize $a^b(w, w) - 2F(u, w)$ on the subspace $\gamma(u, w) = 0$, whereas the limit problem in the membrane dominated case amounts to minimize $a^m(\gamma, \gamma) - 2F(u, w)$ on the whole space $V \times W$.

2. Discretization

In this Section we consider the problems related to the finite element discretization of the continuous problem. For this we assume that we are given a sequence $\mathcal{T}_h$ of triangulations of Ω, and, for each triangulation, finite dimensional subspaces V_h and W_h of V and W respectively, made of functions that are piecewise polynomials in Ω.

2.1. Discretized problems and reduction operators

Once the spaces V_h and W_h have been chosen, the simple minded approach would lead to minimize Π_t on $V_h \times W_h$. Since it is well known that this does not work in general (worse than that, we do not know of any meaningful example in which this works) it is more convenient to shift to a modified problem: minimize on $V_h \times W_h$ the modified energy

$$2\Pi_t^h = a^b(w,w) + t^{-2}a^m(P^h(\gamma), P^h(\gamma)) - 2t^c F(u,w) \tag{10}$$

where P^h is a suitable interpolation (or projection) operator that maps Σ (or a more regular subset of it, containing at least $\gamma(V_h, W_h)$) into a piecewise polinomial subspace Σ_h of the space Σ. In order to provide a method which is good in both bending dominated and membrane dominated cases, the operator P^h must satisfy the following conditions.

First of all, to deal properly with the bending dominated case, we should be able to find, for every (smooth) pair (u,w) in the kernel of γ, a corresponding pair (u^I, w^I) in $V_h \times W_h$, close to (u,w) and such that

$$P^h(\gamma(u^I, w^I)) = 0. \tag{11}$$

Then P^h should also be a suitable approximation of the Identity operator: in particular we require

$$(P^h\gamma - \gamma, \delta) = 0 \qquad \forall\delta \ \textit{ piecewise constant.} \tag{12}$$

Following [2] we can then prove that the minimizing argument of Π_t^h converges to the minimizing argument of Π_t in the bending dominated case.

On the other hand, if we want to have a method that also works in the membrane dominated case, we have to require that $a^m(P^h(\gamma), P^h(\gamma))$ is coercive in $V_h \times W_h$ with the norm of $V \times L^2(\Omega)$, at least when condition (9) holds. For this it is clearly quite convenient that

$$P^h(\varepsilon(u^h)) = \varepsilon(u^h) \qquad \forall u^h \in V^h \tag{13}$$

where $\varepsilon(u)$ is still defined as in Eq. (5). This, unfortunately, might not be sufficient, as the control that we are going to have on w will be reduced, at best, to the L^2 norm of $P^h(\mathbf{b}w)$ where $\mathbf{b}$ is the tensor with components given by (1). This might force us to introduce in (10) an additional term, minimizing

$$\tilde{\Pi}_t^h = \Pi_t^h + \tau||\mathbf{b}w - P^h(\mathbf{b}w)||_{L^2(\Omega)} \tag{14}$$

were τ is scaled as $||t^c F||$ (that is, it will mainly act only in the membrane dominated case.) This *is* ugly, but I cannot do better for the moment. It is clear that (14) can be substituted by several alternative terms, all of them with various degrees of ugliness.

2.2. Guidelines for the construction of the elements

It is clear that the main difficulty will be to satisfy Eq. (11). However, if we start from a given space V_h (to fix the ideas, piecewise quadratics and continuous on a triangular grid) we can first design a space Σ_h (piecewise linear discontinuous symmetric tensors) and an interpolation operator P^h with range in Σ_h such that Eq. (13) holds. In the case of our example, this could be obtained by assigning: the mean value of each component (3 conditions) and the moments of order ≤ 1 (on each edge of each triangle) of the double tangential component ($3\times2=6$ conditions). It can be checked that these 9 conditions determine uniquely a linear symmetric tensor, and that Eq. (13) holds true. Other possibilities are obviously available. However, it is better to keep the three mean values in vue of Eq. (12). We also notice that, for a given smooth vector u, if we define u^I as the piecewise quadratic that agrees with u at the vertices and is such that each component has the same mean value of u on each edge of each triangle, we have that

$$P^h(\varepsilon(u-u^I))=0 \tag{15}$$

which, as we shall see, is remarkably useful. To check Eq. (15), use first the mean value of $u-u^I$ on edges to verify that the mean value of $\varepsilon(u-u^I)$ on each triangle is zero; then use the fact that $u=u^I$ at vertices to verify that the integral of $\varepsilon_{tt}(u-u^I)$ (which is the tangential derivative of the tangential component of $u-u^I$), on each edge, is zero; finally, use both conditions on $u-u^I$ to show that $\varepsilon_{tt}(u-u^I)$ is orthogonal, on each edge, to linear functions. We are left with the construction of w^I satisfying Eq. (11). Assume therefore that we had a pair (u,w) of smooth functions satisfying Eq. (8). Using Eqs. (6) and (15) we see that Eq. (11) reduces to

$$P^h(\mathrm{b}(\chi^\delta(u_\delta-u^I_\delta)+w-w^I))=0. \tag{16}$$

As we are dealing with a Koiter shell, w^I will have to be a C^1 piecewise polynomial, and there is no hope to use, for W_h, the same set of degrees of freedom that we are using for V_h. Hence we have, in a sense, much more freedom. If the coefficients $b_{\alpha\beta}$ are constant, Eq.(16) will simply require that the quantity

$$\chi^\delta(u_\delta-u^I_\delta)+w-w^I \tag{17}$$

has zero mean value on each triangle and zero moments of order ≤ 1 on each edge. This can be easily done whenever the degrees of freedom in W_h include, essentially, two degrees of freedom for each edge and one degree of freedom internal to each triangle. If the $b_{\alpha\beta}$'s are not constant, this will not affect the edge degrees of freedom (where only b_{tt} counts), but will require three internal nodes instead of one. As we see, the outcome is pretty ugly. However, at least, condition (16) can be fulfilled even if some of the $b_{\alpha\beta}$'s vanishes here and there, since they appear as multipliers of the *whole* equation. Still, it is clear that *much* work is needed in order to have a satisfactory shell element by this strategy.

3. Acknowledgements

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4. References

1. M. Bernadou, *Méthodes d'éléments finis pour les problèmes de coques minces* (Masson, Paris, RMA 33, 1994).
2. F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods* (Springer-Verlag, New York, SCM 15, 1991).
3. J. Pitkäranta, The problem of membrane locking in finite element analysis of cylindrical shells, *Numer. Math.* **61** (1992) 523–542.

CONVERGENCE RESULTS FOR ASYMPTOTIC ANALYSIS OF UNCOUPLED AND COUPLED KOITER'S SHELLS

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ABSTRACT

In this paper, is given a generalisation of the usual Koiter's shell model to non symmetrical shells, it couples stretching and bending effects. Then the deformation problems for these models are put into an abstract formulation the limit of which is studied when the thickness of the shell tends to zero. This paper gather the results presented in [2] and [3].

1. Coupled and Uncoupled Koiter's Shell Models

Koiter's model is a shell model involving the stretching effects of the shell surface and the bending effects, it can be derived from 3D linear elasticity using heuristic assumptions. The elastic energy of a displacement field v for the usual (or uncoupled) Koiter's model is (see for instance [1]) :

$$\frac{1}{2}\int_S \varepsilon\, A^{\alpha\beta\mu\nu}\left[\gamma_{\mu\nu}(v)\ \gamma_{\alpha\beta}(v) + \frac{\varepsilon^2}{12}\rho_{\mu\nu}(v)\ \rho_{\alpha\beta}(v)\right] ds$$

$\gamma_{\alpha\beta}(v)$ and $\rho_{\alpha\beta}(v)$ are the tensors characterising the changes of surface metric and of curvature in the displacement v. S is the two-dimensionnal domain used to parametrise the shell, ε the thickness of the shell.

When the shell is not symmetrical, stretching and bending are more intrically coupled. Indeed, the elastic energy of the generalised (or coupled) Koiter's model derived for such shells from 3D elasticity is :

$$\frac{1}{2}\varepsilon\left[\int_S A_{11}^{\alpha\beta\mu\nu}\gamma_{\mu\nu}(v)\ \gamma_{\alpha\beta}(v)\, ds - \varepsilon\int_S A_{12}^{\alpha\beta\mu\nu}\rho_{\mu\nu}(v)\ \gamma_{\alpha\beta}(v)\, ds\right.$$
$$\left. - \varepsilon\int_S A_{21}^{\alpha\beta\mu\nu}\gamma_{\mu\nu}(v)\ \rho_{\alpha\beta}(v)\, ds + \varepsilon^2\int_S A_{22}^{\alpha\beta\mu\nu}\rho_{\mu\nu}(v)\ \rho_{\alpha\beta}(v)\, ds\right]$$

The usual Koiter's shell model is clearly a particular case of the generalised one.

Both models involve the thickness ε of the shell which is assumed to be small, then a deformation problem for a Koiter's shell (coupled or not) can be seen as a perturbation problem. That is this perturbation problem, which is singular in general, that is studied in this paper.

Deriving the limit problem for the Koiter's models as the thickness ε tends to zero is of interest by its own but it is of first importance when studying the convergence process of a finite element computation as locking phenomenæ are most likely to occur.

2. Abstract Perturbation Problem

The weak formulation of a static deformation problem for a generalised Koiter's shell can be put into the abstract formulation :

$$\begin{gathered} \textit{Find } u^\varepsilon \in V \ s.t. \\ \forall\, v \in V\,,\ a^\varepsilon(u^\varepsilon,v) = F^\varepsilon(v) \end{gathered} \tag{1}$$

where V is a Hilbert space, l_1 and l_2 are linear continuous mappings from V to respectively two Hilbert spaces E_1 and E_2, F^ε is a linear form on V and $a^\varepsilon(u,v)$ is the following bilinear form :

$$a^\varepsilon(u,v) = \frac{1}{\varepsilon^2}\, a_{11}(l_1(u),l_1(v)) + \frac{1}{\varepsilon}\, a_{12}(l_1(u),l_2(v)) + \frac{1}{\varepsilon}\, a_{21}(l_2(u),l_1(v)) + a_{22}(l_2(u),l_2(v))$$

It is assumed that the norm on V is :

$$\|v\|_V = \left[\|l_1(v)\|_1^2 + \|l_2(v)\|_2^2\right]^{1/2}$$

and that the bilinear forms on $E_\alpha \times E_\beta$ $a_{\alpha\beta}$ fulfill the coercivity condition :

$$\begin{gathered} \exists\ \alpha > 0 \ s.t.\ \forall\ (\eta_1,\eta_2) \in E_1 \times E_2 \\ \alpha\left[\|\eta_1\|_1^2 + \|\eta_2\|_2^2\right] \le (a_{11}(\eta_1,\eta_2) + a_{11}(\eta_1,\eta_1) + a_{12}(\eta_1,\eta_2) + a_{21}(\eta_2,\eta_1) + a_{22}(\eta_2,\eta_2) \end{gathered}$$

For Koiter's shell models, E_1 and E_2 are equal to $(L^2(S))^3$ and

$$l_1(v) = \{\gamma_{\alpha\beta}(v)\ ;\ \ \alpha,\ \beta = 1{,}2\}\ ;\ l_2(v) = \{\rho_{\alpha\beta}(v)\ ;\ \ \alpha,\ \beta = 1{,}2\}$$

Let G_1 be the null space of l_1 and $G_1^\perp$ be its orthogonal space, that is :

$$G_1 = \{v \in V\, s.t.\ l_1(v) = 0\}$$

and

$$G_1^\perp = \{v \in V\, s.t.\ \forall\, w \in G_1\ ,\ (v,w)_V = 0\}$$

The space V can be split in $V = G_1 \approx G_1^\perp$, that is :

$$\forall\, v \in V \ ,\ v = v_1 + v_1^{\perp}\ ;\ v_1 \in G_1\ ,\ v_1^{\perp} \in G_1^{\perp}$$

It is worth noticing that $\|v\|_1 = \|l_1(v)\|_1$ is a norm on $G_1^{\perp}$. This norm is prehilbertian but, in general, $G_1^{\perp}$ is not complete for it, let $\overline{G_1}^{\perp}$ be the closure of $G_1^{\perp}$ for the norm $\|v\|_1$.In the topology of $\overline{G_1}^{\perp}$, the convergence of sequences is given by :

$$\{v_n \to v \ in\ \overline{G_1}^{\perp}\} \Leftrightarrow \{l_1(v_n) \to l_1(v)\ in\ E_1\}$$

3. Convergence Results

3.1 General case

Proposition

Let F be a linear continuous form on V and $F_{\perp}$ a linear continuous form on $\overline{G_1}^{\perp}$ which means :

$$\forall\ v_1^{\perp} \in G_1^{\perp}\ ,\ |F_{\perp}(v_1^{\perp})| \le K\, \|l_1(v_1^{\perp})\|$$

and set $F^{\varepsilon}(v) = F(v) + \dfrac{1}{\varepsilon} F_{\perp}(v_1^{\perp})$. Then, when ε tends to zero :

$$u^{\varepsilon} \to u_1 \ in\ V\ strong\ ,\ u_1 \in G_1$$
$$\frac{u_1^{\varepsilon\perp}}{\varepsilon} \to u_1^{\perp}\ in\ \overline{G_1}^{\perp}\ strong$$

with $(u_1, u_1^{\perp})$ being the unique solution in $G_1 \times \overline{G_1}^{\perp}$ of the equation :

$$\forall\ v = (v_1, v_1^{\perp}) \in G_1 \times \overline{G_1}^{\perp}$$
$$a_{11}(l_1(u_1^{\perp}), l_1(v_1^{\perp})) + a_{12}(l_1(u_1^{\perp}), l_2(v_1)) + a_{21}(l_2(u_1), l_1(v_1^{\perp})) + a_{22}(l_2(u_1), l_2(v_1)) = F(v_1) + F_{\perp}(v_1^{\perp})$$

In the case when $F_{\perp} \equiv 0$ and moreover $G_1 = \{0\}$ or $F \equiv 0$ on G_1, then $u^{\varepsilon} \to 0$, $\dfrac{u_1^{\varepsilon\perp}}{\varepsilon} \to 0$. In this case, the proposition gives more than that when F is continuous on $\overline{G_1}^{\perp}$, indeed it shows that :

$$\frac{u^{\varepsilon}}{\varepsilon} \to \tilde{u}_1 \ in\ V\ strong\ ,\ \tilde{u}_1 \in G_1 \quad \text{and} \quad \frac{u_1^{\varepsilon\perp}}{\varepsilon^2} \to \tilde{u}_1^{\perp}\ in\ \overline{G_1}^{\perp}\ strong$$

$(\tilde{u}_1, \tilde{u}_1^{\perp})$ being solution of the equation of proposition 1 with $F(v_1^{\perp})$ as right hand side.

If F is not continuous on $\overline{G_1}^{\perp}$, then the previous result does not give the way u^{ε} and $\dfrac{u_1^{\varepsilon\perp}}{\varepsilon}$ tends to zero.

A more precise result can be then established in the uncoupled case.

3.2 Uncoupled case

It is classical to introduce A_{11} and A_{22} the linear and continuous mappings from V to V defined by:

$$(A_{11}(u).v)_V = a_{11}(l_1(u),l_1(v)) \text{ and } (A_{22}(u).v)_V = a_{22}(l_2(u),l_2(v))$$

It is easy to verify that $\|v\|_2 = \|A_{11}(v)\|_V$ define a prehilbertian norm on $G_1^\perp$, in general $G_1^\perp$ is not complete for it, so let $\overline{\overline{G_1}}^\perp$ be closure of $G_1^\perp$ for the norm $\|v\|_2$. In the topology of $\overline{\overline{G_1}}^\perp$, the convergence of sequences is given by :

$$\{v_n \rightarrow v \ in \ \overline{\overline{G_1}}^\perp\} \Leftrightarrow \{A_{11}(v_n) \rightarrow A_{11}(v) \ in \ V\}$$

Let consider the problem (1) in the uncoupled case and with $F^\varepsilon(v) = F(v)$, that is :

$$\begin{gathered} Find \ u^\varepsilon \in V \ s.t. \\ \forall v \in V \ , \ \frac{1}{\varepsilon^2} a_{11}(l_1(u^\varepsilon),l_1(v)) + a_{22}(l_2(u^\varepsilon),l_2(v)) = F(v) \end{gathered} \tag{2}$$

The results of proposition 1 hold true and are here :

$$u^\varepsilon \rightarrow u_1 \ in \ V \ strong \ , \ u_1 \in G_1 \text{ and } \frac{u_1^{\varepsilon\perp}}{\varepsilon} \rightarrow 0 \ in \ \overline{G_1}^\perp \ strong$$

u_1 of G_1 being the unique solution of the equation :

$$\forall v \in G_1 \ , \ a_{22}(l_2(u_1),l_2(v)) = F(v)$$

Moreover, if $a_{22}(l_2(u_1),l_2(v))$ and $F(v)$ are continuous on $\overline{G_1}^\perp$, then $\frac{u_1^{\varepsilon\perp}}{\varepsilon^2}$ tends to $\widetilde{u_1}^\perp$ in $\overline{G_1}^\perp$ strong with $\widetilde{u_1}^\perp$ being the unique solution of the equation :

$$\forall v \in \overline{G_1}^\perp \ , \ a_{11}(l_1(\widetilde{u_1}^\perp),l_1(v)) = F(v) - a_{22}(l_2(u_1),l_2(v))$$

When the only assumptions on F are linearity and continuity on V, it is possible to establish the following convergence result.

Proposition 2

Let u^ε be the solution of (2) with F linear and continuous on V and let define $f \in V$ by $(f.v)_V = F(v)$, then :

$$u^{\varepsilon} \to u_1 \; in\; V\, strong \;\; , u_1 \in G_1 \text{ and } \frac{u_1^{\varepsilon\perp}}{\varepsilon^2} \to \hat{u}_1^{\perp} \; in \; \overline{\overline{G}}_1^{\perp} \; strong$$

with $u_1^{\perp}$ being the unique solution of the equation :

$$A_{11}(\hat{u}_1^{\perp}) = f - A_{22}(u_1)$$

4. References

1. M. Bernadou, *Méthodes d'Eléments Finis pour les Problèmes de Coques Minces* (Masson Recherches en Mathématiques Appliquées, Paris, 1994).

2. D. Caillerie, *Comptes-Rendus de l'Académie des Sciences*, **t. 323 Série I** (1996) 835-840.

3. D. Caillerie and E. Sanchez-Palencia, *Mathematical Models and Methods in Applied Sciences*, **V5 n°1** (1995) 47-66.

ABOUT DISTRIBUTED PARAMETER CONTROL PROBLEM APPLICATION TO SHAPE OPTIMIZATION OF SHELLS

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ABSTRACT

This is a general survey of questions arising in the optimization of the midsurface of a shell using a descent algorithm which needs the gradient of the criterion which is minimized. It fits in the general setting of distributed parameters controle problems.

1. Introduction

The results we present have been obtained by a group of several researchers in mathematics and in computer science [2][1][4][6][7][8]. The subject is the following:search the shape of a shell working in linear elasticity conditions, having the best possible behaviour with respect to a given criterion. This fits in the following setting:

- ϕ is a design variable, it is the unknown. For a given ϕ belonging to a proper space Λ, $u^\phi \in V$ is the unique solution of the following equation:

$$u^\phi \in V \qquad a(\phi; u^\phi, v) = l(\phi; v) \qquad \forall v \in V \tag{1}$$

  where V is a Hilbert space. It depends on the design parameter ϕ.

- Min $j(\phi) = J(\phi, u^\phi)$ where $J(\phi, v)$ is a given functional defined on $\Lambda \times V$.

In our purpose, ϕ is the shape of the shell (particularly its midsurface), and equation (1) is the linear elastic response equation of the shell. For instance, one can be interested in minimizing the elastic energy, the average displacement, the maximum displacement, stresses, or eigenvalues.

We want to use a descent algorithm in order to give a numerical solution to this problem. The crux is the computation and discretization of the differential of J. We give some differentiability results for the mapping $\phi \to u^\phi$ for a proper choice of the functional setting, and we give an explicit expression of the differential of j in a given direction.

In order to get a numerical solution, one needs to discretize the problem. As usual, two methods are possible: the control problem can be discretized first, then differentiated (discrete gradient or DG method), or the differential of the continuous problem can be discretized (DCG method). We briefly recall some mathematical results concerning the comparison of both methods, then we discuss the software structures which go together.

Some numerical results have been obtained in the optimization of arches and cooling towers. Some remarks will be done about numerical locking.

2. Optimal control setting

Let Λ be a Banach space, V a Hilbert space, and Φ an open subset of Λ. We consider

$a : \Phi \times V \times V \longrightarrow \mathbf{R}$ which for each $\phi \in \Phi$ is bilinear continuous, symmetric, coercive, and $l : \Phi \times V \longrightarrow \mathbf{R}$ which is linear continuous on V. We denote by $u^\phi \in V$ the unique solution of:

$$\forall v \in \mathbf{V} \qquad a(u^\phi, v) = l(\phi, v).$$

For a given functional $J : \Phi \times V \longrightarrow \mathbf{R} : \phi, v \mapsto J(\phi, v)$ we consider $j(\phi) = J(\phi, u^\phi)$. If J is $\mathcal{C}^1$, one can prove that j is $\mathcal{C}^1$ and:

$$\frac{d}{d\phi} j(\phi).\psi = \frac{\partial J}{\partial \phi}(\phi, u^\phi).\psi - \frac{\partial a}{\partial \phi}(\phi; u^\phi, p^\phi).\psi + \frac{\partial l}{\partial \phi}(\phi, p^\phi).\psi \tag{2}$$

where p^ϕ is the unique solution of the following adjoint state equation:

$$p^\phi \in V \qquad a(\phi; p^\phi, w) = \frac{\partial J}{\partial v}(\phi; u^\phi).w \qquad \forall w \in V \tag{3}$$

3. The case of shell equations [3]

We consider the Koiter model. The midsurface is the graph of a regular mapping $\phi : \hat{\omega} \subset \mathbf{R}^2 \rightarrow \mathbf{R}^3 : (\xi_1, \xi_2) \mapsto \phi(\xi_1, \xi_2)$. We take it as the design variable. The equilibrium equation is of the form (1) with $V = H^1(\hat{\omega}) \times H^1(\hat{\omega}) \times H^2(\hat{\omega})$+boundary conditions (simply supported shell or clamped shell on a non-zero measure part of $\partial\hat{\omega}$). The linear functional l measures the virtual work of the external forces in the virtual displacement v.

The design parameter ϕ interferes in a through *the metric tensor, the curvature tensor and its derivatives with respect to* ξ, *and the Christoffel symbols.* In order to get $\frac{\partial a}{\partial \phi}(\phi; u, v).\psi$, one needs to differentiate these quantities with respect to ϕ, with values in $L^\infty(\hat{\omega})$. All these depend on ϕ and its three first derivatives with respect to ξ. If ϕ is chosen in $W^{3,\infty}(\hat{\omega})$, then a is well defined, and Frechet-differentiable with respect to ϕ in the space of bilinear continuous functionals defined on V, equiped with its usual norm. Formula (2) gives an explicit expression of $\frac{d}{d\phi} j(\phi).\psi$ for any ψ in Λ.

4. Discretizations

4.1. The discrete gradient method (DG method)

The discretization is done before the differentiation. Equation (1) is discretized for instance with a finite element procedure, which deals with geometrical degrees of freedom (approximation of ϕ) and kinematical degrees of freedom (approximation of u). The functional J is approximated by J_h which depends on this finite number of degrees of freedom. Formula (2) is used in finite dimensional spaces. It gives the *exact gradient of the approximated functional.*

4.2. The discretized continuous gradient method (DCG method)

Another way of handling formula (2) is the following.

First, the space Λ is approximated by a finite dimensional subspace Λ_h which is spanned by functions $\{s_i, i = 1, ...k\}$. One has to approximate $\frac{d}{d\phi} j(\phi).s_i$ for each i. This requires an approximation of u^ϕ and p^ϕ. This is got from any kind of approximation of the variational equation (1) , for instance a finite element procedure. It gives approximations u_h^ϕ and p_h^ϕ. One can set:

$$[\frac{d}{d\phi}j(\phi).s_i]_h = \frac{\partial J}{\partial \phi}(\phi, u_h^\phi).s_i - \frac{\partial a}{\partial \phi}(\phi; u_h^\phi, p_h^\phi).s_i + \frac{\partial l}{\partial \phi}(\phi; p_h^\phi).s_i \qquad (4)$$

which a priori gives *an approximate value of the exact differential of the exact functional.*

5. Comparison between the DG and the DCG method

5.1. Mathematical point of vue

In both methods, if things are handled in a natural way, the discretized functional is the same.

It has been proved in [4] that both discretized gradients are mathematically the same if V_h, approximation of V in the finite element procedure, does not depend on the design approximation. This is true in a wide class of problems.

If not, it may happen that $DG \neq DCG$. Examples will be shown, particularly the shape optimization of an arch in which one has $DG_h \neq DCG_h$ and $||DG_h - DCG_h|| \to 0$ when $h \to 0$.

5.2. Numerical aspects

Of course if $DG = DCG$ both methods behave the same. If not, in the DG method, optimization algorithms are expected to behave better because the gradient which is used is the exact gradient of the approximated functional. This is not the case in the DCG method. We have made numerical experiments in the arch case. As one has $DG_h \neq DCG_h$ and $||DG_h - DCG_h|| \to 0$ when $h \to 0$, in order to see a difference, one needs to chose h big enough. We found out that the difference could be seen for very rough finite elements. The DG method gave significant results, when the DCG method did not reach convergence. So if rough results are required, *the DG method behaved better.* (See [5]).

We have observed another phenomenon [8]. When the thickness of the arch was too small the finite elements degenerated: there was a numerical locking in the finite element procedure we used. The DCG method did not give satisfactory results which indicated the problem. The DG method has no problem converging althoug the results are faulse.

5.3. Software aspects

This is one important interest of the DCG method [5], [6].

In the DG method, if one wants to change the discrete design variables or the finite element procedure, the gradient software has to be rewritten (eventually automatically with a differentiation procedure).

In the DCG method, the finite element package and the gradient package can be treated separately. Each package can be used as a blakbox by the other one. If one wants to change the discrete parameters, only values of input parameters have to be changed in the gradient package.

Nevertheless the output files of the finite element package have to be settled in a proper way to become input files for the gradient package. So an interface needs to be written. The datas which are needed in the gradient package should be easy to get from the finite element packages: for instance, the nodes, the degrees of freedom, the integration scheme which have been used, which must be the same in both packages. Usually, in industrial finite element codes, they can be found in the input-output files or in the user manual. Another kind of datas are less usual, namely derivatives of displacements at Gauss points. If the finite element shape functions are not given in the user manual, these can be rebuilt from the stresses which are usually given. Let us notice that the interface can be written by the "finite element group" who does not know optimal controle techniques. The "gradient group" asks for appropriate datas. For more details see [5][6] .

6. Numerical locking

Some remarks will be done about numerical locking in finite elements computing the behaviour of an arch. When the thickness of the arch becomes too small, problems may occur.

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8. References

1. D. Chenais, Discrete gradient and discretized continuum gradient in shape optimization of shells,*Mechanics of Structures and Machines*, **22 (1)**, (1994)
2. D.Chenais, B. Rousselet, R. Benedict, Design sensitivity for arch structures, *Journal of Optimization Theory and Applications*, **58 (2)**, (1988), 225-239
3. D. Chenais, Optimal design of midsurface of shells: differentiability proof and sensitivity computation,*Applied Mathematics and Optimization*, **16**, (1987), 93-133
4. V.Lods, Gradient discret et gradient continu discrétisé en contrôle optimal à paramètres distribués, *Thèse de l'Université de NICE, France*, (1992)
5. D.Chenais, C.Knopf-Lenoir, From design sensitivity analysis to code structure in arch optimization, *Proceedings First Int. Conf. on Computer Aided Optimal Design of Structures*, (Ed. Brebbia-Hernandez, Springer Verlag, (1989) 23-32
6. D.Chenais, A.Habbal, C.Knopf-Lenoir, Un logiciel d'optimisation de structures interfaçable avec des codes standards d'éléments finis,*Actes du Colloque National en Optimisation de Structures GIENS, France* (1993)
7. S.Moriano, Optimisation de forme de coques, *Thèse de l'Université de NICE, France*, (1988)
8. A.Habbal, Optimisation non differentiable de forme d'arche,*Thèse de l'Université de NICE, France*, (1990)

ASYMPTOTIC ANALYSIS OF ELASTIC SHELLS*

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The asymptotic analysis of elastic shells is a relatively recent subject. After the landmark attempt of Goldenveizer [1963], a major step for *linearly elastic shells* was achieved by Destuynder [1980] in his Doctoral Dissertation, where a convergence theorem for *"membrane shells"* was "almost proved"; another major step was achieved by Sanchez-Palencia [1990], who clearly delineated the kinds of geometries of the middle surface and boundary conditions that yield either two-dimensional *membrane*, or two-dimensional *flexural*, equations when the method of formal asymptotic expansions is applied to the variational equations of three-dimensional linearized elasticity (see also Caillerie & Sanchez-Palencia [1995] and Miara & Sanchez-Palencia [1996]).

Then Ciarlet & Lods [1996a, 1996b] and Ciarlet, Lods & Miara [1996] carried out an *asymptotic analysis of linearly elastic shells that covers all possible cases*: Under three distinct sets of assumptions on the geometry of the middle surface, the boundary conditions, and on the order of magnitude of the applied forces, they established *convergence theorems* in H^1, in L^2, or in *ad hoc* completion spaces, that justify either the linear two-dimensional equations of a *"membrane shell"*, or those of a *"generalized membrane shell"*, or those of a *"flexural shell"*.

More specifically, consider a *family of linearly elastic shells of thickness* 2ε, all having the *same* middle surface $S = \boldsymbol{\theta}(\overline{\omega}) \subset \boldsymbol{R}^3$, where $\omega \subset \boldsymbol{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary γ, and $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \boldsymbol{R}^3)$. The shells are *clamped* on a portion of their lateral face, whose middle line is $\boldsymbol{\theta}(\gamma_0)$, where γ_0 is a *fixed* portion of γ with *length* $\gamma_0 > 0$. Let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$$

denote the covariant components of the *linearized change of metric tensor of* S, where $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols of S, and $b_{\alpha\beta}$ are the covariant components of the *curvature tensor* of S. In Ciarlet, Lods and Miara [1996], a geometrical assumption is made on the middle surface S and on the set γ_0, which asserts that *the space of inextensional displacements* (introduced by Sanchez-Palencia [1989a])

$$\begin{aligned}\boldsymbol{V}_F(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0,\ \gamma_{\alpha\beta}(\eta) = 0 \text{ in } \omega\}\end{aligned}$$

contains non-zero functions. This assumptions is satisfied in particular if S is a portion of a cylinder and $\boldsymbol{\theta}(\gamma_0)$ is contained in a generatrix of S, or if S is contained in a plane, i.e., the shells are *plates.*

It was then showed that, if the applied body force density is $O(1)$ with respect to ε, *the field* $\varepsilon^2\boldsymbol{u}(\varepsilon) = (\varepsilon^2 u_i(\varepsilon))$, where $u_i(\varepsilon)$ denote the three covariant components of the displacement of the points of the shell given by the equations of three-dimensional elasticity, once appropriately *"scaled"* so as to be defined over the fixed domain $\Omega = \omega \times]-1, 1[$, *converges in* $\boldsymbol{H}^1(\Omega)$ to a limit $\boldsymbol{u}^{-2}$, *which is independent of the transverse variable. Furthermore, the average* $\boldsymbol{\zeta} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}^{-2} dx_3$, *which belongs to the space* $\boldsymbol{V}_F(\omega)$, *solves the (scaled) two dimensional*

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equations of a "flexural shell", viz.,

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy = \int_\omega \left\{\int_{-1}^{1} f^i dx_3\right\}\eta_i\sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega)$, where

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{(\lambda+2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

are the contravariant components of the *elasticity tensor of the surface S*,

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) &= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 + b^\sigma_\beta(\partial_\alpha\eta_\sigma - \Gamma^\tau_{\alpha\sigma}\eta_\tau)\\ &\quad + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\sigma_\alpha|_\beta\eta_\sigma - c_{\alpha\beta}\eta_3\end{aligned}$$

are the covariant components of the *linearized change of curvature tensor of S*, b^β_α are the mixed components of the curvature tensor of S, $\sqrt{a}\,dy$ is the area element along S, and f^i are the scaled contravariant components of the applied body force. *If $\boldsymbol{V}_F(\omega) \neq \{\boldsymbol{0}\}$, the two-dimensional equations of a "flexural shell" are therefore justified.*

If $\boldsymbol{V}_F(\omega) = \{\boldsymbol{0}\}$, the above convergence result still applies. However, the only information it provides is that $\varepsilon^2\boldsymbol{u}(\varepsilon) \to 0$ in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon \to 0$. Hence a more refined asymptotic analysis of the scaled field $\boldsymbol{u}(\varepsilon)$ is needed in this case. A first instance of such a refinement was given by Ciarlet and Lods [1996a], where it was assumed that $\gamma_0 = \gamma$ *and* that the surface S is regular and *"elliptic"* in the sense that its Gaussian curvature is > 0 everywhere. As shown in Ciarlet and Lods [1996d] and Ciarlet and Sanchez-Palencia [1996], these two conditions, together with *ad hoc* regularity assumptions, indeed imply that $\boldsymbol{V}_F(\omega) = \{\boldsymbol{0}\}$.

It was then showed that, if the applied body force density is again $O(1)$ with respect to ε, *the field $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ converges in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ as $\varepsilon \to 0$ to a limit $\boldsymbol{u}$, which is independent of the transverse variable. Furthermore, the average $\boldsymbol{\zeta} = \frac{1}{2}\int_{-1}^{1}\boldsymbol{u}dx_3$, which belongs to the space*

$$\boldsymbol{V}_M(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega),$$

solves the (scaled) two-dimensional equations of a "membrane shell", viz.,

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy = \int_\omega \left\{\int_{-1}^{1} f^i dx_3\right\}\eta_i\sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega)$ where the functions $a^{\alpha\beta\sigma\tau}$, $\gamma_{\alpha\beta}(\boldsymbol{\eta})$, a, and f^i have the same meanings as above. If $\gamma_0 = \gamma$ *and S is elliptic, the two-dimensional equations of a "membrane shell" are therefore justified.*

Finally, Ciarlet and Lods [1996c] studied *all the "remaining" cases where $\boldsymbol{V}_F(\omega) = \{\boldsymbol{0}\}$*, e.g., when S is elliptic but *length* $\gamma_0 <$ *length* γ, or when S is a portion of a hyperboloid of revolution, etc. To give a flavor of their results, consider *the important special case where the space $\{\boldsymbol{\eta} \in \boldsymbol{H}^1(\omega);\ \boldsymbol{\eta} = \boldsymbol{0}$ on $\gamma_0,\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in $\omega\}$*, which contains $\boldsymbol{V}_F(\omega)$, *"already" reduces to $\{\boldsymbol{0}\}$, or, equivalently, when the semi-norm*

$$|\cdot|^M_\omega : \boldsymbol{\eta} = (\eta_i) \to |\boldsymbol{\eta}|^M_\omega = \left\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2(\omega)}\right\}^{1/2}$$

becomes a norm over the space

$$\boldsymbol{V}(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{H}^1(\omega);\ \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0\}.$$

In this case, if the applied forces are *"admissible"* in a specific sense, and if the applied body force density is again $O(1)$ with respect to ε, the average $\frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)dx_3$ *converges as* $\varepsilon \to 0$ *in the space*

$$\boldsymbol{V}_M^\sharp(\omega) = \text{ completion of } \boldsymbol{V}(\omega) \text{ with respect to } |\cdot|_\omega^M.$$

A convergence result also holds for the field $\boldsymbol{u}(\varepsilon)$ itself, but it is too technical to be reproduced here. *Furthermore, the limit* $\boldsymbol{\zeta} \in \boldsymbol{V}_M^\sharp(\omega)$ *solves a "limit" variational problem of the form*

$$B_M^\sharp(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L_M^\sharp(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \boldsymbol{V}_M^\sharp(\omega),$$

where $B_M^\sharp$ is the unique extension to $\boldsymbol{V}_M^\sharp(\omega)$ of the bilinear form $B_M : \boldsymbol{V}(\omega) \times \boldsymbol{V}(\omega) \to \boldsymbol{R}$ found above for a "membrane shell", and $L_M^\sharp : \boldsymbol{V}(\omega) \to \boldsymbol{R}$ is an *ad hoc* linear form, determined by the behavior as $\varepsilon \to 0$ of the admissible forces.

In the "last" case, where $\boldsymbol{V}_F(\omega) = \{\boldsymbol{0}\}$ but $|\cdot|_\omega^M$ is a "genuine" semi-norm over the space $\boldsymbol{V}(\omega)$, a similar convergence result can be established, but only in the completion $\dot{\boldsymbol{V}}_M^\sharp(\omega)$ with respect of $|\cdot|_\omega^M$ of the *quotient* space $\dot{\boldsymbol{V}}_M(\omega) = \boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$, where $\boldsymbol{V}_0(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{V}(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$.

All these "remaining" cases where $\boldsymbol{V}_F(\omega) = \{\boldsymbol{0}\}$ *correspond to "generalized mambrane shells", whose two-dimensional equations are therefore justified.*

All the above convergence results rely in a crucial way on two *Korn's inequalities* on surfaces. The first one, valid on a general surface, is due to Bernadou & Ciarlet [1976] (see also Bernadou, Ciarlet & Miara [1994]): It expresses that, for a "general" surface S, the $\boldsymbol{L}^2(\omega)$-norm of the *linearized change of metric tensor* $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$, *plus* the $\boldsymbol{L}^2(\omega)$-norm of the *linearized change of curvature tensor* $(\rho_{\alpha\beta}(\boldsymbol{\eta}))$, associated with displacement fields of S vanishing together with their normal component along the same portion, with *length* > 0, of the "boundary" of S, is equivalent to the $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$-norm of these fields, expressed here in curvilinear coordinates (both tangential components are in $H^1(\omega)$ and the normal component is in $H^2(\omega)$).

The second Korn inequality, due to Ciarlet & Lods [1996d] and Ciarlet & Sanchez-Palencia [1996], is only valid for special surfaces and special boundary conditions. It expresses that, if the surface S is "*elliptic*", the $\boldsymbol{L}^2(\omega)$-norm of the *linearized change of metric* $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$ *alone* is "already" equivalent to the $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$-norm of these fields. Note however that in this case the $H^2(\omega)$-norm of the normal component has to be replaced by its $L^2(\omega)$-norm.

Combining these convergences with results of Destuynder [1985] and Sanchez-Palencia [1989a, 1989b, 1992] (see also Sanchez-Hubert & Sanchez-Palencia [1997]), Ciarlet & Lods [1996b, 1996c] have also justified the well-known two-dimensional *Koiter equations of a linearly elastic shell* (Koiter [1970]), again in all possible cases.

The *formal asymptotic method* has been successfully applied by Miara [1997] and Lods & Miara [1997] to *nonlinearly elastic shells.* They showed through a series of delicate computations (especially in the flexural case) that the leading term of the asymptotic expansion of the scaled three-dimensional displacement, again in terms of the thickness as the "small" parameter, can be identified with the solution of *nonlinear two-dimensional membrane, or flexural, shell equations*, according to the geometry of the middle surface and the boundary conditions as in the linear case.

The *first convergence theorem* for *nonlinearly elastic shells* has been obtained by Le Dret & Raoult [1996]. They use in a clever Γ-*convergence theory* for justifying a *nonlinear "membrane" shell model* (which coincides with that obtained by Miara [1997] only for specific classes of deformations).

We refer to Ciarlet [1998] for a detailed analysis of the asymptotic analysis of elastic shells and for an extensive list of references (for lack of space, the appended list is far from being exhaustive!).

References

BERNADOU, M.; CIARLET, P. G. [1976]: Sur l'ellipticité du modèle linéaire de coques de W. T. Koiter, in *Computing Methods in Applied Sciences and Engineering* (R. Glowinski & J. L. Lions, Eds.), 89-136, Lecture Notes in Economics and Mathematical Systems **134**, Springer-Verlag, Heidelberg.

BERNADOU, M.; CIARLET, P. G.; MIARA, B. [1994]: Existence theorems for two-dimensional linear shell theories, *J. Elasticity* **34**, 645-667.

CAILLERIE, D.; SANCHEZ-PALENCIA, E. [1995]: Elastic thin shells: Asymptotic theory in the anisotropic and heteregeneous cases, *Math. Models Methods Appl. Sci.* **5**, 473-496.

CIARLET, P. G. [1998]: *Mathematical Elasticity, Volume III: Theory of Shells*, North-Holland, Amsterdam.

CIARLET, P. G.; LODS, V. [1996a]: Asymptotic analysis of linearly elastic shells. I. Justification of membrane shell equations, *Arch. Rational Mech. Anal.* **136**, 119-161.

CIARLET, P. G.; LODS, V. [1996b]: Asymptotic analysis of linearly elastic shells. III. Justification of Koiter's shell equations, *Arch. Rational Mech. Anal.* **136**, 191-200.

CIARLET, P. G.; LODS, V. [1996c]: Asymptotic analysis of linearly elastic shells: "Generalized membrane shells", *J. Elasticity* **43**, 147-188.

CIARLET, P. G.; LODS, V. [1996d]: An existence and uniqueness theorem for the two-dimensional linear membrane shell equations, *J. Math. Pures Appl.* **75**, 51-67.

CIARLET, P. G.; LODS, V.; MIARA, B. [1996]: Asymptotic analysis of linearly elastic shells. II. Justification of flexural shell equations, *Arch. Rational Mech. Anal.* **136**, 163-190.

CIARLET, P. G.; SANCHEZ-PALENCIA, E. [1996]: On the ellipticity of linear membrane shell equations, *J. Math. Pures Appl.* **75**, 107-124.

DESTUYNDER, P. [1980]: *Sur une Justification des Modèles de Plaques et de Coques par les Méthodes Asymptotiques*, Doctoral Dissertation, Université Pierre et Marie Curie, Paris.

DESTUYNDER, P. [1985]: A classification of thin shell theories, *Acta Applicandæ Mathematicæ* 4, 15-63.

GOLDENVEIZER, A. L. [1963]: Derivation of an approximate theory of shells by means of asymptotic integration of the equations of the theory of elasticity, *Prikl. Mat. Mech.* **27**, 593-608.

KOITER, W. T. [1970]: On the foundations of the linear theory of thin elastic shells, *Proc. Kon. Ned. Akad. Wetensch.* **B73**, 169-195.

LE DRET, H.; RAOULT, A. [1996]: The membrane shell model in nonlinear elasticity: A variational asymptotic derivation, *J. Nonlinear Sci.* **6**, 59-84.

LODS, V.; MIARA, B. [1997]: Nonlinearly elastic shell models: A formal asymptotic approach. II. The flexural model, *Arch. Rational Mech. Anal.*, to appear.

MIARA, B.; SANCHEZ-PALENCIA, E. [1996]: Asymptotic analysis of linearly elastic shells, *Asymptotic Anal.* **12**, 41-54.

MIARA, B. [1997]: Nonlinearly elastic shell models: A formal asymptotic approach. I. The membrane model, *Arch. Rational Mech. Anal.*, to appear.

SANCHEZ-HUBERT, J.; SANCHEZ-PALENCIA, E. [1997]: *Coques Elastiques Minces : Propriétés Asymptotiques*, Masson, Paris.

SANCHEZ-PALENCIA, E. [1989a]: Statique et dynamique des coques minces. I. Cas de flexion pure non inhibée, *C. R. Acad. Sci. Paris, Sér. I*, **309**, 411-417.

SANCHEZ-PALENCIA, E. [1989b]: Statique et dynamique des coques minces. II. Cas de flexion pure inhibée - Approximation membranaire, *C. R. Acad. Sci. Paris, Sér. I*, **309**, 531-537.

SANCHEZ-PALENCIA, E. [1990]: Passage à la limite de l'élasticité tri-dimensionnelle à la théorie asymptotique des coques minces, *C. R. Acad. Sci. Paris, Sér. II*, **311**, 909-916.

SANCHEZ-PALENCIA, E. [1992]: Asymptotic and spectral properties of a class of singular-stiff problems, *J. Math. Pures Appl.* **71**, 379-406.

INTRINSIC METHODS IN LINEAR THIN SHELLS *

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ABSTRACT

The marriage of work on the oriented (algebraic or signed) distance functions for shape and geometric optimization and the tangential calculus has resulted in the recent development of a simple intrinsic differential calculus on $C^{1,1}$ submanifolds of $\mathbf{R}^N$. In order to demonstrate and develop the potential of this calculus and the associated intrinsic methods, a first attempt was made in [5] (1992) to apply it to a very elementary model of thin shells. Independent interest and motivation in developing more tractable methods to deal with issues in control and design of thin shells further motivated the continuation of this work. The object of this paper is to summarize some of the results to date. We consider the intrinsic linear $P(1,1)$ and $P(2,1)$ models specified by a midsurface with Lipschizian boundary in a $C^{1,1}$ submanifold of $\mathbf{R}^N$. The two models are similar, but present fundamentally different characteristics. In order to validate and compare them with the generally accepted models in the literature, we have done an asymptotic analysis. Under a simple assumption on the continuity constant of the right-hand side of the equations as the thickness goes to zero, we show that their solutions strongly converge in their naturally associated space or quotient space to the solutions of asymptotic models consisting of two coupled variational equations. The asymptotic $P(2,1)$ model corresponds to models obtained by asymptotic analysis of the three-dimensional problem. The results are true for shells without boundary or shells with homogeneous Neumann (quotient space) or partial Dirichlet boundary conditions.

1. Intrinsic geometry and differential calculus on submanifolds

The oriented (algebraic or signed) distance function provides a *level sets description* of an arbitrary set in the Euclidean space and the zero level corresponds to the boundary of the set. In the smooth case the boundary is a smooth submanifold of codimension one and all the associated intrinsic geometrical objects can be obtained from the derivatives of the oriented distance function in a small neighborhood of the boundary. In particular the gradient of the oriented distance function coincides with the normal and the eigenvalues of its Hessian matrix are the principal curvatures of the boundary plus zero (cf. for instance [14], [4]).

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The terminology *oriented* underlines the fact that the oriented distance function specifies the orientation of the normal to the boundary. Furthermore the *projection of a point* onto the submanifold can also be explicitly expressed in term of the gradient of the distance function. This topic has been investigated by many researchers for different purposes and in different contexts (cf. for instance the wonderful work of H. Federer [13] in 1959 where he extended the Steiner-Minkowski formula to *sets with positive reach*). In another direction distance and oriented distance functions have been systematically investigated to define topologies on equivalence classes of sets in order to characterize compact families of equivalence classes and continuity of shape functionals (cf. [4]). In the nonsmooth case it is natural to consider sets for which the matrix of second order derivatives is a matrix of bounded measures.

Another essential ingredient of the theory of thin shells is the differential calculus in the underlying midsurface. This can be done by *parameterizing* the submanifold with local maps and introducing covariant/contravariant coordinates and derivatives and Christoffel symbols. Local maps can be avoided by going to the more abstract notion of *tangential derivatives* defined through extensions of functions (or vector functions) on the submanifold to a small neighborhood of the submanifold. For instance the *tangential gradient* is defined by orthogonal projection onto the tangent plane to the submanifold of the gradient of the extension (cf. for instance [7], [10], [6]). This projection is independent of the choice of the extension. The tangential gradient is an *intrinsic entity* which is independent of the choice of local bases in the submanifold and the ambient Euclidean space. The notion is fundamental but the associated calculus can become quite heavy and delicate. The very nice idea which naturally came up to make the tangential calculus *fully operational* was to choose among all extensions the *canonical* one obtained by composition with the projection onto the submanifold ([7], [10], [6]). All the computations are performed in the ambient Euclidean neighborhood of the submanifold and the final expression is obtained by restriction to the submanifold. Doing differential calculus on the submanifold becomes as easy as its counterpart in N-dimensional Euclidean spaces. The effect of the curvature of the underlying submanifold in the final expressions naturally comes through derivatives of the projection and hence of the gradient of the associated oriented distance function.

2. Polynomial approximations of thin shells

We have considered $P(k,\ell)$ models of linear thin shells of thickness $2h$ around a midsurface ω with Lipschizian boundary γ which is contained in a $C^{1,1}$ submanifold Γ of $\mathbf{R}^N$. The terminology $P(k,\ell)$ means a k-th order polynomial approximation of the displacement vector with respect to the variable z normal to Γ, and an ℓ-th order approximation of the associated linear strain tensor. Such models come from expressions obtained through infinite expansions of the displacement vector and the linear strain tensor ([7], [10], [6]). They are somewhat similar to the mixed approximations studied in [1] for plates. We concentrate on the $P(1,1)$ and $P(2,1)$ models which both yield similar variational equations ([11], [12]). The difference is in the nature of the function spaces involved. For the $P(1,1)$ model the two components (v_h^0, v_h^1) in the expansion of the displacement vector belong to the space $H^1(\omega)^3 \times H^1(\omega)^3$ and the associated bilinear form generates an equivalent norm on that space. For the $P(2,1)$ model the three components (v_h^0, v_h^1, v_h^2) in the expansion of the displacement vector can be chosen in the space $H^1(\omega)^3 \times H^1(\omega)^3 \times L^2(\omega)^3$. The bilinear form still generates a norm on that space, but the space is not complete for the associated norm and we are naturally led to work in the completion of that space. This phenomenon is also encountered in higher orders $P(k,k)$ and $P(k+1,k)$ models (including the $P(1,0)$ model).

3. Functional analysis on submanifolds

To complement this intrinsic differential calculus, it is essential to have an intrinsic theory of Sobolev spaces, integration by part formulae, Korn's and Poincaré's inequalities on a submanifold Γ which is the boundary of a $C^{1,1}$ domain in $\mathbf{R}^N$. Even if a considerable amount of material exists, it is usually not expressed in ready to use language and notation without an important investment. This is probably one of the main reasons why the theory of thin shells is still expressed through local mappings in the 2-dimensional plane where the standard functional analysis applies. Working directly in domains which live in the submanifold raises challenging and fundamental issues. For instance how to define the smoothness of the boundary γ of a domain ω in Γ without reference to the two-dimensional plane.

A special effort has been done in [10] and [7] to put together a minimal amount of basic material in a lecture notes style. Some results have been obtained through local maps, but others such as the Korn's inequalities have been proved by completely intrinsic methods. An important amount of work still remains to be done to firmly anchor the fundamental material which is required in the intrinsic framework.

4. Asymptotic $P(1,1)$ and $P(2,1)$ models

Several models have been considered in our first papers ([5], [6]) on linear thin shells, but the emphasis was more on showing the potential of intrinsic methods than one the pertinence of the models. Furthermore at that time there was no *bridge* to mathematically compare our models with the ones generally accepted in the literature. Explicit relationships between intrinsic tangential derivatives and covariant/contravariant derivatives came later in [7]. It became possible to address the issue of the validity of the models and this prompted us to move to the $P(1,1)$ and more recently to the $P(2,1)$ model which both only involve approximations with respect to the normal variable to the midsurface. There is no explicit assumption on the plane normal constraint.

One way to validate a model is to look at the resulting asymptotic model as the thickness goes to zero and compare it with models currently available from an asymptotic analysis of the three-dimensional model. We first looked at the asymptotic $P(1,1)$ and $P(2,1)$ models under a natural assumption on the constant of continuity of the right-hand side of the variational equation as the thickness goes to zero ([11], [12]). Under that assumption we get two asymptotic models which again have the same structure: a system of two coupled variational equations. For both models the first equation coincides with the asymptotic $P(1,0)$ model which yields the generally accepted *membrane shell equation* and the *Love-Kirchhoff condition*. The second equation is a generalization of the equation obtained in the bending dominated case, but with an additional coupling term which is zero in the case of the plate or when the extended membrane energy is zero. To our knowledge this second equation is new for shells and explicits the role of the *mean* and *Gauss curvatures* in the right-hand side of the equation. But there are slight differences in the second asymptotic equation between the two models. For the constitutive law $\sigma = 2\mu\,\varepsilon + \lambda\,\mathrm{tr}\,\varepsilon I$, the $P(2,1)$ model yields the generally accepted coefficients 2μ and $2\mu\lambda/(2\mu+\lambda)$ (cf. for instance [3] [2]), while the $P(1,1)$ model yields 2μ and λ. Strong convergence is proved in a norm generated by the associated bilinear forms. All this is true for shells without boundary or shells with a Lipschitzian boundary γ and homogeneous Neumann (quotient space with respect to the rigid displacements) or homogeneous Dirichlet boundary condition on a part γ_0 of or the whole boundary γ.

1. S.M. Alessandrini, D.N. Arnold, R.S. Falk, and A. Madureira, *Derivation and justification of plate models by variational methods*, in "Proc. Plates and shells: from theory to practice", CRM Proc. and Lect. Notes Series, AMS Publications, Providence, R.I., to appear.
2. Ph. G. Ciarlet and V. Lods, *Asymptotic analysis of linearly elastic shells.* IV. *Generalized membrane shells*, Université de Paris VI, Report 1996.
3. Ph. G. Ciarlet, V. Lods, and B. Miara, *Analyse asymptotique des coques linéairement élastique.* II. *Coques en flexion*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), 95–100; *Asymptotic analysis of linearly elastic shells.* II. *Justification of flexural shell equations*, Arch. Rational Mech. Anal., to appear.
4. M.C. Delfour and J.-P. Zolésio, *Shape analysis via distance functions*, J. Funct. Anal. **123** (1994), 129–201; *Shape analysis via distance functions: local theory*, CRM Report 2299, March 1996, Université de Montréal.
5. ______ *On a variational equation for thin shells*, in "Control and Optimal Design of Distributed Parameter Systems", J. Lagnese, D.L. Russell, L. White, eds, pp. 25–37 Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1994.
6. ______ *A boundary differential equation for thin shells*, J. Differential Equations **119** (1995), 426–449; *Tangential differential equations for dynamical thin/shallow shells*, J. Differential Equations 128 (1996), 125-167.
7. ______ *Differential equations for linear shells: comparison between intrinsic and classical models*, in " Advances in the Mathematical Sciences - CRM's 25 years, (Luc Vinet, ed.), CRM Proc. Lect. Notes, AMS, Providence, R.I., to appear.
8. ______ *On the design and control of systems governed by differential equations on submanifolds*, Control and Cybernetics 25 (1996), 497-514.
9. ______ *Hidden boundary smoothness for some classes of differential equations on submanifolds*, in "Proc. 1996 Joint Summer Research Conference on Optimization Methods in PDE's", I. Lasiecka and S. Cox, Contemp. Math., AMS Publications, Providence, R.I., to appear.
10. ______ *Intrinsic differential geometry and theory of thin shells*, Lecture Notes, Scuola Normale Superiore, Pisa (Italy), August 1996.
11. ______ *Convergence to the asymptotic model for linear thin shells*; *Hidden boundary smoothness for some classes of differential equations on submanifolds*, in "Proc. 1996 Joint Summer Research Conference on Optimization Methods in PDE's", I. Lasiecka and S. Cox, Contemp. Math., AMS Publications, Providence, R.I., to appear.
12. ______ *Convergence of the linear P(1,1) and P(2,1) thin shells to the asymptotic shells*, in "Proc. Plates and Shells: from theory to practice", M. Fortin, ed., CRM Proc. Lect. Notes ser., AMS Publications, Providence, R.I. 1997 (1996 CMS Annual Seminar) to appear.
13. H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–419.
14. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

EXPLICIT ERROR BOUNDS IN SHELL MODELLING

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ABSTRACT

From Prager-Synge inequality we derive an error bound between the three dimensional solution of a linear elastic model and the one of several shell models. Mainly Kirchhoff-Love and Koiter models. But the asymptotic behavior of this error bound with respect to the thickness of the shell is necessary in order to prove the consistency of this method. Therefore we use the results of the asymptotic analysis for shells. Finally the Prager-Synge equality is used is order to obtain a lower-bound of the error.

1.The three dimensional model

Let us consider a smooth surface ω embedded into $\mathbf{R}^3$. Then we associate with ω a three dimensional open set – say Ω^ε – by :

$$(1)\ \Omega^\varepsilon = \left\{x = \left(x_1, x_2, x_3\right) \in \mathbf{R}^3,\ x = \mathrm{m} + \xi\,\mathrm{N(m)},\ \mathrm{m} \in \omega,\ |\xi| \le \varepsilon\right\}$$

N(m) being the unit normal to ω at the point m (one orientation is chosen). Then we define the spaces :

$$(2)\ \begin{cases} \Sigma^\varepsilon = \left\{\tau = (\tau_{ij});\, i,j \in \{1,2,3\},\ \tau_{ij} = \tau_{ji} \in L^2\,(\Omega^\varepsilon)\right\} \\ V^\varepsilon = \left\{v = (v_i), v_i \in H^1\,(\Omega^\varepsilon); v_i = 0\ \ on\ \Gamma_o^\varepsilon = \partial\omega \times \left]-\varepsilon, \varepsilon\right[\right\}. \end{cases}$$

In order to ensure that Ω^ε is equivalent to $\omega \times \left]-\varepsilon,\varepsilon\right[$, it is necessary the formulate the following hypothesis :

$$\varepsilon \max\left(\frac{1}{|R_1|}, \frac{1}{|R_2|}\right) \le 1$$

where R_i are the two main radii of curvature of the surface ω. Let us now introduce two bilinear forms which are respectively defined on $\Sigma^\varepsilon \times \Sigma^\varepsilon$ and $\Sigma^\varepsilon \times V^\varepsilon$ by (E is the Young modulus and ν the Poisson coefficient):

$$(3)\begin{cases}\forall\,\tau^1,\tau^2\in\Sigma^\varepsilon\,,\; A^\varepsilon\left(\tau^1,\tau^2\right)=\int_{\Omega^\varepsilon}\dfrac{1+\nu}{E}\,Tr\left(\tau^1\,.\,\tau^2\right)-\dfrac{\nu}{E}\,Tr\left(\tau^1\right)Tr\left(\tau^2\right)\\ \forall\,(\tau,v)\in\Sigma^\varepsilon\times V^\varepsilon,\; B^\varepsilon(\tau,v)\;=\;-\int_{\Omega^\varepsilon}Tr\left(\tau\,.\,\dfrac{\partial v}{\partial x}\right)\end{cases}$$

where Tr (•) denotes the trace of an endomorphism and $\dfrac{\partial v}{\partial x}$ is the derivative of v (vector) with respect to x (coordinates). Finally the three dimensional linear elasticity model can be formulated as follows

$$find\;\left(\sigma^\varepsilon,u^\varepsilon\right)\in\Sigma^\varepsilon\times V^\varepsilon\quad such\;that$$

$$(4)\begin{cases}\forall\,\tau\in\Sigma^\varepsilon,\quad A^\varepsilon\left(\sigma^\varepsilon,\tau\right)+B^\varepsilon\left(\tau,u^\varepsilon\right)=0,\\ \forall\,v\in V^\varepsilon,\quad B^\varepsilon\left(\sigma^\varepsilon,v\right)=\ell(v).\end{cases}$$

The first equation is the constitutive relationship and the second one is the equilibrium one. The right hand side $\ell\,(.)$ represents the external loadings applied to the shell. Existence and uniqueness of a solution to (4) is very classical as soon as $\ell\,(.)$ is linear and continuous on V^ε(Duvaut-Lions [5]). The equations (4) are known as Hellinger-Reissner variational principle.

2. The shell models

The most convenient way to derive shell models from the three dimensional consists in introducing closed subspaces of $\Sigma^\varepsilon\times V^\varepsilon$. The first idea due to Kirchhoff and Love is to set (see Destuynder [1985]) :

$$(5)\begin{cases}K^{KL}=\left\{\tau\in\Sigma^\varepsilon,\tau\,.\,N=0\right\},\\ V^{KL}=\left\{v\in V^\varepsilon,\gamma(v)\,.\,N=0\right\},\end{cases}$$

where γ (v) is the linearized strain operator defined by :

$$(6)\;\gamma(v)=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{{}^t\partial v}{\partial x}\right)\quad\left({}^t(.)\;is\;the\;transposed\right).$$

It is obviously possible to characterize precisely the previous restrictions on the stress and the strain fields. From the mechanical point of view, they traduce that the transverse shear strains and stresses are zero. These assumptions are motivated by a classical understanding of the mechanical behavior of the shell. Then the Kirchhoff-Love (say K.L. to be short) model is the following :

$$\text{(7)}\quad \begin{cases} \textit{find } \left(\sigma^{KL}, u^{KL}\right) \in \Sigma^{KL} \times V^{KL} \quad \textit{such that} \\ \forall \tau \in \Sigma^{KL}, \quad A^{\varepsilon}\left(\sigma^{KL}, \tau\right) + B^{\varepsilon}\left(\tau, u^{KL}\right) = 0 \\ \forall v \in V^{KL}, \quad B\left(\sigma^{KL}, v\right) = \ell^{\varepsilon}(v). \end{cases}$$

Existence and uniqueness of a solution is not so obvious as far as we deal with a mixed approximation. Hence a compatibility relationship is required between the spaces Σ^{KL} and V^{KL}. But it can be easily checked because of the particular expression of Σ^{KL} and V^{KL}. Unfortunately the K.L. model is still complicated to be solved numerically. Therefore additional simplifications are wellcome. This leads to Koiter or Budiansky models as far as the simplifications are restricted to the bilinear forms A^{ε} and B^{ε}. But they are done assuming that ε is small enough compared the *smallest radius of curvature of the shell.* More precisely if we consider the operator:

$$I + \xi \frac{\partial N}{\partial m}, \qquad \left(\frac{\partial N}{\partial m} \text{ is the curvature operator } \right)$$

from the tangent plan into itself, we use the following approximation (when justified !) :

$$\left(I + \xi \frac{\partial N}{\partial m}\right)^{-1} = I - \xi \frac{\partial N}{\partial m} + \xi^2 \left(\frac{\partial N}{\partial m}\right)^2 .$$

Therefore the error induced between the K.L. and Koiter or Budiansky-Sanders model is not difficult to handle. But the asymptotic behavior with respect to ε is more tricky because it involves the asymptotic analysis of the simplified models with respect to ε (see Destuynder [1985] and Ciarlet-Lods [1994]). The method is similar to the one used for the error analysis of Reissner-Mindlin plate model in Destuynder [1997].

3. The Prager-Synge relations

Let us introduce the set of stress fields which equilibrate the external loadings applied to the shell by :

$$\text{(8)}\quad H_{\ell}\,(div, \Omega^{\varepsilon}) = \left\{\tau \in \Sigma^{\varepsilon},\ \forall\, v \in V^{\varepsilon},\ B^{\varepsilon}(\tau, v) = \ell^{\varepsilon}(v)\right\}.$$

It is worth noticing that H_{ℓ} (div, Ω^{ε}) is not empty because σ^{ε} solution to the three dimensional model belongs to this set. Then we define by R the stiffness tensor of the material (Hooke law in our case), one has the following basic result.

PROPOSITION 1 (from Prager-Synge [1947]). *Let* $(\sigma^{\varepsilon}, u^{\varepsilon})$ *be the three dimensional* *solution of the model* (4). *Then one has :*

$$\forall\, \tau \in H_\ell\,(\mathrm{div},\Omega^\varepsilon),\quad \forall\, v \in V^\varepsilon,$$
$$A^\varepsilon\left(\tau - R:\gamma(v),\ \tau - R:\gamma(v)\right) = A^\varepsilon\left(\tau-\sigma^\varepsilon, \tau-\sigma^\varepsilon\right) + A^\varepsilon\left(\sigma^\varepsilon - R:\gamma(v),\ \sigma^\varepsilon - R:\gamma(v)\right)$$

where A^ε *is the bilinear form defined at* (2). ❑

Then from the explicit expression of R, we deduce from Proposition 1 the following statement.

PROPOSITION 2 (from Prager-Synge [1947]). *With the same notations as in Propositon 1, one has :*

$$\forall\, \tau \in H_\ell\,(\mathrm{div},\Omega^\varepsilon),\quad \forall\, v \in V^\varepsilon,$$
$$\begin{cases} \dfrac{E}{1+\nu}\left\|\gamma(u^\varepsilon)-\gamma(v)\right\|^2_{\Sigma^\varepsilon} + \dfrac{1-2\nu}{E}\left\|\tau-\sigma^\varepsilon\right\|^2_{\Sigma^\varepsilon} \le \dfrac{1+\nu}{E}\left\|\tau - R:\gamma(v)\right\|^2_{\Sigma^\varepsilon} \\ \dfrac{E}{1-2\nu}\left\|\gamma(u^\varepsilon)-\gamma(v)\right\|^2_{\Sigma^\varepsilon} + \dfrac{1+\nu}{E}\left\|\tau-\sigma^\varepsilon\right\|^2_{\Sigma^\varepsilon} \ge \dfrac{1-2\nu}{E}\left\|\tau - R:\gamma(v)\right\|^2_{\Sigma^\varepsilon} \end{cases}$$ ❑

From the two inequalities of Proposition 2 it is possible to derive an upper and a lower bound for the error between u^ε and the solution v of a shell model. The basic point is to derive

from this last one a three dimensional stress field which belongs to the set H_ℓ (div, Ω^ε) and for which the term $\tau - R:\gamma(v)$ will be small enough.

4. The extension of a shell stress field to H_l (div, Ω^ε)

Let us focus our explanations on the K.L. model given at (7). One can prove (regularity property of the shell solution is assumed) directly from Babusk\o(a;\s\up7(v)) bilinear lemma that there exists a unique term – say T^{KL} – lying in the set H_ℓ (div, Ω^ε) and such that :

$$(9)\quad \Pi \cdot T^{KL} \cdot \Pi = \Pi \cdot \sigma^{KL} \cdot \Pi$$

where Π is the orthogonal projection from $\mathbb{R}^3$ onto the tangent plan to ω (at each point m of ω). As a matter of fact because $\sigma^{KL} \bullet N = 0$, it is possible to identify σ^{KL} with $\Pi\ \sigma^{KL}\ \Pi$. Then choosing $\tau = T^{KL}$ and $v = u^{KL}$ in the first inequality of proposition 2, we derive the upper bound :

$$\left\|\gamma(u^{\varepsilon}) - \gamma(u^{KL})\right\|_{\Sigma^{\varepsilon}} \leq \left(\frac{1+\nu}{E}\right) \left\|T^{KL} - R : \gamma(u^{KL})\right\|_{\Sigma^{\varepsilon}} .$$

The lower bound is deduced in a slightly more complicated way from the second inequality. Finally the asymptotic analysis (with respect to ε) of K.L model (and also of T^{KL} which depends of u^{KL}), enables one to derive the asymptotic behavior of the error bounds when ε tends to zero.

5. References

[1] P.G. Ciarlet - V. Lods [1994]. *Analyse asymptotique des coques linéairement élastiques (III)*. Une justification du modèle de W.T. Koiter. C.R. Acad. Sci. Paris. Série I, 319, p 299-304.

[2] P.G. Ciarlet [1991]. *Mathematical Modelling and numerical analysis of linearly elastic shells*. Birkhauser Verlag, Basel. Proceeding of the international congress of Mathematicians. Zürich.

[3] Ph. Destuynder [1985]. *A classification of thin shell theories*. Acta Applicandae. Mathematical n° 4, p 15-63.

[4] Ph. Destuynder [1997]. *Estimations d'erreur explicites pour les modèles de plaques de Kirchhoff-Love et Reissner-Mindlin* - Note aux CRAS à paraître.

[5] G. Duvaut - J.L. Lions [1972]. *Les inéquations en mécanique et en physique* - Dunod - Paris.

[6] W.T. Koiter [1970]. *On the foundation of the linear theory of thin elastic shells*. Konnikl Nederl. Academic Van Weten Shappen. Amsterdam. Proc. série B 73, n° 3, p 169-195.

[7] P. Ladevèze [1975]. *Comparaison de modèles de mécanique des milieux continus*. Thèse. Université Pierre et Marie Curie. Paris VI.

[8] W. Prager - J.L Synge [1947]. *Approximation in elasticity based on the concept of function space*. Quat. Appl. Math. Vol. V, n° 3, p 241-269.

ANISOTROPIC SHELLS

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ABSTRACT

We show that the bilinear forms associated to the linear thin shell models of Koiter and Naghdi, for a nonhomogeneous and anisotropic material are elliptic. We essentially use the fact that these bilinear forms are deduced from the three dimensional elasticity tensor, which is positive defined and also some techniques already used for homogeneous and isotropic shells ([2], [3], [4], [5]). The ellipticity assures the existence and uniqueness of solution for these models.

1. Notations

We denote by $\bar{\Omega}$ the middle surface of the shell, which is the image of an open connected bounded subset $\bar{\omega}$ in the plane $\mathbb{R}^2$, by a mapping $\vec{\Phi}$ smooth enough. The boundary of ω will be denoted by Γ and $\partial\Omega = \vec{\Phi}(\Gamma)$. We assume that, at any point $\vec{\Phi}(\xi^1, \xi^2)$ of $\bar{\Omega}$, the vectors $\vec{a}_\alpha = \frac{\partial \vec{\Phi}}{\partial \xi^\alpha}$, where $\alpha = 1, 2$, are lineary independent and so they span the tangent plane to the surface $\bar{\Omega}$ at the point $\vec{\Phi}(\xi^1, \xi^2)$. We also define the unit normal to $\bar{\Omega}$ by $\vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}$.

We introduce the function $e(\xi^1, \xi^2)$ for any $(\xi^1, \xi^2) \in \bar{\omega}$ which is the thickness function, smooth enough and such that $e(\xi^1, \xi^2) \geq e_0 > 0$. The shell S will be defined by

$$S = \Big\{ M \in \mathbb{R}^3 : \ \vec{OM} = \vec{\Phi}(\xi^1, \xi^2) + \xi^3 \vec{a}_3, \ \ (\xi^1, \xi^2) \in \bar{\omega}, \ \ -\frac{1}{2} e(\xi^1, \xi^2) \leq \xi^3 \leq \frac{1}{2} e(\xi^1, \xi^2) \Big\}.$$

In the following greek indexes, $\alpha, \beta, \gamma, \ldots$ will belong to the set $\{1, 2\}$ and the latin indexes $i, j, k, \ldots$ will belong to the set $\{1, 2, 3\}$. The usual summation convention will be adopted.

In the sequel we will suppose that the shell is : clamped on $\partial\Omega_0 = \vec{\Phi}(\Gamma_0) \times \left[-\frac{e}{2}, \frac{e}{2} \right]$, where $\Gamma_0 \subset \Gamma$ and measure$(\Gamma_0) > 0$; subjected to the action of volumic external applied forces, whose resultant is $\vec{p}$ on the middle surface $\bar{\Omega}$ and the resultant moment is null on $\bar{\Omega}$; subjected to the action of surface forces on $\partial\Omega_1 = \partial\Omega - \partial\Omega_0$, whose resultant is $\vec{N}$ over Γ_1 and the resultant moment is $\vec{M} \stackrel{\text{def}}{=} M^\alpha \vec{a}_\alpha \times \vec{a}_3$.

We suppose that the shell S is made of an elastic, nonhomogeneous and anisotropic material and that the constitutive equation is the generalized Hooke's law, that is, $\sigma^{ij} = C^{ijkl} \varepsilon_{kl}$ where where ε_{ij} are the covariant components of the deformation tensor of the shell and C^{ijkl} are the elastic coefficients that depend on (ξ^1, ξ^2, ξ^3), are regular enough and verify the following symmetric and elliptic conditions

(i) $C^{ijkl} = C^{jikl} = C^{ijlk} = C^{lkij}$,

(ii) $\exists c_0 > 0 : C^{ijkl}\tau_{ij}\tau_{kl} \geq c_0 \sum_{i,j=1}^{3} |\tau_{ij}|^2$, for all second order symmetric tensor (τ_{ij}).

In addition, we also assume that the shell has elastic symmetry with respect to the surface $\xi^3 =$ const. (see Naghdi [9], Green and Zerna [7], p.157), so we have $C^{\alpha\beta\lambda 3} = C^{\alpha 333} = 0$.

In both models, Koiter and Naghdi, it is assumed that the effect of the transverse normal stress is excluded, so $\sigma^{33} = 0$ and we obtain

$$\begin{aligned} \varepsilon_{33} &= -\frac{C^{33\alpha\beta}}{C^{3333}}\varepsilon_{\alpha\beta} \\ \sigma^{\alpha\beta} &= \Big(C^{\alpha\beta\gamma\mu} - \frac{C^{\alpha\beta 33}C^{33\gamma\mu}}{C^{3333}}\Big)\varepsilon_{\gamma\mu} \overset{\text{def}}{=} A^{\alpha\beta\gamma\mu}\varepsilon_{\gamma\mu}. \end{aligned} \tag{1}$$

Lemma 1 *The reduced elastic coefficients $A^{\alpha\beta\gamma\mu}$ verify the following symmetric and elliptic conditions:*

(i) $A^{\alpha\beta\gamma\mu} = A^{\beta\alpha\gamma\mu} = A^{\alpha\beta\mu\gamma} = A^{\mu\gamma\alpha\beta}$,

(ii) $\exists a_0 > 0 :\ A^{\alpha\beta\gamma\mu}\tau_{\alpha\beta}\tau_{\gamma\mu} \geq a_0 \sum_{\lambda,\mu=1}^{2} |\tau_{\gamma\mu}|^2$, *for all second order symmetric tensor* $(\tau_{\alpha\beta})$.■

2. Existence results for the Koiter and Naghdi models

Let us define the quantities

$$\begin{aligned} \mathcal{A}^{\alpha\beta\lambda\mu} &= \int_{-\frac{e}{2}}^{\frac{e}{2}} A^{\alpha\beta\lambda\mu}\, d\xi^3; & \mathcal{B}^{\alpha\beta\lambda\mu} &= \int_{-\frac{e}{2}}^{\frac{e}{2}} \xi^3 A^{\alpha\beta\lambda\mu}\, d\xi^3; \\ \mathcal{C}^{\alpha\beta\lambda\mu} &= \int_{-\frac{e}{2}}^{\frac{e}{2}} (\xi^3)^2 A^{\alpha\beta\lambda\mu}\, d\xi^3; & \mathcal{D}^{\alpha\lambda} &= \int_{-\frac{e}{2}}^{\frac{e}{2}} C^{\alpha 3\lambda 3}\, d\xi^3, \end{aligned} \tag{2}$$

which are the elastic coefficients of the middle surface, depending on ξ^1 and ξ^2.

We also introduce the following spaces of admissible displacements

$$\vec{V}^K = \Big\{\vec{v} = (v_1, v_2, v_3) \in \Big(H^1(\omega)\Big)^2 \times H^2(\omega) :\quad \vec{v}_{|\Gamma_0} = \vec{0},\ \frac{\partial v_3}{\partial n}_{|\Gamma_0} = 0\Big\},$$

$$V = \{v \in H^1(\omega) :\ v_{|\Gamma_0} = 0\} \quad \text{and} \quad \vec{V}^N = V^5.$$

The variational formulation of the Koiter's model is

$$\begin{cases} \text{Find } \vec{u}^K \in \vec{V}^K, \text{ such that} \\ \\ a^K(\vec{u}^K, \vec{v}) = f^K(\vec{v}), \quad \forall \vec{v} \in \vec{V}^K, \end{cases} \tag{3}$$

and the variational formulation of the Naghdi's model is

$$\begin{cases} \text{Find } (\vec{u}, \underline{\beta}) \in \vec{V}^N, \text{ such that} \\ \\ a^N\left[(\vec{u}, \underline{\beta}), (\vec{v}, \underline{\delta})\right] + b^N\left[(\vec{u}, \underline{\beta}), (\vec{v}, \underline{\delta})\right] = f^N(\vec{v}, \underline{\delta}), \quad \forall (\vec{v}, \underline{\delta}) \in \vec{V}^N. \end{cases} \tag{4}$$

The bilinear forms $a^K(.,.)$, $a^N(.,.)$ and $b^N\left[(.,.),(.,.)\right]$ take the form

$$\begin{cases} a^K(\vec{u},\vec{v}) = \int_\omega \Big[\mathcal{A}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \mathcal{C}^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}(\vec{u})\rho_{\lambda\mu}(\vec{v})\Big]\sqrt{a}\,d\xi^1 d\xi^2 \\ + \int_\omega \mathcal{B}^{\alpha\beta\lambda\mu}\Big(\gamma_{\alpha\beta}(\vec{u})\rho_{\lambda\mu}(\vec{v}) + \gamma_{\lambda\mu}(\vec{v})\rho_{\alpha\beta}(\vec{u})\Big)\sqrt{a}\,d\xi^1 d\xi^2, \qquad \forall \vec{u},\vec{v}\in\vec{V}^K, \end{cases} \tag{5}$$

$$\begin{cases} a^N\left[(\vec{u},\underline{\beta}),(\vec{v},\underline{\delta})\right] = \int_\omega \Big[\mathcal{A}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \mathcal{C}^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(\vec{u},\underline{\beta})\,\chi_{\lambda\mu}(\vec{v},\underline{\delta})\Big]\sqrt{a}\,d\xi^1 d\xi^2 \\ + \int_\omega \mathcal{B}^{\alpha\beta\lambda\mu}\Big(\gamma_{\alpha\beta}(\vec{u})\chi_{\lambda\mu}(\vec{v},\underline{\delta}) + \gamma_{\lambda\mu}(\vec{v})\chi_{\alpha\beta}(\vec{u},\underline{\beta})\Big)\sqrt{a}\,d\xi^1 d\xi^2, \quad \forall\,(\vec{u},\underline{\beta}),\ (\vec{v},\underline{\delta})\ \in \vec{V}^N, \end{cases} \tag{6}$$

and

$$b^N\left[(\vec{u},\underline{\beta}),(\vec{v},\underline{\delta})\right] = \frac{1}{4}\int_\omega \mathcal{D}^{\alpha\mu}\big(\varphi_\alpha(\vec{u})+\beta_\alpha\big)\big(\varphi_\mu(\vec{u})+\delta_\mu\big)\sqrt{a}\,d\xi^1 d\xi^2, \quad \forall\,(\vec{u},\underline{\beta}),\ (\vec{v},\underline{\delta})\ \in \vec{V}^N. \tag{7}$$

The potential energies of the forces $f^K(.)$ and $f^N(.)$ are defined by

$$f^K(\vec{v}) = \int_\omega \vec{p}\vec{v}\sqrt{a}\,d\xi^1 d\xi^2 + \int_{\Gamma_1}\Big[\vec{N}\cdot\vec{v} + M^\alpha(v_{3,\alpha}+b^\lambda_\alpha v_\lambda)\Big]\,d\Gamma, \qquad \forall \vec{v}\in\vec{V}^K. \tag{8}$$

and

$$f^N(\vec{v},\underline{\delta}) = \int_\omega \vec{p}\vec{v}\sqrt{a}\,d\xi^1 d\xi^2 + \int_{\Gamma_1}\Big(\vec{N}\cdot\vec{v} - M^\alpha\delta_\alpha\Big)d\Gamma, \quad \forall\,(\vec{v},\underline{\delta})\ \in\vec{V}^N. \tag{9}$$

The expressions $\gamma_{\alpha\beta}$, $\rho_{\alpha\beta}$, $\chi_{\alpha\beta}$ represent the covariant components of the deformation tensor and the change of curvature tensors of the middle surface (see e.g. Bernadou [2], for the definitions). The functions $\sqrt{a}$, b^λ_α, β_α are related to the geometry of the shell and the components φ_α define the rotations (see e.g. Bernadou [2], for the definitions).

The properties of the elastic coefficients defined by (2) and suitable matrix computation lead to the proof of the following result.

Theorem 1 *Let $\vec{\Phi}\in\Big(C^3(\bar{\omega})\Big)^3$. Then there exist constants $c_1>0$, $c_2>0$ such that*

$$a^K(\vec{v},\vec{v}) \ge c_1\Big\{\sum_{\alpha,\beta=1}^{2}\|\gamma_{\alpha\beta}(\vec{v})\|^2_{L^2(\omega)} + \sum_{\alpha,\beta=1}^{2}\|\rho_{\alpha\beta}(\vec{v})\|^2_{L^2(\omega)}\Big\}, \qquad \forall\vec{v}\in\vec{V}^K. \tag{10}$$

$$\begin{cases} a^N\left[(\vec{v},\underline{\delta}),(\vec{v},\underline{\delta})\right] + b^N\left[(\vec{v},\underline{\delta}),(\vec{v},\underline{\delta})\right] \ge \\ \ge c_2\Big\{\sum_{\alpha,\beta=1}^2\Big(\|\gamma_{\alpha\beta}(\vec{v})\|^2_{L^2(\omega)} + \|\chi_{\alpha\beta}(\vec{v},\underline{\delta})\|^2_{L^2(\omega)}\Big) + \sum_{\alpha=1}^2\|\gamma_{\alpha 3}(\vec{v},\underline{\delta})\|^2_{L^2(\omega)}\Big\}, \\ \forall(\vec{v},\underline{\delta})\in\vec{V}^N. \blacksquare \end{cases} \tag{11}$$

We remark that this theorem is enough to assure the coercivity of the bilinear forms $a^K(.,.)$ and $a^N(.,.)$ for nonhomogeneous and anisotropic shells. In fact, with the above therorem and in order to complete the proof of the V^K-ellipticity for $a^K(.,.)$ and the V^N-ellipticity for $a^N(.,.)$ we can argue exactly as in Bernadou [2], (part I, 5 and 6) for the case of an isotropic and homogeneous shell.

The existence and uniqueness of solutions for the Koiter and Naghdi models, cf. (3) and (4), are then justified by the Lax-Milgram theorem because the linear forms $f^K(.)$ and $f^N(.)$ are continuous in $\vec{V}^K$ and $\vec{V}^N$, respectively.

We also refer to [2], [3], [4] and [5] for the proofs of existence and uniqueness of solutions in the case of homogeneous and isotropic shells.

3. Acknowledgements

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4. References

1. J.A. Alvarez-Dios and J.M. Viaño, *An asymptotic general theory for linear elastic nonhomogeneous anisotropic rods*, in Numerical Methods in Engineering'92, 511-518, ed. C. Hirsh et al. (Elsiver, Amsterdam, 1992).
2. M. Bernadou, *Méthodes d'éléments finis pour les problèmes de coques minces*, (Masson, Paris, 1994).
3. M. Bernadou and P. G. Ciarlet, *Sur l'ellipticité du modèle linéaire de coques de W. T. Koiter*, in Computing Methods in Applied Sciences and Engineering ed. R. Glowinski and J.L. Lions, Lectures Notes in Economics and Mathematical Systems, **134**, 89-136 (Springer-Verlag, Berlin, 1976).
4. M. Bernadou, P. G. Ciarlet and B. Miara, *Existence theorems for two-dimensional linear shell theories*, in J. Elasticity (1995).
5. P. G. Ciarlet and B. Miara, *Une démonstration simple de l'ellipticité des modèles de coques de W.T. Koiter et de P.M. Naghdi*, C. R. Acad. Sci., Paris, Sé.I Math. **312** (1991) 411-415.
6. W. T. Koiter, *On the foundations of the linear theory of thin elastic shells*, in Proc. Kon. Nederl. Akad. Wetensch. **B73** (1970) 169-195.
7. A. E. Green and W. Zerna, *Theoretical elasticity* (Oxford University Press, 2nd Edition, 1968).
8. E. A. Lekhnitskii, *Theory of elasticity of an anisotropic body* (Mir Publishers, Moscow, 1977).
9. P. M. Naghdi, *Foundations of elastic shell theory*, in Progress in Solid Mechanics, 4 (1963) 1-90 (North-Holland, Amsterdam).
10. L. M. Trabucho and J. M. Viaño, *Mathematical modelling of rods*, in Handbook of Numerical Analysis, ed. P.G. Ciarlet and J.L. Lions, 4 (Elsevier Science Publishers B.V., North-Holland, 1996).

ON THE DYNAMICS OF A THIN STRESS-FREE RING

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and

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ABSTRACT

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1. Introduction

In a recently published paper, Ge, Kruse and Marsden [4] have investigated the limits of Hamiltonian structures, which can be obtained as one or two characteristic dimensions of a three–dimensional elastic body tend to zero. The analysis therein is based on the theory of Hamiltonian systems, whose underlying symplectic forms are defined over infinite dimensional manifolds, for which the application of the machinery commonly used in Hamiltonian mechanics is not straightforward, but is considerably restricted, even though the calculus holds, at least formally. Despite these problems, we are at least able to analyze an example explicitly. It is shown that the main ideas of that approach can be realized in all details and resolved rigorously.

Our point of departure is a geometrically nonlinear time-dependent model for the deformation of a ring, which is stress–free along its boundary and which is made of a 2D Saint–Venant material.

The boundary conditions define, in the basic space of functions, a manifold $\mathcal{N}_\epsilon$, where the parameter ϵ denotes the small thickness of the ring. As ϵ tends to 0 one obtains a limit-manifold $\mathcal{N}_0$ consisting of functions not depending on the radial variable r, but still satisfying the boundary conditions for $\epsilon = 0$. With the help of a hard implicit function theorem we show that $\mathcal{N}_\epsilon$ can be locally trivialized, i.e. the functions in $\mathcal{N}_\epsilon$ are power–series in r. Now, given a Taylor–jet of a certain order k, say, this jet defines a vector field on $\mathcal{N}_0$. Inserting this vector field into the Hamiltonian yields a k-th order approximation of a reduced problem, i.e. of a complete Hamiltonian structure living on manifolds of 2π-periodic functions of one variable. In this paper we discuss convergence with respect to ϵ and the influence of the approximation–order on the validity in time. Special solutions are given.

2. The Ring Problem

2.1. Formulation of the Problem

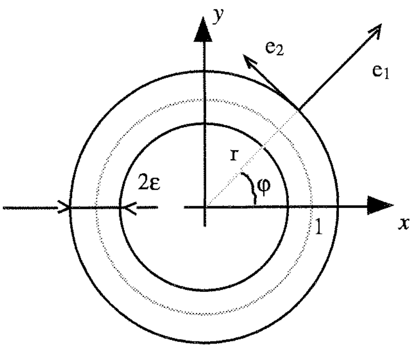

Fig. 1: Thin ring

Let a two dimensional Saint–Venant ring with characteristic radii given in the figure be initially displaced by $\hat{u}_0(x, y)$ from the natural stress–free reference configuration

$$B_\epsilon := \left\{(x, y) \in I\!R^2 | 1 - \epsilon \leq \sqrt{x^2 + y^2} \leq 1 + \epsilon\right\}.$$

The basic problem consists of finding a time dependent vector field $u(t) : B_\epsilon \to I\!R^2$ governing the deformation of the ring such that $\hat{u}(0)(x, y) = \hat{u}_0(x, y)$, and such that $u(t)$ satisfies $Sn = 0$ along ∂B_ϵ for all t where a solution exists. Here S denotes the stress tensor and n is the outer unit normal to the boundary of B_ϵ.

As a first step find a vector field: $\xi(t) : S^1 \to I\!R^2$, such that for small ϵ, the vector field $\xi(t)$, being independent of r is a good approximation to $u(t)$.

2.2. Notations

Introduce polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ and the corresponding covariant basis given by $e_1 = (\cos\varphi, \sin\varphi)^\top$ and $e_2 = (-r\sin\varphi, r\cos\varphi)^\top$ for all r and φ with $1 - \epsilon \leq r \leq 1 + \epsilon$, $0 \leq \varphi < 2\pi$. The metric tensor and the Christoffel symbols are easily calculated as

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \Gamma^1_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix}, \quad \Gamma^2_{\alpha\beta} = \begin{pmatrix} 0 & 1/r \\ 1/r & 0 \end{pmatrix}.$$

The potential–energy–density is evaluated using the nonlinear strain tensor

$$E = \frac{1}{2}(\nabla\hat{u}^\top\nabla\hat{u} + \nabla\hat{u}^\top + \nabla\hat{u}) \text{ with } \hat{u} = u^\alpha e_\alpha$$

and the equations

$$\hat{u}_{,x} = (u^\alpha_{,1}\cos\varphi - u^\alpha_{,2}\frac{\sin\varphi}{r})e_\alpha + u^\alpha(\Gamma^\beta_{\alpha 1}e_\beta\cos\varphi - \Gamma^\beta_{\alpha 2}\frac{\sin\varphi}{r}e_\beta)$$

$$\hat{u}_{,y} = (u^\alpha_{,1}\sin\varphi + u^\alpha_{,2}\frac{\cos\varphi}{r})e_\alpha + u^\alpha(\Gamma^\beta_{\alpha 1}e_\beta\sin\varphi + \Gamma^\beta_{\alpha 2}\frac{\cos\varphi}{r}e_\beta)$$

where $u^\alpha_{,1} = \partial_r u^\alpha$ and $u^\alpha_{,2} = \partial_\varphi u^\alpha$. Furthermore, λ and μ denote, as usual, the Lamé constants of the material and for brevity λ_n stands for $\lambda + n\mu$, where $n \in I\!N$. Note that from now on the unknowns of the problem are $U := u^1$ and $V := ru^2$.

2.3. Hamiltonian Structure

The configuration space of the body is defined as $M^\epsilon := \{u | u : B_\epsilon \to I\!R^2 \text{ is an embed. }\}$. Then the tangent bundle of M^ϵ is given by $TM^\epsilon := \{(u, \dot{u}) | u \in M^\epsilon \text{ and } \dot{u} : B_\epsilon \to I\!R^2\}$. The phase space T^*M^ϵ is identified with TM^ϵ using the metric given by the kinetic energy. Hamilton–equations, governing the motion of the ring will be obtained by using the canonical (weak) symplectic form on T^*M^ϵ, which is actually the imaginary part of an inner product, defined on a Hilbert space, involved in this problem. Now, the Hamiltonian, which is here the total energy of the ring, is given by

$$H(U,V) = \frac{1}{2} \int_0^{2\pi} \int_{1-\epsilon}^{1+\epsilon} (\dot{u}^\alpha g_{\alpha\beta} \dot{u}^\beta + W_2 + W_3 + W_4) r dr d\varphi, \tag{1}$$

where W_k denotes k-th order monomials of the potential–energy in U and V and their derivatives. A straightforward calculation leads to

$$\begin{aligned} W_2 &= \lambda_2 U_r^2 + \mu V_r^2 + \frac{2}{r}(\lambda U_r K_+ + \mu V_r K_-) + \frac{1}{r^2}(\mu K_-^2 + \lambda_2 K_+^2) && (2) \\ W_3 &= (\frac{1}{r}K_+ + U_r)\big(\lambda_2[\frac{1}{r^2}(K_+^2 + K_-^2) + U_r^2 + V_r^2] - \frac{2\mu}{r}(U_r K_+ - V_r K_-)\big) && (3) \\ W_4 &= \frac{\lambda_2}{4}\big(\frac{1}{r}(K_+^2 + K_-^2) + U_r^2 + V_r^2\big)^2 - \frac{\mu}{r^2}(U_r K_+ - V_r K_-)^2 && (4) \end{aligned}$$

where $K_+ := U + \partial_\varphi V$ and $K_- := \partial_\varphi U - V$.

2.4. Boundary conditions

The nonlinear boundary conditions are originally obtained as $D(\varphi)(a,b)^\top = 0$, where $D(\varphi) \in \mathrm{SO}(2)$. Then the boundary condition $D(\varphi)(a,b)^\top = 0$ is satisfied, if and only if $a = b = 0$. Thus, again after some calculations

$$\begin{aligned} a := \lambda_2 U_r + \frac{\lambda}{r}K_+ + \frac{\lambda}{2r^2}(K_+^2 + K_-^2) + \frac{\lambda_2}{2}(U_r^2 + V_r^2) &= 0 && (5) \\ b := V_r + \frac{1}{r}K_- + \frac{1}{r}(U_r K_- + V_r K_+) &= 0 && (6) \end{aligned}$$

is obtained for $r = 1 \pm \epsilon$.

3. Reduction procedure

Assume now for simplicity that U and V belong to a space $\mathcal{P}$ of real analytical functions with respect to r. Then the nonlinear manifold $\mathcal{N}_\epsilon \subset \mathcal{P}$ defined through (5) and (6) reduces at $\epsilon = 0$ to

$$\mathcal{N}_0 = \big\{(\xi_1, \xi_2, w_1, w_2) \in X_0 | w_1 = -\frac{\lambda}{\lambda_2}(\xi_1 + \xi_2'),\ w_2 = \frac{(\xi_2 - \xi_1')(1 - \lambda/\lambda_2(\xi_1 + \xi_2'))}{1 + \xi_1 + \xi_2'}\big\},$$

where the prime denotes derivatives with respect to φ and X is a space of smooth 2π-periodic functions. Note, that at $\epsilon = 0$, $\mathcal{N}_0$ is tangent to $\mathcal{N}_\epsilon$. Thus, a first order approximation to a local trivialization of $\mathcal{N}_\epsilon$ is given by

$$\mathcal{F}_1 = \big\{(\xi_1 + (r-1)w_1, \xi_2 + (r-1)w_2) | (\xi_1, \xi_2, w_1, w_2) \in \mathcal{N}_0\big\}.$$

Now, the complete Hamiltonian structure is easily pulled back on $\mathcal{F}_1$. So, the Hamilton–equations for the reduced problem read

$$\dot{\xi}_1 = p_1, \tag{7}$$

$$\dot{p}_1 = -\frac{4\mu\lambda_1}{\lambda_2}(\xi_1 + \xi_2') + \mathcal{O}(\| \xi, D\xi \|^2), \tag{8}$$

$$\dot{\xi}_2 = p_2, \tag{9}$$

$$\dot{p}_2 = \frac{4\mu\lambda_1}{\lambda_2}(\xi_1' + \xi_2'') + \mathcal{O}(\| \xi, D\xi \|^2). \tag{10}$$

4. Acknowledgement

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5. References

1. **R. Abraham and J.E. Marsden**, Foundations of Mechanics, *Addison–Wesley* (1985).
2. **S. S. Antman**, Nonlinear Problems in Elasticity, *Springer* (1995).
3. **P. G. Ciarlet**, Mathematical Theory of Elasticity, Vol. 1, *North Holland* (1988).
4. **Z. Ge, H.P. Kruse and J.E. Marsden**, The Limits of Hamiltonian Structures in Three–Dimensional Elasticity, Shells, and Rods,*J. Nonlinear Sci.*, **Vol. 6**, 19–57, (1996).
5. **R.S. Hamilton**, The inverse function theorem of Nash and Moser, *Preprint Cornell University* (1974).
6. **S. Lang**, Introduction to Differentiable manifolds, *John Wiley & Sons* (1962).
7. **J.E. Marsden and T.J.R Hughes**, Foundations of Elasticity, *Dover* (1983).
8. **J.E. Marsden and T.S. Ratiu**, Introduction to Mechanics and Symmetry, *Springer* (1994).
9. **A. Mielke**, Saint–Venant's problem and semi–inverse solutions in nonlinear elasticity, *Arch. Rat. Mech. Anal.*, **102**, 205–229, (1988).

MULTI-LEVEL MODELLING OF DAMAGE PROCESSES OF SHELL STRUCTURES

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ABSTRACT

The simulation of structural damage evolution processes like the crack-damage study on Fig. 1 requires a computational treatment in a 3-dimensional environment $\{\sigma^{ij}, \gamma_{ij}; i,j = 1, 2, 3\}$ in which damage phenomena used to be formulated. Thus the application of classical structural models of reduced internal dimensions, such as rods or shells, has to be restricted to the kinematic mapping. Since the material description remains in the E3, the computational strategy calls for a rather sophisticated concept as described in the following paper.

1. Multi-Layered Shell Continua

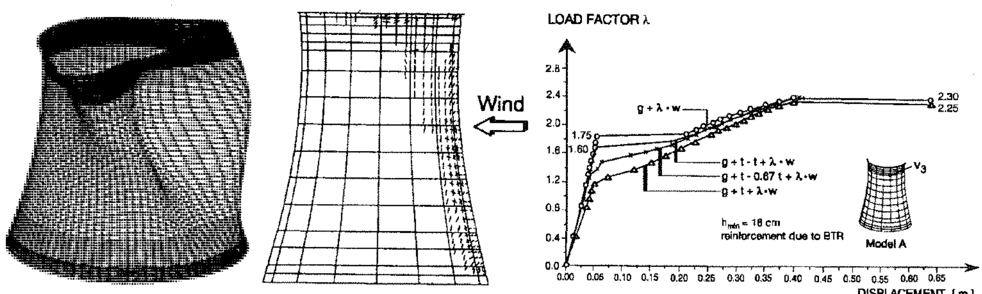
1.1 General Nonlinear Shell Theories

There exist different levels of nonlinear shell theories capable for the simulation of damage processes [1,2]. In general, the necessary materially nonlinear description is supplemented also by a geometrically nonlinear one.

Suitable shell theories possess the following physical structure. Loads $\mathbf{p}$ and stress resultants $\boldsymbol{\sigma}$ are connected by the dynamic equilibrium conditions

$$-\mathbf{p} = \mathbf{D}_e \cdot \boldsymbol{\sigma} = (\mathbf{D}_{el} + \mathbf{D}_{eN}(\mathbf{u})) \cdot \boldsymbol{\sigma} , \quad \{\mathbf{p}, \boldsymbol{\sigma}\} \in F , \tag{1.1}$$

in which $\mathbf{D}_{eL}$ describes a linear, $\mathbf{D}_{eN}$ a nonlinear partial differential operator and linear functional of $\mathbf{u}$. $\mathbf{p}$ may include D'ALEMBERT forces. Displacement variables $\mathbf{u}$ and strain measures $\boldsymbol{\varepsilon}$ are linked by the kinematic relations



Deformed shape at g+2.30w — Outer face crack-pattern — Load-displacment plot

Fig. 1: Crack-damage of a cooling tower shell

$$\boldsymbol{\varepsilon} = \mathbf{D}_k \cdot \mathbf{u} = (\mathbf{D}_{kL} + \mathbf{D}_{kN}(\mathbf{u})) \cdot \mathbf{u} \ , \quad \{\mathbf{u}, \boldsymbol{\varepsilon}\} \in \mathrm{F} \tag{1.2}$$

with the kinematic operator $\mathbf{D}_k$ consisting of the linear part $\mathbf{D}_{kL}$ and the nonlinear $\mathbf{D}_{kN}$, the latter also a linear functional of **u**. Prescribed displacements **r** are related to **u** by the kinematic boundary conditions and prescribed external force variables **t** by the dynamic boundary conditions [3]:

$$\mathbf{t} = \overset{o}{\mathbf{t}} = \mathbf{R}_t \cdot \boldsymbol{\sigma}, \quad \mathbf{r} = \overset{o}{\mathbf{r}} = \mathbf{R}_r \cdot \mathbf{u} \ , \quad \{\mathbf{t}, \mathbf{r}\} \in \mathrm{C}, \ \overset{o}{\mathbf{t}} \in \mathrm{C}_t, \ \overset{o}{\mathbf{r}} \in \mathrm{C}_r \ . \tag{1.3}$$

For general nonlinear behaviour the standard constitutive law is based on rate formulations in the E3, which have to be transformed into shell space by numerical integrations yielding

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} \cdot \dot{\boldsymbol{\varepsilon}} \rightarrow \boldsymbol{\sigma} = \int \dot{\boldsymbol{\sigma}} \, dt = \int \mathbf{C} \cdot \dot{\boldsymbol{\varepsilon}} \, dt \ ,$$
$$\boldsymbol{\varepsilon} = \int \dot{\boldsymbol{\varepsilon}} \, dt = \int \mathbf{C}^{-1} \cdot \dot{\boldsymbol{\sigma}} \, dt \ . \tag{1.4}$$

In time-independent processes this is substituted by incremental relations.

1.2 Transverse Shell Modelling

The REISSNER-MINDLIN shell theory as the most conventional analysis concept consists of a transversely uniform kinematic model, in which the strains in each sub-layer over the thickness can be derived from the 1st ($\alpha_{\alpha\beta}$) and 2nd ($\beta_{\alpha\beta}$) strain tensors and from the shear distorsion ($\gamma_{\alpha 3}$) of the reference surface:

$$\gamma_{\alpha\beta} = \alpha_{\alpha\beta} + \Theta^3 \beta_{\alpha\beta} \ , \quad \gamma_{\alpha 3} = \gamma_\alpha \ . \tag{1.5}$$

In this case of a singlelayer, singledirector representation, the deformations of all sub-layers of the shell are solely controlled by variables of the reference surface.

However, if large differences in thickness or material properties of sub-layers occur or if 3-dimensional damage effects have to be assessed, so-called multidirector theories are in use [4]. Their most simple representation is a multilayer model with single directors each [5,6,7], with or without director stretch, but also multilayer models with several directors for each sub-layer have been derived [8,9] leading to higher order strain approximations than customary in shell theory.

1.3 Total and Incremental Principle of Virtual Work

The principle of virtual work

$$-\int_{\mathrm{F}} \overset{o}{\rho} \, \ddot{\mathbf{u}}^T \delta \mathbf{u} \, d\overset{o}{\mathrm{F}} + \int_{\mathrm{F}} \mathbf{p}^T \delta \mathbf{u} \, d\overset{o}{\mathrm{F}} + \int_{\mathrm{C}_t} \overset{o}{\mathbf{t}}^T \delta \mathbf{r} \, d\overset{o}{\mathrm{C}}_t - \int_{\mathrm{F}} \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon} \, d\overset{o}{\mathrm{F}} = 0 \tag{1.6}$$

with the mass density $\overset{o}{\rho}$ of the undeformed reference surface $\overset{o}{\mathrm{F}}$ with boundary $\overset{o}{\mathrm{C}}$ governs the damage-inducing load response of the structure. Aiming at a later incremental-iterative solution process, all fields of response variables, such as

$$\mathbf{u} = \overset{o}{\mathbf{u}} + \bar{\mathbf{u}} + \overset{+}{\mathbf{u}} \tag{1.7}$$

due to Fig. 2 with $\overset{o}{\dots}$ as imperfections or prestresses, $\bar{\dots}$ as fundamental state variables and $\overset{+}{\dots}$ as incremental changes from the fundamental to the adjacent state, have to be incremented and substituted

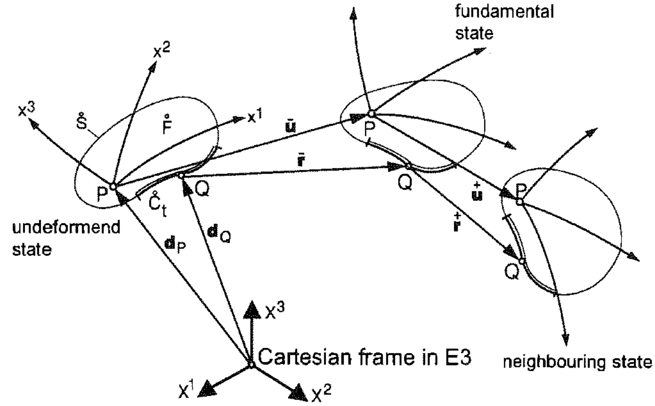


Fig. 2: Different states of deformation

into (1.6). After some tedious transformations we receive the following incremental principle of virtual work:

$$\int_F \overset{o}{\rho}\, \overset{+}{\ddot{\mathbf{u}}}{}^T \delta\mathbf{u}\, d\overset{o}{F} + \int_F \left[(\overset{o}{\boldsymbol{\sigma}} + \bar{\boldsymbol{\sigma}})^T \frac{1}{2}\, \delta\overset{++}{\boldsymbol{\varepsilon}} + \overset{+}{\boldsymbol{\sigma}}{}^T \delta\overset{+}{\boldsymbol{\varepsilon}} \right] d\overset{o}{F}$$

$$= \delta \left[\int_F \mathbf{p}^T \delta\bar{\mathbf{u}}\, d\overset{o}{F} + \int_{C_t} \bar{\mathbf{t}}^T \delta\bar{\mathbf{r}}\, d\overset{o}{C}_t - \int_F \overset{o}{\rho}\, \ddot{\bar{\mathbf{u}}}{}^T \delta\bar{\mathbf{u}}\, d\overset{o}{F} - \int_F (\overset{o}{\boldsymbol{\sigma}} + \bar{\boldsymbol{\sigma}})^T \delta\bar{\boldsymbol{\varepsilon}}\, d\overset{o}{F} \right]$$

$$+ \int_F \overset{+}{\mathbf{p}}{}^T \delta\overset{+}{\mathbf{u}}\, d\overset{o}{F} + \int_{C_t} \overset{+}{\mathbf{t}}{}^T \delta\overset{+}{\mathbf{r}}\, d\overset{o}{C} = 0 \qquad (1.8)$$

with $\quad \delta\overset{++}{\varepsilon} = \mathbf{D}_{kN}(\overset{+}{\mathbf{u}}) \cdot \overset{+}{\mathbf{u}}$

as basis of all future simulations.

2. The Discretized System

2.1 Displacement Discretization

If the shell is modelled as a singlelayer, singledirector continuum, the displacement discretization process for the p-th element is standard:

$$\mathbf{u}^p = \boldsymbol{\Omega}^p \cdot \mathbf{v}^p \,, \quad \boldsymbol{\varepsilon}^p = \mathbf{D}_k \cdot \mathbf{u}^p = \mathbf{D}^p_{kL} \cdot \mathbf{v}^p + \mathbf{D}^p_{kN}(\mathbf{v}^p) \cdot \mathbf{v}^p \,. \qquad (2.1)$$

The resulting 5 parametric theory delivers at least 5 degrees of freedom at each element node of the reference surface, from which the kinematic variables of all sub-layers can be evaluated.

For a multilayer model with single directors each due to Fig. 3, each layer adds 2 rotational degrees of freedom to those of the reference surface for suppressed transverse stretching, since the inplane displacements are coupled at the layer bounds. If transverse stretching is taken into account, each layer adds another transverse displacement degree of freedom.

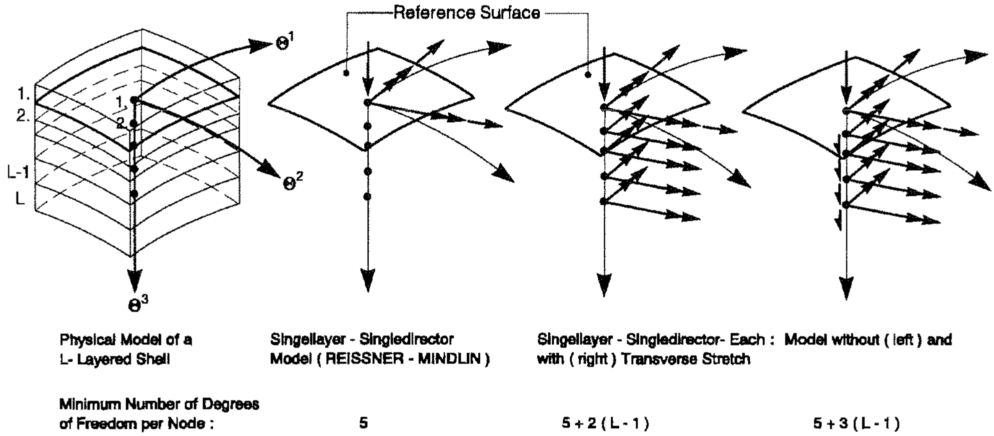


Fig. 3: Possible discretized shell representations

2.2 Tangential Equation of Motion

Accomplishing the discretization process for all elements and assembling all element contributions [3] together, the global tangential equation of motion is received:

$$\mathbf{M}\cdot\overset{\ddot{+}}{\mathbf{V}} + \mathbf{C}_T\cdot\overset{\dot{+}}{\mathbf{V}} + \mathbf{K}_T\cdot\overset{+}{\mathbf{V}} = \mathbf{P} - \mathbf{F}_i \quad . \tag{2.2}$$

With the mass matrix $\mathbf{M}$, the tangential damping $\mathbf{C}_T$ and stiffness $\mathbf{K}_T$ matrices, with the total applied load $\mathbf{P}$ and the internal forces $\mathbf{F}_i$, and with the increments $\overset{+}{\mathbf{V}}$, $\overset{\dot{+}}{\mathbf{V}}$, $\overset{\ddot{+}}{\mathbf{V}}$ of the nodal degrees of freedom, their velocities and their accelerations, (2.2) describes the gobal response properties of the considered loading process in a tangential subspace to the fundamental state.

3. Damage-Included Material Modelling

3.1 Standard Material Description

Damage processes require a treatment in a materially nonlinear range. In contrast to state-dependent elastic behaviour, inelastic materials deliver path-dependent responses. Applying descriptions like

$$\boldsymbol{\varepsilon} = \mathbf{f}(\boldsymbol{\sigma}) \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{g}(\boldsymbol{\varepsilon}) \ , \tag{3.1}$$

complicated implicite functional operators enter the computational strategy.

Advantageously, most engineering materials can be properly described by 1st order differential equations. If we introduce the following rate-abbreviations

$$\dot{\boldsymbol{\varepsilon}}^* = \begin{bmatrix} \dot{\gamma}_{ij} \\ \dot{h}_i \end{bmatrix} , \quad \dot{\boldsymbol{\sigma}}^* = \begin{bmatrix} \dot{\sigma}^{ij} \\ \dot{h}^i \end{bmatrix} , \quad \dot{\boldsymbol{\varepsilon}} = [\,\dot{\gamma}_{ij}\,] \ , \quad \dot{\boldsymbol{\sigma}} = [\,\dot{\sigma}^{ij}\,] \ , \tag{3.2}$$

in which h_i denotes a set of suitable internal variables and the dot a material time derivative, engineering materials fit into rate-form laws, like [10]

$$\dot{\boldsymbol{\varepsilon}}^* = \mathbf{I}\cdot\dot{\boldsymbol{\sigma}} + \mathbf{b} \ , \quad \dot{\boldsymbol{\sigma}}^* = \mathbf{C}\cdot\dot{\boldsymbol{\varepsilon}} + \mathbf{d} \ . \tag{3.3}$$

(3.3) form initial value problems in each GAUSS point of each sub-layer, for time-invariant responses they are substituted by increments.

3.2 Modelling of Structural Steel Components

For metals, many well-recommended classical elasto-plastic material descriptions can be transformed into (3.3). To capture damage phenomena, rather realistic models should be employed [11], in which additionally microdamage has to be considered [12,13]. Future simulations require tracing techniques for the growth of micro-voids [14] to final macro-scale defects [15]. Especially for cyclic loading, hysteretic behaviour has to be described properly as well as the BAUSCHINGER effect.

3.3 Modelling of Reinforced Concrete Components

Nonlinear behaviour of concrete is usually modelled by more empirical elasto-plastic theories [16], which have to be fitted into the frame of (3.3). Introducing tension-cracking criteria, inelastic bond properties between concrete and steel [17] and elasto-plastic steel models, main structural damage effects of reinforced concrete shells can be assessed in a homogenized scale [18].

A rather sophisticated but efficient concrete model has been proposed in [19]. Its micro- cracking component describing the de-strengthening phase after exceedance of compression strength gives rise to damage formulations also in the concrete compression range.

4. Simulation Strategy

After all, the final simulation strategy for time-invariant responses is summarized on Fig. 4. Incremental-iterative strategies are applied to the global tangential stiffness equation, the time-independent part of (3.2). With a new increment $\overset{+}{V}_n$ of the global degrees of freedom, one enters each finite element on all GAUSS points and proceeds with the kinematic model (1.5) to each material sub-point of the L sub-layers. Here, the incremental constitutive law (3.3) is evaluated and its result integrated numerically over all sub-layers arriving back at the original GAUSS point of the (layered) shell level, e.g.:

$$\overset{+}{n}{}^{\alpha\beta} = \overset{(1)}{E}{}^{\alpha\beta\lambda\mu}\,\overset{+}{\alpha}_{\lambda\mu} + \overset{(2)}{E}{}^{\alpha\beta\lambda\mu}\,\overset{+}{\beta}_{\lambda\mu}\,,$$
$$\overset{+}{m}{}^{\alpha\beta} = \overset{(1)}{E}{}^{\alpha\beta\lambda\mu}\,\overset{+}{\alpha}_{\lambda\mu} + \overset{(3)}{E}{}^{\alpha\beta\lambda\mu}\,\overset{+}{\beta}_{\lambda\mu} \tag{4.1}$$

with

$$\overset{(l)}{E}{}^{\alpha\beta\lambda\mu} = \int_h C^{\alpha\beta\lambda\mu}\,(\Theta^3)^{l-1}\,d\Theta^3 \tag{4.2}$$

from the original material law (4.3):

$$\overset{+}{\sigma}{}^{ij} = C^{ijkl}\,\overset{+}{\gamma}_{kl}\,,\quad d^{ij} \equiv 0\,. \tag{4.3}$$

This step requires the integration of the imbedded initial value problem (3.3) in best possible quality, solving for the stress-increments, then for the total strains, stresses and damage parameters.

From here we proceed further to the element level, and then to the global level for the evaluation of the next increment $\overset{+}{V}_{n+1}$.

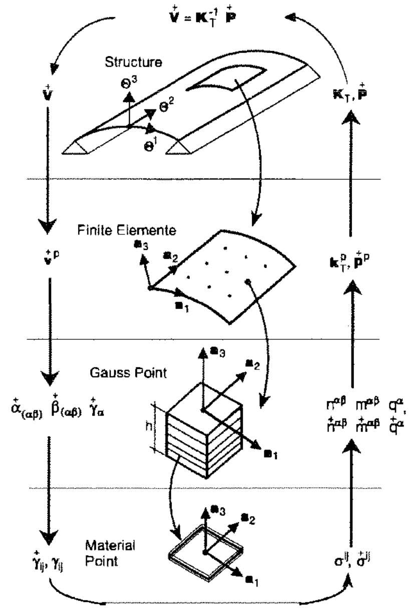


Fig. 4: Simulation strategy

5. References

1. Y. Basar, W.B. Krätzig, *Mechanik der Flächentragwerke*, Friedr. Vieweg & Sohn, Braunschweig 1985.
2. Y. Basar, W.B. Krätzig, A consistent shell theory for finite deformations, *Acta Mechanica*, 76 (1988), pp. 73-87.
3. W.B. Krätzig, Fundamentals of Numerical Algorithms for Static and Dynamic Instability Phenomena, In: W.B. Krätzig, E. Onate (Eds.): *Computational Mechanics of Nonlinear Response of Shells*. Springer-Verlag, Berlin 1990.
4. J.N. Reddy, A simple higher-order theory for laminated composite plates, *J. Appl. Mech.* 51 (1984), pp. 745-752.
5. M. Braun, M. Bischoff, E. Ramm, Nonlinear shell formulations for complete three-dimensional constitutive laws including composites and laminates, *Computational Mechanics* 15, (1994), pp. 1-18.
6. K. Dorninger, A nonlinear layered shell finite element with improved transverse shear behaviour, *Composites Engineering* 1 (1991), pp. 211-224.
7. W. Wagner, F. Gruttmann, A simple finite rotation formulation for composite shell elements, *Eng. Computations* 11 (1994), pp. 145-176.
8. Y. Basar, Y. Ding, R. Schultz, Refined Shear-Deformation Models for Composite Laminates with Finite Rotations, *Int. J. Solids Structures*, 30 (1993), pp. 2611-2638.
9. W.B. Krätzig, Best transverse shearing and stretching shell theory for nonlinear finite element simulations, *Comp. Meth. Appl. Mech. Engg.* 103 (1993), pp. 135-160.
10. H. Bergander, Zur Formulierung von Materialgleichungen in FEM-Programmen, In: *Schiffsfestigkeit*, pp. 203-221, Universität Rostock 1982.

THE KOITER MODEL FOR SHELLS WITH LITTLE REGULARITY

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ABSTRACT

We give a simple proof of existence and uniqueness of the solution of the Koiter model for linearly elastic thin shells whose midsurfaces can have charts with discontinuous second derivatives. The proof is based on new expressions for the linearized strain and change of curvature tensors. It also makes use of a new version of the rigid displacement lemma under hypotheses of regularity for the displacement and the midsurface of the shell that are weaker than those required by earlier proofs.

1. Introduction

We consider here the Koiter shell model in linearized elasticity introduced by Koiter in [7]. Existence and uniqueness for Koiter's model were first established by Bernadou and Ciarlet [1] by means of a particularly technical proof. Ciarlet and Miara [4] were later able to give a simpler existence and uniqueness proof.

The purpose of this work is to provide an even simpler proof of existence and uniqueness for Koiter's model. Moreover, our result is valid for shells whose midsurface can have discontinuous curvature. We thus improve—and significantly simplify—the earlier proofs of Bernadou and Ciarlet, see also Bernadou, Ciarlet and Miara [2], which all assumed midsurfaces of class at least C^3.

Our basic idea is to reconsider the commonly accepted framework of working with the covariant components of the displacement. In effect, such covariant components are scalar products of the displacement itself by vectors which depend on the chosen midsurface chart. In this way, the regularity of the displacement gets somehow *mixed up* with that of the chart. Instead, we simply consider displacements as $\mathbb{R}^3$-valued functions, $\mathbb{R}^3$ being the ambient physical space in which the shell deforms itself, which is indeed a more intrinsic approach.

Using this simple idea, we obtain expressions for the linearized strain and change of curvature tensors of a displacement of the midsurface that are new, or at least previously unnoticed in this context. The crucial point for our purposes here is that these expressions *do not* involve any derivatives of the second fundamental form of the midsurface, as opposed to the classical expressions in terms of covariant components. Such derivatives thus *actually* do not enter in the change of curvature tensor and this is what allows us to weaken the customary regularity requirements for the midsurface. Indeed, the new expressions are valid for midsurfaces of class

$W^{2,\infty}$ and are defined in general as distributions. A further consequence of this approach is that our expressions for the strain and change of curvature tensors are considerably simpler than the classical ones.

¿From there on, the argument is fairly standard. The new expressions are used to prove the rigid displacement lemma for a shell under hypotheses of regularity for the displacement and midsurface that are significantly weaker than those required by earlier proofs.

We then establish the ellipticity of the bilinear form associated with the Koiter model over an appropriate Hilbert space in the case of simple support on the boundary and clamping on one part of the boundary and applied forces and moments on the remaining part. The existence and uniqueness result then follows from the Lax-Milgram lemma. Again, this is made possible by our new expressions for the linearized strain and change of curvature tensors. Let us note that the result is especially interesting insofar as it allows quite common situations, such as an egg-shaped C^1-shell made of spherical part and an ellipsoidal part, which are excluded by the usual hypothesis that the midsurface be of class C^3.

Let us sum up by emphasizing again that the main novelty of this work, in addition to a significant simplification of the proofs, is that it allows shells whose midsurfaces may have curvature discontinuities. Note that Destuynder and Salaün [5] obtained a mixed formulation of the Koiter model that is also valid for a $W^{2,\infty}$ shell in that third order derivatives of the chart do not appear in the final model. However, the existence of third order derivatives seem to be required to derive their formulation and existence and uniqueness are obtained as a consequence of Bernadou and Ciarlet's result. The idea of forgoing covariant components was also used by Le Tallec and coworkers, see *e.g.* [8], in the linear and nonlinear cases. The full text of this abstract is given in Blouza and Le Dret [3]. Numerical experiments on our formulation of the model will appear in Kerdid and Mato-Eiroa [6].

2. Shell strain tensors and the rigid displacement lemma revisited

In the sequel, Greek indices and exponents always belong to the set $\{1,2\}$, while Latin indices and exponents belong to the set $\{1,2,3\}$. We use the Einstein summation convention.

Let ω denote a Lipschitz domain of $\mathbb{R}^2$. We consider a shell of midsurface $S=\varphi(\overline{\omega})$, where $\varphi \in W^{2,\infty}(\omega;\mathbb{R}^3)$ is an injective mapping such that the two vectors $a_\alpha(x)=\partial_\alpha\varphi(x)$ are linearly independent at each point $x\in\overline{\omega}$. We let $a_3(x)=\frac{a_1(x)\wedge a_2(x)}{|a_1(x)\wedge a_2(x)|}$ be the unit normal vector on the midsurface at point $\varphi(x)$. The vectors $a_i(x)$ define the covariant basis at point $\varphi(x)$. The regularity of the midsurface chart and the hypothesis of linear independence on $\bar{\omega}$ imply that the vectors a_i belong to $W^{1,\infty}(\omega;\mathbb{R}^3)$. The contravariant basis $a^i(x)$, defined by the relations $a^i(x)\cdot a_j(x)=\delta^i_j$, also belongs to $W^{1,\infty}(\omega;\mathbb{R}^3)$. We let $a(x)=|a_1(x)\wedge a_2(x)|^2$, so that $\sqrt{a}$ is the area element of the midsurface expressed in the chart φ.

We denote by $a_{\alpha\beta}$ and $b_{\alpha\beta}$ the covariant components of first and second fundamental forms of the surface and by $\Gamma^\rho_{\alpha\beta}$ its Christoffel symbols. Since $W^{1,\infty}(\omega;\mathbb{R}^3)$ is a Banach algebra, it follows that $a_{\alpha\beta}\in W^{1,\infty}(\omega)$ and $b_{\alpha\beta}\in L^\infty(\omega)$ and the same holds for their contravariant and mixed components. Similarly, $\Gamma^\rho_{\alpha\beta}$ belongs to $L^\infty(\omega)$.

We begin by recalling the classical definitions for the shell strain tensors. Let u be a displacement of the midsurface, *i.e.*, a regular mapping from $\overline{\omega}$ into $\mathbb{R}^3$ given in the contravariant basis by $u(x)=u_i(x)a^i(x)$, whence $u_i=u\cdot a_i$. In the classical approach, the displacement is identified with the triple (u_i) of its covariant components. The linearized strain tensor $\gamma(u)$ is given in covariant components by $\gamma_{\alpha\beta}(u)=\frac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-b_{\alpha\beta}u_3$ and the linearized change of curvature tensor $\Upsilon(u)$ by $\Upsilon_{\alpha\beta}(u) = u_{3|\alpha\beta} - b_{\alpha\rho}b^{\rho}_{\beta}u_3 + b^{\rho}_{\beta}u_{\rho|\alpha} + b^{\rho}_{\alpha}u_{\rho|\beta} + b^{\rho}_{\beta|\alpha}u_{\rho}$, where all $|$ subscripts denote covariant derivatives. This last definition restricts the regularity of the chart in the classical approach since it involves third derivatives of the chart in the term $b^{\rho}_{\beta|\alpha}$.

We now change points of view and instead of identifying the displacement u with its covariant components, we consider it as a mapping from ω into $\mathbb{R}^3$. The partial derivatives $\partial_\alpha u$ and $\partial_{\alpha\beta}u$ also are considered as mappings from ω into $\mathbb{R}^3$.

LEMMA 1. *If $u \in H^1(\omega;\mathbb{R}^3)$ and $\varphi \in W^{2,\infty}(\omega;\mathbb{R}^3)$, then the expressions*

$$\gamma^{\text{new}}_{\alpha\beta}(u) = \frac{1}{2}(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha) \tag{1}$$

define functions of $L^2(\omega)$ which coincide with the covariant components of the strain tensor when u and φ belong to $C^3(\bar{\omega};\mathbb{R}^3)$. The expressions

$$\Upsilon^{\text{new}}_{\alpha\beta}(u) = \left(\partial_{\alpha\beta}u - \Gamma^{\rho}_{\alpha\beta}\partial_\rho u\right) \cdot a_3 \tag{2}$$

define distributions of $H^{-1}(\omega)$ which coincide with the covariant components of the change of curvature tensor when u and φ belong to $C^3(\bar{\omega};\mathbb{R}^3)$.

Remarks. Lemma 1 gives two expressions for the linearized strain and change of curvature tensors that are simpler and more intrinsic than the classical ones. Note in particular that the derivatives of the second fundamental form are absent from the change of curvature tensor. We also prove a convergence result which shows that expressions (1) and (2) provide natural extensions for the strain and change of curvature tensors to our less smooth situation. We will thus remove the "new" exponent from the notation thereafter.

THEOREM 2. (infinitesimal rigid displacement lemma) *Assume that $\varphi \in W^{2,\infty}(\omega;\mathbb{R}^3)$. Let $u \in H^1(\omega;\mathbb{R}^3)$ be a displacement of the surface S. If u satisfies $\gamma(u) = 0$ and $\Upsilon(u) = 0$ then there exists two constant vectors ψ and c in $\mathbb{R}^3$ such that for all $x \in \omega$*

$$u(x) = c + \psi \wedge \varphi(x). \tag{3}$$

The proof is based on the existence of the infinitesimal rotation vector ψ, which follows here from elementary arguments of vector analysis in $\mathbb{R}^3$ recast in a distributional framework. It makes essential use of expressions (1) and (2).

3. Existence and uniqueness for the Koiter model

The variational formulation of the Koiter model requires that the strain and change of curvature tensors of competing displacements be square-integrable. In view of the new expressions for these tensors, we are thus led to introduce the Hilbert space (this is for simple support, for simplicity)

$$V = \left\{v \in H^1_0(\omega;\mathbb{R}^3),\quad \partial_{\alpha\beta}v \cdot a_3 \in L^2(\omega)\right\}, \tag{4}$$

which we equip with the natural norm $\|v\|_V = \left(\|v\|^2_{H^1(\omega;\mathbb{R}^3)} + \sum_{\alpha,\beta}\|\partial_{\alpha\beta}v \cdot a_3\|^2_{L^2(\omega)}\right)^{1/2}$ (it is easy to see that for $v \in H^1(\omega;\mathbb{R}^3)$ then $\partial_{\alpha\beta}v \cdot a_3$ is a distribution belonging to $H^{-1}(\omega)$). It is not difficult to check that if φ is C^3, the space V is isomorphic to the space $H^1_0(\omega) \times H^1_0(\omega) \times (H^1_0(\omega) \cap H^2(\omega))$ that corresponds to a simply supported shell in Bernadou and Ciarlet's approach and their existence and uniqueness result in the regular case is thus recovered.

Given a uniformly positive elasticity tensor $a^{\alpha\beta\rho\sigma}$ and a thickness $e > 0$, we then show:

THEOREM 3. *Let $\varphi \in W^{2,\infty}(\omega;\mathbb{R}^3)$ and let $P \in L^2(\omega;\mathbb{R}^3)$ be a given force resultant density. Then there exists a unique solution to the variational formulation of Koiter's model: Find $u \in V$ such that*

$$\forall v \in V, \quad \int_\omega e a^{\alpha\beta\rho\sigma}\Big(\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{e^2}{12}\Upsilon_{\alpha\beta}(u)\Upsilon_{\rho\sigma}(v)\Big)\sqrt{a}\,dx = \int_\omega P\cdot v\,\sqrt{a}\,dx. \tag{5}$$

The proof consists in showing that the bilinear form is V-elliptic by the standard contradiction argument, using on the one hand Rellich's lemma for compactness and on the other hand the two-dimensional Korn inequality, in conjunction with expressions (1) and (2). Again, these expressions make the proof quite elementary.

The same argument can be applied to shells that are clamped on a part γ_0 of their boundary and submitted to applied tractions and moments on the remaining part. The variational space to be used in this case is the space

$$\tilde{V} = \left\{v \in H^1(\omega;\mathbb{R}^3), \partial_{\alpha\beta}v\cdot a_3 \in L^2(\omega), v = \partial_\alpha v \cdot a_3 = 0 \quad \text{on} \quad \gamma_0\right\}, \tag{6}$$

which is a well-defined Hilbert space with the same norm as before. Some care has to be exerted concerning applied moments, which involve the trace of the tangential part of the infinitesimal rotation vector on the boundary. We proceed in the same spirit, by rewriting the infinitesimal rotation vector in a fashion similar to that of the strain tensor. Indeed, the strain tensor is the symmetric part of the displacement gradient, whereas the infinitesimal rotation vector is its antisymmetric part. Thus the same ideas apply and the above functional setting remains perfectly adequate for these boundary conditions, see Blouza and Le Dret [3].

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5. References

1. M. Bernadou and P.G. Ciarlet, in *Computing Methods in Sciences and Engineering*, R. Glowinski, J.-L. Lions (eds.), Lecture Notes in Economics and Math. Systems, vol. 134, (Springer-Verlag, Berlin, 1976).
2. M. Bernadou, P.G. Ciarlet and B. Miara, *J. Elasticity* **34** (1994) 111–138.
3. A. Blouza and H. Le Dret, Existence and uniqueness for the linear Koiter model for shells with little regularity, to appear.
4. P.G. Ciarlet and B. Miara, *Z. Angew. Math. Phys.* **43** (1992) 243–253.
5. P. Destuynder and M. Salaün, *Mat. Apl. Comput.* **10** (1991) 161–190.
6. N. Kerdid and P. Mato Eiroa, to appear.
7. W.T. Koiter, *Proc. Kon. Ned. Akad. Wetensch.* **B73** (1970) 169–195.
8. P. Le Tallec and S. Mani, Analyse numérique d'un modèle de coques de Koiter discrétisé en base cartésienne par éléments finis DKT, INRIA preprint #3073 (1996).

DKT FINITE ELEMENT APPROXIMATION OF GEOMETRICALLY EXACT SHELL MODELS

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1 Introduction

Many technical applications require a geometrically accurate description of the large motion of the structures which are involved. Usually, these structures are thin and flexible and can be viewed as shells subjected to large displacements. Standard shell models turn out to be rather inappropriate for such flexible structures, because their formulation and numerical solution use local basis which are difficult to update in large displacements. In contrast, geometrically exact shell models [6] not only use an accurate description of the strain tensors within the shell, but can be also completely formulated without ever referring to any local basis.

The purpose of this work is then to develop finite element models for thin geometrically exact nonlinear shells. The originality of our approach is to always work in a fixed cartesian basis. After a brief introduction of the shell model, the paper presents a finite elements approximation specially developed for this problem and based on nonlinear Discrete Kirchhoff Triangles (DKT). This choice of finite elements is then analysed within the linear functional framework of [1]. By deriving specific inverse inequalities, we can prove consistency and optimal convergence with very little regularity assumptions on the initial shape of the shell, and without apriori conditions on the mesh size.

2 Geometrically Exact Shells.

The basic idea of geometrically exact shell models is to describe the motion of the structure by the motion of its midsurface and of an associate field of directors, to compute the exact strain field generated by this motion, and to reduce the constitutive response of the structure to membrane, flexion and shear. Representing any material point x_0 of the shell by its position (m_0, ξ) in a fixed reference configuration, characterised by its projection m_0 onto the shell reference midsurface and its distance ξ to m_0, such models assume that the final position x of x_0 is given by

$$x(x_0) = 3D\varphi(m_0) + \xi\vec{t}(m_0),$$

with $\varphi(m_0)$ the final position of m_0 and where the unit vector $\vec{t}, \|\vec{t}\| = 3D1$ is the director of the normal fiber after deformation. The (virtual) velocity field $\hat{U}$ is then characterized by the (virtual) velocity field $\hat{\varphi}$ of the midsurface and by the rate of variation $\hat{t}$ ($\vec{t} \cdot \hat{t} = 3D0$) of the director

$$\hat{U} = 3D\hat{\varphi}(m_0) + \xi\hat{t}(m_0).$$

In this framework, the total strains are simply measured by three strain tensors characterising the shell motion within a rigid body motion

$$C = 3D\nabla\varphi^T \cdot \nabla\varphi, \varepsilon = 3D1/2(\nabla\varphi^T \cdot \nabla\varphi - \nabla\varphi_0^T \cdot \nabla\varphi_0) \text{ membrane},$$

$$K = 3D\nabla\varphi^T \cdot \nabla\vec{t}; \quad \rho = 3DK - K^0 \text{ flexion},$$
$$R = 3D\vec{t}\cdot\nabla\varphi, \quad \gamma = 3DR - R^0 \text{ shear}.$$

Within this kinematic description, the stress contribution can then be decomposed as usual in three parts, representing the resulting action of the local stresses against membrane, flexion and shear variation. Indeed, introducing the variations of strains in any given virtual motion

$$(\hat{\varepsilon}, \hat{\rho}, \hat{\gamma}) = 3D\frac{\partial(\varepsilon, \rho, \gamma)}{\partial(\varphi, \vec{t})} \cdot (\hat{\varphi}, \hat{t}),$$

a direct integration through the thickness yields

$$\int_\Omega \sigma : \frac{\partial \hat{U}}{\partial x} = 3D\int_{\Omega_0} (J\sigma \cdot F^{-T}) : \frac{\partial \hat{U}}{\partial = x_0} = 3D\int_{\omega_0} n : \hat{\varepsilon} + m : \hat{\rho} + q \cdot \hat{\gamma},$$

where the efforts (n, m, q) are defined by appropriate integration of the three-dimensional Cauchy stress tensor σ through the thickness. If the internal efforts only depend on the local (3D) deformation and if we assume hyperelasticity (no work in cyclic motion), on can also prove that the generalised stresses (n, m, q) are necessarily given by [3]

$$(n, m, q) = 3D\frac{\partial\psi}{\partial(\varepsilon, \rho, \gamma)}(\varepsilon, \rho, \gamma).$$

With such hyperelastic constitutive laws, shell problems are then simply governed by the following variational formulation with unknown $(\varphi, \vec{t}) : \omega_0 \to R^3 \times S^2$

$$\int_{\omega_0} \frac{\partial\psi(\varepsilon, \rho, \gamma)}{\partial(\varepsilon, \rho, \gamma)} \cdot \frac{\partial(\varepsilon, \rho, \gamma)}{\partial(\varphi, \vec{t})} \cdot = (\hat{\varphi}, \hat{t}) = 3D\int_{\omega_0} f \cdot \hat{\varphi} + m_{ext} \cdot \hat{t}, \text{ for all admissible} (\hat{\varphi}, \hat{t}).$$

This model has been constructed without using any mathematical artefact such as a choice of a local basis or a strong regularity assumption on the deformation φ. It is perfectly objective, uses an exact, highly nonlinear, quadratic definition of the strains based on the introduction of three basic tensors which fully characterise the motion within a rigid body, and can be transported on any other reference configuration $\hat{\omega}$. All these properties must now be respected in the discretisation process.

3 Finite Element Discretisation.

We now consider a thin shell model where all shear is forbidden. In order to construct an efficient finite element approximation which does not use any local basis and stays objective, we introduce a fixed cartesian basis to measure the deformation φ and the associated director $\vec{t}$, and choose once for all a reference configuration $\hat{\omega}$ equipped with a triangulation $\hat{\omega} = 3D \cup T$. In practice, the reference configuration $\hat{\omega}$ corresponds to the union of the triangular faces T describing the structure in a CAD file, and the smooth stress free configuration ω_0 is automatically built from this triangulation by interpolation. Once the triangulation is known, we approximate each cartesian component of the deformation $(\varphi, \vec{t})$ by DKT finite elements defined on this triangulation.

In other words, the kinematically admissible discrete deformations are characterised by

$$\varphi_h \in C^0(\hat{\omega}, R^3), (\varphi_h)_{|T} \in P'^3,$$
$$\nabla\varphi_h \text{ continuous at vertices},$$
$$\vec{t} = 3D\frac{(\nabla_h\varphi)_1 \wedge (\nabla_h\varphi)_2}{\|(\nabla_h\varphi)_1 \wedge (\nabla_h\varphi)_2\|}.$$

Here $\nabla_h\varphi_h$ is a DKT continuous [2] approximation of the local GRADIENTS $\nabla\varphi_h$ (and not of the rotations)

$$\nabla_h v_h = 3D\sum_i (\nabla v_h(b_i)\lambda_i + 6\nabla_{i+1\ i+2} v_h \otimes t_{i+1\ i+2}|b_{i+1}b_{i+2}|\lambda_{i+1}\lambda_{i+2})$$

$$\nabla_{ij} v_h = 3D \frac{1}{|b_i b_j|} \left[v_h(b_i) - v_h(b_j) - = \frac{1}{2} [\nabla v_h(b_i) + \nabla v_h(b_j)] \cdot b_j b_i . \right]$$

Such elements are free of shear locking because of the DKT assumption. This is confirmed by the convergence analysis made below. They never involve local basis to be incrementally updated during the large displacement process. Moreover, the use of the discrete gradient $\nabla_h \varphi_h$ as local degrees of freedom instead of rotations avoids the introduction of any artificial drilling degree of freedom. The price to pay is that we approximate all three components of the position in the same DKT space, and that we use two gradient components by cartesian direction instead of two global rotations per node.

This finite element discretisation finally reduces the equilibrium problem=

to a nonlinear algebraic system whose unknown are the values of the three cartesian components of φ_h and of their gradients at each node (9 degrees of freedom by node). This system is solved by a Newton solver [4].

4 Convergence analysis of the linearised problem

The linearised thin shell problem consists in studying the small= displacements of a shell around a given initial configuration φ_0 with unit normal vector $\vec{t_0} = 3Da_3$. The linearised unknown are then the displacement $u = 3D\varphi - \varphi_0$ of the midsurface and the variation of director $\beta = 3D\vec{t} - \vec{t_0}$. As proved in [1], the natural topology for studying the linearised problem is

$$Z = 3D\{(v, \delta) \in H^1(\Omega, \mathbb{R}^3 \times \mathbb{R}^3), \delta = 3D - a_3 \cdot \nabla(v) \cdot F^{-1}, + \text{ boundary cond.}\}.$$

In this framework, the finite element space introduced before reduces to the space

$$\begin{aligned} Z_h \quad = 3D \quad & \{(v_h, \delta_h) \in H^1(\Omega, \mathbb{R}^3 \times \mathbb{R}^3), v_h|_T \in P_3'(T, \mathbb{R}^3), \\ & \nabla v_h \text{ cont. at vertices}, \\ & \delta_h = 3D - a_3 \cdot \nabla_h(v_h) \cdot F^{-1}, + \text{ boundary cond.}\}. \end{aligned}$$

For any $(u_h, \beta_h) \in Z_h$, we then introduce the linearized strains

$$\begin{aligned} \varepsilon(u_h) &= 3D\frac{1}{2}(\nabla u_h^t \cdot \nabla\varphi + \nabla\varphi^t \cdot \nabla u_h), \\ \beta(u_h) &= 3D - a_3 \cdot \nabla u_h \cdot F^{-1}, \\ \chi_h(u_h, \beta_h) &= 3Da_3 \cdot (\nabla u_h - \beta_h), \\ \rho(u_h)_{|T} &= 3D\frac{1}{2}(\nabla u_h^t \cdot \nabla = a_3 + \nabla a_3^t \cdot \nabla u_h + \nabla\beta^t \cdot \nabla\varphi + \nabla\varphi^t \cdot \nabla\beta)_{|T} \\ \rho_h(u_h, \beta_h) &= 3D\frac{1}{2}(\nabla = u_h^t \cdot \nabla a_3 + \nabla a_3^t \cdot \nabla u_h + \nabla\beta_h^t \cdot \nabla\varphi + \nabla\varphi^t \cdot \nabla\beta_h) \end{aligned}$$

and the linearized energy forms (with arbitrary constant C_{h_T})

$$a((u, \beta), (v, \delta)) = 3D \int \{eE\varepsilon(u) : \varepsilon(v) + \frac{e^3}{12} E\rho(u, \beta) : \rho(v, \delta)\}$$

$$a_h((u_h, \beta_h), (v_h, \delta_h)) = 3D \int \{eE\varepsilon(u_h) : \varepsilon(v_h)$$

$$+ \frac{e^3}{12} E\rho_h(u_h, \beta_h) : \rho_h(v_h, \delta_h) + C_{h_T}\chi_h(u_h, \beta_h) \cdot \chi_h(v_h, \delta_h)\} \, d\xi$$

Associated to the continuous linearised problem analyzed by [1]

$$(P) \begin{cases} \text{Find } (u, \beta) \in Z \text{ such that} \\ a((u, \beta), (v, \delta)) = 3Df(v, \delta), \quad \forall (v, \delta) \in Z, \end{cases}$$

we then obtain a discrete problem

$$(P_h)\left\{\begin{array}{l}\text{Find } (u_h,\beta_h)\in Z_h \text{ such that}\\ \\ a_h((u_h,\beta_h),(v_h,\delta_h)) = 3Df(v_h,\delta_h), \quad \forall (v_h,\delta_h)\in Z_h.\end{array}\right. ,$$

on which we can prove

Theorem 1 *If $\varphi \in W^{2,\infty}(\Omega)$, then the error between the continuous and the discrete solutions is bounded by*

$$\begin{aligned}&\|(u-u_h,\beta-\beta_h)\|_Z\\ &\quad\leq Ch\|\nabla\varphi.q\|_{2,\Omega} + \inf_{(v_h,\delta_h)\in Z_h}\|(u-v_h,\beta-\delta_h)\|_Z,\\ &\quad\leq Ch(\|\nabla\varphi.q\|_{2,\Omega}+\|u\|_{3,2,\Omega}).\end{aligned}$$

This convergence theorem uses standard tools of finite element analysis, but in addition involves three key technical lemmas [5]. The first one proves basic inverse inequalities on the DKT triangles by mapping all triangles to a fixed reference triangle $\hat{T}$.

Lemma 1 *Let a_3^{moy} be the average of the normal vector a_3 on the triangle Then, for all $(w_h,\delta_h = 3D - a_3\cdot\theta_h\cdot F^{-1})\in Z_h$ and for all regularly shaped triangles T, we have*

$$\begin{aligned}\|D^3 w_h\|_{\infty,T} &\leq \tfrac{C}{h_T^3}\,|w_h|_{1,2,T},\\ |\theta_h|_{m,2,T} &\leq C\,h_T^{-m}|w_h|_{1,2,T}, \quad \forall m = 3D = 0,1,\\ \|a_3^{moy}\cdot D^3 w_h\|_{\infty,T} &\leq \tfrac{C}{h_T^2}\,\|a_3^{moy}\cdot D\theta_h\|_{2,T}.\end{aligned}$$

The next key lemma bounds the error made in the calculation of the shell curvature when using a DKT interpolation. More precisely, by construction, the error between the exact variation of normal $\beta(w_h) = 3D - a_3.\nabla w_h.F^{-1}$ and its DKT estimate $\beta_h = 3D - a_3.\nabla_h(w_h).F^{-1}$ is given by

$$\beta(w_h)-\beta_h = 3D - \sum_{i<j}\lambda_i\lambda_j a_3.D^3 w_h(b_j b_i, b_j b_i,\nu_{ij})\nu_{ij}.F^{-1},$$

which, with the help of the above inverse inequalities, directly leads to the following bounds

Lemma 2 *If φ belongs to $W^{2,\infty}(\Omega)$, then for all (w_h,β_h) in Z_h, we have*

$$|\beta(w_h)-\beta_h|_{m,2,T} \leq Ch^{1-m}\|(w_h,\beta_h)\|_{Z,T},$$

$$\|\chi_h(w_h,\beta_h)\|_{2,T} \leq Ch\|(w_h,\beta_h)\|_{Z,T}.$$

The final technical lemma controls the consistency error.

Lemma 3 *Assume $\varphi\in W^{2,\infty}(\Omega)$ and let (u,β) be the solution of the continuous problem, and q be the associated shear resultant. We then have*

$$\sup_{w_h\in Z_h}\frac{|a((u,\beta),(w_h,\beta_h)) - f(w_h,\beta_h)|}{\|(w_h,\beta_h)\|_Z} \leq Ch\|\nabla\varphi.q\|_{2,\Omega}.$$

Proof. By construction, the consistency error is given by

$$\begin{aligned}&a((u,\beta),(w_h,\beta_h(w_h))) - f(w_h,\beta_h(w_h))\\ &\quad = 3D\sum_T\int_{\partial T}(\nabla\varphi.m.\nu_T).\beta d\sigma + \int_T \nabla\varphi.m:(\nabla\beta_h-\nabla\beta)d\xi\\ &\quad -\int_{\Gamma_1} M\cdot\beta_h d\sigma.\end{aligned}$$

By continuity of the approximate gradients and hence of the discrete variation β_h across the edges, we have after integration by parts

$$a((u,\beta),(w_h,\beta_h)) - f(w_h,\beta_h) = 3D - \sum_T \int_T div(\nabla\varphi.m).(\beta_h - \beta)d\xi.$$

We then conclude by our above control of curvature error

$$|\beta_h - \beta|_{m,2,T} \leq Ch^{1-m}\|(w_h,\beta_h)\|_{1,2,T}.$$

5 Conclusion

The above approach and analysis proves that it is possible to write, approximate, implement and completely analyse a shell problem without ever introducing a local basis or a Christoffel symbol.

This guarantees a better respect of mechanical invariants which are also defined independently of any local basis, and a simple and direct numerical implementation which avoids any update of local basis during a large displacement analysis.

This also leads to existence and convergence results requiring less regularity ($\varphi \in W^{2,\infty}(\Omega)$). Nevertheless a major restriction remains. Membrane locking has not been overcome, and all coercivity constants are still thickness dependant.

References

[1] A. Blouza, H. Le Dret [1994], "Existence et Unicité pour le Modèle de Koiter pour une Coque peu Régulière", C.R. Acad. Sci., Paris, 319, pp. 1127-1132.

[2] G.S. Dhatt [1970], "An Efficient triangular shell element", AIAA J., 8, pp. 2100-2102.

[3] M. Carrive [1995], "Modélisation intrinsèque et analyse numérique d'un problème de coque mince en grands déplacements". Thèse de l'Université Paris-Dauphine, Juin 1995.

[4] M. Carrive, P. Le Tallec, J. Mouro [1995], "Approximation par éléments finis d'un modèle de coques minces géométriquement exact", Revue Européenne des Eléments Finis 4, 5-6, pp. 633-662.

[5] P. Le Tallec, S. Mani [1996], "Analyse Numérique d'un modèle de coques de Koiter discrétisé en base cartésienne par éléments finis DKT." to appear in RAIRO. Also INRIA research report 3073, décembre 1996.

[6] J.C. Simo, D.D. Fox, M.S. Rifai [1989], "On a stress resultant geometrically exact shell model, Part I : Formulation and Optimal parametrization", Comput. Methods Appl. Mech. Eng. 72, pp. 267-304.

ABOUT THE FORMAL EXPANSIONS OF THE DISPLACEMENT VECTOR OF A LINEARLY ELASTIC SHELL

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ABSTRACT

The Koiter's model in linearized elasticity consists in the approximation of the displacement vector of a loaded shell by an affine function of the transverse variable. The first term solves the two-dimensional Koiter's equations. Under some assumptions on the loads, P.G. Ciarlet and V. Lods (cf. [3]) proved that the average across the thickness of the shell of the three-dimensional vector has the same limit as the Koiter's displacement, when the thickness of the shell goes to zero, in appropriate spaces. This convergence is also satisfied in Budiansky–Sanders's or Novozhilov's models.
Here, we formally justify the affine approximation done in the three-dimensional Koiter's model, based on the Kirchhoff–Love assumptions. To this end, we calculate the second term of the for mal expansion across the thickness of the displacement vector of the shell. This term is equal to the second term of the formal expansion of the solution of Koiter's model. For an elliptic shell clamped along its entire boundary, this affine approximation is reduced to zero. Moreover, in this case, we can calculate the first two non–zero terms of the formal expansion of the three-dimensional vector, which allows us to obtai n an affine approximation. This one is not equal to the approximation done by W.T. Koiter.

1. The three-dimensional shell problem

A shell is a three-dimensional body, a dimension of which is small compared to the others. Thus, a shell can be characterized by a surface S, called the middle surface, and by its thickness 2ε, assumed to be constant for the sake of simplicity. The surfac e S is parametrized by a mapping denoted by $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$, where ω is a bounded, open, and connected subset of $\mathbf{R}^2$, with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ. Let $y = (y_\alpha)$ denote a generic point in the set $\overline{\omega}$, and let $\partial_\alpha := \partial/\partial y_\alpha$. We assume that $\boldsymbol{\varphi}$ is injective, of class $\mathcal{C}^3$ and that the two vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\varphi}(y)$ are linearly independent at all points $y \in \overline{\omega}$. They form the *covariant basis* of the tangent plane to the surface S at the point $\boldsymbol{\varphi}(y)$, the two vectors $\boldsymbol{a}^\alpha(y)$ of the same tangent plane defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$ constitute its *contravariant basis*. The normal vector to S at the point $\boldsymbol{\varphi}$ is denoted by

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)|}.$$

The shell, of middle surface S and thickness 2ε, is then parametrized by the mapping $\boldsymbol{\Phi}$: $\overline{\Omega}^\varepsilon \to \mathbf{R}^3$, with $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$ defined by

$$\boldsymbol{\Phi}(x^\varepsilon) := \boldsymbol{\varphi}(y) + x_3^\varepsilon \boldsymbol{a}_3(y) \text{ for all } \boldsymbol{x}^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon.$$

P.G. Ciarlet & J. C. Paumier [1] proved that the physical problem described below is meaningful for ε small enough, in the sense that there exists $\varepsilon_0 > 0$, here chosen less than 1, such that the mapping $\boldsymbol{\Phi} : \overline{\Omega}^\varepsilon \to \mathbf{R}^3$ is injective for all $0 < \varepsilon \le \varepsilon_0$, and moreover, the three vectors $\boldsymbol{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)$ are linearly independant at all points $x^\varepsilon \in \overline{\Omega}^\varepsilon$. With the basis $(\boldsymbol{g}_i^\varepsilon(x^\varepsilon))$, called *covariant basis*, we associate the contravariant basis, constitued by the three vectors $\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ defined by $\boldsymbol{g}^{j,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j$. To detail the equilibrium equations in curviligne coordinates, we introduce the *metric tensor* (g_{ij}^ε) or $(g^{ij,\varepsilon})$ (in covariant or contravariant components) and the *Christoffel symbols* of t he manifold $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ by letting:

$$g_{ij}^\varepsilon := \boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}_j^\varepsilon, \qquad g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}, \quad \Gamma_{ij}^{p,\varepsilon} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial_i^\varepsilon \boldsymbol{g}_j^\varepsilon.$$

The *volume element* in the set $\boldsymbol{\Phi}(\Omega^\varepsilon)$ is$\sqrt{g^\varepsilon}dx^\varepsilon$, where $g^\varepsilon := \det(g_{ij}^\varepsilon)$.

We can now recall the elasticity equations. We assume that, for each $0 < \varepsilon \le \varepsilon_0$, the set $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is the reference configuration of an *elastic shell* in a natural state, and that the material constituting the shell is homogeneous and isotropic. Consequently, the material is characterized by its two *Lamé constants* $\lambda > 0$ and $\mu > 0$, independent of ε. When the shell is loaded, it appears a vector displacement, denoted by $\overrightarrow{\boldsymbol{u}^\varepsilon}(x^\varepsilon)$ at each point $\boldsymbol{\Phi}(x^\varepsilon)$. The unknowns of the variational equations (1.1) are the three covariant components $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbf{R}^3$ of the displacement vector on the basis $(\boldsymbol{g}^{i,\varepsilon})$. To simplify the notations, we impose that the shell is *clamped* along its whole "lateral" face $\boldsymbol{\Phi}(\Gamma^\varepsilon)$, where $\Gamma^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$; which means that the displacement vector vanishes there.

The variational formulation of the three-dimensional problem of linearized elasticity can be written in the curvilinear coordinates as follows. The function
$\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ solves the elliptic problem

$$\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon);\ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma^\varepsilon\},$$

$$\int_{\Omega^\varepsilon} A^{ijk\ell,\varepsilon} e_{k\|\ell}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}dx^\varepsilon \text{ for all } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon), \tag{1.1}$$

where

$$A^{ijk\ell,\varepsilon} := \lambda g^{ij,\varepsilon} g^{k\ell,\varepsilon} + \mu(g^{ik,\varepsilon} g^{j\ell,\varepsilon} + g^{i\ell,\varepsilon} g^{jk,\varepsilon})$$

designate the contravariant components of the *three-dimensional elasticity tensor*,

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) := \frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon$$

designate the covariant components of the *linearized strain tensor* associated with an arbitrary displacement field $v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the surface S and $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ are the contravariant components of the *applied body force density*. To simplify the calculations,

we consider that there are no surface forces acting on the upper and lower faces of the shell.

2. The "scaled" three-dimensional shell problem over a domain independent of ε

To study the behavior of the displacement vector, we introduce the sets $\Omega = \omega \times]-1, 1[$, $\Gamma = \gamma \times [-1, 1]$, which are independent of ε. Let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$, and let $\partial_i = \partial/\partial x_i$. With $x^\varepsilon = (x_i^\varepsilon) \in \overline{\Omega}^\varepsilon$, we associate the point $x = (x_i) \in \overline{\Omega}$ defined by $x_\alpha = x_\alpha^\varepsilon (= y_\alpha)$ and $x_3 = (1/\varepsilon)x_3^\varepsilon$.

Classically, we then define the *scaled unknown* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbf{R}^3$ by

$$u_i(\varepsilon)(y, x_3) = u_i^\varepsilon(y, \varepsilon x_3) \text{ for all } (y, x_3) \in \overline{\Omega}.$$

Thus, $\boldsymbol{u}(\varepsilon)$ belongs to the space

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma\}.$$

To study the behavior of $\boldsymbol{u}(\varepsilon)$ as ε goes to zero, we have to do some *assumptions on the data*, i.e., on the *loads*: there exist functions $f^{i,0}, f^{i,1} \in L^2(\omega)$ *independent of* ε such that

$$f^{i,\varepsilon}(y, \varepsilon x_3) = \varepsilon^2\{f^{i,0}(y) + \varepsilon x_3 f^{i,1}(y)\} \text{ for all } (y, x_3) \in \Omega.$$

We now assume that the scaled displacement $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ can be written on the following way

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 + \varepsilon \boldsymbol{u}^1 + \varepsilon^2 \boldsymbol{u}^2 + ..., \tag{2.1}$$

where the functions $\boldsymbol{u}^0 = (u_i^0), \boldsymbol{u}^1 = (u_i^1)$ and $\boldsymbol{u}^2 = (u_i^2)$ which are independent of the parameter ε, are such that

$$\boldsymbol{u}^1 \in \boldsymbol{V}(\Omega), \quad u_\alpha^2 \in H^1(\Omega), u_\alpha^2 = 0 \text{ on } ;\gamma, \quad u_3^2 \in L^2(\Omega). \tag{2.2}$$

The function $\boldsymbol{u}^0$, which is independent of the transverse variable x_3, solves the two-dimensional flexural equations, from the results of convergence proved in [2]. The aim of this paper is to compare the second term $\boldsymbol{u}^1$ with the second term of the formal expansion of the solution of Koiter's model.

3. The formal asymptotical expansions

3.1. Calculation of the formal expansion of the displacement of the shell

After apropriate algebraic manipulations, we can check the following result

Theorem *The formal asymptotical expansion (2.1) of the components* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ *of the displacement vector on the basis* $(\boldsymbol{g}^i(\varepsilon))$ *is given by*

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 - \varepsilon x_3 \boldsymbol{\theta}^0 ....,$$

where $\boldsymbol{u}^0 = (u_i^0)$ *solves the flexural equations (cf. e.g. [2]) and* $\boldsymbol{\theta}^0 = (\theta_i^0)$ *is given by*

$$\theta_\alpha^0 = \partial_\alpha u_3^0 + 2b_\alpha^\beta u_\beta^0, \quad \theta_3^0 = 0$$

where (b_α^β) *are the mixed components of the curvature tensor of the middle surface* S *of the shell, defined by*

$$b_\alpha^\beta = \partial_\alpha \boldsymbol{a}^\beta \cdot \boldsymbol{a}_3.$$

These formulas can be rewritten on the following way

$$\overrightarrow{\boldsymbol{u}^\varepsilon} = \overrightarrow{\boldsymbol{u}^0} - x_3^\varepsilon \overrightarrow{\boldsymbol{\theta}^0} ....$$

where $\overrightarrow{u^\varepsilon} = u_i^\varepsilon g^{i,\varepsilon}$, $\overrightarrow{u^0} = u_i^0 a^i$, $\overrightarrow{\theta^0} = (\partial_\alpha u_3^0 + b_\alpha^\beta u_\beta^0) a^\alpha$.

3.2. Comparison with the three-dimensional Koiter's model

The Koiter's model consists in calculating the displacement vector $\overrightarrow{u_K^\varepsilon}$ defined by

$$\overrightarrow{u_K^\varepsilon}(y, x_3^\varepsilon) = \overrightarrow{\zeta^\varepsilon}(y) - x_3^\varepsilon \overrightarrow{\theta^\varepsilon}(y) \text{ for all } (y, x_3^\varepsilon) \in \Omega^\varepsilon,$$

where $\overrightarrow{\zeta^\varepsilon} = \zeta_i^\varepsilon a^i$ solves the two dimensional Koiter's model (cf. e.g. [3]) and the normal rotation $\overrightarrow{\theta^\varepsilon} = \theta_\alpha^\varepsilon a^\alpha$ is given by

$$\theta_\alpha^\varepsilon := \partial_\alpha \zeta_3^\varepsilon + b_\alpha^\beta \zeta_\beta^\varepsilon.$$

It is easy to calculate the following formal expansion

$$\overrightarrow{u_K}(\varepsilon) = \overrightarrow{u^0} - \varepsilon\, x_3 \overrightarrow{\theta^0} + \ldots$$

where $\overrightarrow{u_K}(\varepsilon)$ is the scaled displacement defined by $\overrightarrow{u_K}(\varepsilon)(y, x_3) = \overrightarrow{u_K^\varepsilon}(y, \varepsilon x_3)$ for all $(y, x_3) \in \overline{\Omega}$. We can obtain some results of weak convergence of the function $\dfrac{\overrightarrow{u_K}(\varepsilon) - \overrightarrow{u^0}}{\varepsilon}$ under some asumptions on the geometry of the shell and on the loads.

Consequently, the affine approximation done by W.T. Koiter is justified in the sense that the first two terms of the asymptotical expansions of $\overrightarrow{u^\varepsilon}$ and $\overrightarrow{u_K^\varepsilon}$ are equal.

3.3. The case of uniformly elliptic shells

If the middle surface S is uniformly elliptic, like a portion of sphere for instance, and clamped along its entire boundary, the functions u^0 and u^1 are equal to zero, since the function $\dfrac{u(\varepsilon)}{\varepsilon^2}$ converges in $L^2(\Omega)$ as ε g oes to zero (cf. eg. [4], [5], [6], [7]). So, we have calculated the second term of the asymptotical expansion of $\dfrac{u(\varepsilon)}{\varepsilon^2}$. Since the normal compone nt of this second term is not equal to zero, we cannot obtain the affine approximation of W.T. Koiter in this case.

4. Conclusion

The three-dimensional Koiter's model is thus formally justified. To obtain the corresponding results of convergence of the three-dimensional vector and of the solution of Koiter's model, some assumptions on the loads and on the shell seem necessary.
Others models, like Budiansky Sanders's or Novozhilov's, also based on Kirchhoff–Love assumptions, can be justified in the same way. However, notice that this method is not able to make a two by two comparison of the above models.

5. References

1. P. G. Ciarlet and J. C. Paumier, *Computational Mechanics* **1** (1986) 177–202.
2. P.G. Ciarlet, V. Lods, and B. Miara, *C. R. Acad. Sci. Paris, Série I* **319** (1994) 95–100.
3. P.G. Ciarlet and V. Lods, *C. R. Acad. Sci. Paris, Série I* **319** (1994) 299–304.

4. P.G. Ciarlet and V. Lods, *C. R. Acad. Sci. Paris, Série I* **318** (1994) 863–868.

5. Ph. Destuynder, *Sur une Justification des Modèles de Plaques et de Coques par les Méthodes Asymptotiques*, (Doctoral Dissertation, Université Pierre et Marie Curie, Paris, 1980).

6. Ph. Destuynder, *Acta Applicandæ Mathematicæ* 4, (1985) 15–63.

8. E. Sanchez-Palencia, *C. R. Acad. Sci. Paris, Série II* **311** (1990) 909–916.

A DOMAIN DECOMPOSITION METHOD FOR BONDED PLATES

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ABSTRACT

We present a domain decomposition type algorithm for dealing with the numerical solution of bonded plates

1. Introduction

Since a pioneering work by Goland and Reissner in 1944 [6] the bonding of two elastic three dimensional structures by an adhesive layer is treated with asymptotic analysis. (See, e.g., [1],[2],[5],[8].) In the resulting limit problem the adhesive disappears from a geometrical point of view but it gives rise to suitable transmission conditions. In [3] we introduced and analyzed a domain decomposition type procedure to deal with the limit problem numerically. In the present paper we apply the same technique to the bending of two thin elastic plates (Love-Kirchhoff), bonded in their common plane by an adhesive layer. This layer is also treated as a Love-Kirchhoff plate having, in its plane, a small dimension with respect of those of the two adherent plates. Let ε donote the smallness ratio. The type of transmission conditions in the limit problem depends on the ratio of the bending rigidity coefficients. We refer to [4] for the derivation of the limit problem in the different cases. In what follows we shall consider the case where the bending rigidity coefficient of the glue is given by $\varepsilon^3 D_0$, D_0 being of the same order of magnitude of D^+, D^-, the bending coefficients of the adherents.

2. Position of the problem

Let Ω^+ and Ω^- denote the two plates, that we assume to be open connected subsets of $\mathbf{R}^2$ with boundaries $\partial\Omega^+$ and $\partial\Omega^-$ piecewise of class C^2, and let $S = \partial\Omega^+ \cap \partial\Omega^-$ be a non empty regular curve of positive measure. Let Ω be the union of Ω^+ and Ω^-, with boundary $\partial\Omega$, and let $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$, $\Gamma^- = \partial\Omega^- \cap \partial\Omega$. For simplicity, assume that the plate is clamped on $\partial\Omega$. For a function v defined on Ω, let v^+ (resp. v^-) denote the restriction of v to Ω^+

(resp. Ω^-). The local equations are (see [4])

$$\left\{\begin{array}{rcll} D^+\Delta^2 w^+ & = & p^+ & \text{in } \Omega^+ \\ D^-\Delta^2 w^- & = & p^- & \text{in } \Omega^- \\ w^+ = \frac{\partial w^+}{\partial n} & = & 0 & \text{on } \Gamma^+ \\ w^- = \frac{\partial w^-}{\partial n} & = & 0 & \text{on } \Gamma^- \end{array}\right. \tag{1}$$

with the transmission conditions on S

$$\left\{\begin{array}{rcll} M_n(w^+) = M_n(w^-) & = & 0 & \text{on } S \\ K_n(w^+) & = & -12D_0(w^+ - w^-) & \text{on } S \\ K_n(w^-) & = & 12D_0(w^+ - w^-) & \text{on } S \end{array}\right. \tag{2}$$

where p^+, p^- are the applied external loads, $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the outward unit normal to Ω^+ (resp. Ω^-), M_n is the normal bending moment, and K_n the normal Kirchhoff shear force. In order to apply a domain decomposition type procedure, we observe that the boundary conditions (2) can be rewritten as

$$\left\{\begin{array}{rcll} M_n(w^+) = M_n(w^-) & = & 0 & \text{on } S \\ K_n(w^+) & = & -K_n(w^-) & \text{on } S \\ K_n(w^+) + 24D_0 w^+ & = & K_n(w^-) + 24D_0 w^- & \text{on } S \end{array}\right. \tag{3}$$

Next, for $g \in L^2(S)$, consider the following problems

$$\left\{\begin{array}{rcll} D^+\Delta^2 w^+ & = & p^+ & \text{in } \Omega^+ \\ w^+ = \frac{\partial w^+}{\partial n} & = & 0 & \text{on } \Gamma^+ \\ M_n(w^+) & = & 0 & \text{on } S \\ K_n(w^+) + 24D_0 w^+ & = & g & \text{on } S \end{array}\right. \quad \left\{\begin{array}{rcll} D^-\Delta^2 w^- & = & p^- & \text{in } \Omega^- \\ w^- = \frac{\partial w^-}{\partial n} & = & 0 & \text{on } \Gamma^- \\ M_n(w^-) & = & 0 & \text{on } S \\ K_n(w^-) + 24D_0 w^- & = & g & \text{on } S \end{array}\right. \tag{4}$$

For any given $g \in L^2(S)$, $p^+ \in L^2(\Omega^+)$, $p^- \in L^2(\Omega^-)$ problems (4) have a unique solution $w^+ \in H^2(\Omega^+)$, and $w^- \in H^2(\Omega^-)$ respectively. (Note that the boundary conditions (4) actually induce more regularity on the solutions.) Due to linearity, these solutions can be split as

$$w^+ = w_p^+ + w_g^+, \qquad w^- = w_p^- + w_g^-, \tag{5}$$

with w_p^+, w_p^- solutions of (4) with $g = 0$, and w_g^+, w_g^- solutions of (4) with $p^+ = 0$, $p^- = 0$. We can then define the linear continuous operators T_p^+, T_p^-, T_g^+, T_g^-

$$\begin{array}{c} p^+ \in L^2(\Omega^+) \longrightarrow w_p^+ = T_p^+(p^+), \quad p^- \in L^2(\Omega^-) \longrightarrow w_p^- = T_p^-(p^-), \\ g \in L^2(S) \longrightarrow w_g^+ = T_g^+(g), \quad w_g^- = T_g^-(g), \end{array} \tag{6}$$

so that (5) becomes

$$w^+ = T_p^+(p^+) + T_g^+(g) \qquad w^- = T_p^-(p^-) + T_g^-(g). \tag{7}$$

Next, let $\mathcal{A}$ be the operator from $L^2(S)$ in itself defined as

$$g \in L^2(S) \longrightarrow \mathcal{A}g \;=\; (w_g^+ + w_g^-)_{|S} \equiv (T_g^+(g) + T_g^-(g))_{|S}. \tag{8}$$

It is immediate to check that $\mathcal{A}$ is linear and continuous. Moreover, thanks to the trace theorem (see, e.g., [7]), we have in particular $w^+_{g\,|S} \in H^{3/2}_{00}(S)$, $w^-_{g\,|S} \in H^{3/2}_{00}(S)$, so that $\mathcal{A}$ is linear and continuous from $L^2(S)$ into $H^{3/2}_{00}(S)$.

Going back to formulation (4), note that the continuity condition on K_n in (3) is not taken into account. Hence, we must find a suitable g such that the solutions of (4) verify (3). Since from (4) it follows that $K_n(w^+) + K_n(w^-) = 2(g - 12D_0(w^+ + w^-))$, such a g will be the solution of the following minimization problem

$$\text{Find } g^* \in L^2(S) \;:\; 0 = J(g^*) \le J(g) \qquad \forall g \in L^2(S), \tag{9}$$

for the quadratic functional

$$J(g) := ||g - 12D_0(w^+ + w^-)||^2_{0,S}. \tag{10}$$

Using the notation introduced in (7)-(8) we have

$$12D_0(w^+ + w^-)_{|S} = F + 12D_0\mathcal{A}g, \qquad \text{having set } F := 12D_0(T_p^+(p^+) + T_p^-(p^-))_{|S}, \tag{11}$$

so that (10) can be written as

$$J(g) = ||g - (F + 12D_0\mathcal{A}g)||^2_{0,S}. \tag{12}$$

It is easy to check that $J(g)$ is strictly convex, so that problem (9) has a unique solution g^*, which verifies

$$g^* = F + 12D_0\mathcal{A}g^*. \tag{13}$$

In order to write the variational formulation of (4) we set

$$V^+ := \{v \in H^2(\Omega^+)\,,\, v = \partial v/\partial n = 0 \text{ on } \Gamma^+\}, \tag{14}$$

$$V^- := \{v \in H^2(\Omega^-)\,,\, v = \partial v/\partial n = 0 \text{ on } \Gamma^+\}, \tag{15}$$

$$a^+(v,w) = D^+\int_{\Omega^+} (v_{/11}w_{/11} + 2(1-\nu)v_{/12}w_{/12} + v_{/22}w_{/22} + \nu(v_{/11}w_{/22} + v_{/22}w_{/11})\,dx \tag{16}$$

$$a^-(v,w) = D^-\int_{\Omega^-} (v_{/11}w_{/11} + 2(1-\nu)v_{/12}w_{/12} + v_{/22}w_{/22} + \nu(v_{/11}w_{/22} + v_{/22}w_{/11})\,dx \tag{17}$$

$$A^+(w,v) = a^+(w,v) + 24D_0 \textstyle\int_S v\,w\,ds, \tag{18}$$

$$A^-(w,v) = a^+(w,v) + 24D_0 \textstyle\int_S v\,w\,ds. \tag{19}$$

The variational formulation of problems (4) is then

$$\left\{ \begin{array}{l} \text{Find } w^+ \in V^+ \text{ such that :} \\ A^+(w^+, v) = (p^+, v) + (g, v)_S \qquad \forall v \in V^+, \end{array} \right. \tag{20}$$

$$\left\{ \begin{array}{l} \text{Find } w^- \in V^- \text{ such that :} \\ A^-(w^-, v) = (p^-, v) + (g, v)_S \qquad \forall v \in V^-. \end{array} \right. \tag{21}$$

Existence, uniqueness and a-priori error bounds for the solutions of (20)-(21) are ensured by the continuity and coercivity properties of the bilinear forms $A^+,\ A^-$.

3. The Algorithm

We shall now present a domain decomposition type algorithm, based on the variational formulations (20)-(21) and the minimum problem (9), for which we shall prove convergence. Compute $w_p^+ = T_p^+(p^+),\ w_p^- = T_p^-(p^-)$ solutions of

$$w_p^+ \in V^+ : \quad A^+(w_p^+, v) = (p^+, v) \qquad \forall v \in V^+, \tag{22}$$

$$w_p^- \in V^- : \quad A^-(w_p^-, v) = (p^-, v) \qquad \forall v \in V^-, \tag{23}$$

and set

$$g^0 = 12D_0(w_p^+ + w_p^-)_{|S}\ (= F). \tag{24}$$

For $m \geq 0$ compute the solutions $w_m^+ = T_g^+(g^m)$, $w_m^- = T_g^-(g^m)$ of the problems

$$w_m^+ \in V^+ \ : A^+(w_m^+, v) = (g^m, v)_S \qquad \forall v \in V^+, \tag{25}$$

$$w_m^- \in V^- \ : A^-(w_m^-, v) = (g^m, v)_S \qquad \forall v \in V^-. \tag{26}$$

Then set

$$\tilde{g}^m := g^m - 12D_0(w_m^+ + w_m^-)_{|S}, \tag{27}$$

$$g^{m+1} := g^m - \rho(\tilde{g}^m - g^0), \tag{28}$$

and compute the solutions $w_{m+1}^-,\ w_{m+1}^+$ of (25)-(26) with the new datum g^{m+1}. In (28) $\rho > 0$ is a parameter to be chosen in order to have convergence of g^m to g^*, as $m \to \infty$, where g^* is defined in (13). In order to prove convergence we shall use the following result

Theorem 1 *$\mathcal{A}$ is a compact operator. Moreover, the eigenvalues z of $12D_0\mathcal{A}$ are all real and verify*

$$\exists C_1 > 0 \ \text{ such that } \qquad 0 \leq z \leq 1 - C_1 < 1 \qquad \forall z. \tag{29}$$

Proof The proof is a slight modification of that given in [3] and we shall not report it here. We can now prove the following convergence theorem.

Theorem 2 *There exists a $\rho_0 \geq 1$ such that, for $\rho \in]0, \rho_0[$ we have*

$$\lim_{m\to\infty} g^m = g^*, \tag{30}$$

where g^m is the sequence defined in (22)-(28), and g^* is defined in (13).

Proof Note that, according to definition (8), (27) can be rewritten as

$$\tilde{g}^m = (I - 12D_0\mathcal{A})g^m. \tag{31}$$

From (28) and (31), using (24) and (13) we then have

$$\begin{aligned} g^{m+1} - g^* &= (1-\rho)g^m + \rho 12D_0\mathcal{A}g^m + \rho g^0 - g^* + \rho g^* - \rho g^* \\ &= (1-\rho)g^m + \rho 12D_0\mathcal{A}g^m + \rho(g^0 - g^*) - (1-\rho)g^* \\ &= (1-\rho)(g^m - g^*) + \rho 12D_0\mathcal{A}(g^m - g^*) \\ &= ((1-\rho)I + \rho 12D_0\mathcal{A})(g^m - g^*). \end{aligned} \tag{32}$$

Recursive application of (32) yields

$$g^{m+1} - g^* = ((1-\rho)I + \rho 12 D_0 \mathcal{A})^{m+1}(g^0 - g^*), \quad \text{with } g^0 - g^* = -12 D_0 \mathcal{A} g^*. \tag{33}$$

Convergence will be proved if we can show that

$$\lim_{m\to\infty} |||((1-\rho)I + \rho 12 D_0 \mathcal{A})^{m+1}||| = 0, \tag{34}$$

where $|||L|||$ denotes the norm of the operator L. From a theorem by Gelfand, if L is bounded then $\lim_{n\to\infty} |||L^n|||^{1/n} = \sup\{|\lambda|,\ \lambda \in \sigma(L)\}$, $\sigma(L)$ being the spectrum of L. Thanks to Theorem 1, the spectrum of the operator $(1-\rho)I + \rho 12 D_0 \mathcal{A}$ is given by $1-\rho$ and

$$\lambda_j = (1-\rho) + \rho z_j, \tag{35}$$

z_j being the eigenvalues of $12 D_0 \mathcal{A}$. Proving (34) amounts then to prove that

$$\max_j |\lambda_j| < 1, \quad \text{i.e.,} \ -2 < \rho(z_j - 1) < 0 \qquad \forall j. \tag{36}$$

This inequality is verified for all the values $\rho \in]0, \frac{2}{1-z_{max}}[$ and the proof is completed. ∎
Remark Notice that, since $\frac{2}{1-z_{max}} > 1$, the value $\rho = 1$ can always be taken. The optimal value for ρ is $\rho_{opt} = \frac{1}{1-z_{max}}$.

4. References

All references should include initials and last name of the author(s), title of publication (in italics), volume (in bold), year of publication of paper in the journal/book, and page numbers, e.g.,

1. P. Destuynder, F. Michavila, A. Santos and Y. Ousset, Some theoretical aspects in computational analysis of adhesive lap joints, *Internat. J. Numer. Methods Engrg.* **35** (1992) 1237–1262.
2. G. Geymonat, F. Krasucki and S. Lenci, Analyse asymptotique du comportement d'un assemblage collé, *C.R. Acad. Sci. Paris* **322, Série I** (1996) 1107–1112.
3. G. Geymonat, F. Krasucki, D. Marini and M. Vidrascu (to appear)
4. G. Geymonat and F. Krasucki, (to appear)
5. Y. Gilibert and A. Rigolot, Analyse asymptotique des assemblages collés à double recouvrement sollicités au cisaillement en traction, *J. Mec. Appl.* **3** (1979) 341–372.
6. M. Goland and E. Reissner, The stresses in cemented joints, *J. Appl. Mech. ASME* **11** (1944) A.17–A.27.
7. P. Grisvard, *Elliptic problems in nonsmooth domains* (Pitman, London) (1985).
8. A. Klarbring, Derivation of a model of adhesively bonded joints by the asymptotic expansion method, *Int. J. Engrg. Sci.* **29** (1991) 493–512.

EXPLICIT FORMS OF THE LIMIT STRESSES FOR ELASTIC SHELLS

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ABSTRACT

We show how to find the explicit forms of the limit stresses in a linearly or nonlinearly elastic shell by means of the formal asymptotic expansion of the displacement field. In the linear case, we are in a position to establish convergence theorems.

1. Piola-Kirchhoff stress tensor in three-dimensional curvilinear coordinates

Let $\varepsilon > 0$ and ω be a domain in $\mathbb{R}^2$ with a Lipschitz boundary γ. Let $x^\varepsilon = (x_i^\varepsilon) \in \overline{\Omega}^\varepsilon = \overline{\omega} \times [-\varepsilon, +\varepsilon]$ be a system of curvilinear coordinates associated to the injective mapping $\boldsymbol{\Phi} \in \boldsymbol{C}^3(\overline{\Omega}^\varepsilon; \mathbb{R}^3)$, where $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is the *configuration reference* of the shell under study. We consider a shell made with a *Saint-Venant Kirchhoff material* with Lamé constants $\lambda > 0$ and $\mu > 0$. When subjected to *applied body forces* with scaled contravariant components $\boldsymbol{f}(\varepsilon) = (f^i(\varepsilon)) \in \boldsymbol{L}^2(\Omega)$ and *surface forces* with scaled contravariant components $\boldsymbol{h}(\varepsilon) = (h^i(\varepsilon)) \in \boldsymbol{L}^2(\Gamma_+ \cup \Gamma_-)$, $\Gamma_\pm = \omega \times \{\pm 1\}$, the shell, *clamped* on a portion $\boldsymbol{\Phi}(\Gamma_0^\varepsilon)$ ($\Gamma_0^\varepsilon = \gamma_0 \times (-\varepsilon, +\varepsilon)$, $\gamma_0 \subset \gamma$ with *length* $\gamma_0 > 0$) *of its lateral surface*, undergoes a *displacement field* whose scaled covariant components $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ are a *critical point* of the scaled energy $J(\varepsilon)$ defined on $\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,4}(\Omega); \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0 = \gamma_0 \times (-1, +1)\}$ by:

$$J(\varepsilon)(\boldsymbol{v}) = \frac{1}{2} \int_\Omega A^{ijk\ell}(\varepsilon) E_{i||j}(\varepsilon)(\boldsymbol{v}) E_{k||l}(\varepsilon)(\boldsymbol{v}) \sqrt{g(\varepsilon)} dx - L(\varepsilon)(\boldsymbol{v}), \tag{1}$$

where $A^{ijk\ell}(\varepsilon)$ is the three-dimensional scaled elasticity tensor in curvilinear coordinates, $E_{i||j}(\varepsilon)$ are the scaled covariant components of the *change of metric tensor* and $L(\varepsilon)$ is a linear form depending on $\boldsymbol{f}(\varepsilon)$ and $\boldsymbol{h}(\varepsilon)$. We refer to [3] or [4] for all the notations not recalled here. The associated *scaled contravariant components of the first Piola-Kirchhoff stress tensor* read:

$$T^{k\ell}(\varepsilon) \overset{\text{def}}{=} T^{k\ell}(\boldsymbol{u}(\varepsilon)) = (g^{ijk\ell}(\varepsilon) + g^{ij\ell m}(\varepsilon) g^{nk}(\varepsilon) u_{n||m}(\varepsilon)) E_{i||j}(\varepsilon)(\boldsymbol{u}(\varepsilon))$$

and they satisfy the scaled equations of equilibrium

$$\left\{ \begin{array}{rcl} -T^{ij}_{\ ||j}(\varepsilon) &=& f^i(\varepsilon) \text{ in } \Omega \quad , \\ \pm T^{i3}(\varepsilon) &=& h^{i,\pm}(\varepsilon) \text{ on } \Gamma_\pm \quad , \end{array} \right.$$

where the covariant derivatives are given by:

$$\left\{ \begin{array}{rcl} T^{ij}_{\ ||\alpha}(\varepsilon) &=& \partial_\alpha T^{ij}(\varepsilon) + T^{rj}(\varepsilon)\Gamma^i_{r\alpha}(\varepsilon) + T^{ir}(\varepsilon)\Gamma^j_{r\alpha}(\varepsilon) \quad , \\ T^{ij}_{\ ||3}(\varepsilon) &=& \dfrac{1}{\varepsilon}\partial_3 T^{ij}(\varepsilon) + T^{rj}(\varepsilon)\Gamma^i_{r3}(\varepsilon) + T^{ir}(\varepsilon)\Gamma^j_{r3}(\varepsilon) \quad , \end{array} \right.$$

and Γ_{ij}^k are the scaled Christoffel symbols. Assuming that there exists *an asymptotic expansion of the unknown* $\boldsymbol{u}(\varepsilon) = \sum_{k\geq 0} \varepsilon^k \boldsymbol{u}^k$, with $\boldsymbol{u}^k$ a three-dimensional field independent of ε and a *Taylor expansion of the geometric data* around $x_3^\varepsilon = \varepsilon x_3 = 0$ (for example for the covariant basis vector $\boldsymbol{g}_i(\varepsilon)(x) = \boldsymbol{a}_i(x_1, x_2) + \sum_{k\geq 1} (\varepsilon x_3)^k \boldsymbol{g}_i^k(x_1, x_2)$, with $\boldsymbol{g}_i^k$ a two-dimensional field independent of x_3 and ε) and by identification of the successive powers of ε in (1) we showed that *for specific assumptions on the applied forces* (with respect to ε) and on *the geometry of the shell*, the leading term $\boldsymbol{u}^0$ solves either a "membrane" model [3] or a "flexural" model [4]. We therefore use these results to identify the leading terms of the corresponding expansion $T^{ij}(\varepsilon) = \sum_{k\geq 0} \varepsilon^k T^{ij,k}$ where $T^{ij,k}$ is a function independent of ε. (The proofs of the Theorems can be found in [2]).

2. The "membrane" model

To obtain the membrane shell model , the following assumptions on the forces and on the geometry of the shell have to be made: There exist $\boldsymbol{f}$ and $\boldsymbol{h}$ *independent of* ε such that:

$$\boldsymbol{f}(\varepsilon) = \boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \quad , \quad \boldsymbol{h}(\varepsilon) = \varepsilon \boldsymbol{h} \in \boldsymbol{L}^2(\Gamma_+ \cup \Gamma_-) \quad , \tag{2}$$

and the space of "inextensional displacements" $\boldsymbol{Y}_F(\omega)$ reduces to $\{\boldsymbol{0}\}$:

$$\boldsymbol{Y}_F(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{W}^{1,4}(\omega);\ \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0,\ E_{\alpha||\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} = \{\boldsymbol{0}\}. \tag{3}$$

2.1. The nonlinear case

Under assumptions (2) and (3) we showed [3] that $\boldsymbol{u}^0$ is a critical point of the "membrane" energy J_M defined on the space $\boldsymbol{V}_M(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{W}^{1,4}(\omega); \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0\}$ by:

$$J_M(\boldsymbol{\eta}) = \int_\omega b^{\alpha\beta\sigma\tau} E^0_{\alpha||\beta}(\boldsymbol{\eta}) E^0_{\sigma||\tau}(\boldsymbol{\eta}) \sqrt{a} d\omega - L(\boldsymbol{\eta})$$

where $b^{\alpha\beta\sigma\tau}$ is the two-dimensional elasticity tensor, $(E^0_{\alpha||\beta})$ is the two-dimensional *change of metric* tensor, $L(\boldsymbol{\eta}) = \int_\omega \left(\int_{-1}^1 f^i(t)dt + h^{i,+} + h^{i,-} \right) \boldsymbol{\eta}_i d\omega$. After lengthy computations, we can then deduce the following result:

Theorem 1
For nonlinearly elastic membrane shells, i.e., under assumptions (2) *and* (3), *the first non-vanishing terms of the expansion of* $T^{ij}(\varepsilon)$ *are given by:*

$$\begin{aligned}
T^{\sigma\tau,0} &= (b^{\alpha\beta\sigma\tau} + b^{\alpha\beta\tau\delta} a^{\gamma\sigma} u^0_{\gamma||\delta}) E^0_{\alpha||\beta}(\boldsymbol{u}^0) \quad , \\
T^{3\sigma,0} &= b^{\alpha\beta\sigma\tau} u^0_{3||\tau} E^0_{\alpha||\beta}(\boldsymbol{u}^0) \quad , \\
T^{\sigma 3,1} &= -h^{\sigma,-} - \int_{-1}^{x_3} f^\sigma(t)dt - (x_3+1)(\partial_\tau T^{\sigma\tau,0} + T^{\sigma\tau,0}\Gamma^{\alpha *}_{\alpha\tau} + T^{\alpha\beta,0}\Gamma^{\sigma *}_{\alpha\beta} - T^{3\tau,0} b^\sigma_\tau) \quad , \\
T^{33,1} &= -h^{3,-} - \int_{-1}^{x_3} f^3(t)dt - (x_3+1)(\partial_\sigma T^{3\sigma,0} + T^{3\sigma,0}\Gamma^{\alpha *}_{\sigma\alpha} + T^{\sigma\tau,0} b_{\sigma\tau}) \quad .
\end{aligned}$$

where $\Gamma^{\gamma *}_{\alpha\beta}$ *are the bi-dimensional Christoffel symbols of the middle surface* S *of the shell.*

2.2. The linear case

For linearly elastic shells Ciarlet and Lods [5] showed that under the following additional assumptions on the forces and on the geometry of S:

(i) $h_3 \in H^1(\Gamma_+ \cup \Gamma_-)$,

(ii) the middle surface S is elliptic,

(iii) the shell is *completely* clamped on its lateral face,

the vector $\boldsymbol{u}(\varepsilon)$ converges in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ towards the unique minimum $\boldsymbol{u}^0$ of the "linearized" membrane energy defined on $\boldsymbol{V}_M^{lin}(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$. In this case we can show the new result [1]:

Theorem 2
For linearly elastic membrane shells, under the assumptions (2), (i), (ii) *and* (iii), *the contravariant components of the first Piola-Kirchhoff stress tensor, once scaled, converge as follows:*

$$
\begin{aligned}
&T^{\sigma\tau}(\varepsilon) \to T^{\sigma\tau,0} = b^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{u}^0) \ \ in\ L^2(\Omega),\\
&\frac{T^{\sigma 3}(\varepsilon)}{\varepsilon} \to T^{\sigma 3,1} = -h^{\sigma -} - \int_{-1}^{x_3} f^\sigma(t)dt - (x_3+1)(\partial_\tau T^{\sigma\tau,0} + T^{\sigma\tau,0}\Gamma^{\alpha *}_{\alpha\tau} + T^{\alpha\beta,0}\Gamma^{\sigma *}_{\alpha\beta}) \quad ,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad in\ H^1(-1;1;H^{-1}(\omega)) \quad ,\\
&\frac{T^{33}(\varepsilon)}{\varepsilon} \to T^{33,1} = -h^{3,-} - \int_{-1}^{x_3} f^3(t)dt - (x_3+1)T^{\sigma\tau,0}b_{\sigma\tau} \ \ in\ H^1(-1;1;H^{-1}(\omega)).
\end{aligned}
$$

where $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ *is the "linearized" part of* $E^0_{\alpha||\beta}(\boldsymbol{\eta})$.

3. The "flexural" model

The flexural model is obtained under the following assumptions on the forces and on the geometry of the middle surface S: There exists $\boldsymbol{f}$ and $\boldsymbol{h}$ independent of ε such that:

$$\boldsymbol{f}(\varepsilon) = \varepsilon^2 \boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \quad , \quad \boldsymbol{h}(\varepsilon) = \varepsilon^3 \boldsymbol{h} \in \boldsymbol{L}^2(\Gamma_+ \cup \Gamma_-) \tag{4}$$

and the space of "inextensional displacements" $\boldsymbol{Y}_F(\omega)$ does not reduce to $\{\boldsymbol{0}\}$:

$$\boldsymbol{Y}_F(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{W}^{1,4}(\omega);\ \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0,\ E^0_{\alpha||\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \neq \{\boldsymbol{0}\}. \tag{5}$$

3.1. The nonlinear case

Under assumptions (4), (5) Lods and Miara showed [4] that $\boldsymbol{u}^0$ is a critical point of the "flexural" energy J_F defined on the space

$$\boldsymbol{V}_F(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{W}^{2,4}(\omega);\ \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \boldsymbol{0} \ \text{ on } \ \gamma_0 \ ,\ E^0_{\alpha||\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

by:

$$J_F(\boldsymbol{\eta}) = \frac{1}{3}\int_\omega b^{\alpha\beta\sigma\tau}\hat{E}^1_{\alpha||\beta}(\boldsymbol{\eta})\hat{E}^1_{\sigma||\tau}(\boldsymbol{\eta})\sqrt{a}d\omega - L(\boldsymbol{\eta}),$$

where $(\hat{E}^1_{\alpha||\beta})$ is the two-dimensional *change of curvature tensor*. After lengthy computations, we likewise get the next result :

Theorem 3
For flexural shells, under assumptions (4), (5) *the first non-vanishing terms of the expansion of* $T^{ij}(\varepsilon)$ *are given by:*

$$\begin{aligned}
T^{\sigma\tau,1} &= -x_3(b^{\alpha\beta\sigma\tau} + b^{\alpha\beta\tau\delta}a^{\sigma\gamma}u^0_{\gamma||\delta})\hat{E}^1_{\alpha||\beta}(\boldsymbol{u}^0) = x_3\tilde{T}^{\sigma\tau,1} \quad ,\\
T^{3\sigma,1} &= -x_3 b^{\alpha\beta\sigma\tau}u^0_{3||\tau}\hat{E}^1_{\alpha||\beta}(\boldsymbol{u}^0) = x_3\tilde{T}^{3\sigma,1} \quad ,\\
T^{\sigma 3,2} &= -\frac{x_3^2-1}{2}(\partial_\tau\tilde{T}^{\sigma\tau,1} + \tilde{T}^{\sigma\tau,1}\Gamma^{\alpha *}_{\alpha\tau} + \tilde{T}^{\alpha\beta,1}\Gamma^{\sigma *}_{\alpha\beta} - \tilde{T}^{3\tau,1}b^\sigma_\tau) \quad ,\\
T^{33,2} &= -\frac{x_3^2-1}{2}(\partial_\sigma\tilde{T}^{3\sigma,1} + \tilde{T}^{3\sigma,1}\Gamma^{\alpha *}_{\sigma\alpha} + b_{\alpha\beta}\tilde{T}^{\alpha\beta,1}) \quad .
\end{aligned}$$

3.2. The linear case

For linearly elastic shells Ciarlet, Lods and Miara [6] showed that under assumption (4) and if the space of inextensional displacements $\boldsymbol{V}^{lin}_F(\omega)$ does not reduces to $\{\boldsymbol{0}\}$:

$$\boldsymbol{V}^{lin}_F(\omega) = \{\boldsymbol{\eta} \in H^1(\omega)\times H^1(\omega)\times H^2(\omega);\ \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha||\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \neq \{\boldsymbol{0}\}, \tag{6}$$

the vector $\boldsymbol{u}(\varepsilon)$ converges in $\boldsymbol{H}^1(\Omega)$ to the unique minimum $\boldsymbol{u}^0$ of the "linearized" flexural energy defined on $\boldsymbol{V}^{lin}_F(\omega)$. In this case we get the next convergence theorem [1].

Theorem 4
For linearly elastic flexural shells, under the assumptions (4), (6), *the contravariant components of the first Piola-Kirchhoff stress tensor, once scaled, converge as follows:*

$$\begin{aligned}
&\frac{1}{\varepsilon}T^{\sigma\tau}(\varepsilon) \to T^{\sigma\tau,1} = -x_3 b^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{u}^0) = x_3\tilde{T}^{\sigma\tau,1} \text{ in } L^2(\Omega),\\
&\frac{1}{\varepsilon^2}T^{\sigma 3}(\varepsilon) \to T^{\sigma 3,2} = -\frac{x_3^2-1}{2}(\partial_\tau\tilde{T}^{\sigma\tau,1} + \tilde{T}^{\sigma\tau,1}\Gamma^{\alpha *}_{\alpha\tau} + \tilde{T}^{\alpha\beta,1}\Gamma^{\sigma *}_{\alpha\beta}) \quad , \text{ in } H^1(-1;1;H^{-1}(\omega)) \quad ,\\
&\frac{1}{\varepsilon^2}T^{33}(\varepsilon) \to T^{33,2} = -\frac{x_3^2-1}{2}b_{\alpha\beta}\tilde{T}^{\alpha\beta,1} \text{ in } H^1(-1;1;H^{-1}(\omega)) \quad .
\end{aligned}$$

where $\rho_{\alpha\beta}(\boldsymbol{\eta})$ *is the linearized part of* $\hat{E}^1_{\alpha||\beta}(\boldsymbol{\eta})$.

4. References

1. C. Collard and B. Miara, Analyse asymptotique des coques linéairement élastiques : convergence des contraintes, C.R. Acad. Sci. Paris, Série I, **322**, 699-702, (1996).
2. C. Collard and B. Miara, Analyse asymptotique des coques élastiques. Étude du tenseur des contraintes de Piola-Kirchhoff, to appear.
3. B. Miara, Analyse asymptotique des coques membranaires non linéairement élastiques, C.R. Acad. Sci. Paris, Série I, **318**, 689-694, (1994).
4. V. Lods and B. Miara, Analyse asymptotique des coques en flexion non linéairement élastiques, C. R. Acad. Sci. Paris, Série I, **321**, 1097-1102, (1995).
5. P. G. Ciarlet, V. Lods, Asymptotic analysis of linearly elastic shells, I. Membrane shells, Arch. Rational Mech. Anal., **136**, 119-161, (1996).
6. P. G. Ciarlet, V. Lods, and B. Miara, Asymptotic analysis of linearly elastic shells, II. Flexural shells, Arch. Rational Mech. Anal., **136**, 163-190, (1996).

SOME REMARKS ON ELASTOPLASTIC MODELS FOR SHELLS

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ABSTRACT

One essential difference between how elastic and plastic behaviour can be modeled is the way in which the distribution of stress through the cross-sectional fiber is considered. Elastoplastic models for shells should be capable of simulating the spread of plasticity through the thickness and, at the same time, such models should be simple enough to allow efficient numerical treatment. Thus far, two different approaches toward a satisfactory elastoplastic shell model have been pursued: (i) Use of models formulated in terms of stress resultants by introducing ajusted constitutive response functions; (ii) Use of $2D/3D$ theories with first order strains and higher order stresses. In this work, the attention is mainly focussed on the analysis of different formulations of flow rules and in the theoretical implementations of classical methods as Augmented Lagrangian and Consistent Tangent Matrix within the framework of the second such approach. For the sake of simplicity the discussion is restricted to the case of infinitesimal displacements and strains.

1. Shell bodies

Let ω be a smooth surface, oriented by the unit normal field $\vec{n}$ in the euclidean space E^3. The reference configuration of the shell body is represented as

$$\mathcal{B} = \{X \in E^3 : X = X_0 + s\vec{n}(X_0), X_0 \in \omega, |s| \leq \frac{h}{2}\}$$

in which h is the constant thickness of the shell. The natural map X, defined by

$$(X_0, s) \in \omega \times [-\frac{h}{2}, \frac{h}{2}] \mapsto X = X_0 + s\vec{n}(X_0) \in E^3$$

transforms, for each fixed number s, the midsurface ω into a parallel surface at distance s. We assume that the mapping X is one-to-one and the tangent map dX to X at the point (X_0, s) is an isomorphism from $T_{X_0}\omega \times R$ onto R^3_X ($T_{X_0}\omega$ denotes the tangent space to ω at X_0) given by

$$dX : (\vec{w}, r) \longmapsto (I - sB)\vec{w} + r\vec{n}$$

in which B is the Weingarten map (or shape operator) defined by $B\vec{w} = -\nabla_{\vec{w}}\vec{n}$ for all $\vec{w} \in T_{X_0}\omega$. By consequence, it follows that the shell shifter $\mu_s = I - sB$ is an isomorphism and the inverse operator has the Neumann expansion $\mu_s^{-1} = I + sB + s^2B^2 + \cdots$. Also μ_s^{-1} is a solution of the differential equation $\frac{d}{ds}(s\phi) = \phi^2$ in the space of the linear operators $\mathcal{L}(T_{X_0}\omega, T_{X_0}\omega)$.

The evolution of the shell is represented by a map $x_t : \mathcal{B} \longmapsto \mathcal{B}_t \subset E^3$, depending on a well chosen parameter $t \in [0, T]$. In terms of the displacement, we have $x_t = X + \vec{u}(X)$.

The deformation gradient is given by $F = I + \nabla\vec{u} : R^3_X \longmapsto R^3_X$. Pull-backing the vector displacement to $\omega \times [-\frac{h}{2}, \frac{h}{2}]$ we obtain a vector field $\vec{u}_0$ such that $\vec{u} = \mu_s \vec{u}_0 + u^3\vec{n}$. The associated operator matrix to the deformation gradient in $\omega \times [-\frac{h}{2}, \frac{h}{2}]$ is

$$\begin{pmatrix} \mu_s^{-1}(D\mu_s\vec{u}_0 - u^3 B) & \frac{\partial \vec{u}_0}{\partial s} - \mu_s^{-1}B\vec{u}_0 \\ (\mu_s^{-1}B\vec{u}_0 + \mu_s^{-2}Du^3)\cdot & \frac{\partial u^3}{\partial s} \end{pmatrix}$$

where D denotes the covariant derivative on the surface and $\cdot$ the dot product induced by the natural map. Thus, shear and normal components of the linear deformation tensor are given by

$$\vec{\gamma}_s = \frac{\partial \vec{u}_0}{\partial s} + \mu_s^{-2}Du^3, \qquad \gamma_n = \frac{\partial u^3}{\partial s}$$

In the context of the classical theories of shells, the following kinematical hypotheses of shells are assumed:

- *The normal deformation is null.* By consecuence, we have

$$u^3 = u^3(X_0), \qquad \vec{\gamma}_s = \frac{\partial}{\partial s}(\vec{u}_0 + s\mu_s^{-1}Du^3)$$

- Shear assumptions are:
  - Kirchhoff-Love hypothesis, *The shear deformation is null.* In this case, we have $\vec{u}_0 + s\mu_s^{-1}Du^3 = \vec{z}_0$ being $\vec{z}_0$ a vector field independent of s. By consequence, the tangential component of the displacement takes the form

$$\vec{u} = \mu_s\vec{z}_0 + u^3\vec{n} - sDu^3$$

  - Mindlin-Reissner hypothesis, *The shear deformation is independent of s.* In this case, we have $\mu_s\vec{u}_0 = \mu_s(\vec{z}_0 + s\vec{\gamma}_s) - sDu^3 = \vec{z}_0 + s(\vec{\gamma}_s - B\vec{z}_0 - Du^3) + s^2 B\vec{\gamma}_s$ being $\vec{z}_0$ a vector field independent of s. Neglecting the second order terms in s we obtain

$$\vec{u} = \vec{z}_0 + u^3\vec{n} + s\vec{\theta}$$

    where $\vec{\theta}$ is a vector field independent of s.

In what follows, we denote by V the space of admissible virtual displacements, constrained by the previous kinematical assumptions (cf. Destuynder[2]).

2. Hellinger-Reissner formulation

We introduce a bilinear form b defined over the space of stresses Q and the space of admissible displacements V by

$$b(\tau, \vec{v}) = \int_{\mathcal{B}} \tau : \nabla\vec{v}\, dV$$

If we denote by $F(\vec{v})$ the external forces work, then the principle of virtual work reads as

$$b(\sigma, \vec{v}) = F(\vec{v}) \qquad \text{for all } \vec{v} \in V$$

where σ represents the stress field on $\mathcal{B}$.

In the classical theories of shells it is usual to reduce the dimension of this variational equation, as may be expected, by introducing stress resultants and moments, without additional assumptions regarding the stresses.

To formulate the constitutive relations, we consider the bilinear form defined over $Q \times Q$ by

$$a(\sigma, \tau) = \int_{\mathcal{B}} A\sigma : \tau dV$$

where A^{-1} denotes the elasticity tensor. In the Elasticity theories, the following variational constitutive equation

$$a(\sigma, \tau) - b(\tau, \vec{u}) = 0 \qquad \text{for all } \tau \in Q$$

completes the Hellinger-Reissner principle. The reduction from the three-dimensional equations can be achieved in a range of different possibilities; however, theories dues to Koiter and Naghdi are the most widely accepted. Although the reduction of the dimension is a reasonable way to apply elasticity theories in thin bodies such as plates or shells, these procedures cannot be used in so simple a way for elastoplastic models because the non-linear behaviour of the stress distribution through a cross-sectional fiber. The use of ajusted constitutive response functions has been propossed by Shapiro, Ivanov, Robinson, Crisfield and other authors (cf. Simo and Kennedy[3]) with the main goal of preserving the two-dimensional nature of the problem. In an opposite side, in the case of a plate, the asymptotic analysis show us that the limit model is still three-dimensional in the constitutive relations (cf. Destuynder[1]). In the case of a shell, there is not an unique limit model (as far as the author knows this analysis has not yet been developed, in the plastic case) but it seems reasonable to suppose that the three-dimensional nature cannot be bypassed.

3. Elasto-plastic constitutive relations

Three-dimensional isothermal elasto-plastic theories relate stresses, strains and additional plastic variables at every point of the shell. In this work, we consider a model in which the plastic behaviour is described by a symmetric second order tensor field ε^p representing the elastic deformation and a scalar field r representing the hardening parameter. The basic assumption is the existence of two scalar functions $\hat{W} = \hat{W}(\varepsilon, \varepsilon^p, r)$ and $\hat{H} = \hat{H}(\sigma_r, \xi, \dot{\varepsilon}^p, \dot{r})$ which give the free energy and the potential of dissipation at each point, being $(\dot{\varepsilon}^p, \dot{r})$ the plastic rates and (σ_r, ξ) their energy conjugate variables. However, for the sake of simplicity in this summary, we consider a simple situation where the constitutive relations are reduced to the following:

- $\dot{\varepsilon}(\vec{u}) = A\dot{\sigma} + \dot{\varepsilon}^p$
- If the stress is on the yield surface $F(\sigma) = 0$, the plastic deformation rate is a tensor normal to it.

It seems that a correct way to express the flow rules is to maintain a geometrical point of view by using Convex Analysis tools, such as the normal cone or the subdifferential calculus. Using these tools, we can express the flow rule in the following compact form

$$\dot{\varepsilon}^p \in \partial I_K(\sigma)$$

where K denotes the convex set $K = \{\tau \in M_s : F(\sigma) \leq 0\}$. Classical methods take a less abstract geometric thinking and they avoid the multivalued relation by using a plastic multiplier $\dot{\lambda}$ in the form

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial F}{\partial \sigma}$$

A shortcoming of this formulation lies in the fact that the yield surface can have corners, because it could be piecewise linear or it could be defined by multiple yield surfaces intersecting in a nosmooth fashion. In this case, in presence of multiple plastic multipliers the theory become not so simple. An intermediate way to express the flow rules makes use of a return map to the convex set K. It is subject to a similar criticism but it retains better the geometrical issues of the problem. Let P_K be the closest point projection onto the convex set K in the stress space with the metric $< \sigma, \tau >= A\sigma : \tau$. It is easy to recast the flow rule as

$$\sigma = P_K(\sigma + \lambda A^{-1}\dot{\varepsilon}^p)$$

for an arbitrary $\lambda > 0$. If we denote by $p = A^{-1}\dot{\varepsilon}^p$ the plastic stress rate, the flow rule is given by

$$p = \frac{I - P_K}{\lambda}(\sigma + \lambda p)$$

for an arbitrary λ. It is very important to keep in mind that it is not an approximation. It is an exact representation of the flow rule. Other posibilities can be envisaged as alternative to the penalization operator $\frac{I-P_K}{\lambda}$, such as the Norton-Hoff approximation (in this case, λ is a nonlinear function of σ and p). This formulation suggests that the plasticity problem might be appropriately approached if one find up efficient approximations to the above equation.

4. Augmented Lagrangian-like methods

Let Π be the projection from Q onto the subspace of the tensors of first order in s. Then, stresses and plastic stress rates [a]can be decomposed into a first order term and a residual term. Now, computational strategies can be designed at many differing levels of approximation. At the lowest, we consider the transformation T defined as follows:

1. Elastic prediction: Solve the two-dimensional reducible linear equations

$$a(\Pi\dot{\sigma}, \tau) - b(\tau, \vec{u},) = -a(\Pi p, \tau) \qquad \text{for all } \tau \in Q$$

$$b(\sigma, \vec{v}) = F(\vec{v}) \qquad \text{for all } \vec{v} \in V$$

2. Plastic correction: Solve the three-dimensional ordinary differential equations

$$(I - \Pi)\dot{\sigma} = (I - \Pi)p$$

and calculate the transformed plastic stress rate

$$Tp = \frac{I - P_K}{\lambda}(\sigma + \lambda p)$$

[a]We suppose the same smoothness as we have in the hardening processes.

Iteration with the transformation T is a robust method allowing large time step sizes. However, in high accuracy computations the convergence can be very slow because it is related to the condition number of the rigidity matrix. The choice of the parameter λ is rather much important for the convergence rate.

5. Consistent tangent matrix-like method

By using a fully implicit Euler scheme to the constitutive equation, we obtain

$$a(\frac{\sigma - \sigma^n}{\Delta t}, \tau) - b(\tau, \vec{u}) = -a(p, \tau) \qquad \text{for all } \tau \in Q$$

If the yield surface is smooth, then we consider the tangent map to P_K at $\sigma^n + \lambda p^n$, modified by $dP_K^n = I$ if $\sigma^n + \lambda p^n \in K$, to avoid the nodifferentiability on the boundary. Thus we take the following Newton approximation

$$p \approx p^{n+\frac{1}{2}} + \frac{I - dP_K^n}{\lambda}(\sigma + \lambda p - \sigma^n - \lambda p^n)$$

where $p^{n+\frac{1}{2}}$ denotes

$$p^{n+\frac{1}{2}} = \frac{I - P_K}{\lambda}(\sigma^n + \lambda p^n)$$

Taking $\lambda = \Delta t$ and eliminating p we obtain

$$a(\frac{\sigma - \sigma^n}{\Delta t}, \tau) - b((I - A^{-1}M^t A)\tau, \vec{u}) + = a((M - I)p^n - p^{n+\frac{1}{2}}, \tau)$$

being $M = I - \delta P_K^n$. Again, this equation can be decomposed into one two-dimensional reducible equation and one local three-dimensional equation. Although this algorithm needs small time step sizes, much higher convergence rates should be obtained.

1. P. Destuynder, *Sur les modeles de plaques minces en elasto-plasticite.* J. de Mecanique theorique et appliquee, *1* 1 (1982) 73-80.
2. P. Destuynder, *A classification of thin shell theories.* Acta Applicandae Mathematicae. 4 (1985) 15-63.
3. J.C. Simo and J.G.Kennedy, *On a stress resultant geometrically exact shell model. Part V. Nonlinear plasticity: formulation and integration algorithms.* Computer Methods in Applied Mechanics and Engineering, **96** (1992) 133-171.

FINITE ELEMENT METHODS FOR SOME PROBLEMS OF THIN ELASTIC SHELLS

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ABSTRACT

This paper is essentially based upon some of our numerical results concerning the approximation of some variants of Koiter's linear model of thin elastic shells by conforming and non conforming finite element methods. Some numerical questions concerning the membrane locking phenomenon are also considered.

1. The continuous problem.

Let Ω be a bounded subset in the plane $\mathcal{R}^2$ with $\Gamma = \partial\Omega$. We define the middle surface $\mathcal{S}$ of a shell $\mathcal{C}$ as the image of $\bar{\Omega}$ by a regular mapping $\vec{\phi} \in \mathcal{C}^3(\bar{\Omega};\mathcal{R}^3)$. For all points $\xi \in \bar{\Omega}$, we denoting by $\{\vec{a}_\alpha\}$ and $\{\vec{a}^\alpha\}$ the covariant and contravariant basis of the tangent plane, and by $\vec{a}_3 = \vec{a}^3$ the normal vector to the middle surface.

The thickness e of the shell $\mathcal{C}$ can be viewed as an application of $\mathcal{C}^0(\bar{\Omega};\mathcal{R}_+)$. Then the shell $\mathcal{C}$ is the set

$$\mathcal{C} = \left\{ M \in \mathcal{R}^3 : \vec{OM} = \vec{\phi}(\xi) + \xi^3\,\vec{a}_3\,, \xi \in \bar{\Omega}\,, |\xi^3| \leq \frac{1}{2}\,e(\xi) \right\} \tag{1}$$

Subsequently, we use the linear model of Koiter [5], who reduce the study of the deformation of a thin shell to the determination of the displacement field $\vec{u} = u_i\,\vec{a}^i$ of the middle surface; in this way, Koiter uses the two hypotheses: (i) conservation of the normals, (ii) plane stresses.

In what follows, we assume that the shell is: (i) loaded by a distribution of forces whose resultant is $\vec{p}$ on the middle surface and whose resultant moment is null (more general forces are also considered), (ii) clamped on the part Γ_0 of its boundary Γ, (iii) free on the complementary part $\Gamma_1 = \Gamma - \Gamma_0$ of its boundary.

Then, the strain energy of the shell is associated to the bilinear form

$$a(\vec{u},\vec{v}) = \int_\Omega e\,E^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}(\vec{v})\,\gamma_{\lambda\mu}(\vec{u}) + \frac{e^2}{12}\,\bar{\rho}_{\alpha\beta}(\vec{v})\,\bar{\rho}_{\lambda\mu}(\vec{u})\right)\sqrt{a}\,d\xi\,, \tag{2}$$

where the elastic modulus tensor $E^{\alpha\beta\lambda\mu}$, the strain tensor $\gamma_{\alpha\beta}$ and the change of curvature tensor $\bar{\rho}_{\alpha\beta}$ are given by

$$\begin{aligned}
E^{\alpha\beta\lambda\mu} &= \frac{E}{2\,(1+\nu)}\left(a^{\alpha\lambda}\,a^{\beta\mu} + a^{\alpha\mu}\,a^{\beta\lambda} + \frac{2\,\nu}{1-\nu}\,a^{\alpha\beta}\,a^{\lambda\mu}\right)\,,\\
\gamma_{\alpha\beta}(\vec{v}) &= \frac{1}{2}\,\left(v_{\alpha|\beta} + v_{\beta|\alpha}\right) - b_{\alpha\beta}\,v_3\,,\\
\bar{\rho}_{\alpha\beta}(\vec{v}) &= v_{3|\alpha\beta} - b_\alpha^\lambda\,b_{\lambda\beta}\,v_3 + b_\alpha^\lambda\,v_{\lambda|\beta} + b_\beta^\lambda\,v_{\lambda|\alpha} + b_{\alpha|\beta}^\lambda\,v_\lambda\,.
\end{aligned}$$

The potential energy of exterior forces is associated to the linear form

$$f(\vec{v}) = \int_\Omega p^i\, v_i\, \sqrt{a}\, d\xi\,, \tag{3}$$

Finally, the admissible displacement space is defined by

$$\vec{V} = \left\{\vec{v} = v_i\, \vec{a}^i \in (H^1(\Omega))^2 \times H^2(\Omega) : v_i = v_{3,\vec{n}} = 0 \text{ on } \Gamma_0\right\} \tag{4}$$

In [2] is proved the existence and uniqueness of a solution for the corresponding variational problem:

$$\text{Find } \vec{u} \in \vec{V} \text{ such that } a(\vec{u}, \vec{v}) = f(\vec{v})\,, \text{ for ¿all } \vec{v} \in \vec{V}\,. \tag{5}$$

2. The discrete problem

¿From now on, we shall assume that the set $\bar{\Omega}$ is a polygon; then, we may exactly cover by regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$. To each triangulation $\mathcal{T}_h$ we associate a product of finite element spaces $\vec{X}_h = (X_{h_1})^2 \times X_{h_2}$; next we define a suitable subspace $\vec{V}_h = (V_{h_1})^2 \times V_{h_2}$ of $\vec{X}_h$ taking into account the boundary conditions which appear in the definition of the space $\vec{V}$. If $\vec{V}_h \subset \vec{V}$ is true, we have a conforming finite element method; else we have a non-conforming finite element method.

Now, the discrete problem, taking into account the use of a numerical integration scheme is:

$$\text{Find } \vec{u}_h \in \vec{V}_h \text{ such that } a_h(\vec{u}_h, \vec{v}_h) = f_h(\vec{v}_h)\,, \text{ for all } \vec{v}_h \in \vec{V}_h\,, \tag{6}$$

where $a_h(.,.)$ and $f_h(.)$ are defined in a natural way. The next step is show that the problem (6) has a unique solution; this is achieved by showing that the bilinear form $a_h(.,.)$ is $\vec{V}_h$-elliptic, uniformly with respect to h (see [1]).

V_{h_1}	V_{h_2}	$\Vert\vec{u} - \vec{u}_h\Vert$	Num. Int.	Degrees of freedom
Ganev-Dimitrov	Argyris	$O(h^4)$	Exact P_6	15+15+21
P_3-Hermite	Bell	$O(h^3)$	Exact P_6	10+10+18
P_2-Lagrange	Complet H.T.C.	$O(h^2)$	Exact P_4	6+6+12
P_1-Lagrange	Reduced H.T.C.	$O(h^1)$	Exact P_2	3+3+9
P_2-Lagrange	Sander.	$O(h^2)$	Exact P_4	6+6+12
P_1-Lagrange	Morley.	$O(h^1)$	Exact P_0	3+3+6

Table 1: Finite elements for thin elastic shells

In [1], pages 72–73 and 141, we can see some combinations of conforming and non-conforming finite elements which leads to suitable spaces $\vec{V}_h$. In Table (1) are summarized different optimal choices; the results for these combinations seem to be the best with respect to: the error estimate $\Vert\vec{u} - \vec{u}_h\Vert$, the cost and the implementation (to define the space V_{h_1} we use a finite element having almost the totality of its degrees of freedom included in the set of degrees of freedom of the finite element used to define the space V_{h_2}). We note that in all the cases, the criteria observed in the choice of the numerical integration ensure the same asymptotic error estimate as in the case of exact integrations.

In order to illustrate the previous considerations, we will present some numerical experiments derived using these combinations of finite elements. These tests have been obtained with the aid of the MODULEF library (see [3]), in which we have implemented the previous methods. We refer to [6] for some interesting comparisons between conforming and non-conforming finite element methods over classical bench-marks.

3. Variants of Koiter's linear model

In [5] Koiter introduce a modified model of thin elastic shells by making the following substitution

$$\rho_{\alpha\beta}(\vec{v}) = \bar{\rho}_{\alpha\beta}(\vec{v}) - \frac{1}{2}\, b^{\lambda}_{\alpha}\, \gamma_{\lambda\beta}(\vec{v}) - \frac{1}{2}\, b^{\lambda}_{\beta}\, \gamma_{\lambda\alpha}(\vec{v})\,; \tag{7}$$

this is, in a certain way, an arithmetic mean among the expression used in the classic model of Koiter and the formulas deduced as a consequence of the Kirchhoff-Love assumptions.

Nevertheless, we can prove that the addition to $\bar{\rho}_{\alpha\beta}(\vec{v})$ of terms of the form $C\, b^{\lambda}_{\alpha}\, \gamma_{\lambda\beta}(\vec{v})$ (C small constant) produces in (2) addends whose order of magnitude is small (proportional to e or e^2) with respect to the already existing (see[7] for the details); by recalling the thinness hypothesis of the shells, this variety of formulations is justified.

Thickness	$U_3(A)$ clas.	$U_3(A)$ modif.	% Relat. error	% Change
$e/3$	0.0416920940	0.0416888700	0.0077326890	0.480
$e/2$	0.0297627322	0.0297600199	0.0091130410	0.566
e	0.0242857928	0.0242897008	0.0160917538	1.000
$2\,e$	0.0152566484	0.0152673617	0.0702202060	4.363
$3\,e$	0.0092417958	0.0092529293	0.1204689351	7.486
$4\,e$	0.0057903262	0.0057998190	0.1639437514	10.188
$5\,e$	0.0037729762	0.0037804555	0.1982337968	12.318

Table 2: *Hyperbolic paraboloid: results*

In this way, we will show also experiment over the classic test of a hyperbolic paraboloid clamped or a cylindrical roof. For example, in Table (2) we give the results obtained for the normal displacement in the center of the paraboloid (point A); as can be observed, the change is very small, but it is sensitive to the increase in the thickness.

4. The Locking Phenomenon

The locking phenomenon consists in a loss of meaning of numerical results when the computations are made for some very small values of a parameter, in our case the thickness of the shells, while the results are valid for the rest of the values.

In [4] we have proved that any finite element internal approximation made of functions which are piecewise polynomials leads to locking as $e \to 0$ for certain non-inhibited shells (as the hyperbolic paraboloid or the straight helycoid, with appropriate boundary conditions).

In this case, the computations were done with different finite element methods (conformal and non-conformal, using high and lower order of polynomials, etc.) and several different integration schemes were considered.

The numerical results show the better behavior of the higher polynomials; this may be understood as a better ability of the high degree polynomials for approximating the very peculiar functions of the limit space of pure bendings. Otherwise, very little influence of the integration schemes was observed (this is contrary to the wide-spread opinion that reduced integration diminishes locking effects) and only for e very small we find some differences.

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6. References

1. M. Bernadou, *Méthodes d'éléments finis pour les problèmes de coques minces*, (Masson, Paris,1994).
2. M. Bernadou and P. G. Ciarlet, "Sur l'ellipticité du modèle linéaire de coques de W. T. Koiter" in *Computing Methods in Applied Sciences and Engineering*, ed. R. Glowinski and J. L. Lions (Springer-Verlag, Berlin, 1976).
3. M. Bernadou, P. L.George, A. Hassim, P. Joly, P. Laug, B. Muller, A. Perronnet, E. Saltel, D. Steer, G. Vanderbork and M. Vidrascu, *MODULEF: une bibliothèque modulaire d'éléments finis*, (Editions INRIA, Rocquencourt, 1988).
4. D. Choi, F. J. Palma, E. Sánchez-Palencia and M. A. Vilariño, "Membrane Locking in the Finite Element Computation of very Thin Elastic Shells", *RAIRO, Math. Model. and Numer. Anal.*, to appear.
5. W. T. Koiter, "On the Foundations of the Linear Theory of Thin Elastic Shells", *Proc. Kon. Ned. Akad. Wetensch.*, **B69** (1970) 169–195.
6. F. J. Palma and M. A. Vilariño, "Linear Analysis of Thin Elastic Shells by two Non-conforming F.E.M." *Rev. Europ. Éléments Finis*, **6** (1997) 7–22.
7. M.A. Vilariño, "Analytical and Numerical Study of some Variants of Koiter's Linear Model of Thin Shells", *Publ. Matem.*, **40** (1996) 215–228.

ON THE NUMERICAL ANALYSIS OF THE NONLINEAR BUCKLING IN THE SHALLOW SHELL THEORY

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ABSTRACT

We consider an approximation of a nonlinear shallow shell model using conforming finite elements for the displacements and for the geometry. Under an assumption on the existence of a snap buckling point, we perform a nonlinear numerical analysis of a branch of solutions containing it. Moreover we give error estimates.

1. Introduction

The nonlinear shallow shell model of Koiter [6] is a sample model providing nonlinear buckling phenomena. For example, in a 1-dimensional and constant curvature version of this model, it is possible to develop a formal calculus with a complete scenario of bifurcation path: if the thickness is small enough, the shell buckles when the loading parameter increases from the trivial equilibrium position (see [7]). Otherwise, a result of existence of a solution to a general shallow shell Koiter's model can be found in [2].

We consider a sample hypothesis of buckling: in the loading process of the shell with a load parameter λ, we assume that there exists a snap point at an extremal value $\lambda = \lambda_c$ of a regular solution path $\lambda \mapsto (\lambda, u(\lambda))$ such that $u(0) = 0$. For the nonlinear shallow shell model of Koiter, we use classical results on the numerical analysis of linear shells problems (see [4] and [1]) with conforming finite element methods for the displacement *and* for the geometry. Under a classical property on the regularity of the solution of the associated linear shallow shell model, we perform the nonlinear analysis of the finite dimensional approximation in the vicinity of the value (λ_c, u_c) (in the framework of [3] or [8]). This analysis end up in a result of local existence and uniqueness of a family of discrete paths which converges toward the continuous path. Then, we deduce the existence of discrete snap points (λ_h^c, u_h^c). At last, we give error estimates between the buckling path and the discrete path.

2. The nonlinear shallow shell model

We shall use Greek letter: α, β, μ, $\ldots$, for indices belonging to the set $\{1, 2\}$, while Latin letters: i, j, k, $\ldots$, will be used for indices belonging to the set $\{1, 2, 3\}$. We shall use convention for summation on repeated indices. The symbols ∂_α, $\partial_{\alpha\beta}$ denote the partial derivative $\frac{\partial}{\partial\xi^\alpha}$, $\frac{\partial^2}{\partial\xi^\alpha\partial\xi^\beta}$, where ξ^1 and ξ^2 are the components of $\xi \in \Omega$, a given 2-dimensional bounded connected open set with a lipschitz continuous boundary $\partial\Omega$.

In the following we consider a map $\vec{\varphi} : \bar{\Omega} \to \mathcal{E}^3$, where $\mathcal{E}^3$ is the 3-dimensional usual euclidian space.

Assume that $\vec{\varphi}$ is injective and C^∞ on $\bar{\Omega}$ and let $\vec{a}_\alpha = \partial_\alpha \vec{\varphi}$. We also assume that the surface $\mathcal{S} := \varphi(\bar{\Omega})$ (the middle surface of the shell) is regular in the sense that $\vec{a}_1$ and $\vec{a}_2$ are linearly independent. Then we denote by $\vec{a}_3$ the normal unit vector to the middle surface $\mathcal{S} = \vec{\varphi}(\bar{\Omega})$. While the set $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ corresponds to the *local covariant basis*, the *local contravariant basis* is denoted by $\{\vec{a}^1, \vec{a}^2, \vec{a}^3\}$. In the following we denote by $a_{\alpha\beta}$ the *first fundamental form* of the surface and its determinant is noted $a = a_{11}a_{22} - |a_{12}|^2$. The matrix $a^{\alpha\beta}$ will denote the inverse of this form. We also denote by $b_{\alpha\beta}$ the *second fundamental form* and by $\Gamma^i_{\alpha\beta}$ the set of *Christoffel symbols* with $\Gamma^3_{\alpha\beta} := b_{\alpha\beta}$.

The displacements $\vec{u}(\xi)$ of the points $x = \vec{\varphi}(\xi)$ of the middle surface have covariant components $u_i(\xi)$, that is to say: $\vec{u} = u_i\, \vec{a}^i$. We will assume that the set u of components (u_1, u_2, u_3) lives in a closed subspace $V = H^1_0(\Omega) \times H^1_0(\Omega) \times V_3$ such that V_3 is closed in $H^2(\Omega)$ and $H^2_0(\Omega) \subset V_3 \subset H^2(\Omega) \cap H^1_0(\Omega)$. For $V_3 = H^2_0(\Omega)$ we recognize a *clamped shell* and for $V_3 = H^2(\Omega) \cap H^1_0(\Omega)$ a *simply supported shell.*

The *nonlinear simplified strain tensor* γ of the shallow shell theory is given by

$$\gamma_{\alpha\beta}(u) = \theta_{\alpha\beta}(u) + \frac{1}{2}\chi_{\alpha\beta}(u,u) \tag{1}$$

where χ is the *nonlinear part* such that: $\chi_{\alpha\beta}(u,v) := \partial_\alpha u_3\, \partial_\beta v_3$ and where θ is the *simplified linear strain tensor*: $\theta_{\alpha\beta}(u) = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^i_{\alpha,\beta}\, u_i$.

The *linear simplified change of curvature tensor* ρ reads

$$\rho_{\alpha\beta}(u) = \partial_{\alpha\beta}\, u_3 - \Gamma^\lambda_{\alpha\beta}\, \partial_\lambda u_3. \tag{2}$$

Let $E > 0$ and $\nu \in]0, \frac{1}{2}[$ be respectively the *Young modulus* and the *Poisson coefficient*; we introduce the contravariant *elastic modulus tensor* $\bar{E}^{\alpha\beta\lambda\mu}$:

$$\bar{E}^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}).$$

Let $\tilde{E}^{\alpha\beta\lambda\mu} = e\sqrt{a}\, \bar{E}^{\alpha\beta\lambda\mu}$ and $\varepsilon = \frac{e^2}{12}$ be coefficients where e is the (constant) thickness of the shell. Then the *energy* of the nonlinear shallow shell model (see [6]) is given by

$$J(u) = \frac{1}{2}\int_\Omega \tilde{E}^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}(u)\, \gamma_{\lambda\mu}(u) + \varepsilon\, \rho_{\alpha\beta}(u)\, \rho_{\lambda\mu}(u))\, d\xi - \lambda \int_\Omega f^i\, u_i\, d\xi, \tag{3}$$

where $f(u) = \int_\Omega f^i\, u_i\, d\xi$ is the potential energy of the external forces and where λ is the load parameter. For the sake of simplicity, we will assume that the f_i are polynomials in ξ^1, ξ^2.

We are looking for $u \in V$ such that $J(u) \leq J(v), \quad \forall v \in V$.

The Euler equation $J'(u) \cdot v = 0, \quad \forall v \in V$ of this minimization problem reads

$$\text{find} \quad u \in V, \quad \text{such that} \quad a(u,v) + g(u,v) = \lambda f(v), \quad \forall v \in V, \tag{4}$$

where we have noted $g(u,v) := b(u,v,u) + \frac{1}{2}b(u,u,v) + \frac{1}{2}c(u,u,u,v)$ and

$$\begin{aligned}
a(u,v) &= \int_\Omega \tilde{E}^{\alpha\beta\lambda\mu}\{\theta_{\alpha\beta}(u)\, \theta_{\lambda\mu}(u) + \varepsilon\, \rho_{\alpha\beta}(u)\, \rho_{\lambda\mu}(u)\}\, d\xi, \\
b(u,v,w) &= \int_\Omega \tilde{E}^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(u,v)\, \theta_{\lambda\mu}(w)\, d\xi, \\
c(u,v,u',v') &= \int_\Omega \tilde{E}^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}(u,v)\, \chi_{\alpha\beta}(u',v')\, d\xi.
\end{aligned}$$

In the following, we assume that *the bilinear for* $a(\cdot,\cdot)$ *is* V*-elliptic* (this is true if the shell is shallow enough, see [1]). Then the variational equation (4) can be written as an equation in the space V: $u + G(u) = \lambda F$.

A sample application of the implicit function theorem on this equation provides a unique regular solution branche $\lambda \mapsto u(\lambda)$ defined on a open interval Λ containing $\lambda = 0$ at which $u(0) = 0$. We take Λ as the *maximal*=D2=DD interval satisfaying this property.

We assume that the interval Λ has an extremal value λ_c such that $\lim_{\lambda \to \lambda_c} u(\lambda) = u_c$ exists and is *a sample limit point* (called a *snap point* in [5]), that is to say: the kernel space $W_c = \{w \in V;\ w + G'(u_c)\cdot w = 0\}$ is 1-dimensional and $a(F, w_c) \neq 0$, where w_c notes a basis of W_c and $G'(u)$ the Fréchet-derivative of the map G at $u \in V$.

Then, we add to the equation $u + G(u) = \lambda F$ the new scalar equation $a(u - u_c, w_c) = t$, where t is a parameter. Applying the implicit function theorem to this system, we find a unique regular buckling path $t \mapsto (\lambda(t), u(t))$ defined on the interval $[-t_c, t_c]$ with $t_c > 0$, satisfaying $(\lambda(0), u(0)) = (\lambda_c, u_c)$ and

$$(5)\ \forall t \in [-t_c, t_c]:\quad a(u(t) - u_c, w_c) = t \quad\text{and}\quad a(u(t), v) + g(u(t), v) = \lambda(t)\, f(v), \quad \forall v \in V.$$

Let $\tilde{v} = (\tilde{v}_{(1)}, \tilde{v}_{(2)}, \tilde{v}_{(3)})$ be a vector with: $\tilde{v}_{(3)} = (v_3, \partial_1 v_3, \partial_2 v_3, \partial_{11} v_3, \partial_{12} v_3, \partial_{22} v_3)$ and $\tilde{v}_{(\alpha)} = (v_\alpha, \partial_1 v_\alpha, \partial_2 v_\alpha)$. As a matter of fact, it is convenient to note the twelve components of the vector $\tilde{v}$ as $\tilde{v}_I$ for $I = 1, \ldots, 12$ (indices without parenthesis). Therefore, the bilinear form $a(\cdot,\cdot)$ (see [4]), $b(\cdot,\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot,\cdot)$ reads as the following matrix form:

$$\begin{aligned} a(u,v) &= \sum_{I,J=1}^{12} \int_\Omega A_{IJ}\, \tilde{u}_I\, \tilde{v}_J\, d\xi, \qquad b(u,v,w) = \sum_{I,J,K=1}^{12} \int_\Omega B_{IJK}\, \tilde{u}_I\, \tilde{v}_J\, \tilde{w}_K\, d\xi, \\ c(u,v,u',v') &= \sum_{I,J,K,L=1}^{12} \int_\Omega C_{IJKL}\, \tilde{u}_I\, \tilde{v}_J\, \tilde{u}'_K\, \tilde{v}'_L\, d\xi. \end{aligned}$$

3. Finite element approximation

Assuming that $\bar{\Omega}$ is a polygon, we consider a regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$. We shall assume that there exists a family of finite element space $\{\Phi_h\}_{h>0}$ and an integer $m \geq 3$ such that, $\forall K \in \cup_{h>0}\mathcal{T}_h$ we have $P_m(K) \subset P(K) \subset C^3(K)$ where $P_m(K)$ is the space of (the restriction to K of) all polynomials of degree $\leq m$ in the variables ξ^1 and ξ^2 and $P(K)$ is the associated local interpolation space. Notice that the inclusion $\Phi_h \subset C^0(\bar{\Omega})$ *is not* required. Now, we note by φ_{ih} the Φ_h-interpolate of φ_i. Then, as in [4], the *approximation of the geometry* is performed by computing the coefficients A_{IJ}, B_{IJL} and C_{IJKL} with the functions φ_{ih} instead of φ_i. Thus, we get the coefficients A^h_{IJ}, B^h_{IJL} and C^h_{IJKL} defining the new approximate forms

$$\begin{aligned} a_h(u,v) &= \sum_{I,J=1}^{12} \int_\Omega A^h_{IJ}\, \tilde{u}_I\, \tilde{v}_J\, d\xi, \qquad b_h(u,v,w) = \sum_{I,J,K=1}^{12} \int_\Omega B^h_{IJK}\, \tilde{u}_I\, \tilde{v}_J\, \tilde{w}_K\, d\xi, \\ c_h(u,v,u',v') &= \sum_{I,J,K,L=1}^{12} \int_\Omega C^h_{IJKL}\, \tilde{u}_I\, \tilde{v}_J\, \tilde{u}'_K\, \tilde{v}'_L\, d\xi. \end{aligned}$$

To approximate the components of the displacement, we consider three spaces of finite elements $\{V_{hi}\}_{h>0}$, $1 \le i \le 3$ with $V_{h1} = V_{h2}$, such that we have $P_k(K) \subset P_K \subset H^1(K)$ and $P_l(K) \subset Q_K \subset H^2(K)$, $\forall K \in \cup_{h>0}\mathcal{T}_h$, for given integers $k \ge 1$ and $l \ge 2$, where P_K and Q_K denote local interpolation spaces associated respectively to V_{h1} and V_{h3}. We now define the space $V_h := V_{h1} \times V_{h2} \times V_{h3}$ and *we shall assume that* $V_h \subset V$.

The approximate problem associated to (4) is:

$$\text{(6)} \qquad \text{find } u_h \in V_h, \text{ such that } a_h(u_h, v_h) + g_h(u_h, v_h) = \lambda f(v_h), \quad \forall v_h \in V_h,$$

where we have noted: $g_h(u,v) := b_h(u,v,u) + \frac{1}{2}b_h(u,u,v) + \frac{1}{2}c_h(u,u,u,v)$.

The numerical analysis is performed as in [8], meaning with the implicit function theorem using h as the small parameter. For this, we assume a classical regularity of the solution $Tw := u$ of the linear problem: $\{\ u \in V$ and $a(u,v) =< w, v >, \quad \forall v \in V\ \}$ (in fact, this assumption concerns the map $w \mapsto Tw$). In this way we show the local existence and uniqueness of a family of discrete regular paths $\{[-t_c, t_c] \ni t \mapsto (\lambda_h(t), u_h(t)),\ \text{for } 0 < h < h_0\}$ which converges uniformly on $[-t_c, t_c]$ to the buckling path $[-t_c, t_c] \ni t \mapsto (\lambda(t), u(t))$ and satisfying:

$$\forall t \in [-t_c, t_c] \ \ a_h(u_h(t) - u_c, w_c) = t \text{ and } a_h(u_h(t), v_h) + g_h(u_h(t), v_h) = \lambda_h(t)\, f(v_h),\ \forall v_h \in V_h.$$

Otherwise, there exists a constant C independent of h such that, we have the error estimate:

$$\sup_{|t| \le t_0} \{|\lambda(t) - \lambda_h(t)| + \|u(t) - u_h(t)\|_V\} \le Ch^\sigma,$$

where σ is an integer depending of k, l, m and of the quality of the map $w \mapsto Tw$. At last, if the snap point (λ_c, u_c) is non degenerate (a *turning point*), for each $h < h_0$ there exists a unique non degenerate snap point(λ_h^c, u_h^c) belonging to the discrete path.

4. References

1. Bernadou, M., *Méthodes des éléments finis pour les problèmes de coques minces*, Recherches en Mathématiques Appliquées, Masson (1994)
2. Bernadou, M. & Oden, J. T., *An existence theorem for a class of nonlinear shallow shell problems*, J. Math. Pures Appl., **60** (1981), 285–308
3. Brezzi, F., Rappaz, J. & Raviart, P. A., *Finite dimensional approximation of nonlinear problems*, Part II: *Limit points*, Numer. Math. **37** (1980), 1–28
4. Ciarlet, P. G., *The finite element method for elliptic problems*, Studies in mathematics and its applications, North-Holland (1978)
5. Fujji, H. & Yamaguti, M. *Structure of singularities and its numerical realization in nonlinear elasticity*, Math. Kyoto Univ., **20**, (1980) 489–590
6. Koiter, W.T., *On the nonlinear theory of thin elastic shells*, Proc. Kon. Nederl. Akad. Wetensch. **B69** (1966) 1–54
7. Ould Ely Telmoudy, E., *Étude qualitative et quantitative de problèmes de coques élastiques non linéaires*, Thèse de doctorat, université Joseph Fourier, Grenoble (1994)
8. Paumier, J.-C., *Bifurcations et méthodes numériques, applications aux problèmes semi-linéaires elliptiques*, to appear in: Recherches en Mathématiques Appliquées, Masson (1997)

ON THE FINITE ELEMENT APPROXIMATION OF PLATE AND SHELL BOUNDARY LAYERS

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ABSTRACT

We list some of the main issues that arise when approximating boundary or interior layers of plate/shell deformations by finite elements. In shell problems, the main issue is the numerical locking of low-order elements at layers with wide ranges. On the other hand, if the strain reduction approach is taken, there arises another enemy in the form of uncontrolled consistency errors — a layer-related phenomenon that may be seen even in plate bending problems. So far, raising the order within the standard FE framework appears the most reliable approach for shells. At layers, however, careful mesh design is still needed, as demonstrated here by an example.

1. Introduction

In linear deformation states of plates and shells, the displacement field usually consists of a (more or less) smooth part and a layer part which decays exponentially, in some characteristic length scale depending on the thickness (d) of the body, from certain points and lines. In both plates and shells, layers occur at the boundaries and at lines or points where the load is irregular. Moreover, in parabolic and hyperbolic shells, layers occur also at characteristic lines passing through the irregularity of the load or boundary line.

In plate problems, layers appear only in the short length scale proportional to d. The layer is present in the actual 3D displacement field, as well as in dimension reduction models except the asymptotic Kirchhoff model. In shells this layer is present as well, but here much stronger layers arise from curvature effects. These vary in length scales $l \sim d^{1/m}$ where $m \in \{2, 3, 4\}$ (depending on case), so that layers have much wider range than plate layers. They also have strong amplitudes typically, so their numerical capturing can be as important, sometimes even more important than the accurate approximation of the smooth part of the displacement/stress field.

2. Layer approximation by FEM: The main issues

Finite element analysis of plates and shells is often based on the Reissner-Mindlin plate model or the analogous (Naghdi) model for shells. Here the energy to be minimized takes the form

$$F(u) = d^{-2}a(u,u) + b(u,u) - q(u), \tag{1}$$

where the first (quadratic) term represents the deformation energy due to transverse share and membrane deformation, the second term is the bending defromation energy, and the last term is the linear load potential. When the finite element approach is taken, the main issue is the possible error growth as $d \to 0$, a phenomenon known as shear-membrane locking.

To prevent, os at least reduce, the FE error growth as $d \to 0$, there are basically two strategies. First, one may use elements of relatively *high order*, say cubic or quartic at least, to reduce the error growth. At practical (positive) values of the thickness, this turns out to be a rather efficient approach (For the reasoning, see [2]). This is also a "no tricks" approach in the sense that the energy is minimized just as given. Another, historically more popular approach has been to maintain low orders but to modify the energy before minimizing it. In practise this is done by imposing various *strain reduction* operators on shear and membrane strains. This has been a very successful approach in plate bending problems (c.f. [4] and the references therein) but less so in shell problems.

Whichever strategy is chosen, some basic numerical issues have to be confronted when it comes to approximation of layers.

1 *Mesh refinement.* In order to numerically capture a layer with a small range, local mesh refinement becomes necessary, for obvious approximation theory reasons. In shell problems, proper mesh design for layer capturing is important even if the high-order approach is taken. Here mesh alignment is also important [1].
2 *Locking in layer approximation.* In the FE approximation of a layer of length scale $l = l(d)$, error amplification by factor l/d arises when the "no tricks" approach is taken [2]. This is analogous to the known locking effects encountered in the approximation of smooth bending-dominated deformations. In shell problmes, reduction operators that would simultaneously prevent the error growth for all smooth and layer components of possible deformations very likely do not exist. If so, then problem dependent (possibly adaptive) strain reduction techniques remain the only chance within the low-order framework.

If the low-order strain-reduction approach is anyhow taken, there arises still another numerical difficulty. So far this has been mostly overlooked in the mathematical error analysis, though, it is present even in plate bending problem [4]. In shells, this issue is perhaps the main obstacle of strain reduction approaches.

3 *Strain reduction consistency errors.* In both plate and shell problmes, strain reductions are usually designed so that locking is avoided in the smooth bending-dominated part of the solution. This may leave layer locking problems invariant, but actually a more severe problem arises: Strain reductions cause consistency errors that may be very difficult to control, especially in shell problems. In general, when a strain γ_α is replaced by a numerically reduced strain, say $R_h\gamma_\alpha$, there arises a consistency error, the leading part of which can be estimated as the norm of the linear functional

$$< \phi, v >= (\sigma_\alpha(u), (I - R_h)\gamma_\alpha(v)), \tag{2}$$

over the finite element space. Here $(\cdot, \cdot)$ is the L_2 inner product over the the plate/shell midsurface and σ_α is a stress component. We may interpret σ_α as the Lagrange multiplier that imposes the constraint $\gamma_\alpha = 0$ at $d = 0$. In the presence of boundary layers (at $d > 0$), problmes now arise since the layer in σ_α can be very strong. In fact, it can be so strong that the layer part of σ_α does not converge to zero as $d \to 0$, even if the layer in the displacement field vanishes asymptotically. As a result, the finite element scheme may then suffer from layer-related consistency errors even at $d = 0$, where the displacement field often carries no sign of the layer. In [4], this effect is demonstrated in a plate bending problem when the boundary of the plate is free. (The effect does not arise if the boundary is clamped.) In shell problems, layer-related consistency errors seem indeed very difficult to control.

3. Approximation of a shell layer: Example

We consider the benchmark problem of a "pinched" hyperbolic shell introduced in [3]. Here a shell of revolution, with both ends clamped, is loaded by two opposite, concentric point loads. Using the symmetries and natural curvilinear coordinates we reduce the computational problem to a rectangle $\omega = \{(x,y) \,|\, 0 < x < 1,\ 0 < y < \pi/2\}$, where the load acts at the origin. (For more details of the setup, see [3].) Since the shell is hyperbolic, two characteristic lines pass through the origin, one of which is seen on the reduced domain ω. In the geometry assumed, the characteristic is nearly straigth on ω and tangent to the line $y = x$ at the origin.

When the shell is loaded, the deformation field is dominated by strong interior layers decaying away from the characteristic lines (in the length scale $l \sim d^{1/3}$). We choose $d = 1/100$ and try to capture the layer first by a uniform grid obtained by subdividing ω into 30 rectangles and these further into triangles by lines parallel either with $y = \pi x/2$ (NE) or $y = -\pi x/2$ (NW) so that we end up with 60 triangular elements. Apparantely the north-east (NE) option gives better grid alignment for layer capturing in this case. This is also clearly seen in the numerical results (Figs. 1(a), (b)). In Fig. 1(c) we have chosen an even better, locally refined and aligned layer-capturing grid with only 44 triangular elements [1]. The computations were based on standard finite elements (i.e., no strain reductions) of degree $p = 6$. The example demonstrates that in shell problmes, just raising the degree is not always sufficient alone. When capturing layers, good mesh design is needed as well. Further examples of this kind are given in [1].

4. References

1. H. Hakula, *High-Order Finite Element Tools for Shell Problems*, (Doctoral Thesis, Helsinki University of Technology, 1997).
2. H. Hakula, Y. Leino and J. Pitkäranta, *Comput. Methods Appl. Mech. Engrg.*, **133** (1996) 157-182.
3. H. Hakula and J. Pitkäranta, in *Proc. of the Third International Conference on Spectral and High-Order Methods (ICOSAHOM95)* ed. A.V. Ilin and L.R. Scott (1995) 193-201.
4. J. Pitkäranta and M. Suri, *Numer. Math.* **75** (1996) 223-266.

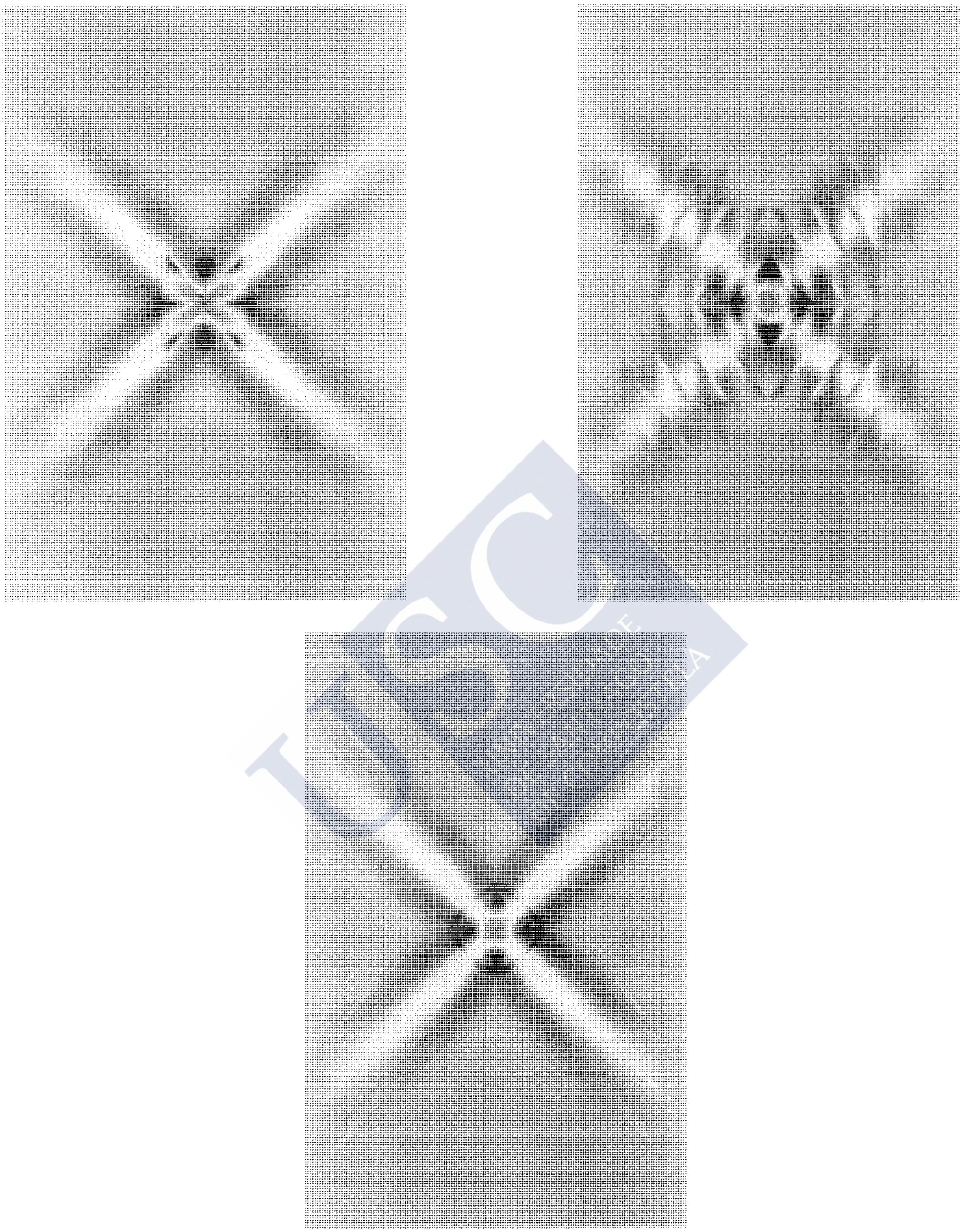

Figure 1: Capturing a layer in the deformation of a hyperbolic shell using high-order elements: From top left (a) Uniform mesh (NE), (b) Uniform mesh (NW), (c) Layer capturing mesh. Here shown: $\omega = [-1, 1] \times [-\pi/2, \pi/2]$

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STABILIZATION OF A PLATE EQUATION WITH DYNAMICAL BOUNDARY CONTROL

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ABSTRACT We consider a system composed of a plate equation and two ordinary differential equations. We prove that the system is not uniformly stable with the usual boundary feedbacks. Using a new approach, we show that the smooth solution has a rational decay rate. Finally we establish the uniform energy decay rate for a simplified system.

1. Introduction

The purpose of this work is to investigate the stabilization of a thin elastic plate rimmed along one part of the edge with a flange that has mass and moment of inertia of the boundary. Let $\Omega \subset R^2$ denote a bounded open set, with smooth boundary Γ consisting of two disjoint pieces : Γ_0 the clamped part, and Γ_1 the rimmed part whereon are applied the boundary controls. The vibration y of the plate is governed by the plate equation associated with two dynamical boundary conditions

$$\begin{cases} y'' + \Delta^2 y = 0 & \text{in } \Omega\times]0,+\infty[, \\ y = \partial_\nu y = 0 & \text{on } \Gamma_0\times]0,+\infty[, \\ J\partial_\nu y'' + \Delta y + (1-\mu)B_1 y = m & \text{on } \Gamma_1\times]0,+\infty[, \\ \rho y'' - \partial_\nu \Delta y - (1-\mu)\partial_\tau B_2 y = f & \text{on } \Gamma_1\times]0,+\infty[, \end{cases} \tag{1}$$

where $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector and $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector, $0 < \mu < 1/2$ is the Poisson's coefficient, $\rho > 0$ is the boundary density and $J > 0$ is the bending moment of inertia of the boundary. The boundary operators B_1, B_2 associated with the plate equation are defined by

$$\begin{cases} B_1 y = 2\nu_1\nu_2\partial_{12}y - \nu_1^2\partial_{22}y - \nu_2^2\partial_{11}y, \\ B_2 y = (\nu_1^2 - \nu_2^2)\partial_{12}y + \nu_1\nu_2(\partial_{11}y - \partial_{22}y). \end{cases} \tag{2}$$

The boundary feedbacks are chosen as

$$m = -Ly', \qquad\qquad f = -L\partial_\nu y' \tag{3}$$

where L is the canonical isomorphism from $H^{-s}(\Gamma_1)$ onto $H^s(\Gamma_1)$ defined as

$$\langle L\xi, \zeta\rangle_{H^s(\Gamma_1)\times H^{-s}(\Gamma_1)} = \langle \xi, \zeta\rangle_{H^{-s}(\Gamma_1)} \qquad s \geq 0. \tag{4}$$

Because of the derivatives of high order, such that $J\partial_\nu y''$ and $\rho y''$, the last two dynamical boundary conditions of the system (1) have to be treated as two ordinary differential equations.

Therefore, we have indeed a system made up of one partial differential equation and two ordinary differential equations, called a hybrid system.

For this kind of problems, a method developed in [4] and [5] is based on the spectrum of the system. Roughly speaking, if the spectrum approaches asymptotically the imaginary axis, then the system loses the uniform energy decay rate. Further, if the eigenvectors form a Riesz basis, then the smooth solution has a rational decay rate. Obviously, this method is limited to one-dimensional problems.

In [8], we introduced a compact perturbation method, which is based on a result of Russell [9]. Unlike the usual method, this method does not need any knowledge of the spectrum of the system. We will apply this method to the hybrid system (1)-(4).

2. Lack of Uniform Stabilization

Let y a smooth solution of the hybrid system (1)-(4). Setting

$$z = y', \qquad \eta = \partial_\nu y'|_{\Gamma_1}, \qquad \xi = y'|_{\Gamma_1} \tag{5}$$

we transform formally the hybrid system (1)-(4) into a system of first order

$$\begin{pmatrix} y \\ z \\ \eta \\ \xi \end{pmatrix}' + \begin{pmatrix} -z \\ \Delta^2 y \\ \frac{1}{J}(\Delta y + (1-\mu)B_1 y) \\ -\frac{1}{\rho}(\partial_\nu \Delta y + (1-\mu)\partial_\tau B_2 y) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J} L\eta \\ \frac{1}{\rho} L\xi \end{pmatrix} = 0 . \tag{6}$$

Denoting the colums by u', Au and Bu, we get an evolutionary equation

$$u' + Au + Bu = 0, \qquad u(0) = u_0. \tag{7}$$

The energy space

$$H = \{u = (y, z, \eta, \xi) \in H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)\}$$

is equiped with the following usual norm

$$\|u\|_H^2 = \int_\Omega \left(a(y,y) + |z|^2\right) dx + J \int_{\Gamma_1} |\eta|^2 d\Gamma + \rho \int_{\Gamma_1} |\xi|^2 d\Gamma$$

where we have put

$$a(y,y) = (\partial_{11} y)^2 + (\partial_{22} y)^2 + 2\mu \partial_{11} y \partial_{22} y + 2(1-\mu)(\partial_{12} y)^2. \tag{8}$$

Next, we introduce the linear operators

$$Bu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J} L\eta \\ \frac{1}{\rho} L\xi \end{pmatrix}, \qquad Au = \begin{pmatrix} -z \\ \Delta^2 y \\ \frac{1}{J}(\Delta y + (1-\mu)B_1 y) \\ -\frac{1}{\rho}(\partial_\nu \Delta y + (1-\mu)\partial_\tau B_2 y) \end{pmatrix}$$

where the domain of A is defined by

$$D(A) = \begin{pmatrix} u = (y, z, \eta, \xi) \in H^2_{\Gamma_0}(\Omega) \times H^2_{\Gamma_0}(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1) \\ \eta = \partial_\nu z|_{\Gamma_1}, \qquad \xi = z|_{\Gamma_1}, \qquad \Delta^2 y \in L^2(\Omega) \\ \Delta y + (1-\mu)B_1 y \in L^2(\Gamma_1), \quad \partial_\nu \Delta y + (1-\mu)\partial_\tau B_2 y \in L^2(\Gamma_1) \end{pmatrix}.$$

We prove easily that A is maximal monotone and B is continuous and monotone in the energy space H. Therefore, the equation (7) is wellposed in the sense of semigroups of contractions. Moreover if $s > 0$, then using the compact perturbation method [2, 8] we can prove the following result.

Theorem 1. *Assume that $s > 0$. Then the energy $E(t)$ of the hybrid system (1)-(4) has no uniform energy decay rate.*

The abovbe result improves an earlier result of Markus-You in [6]. On the other hand, if $s = 0$ then L becomes the identity of $L^2(\Gamma_1)$, and the control operator B is not compact. Therefore we don't know if Theorem 1 remains valid in that case. Nevethelses let us consider an example where Ω is the unit disc, and Γ_1 is the whole circle. Considering the radial solutions of the system (1)-(4), then the control space $L^2(\Gamma_1) \times L^2(\Gamma_1)$ will be reduced into R^2 and the control operator B becomes again compact. Consequently, the system (1)-(4) actually losses the uniform energy decay rate, even in this geometrically favorable case ([3]). Of course, the uniform stability of the system (1)-(4) remains an open problem in the general case.

3. Energy Decay Rates

¿From now on, we assume that there exists a point $x_0 \in \mathbf{R}^2$ such that, setting $m = x - x_0$, we have :

$$\Gamma_0 = \{x \in \Gamma \ : \quad (m \cdot \nu) \leq 0\}, \qquad \Gamma_1 = \{x \in \Gamma \ : \quad (m \cdot \nu) > 0\}. \tag{9}$$

Theorem 2. *Assume that $s = 0$. Then given any $u_0 \in D(A)$, there exists a constant $M > 0$ depending only on u_0 such that the following rational energy decay rate holds*

$$E(t) \leq E(0)\frac{2M}{M+t} \qquad \forall t > 0 \tag{10}$$

for all smooth solution u of the system (1)-(4).

For most of linear problems, we use the classical multiplier $m \cdot \nabla y$. The idea of the proof of Theorem 2 consists in taking the multiplier $E(t) m \cdot \nabla y$ which is usually used in nonlinear problems.

There is another natural approach based on the spectral theory ([5]). Roughly speaking, let $\lambda_n = \alpha_n + i\beta_n$ be the eigenvalues of the operator $A+B$. Assume that (i) $\alpha_n \geq 1/n^p$ $(p > 0)$, (ii) the associated eigenvectors ϕ_n form a Riesz basis, then the trajectory $S_{A+B}(t)u_0$ has a rational decay rate for smooth initial data u_0. This method was applied to one-dimensional problems such as the Euler-Bernoulli beam model with very careful calculation of the eigenvalues [5]. It seems to be impossible to justify the conditions (i) and (ii) for the plate model of general shape. Here our multiplier method is very simple and can be easily applied to other problems.

Now setting $J = 0$ and $\rho > 0$ in the system (1)-(4), we obtain a simplified hybrid system

$$\begin{cases} y'' + \Delta^2 y = 0 & \text{in } \Omega \times]0, +\infty[, \\ y = \partial_\nu y = 0 & \text{on } \Gamma_0 \times]0, +\infty[, \\ \Delta y + (1-\mu)B_1 y = -\partial_\nu y' & \text{sur } \Gamma_1 \times]0, +\infty[, \\ \rho y'' - \partial_\nu \Delta y - (1-\mu)\partial_\tau B_2 y = -y' & \text{sur } \Gamma_1 \times]0, +\infty[. \end{cases} \tag{11}$$

In this case we define the energy of system (11) $E(t)$ by

$$E(t) = \frac{1}{2} \int_{\Omega} \left(a(y,y) + |y'|^2\right) dx + \rho \int_{\Gamma_1} |y'|^2 d\Gamma.$$

Then using the multiplier method ([1]), we can prove the following uniform stability result.

Theorem 3. *There exist two positive constants M and ω such that*

$$E(t) \leq ME(0)e^{-\omega t} \qquad \forall t > 0 \tag{12}$$

for any solution u of the equation (11).

We have shown that the hybrid system (11) can be uniformly stabilized by means of the usual boundary feedback controls. But if we take $\rho = 0$ and $J > 0$, then contrary to the previous case, the uniform stability of the corresponding hybrid system remains an *open* problem. See also [7] for other uniformly stable hybrid systems.

4. References

1. V. KOMORNIK, *Exact controllability and stabilization. The multiplier method*, Masson John Willey, Paris, 1994.

2. V. KOMORNIK ; B.P. RAO *Stabilisation frontière d'un système d'équations des ondes*, C. R. Acad. Sci. Paris, **320**, Série I, (1995), pp. 833–838.

3. J. E. LAGNESE, *Boundary stabilization of thin plates*, SIAM Publications, Philadelphia, PA 1989.

4. W. LITTMAN ; L. MARKUS, *Stabilization of a hybrid system of elasticity by feedback boundary damping*, Annali di Matematica Pura ed Applicata **152** (1988), pp. 281-330.

5. W. LITTMAN ; L. MARKUS, *Some recent results on control and stabilization of flexible structures*, Proc. COMCON Workshop, Montpellier (1987).

6. MARKUS ; Y. C. YOU, *Dynamical boundary control for elastical plates of general shape*, SIAM J. Control Optim., **31** (1993), pp. 983-992.

7. B. RAO, *Decay estimates of solutions for a hybrid system of flexible structures*, Europen Journal of Applied Mathematics, 4 (1993), 303-319.

8. B.P. RAO, *Uniform stabilization of a hybrid system of elasticity*, SIAM, J. Control Opt., **33** (1995), pp.440-454.

9. D.L. RUSSELL, *Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods*, J. Differential Equations, **19** (1975), pp.344-370.

ASYMPTOTIC CONSISTENCY OF THE POLYNOMIAL APPROXIMATION FOR SLENDER STRUCTURES

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ABSTRACT

A classical way of approximating the three-dimensional displacement of a linearly elastic plate is to construct a polynomial model: the approximated displacement components are assumed to be polynomial with respect to the thickness variable. Similarly, director models are frequently used for approximating large displacements of nonlinearly elastic plates or shells. We study the asymptotic consistency of these models when the thickness goes to zero.

1. Reduced Models for Slender Structures

Models of practical use for thin structures replace the three-dimensional system that describes the behavior of the stucture seen as a body in $\mathbb{R}^3$ by a two-dimensional system. Two standard methods for deriving such models from three-dimensional elasticity are

1) the asymptotic method,

2) the polynomial reduction method in linearized elasticity, or the director method in nonlinear elasticity.

In the former, one searches for an asymptotic behavior of the displacements, or deformations, or stresses, when the thickness goes to zero. In the linearized elasticity framework, complete convergence results can be obtained on the weak formulation of the p.d.e's system. We refer to [6] for a thorough analysis of the method. In the nonlinear elasticity framework, asymptotic formal expansions on (the weak form of) the nonlinear p.d.e's system have been useful in a first attempt to obtain membrane and bending models that are valid for large displacements (see [8]). But, up to our knowledge, rigorous convergence results have been obtained till now only by means of variational tools: the three-dimensional system is written as a minimization problem amenable to a Γ-convergence analysis ([1], [9], [10]). The nonlinear bidimensional energy derived that way partially modifies the formal one.

In the polynomial method (see, *e.g*, [4]) as well as in the director models, one accounts for the smallness of the thickness by assuming that the displacement-vector has a specific structure. Then a model that this simplified displacement satifies must be derived. In the linearized case, classical works in this direction ([11], [14]) describe the procedure as a series of weighted integration through the thickness of the equilibrium equations. It can be interpreted as a projection of the solution on the subspace of displacements that are polynomial with respect to the thickness. Similarly, in the nonlinear framework, Cosserat models assume, roughly speaking, that the three-dimensional position of a point that does not belong to the midsurface can be expressed as the sum of the actual position of a point of the midsurface an of the deformed position of a director vector. For an extensive analysis of general Cosserat models, we refer to [3].

We perform an asymptotic analysis of the polynomial method thus characterizing the minimal polynomial subspaces that are consistent with the Kirchhoff-Love model. The horizontal components of the limit displacement to be recovered are affine and its vertical component is independent of x_3. Nevertheless, our analysis proves that projecting on such a subspace in not correct, and that allowing P_2 horizontal components is mandatory.

Similarly, for the nonlinear system of three-dimensional elasticity written as a minimization problem, and under the hypotheses on the loads that asymptotically lead to a membrane model, we successively restrict the space of admissible deformations to mappings that are independant of the thickness variable, to Cosserat deformations with a director vector constrained to remain of Euclidean norm equal to 1, and to unconstrained Cosserat deformations. We show that the latter only is asymptotically consistent.

Results of Section 2 have been obtained jointly with Jean-Claude Paumier, those of Section 3 have been obtained jointly with Hervé Le Dret.

2. Internal Approximation of the System of Linearized Three-Dimensional Elasticity

We first recall the classical asymptotic procedure used to justify the Kirchhoff-Love plate model. We consider a plate with thickness 2ε and reference configuration $\Omega^\varepsilon := \omega\times]-\varepsilon,\varepsilon[$ made of an elastic material with Lamé constants λ and μ. The plate is clamped along the portion $\Gamma_0^\varepsilon = \gamma_0\times]-\varepsilon,\varepsilon[$ of its lateral boundary and subject to applied body forces with densities $f^\varepsilon = (f_i^\varepsilon) \in L^2(\Omega^\varepsilon)$. Convenient scalings consist in assuming that

$$f_\alpha^\varepsilon(x_1,x_2,\varepsilon x_3) = \varepsilon^2 f_\alpha(x_1,x_2,x_3),\ f_3^\varepsilon(x_1,x_2,\varepsilon x_3) = \varepsilon^3 f_3(x_1,x_2,x_3) \text{ for all } x\in\Omega,$$

and in defining the new unknowns $u_i(\varepsilon), i=1,2,3$, by

$$u_\alpha^\varepsilon(x_1,x_2,\varepsilon x_3) = \varepsilon^2 u_\alpha(\varepsilon)(x_1,x_2,x_3),\ u_3^\varepsilon(x_1,x_2,\varepsilon x_3) = \varepsilon u_3(\varepsilon)(x_1,x_2,x_3) \text{ for all } x\in\Omega.$$

Let $V := (H^1_{\Gamma_0}(\Omega))^3$ where $\Gamma_0 = \gamma_0\times]-1,1[$. Then, $u(\varepsilon)$ solves the variational system

$$\text{Find } u(\varepsilon)\in V \text{ such that } a(\varepsilon)(u(\varepsilon),v) = l(v) \text{ for all } v\in V \qquad (P(\varepsilon,V))$$

where $l(v) = \int_\Omega f_i v_i\,dx$ and where the bilinear form $a(\varepsilon)$ is the sum of three terms with different weights $a(\varepsilon)(u,v) = \frac{1}{\varepsilon^4}a_4(u,v) + \frac{1}{\varepsilon^2}a_2(u,v) + a_0(u,v)$.

Let $V_{KL} := \{v=(v_i); v_3\in H^2_{\gamma_0}(\omega),\ v_\alpha = \eta_\alpha - x_3\partial_\alpha v_3,\ \eta_\alpha\in H^1_{\gamma_0}(\omega)\}$. It was proved in [7] that the sequence $u(\varepsilon)$ converges to a limit u that solves the adimensional Kirchhoff-Love problem

$$\text{Find } u\in V_{KL} \text{ such that } a_0^\star(u,v) = l(v) \text{ for all } v\in V_{KL} \qquad (P_{KL}(V))$$

where

$$a_0^\star(u,v) := \int_\Omega \{2\mu\, e_{\alpha\beta}(u)\, e_{\alpha\beta}(v) + \lambda^\star\, e_{\gamma\gamma}(u)\, e_{\gamma\gamma}(v)\}\,dx, \text{ with } \lambda^\star := \frac{2\mu\lambda}{2\mu+\lambda}.$$

We now perform an internal approximation of problem $(P(\varepsilon,V))$. Let W be a closed subspace of V, that may be finite-dimensional or not. The internal approximation of problem $(P(\varepsilon,V))$ is obtained by projecting its solution $u(\varepsilon)$ on W. In other words, we define $w(\varepsilon)$ as the unique solution of

$$\text{Find } w(\varepsilon)\in W \text{ such that } a(\varepsilon)(w(\varepsilon),v) = l(v) \text{ for all } v\in W. \qquad (P(\varepsilon,W))$$

Cases of practical interest are

i) the finite element approximation. A preliminary version of the following result valid for this specific setting (the approximation space is finite dimensional) can be found in [12].

ii) the polynomial approximation. There, we assume that

$$W = P_m(H^1_{\gamma_0}(\omega)) \times P_m(H^1_{\gamma_0}(\omega)) \times P_n(H^1_{\gamma_0}(\omega)),$$

where $P_k(H^1_{\gamma_0}(\omega)) := \{w : \Omega \mapsto R;\ w(x_1,x_2,x_3) := \sum_{j=0}^{j=k} x_3^j\, w_j(x_1,x_2),\ w_j \in H^1_{\gamma_0}(\omega)\}$.

Our purpose is to identify the limit behavior of the sequence $w(\varepsilon)$ when ε goes to zero. Let Q be the inner projection in $L^2(\Omega)$ on $\bar{D}_3$ where $D_3 := \partial_3 W_3 = \{\partial_3 w_3,\ w_3 \in W_3\}$. We introduce the linear subspace $W_{KL} := V \cap V_{KL}$ of W and a bilinear form $b_0^\star$ on W by

$$b_0^\star(w,v) := \int_\Omega \{2\mu\, e_{\alpha\beta}(w)\, e_{\alpha\beta}(v) + \Lambda^\star\,(e_{\gamma\gamma}(w))\ e_{\gamma\gamma}(v)\}\, dx, \text{ where } \Lambda^\star := \lambda(Id - \frac{\lambda}{2\mu+\lambda}Q).$$

Theorem.

The sequence $w(\varepsilon)$ strongly converges in $(H^1(\Omega))^3$ to w when ε goes to zero, where w is the unique solution of the variational problem

$$\textit{Find } w \in W_{KL} \textit{ such that } b_0^\star(w,v) = l(v) \textit{ for all } v \in W_{KL}. \qquad (P_{KL}(W))$$

For an arbitrary space W, by comparing $(P_{KL}(V))$ and $(P_{KL}(W))$, one easily deduces from this Theorem consistency conditions ensuring that the limits u and w coincide. In case ii), the conditions reduce to $m \geq 1$, $n \geq 2$. Therefore the naive idea of projecting onto $W = P_1(H^1_{\gamma_0}(\omega)) \times P_1(H^1_{\gamma_0}(\omega)) \times P_0(H^1_{\gamma_0}(\omega))$ is not correct. It explains why an ad-hoc change of the Lamé constant can be seen in the engineering literature.

For a treatment of related singular perturbation problems that arise in the asymptotic analysis of plates or shells, we refer to [5]. Let us mention that the above analysis sheds some light on the Reissner-Mindlin model [13]. An important work leading to an obtention of the Reissner-Mindlin model, which also uses some projections on polynomial subspaces, but in the displacement-stress mixed formulation, is [2].

3. A Cosserat Approximation of the Nonlinear Membrane Model

Let us first recall the main result of [9] which consists in a rigorous justification of the nonlinear membrane model. It was first obtained for membranes with planar reference configuration, then extended to shell membranes ([10]). With notations of Section 2, Ω^ε is the reference configuration of a hyperelastic homogeneous three-dimensional structure whose stored energy density is a function W that satisfies appropriate growth and coerciveness conditions. For the sake of brevity, we assume that the structure is clamped along its lateral surface and that it is submitted to the action of dead loading body force densities f^ε. Here, the assumed order of magnitude is $f^\varepsilon(x_1,x_2,\varepsilon x_3) = f(x_1,x_2,x_3)$. In other words the loads are order 1, note that it is shown below that the induced displacements are of order 1 as well. The equilibrium problem of nonlinear elasticity may be formulated as a minimization problem whose unknown is the deformation ϕ^ε defined on Ω_ε. We transport this problem on Ω by the immediate change of unknowns $\phi(\varepsilon)(x_1, x_2, x_3) = \phi^\varepsilon(x_1, x_2, \varepsilon x_3)$. Then, $\phi(\varepsilon)$ minimizes, or almost minimizes as the case may be, the energy

$$I(\varepsilon)(\psi) = \int_\Omega W\Big(\Big(\partial_1\psi\Big|\partial_2\psi\Big|\frac{\partial_3\psi}{\varepsilon}\Big)\Big)\, dx - \int_\Omega f(\varepsilon)\cdot\psi\, dx, \qquad (NL(\varepsilon))$$

over the set

$$Ad(\varepsilon) = \{\psi \in W^{1,p}(\Omega; I\!R^3); \psi(x) = (x_1, x_2, \varepsilon x_3) \text{ on } \partial\omega\times]-1,1[\}.$$

It has been proved in [9] by means of Γ-convergence arguments that the limit points φ of $\phi(\varepsilon)$ are independent of x_3 and solve the bidimensional minimization problem

$$\inf\{2\int_\omega QW_0(\nabla\varphi)\, dx_1 dx_2 - \int_\omega \mathcal{F}\cdot\varphi\, dx_1 dx_2, \text{ with } \mathcal{F}(x_1, x_2) = \int_{-1}^{1} f(x_1, x_2, x_3)\, dx_3, \quad (M)$$

over the set $\{\varphi \in W^{1,p}(\omega; I\!R^3),\ \varphi(x_1, x_2) = (x_1, x_2, 0)^T\}$ on $\partial\omega$. Equation (M) is a nonlinear membrane problem. Computing the membrane energy QW_0 requires two steps: First, $W_0\colon M_{3,2} \to I\!R$ is defined by $W_0(\bar{F}) = \inf_{z\in I\!R^3} W((\bar{F}|z))$, then, QW_0 denotes its quasiconvex envelope.

Keeping in mind Cosserat models, we now perform an internal approximation of the minimization problem $(NL(\varepsilon))$. More precisely, we concentrate on three natural choices and replace the set $Ad(\varepsilon)$ of admissible deformations by one of the following subsets

i) $P_0^\varepsilon = \{\phi^{\varepsilon a}(x_1, x_2, y_3) = \varphi^\varepsilon(x_1, x_2), \varphi^\varepsilon_{|\partial\omega} = id\} \cap W^{1,p}(\Omega; I\!R^3)$,

ii) $P_{1c}^\varepsilon = \{\phi^{\varepsilon a} = \varphi^\varepsilon + y_3\, d^\varepsilon, |d^\varepsilon(x_1, x_2)|_{I\!R^3} = 1, \varphi^\varepsilon_{|\partial\omega} = id, d^\varepsilon_{|\partial\omega} = e_3\} \cap W^{1,p}(\Omega; I\!R^3)$

iii) $P_1^\varepsilon = \{\phi^{\varepsilon a}(x_1, x_2, y_3) = \varphi^\varepsilon(x_1, x_2) + y_3\, d^\varepsilon(x_1, x_2), \varphi^\varepsilon_{|\partial\omega} = id, d^\varepsilon_{|\partial\omega} = e_3\} \cap W^{1,p}(\Omega; I\!R^3)$.

Using tools similar to those of [9], we study the limit behavior of these reduced models. We show, that although the limit deformation of problem $(NL(\varepsilon))$ has been proved to belong to P_0^ε, neither approximation P_0^ε, nor approximation P_{1c}^ε are consistent. On the contrary, the limit problem in φ^ε of the unconstrained approximation P_1^ε coincides with problem (M).

4. References

1. E. Acerbi, G. Buttazzo and D. Percivale, *J. Elasticity* **25**, 1991, 137–148.
2. S.M. Alessandrini, D.N. Arnold, R.S. Falk and A.L. Madureira, to appear in *Proceedings of the Summer Seminar on Plates and Shells, Québec, 1996.*
3. S.S. Antman, *Nonlinear Problems of Elasticity* (Springer-Verlag, New-York, 1994).
4. I. Babuška, J.M. d'Harcourt and C. Schwab, *Math. Model. Sci. Comput.* **1-2**, 1993, 1–30.
5. D. Caillerie and E. Sanchez-Palencia, *Math. Models Methods Appl. Sci.* **5**, 1995, 47–66.
6. P.G. Ciarlet, *Plates and Junctions in Elastic Multi-Structures: An Asymptotic Analysis* (Masson/Springer-Verlag, Paris, 1991).
7. P.G. Ciarlet and P. Destuynder, *J. Mécanique* **18**, 1979, 315–344.
8. D.D. Fox, A. Raoult and J.C. Simo, *Arch. Rational Mech. Anal.* **124**, 1993, 157–199.

9. H. Le Dret and A. Raoult, *J. Math. Pures Appl.* **74**, 1995, 549–578.
10. H. Le Dret and A. Raoult, *J. Nonlinear Sci.* **6**, 1996, 59–84.
11. R.D. Mindlin, *J. Appl. Mech.* **18**, 1951, 31–38.
12. J.-C. Paumier, Technical Report LMC-IMAG, RT 76, 1992.
13. J.-C. Paumier and A. Raoult, in *ESAIM Proceedings: Elasticity, Viscoelasticity and Optimal Control*, ed. J. Baranger, J. Blum and A. Raoult (1997).
14. E. Reissner, *J. Math. and Phys.* **23**, 1944, 184–191.

A 3D-2D MODEL FOR A TURBINE BLADE*

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ABSTRACT

In this work we obtain a limit model for a turbine blade clamped in a 3D-solid using the asymptotic analysis. This model consists on a three-dimensional elasticity problem in the 3D-part and a two-dimensional shallow shell problem in the 2D-part (the turbine blade), with junction conditions in the clamped zone of the shallow shell.

1. Formulation of the problem

This work justifies, in a rigorous mathematical way, a 3D-2D coupled model for a turbine blade under high centrifuge and pressure forces. In order to do that, we will suppose that the turbine blade can be modeled by a *shallow shell* as they are defined in Ciarlet and Miara[4] and, as in [2] and [3], we will suppose that the shallow shell is clamped into a massif solid, in such a way that a portion of the shallow shell is "introduced" into it.

We part from the linearly elasticity three-dimensional equations in its variational formulation in the "composed" solid. We define the shallow shell so that its thickness is given by a parameter ε and we make this parameter tend to zero, so we will obtain a limit model by asymptotic analysis.

Let $\mathcal{S}^\varepsilon = \mathcal{O} \cup \hat{\Omega}^\varepsilon \subset \mathbb{R}^3$ be the three-dimensional solid composed by the turbine blade $\hat{\Omega}^\varepsilon$ and the massif solid $\mathcal{O}$. Let us suppose that the turbine blade is clamped into $\mathcal{O}$ in the following way: $\hat{\Omega}^\varepsilon_\beta = \mathcal{O} \cap \hat{\Omega}^\varepsilon \neq \emptyset$. We suppose that $\hat{\Omega}^\varepsilon$ is a shallow shell in the sense of [4], that is, it can be described in the following way (we profit to define other sets that we will use later):

$$
\begin{aligned}
&\omega \subset \mathbb{R}^2, \quad \Omega = \omega \times]-1,1[, \quad \Omega^\varepsilon = \omega \times]-\varepsilon,\varepsilon[,\\
&\theta : (x_1,x_2) \in \overline{\omega} \longrightarrow \mathbb{R}, \quad \theta \in C^3(\overline{\omega}), \quad \theta^\varepsilon(x_1,x_2) = \varepsilon\theta(x_1,x_2),\\
&\alpha^\varepsilon(x_1,x_2) = 1 + \mid \partial_1\theta^\varepsilon \mid^2 + \mid \partial_2\theta^\varepsilon \mid^2, \quad \hat{d}^\varepsilon(x_1,x_2) = (\alpha^\varepsilon)^{-1/2}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1),\\
&\pi^\varepsilon : \mathbf{x} \in \overline{\Omega} \longrightarrow \mathbf{x}^\varepsilon = \pi^\varepsilon(\mathbf{x}) = (x_1,x_2,\varepsilon x_3) \in \overline{\Omega}^\varepsilon,\\
&\Theta^\varepsilon : \mathbf{x}^\varepsilon \in \overline{\Omega}^\varepsilon \longrightarrow \hat{\mathbf{x}}^\varepsilon = \Theta^\varepsilon(\mathbf{x}^\varepsilon) \in \overline{\Theta^\varepsilon(\Omega^\varepsilon)},\\
&\Theta^\varepsilon(\mathbf{x}^\varepsilon) = (x_1,x_2,\theta^\varepsilon(x_1,x_2)) + x_3^\varepsilon \hat{d}^\varepsilon(x_1,x_2),\\
&\hat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon), \quad \omega_\beta = \{(x_1,x_2) \in \omega : (x_1,x_2,0) \in \mathcal{O}\},\\
&\omega^* = \omega - \overline{\omega}_\beta, \quad \gamma^* = \overline{\omega}^* \cap \overline{\omega}_\beta, \quad \mathcal{O}^\varepsilon_\beta = \mathcal{O} - \overline{\hat{\Omega}^\varepsilon_\beta}.
\end{aligned}
$$

As we had said, we part from three-dimensional variational formulation of the linear

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elasticity problem on $\mathcal{S}^\varepsilon$, that is, we look for the solution $\hat{\mathbf{u}}^\varepsilon \in V^\varepsilon$ of the problem

$$\begin{array}{l} \int_{\mathcal{S}^\varepsilon} \left\{ \hat\lambda^\varepsilon \hat e^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon) \hat e^\varepsilon_{qq}(\hat{\mathbf{v}}^\varepsilon) + 2\hat\mu^\varepsilon \hat e^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)\hat e^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon) \right\} d\hat{\mathbf{x}}^\varepsilon \\ \quad = \int_{\mathcal{S}^\varepsilon} \hat f^\varepsilon_i \hat v^\varepsilon_i \, d\hat{\mathbf{x}}^\varepsilon + \int_{\hat\Gamma^\varepsilon_+} \hat g^\varepsilon_i \hat v^\varepsilon_i \, d\hat a^\varepsilon, \qquad \forall\, \hat{\mathbf{v}}^\varepsilon \in V^\varepsilon, \end{array} \tag{1}$$

where $V^\varepsilon = \left\{ \hat{\mathbf{v}}^\varepsilon \in H^1(\mathcal{S}^\varepsilon)^3 : \hat{\mathbf{v}}^\varepsilon = 0 \text{ on } \Gamma_0 \right\}$, $\hat e^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon) = \frac{1}{2}(\hat\partial^\varepsilon_j \hat v^\varepsilon_i + \hat\partial^\varepsilon_i \hat v^\varepsilon_j)$, $\hat\lambda^\varepsilon$ and $\hat\mu^\varepsilon$ are the Lamé coefficients of the material, Γ_0 is the part of $\mathcal{O}$ that is fixed, $\hat\Gamma^\varepsilon_+$ is the part of the surface of the turbine blade $\hat\Omega^\varepsilon$ "exposed" to the pressure $\hat{\mathbf{g}}^\varepsilon$, and $\hat{\mathbf{f}}^\varepsilon$ is the force centrifuge applying on $\mathcal{S}^\varepsilon$. We also use the repeated index summation convention, where latin indexes take values in set $\{1,2,3\}$ and greek indexes take values in set $\{1,2\}$, as in the following example: $x_i x_i = x_1^2 + x_2^2 + x_3^2$, $x_\alpha x_\alpha = x_1^2 + x_2^2$.

The following step is to change from formulation (1) to a formulation in a reference domain. An usual technic (see [2], [3], [5]) consist on consider two different reference domains, one for $\mathcal{O}$ and another for $\hat\Omega^\varepsilon$. In this way, let us consider $\tilde{\mathcal{O}} = \mathcal{O} + \mathbf{r}$, where $\mathbf{r} \in \mathbb{R}^3$, as the reference domain for $\mathcal{O}$, and let us consider Ω as the reference domain for $\hat\Omega^\varepsilon$. We choose $\mathbf{r}$ such that $\tilde{\mathcal{O}} \cap \Omega = \emptyset$.

In order to obtain a variational formulation in domains independent of the parameter ε, we will apply to (1) the following changes of variable

$$\mathbf{x} \in \Omega \leftrightarrow \hat{\mathbf{x}}^\varepsilon = \Theta^\varepsilon(\pi^\varepsilon(\mathbf{x})) \in \hat\Omega^\varepsilon, \qquad \hat{\mathbf{x}}^\varepsilon \in \mathcal{O} \leftrightarrow \tilde{\mathbf{x}} = \hat{\mathbf{x}}^\varepsilon + \mathbf{r} \in \tilde{\mathcal{O}}. \tag{2}$$

Let us consider the "scaled" solution

$$\begin{array}{l} \hat u^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 \tilde u_i(\varepsilon)(\tilde{\mathbf{x}}), \quad \hat{\mathbf{x}}^\varepsilon \in \mathcal{O}, \\ \hat u^\varepsilon_\alpha(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(\mathbf{x}), \quad \hat u^\varepsilon_3(\hat{\mathbf{x}}^\varepsilon) = \varepsilon u_3(\varepsilon)(\mathbf{x}), \quad \hat{\mathbf{x}}^\varepsilon \in \hat\Omega^\varepsilon. \end{array} \tag{3}$$

Let us suppose the following asymptotic behaviour of the forces and the Lame's constants:

$$\begin{array}{l} \hat f^\varepsilon_1(\hat{\mathbf{x}}^\epsilon) = \varepsilon^{-\gamma} \tilde f_1(\tilde{\mathbf{x}}), \ \hat f^\varepsilon_3(\hat{\mathbf{x}}^\epsilon) = \varepsilon^{-\gamma} \tilde f_3(\tilde{\mathbf{x}}), \quad \hat{\mathbf{x}}^\epsilon \in \mathcal{O}, \\ \hat f^\varepsilon_1(\hat{\mathbf{x}}^\epsilon) = \varepsilon^{-\gamma} f_1(\mathbf{x}), \ \hat f^\varepsilon_3(\hat{\mathbf{x}}^\epsilon) = \varepsilon^{-\gamma+1} f_3(\mathbf{x}), \quad \hat{\mathbf{x}}^\epsilon \in \hat\Omega^\varepsilon, \\ \hat f^\varepsilon_2(\hat{\mathbf{x}}^\epsilon) = 0, \quad \hat{\mathbf{x}}^\epsilon \in \mathcal{S}^\epsilon, \\ \hat g^\varepsilon_\alpha(\hat{\mathbf{x}}^\epsilon) = \varepsilon^{-\eta+1} \left(g_\alpha \chi(\Gamma_+) \right)(\mathbf{x}), \quad \hat g^\varepsilon_3(\hat{\mathbf{x}}^\epsilon) = \varepsilon^{-\eta} \left(g_3 \chi(\Gamma_+) \right)(\mathbf{x}), \quad \hat{\mathbf{x}}^\epsilon \in \hat\Gamma^\varepsilon_+, \end{array} \tag{4}$$

and

$$\hat\lambda^\varepsilon = \varepsilon^{-t} \hat\lambda, \quad \hat\mu^\varepsilon = \varepsilon^{-t} \hat\mu, \tag{5}$$

where $\gamma > 0, \eta > 0, t > 0$ and $\chi(A)$ is the characteristic function of the set A.

Remark 1 The hypothesis (4) is consistent with the fact that $\mathbf{f}$ and $\mathbf{g}$ are centrifuge and pressure forces, respectively (see [6]). We obtain $\hat f^\varepsilon_2 = 0$ if we suppose that $\mathcal{S}^\varepsilon$ turn about the axis Ox_2.

Now we are able to apply changes (2) and scalings (3)-(5) to the formulation (1). If we make the choice $\eta > 0, t = 4 + \eta, \gamma = 2 + \eta$, we obtain that the solution is a pair $(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)) \in V(\varepsilon)$ such that satisfies the following variational problem

$$\frac{1}{\varepsilon} \mathcal{A}^\varepsilon(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}) + \mathcal{B}^\varepsilon(\mathbf{u}(\varepsilon), \mathbf{v}) = \mathcal{L}^\varepsilon(\mathbf{v}), \qquad \forall\, (\tilde{\mathbf{v}}, \mathbf{v}) \in V(\varepsilon), \tag{6}$$

where

$$\begin{array}{l} V(\varepsilon) = \Big\{ (\tilde{\mathbf{v}}, \mathbf{v}) \in H^1(\tilde{\mathcal{O}})^3 \times H^1(\Omega)^3 : \ \tilde{\mathbf{v}} = 0 \text{ on } \tilde\Gamma_0 \text{ and} \\ \qquad \tilde v_\alpha(\tilde{\mathbf{x}}) = v_\alpha(\mathbf{x}), \varepsilon \tilde v_3(\tilde{\mathbf{x}}) = v_3(\mathbf{x}) \text{ a.e. } \hat{\mathbf{x}}^\varepsilon \in \hat\Omega^\varepsilon_\beta \quad (\tilde{\mathbf{x}} = \hat{\mathbf{x}}^\varepsilon + \mathbf{r}, \hat{\mathbf{x}}^\varepsilon = \Theta^\varepsilon(\pi^\varepsilon(\mathbf{x}))) \Big\}, \end{array} \tag{7}$$

and $\tilde{\Gamma}_0 = \Gamma_0 + \mathbf{r}$. The functionals $\mathcal{A}^\varepsilon(\cdot,\cdot)$, $\mathcal{B}^\varepsilon(\cdot,\cdot)$ and $\mathcal{L}^\varepsilon(\cdot)$ are defined in the following way

$$\begin{aligned}
\mathcal{A}^\varepsilon(\tilde{\mathbf{u}},\tilde{\mathbf{v}}) &= \int_{\tilde{\mathcal{O}}} \left(\chi(\tilde{\mathcal{O}}^\varepsilon_\beta) + \tfrac{1}{2}\chi(\tilde{\Omega}^\varepsilon_\beta)\right) \mathbf{A}e(\tilde{\mathbf{u}}) : e(\tilde{\mathbf{v}})d\tilde{\mathbf{x}} - \int_{\tilde{\mathcal{O}}} \chi(\tilde{\mathcal{O}}^\varepsilon_\beta) \left[\tilde{f}_1\tilde{v}_1 + \tilde{f}_3\tilde{v}_3\right] d\tilde{\mathbf{x}},\\
\mathcal{B}^\varepsilon(\mathbf{u},\mathbf{v}) &= \int_\Omega \left(\tfrac{1}{2}\chi(\Omega_\beta(\varepsilon)) + \chi(\Omega - \overline{\Omega_\beta(\varepsilon)})\right) (1+\varepsilon^2\delta^\sharp(\varepsilon,\theta))\mathbf{A}P^{\theta,\varepsilon}(\mathbf{u}) : P^{\theta,\varepsilon}(\mathbf{v})d\mathbf{x},\\
\mathcal{L}^\varepsilon(\mathbf{v}) &= \int_\Omega (1+\varepsilon^2\delta^\sharp(\varepsilon,\theta))\left[f_1v_1 + f_3v_3\right] d\mathbf{x}\\
&\quad + \int_{\omega\times\{1\}} \chi(\Gamma_+(\varepsilon))(1+\varepsilon^2\delta^\sharp(\varepsilon,\theta))(1+\varepsilon^2B^\sharp(\varepsilon,\theta))\left[\varepsilon^2 g_\alpha v_\alpha + g_3v_3\right] dx_1dx_2,\\
&\qquad \forall\, (\tilde{\mathbf{u}},\mathbf{u}),(\tilde{\mathbf{v}},\mathbf{v}) \in V(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{A}\ \mathbf{a} : \mathbf{b} = \left(\hat{\lambda}\delta_{ij}\delta_{kl} + \hat{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right) a_{ij}b_{ij},\\
&P^{\theta,\varepsilon}_{\alpha\beta}(\mathbf{v}) = e^\theta_{\alpha\beta}(\mathbf{v}) + \varepsilon^2 e^\sharp_{\alpha\beta}(\varepsilon,\theta;\mathbf{v}), \quad P^{\theta,\varepsilon}_{\alpha3}(\mathbf{v}) = \tfrac{1}{\varepsilon}\left(e^\theta_{\alpha3}(\mathbf{v}) + \varepsilon^2 e^\sharp_{\alpha3}(\varepsilon,\theta;\mathbf{v})\right),\\
&P^{\theta,\varepsilon}_{33}(\mathbf{v}) = \tfrac{1}{\varepsilon^2}\partial_3v_3 + \partial_\alpha\theta\partial_\alpha v_3 + b^\sharp_{33}(\varepsilon,\theta)\partial_3v_3 + \varepsilon^2 e^\flat_{33}(\varepsilon,\theta;\mathbf{v}),\\
&e^\theta_{\alpha\beta}(\mathbf{v}) = e_{\alpha\beta}(\mathbf{v}) - \tfrac{1}{2}\left(\partial_\beta\theta\partial_3v_\alpha + \partial_\alpha\theta\partial_3v_\beta\right),\\
&e^\theta_{\alpha3}(\mathbf{v}) = e^\theta_{3\alpha}(\mathbf{v}) = e_{\alpha3}(\mathbf{v}) - \tfrac{1}{2}\partial_\alpha\theta\partial_3v_3, \quad e^\theta_{33}(\mathbf{v}) = e_{33}(\mathbf{v}),\\
&e_{ij}(\mathbf{v}) = \tfrac{1}{2}\left(\partial_jv_i + \partial_iv_j\right),\\
&\tilde{\mathcal{O}}^\varepsilon_\beta = \mathcal{O}^\varepsilon_\beta + \mathbf{r}, \quad \tilde{\Omega}^\varepsilon_\beta = \hat{\Omega}^\varepsilon_\beta + \mathbf{r}, \quad \tilde{\omega}_\beta = \omega_\beta + \mathbf{r},\\
&\Omega_\beta(\varepsilon) = \left(\Theta^\varepsilon\circ\pi^\varepsilon\right)^{-1}\left(\hat{\Omega}^\varepsilon_\beta\right), \quad \Gamma_+(\varepsilon) = \left(\Theta^\varepsilon\circ\pi^\varepsilon\right)^{-1}\left(\hat{\Gamma}^\varepsilon_+\right),
\end{aligned}$$

and functions $\delta^\sharp(\varepsilon,\theta)$, $B^\sharp(\varepsilon,\theta)$, $e^\sharp_{ij}(\varepsilon,\theta;\mathbf{v})$, $e^\flat_{33}(\varepsilon,\theta;\mathbf{v})$ and $b^\sharp_{33}(\varepsilon,\theta)$, can be bounded independently of ε.

2. Obtention of an uniform bound

Our objective in this section is to obtain an uniform bound

$$\|\, \tilde{\mathbf{u}}(\varepsilon)\,\|_{1,\tilde{\mathcal{O}}} + \|\, \mathbf{u}(\varepsilon)\,\|_{1,\Omega} \le C,$$

where C is independent of ε and $\|\cdot\|_{1,\tilde{\mathcal{O}}}$ and $\|\cdot\|_{1,\Omega}$ are, respectively, the usual norms in $H^1(\tilde{\mathcal{O}})^3$ and $H^1(\Omega)^3$.

We do that in two steps. First of all we build a function $(\tilde{\mathbf{w}}^\varepsilon_0, \mathbf{w}^\varepsilon_0) \in V(\varepsilon)$ such that

$$\begin{aligned}
&\mathcal{A}^\varepsilon(\tilde{\mathbf{w}}^\varepsilon_0,\tilde{\mathbf{v}}) = 0, \quad \forall\, \tilde{\mathbf{v}} \in \tilde{V} = \{\tilde{\mathbf{v}} \in H^1(\tilde{\mathcal{O}}) : \tilde{\mathbf{v}} = 0 \text{ on } \tilde{\Gamma}_0\},\\
&|\,(\mathbf{w}^\varepsilon_0)_\alpha\,|_{0,\Omega} \le C, \quad |\,(\mathbf{w}^\varepsilon_0)_3\,|_{0,\Omega} \le C\varepsilon, \quad |\,P^{\theta,\varepsilon}(\mathbf{w}^\varepsilon_0)\,|_{0,\Omega} \le C,
\end{aligned}$$

where C is independent of ε. As second step, we can use then the fact that

$$\frac{1}{\varepsilon}\left(\mathcal{A}^\varepsilon(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{u}}(\varepsilon)-\tilde{\mathbf{w}}^\varepsilon_0) - \mathcal{A}^\varepsilon(\tilde{\mathbf{w}}^\varepsilon_0,\tilde{\mathbf{u}}(\varepsilon)-\tilde{\mathbf{w}}^\varepsilon_0)\right) + \mathcal{B}^\varepsilon(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)-\mathbf{w}^\varepsilon_0) = \mathcal{L}^\varepsilon(\mathbf{u}(\varepsilon)-\mathbf{w}^\varepsilon_0), \quad (8)$$

to obtain the following bound

$$\frac{1}{\varepsilon}\|\,\tilde{\mathbf{u}}(\varepsilon)-\tilde{\mathbf{w}}^\varepsilon_0\,\|^2_{1,\tilde{\mathcal{O}}} + \|\,\tilde{\mathbf{u}}(\varepsilon)\,\|^2_{1,\tilde{\mathcal{O}}} + \|\,\mathbf{u}(\varepsilon)\,\|^2_{1,\Omega} \le C\left(1 + \|\,\mathbf{u}(\varepsilon)\,\|_{1,\Omega}\right), \quad (9)$$

where C is independent of ε. Finally, we deduce that we have the convergences when $\varepsilon \to 0$

$$\tilde{\mathbf{u}}(\varepsilon) \longrightarrow \tilde{\mathbf{u}} \text{ in } H^1(\tilde{\mathcal{O}})^3 - \text{strong}, \qquad \mathbf{u}(\varepsilon) \rightharpoonup \mathbf{u} \text{ in } H^1(\Omega)^3 - \text{weak}, \quad (10)$$

where we must determine the limits $\tilde{\mathbf{u}}$, $\mathbf{u}$.

3. Determination of the limits

Using techniques similar to those applied in [2], [3] and [4] (see [6] for a complete description of how to apply these techniques to this particular case), we determine the limits $\tilde{\mathbf{u}}$ and $\mathbf{u}$.

We can see that $\tilde{\mathbf{u}}$ is the solution of the three-dimensional linear elasticity problem in $\tilde{\mathcal{O}}$:

$$\tilde{\mathbf{u}} \in \tilde{V}, \qquad \int_{\tilde{\mathcal{O}}} \mathbf{A}e(\tilde{\mathbf{u}}) : e(\tilde{\mathbf{v}}) d\tilde{\mathbf{x}} = \int_{\tilde{\mathcal{O}}} \left(\tilde{f}_1 \tilde{v}_1 + \tilde{f}_3 \tilde{v}_3\right) d\tilde{\mathbf{x}}, \quad \forall\, \tilde{\mathbf{v}} \in \tilde{V}. \tag{11}$$

We can also see that $\mathbf{u}$ verifies a.e. $\mathbf{x} \in \Omega$:

$$\begin{array}{l} u_\alpha(x_1,x_2,x_3) = \zeta_\alpha(x_1,x_2) - x_3\partial_\alpha\zeta_3(x_1,x_2), \quad u_3(x_1,x_2,x_3) = \zeta_3(x_1,x_2), \\ \text{with } \ \zeta_\alpha \in H^1(\omega),\ \zeta_3 \in H^2(\omega), \end{array} \tag{12}$$

where $\zeta_{\alpha|\omega_\beta} = \tilde{u}_{\alpha|\tilde{\omega}_\beta}$, $\zeta_{3|\omega_\beta} = 0$.

Function $(\zeta)_{i=1}^3$ is the solution of the usual shallow shell problem in ω^*:

$$\begin{array}{l} -\partial_{\alpha\beta} m^\theta_{\alpha\beta}(\zeta) - \partial_\beta\left(n^\theta_{\alpha\beta}(\zeta)\partial_\alpha\theta\right) = \int_{-1}^1 f_3 dx_3 + g_3 + \partial_1\left(\int_{-1}^1 x_3 f_1 dx_3\right) \text{ in } \omega^*, \\ -\partial_\beta n^\theta_{1\beta}(\zeta) = \int_{-1}^1 f_1 dx_3 \text{ in } \omega^*, \quad -\partial_\beta n^\theta_{2\beta}(\zeta) = 0 \text{ in } \omega^*, \\ \zeta_{\alpha|\gamma^*} = (\tilde{u}_{\alpha|\tilde{\omega}_\beta})_{|\gamma^*}, \quad \zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma^*, \\ (\partial_\alpha m^\theta_{\alpha\beta}(\zeta))\nu_\beta + \partial_\tau(m^\theta_{\alpha\beta}(\zeta)\nu_\alpha\tau_\beta) + n^\theta_{\alpha\beta}(\zeta)\partial_\alpha\theta\nu_\beta \\ \qquad = -(\int_{-1}^1 x_3 f_1 dx_3)\nu_1 \text{ on } \gamma_1^* = \partial\omega^* - \gamma^*, \\ m^\theta_{\alpha\beta}(\zeta)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_1^*, \quad n^\theta_{\alpha\beta}(\zeta)\nu_\beta = 0 \text{ on } \gamma_1^*, \end{array} \tag{13}$$

where

$$\begin{array}{l} m^\theta_{\alpha\beta}(\zeta) = -\left\{\frac{4\hat{\lambda}\hat{\mu}}{3(\hat{\lambda}+2\hat{\mu})}\Delta\zeta_3\delta_{\alpha\beta} + \frac{4}{3}\hat{\mu}\partial_{\alpha\beta}\zeta_3\right\}, \\ n^\theta_{\alpha\beta}(\zeta) = \frac{4\hat{\lambda}\hat{\mu}}{\hat{\lambda}+2\hat{\mu}} e^\theta_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 4\hat{\mu} e^\theta_{\alpha\beta}(\zeta), \\ e^\theta_{\alpha\beta}(\eta) = \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) + \frac{1}{2}(\partial_\alpha\theta\partial_\beta\eta_3 + \partial_\beta\theta\partial_\alpha\eta_3), \end{array}$$

and where $\nu = (\nu_1, \nu_2)$ is the unit normal vector outward to ω^* and $\tau = (\tau_1, \tau_2)$ is the unit tangent.

We remark that equation $\zeta_{\alpha|\gamma^*} = (\tilde{u}_{\alpha|\tilde{\omega}_\beta})_{|\gamma^*}$ establish the relation between the three-dimensional problem and the two-dimensional problem.

4. References

1. P. G. Ciarlet, *Plates and junctions in elastic multi-structures*, (Masson, Paris, 1990).
2. P. G. Ciarlet, H. Le Dret, *Asymptotic Analysis* **2** (1989), 257-277.
3. P. G. Ciarlet, H. Le Dret, R. Nzengwa, *J. Math. Pures Appl.* **68** (1989), 261-295.
4. P. G. Ciarlet, B. Miara, *Comm. Pure Appl. Math.* **45** (1992), 327-360.
5. H. Le Dret, *Problèmes Variationnels dans le Multi-Domaines: Modélisation des Jonctions et Applications*, (Masson, Paris, 1991).
6. J. M. Rodríguez, *Analyse asymptotique des pales de turbines et des poutres à profil mince* (Thèse de Doctorat de l'Université Paris 6, 1997).

APPLICATION OF GLOBAL NUMERICAL PROCEDURES TO THE ANALYSIS OF SHELLS

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Abstract

Examples of global solutions of the shell equations are presented, such as the ones based on the well known Levy series expansion. Also discussed are some natural extensions of the Levy method as well as the inherent limitations of these methods concerning the shell model assumptions, boundary conditions and geometric regularity. Finally, some open additional design questions are noted mainly related to the simultaneous use in analysis of these global techniques and the local methods (like the finite elements) to finding the optimal shell shape, and to determining the reinforcement layout.

1. Introduction

The shallow curved plate theory of linear elastic thin shells is assumed. Details can be seen in [1]. A right hand cartesian axis (x_1, x_2, z) is used to describe the middle shell surface by means the following parametric equations:

$$x_1 = x_1(\alpha_1, \alpha_2); x_2 = x_2(\alpha_1, \alpha_2); z = z(\alpha_1, \alpha_2)$$

The following notation is used:

- Indexes i and j vary between 1 and 2 $(i \neq j)$.
- δ_{ij} is the Kronecker delta.
- Einstein summation convention applies unless the contrary is explicitly stated.
- Comma notation with index i is used to represent partial derivatives respect to α_i.
- X_i, Z are the pressure force components on the middle surface.
- u_i, w are the displacement components at a point of the middle surface.
- n_{ij} and q_i are the force stress-resultants of the stresses σ_{ij} and σ_{iz}.
- m_{ij} are the couple stress-resultants of the stresses σ_{ij}.
- A_{ij} are the coefficients of the first fundamental form of the middle surface. It is assumed $A_{ij} = 0$.
- K_{ij} are the curvatures of the middle surface.
- h is the constant shell thickness, E the Youngs modulus, ν the Poisson ratio, $D = \frac{Eh^3}{12(1-\nu^2)}$ and $K = \frac{Eh}{(1-\nu^2)}$.

The equilibrium equations of a differential shell element are:

$$\begin{aligned} n_{ij,j} + X_i &= 0 \\ m_{ij,j} - q_i &= 0 \\ K_{ij}n_{ij} + q_{i,j} + Z &= 0 \end{aligned}$$

The following main relations between the different shell variables hold:

- Strains/ displacements

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - 2K_{ij}w)$$

$$k_{ij} = -w_{,ij}$$

- Stress-resultants/strains

$$n_{ij} = K[(1-\nu)e_{ij} + \nu\delta_{ij}e_{rr}$$

$$m_{ij} = D[(1-\nu)k_{ij} + \nu\delta_{ij}k_{rr}]$$

- Stress-resultants/ displacements

$$n_{ii} = K[u_{i,i} + \nu u_{j,j} - (K_{ii} + \nu K_j j)w]$$

$$n_{ij} = K[u_{i,j} + u_{j,i} - 2K_{ij}w]$$

$$q_i = -D\nabla^2 w_{,j}$$

$$r_i = q_i + m_{ij,j} = -D[w_{,iii} + (2-\nu)w_{,jjj}]$$

in which r_i are the Kirchhoff shears and $\nabla^2 = \frac{\partial^2}{\partial\alpha_1^2} + \frac{\partial^2}{\partial\alpha_2^2}$.

The governing differential equations expressed in terms of the normal displacement w and the Pücher stress function Φ, defined by the expression (not summed) $n_{ij} = (-)^{i+j}\Phi_{ij} - \delta_{ij}\int X_i d\alpha_i$, are:

$$D\nabla^4 w - \nabla_K^2\Phi = Z - K_{ii}\int X_i d\alpha_i$$

$$\nabla^4\Phi - Eh\nabla_K^2 w = \int X_{i,jj}d\alpha_i + \nu X_{i,i}$$

in which $\nabla_K^2 = K_{ii}\frac{\partial^2}{\partial\alpha_j^2} - K_{ij}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}$.

In the case of $\nabla_K^2 \neq \nabla^2$ the Ambartsuyam function W is normally introduced [3] to obtain the complementary solution by solving a single differential equation:

$$\nabla^8 W + \frac{12(1-\nu^2)}{h^2}\nabla_K^4 W = 0$$

in which $w = \nabla^4 W; \Phi = -Eh\nabla_K^2 W$.

In the other case ($K_{11} = K_{22}; K_{12} = 0$ i.e. spherical shallow curved plate) the Mishonov function is used instead, defined as: $\nabla^2 w = \nabla^2 W; \nabla^2\Phi = -EhK\nabla_K^2 W$, and the same single differential equation as before is reached except the coefficient of second term is now multiplied by K.

2. Description of the Levy solution.

Rectangular planform, curvature lines ($K_{12} = 0$) and normal gable boundary conditions along two opposite edges of the shell are assumed, i.e.:

$$n_{11} = 0; u_2 = 0; w = 0; m_{11} = 0 \text{ along } \alpha_1 = 0, L_1$$

For brevity only the case with different curvatures is shown. The spherical solution $K_{11} = K_1 = K_{22} = K_2$ follows a similar pattern.

The boundary conditions along the two other edges $\alpha_2 = 0, L_2$ can be quite arbitrary.

The solution is expressed by sum of harmonic terms. For each term a vector R of dimension 15x1 containing the results of interest in the analysis is defined as follows:

$$R = (u_1, u_2, w, w_{,1}, w_{,2}, n_{11}, n_{22}, n_{12}, m_{11}, m_{22}, m_{12}, q_1, q_2, r_1, r_2)^T$$

in which each element is a function of α_2 and varies along the direction α_1 as $\sin\lambda\alpha_1$ except the terms 1, 4, 8, 11, 12 and 14 which vary as $\cos\lambda\alpha_1$, $\lambda = n\frac{\pi}{L_1}$ corresponds to the n-th expansion term. The expression of this vector of results is given as sum of a particular and the complementary solutions [2]:

$$R = R_0 + R_c = R_0 + G[C_1 P(\alpha_2) A + C_2 P(\beta_2) B]$$

in which G is a 15×8 matrix shown in table 1. $C_i = [C_1^i, C_2^i]$ is a partitioned matrix of dimension 8×4 and the k-th row of the 8×2 submatrix C_j^i is $\epsilon_{jk}[\rho_i^k \cos k\varphi_i, \rho_i^k \sin k\varphi_i]$ with $\epsilon_{1k} = (-1)^k, \epsilon_{2k} = 1; k = 0, 1, ..., 7, \rho_i = (r_i^2 + s_i^2)^{\frac{1}{2}}$ and r_i and s_i are constants depending on the real and imaginary parts of the roots of the characteristic equation,:

$$r_i = \left[\frac{a_i + \sqrt{a_i^2 + b_i^2}}{2}\right]^{\frac{1}{2}}; s_i = \left[\frac{-a_i + \sqrt{a_i^2 + b_i^2}}{2}\right]^{\frac{1}{2}}$$

$$a_i = \lambda^2 - (-1)^i \mu \sqrt{\frac{-K_1^2 + \Delta}{2}}; b_i = \mu\left[\frac{K_1^2 + \Delta}{2} + (-1)^i K_1 \frac{K_2 - K_1}{|K_2 - K_1|}\right]$$

$$\mu = \frac{\sqrt{3(1-\nu^2)}}{h}; \qquad \Delta = \sqrt{K_1^4 + 4(K_1 - K_2)^2 \frac{\lambda^4}{\mu^2}}$$

The square matrix $P(x) = [P_{ij}(x)], x = \alpha_2, \beta_2 = L_2 - \alpha_2$ of dimension 4 is partitioned in 4 submatrices of dimension 2×2. Each of these submatrices has the following expression:

$$P_{ii}(x) = \begin{bmatrix} p_{i1}(x) & p_{i2}(x) \\ -p_{i2}(x) & p_{i1}(x) \end{bmatrix}; P_{ij}(x) = 0; \quad p_{i1} = e^{-r_i x}\cos s_i x; \quad p_{i2} = e^{-r_i x}\sin s_i x$$

The eight arbitrary constants A_{ij}, B_{ij} are contained in the two 4×1 column matrices

$$A = (A_{11}, A_{12}, A_{21}, A_{22})^T; B = (B_{11}, B_{12}, B_{21}, B_{22})^T$$

These constants can be found by imposing the arbitrary boundary conditions along the edges $\alpha_2 = 0, L_2$

TABLE 1. Matrix $G = [G_1, G_2]$

$$G_1 = \begin{bmatrix}
(-K_1 + \nu K_2)\lambda^3 & 0 & -K_2\lambda + (2+\nu)K_1\lambda & 0 \\
0 & K_1\lambda^2 - (2+\nu)K_2\lambda^2 & 0 & K_2 + \nu K_1 \\
\lambda^4 & 0 & -2\lambda^2 & 0 \\
\lambda^5 & 0 & -2\lambda^3 & 0 \\
0 & \lambda^4 & 0 & -2\lambda^2 \\
0 & 0 & K_2^2\lambda^2 Eh & 0 \\
-K_2\lambda^4 Eh & 0 & K_1\lambda Eh & 0 \\
0 & -K_2\lambda^3 Eh & 0 & K_1\lambda Eh \\
\lambda^6 & 0 & -(2+\nu)\lambda^4 D & 0 \\
\nu\lambda^6 D & 0 & -(1+2\nu)\lambda^4 D & 0 \\
0 & -(1+\nu)\lambda^5 D & 0 & 2(1-\nu)\lambda^3 D \\
\lambda^7 D & 0 & -3\lambda^5 D & 0 \\
0 & \lambda^6 D & 0 & -3\lambda^4 D \\
\lambda^7 D & 0 & -(4-\nu)\lambda^5 D & 0 \\
0 & (2-\nu)\lambda^6 D & 0 & -(5-2\nu)\lambda^4 D
\end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -K_1Eh & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1+2\nu)\lambda^2 D & 0 & -\nu D & 0 \\ (2+\nu)\lambda^2 D & 0 & -D & 0 \\ 0 & -(1-\nu)\lambda D & 0 & 0 \\ 3\lambda^3 D & 0 & -\lambda D & 0 \\ 0 & 3\lambda^2 D & 0 & -D \\ (5-2\nu)\lambda^3 D & 0 & -(2-\nu)\lambda D & 0 \\ 0 & (4-\nu)\lambda^2 D & 0 & -D \end{bmatrix}$$

3. Extensions of the Levy process.

The Levy formulation just described can be extended [2] to treat quite general systems of shallow shell structures, with transversally variable thickness or curvature changes, by introducing the standard methods of matrix analysis of structures, such as the stiffness method, along the transversal shell direction, i.e. the α_2 direction. By a suitable selection of columns and rows of the matrix G and the vector R_0 the two following relations can be reached:

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = d_0 + G_d \begin{bmatrix} A \\ B \end{bmatrix}; \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_0 + G_p \begin{bmatrix} A \\ B \end{bmatrix}$$

in which p_i and d_i are the vectors of the forces and displacements along the border $\alpha_2 = 0$ and $\alpha_2 = L_2$ respectively, i.e., the vectors with components $n_{12}, n_{22}, r_1, m_{22}$ and $u_1, u_2, w, w_{,2}$. By elimination of the constants A and B the stiffness fundamental equation is obtained:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_0 - kd_0 + k \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

with $k = G_pG_d^{-1}$ the 8×8 stiffness matrix.

To treat other boundary conditions than the normal gables in the two opposite edges $\alpha_1 = 0, L_1$ several attempts have been made. The first introduce further approximations in the theory [4], in such a way that the derivatives of order 6 and 2 in the direction α_1 disappear. Some typical approximations are summarized in table 2. In these cases the trigonometric functions $\sin \lambda\alpha_1$ and $\cos \lambda\alpha_1$ used in the Levy process can be replaced by the new orthogonal set of functions in $[0, L_1]$, known as Raleigh functions, $\phi(\alpha_1)$. These functions satisfy more general boundary conditions than the normal gable ones along the two opposite shell edges and they are defined by the eigenvalue problem: $\phi_{,1111} + \lambda^4\phi = 0$ and the general homogenous boundary conditions. In the formulation of the equations of table 2, the static-kinematic analogy of Goldenweizer [5] has been considered.

Table 2. Approximations of the shell equations.

Approximations		General Plate			Shallow Curved Plate		
State	Functions neglected	Name	Order derivatives		Name	Order derivatives	
			α_1	α_2		α_1	α_2
1	None	-	8,6,4,2	8,6,4,2	Donell Jenkins	8,6,4,2	8,6,4,2
2	m_{12}, ϵ_{12}	-	8,4	8,6,4,2	-	8,4	8,4
3	m_{12}, ϵ_{12} m_{11}, ϵ_{22}	Vlasov tvb	4	8,6,4	Schorer	4	8

Another possibility to extend the Levy process has been investigated by Gunasekera [6]. It consists of using three groups of series expansions. The first corresponds to the particular solution and may be a double trigonometric series i.g. the Navier solution. The complementary solution is constructed of two groups of linear independent Levy solutions, along of each the directions α_1 and α_2. The procedure combines suitable sets of Levy solutions with the particular solution to satisfy all the boundary conditions simultaneously. Good convergence has been reported mainly for kinematic boundary conditions.

Finally Michael [7] has developed several strategies to extend the Levy process to more general boundary conditions. The line techniques has been applied to the direct differential equations either in the two variables (normal displacement and stress function) or in the three displacements. Indirect solutions i.e. those using a variational approach or a Raleigh-Ritz method with or without Lagrange multipliers have proven to be efficient even with "difficult" boundary conditions. In table 3 the functions selected in these methods are shown. The suitable selection of these approximating functions is essential for the convergence of the methods.

Table 3.- Approximating functions

Group		Functions
I	A	$\sin m\pi\xi_2$ or $\cos m\pi\xi_2$
	B	$\sin m\pi\xi_2$, $\cos m\pi\xi_2$
	C	$1, (1-2\xi_2)\sin m\pi\xi_2$
	D	$[\cos n\pi\xi_2 - \cos(n+2)\pi\xi_2]$
	E	$[\cos n\pi\xi_2 - \cos(n+2)\pi\xi_2], [\sin m\pi\xi_2 - \frac{m}{m+2}\cos(m+2)\pi\xi_2]$
II	A	Raleigh functions for the clampled-campled case
	B	$1, (1-2\xi_2), F_m$ where F_m are Raleigh functions for the free-fre case
$m = 1, 2, 3, \cdots n = 0, 1, 2, 3, \cdots \xi_2 = \frac{\alpha_2}{L_2}$		

The functions IA, IB, IC and IIB have been used by Chuang and Veletsos, while Noor and Veletsos have used IB and ID. The functions IA, IIA and IIB are orthogonal functions. The functions ID satisfy the clamped boundary conditions but produce zero normal and Kirchhoff shears on the boundary. They have been proposed in [8] when they are referred as *almost orthogonal functions.* The functions IE have been obtained by modifying ID to avoid the zero shears along the boundaries. However, the shape of the subgroups of IE are similar, and numerical difficulties are foreseen if more terms are considered. The functions IIA and IIB have been used with the three displacements formulation of the shell equations. Some essential or kinematic boundary conditions are not satisfied by some groups of these functions and in these cases Lagrange multipliers have been used.

4. Conclusions.

Shown above are several global numerical techniques that have been successively applied during the past for the analysis of shells. However, with the advent of the FEM and related numerical methods, global solutions had diminished use in shell analysis. Nevertheless, in the author's opinion, the simultaneous use of these solution techniques for the regular or smooth part of the shell structure and the application of a discretized model for the borders and the most irregular part of the shell can be a an efficient compromise.

Finally it is important to point out that shell analysis represents only a part of the more comprehensive task of the shell design. Problems of optimal shell shape finding, reinforcement of concrete shells and construction procedures must also be considered.

1. V. Z. Vlasov *General Theory of Shells and its Applications to Engineering* NASA. TTF-99 (1964).
2. A. Q. Samartin and J. Munro Dynamic *Analysis of Translational Shells* CSTR 67/2. Imperial College(London 1967).

3. S. A. Ambartsumyan *On the calculation of shallow shells* NACA. TM-1425 (English translation from Prikjanaya Matematika i Mekhankia, Vol **11** (1947).
4. J. Munro *The Linear analysis of thin shallow shells* Proc. Inst. Civil Engineers, Vol **19**. (1961).
5. A. L. Goldenweizer *Theory of elastic thin shells* Pergamon Press (1961).
6. D. A. Gunasekera *Numerical analysis of thin shells* PhD Thesis University of London (1967).
7. K. C. Michael and J. Munro *Approximating functions and indirect solutions of shell problems* Proc. of the IASS Conference (Mexico, 1967)
8. M. M. Filonenko-Boroditch *On a system of functions and its applications in the theory of elasticity* Prikjanaya Matematika i Mekhankia, Vol 10 (1963).

EXAMPLE OF SENSITIVITY IN SHELLS WITH EDGES

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ABSTRACT

We give several examples of shells with edges which are sensitive, i.e. there are very smooth loadings (belonging to the space $\mathcal{D}$ of test functions of distributions) such that the corresponding solutions go out of the energy space.

1. - Introduction

It is known [1,2] that the limit behaviour (as the thickness ε tends to zero) of thin shells which are geometrically rigid is given by the membrane model. The corresponding energy space V^a is obtained by completion of a space $\mathcal{V}$ of "smooth functions" for the membrane energy norm. The *sensitivity phenomenon* [3, 4, 5] appears when V^a is "very large", going out of the space $\mathcal{D}'$ of distributions. Correspondingly, the dual $V^{a\prime}$ is "small" and does not contain the space $\mathcal{D}$ of test functions of distributions. This implies an instability phenomenon : there are very small and smooth perturbations of the loading such that the corresponding perturbation of the solution goes out of the energy space V^a.

Numerical computation of sensitive problems is practically impossible [6]. This impossibility is only concerned with the limit behavior as the thickness ε of the shell tends to zero. Real shells with $\varepsilon > 0$ are calculable, usually in spaces $H^1 \times H^1 \times H^2$ [7]. The convergence of the finite element schemes hold true for $\varepsilon > 0$, but this concergence can only be uniform with respect to ε in the topology of the "large" space V^a. In any "usual" space the convergence is not uniform. This is the locking phenomenon as defined in [8], which also appears in shells which are not geometrically rigid [9].

In this communication we give several examples of shells with edges. The middle surface S is defined by a function $\mathbf{r}$

$$(y^1, y^2) \to \mathbf{r}(y^1, y^2) \tag{1}$$

from Ω into $\mathbb{R}^3$, where Ω denote a domain of the plane $\mathbb{R}^2$ of the parameters. The function $\mathbf{r}$ is piecewise smooth and everywhere continuous. It may have discontinuities of the first order derivatives along certain curves Γ_2 (the "edges"). It is then supposed that the angle 2ψ of the tangent planes corresponding to both sides of Γ_2 is different from 0 and π. It is a smooth function defined along the edge.

The displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ is referred to the local basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ where $\mathbf{a}_\alpha$ $(\alpha = 1, 2)$ are the tangent vectors associated with the parametrization (1), and $\mathbf{a}_3$ is the unit normal vector, which is discontinuous across the edges.

The shell is fixed (or clamped) by a part Γ_0 of its boundary, and free by the remaineder part Γ_1. The kinematical boundary conditions are

$$u_1 = u_2 = 0 \quad \text{on } \Gamma_0 \tag{2}$$

$$u_3 = 0 \quad \text{on } \Gamma_0 \tag{3}$$

and there are no boundary conditions on Γ_1. Moreover, the transmission condition on the edge Γ_2 are

$$u_2^+ = u_2^- \quad \text{on } \Gamma_2 \tag{4}$$

$$u_3^+ = \frac{cos^2\psi}{sin2\psi}(u_1^+ - u_1^-) - \frac{sin^2\psi}{sin2\psi}(u_1^+ + u_1^-) \tag{5}$$

$$u_3^- = \frac{cos^2\psi}{sin2\psi}(u_1^+ - u_1^-) + \frac{sin^2\psi}{sin2\psi}(u_1^+ + u_1^-) \tag{6}$$

where + and - denote both sides of the edge. Equations (4)-(5) were written in the case when the edge is defined by $y^1 = const.$ and the tangent vectors associated with (1) are orthonormal on Γ_2. We shall see that (3), (5), (6) are irrelevant, thus (4) amonts to the equality of traces of the tangent componentn of $\mathbf{u}$.

Then, $\mathcal{V}$ is defined as the space of the vectors which are piecewise (i.e. under on Γ_2) in $H^1 \times H^1 \times H^2$ and satisfy (2)-(6). The membrane energy norm is

$$\| \mathbf{u} \|_a = \left[\int_\Omega \sum_{\alpha\beta} \| \gamma_{\alpha\beta}(\mathbf{u}) \|_0^2 \right]^{1/2} \tag{7}$$

i.e. the L^2-norm of the strain :

$$\begin{cases} \gamma_{11} \equiv D_1 u_1 - b_{11} u_3 \\ \gamma_{22} \equiv D_2 u_2 - b_{22} u_3 \\ \gamma_{12} \equiv \gamma_{21} = \frac{1}{2}(D_2 u_1 + D_1 u_2) - b_{12} u_3 \end{cases} \tag{8}$$

where $b_{\alpha\beta}$ is the second fundamental form of the surface and $\mathcal{D}$ denotes covariant differentization

$$D_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\lambda_{\alpha\beta} v_\lambda \tag{9}$$

It is then supposed that the surface with the boundary conditions (2)-(6) is geometrically rigid, i.e. $\mathbf{u} \in \mathcal{V}$ and $\gamma_{\alpha\beta}(\mathbf{u}) = 0$, $\alpha, \beta = 1, 2$ implies $\mathbf{u} = 0$. It then follows that (7) is a *norm* on $\mathcal{V}$.

The space V^a is defined as the completion of $\mathcal{V}$ with this norm.

Remark 1.1. - Classicaly, traces do not make sense in L^2. Then, when considering smooth functions, we may modify the traces by adding variations of the functions such that the L^2-norm of the variation is as small as desired. Using this property in the components u_3 of the functions, we see that the boundary conditions (3), (5), (6) which involve u_3 are irrelevant : we obtain the same completion V^a by discarding them in the definition of $\mathcal{V}$.■

The shell problem is said to be sensitive if

$$V^a \not\subset \mathcal{D}'(\Omega_+)^3 \times \mathcal{D}'(\Omega_-)^3 \tag{10}$$

or correspondingly

$$V^{a\prime} \not\supset \mathcal{D}(\Omega_+)^3 \times \mathcal{D}(\Omega_-)^3 \tag{11}$$

where Ω_+ and Ω_- denote the smooth parts of Ω (it may exist more than two).

2. - Examples

There is a criterion of sensitivity [4], [5] to prove (11) by contradiction. If (11) is not satisfied, i.e. if its right hand side is contained into the left hand side, then, for any $\varphi \in \mathcal{D}(\Omega_+)^3 \times \mathcal{D}(\Omega_-)^3$, there exist $T^{\alpha\beta} = (T^{11}, T^{22}, T^{12} = T^{21}) \in L^2(\Omega)^3$ and satisfying

$$-D_\beta T^{\alpha\beta} = \varphi^\alpha \quad (\alpha = 1, 2) \tag{12}$$

$$-b_{\alpha\beta} T^{\alpha\beta} = \varphi^3 \tag{13}$$

on Ω_+ and Ω_-, with the boundary conditions on the "free part" of the boundary

$$n_\beta T^{\alpha\beta} = 0 \quad on\ \Gamma_1 \quad (\alpha = 1, 2) \tag{14}$$

and the transmission conditions on the edge :

$$T^{11+} = T^{11-} = 0 \quad \text{on } \Gamma_2 \tag{15}$$

$$T^{12+} - T^{12-} = 0 \quad \text{on } \Gamma_2. \tag{16}$$

A surface will be sensitive if (12)-(16) cannot have solution $T \in (L^2)^3$ for any $\varphi \in \mathcal{D}(\Omega_+)^3 \times \mathcal{D}(\Omega_-)^3$.

Example A - We consider a surface defined shematically as in fig. A. This figure (as well as B and C) is mostly topological, and deformations are allowed. The parts Ω_- and Ω_+ are hyperbolic and elliptic, respectively. The lines in the hyperbolic region denote the characteristics (= asymptotic curves of S) Rigidity follows from the fact that $\gamma_{\alpha\beta}(u) = 0$ is equivalent (by eliminating u_3) to a first order system in u_1, u_2 which is in each region of the same type as the surface. Moreover, (2) are Cauchy conditions on Γ_0, so that $u = 0$ on Ω_-. Then, (4), (5), (6) give again Cauchy conditions for u on Ω_+.

Then, considering the elliptic region Ω_+, (12), (13) and (14) are impossible. Indeed, taking for instance $\varphi_3 = 0$ and eliminating T^{22} from (13) and substituting in (12), we have a Cauchy problem for an elliptic system of 2 equations of first order with the unknowns T^{11}, T^{12}, which is classically impossible. Then the surface is sensitive. Variants are possible. For instance, the edge is not relevant : we may consider a smooth surface changing of type, provided it is geometrically rigid.

Example B - See Fig. B, and note that the hyperbolic part is contained in the determination domain of the edge Γ_2 so that rigidity is easily proved. This surface is sensitive. Indeed, we consider Ω_+. Then (12) (13) and (14) amounts to a Cauchy problem (as in example A, we may eliminate T^{22} to have a hyperbolic first order system in T^{11}, T^{12}), but a supplementary condition $T^{11+} = 0$ must be satisfied on Γ_2 (see (15)) and incompatibilities are easily established. Here also variants are possible : Ω_- may be hyperbolic, and Γ_0 may be disposed otherwise, but ensuring geometric rigidity.

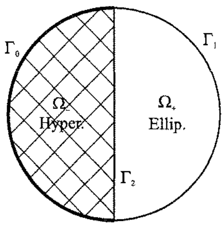

Fig. A

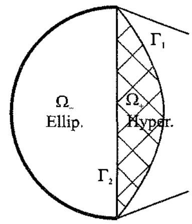

Fig. B

Example C - See Fig. C. It is merely a variant of the previous example. The curves P_1P_2 and $P_1'P_2'$ are identified by periodicity.

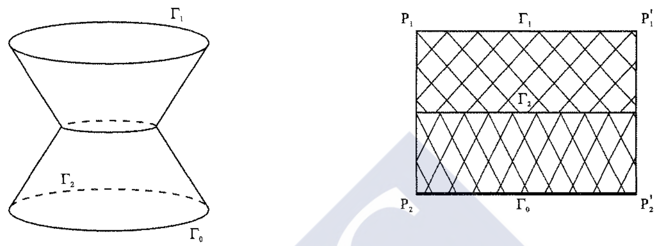

Fig. C

References

1. J. Sanchez-Hubert and E. Sanchez-Palencia, *Coques élastiques minces ; propriétés asymptotiques*, (Masson, Paris, 1997).
2. P.G. Ciarlet, *Mathematical elasticity, vol. III : Shell theory*, (North-Holland, Amsterdam, to appear).
3. J.L. Lions and E. Sanchez-Palencia, in *Partial differential equations and functional analysis in memory of Pierre Grisvard*, ed. C. Céa, D. Chenais, G. Geymonat, J.L. Lions, (Birkhauser, New York, 1996), 206-220.
4. J.L. Lions and E. Sanchez-Palencia, in *Homogenization and application to material sciences*, ed. D. Cioranescu, A. Damalamian, P. Donato (Gakkotosho, Tokyo, 1977, to appear).
5. J.L. Lions and E. Sanchez-Palencia, *Proceedings of the IUTAM symposium on Variation of domains and free-boundary problems in solid mechanics*, (Elsevier, to appear).
6. D. Leguillon and E. Sanchez-Palencia, in *Actes 2ème Colloque Nat. Calcul Structures*, (Hermès, Paris, 1995), 275-280.
7. M. Bernadou, *Méthodes d'éléments finis pour les problèmes de coques minces*, (Masson, Paris, 1994).
8. I. Babuska and M. Suri, *SIAM Jour. Num. Anal.*, **29** (1992) 1261-1293.
9. D. Choï, F.J. Palma, E. Sanchez-Palencia and M.A. Vilarino, *Math. Modelling and Num. Anal.* (to appear).

hp-FEM FOR HIGH RESOLUTION COMPUTATION OF PLATE AND SHELL PROBLEMS

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ABSTRACT

We consider the numerical solution of shell problems by hp-FEM. The standard displacement formulation is used with exact evaluation of the stiffness matrix. A Naghdi-type shell-model and the fully three-dimensional problem is computed. Increasing the polynomial degree p from 1 to 8 on a properly designed mesh resolves the boundary layers and achieves 0.01% relative error in the energy norm, even in the bending dominated case and with unstructured meshes. Theoretical and numerical convergence estimates indicate exponential convergence of the method.

1. Setting of the Problem

We consider a shell with midsurface $S = \varphi(\omega) \subset \mathbb{R}^3$ where the parameter domain $\omega \subset \mathbb{R}^2$ is a bounded polygon and the parametrization φ of S is analytic in $\overline{\omega}$ and injective. Denoting by $\boldsymbol{a}_\alpha(\xi)$ the tangent vectors* to S at $x = \varphi(\xi)$ and by $\boldsymbol{a}_3(\xi)$ the unit normal vector, the **shell of thickness** t is a linearly elastic, three-dimensional body

$$\Omega = \Phi\left(\omega \times \left(-\frac{t}{2}\,,\,\frac{t}{2}\right)\right) \tag{1}$$

where

$$\Phi(\xi_1,\xi_2,\xi_3) = \varphi(\xi_1,\xi_2) + \xi_3\,\boldsymbol{a}_3(\xi_1,\xi_2), \quad |\xi_3| < t/2$$

is analytic and bijective for sufficiently small t.

A **shell model** is a dimensionally reduced approximation to the problem of $3-d$ elasticity in Ω. We are particularly interested in 5-field models of Naghdi-type where the displacements $U_i(\xi_1,\xi_2,\xi_3)$ are linear in ξ_3:

$$U_\alpha = \zeta_\alpha(\xi_1,\xi_2) + \xi_3\, r_\alpha(\xi_1,\xi_2), \;\; U_3 = \zeta_3(\xi_1,\xi_2)\,. \tag{2}$$

Here ζ_2 are tangential displacements, ζ_3 is the normal displacement and r_α are the rotations. Combining the 5 unknown fields into the vector $\vec{U}$, the shell-model reads:

$$\vec{U} \in H : \; B(\vec{U},\vec{V}) = F(\vec{V}) \quad \forall\, \vec{V} \in H\,. \tag{3}$$

*as usual, $\alpha,\beta \in \{1,2\}, i,j,k \in \{1,2,3\}$, etc.

Here $H \subset [H^1(\omega)]^5$ denotes the admissible displacements, and

$$B = t^3 B_f + t(B_m + B_{sh})$$

with B_f, B_m, F_{sh} the flexural, membrane and shear part of the deformation energy of the shell model, respectively, which can be found, e.g. in [1], and $F(\cdot)$ in Eq. (3) represents the external loadings acting on the shell. Since (see e.g. [1])

$$B(\vec{U},\vec{U}) + \| \vec{U} \|^2_{L^2(\omega)} \geq C(t) \| \vec{U} \|^2_{H^1(\omega)} \quad \forall \vec{U} \in [H^1(\omega)]^5 \ . \tag{4}$$

for some $C(t) > 0$, coercivity of $B(\cdot,\cdot)$ on various choices for H follows in the usual way and thus Eq. (3) admits a unique solution if $(F(\vec{V})| \leq C \| \vec{V} \|_{H^1(\omega)} \ \forall \vec{V} \in H$.

2. *hp*-FEM

Let $\mathcal{T} = \{K\}$ denote a regular partition of Ω into triangles and/or quadrilateral elements such that $K \cap K'$ is either empty, a vertex or an entire side for $K \neq K' \in \mathcal{T}$. We assume further that every $K \in \mathcal{T}$ is affine equivalent to a reference element $\hat{K}, K = A_K(\hat{K})$. For each $K \in \hat{K}$, a polynomial degree $p_K \geq 1$ is given. The p_K are combined into the *degree vector* $\boldsymbol{p} = \{p_K : K \in \mathcal{T}\}$. The FE-space is

$$S^{\mathbf{p}}(\omega, \mathcal{T}) := \{u \in H^1(\omega) : u|_K \circ A_K \in \mathcal{S}^{p_K}(\hat{K}),\ K = A_K(\hat{K}) \in \mathcal{T}\} \tag{5}$$

where

$$\mathcal{S}^p(\hat{K}) = \begin{cases} \operatorname{span}\{\xi_1^{\alpha_1}\xi_2^{\alpha_2} : \quad 0 \leq \alpha_1, \alpha_2,\ \alpha_1 + \alpha_2 \leq p\} & \text{if } K \text{ is a triangle} \\ \operatorname{span}\{\xi_1^{\alpha_1}\xi_2^{\alpha_2} : \quad 0 \leq \alpha_1, \alpha_2 \leq p\} & \text{if } K \text{ is quadrilateral .} \end{cases} \tag{6}$$

The *hp*-FE approximation is defined in the usual way:

$$\vec{U_N} \in H_N : \ B(\vec{V_N}, \vec{V}) = F(\vec{V}) \quad \forall \vec{V} \in H_N \tag{7}$$

where $H_N := [S^{\mathbf{p}}(\omega, \mathcal{T})]^5 \cap H$. By Eq. (4), for every $t > 0$, $\vec{U_N}$ exists and is unique and

$$||| \vec{U} - \vec{U_N} ||| = \inf_{\vec{V} \in H_N} ||| \vec{U} - \vec{V} ||| \tag{8}$$

(here $||| \vec{U} ||| = (B(\vec{U},\vec{U}))^{1/2}$ denotes the energy norm).

3. Numerical Integration. *hp*-Surface Fitting

In general $B_m(\cdot,\cdot)$, $B_f(\cdot,\cdot)$ and $B_{sh}(\cdot,\cdot)$ contain first derivatives of $\vec{U}$ and $\vec{V}$, but second derivatives of φ due to the curvature terms. In practice (CAD, NURB-patches) φ is a) (piecewise) analytic, b) not explicitly available. What is usually available are values of φ and possibly of its first derivatives at any given point $\xi \in \omega$. We propose therefore to sample φ and refit it with spectral accuracy by continuous, piecewise polynomial approximations $\widetilde{\varphi}_K$ of degrees q_K. This can be done by sampling $\varphi \circ A_K$ in the Lobatto- or Tchebychev-Points in $\hat{K}$. Owing to the analyticity of φ we find for $k = 0, 1, 2, \ldots$ [2]

$$\|\varphi \circ A_K - \widetilde{\varphi}_K \circ A_K\|_{W^{k,\infty}(\hat{K})} \leq C_k(1 + \ln q_K)\, q_K^{2k} e^{-\sigma q_K} \tag{9}$$

for every $K \in \mathcal{T}$. The corresponding global map $\widetilde{\varphi}$ is continuous, piecewise polynomial, but in general not differentiable. We define

$$\widetilde{B}(\vec{U},\vec{V}) := \sum_{K\in\mathcal{T}} \widetilde{B}_K(\vec{U},\vec{V}), \quad \widetilde{F}(\vec{V}) := \sum_{K\in\mathcal{T}} \widetilde{F}_K(\vec{V}) \tag{10}$$

where $\widetilde{B}_K$ is obtained by replacing φ and its derivatives by $\widetilde{\varphi}_K$ and compute

$$\vec{\widetilde{U}} \in H_N : \ \widetilde{B}(\vec{\widetilde{U}},\vec{V}) = \widetilde{F}(\vec{V}) \quad \forall \vec{V} \in H_N \ . \tag{11}$$

The integration in $\widetilde{B}(\vec{\widetilde{U}},\vec{V})$ can be performed exactly since the integrands are piecewise polynomial, but Eq. (11) is in effect a nonconforming *hp*-FEM, [3].

4. Boundary Layer Resolution

If $\partial\omega$ is smooth, the exact solution $\vec{U}$ of Eq. (3) is also smooth, but may exhibit *boundary layers*, i.e.

$$\vec{U} = \vec{U}_{\text{smooth}} + \vec{U}_{b\ell} + \vec{R} \tag{12}$$

where $\vec{U}_{\text{smooth}}$ has derivatives bounded independent of t, $\vec{U}_{b\ell}\ (x) \approx C(t)\ \exp(-\rho(x)/t^{\beta})$ and $\vec{R}$ is small.

Here $\rho(x) = \text{dist}(x, \partial S)$ and β is the length scale of the layer. We always have terms with $\beta = 1$ (shear)†, $\beta = 1/2$ (membrane) and, depending on the geometry of S, also $\beta = 1/3, 1/4$ in parabolic and hyperbolic shells. We *resolve each boundary layer* length scale by a layer of elements of width $p_{\max} t^{\beta}$ near ∂S. Under these circumstances we get the *general error estimate*, [2], [3]

$$E = \frac{|||\ \vec{U} - \vec{\widetilde{U}}\ |||}{|||\ \vec{U}\ |||} \le C\, t^{-1}\ \exp(-b\sqrt{N}) \tag{13}$$

where b, C are independent of t, and $N = \dim(H_N)$ is the number of degrees of freedom. In special cases, e.g. for membrane dominated problems, the *locking-factor* t^{-1} in Eq. (13) is pessimistic, but estimate Eq. (13) is sharp for general bending dominated situations, as can be seen for the special case of a cylindrical shell [2], [4].

5. Numerical Experiments. Discussion

We present detailed experiments for cylindrical shells of thicknesses $t = 0.1$, 0.05, 0.01, $0.005, \ldots, 0.0005$ in bending- and membrane-dominated cases. Using a quadrilateral 3 element mesh for $1/8$ of S (symmetry!), we achieve a relative energy norm error $E < 0.01\%$ in all cases with 474 DOF.

These results were achieved using the (soon to be commerical) shell code STRESSCHECK [5]. Similar calculations with triangular, irregular meshes yielded analogous results. Results for other geometries and partially (well-) inhibited shell will also be presented. We conclude that the error estimate Eq. (13) exhibits, for *hp*-FEM, a *locking-factor* of $O(t^{-1})$ in general, bending dominated problems. Due to the exponential approximability Eq. (13), however, which is achievable with proper mesh-design and low p [6], the influence of the locking factor

† absent for Koiter-shells

is hardly visible in practice and no "tricks", as e.g. reduced integration etc. need to be used (see Table 1 where the "case 2" example of [4] was computed using a *hp*-FEM).

p	DOF	Total Potential Energy	Convergence	% Error
1	18	-2.290671368938412e+01	0.00	100.00
2	51	-1.754626842719269e+03	0.00	99.63
3	84	-1.756906617963227e+05	1.38	50.16
4	132	-2.328903759740552e+05	3.82	8.94
5	195	-2.347445980450458e+05	5.71	0.96
6	273	-2.347660757807588e+05	6.22	0.12
7	366	-2.347663834135083e+05	4.44	0.03
8	474	-2.347664055475136e+05	4.44	0.01

Table 1: Convergence of *hp*-FEM for Naghdi-type cylindrical shell, free-ends, $t = 0.01$ (3 element mesh, 1/8 of shell discretized).

We conclude that the *hp*-FEM gives accurate approximations of thin-shell problems efficiently with little sensitivity with respect to geometry and meshing, provided that the boundary layer length scales are properly resolved.

A similar behaviour is also exhibited by *3-d hp-FEM applied to the three-dimensional problem* (Table 2).

p	DOF	Total Potential Energy	Convergence	% Error
1	22	-4.612138310935235e+01	0.00	99.99
2	73	-7.696612567237567e+02	0.00	99.84
3	124	-1.440075474801176e+05	0.89	62.18
4	213	-2.301215439151931e+05	2.75	14.07
5	340	-2.346202240610073e+05	3.68	2.52
6	514	-2.347653480868294e+05	4.57	0.38
7	744	-2.347685719343401e+05	4.04	0.09
8	1039	-2.347687318823435e+05	4.04	0.02

Table 2: Convergence of *hp*-FEM for 3-d shell, $t = 0.01$

6. References

1. M. Bernadou, P.G. Ciarlet and B. Miara: Existence Theorems for two dimensional linear shell theories, *J. Elasticity* **34**, (1994), 111-138.
2. K. Gerdes, A.M. Matache and C. Schwab: On membrane locking in *hp* FEM for a cylindrical shell, in preparation.
3. J. Melenk and C. Schwab: *hp*-FEM for shell problems (in preparation).
4. J. Piila, Y. Leino, O. Ovaskainen and J. Pitkäranta: Shell deformation states and the FEM: A benchmark study of cylindrical shells, *Comp. Meth. Appl. Mech. Engg.* **128**, (1995), 81-121.
5. Stresscheck Users Guide: ESRD Research Inc., St. Louis, USA
6. C. Schwab and M. Suri: The *p*- and *hp* versions of the FEM for problems with boundary layers, *Math. Comp.* **65**, (1996), 1403-1429.
7. M. Suri: A reduced constraint *hp* finite element method for shells, *Math. Comp.* **66**, (1997), 15-30.

STABILIZED FINITE ELEMENT METHODS FOR PLATES AND SHELLS

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ABSTRACT

We give a short review of our locking free stabilized finite element methods for the Reissner–Mindlin plate model and Naghdi shell model.

1. Introduction

The aim of our work has been to explore the possibilities of using stabilization techniques for avoiding the "locking" phenomena which appear when the standard finite element method is used for "thin" structures, especially for shells and plates. During the last decade, methods of this type have been shown to be very successful for related problem in incompressible elasticity and fluid flow. For the basic problems in structural engineering the technique has not yet been used as much.

2. Methods for Naghdi shells

For shells there exist two types of locking, "membrane" and "shear" locking, and this appears when the boundary conditions and loading are such that the shell will be in a "bending" dominated state.

In our work [2] we consider this case for the Naghdi shell model. The approach we follow can briefly be explained as follows. First, the variational equations are written in a mixed form in which the shear and membrane forces explicitly appear. In the discretization, properly weighted element-wise least squares forms of the constitutive and equilibrium equations (i.e. the strong forms obtained by integration by part) are added. We consider two alternatives for the element-wise weighting and for both we introduce mesh-dependent norms in which we are able to prove the stability for standard finite element spaces. This enables us to derive error estimates wich are uniform with respect to the "locking parameter", i.e. the thickness of the shell. We present numerical result that confirm the convergence behaviours predicted by the mathematical analysis. It should be remarked that the introduction of the force variables is an intermediate step; they are approximated by discontinuous functions and hence they can be eliminited at the element level. This gives a displacement formulation which is the one structural finite element codes utilize.

The biggest shortcoming of our methods is that they do not work well for a shell in a membrane dominated state. Nevertheless, we hope that this will be a contribution to the search for robust shell elements.

3. Methods for the Reissner–Midlin plate model

When the shell is flat it reduces to a plate and only the "shear" locking is left. For this case it seems that the second of our weightings for the shell elements is preferable, cf. [3]. Then the method obtained is essentially one introduced by Hughes and Franca [4]. This formulation we are able to further develop in a number of ways. As shown in [9] the intermediate mixed form is unnecessary; it is possible to directly write down a stable displacement formulation. This formulation, as well as the one of Hughes and Franca, suffers from the fact that equal interpolation cannot be used for the rotation and the deflection. By combining the stabilization technique with the interpolation technique used in the MITC elements (cf. [1]) it is, however, possible to use equal order interpolation [7]. By the stabilization the bubble degrees of freedom in the original MITC elements are avoided and hence standard finite element spaces can be used. The stiffness matrix obtained is also better conditioned. Furthermore, it is possible to obtain elements with linear (or bilinear) shape functions for all variables. These lowest order elements where already introduced in [1]. Recently Lyly has shown [5] that the linear element is in essence equivalent to methods introduced by Tessler and Hughes [11], and Xu, Taylor and Auricchio [12,10], thus providing a mathematical justification for these elements as well.

Our results have been announced in [6] and will be presented in a forthcoming paper [7]. In [8,6] we give numerical results obtained with the methods.

4. References

1. F. Brezzi, M. Fortin and R. Stenberg, Error analysis of mixed-interpolated elements for Reissner–Mindlin plates. M^3AS **1** (1991) 125–151
2. D. Chapelle and R. Stenberg, Stabilized finite element formulations for shells in a bending dominated state. *INRIA Rapport de Recherche 2941, Juillet 1996.* http://www.inria.fr/RRRT/RR-2941.html
3. D. Chapelle and R. Stenberg, An optimal low-order locking-free finite element method for Reissner–Mindlin plates. *Institut für Mathematik und Geometrie, Universität Innsbruck, Forschungsbericht 1–1997.* To appear in M^3AS 1998
4. T.J.R. Hughes and L.P. Franca, A mixed finite element formulation for Reissner–Mindlin plate theory: Uniform convergence of all higher order spaces. *Comp. Meths. Appl. Mech. Engn.* **67** (1988) 85–96
5. M. Lyly, On the connection between some linear triangular Reissner–Mindlin plate

bending elements. *Numer. Math.* Accepted for publication

6. M. Lyly and R. Stenberg, Stabilized MITC plate bending elements. in *Advances in Finite Element Techniques*, ed. M. Papadrakikis and B.H.V. Topping. (CIVIL-COMP Ltd. Edingburg, Scotland, 1994) 11–16. Also available from: http://www.solid.hut.fi/reports/index.html
7. M. Lyly and R. Stenberg, Stabilized finite element methods for Reissner–Mindlin plates. *In preparation*
8. M. Lyly, R. Stenberg and T. Vihinen, A stable bilinear element for Reissner–Mindlin plates. *Comp. Meths. Appl. Mech. Engn.* **110** (1993) 343–357
9. R. Stenberg, A new finite element formulation for the plate bending problem. *Asymptotic Methods for Elastic Structures,* ed. Ciarlet, Trabucho and Viano (Walter de Gruyter & Co., 1995) 209–221
10. R.L. Taylor and F. Aurichhio, Linked interpolation for Reissner–Mindlin plate elements: Part II - a simple triangle. *Int. J. Num. Meths. Eng.* **36** (1993) 311-335
11. A. Tessler and T.J.R. Hughes. A three-noded Mindlin plate element with improved transverse shear. *Comp. Meths. Appl. Mech. Engn.* **50** (1985) 71–101
12. Z. Xu, A thick-thin triangular plate element. *Int. J. Num. Meths. Eng.* **33** (1992) 963-973

A RESULT IN ADAPTIVE ELASTICITY THEORY WITH RELEVANCE TO APPLICATIONS IN BIOMECHANICS*

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ABSTRACT

In this work we establish an existence and uniqueness result in the theory of adaptive elasticity which generalizes some of the results obtained in [4] and which is relevant for bone remodeling models in the framework of Biomechanics.

1. Introduction

Living bone is continuously adapting itself to external stimuli. This process termed collectively "remodeling" has an enormous effect in the overall behavior and health of the entire body. Most of the existing bone remodeling models are of an empirical and/or experimental nature, with the possible exception of those derived by Cowin and Hegedus, [1], [3], which are obtained from Continuum Mechanics, and constitute a generalization of nonlinear elasticity theory.

In this work we establish an existence and uniqueness results for a linearized elasticity model with a remodeling law exhibiting a quadratic dependance on the linearized strain tensor, which generalizes some of the results obtained in [4] and which is relevant for bone remodeling in the framework of Biomechanics.

2. The physical model

In this section we describe the physical model under consideration. We point out that one of the diffilcuties in the analysis is related to the imposition of the condition that the volume fraction must belong to the interval $[0,1]$; when it takes the unit value one gets classical elasticity but when it takes the zero value one looses ellipticity. This is the reason why we are lead to the use of a truncated and mollified model, which we physically interpret afterwards.

2.1. Notations

Let Ω be an open, bounded, connected subset of $I\!R^n$ ($n = 2$ or 3) of class C^2 and independent of time t. Let $T > 0$ be a real parameter and denote $Q =]0,T[\times\Omega$, $\bar{Q} = [0,T]\times\bar{\Omega}$ and $\Sigma =]0,T[\times\partial\Omega$. Let $\mathcal{R}$ be the set of infinitesimal rigid displacements, $\mathcal{R} = \{v/v = a + b\wedge x;\ a, b \in I\!R^3\}$. Let q be a real number, $1 \leq q \leq \infty$, and m be a positive integer, and define the following spaces: $V^{m,q} = (W^{m,q}(\Omega)/\mathcal{R})^n$, $V^m = (H^m(\Omega)/\mathcal{R})^n$, $\mathcal{L}^q = (L^q(\Omega))^{n\times n}$, $\mathcal{W}^{m,q} = (W^{m,q}(\Omega))^{n\times n}$, $\mathcal{H}^m = (H^m(\Omega))^{n\times n}$, $\mathcal{C}^m = (C^m(\bar{\Omega}))^{n\times n}$ where $W^{m,q}(\Omega)$ and $H^m(\Omega)$ are the classical Sobolev spaces and $C^m(\bar{\Omega})$ is the space of functions m times continuously differentiable in $\bar{\Omega}$. We denote by $C^m([0,T];V)$ the space of functions g such that $g(t) \in V$ for all $[0,T]$ and the function $t \in [0,T] \mapsto g(t) \in V$ is m times continuously

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differentiable with respect to t. If V is a Banach space then $C^m([0,T];V)$ equipped with its usual norm is also a Banach space.

We denote the displacements vector field by $u = (u_i)$, $1 \le i \le n$, the Cauchy stress tensor by $\sigma = (\sigma_{ij})$, $1 \le i,j \le n$, the linearized strain tensor by $\varepsilon = (\varepsilon_{ij})$, $1 \le i,j \le n$ and the measure of the change in volume fraction from a reference configuration by e. They are all functions of time t and of the space variable x.

2.2. Data of the problem

We suppose given:
.the open set Q of $I\!R \times I\!R^n$;
.the density γ of the full elastic material which is supposed to be constant;
.the reference volume fraction ξ_0. It belongs to $C^1(\bar{\Omega})$ and there exist constants ξ_0^{min} and ξ_0^{max} such that: $0 < \xi_0^{min} \le \xi_0(x) \le \xi_0^{max} < 1$ in $\bar{\Omega}$;
.the coefficients of elasticity denoted by $a_{ijkm}(e)$, $1 \le i,j,k,m \le n$. They are continuously differentiable with respect to e, they satisfy the conditions of symmetry : $\forall e$, $a_{ijkm}(e) = a_{jikm}(e) = a_{kmij}(e)$, $1 \le i,j,k,m \le n$, and they also satisfy the following ellipticity condition:

$$(\xi_0 + e)\ a_{ijkm}(e)\varepsilon_{ij}\varepsilon_{km} \ge N\varepsilon_{ij}\varepsilon_{ij} \quad \forall \varepsilon_{ij} \in I\!R^{n\times n} \text{ with } \varepsilon_{ij} = \varepsilon_{ji}, \tag{1}$$

where N is a strictly positive constant (independent of e, t and x). Let us notice that this inequality implies the assumption: $(\xi_0(x) + e(t,x)) > 0 \quad \forall (t,x) \in Q$;
.the body load $f_i \in C^1([0,T])$, $1 \le i \le n$, f_i depends only on t;
.the normal traction on the boundary $\partial\Omega$, $F_i \in C^1([0,T];W^{1-\frac{1}{p},p}(\partial\Omega))$, $1 \le i \le n$ with $p > n$;
.the constitutive function $a(e)$ and the remodeling rate coefficients $A(e) = (A_{km}(e))$ and $B(e) = (B_{ijkm}(e))$, $1 \le i,j,k,m \le n$, which are continuously differentiable with respect to e;
.the initial value of the change in volume fraction $e_0(x)$, which belongs to $C^1(\bar{\Omega})$.

We shall employ the usual summation and differentiation conventions. Moreover, given a function $g(t,x)$ we denote by $\dot{g}$ its partial derivative with respect to t and by $\partial_j g$ its partial derivative with respect to x_j.

2.3. Truncation and mollification

Let $\eta > 0$, η a small parameter, and denote by $\mathcal{P}_\eta(e)$ a truncation operator of class C^1, such that:

$$\mathcal{P}_\eta(e)(x) = \begin{cases} -\xi_0(x) + \frac{\eta}{2} & \text{if} \quad e(x) \le -\xi_0 + \frac{\eta}{2} \\ e(x) & \text{if} \quad \eta - \xi_0 \le e(x) \le 1 - \xi_0 - \eta \\ 1 - \xi_0(x) & \text{if} \quad e(x) \ge 1 - \xi_0 \end{cases} \tag{2}$$

Let the function:

$$w(x) = \begin{cases} e^{\left(\frac{1}{|x|^2 - 1}\right)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

and let $\rho > 0$ be a positive real number. We define, in a classical way, the mollifier $w_\rho(x) = c\frac{1}{\rho^n} w(\frac{x}{\rho})$ with $c = (\int w)^{-1}$.

Let a function $g \in C^0([0,T];C^0(\bar{\Omega}))$, denote by $\bar{g}(t)$ an extension of $g(t)$ to $I\!R^n$ such that $\bar{g}(t) \in C^0(I\!R^n)$ and define the operator M_ρ from $C^0(\bar{\Omega})$ into $C^\infty(I\!R^n)$ such that:

$$M_\rho(g(t)) = w_\rho * \bar{g}(t)$$

where $w_\rho * \bar{g}(t) = \int_{\mathbb{R}^n} w_\rho(x-y)\ \bar{g}(t,y)\ dy$. We then define the non local coefficients of elasticity $c_{ijkm}(e)$, $1 \le k, m \le n$, by:

$$c_{ijkm}(e) = (\xi_0 + M_\rho \circ \mathcal{P}_\eta(e))\ a_{ijkm}(M_\rho \circ \mathcal{P}_\eta(e)) \tag{3}$$

2.4. The model of non local type

The problem for which we prove, in the sequel, the existence and uniqueness of the solution is the following: find (u, e) which satisfy (in the sense of distributions)

$$-\partial_j \sigma_{ij} = \gamma(\xi_0 + \mathcal{P}_\eta(e)) f_i \quad \text{in } Q \tag{4}$$

$$\sigma_{ij} = c_{ijkm}(e)\varepsilon_{km}(u) \tag{5}$$

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6}$$

$$\sigma_{ij} n_j = F_i \quad \text{on } \Sigma \tag{7}$$

$$\dot{e} = a(e) + A_{km}(e)\ \varepsilon_{km}(u) + B_{ijkm}\varepsilon_{km}(u)\varepsilon_{ij}(u) \quad \text{in } Q \tag{8}$$

$$e(x,0) = e_0(x) \quad \text{in } \bar{\Omega} \tag{9}$$

where n_j are the components of the unit outward normal n to $\partial\Omega$. Moreover, we assume that the resultant of the external forces is null:

$$\forall w \in \mathcal{R}, \quad \int_\Omega \gamma(\xi_0 + \mathcal{P}_\eta(e)) f_i(t) w_i\ dx + \int_{\partial\Omega} F_i(t) w_i\ ds \ = \ 0 \quad \text{in } [0,T] \tag{10}$$

Remark 1 *The functions $a_{ijkm}(e)$, $a(e)$, $A_{ij}(e)$ and $B_{ijkm}(e)$ characterize the material properties and there is very few experimental data on these functions. We can make a polynomial approximation of these functions as in [3] and, we point out that, due to the presence of the quadratic term in the strain tensor, in the remodeling equation, the model is sensitive to remodeling under torsion loads and can also include the case where the remodeling equation is a function of $\sigma_{ij}(u)\varepsilon_{ij}(u)$. The case where the coefficients B_{ijkm} are identically zero was studied in [4].*
Truncation is a way of imposing the physical condition that the volume fraction belongs to the interval $]0,1]$. This can also be done using other methods but one needs some regularity in order to study the coupling between equations (4) and (8) and this is the reason for the C^1 truncation. The mollification can be regarded as a nonlocal constitutive law. There is experimental evidence, in bone remodeling, of its validity. The fact that we have a pure Neumann problem is required in order to have some regularity results for the displacement field u, in the framework of elasticity theory, but it also corresponds to a realistic situation.

3. Existence, uniqueness and regularity

With the above notations, the following results follow:

Proposition 1 *Assume that $\varepsilon(u)$ is given in $C^0([0,T];\mathcal{C}^0)$. Then, there exists a unique e in $C^1([0,T];C^0(\bar{\Omega}))$ solution to (8)(9). Furthermore, there exists a positive constant c such that:*

$$\begin{aligned}
\|e\|_{C^1(C^0)} \ \le \ & \{c \ + \ c\|\varepsilon(u)\|_{C^0(C^0)} + c\|\varepsilon(u)\|^2_{C^0(C^0)}\} \ \times \ \{\ \|e_0\|_{C^0(\bar{\Omega})} \\
+ \ & T\ [\ \|a(e_0)\|_{C^0(\bar{\Omega})} \ + \ \|A(e_0)\|_{C^0(\bar{\Omega})}\|\varepsilon(u)\|_{C^0(C^0)} + \qquad (11) \\
+ \ & \|B(e_0)\|_{C^0(\bar{\Omega})}\|\varepsilon(u)\|^2_{C^0(C^0)}\ ] \\
\times \ & \exp[\ T\ (k_1 \ + \ k_2\|\varepsilon(u)\|_{C^0(C^0)} \ + \ k_3\|\varepsilon(u)\|^2_{C^0(C^0)})\ ]\ \} \qquad (12)
\end{aligned}$$

Lemma 1 *Let $e \in C^1([0,T];C^0(\bar\Omega))$, then $c_{ijkm}(e) \in C^1([0,T];C^1(I\!R^n))$, $1 \le i,j,k,m \le n$, and*

$$\|c_{ijkm}(e(t))\|_{C^1(\bar\Omega)} \le c \quad in\ [0,T] \tag{13}$$

where c is a positive constant, which depends notably on $\|w_\rho\|_{W^{1,1}(I\!R^n)}$, $\|\xi_0\|_{C^1(\bar\Omega)}$, $\|a_{ijkm}(f)\|_{C^1([-\xi_0^{max},1-\xi_0^{min}]}$ but is independent of e.

Proposition 2 *Let $e(t,x)$ be given in $C^1([0,T];C^0(\bar\Omega))$. Then, there exists a unique solution $u \in C^1([0,T];V^{2,p})$ to (4)-(7). Furthermore, there exists a positive constant c independent of e such that:*

$$\|u\|_{C^1(W^{2,p})} \le c\ (\gamma\|f\|_{C^1(L^p)} + \|F\|_{C^1(W^{1-\frac{1}{p},p})}) \tag{14}$$

Theorem 1 *Under the conditions of Section 2, problem (4)-(9) possesses a unique solution (u,e) in $C^1([0,T];V^{2,p}) \times C^1([0,T];C^0(\bar\Omega))$.*

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5. References

1. S.C. Cowin and D.H. Hegedus. Bone remodeling i: A theory of adaptative elasticity. *J. of Elasticity*, 6(3):313–326, 1976.
2. S.C. Cowin and R.R. Nachlinger. Bone remodeling iii: uniqueness and stability in adaptative elasticity theory. *J. of Elasticity*, 8(3):285–295, 1978.
3. D.H. Hegedus and S.C. Cowin. Bone remodeling ii: Small strain adaptative elasticity. *J. of Elasticity*, 6(4):337–352, 1976.
4. J. Monnier and L. Trabucho. Existence and uniqueness of solution to an adaptive elasticity model. *Mathematics and Mechanics of Solids*, (to appear).
5. T. Valent. *Boundary value problems of finite elasticity*, volume 31. Springer Tracts in Natural Philosophy, Springer-Verlag, New-York, 1988.

ON THE CONTROLLABILITY AND EVERSION OF THIN SHELLS

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ABSTRACT

In this note we point out some intrinsic phenomena in the shell theory.In particular the loss of exact controlability and the eversion of the shell are connected to its thinness h and occur when we consider the limit case $h \to 0$.

1. The System of Equations

Let $\mathcal{E}^\ni$ be the Euclidean space and let Ω be a bounded open set of boundary $\Gamma$of the plane E^2; the middle surface S of an elastic shell is defined by means of two curvilinear coordinates ξ_1 and ξ_2; it is the image in $\mathcal{E}^\ni$ of Ω by the map $\boldsymbol{\eta} : (\xi_1, \xi_2) \in \overline{\Omega} \to \mathcal{E}^\ni$ moreover let ξ_3 be the normal distance to S.In each point of S we consider two tangent vectors$\mathbf{a}_\alpha = \partial\boldsymbol{\eta}/\partial\xi_\alpha$ $\alpha = 1, 2$ and a normal vector $\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$.The reference configuration of a thin shell of constant thickness $2h$ is the following closed subset of $\mathcal{E}^\ni$

$$C = \{P \in \mathcal{E}^\ni,\ \mathcal{P} = \boldsymbol{\eta}(\xi_\infty, \xi_\in) + \xi_\ni \mathbf{a}_\ni\ ,\quad (\xi_\infty, \xi_\in) \in \overline{\otimes}\ ,\quad -\langle \le \xi_\ni \le \langle\}\ .$$

The summation convention is used, moreover we denote by $f_{,\alpha}$ the partial derivative of f with respect to ξ_α. We introduce the *fundamental forms* $(a_{\alpha\beta})$ $(b_{\alpha\beta})$ and $(c_{\alpha\beta})$ where

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \qquad b_{\alpha\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} \qquad c_{\alpha\beta} = b_\alpha^\lambda \cdot b_{\lambda\beta} \quad \textit{with} \quad b_\beta^\alpha = a^{\alpha\gamma} b_{\gamma\beta} \qquad \alpha, \beta = 1, 2$$

We denote with$(a^{\alpha\beta})$ the inverse matrix of $(a_{\alpha\beta})$ so that the reciprocal basis $\mathbf{a}^\alpha$ is defined by$\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$. Let $\mathbf{v}(\xi_1, \xi_2) = v_1\mathbf{a}^1 + v_2\mathbf{a}^2 + v_3\mathbf{a}^3 = v_i\mathbf{a}^i$ be the displacement vector of the middle surface S, the deformed middle surface is given by $\boldsymbol{\eta} + \mathbf{v}$.We write the *deformation tensor* of the middle surface $\gamma_{\alpha\beta}(\mathbf{v})$ and the *change of curvature tensor* $\rho_{\alpha\beta}(\mathbf{v})$ in this unified form that corresponds to the classic linear and nonlinear shallow shell theory of Koiter respectively for$g_{\alpha\beta} = 0$, and $f_{\alpha\beta} = 0$.

$$\gamma_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}\left(v_{\beta|\alpha} + v_{\alpha|\beta}\right) - b_{\alpha\beta}v_3 + g_{\alpha\beta}(v_3), \qquad g_{\alpha\beta}(v_3) = \frac{1}{2}v_{3,\alpha}v_{3,\beta} \tag{1.1}$$

$$\rho_{\alpha\beta}(\mathbf{v}) = v_{3|\alpha\beta} + f_{\alpha\beta}(\mathbf{v}), \qquad f_{\alpha\beta}(\mathbf{v}) = b^\lambda_{\beta|\alpha}v_\lambda + b^\lambda_\beta v_{\lambda|\alpha} + b^\lambda_\alpha v_{\lambda|\beta} - c_{\alpha\beta}v_3 \tag{1.2}$$

where with the bar $|$ we indicate the covariant derivative defined by means of the Christoffel symbols$\Gamma^\alpha_{\beta\lambda} = \mathbf{a}^\alpha \cdot \mathbf{a}_{\beta,\lambda}$ and

$$v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta}v_\lambda\ , \quad v_{3|\alpha\beta} = v_{3,\alpha\beta} - \Gamma^\lambda_{\alpha\beta}v_{3,\lambda} \quad b^\lambda_{\beta|\alpha} = b^\lambda_{\beta,\alpha} + \Gamma^\lambda_{\alpha\mu}b^\mu_\beta - \Gamma^\mu_{\beta\alpha}b^\lambda_\mu\ .$$

We denote by $\mathbf{u}$ the vector (v_1, v_2) so that $\mathbf{v} = (v_1, v_2, v_3) = (\mathbf{u}, v_3)$ and introduce the spaces $\mathbf{U} \times L^2$ and $\mathbf{U} \times V_3$(with $\mathbf{U} \subset (H^1(\Omega))^2, V_3 \subset H^2(\Omega)$)of functions satisfying suitable boundary conditions. The space $\mathbf{V}$ (depending on the form of S) of admissible displacements is $\mathbf{V} = \{\mathbf{v}; \mathbf{u} \in \mathbf{U}, v_3 \in V_3\ ,\quad \mathbf{v}$ satisfy the kinematic boundary conditions$\}$and $\mathbf{H}$ is the space $(L^2(\Omega))^3$ equipped with the standard scalar product

$$(\mathbf{v}, \tilde{\mathbf{v}})_{\mathbf{H}} = ((\mathbf{v}, \tilde{\mathbf{v}})) = \int_\Omega (a^{\alpha\beta}\mathbf{u}_\alpha \cdot \tilde{\mathbf{u}}_\beta + v_3 \cdot \tilde{v_3})\, \mathrm{d}\, S\ .$$

The energy of deformation of the shell (also in the general case of anisotropy) is defined by two symmetric forms :

$$a^m(\mathbf{v}, \tilde{\mathbf{v}}) = \int_S a^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{v})\gamma_{\lambda\mu}(\tilde{\mathbf{v}})\, \mathrm{d}\, S \qquad a^f(\mathbf{v}, \tilde{\mathbf{v}}) = \int_S a^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}(\mathbf{v})\rho_{\lambda\mu}(\tilde{\mathbf{v}})\, \mathrm{d}\, S \qquad (1.3)$$

where $\mathrm{d}\, S = |\mathbf{a}_1 \times \mathbf{a}_2|\, \mathrm{d}\,\xi_1\, \mathrm{d}\,\xi_2$ and $a^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{(1-\nu)}\, a^{\alpha\beta}a^{\lambda\mu}\right]$is the tensor of "elastic moduli", with E and ν Young modulus and Poisson ratio respectively. **1.1.**

Spectral Analysis For The Linear Case

We assume that $a^m + \frac{h^2}{3}\, a^f$ is *continuous* and *coercive*in $\mathbf{U} \times V_3$, and a^m is *continuous* and *coercive*in $\mathbf{U} \times L^2$.We denote by $\mathbf{A}^m$ (resp. $\mathbf{A}^f$) the differential operator associated to the form a^m (resp. a^f). For $h > 0$ the operator $\mathbf{A} = \mathbf{A}^m + \frac{h^2}{3}\, \mathbf{A}^f$ is a linear system of differential operators of mixed order with indices $m_\alpha = 1(\alpha = 1, 2)$, $m_3 = 2$. It is well known, in this case, that there exists a constant $c(\mathbf{A})$ that does not depend on the choice of the boundary conditions (provided that they satisfy the Shapiro–Lopatinskii condition) such that the number of eigenvalues less than λ is given in first approximation by $c(\mathbf{A})\lambda + o(\lambda)$. Moreover we proved (see [1]) there exist two constants c_1 and c_2 independent on h such that

$$N_\lambda(\mathbf{A}) \geq c_1\lambda + o(\lambda) + \frac{c_2}{h}\sqrt{\lambda} + o(\sqrt{\lambda}/h) \qquad \lambda \to \infty\ . \qquad (1.4)$$

The inequality (1.4) shows that when the thickness of the shell goes to 0 (that is for $h \to 0)N_\lambda(\mathbf{A}) \to \infty$, so an accumulation point for the eigenvalues of the limit problem $h = 0$ may occur.*For* $h = 0$ (the so called membrane approximation) we have a system with indices $m_3 = 0$, $m_\alpha = 1(\alpha = 1, 2)$,$\mathbf{A}^m - \lambda$ is not always Douglis Nirenberg ellliptic and the *essential spectrum* is not empty.

1.2. Characteristic Parameters

There are two characteristic parameters in our shell model : the *thinness* and the*shallowness*. So we can look at two limit problems the *membrane approximation*when the thinness goes to 0 (at fixed shallowness) and the *plate approximation* when the shallowness goes to 0 (at fixed thinness).

2. Exact Controllability For Thin Shells

For studying the controllability problem we restrict our attention to the axially symmetric deformations of a spherical thin shell with opening angle θ_0.We introduce the map $\boldsymbol{\eta} = (Rsin\theta cos\psi, Rsin\theta sin\psi, Rcos\theta)$, where R is the radious of the shell middle surface, and take in (1.1) $g_{\alpha\beta} = 0$(linear case). For spherical shells the thinness and the shallowness are defined respectively by $h/(Rsin\theta_0)$ and $(1-cos\theta_0)/sin\theta_0$. The exact controllability problem requires, *given $T>0$ and an initial state $\{\Phi^o,\Phi^1\}$, to find the control functions $g(t)_1$, $g(t)_2$, $g(t)_3$ such that the unique solution $\Phi(\phi,\psi)$ of*

$$\ddot{\phi}-\mathcal{L}(\phi)+(\infty+\nu)\psi'-\varepsilon\mathcal{L}(\phi+\psi')=\prime \tag{2.1}$$

$$\ddot{\psi}+\frac{\varepsilon}{\sin\theta}\left[\mathcal{L}(\phi+\psi')\sin\theta\right]'-\frac{(\infty+\nu)}{\sin\theta}(\phi\sin\theta)'+\in(\infty+\nu)\psi=\prime \tag{2.2}$$

in $(0,\theta_0)\times(0,T)$ where $\mathcal{L}(\phi)=\phi''+\phi'\cot\theta-\phi(\nu+\cot^{\in}\theta)$, $\varepsilon=h^2/3R^2$ and $\nu\in(-1,1/2)$; with the boundary conditions: $\phi(0,t)=\phi(0,t)+\psi'(0,t)=\mathcal{L}(\phi+\psi)_{\theta=\prime}=\prime$

$$\mathcal{B}_\infty\Phi=\phi(\theta_\prime,\sqcup)=\}_\infty(\sqcup)\qquad \mathcal{B}_\in\Phi=\phi(\theta_\prime,\sqcup)+\psi'(\theta_\prime,\sqcup)=\}_\in\qquad \mathcal{B}_\ni\Phi=\mathcal{L}(\phi+\psi')_{\theta=\theta_\prime}=\}_\ni$$

*and initial conditions $\Phi(0)=\Phi^0$, $\dot{\Phi}(0)=\Phi^1$ satisfies the following conditions: $\Phi(T)=0$, $\dot{\Phi}(T)=0$.*If we show that for T large enough the complementary system $\mathbf{C}_j$ of boundary conditions in θ_0,defines a *norm* on the set of initial data $\{\mathbf{v}^0,\mathbf{v}^1\}$ of the homogeneous problem associated to the problem (2.1)-(2.2), i.e.$\left(\Sigma_{j=1}^3\int_0^T(\mathbf{C}_j\mathbf{v})^2\,\mathrm{d}\,t\right)^{1/2}=\|\{\mathbf{v}^0,\mathbf{v}^1\}\|_{\mathbf{F}}$,then the controllability problem can be solved and we have exact controllability for any $\{\Phi^1,\Phi^0\}\in\mathbf{F}'$.The first result on the exact controllability for thin hemispherical shell is given in [1](see references therein) where we proved the existence of an asimptotic gap (depending on ϵ)for the square root of the eigenvalues of $\mathbf{A}$.Studying the exact controllability by means of spectral properties,we can get to an example of non exact controllability for the limit case $h=0$ we report below in section 2.1.More recent results on the stabilizability (hence controllability) of shallow thin spherical shells are given in [5]. We observe that when the shallowness goes to zero (i.e.$\theta_0\to0$ and $0<a=Rsin\theta_0<\infty$) we find the classical exact controllability results for plate and wave equations.

2.1. Membrane Approximation

For hemispherical shell the orthonormalized (in $\mathbf{H}$) eigenfunctions $\mathbf{v}_n^{\pm}=(u_n^{\pm},w_n^{\pm})$ of $\mathbf{A}^m$ can be easily computed.We proved (see [1],[7]) the limit problem $h=0$ (and hence $\varepsilon=0$) is not exactly controllable for any$\{\Phi^1,\Phi^0\}\in U'\times L^2\times\mathbf{H}$.Indeed we can observe that for all $n{u^-}'_n(\pi/2)-(1+\nu)w_n^-(\pi/2)\neq0$ moreover $\lim_{n\to\infty}{u^-}'_n(\pi/2)-(1+\nu)w_n^-(\pi/2)=0$ .Then if we choose the sequence $\{\mathbf{v}_n^0,\mathbf{v}_n^1\}\in(U\times L^2)\times\mathbf{H}$ of initial data for the homogeneous problem associated tothe E.C. $\varepsilon=0$, in such way $\mathbf{v}_n^0=\mathbf{v}_n^-$, $\mathbf{v}_n^1=0$we have$\|\{\mathbf{v}_n^0,\mathbf{v}_n^1\}\|_{\mathbf{H}\times\mathbf{H}}=1$and $a^m(\mathbf{v}_n^0,\mathbf{v}_n^0)\to(1-\nu^2)$ as $n\to\infty$ hence, since a^m is continous and coercive in $U\times L^2$,$\|\{\mathbf{v}_n^0,\mathbf{v}_n^1\}\|_{U\times L^2\times\mathbf{H}}=\|\mathbf{v}_n^0\|_{U\times L^2}\to const.>0$ as $n\to\infty$ and for any given $T>0$,$\int_0^T[{u^-}'_n(\pi/2)-(1+\nu)w_n^-(\pi/2)]^2\,\mathrm{d}\,t\to0$ as $n\to\infty$ .These last two conditions are in contradiction with the necessary (and sufficient) condition for the controllability:

$$\left(\int_0^T(\mathbf{C}_1\mathbf{v})^2\,\mathrm{d}\,t\right)=\int_0^T[u'(\pi/2,t)-(1+\nu)w(\pi/2,t)]^2\,\mathrm{d}\,t\geq c\|\{\mathbf{v}^0,\mathbf{v}^1\}\|^2_{U\times L^2\times\mathbf{H}}.$$

Since the exact controllability for membrane approximation generally fails,we can look for a relaxed exact controllability result or a partial exact controllability result ([6] [7]).

3. The Eversion of Thin Shells

Another phenomenon depending on the thickness of the shell is the so called "eversion".The analysis for the eversion of thin shells is carried out from theoretical and numerical point of view ([2][3][4][10]) for shallow spherical shells in the static case.The existence of equilibrium configurations, interpretable as everted stressed configurations of the shell,is proved for small values of the thinness parameter ϵ.We consider (1.1) with $f_{\alpha\beta} = 0$ andintroduce the stresses tensors $n^{\alpha\beta}(\mathbf{v}) = a^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{v})$. The equations for the axially symmetric deformations of a spherical cap are expressible in terms of n^{11} and the derivative of the radial displacement v_3.After replacing $cot\theta = 1/\theta$,we have in the new variables $S = n^{11}$ and $f = v_3'/\theta$ the following system

$$S'' + 3\frac{S'}{\theta} = -f(1 + \frac{1}{2}f), \qquad f'' + 3\frac{f'}{\theta} = \frac{S(1+f)}{\epsilon} \tag{3.1}$$

and consider the boundary conditions$S(0) = f'(0) = S(\theta_0) = \theta_0 f'(\theta_0) + (1+\nu)f(\theta_0) = 0$.We denote by $K = \{g;\ \int_0^{\theta_0} g^2(r)r^3dr < \infty\}$ and $K_0^1 = \{g \in K;\quad \frac{dg}{dr} \in K \quad g(\theta_0) = 0\}$.We put $\|g\|_0 = \|\frac{dg}{dr}\|_Kand\|g\|_1 = \|g\|_0 + (1+\nu)\theta_0^2\, g^2(\theta_0)$, then the solutions of the above equations are the stationary points of the following functional

$$J(f) = \frac{1}{2}\|f\|_1^2 + \frac{1}{\epsilon}\|G_0(f(f+2)\|_1^2, \qquad S = G_0(f(f+2)) \tag{3.2}$$

Besides the trivial *stable* solution (absolute minimum for the above functional) $f = S = 0$, for ϵ small enough, we have other two solutions(one of these is stable). The branch of nontrivial everted stable shapes (ϵ, f_ϵ) is extendibleto the right until it reaches a limit point (a simple turning point) $(\epsilon^*, f_{\epsilon^*})$. Moreover for $\epsilon \to 0$ the sequence of everted configurations tends to a configuration $(S = 0, \quad f = -2)$ that is the reflection with the respect to the horizontal plane of the middle surface of the shell in its reference configuration. According to the linear analysis of the transitionshell/plate considered in [8][9],the asymptotic behavior for $\theta_0 \to 0$ of our nonlinear problem has been studied in [10] also under the action of an external load (tensile load or compressive load). The limit problem is actually a circular plate problem and leads, for example under radial compressive forces, to the classical buckling mode of a simply supported circular plate.

4. References

1. G. Geymonat, P. Loreti and V. Valente, in *Spectral Analysis of Complex Struttures*, Travaux en Cours **49**(Hermann, Paris, 1995) 35–57.
2. G. Geymonat, A. Leger, in *Lecture Notes in Pure and Appl. Math.*, **163**(Dekker, New York, 1994) 241–260.
3. G. Geymonat, M. Rosati and V. Valente, in *Computer Methods in Appl. Mech. Eng.*, **75**(1989) 39–52.
4. P. Podio Guidugli, M. Rosati, A. Schiaffino and V. Valente, in *SIAM J. Math Anal.*, **20**(1989) 643–663.
5. I. Lasiecka, R. Triggiani and V. Valente, in *Ad. Diff. Eq.* **1** (1996) 635–674.
6. P. Loreti and V. Valente, in *SIAM J. Control Opt.* (1997).
7. V. Valente, in *J. Math. Pures Appl.* (1997).
8. P. G. Ciarlet, in *C.R. Acad. Sci. Paris, serie I*, **315** (1992) 107–111, 227–233.
9. E. Sanchez–Palencia, in *C.R. Acad. Sci. Paris, serie I*, **318** (1994) 783–790.
10. G. Geymonat, A. Leger, in *C.R. Acad. Sci. Paris, serie I*, **319** (1994) 305–310.

DECAY RATES IN THERMOELASTICITY

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ABSTRACT

In this lecture we will present some recent results on the rate of decay of the energy for some systems of thermoelasticity. First we will address the von Kármán system of thermoelastic plates. We will show how the exponential decay of the energy may be proved by using a suitable Lyapunov function. Then, we will consider the linear system of 3-d thermoelasticity with Dirichlet boundary conditions. We will show that, due to the weak interaction between transversal and longitudinal waves, the decay is not uniform for convex domains.

1. The von Kármán system for thermoelastic plates

Let Ω be a bounded smooth domain of $\mathbb{R}^2$. Let us denote by $u = u(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$ the vertical displacement of the plate and by $\theta = \theta(x,t)$ the temperature.

Let us consider the system:

$$\left\{\begin{array}{lll} u_{tt} - h\Delta u_{tt} + \Delta^2 u + \Delta\theta = [u,v] & \text{in} & \Omega \times (0,\infty) \\ \Delta^2 v = [u,u] & \text{in} & \Omega \times (0,\infty) \\ \theta_t - \Delta\theta + \Delta u_t = 0 & \text{in} & \Omega \times (0,\infty) \\ u = \Delta u = 0,\ v = \Delta v = 0, & \text{on} & \partial\Omega \times (0,\infty) \\ \theta = 0 & \text{on} & \partial\Omega \times (0,\infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} & \Omega \\ \theta(x,0) = \theta^0(x) & \text{in} & \Omega. \end{array}\right. \tag{1}$$

In (1), $h \geq 0$ denotes the constant of rotational inertia of the plate and the bracket $[\cdot,\cdot]$ is defined as follows:

$$[\varphi,\psi] = \frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\psi}{\partial x_2^2} - 2\frac{\partial^2\varphi}{\partial x_1\partial x_2}\frac{\partial^2\psi}{\partial x_1\partial x_2} + \frac{\partial^2\varphi}{\partial x_2^2}\frac{\partial\psi}{\partial x_1^2}. \tag{2}$$

The energy of the system is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[|u_t|^2 + h\,|\nabla u_t|^2 + |\Delta u|^2\right] dx + \frac{1}{4}\int_\Omega |\Delta v|^2\, dx + \frac{1}{2}\int_\Omega \theta^2 dx. \tag{3}$$

For initial data of finite energy and satisfying suitable boundary conditions system (1) has a unique solution.

The energy decreases along trajectories. More precisely,

$$\frac{dE}{dt}(t) = -\int_\Omega |\ \nabla\theta\ |^2\ dx \leq 0. \tag{4}$$

The following result, established in a joint work with G. Perla Menzala [5], guarantees the exponential decay of the energy:

Theorem 1 *There exists* $C > 0$ *and* $\omega > 0$ *such that*

$$E(t) \leq C \exp\left(-\frac{\omega}{1+R^2}t\right) E(0), \forall t \geq 0 \tag{5}$$

for every solution of (1) *such that* $E(0) \leq R$.

Remark 1

(a) The constant C and ω in the statement of Theorem 1 depend on Ω and h but do not depend on the initial data.

(b) The estimate (5) guarantees an exponential decay rate of the order of R^{-2} as $E(0) = R \to \infty$. We do not know whether this estimate is sharp.

■

The method of proof of Theorem 1 consists, roughly, on finding a suitable perturbation F of the energy E for which an inequality f the form

$$\frac{dF}{dt} \leq -cF \tag{6}$$

holds, F being equivalent to E, i.e.

$$\frac{1}{2}F \leq E \leq 2F. \tag{7}$$

It is clear that (6) and (7) provide an exponential decay rate for the energy E.

The Lyapunov function we introduce is of the form

$$F = E + \varepsilon\rho$$

with ε small enough an ρ given by

$$\rho = \int_\Omega \left[hu_t\theta - \frac{h}{2}\theta^2 + u_t(-\Delta)^{-1}\theta + \frac{uu_t}{2} + \frac{h}{2}\nabla u \cdot \nabla u_t \right] dx,$$

where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian.

In the lecture we will discuss the role that ρ plays in the proof and how the dependence of the decay rate on R appears.

2. The 3-d linear system of linear thermoelasticity

Let Ω be a bounded smooth domain of $\mathbb{R}^3$ and consider the following system:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad \Omega \times (0,\infty) \\ u = 0, \quad \theta = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} \quad \Omega \\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{8}$$

This time $u = (u_1, u_2, u_3)$ is a vector field, λ and μ are the Lamé coefficients and $\alpha, \beta > 0$ the coupling parameters.

The energy is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[| u_t |^2 + \mu | \nabla u |^2 + (\lambda+\mu) | \operatorname{div} u |^2 \right] + \frac{\alpha}{2\beta}\int_\Omega \theta^2 \tag{9}$$

and is dissipated along trajectories, i.e.

$$\frac{dE}{dt}(t) = -\frac{\alpha}{\beta}\int_\Omega | \nabla\theta |^2 \leq 0. \tag{10}$$

C. Dafermos [1] proved that the energy of every solution tends to zero as $t \to \infty$ if and only if the following eigenvalue problem has no non–trivial solution:

$$\left\{ \begin{array}{lll} -\Delta\varphi = \gamma^2\varphi & \text{in} & \Omega \\ \operatorname{div}\varphi = 0 & \text{in} & \Omega \\ \varphi = 0 & \text{on} & \partial\Omega. \end{array} \right. \tag{11}$$

When Ω is a ball non–trivial solutions of (11) do exist. However, generically with respect to the domain Ω, the eigenvalues of the Laplacian are simple and in this case the existence of non–trivial solutions of (11) can be easily excluded. We refer to J.-L. Lions and E. Zuazua [4] for other applications of this type of arguments.

In this lecture we will discuss the problem of the uniform decay, i.e. of whether there exist positive constants C and ω such that

$$E(t) \leq C\exp(-\omega t)E(0), \quad \forall t > 0 \tag{12}$$

holds for every solution of (8).

We will give a sketch of the proof of the following result obtained in collaboration with G. Lebeau in [3]:

Theorem 2 *When Ω is convex* (12) *does not holds, i.e. the decay rate of solutions of* (8) *is not uniform.*

The proof of the result combines two ingredients. The first one consists on applying the decoupling method by D. Henry, O. Lopes and A. Perissinotto [2]. This allows to reduce the problem to the analysis of the system of elasticity:

$$\left\{ \begin{array}{lll} u_{tt} - \mu\Delta u - (\lambda+\mu)\operatorname{div} u = 0 & \text{in} & \Omega\times(0,\infty) \\ u = 0 & \text{in} & \Omega\times(0,\infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} & \Omega \end{array} \right. \tag{13}$$

and more precisely to the existence of a time $T > 0$ and a constant $C > 0$ such that

$$\| u^0 \|^2_{(L^2(\Omega))^3} + \| u^1 \|^2_{(H^{-1}(\Omega))^3} \leq C\int_0^T \| \operatorname{div} u \|^2_{H^{-1}(\Omega)}\, dt \tag{14}$$

for every solution of (13).

A geometric optics construction in the spirit of J. Ralston [6] allows to show that (14) does not hold if there exists a ray in Ω that is always reflected perpendicularly on the boundary. This is obviously the case when Ω is a convex smooth domain.

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4. References

1. C. Dafermos, *Proc. AMS.* **48** (2) (1975), 413–418.
2. D. Henry, O. Lopes and A. Perissinotto, *Nonlinear Anal. TMA.* **21** (1993), 65–75.
3. G. Lebeau and E. Zuazua, *C. R. Acad. Sci. Paris*, to appear.
4. J.-L. Lions and E. Zuazua, in *Partial Differential Equations and Applications*, ed. P. Marcellini, G. Talenti and E. Visentini (LNPAS 177, Marcel–Dekker Inc., 1996).
5. G. Perla-Menzala and E. Zuazua, *C. R. Acad. Sci. Paris*, **324** (1997), 49-54.
6. J. Ralston, in *Studies in Partial Differential Equations*, ed. W. Littman, (MAA Studies in Mathematics, vol. **23**, 1982).

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