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RESEARCH ARTICLE

Positive Homoclinic Solutions of n-th Order Difference Equations with Sign-changing Green's Function

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Abstract

In this paper we consider a n-th order nonlinear difference equation with parameter dependence. An exhaustive study of the related Green's function is done. The exact expression of the function is given. The range of parameter for which either it has constant sign or it changes sign is obtained. Some existence results for the nonlinear problem are deduced by using the classical Krasnosel'skii's fixed point theorem on cones and fixed point index theory.

KEYWORDS:
difference equation, homoclinic solutions, sign-changing Green's function, positive solutions, parameter dependence

1 | INTRODUCTION AND PRELIMINARIES

Recently, the existence of positive solutions for boundary value problems has been widely studied by many authors. Obtaining homoclinic solutions is usually based on establishing a proper variational framework and using the critical point theory [4, 9, 13, 15]. In [13], using suitable conditions, the authors proved existence of nontrivial solutions of the fourth-order difference problem

$$\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right)-\Delta\left(q_n\Delta u_{n-1}\right)+r_n u_n=f\left(n,u_{n+1},u_n,u_{n-1}\right),\ n\in\mathbb{Z},$$

where $\{p_n\},\{q_n\}$ and $\{r_n\}$ are real sequences, $f\in C\left(\mathbb{Z}\times\mathbb{R}^3,\mathbb{R}\right)$, T is a given positive integer, $p_{n+T}=p_n>0, q_{n+T}=q_n>0$, $r_{n+T}=r_n>0$ and $f\left(n+T,v_1,v_2,v_3\right)=f\left(n,v_1,v_2,v_3\right)$.

However, in most of the papers that do not use this theory, the results was mainly dependent of the constant sign of the associated Green's function to the periodic or Dirichlet problems [5, 14] in bounded domains and, more recently, for homoclinic solutions of second order equations [6, 12]. In [2, 8] the authors studied the existence of positive solutions to the periodic boundary value problems. In the first one, the authors obtained existence, multiplicity and nonexistence results for the following nonlinear fourth order problem with parameter dependence

$$\begin{cases} u(k+4)+Mu(k)=\lambda g(k)f(u(k))+c(k),\ k\in\{0,\dots,T-1\},\\ u(i)=u(T+i),\ i=0,\dots,3,\end{cases}$$

in which the Green's function is non-negative and attains the zero value at some points of its set of definition.

In the latter one, Graef, Kong and Wang studied the periodic problem

$$\begin{cases} y''+a(t)y=g(t)f(y),\ 0\le t\le 2\pi,\\ y(0)=y(2\pi),\ y'(0)=y'(2\pi),\end{cases}$$

also assuming that the Green's function vanishes at some points of its set of definition imposing that $f:[0,\infty)\to[0,\infty)$ is a continuous, convex and nondecreasing, $g:[0,2\pi)\to[0,\infty)$ is a continuous with $\min_{t\in[0,2\pi]} g(t)>0$ and $\min_{0\le s\le 2\pi}\int_0^{2\pi} G(t,s)dt>0$.

In the recent paper [7] Gao, Zhang and Ma extended these results and obtained existence of positive solution to the periodic problem

$$\begin{cases} u''+\left(\frac{1}{2}+\varepsilon\right)^2 u=\lambda g(t)f(u),\ 0<t<2\pi,\ 0<\varepsilon<\frac{1}{2},\ \lambda>0,\\ u(0)=u(2\pi),\ u'(0)=u'(2\pi),\end{cases}$$

under the assumption that $g:[0,2\pi]\to\mathbb{R}$ is continuous, $g\not\equiv 0$, and there exists a number $k>1$ such that

$$\int_0^{2\pi}(G(t,s)g(s))^+\,ds\ge k\int_0^{2\pi}(G(t,s)g(s))^-\,ds,\ t\in[0,2\pi].$$

Such results have been improved in [3] for the second order Hill's equation

$$u''(t)+a(t)u(t)=f(t,u(t)),\ t\in[0,T],$$

with periodic, Dirichlet, Neumann or mixed conditions.

In this case, existence results have been obtained by allowing to the related Green's function to change its sign on the square of definition and imposing some relationship of function f, that define the nonlinear part of the equation, with the first eigenvalue of the linear part of the equation.

In this paper we are able to prove existence of positive solutions of the following problem

$$u(k+n)+Mu(k)=g\left(k\right)f\left(u(k)\right),\ k\in\mathbb{Z}, \tag{1}$$

where $M \neq 0, \pm 1$ is a real parameter, n is a fixed positive integer, $g \not\equiv 0$ is nonnegative and f is a continuous function. We are looking for nontrivial positive homoclinic solutions of the considered problem, i.e.

$$u(k) > 0 \text{ for all } k \in \mathbb{Z} \ \text{ and } \lim_{|k| \to \infty} u(k) = 0. \tag{2}$$

The arguments are in the basis of the ones given in [3, 7] but, in our case, for an unbounded domain. As far as the authors know, this situation, positive solutions with sign-changing Green's function and unbounded intervals of definition, is new in the literature under this framework.

The paper is developed as follows: in Section 2 we study the Green's function related to the linear problem to be considered. The exact expression is obtained, depending on the value of the real parameter M. Section 3 is devoted to the study of the existence of solution for the considered nonlinear problem by using the classical Krasnosel'skii's fixed point theorem when the Green's function has a constant sign. In Section 4 we use fixed point index theory to obtain existence of positive solutions for the nonlinear problem with sign-changing Green's function.

We illustrate the obtained results with suitable example.

2 | STUDY OF THE GREEN'S FUNCTION

Let us define the space

$$X = \left\{ u : \mathbb{Z} \to \mathbb{R}, \ \max_{k \in \mathbb{Z}} |u(k)| < \infty, \ \lim_{|k| \to \infty} u(k) = 0 \right\},$$

with the norm $\|u\|_X = \max_{k \in \mathbb{Z}} |u(k)|$.

It is not difficult to verify that $(X, \|\cdot\|_X)$ is a Banach space.

First we study the Green's function of problem

$$u(k+n) + M u(k) = h(k), \ k \in \mathbb{Z}, \ M \in \mathbb{R}, \quad \lim_{|k| \to \infty} u(k) = 0, \tag{3}$$

with

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty.$$

That is, we look for a function $G : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ such that u is a solution of problem (3) if and only if

$$u(k) = \sum_{k=-\infty}^{\infty} G(k,s)\, h(s) \quad \text{for all } \ k \in \mathbb{Z}. \tag{4}$$

Following the approach given in ([10], Chapter 6.3), function G must satisfy, for every $k, \ s \in \mathbb{Z}$:

$$G(k+n, s) + M\, G(k,s) = \begin{cases} 1, \text{ if } k = s, \\ 0, \text{ if } k \neq s \end{cases} \tag{5}$$

and

$$\lim_{|k|\to\infty} G(k,s) = 0, \quad \text{for all } \ s \in \mathbb{Z}. \tag{6}$$

Notice that, from (5) the first part of equation (3) is immediately verified by u given in (4). Once we prove that $|G|$ is bounded, the homoclinic boundary condition in (3) holds from the discrete Lebesque dominated convergence Theorem.

We separate the study in two parts, depending on the sign of the real parameter M.

Case 1. Let us suppose that $M < 0$. So we have that $M = -p^n$, for some $p > 0$.

The related Green's function has the following expression

$$G(k,s) = \begin{cases} G_1(k,s), \text{ if } k \le s, \\ G_2(k,s), \text{ if } k > s, \end{cases}$$

where

$$G_1(k,s) = p^k \left(C_1(s)\varpi_1^k + C_2(s)\varpi_2^k + \ldots + C_n(s)\varpi_n^k \right)$$

and

$$G_2(k,s) = p^k \left(C_{n+1}(s)\varpi_1^k + C_{n+2}(s)\varpi_2^k + \ldots + C_{2n}(s)\varpi_n^k \right).$$

Here $\varpi_k = e^{\frac{2k\pi}{n} i}$, $k = 1, 2, \ldots, n$.

We have that

$$G(k+n,s) - p^n G(k,s) = \begin{cases} 1, \text{ if } k = s, \\ 0, \text{ if } k \ne s. \end{cases}$$

This gives us the following system

$$G_2(s+n-j,s) - p^n G_1(s-j,s) = \delta_{0,j} := \begin{cases} 1, \text{ if } j = 0, \\ 0, \text{ if } j \ne 0. \end{cases}$$

It is clear that for $p = 1$ ($M = -1$), the only possibility to ensure that, for any fixed s, the Green's function converges asymptotically to zero is that it is the zero function.

So, we must distinguish two situations:

Case 1.1 If $p > 1$ ($M < -1$) we have that, for any fixed s, $\lim\limits_{k\to-\infty} G_1(k,s) = 0$, for any value of the coefficients $C_j(s)$, $j \in \{1, \ldots, n\}$.

However, $\lim\limits_{k\to\infty} G_2(k,s) = 0$ if and only if $C_j(s) = 0$ for all $j \in \{n+1, \ldots, 2\,n\}$, i.e. $G_2(k,s) \equiv 0$.

Thus, the above system is equivalent to

$$\begin{cases} p^{s+n}\left(C_1(s)\varpi_1^s + C_2(s)\varpi_2^s + \ldots + C_n(s)\varpi_n^s\right) &= -1\\ C_1(s)\varpi_1^{s-1} + C_2(s)\varpi_2^{s-1} + \ldots + C_n(s)\varpi_n^{s-1} &= \ \ 0\\ C_1(s)\varpi_1^{s-2} + C_2(s)\varpi_2^{s-2} + \ldots + C_n(s)\varpi_n^{s-2} &= \ \ 0\\ &\ \ \vdots\\ C_1(s)\varpi_1^{s-n+1} + C_2(s)\varpi_2^{s-n+1} + \ldots + C_n(s)\varpi_n^{s-n+1} &= \ \ 0 \end{cases} \cdot$$

After solving it, we obtain that

$$C_1(s) = \frac{-\varpi_1^{-s}}{np^{s+n}},\ C_2(s) = \frac{-\varpi_2^{-s}}{np^{s+n}},\ \ldots,\ C_n(s) = \frac{-\varpi_n^{-s}}{np^{s+n}}.$$

Then

$$G_1(k,s) = -\frac{1}{n}p^{k-s-n}\left(\varpi_1^{k-s} + \varpi_2^{k-s} + \ldots + \varpi_n^{k-s}\right).$$

Let

$$A_1(k) := \{s = k + d\,n,\ d = 0,1,2,\ldots\}.$$

One can check that

$$\varpi_1^{k-s} + \varpi_2^{k-s} + \ldots + \varpi_n^{k-s} = \begin{cases} n, \text{ if } s \in A_1(k),\\ 0, \text{ otherwise.}\end{cases}$$

Thus, we deduce that

$$G(k,s) = \begin{cases} -p^{k-s-n}, & \text{if } s \in A_1(k),\\ 0, & \text{otherwise.}\end{cases}$$

Case 1.2 If $p < 1$ ($M \in (-1,0)$) then, for any fixed s, $\lim\limits_{k\to\infty} G_2(k,s) = 0$, for any value of the coefficients $C_j(s)$, $j \in \{n+1,\ldots,2\,n\}$.

Moreover, $\lim\limits_{k\to-\infty} G_1(k,s) = 0$ if and only if $G_1(k,s) \equiv 0$.

Using similar arguments as above we obtain that in this case

$$G(k,s) = \begin{cases} p^{k-s-n}, & \text{if } s \in A_2(k),\\ 0, & \text{otherwise,}\end{cases}$$

where

$$A_2(k) := \{s = k - d\,n,\ d = 1,2,\ldots\}.$$

Case 2. Now, let us suppose that $M > 0$ and let $M = p^n$, $p > 0$.

In this case, the related Green's function has the following expression

$$G(k,s)=\begin{cases} G_1(k,s), \text{ if } k\le s,\\ G_2(k,s), \text{ if } k>s,\end{cases}$$

where

$$G_1(k,s)=p^k\left(C_1(s)\varpi_1^k+C_2(s)\varpi_2^k+...+C_n(s)\varpi_n^k\right)$$

and

$$G_2(k,s)=p^k\left(C_{n+1}(s)\varpi_1^k+C_{n+2}(s)\varpi_2^k+...+C_{2n}(s)\varpi_n^k\right).$$

Here $\varpi_k=e^{\frac{(2k+1)\pi}{n}i}$, $k=1,2,\dots,n$.

Now, the following system is fulfilled

$$G_2(s+n-j,s)+p^nG_1(s-j,s)=\delta_{0,j}\ :=\begin{cases}1, \text{ if } j=0,\\ 0, \text{ if } j\neq 0,\end{cases}$$

which is equivalent to

$$G(k+n,s)+p^nG(k,s)=\begin{cases}1, \text{ if } k=s,\\ 0, \text{ if } k\neq s.\end{cases}$$

As in the negative case, we have that there is no nontrivial Green's function for $p=1$ ($M=1$). So, we must consider two cases.

Case 2.1 If $p>1$ ($M>1$) then, in order to ensure the asymptotic behavior of the Green's function, we arrive to the same conclusions as in Case 1.1.

In this case, the above system is as follows

$$\begin{cases} p^{s+n}\left(C_1(s)\varpi_1^s+C_2(s)\varpi_2^s+\ldots+C_n(s)\varpi_n^s\right) &= 1\\ C_1(s)\varpi_1^{s-1}+C_2(s)\varpi_2^{s-1}+\ldots+C_n(s)\varpi_n^{s-1} &= 0\\ C_1(s)\varpi_1^{s-2}+C_2(s)\varpi_2^{s-2}+\ldots+C_n(s)\varpi_n^{s-2} &= 0\\ &\vdots\\ C_1(s)\varpi_1^{s-n+1}+C_2(s)\varpi_2^{s-n+1}+\ldots+C_n(s)\varpi_n^{s-n+1} &= 0\end{cases}\cdot$$

After solving it, we obtain that

$$C_1(s)=\frac{\varpi_1^{-s}}{np^{s+n}},\ C_2(s)=\frac{\varpi_2^{-s}}{np^{s+n}},\ \ldots,\ C_n(s)=\frac{\varpi_n^{-s}}{np^{s+n}}.$$

Then

$$G_1(k,s)=\frac{1}{n}p^{k-s-n}\left(\varpi_1^{k-s}+\varpi_2^{k-s}+...+\varpi_n^{k-s}\right).$$

It is not difficult to verify that

$$\varpi_1^{k-s} + \varpi_2^{k-s} + \ldots + \varpi_n^{k-s} = \begin{cases} 0, & \text{if } s \notin A_1(k), \\ n, & \text{if } d = \frac{s-k}{n} \text{ is even,} \\ -n, & \text{if } d = \frac{s-k}{n} \text{ is odd,} \end{cases}$$

and so, we conclude that

$$G(k,s) = \begin{cases} (-1)^{\frac{s-k}{n}}\, p^{k-s-n}, & \text{if } s \in A_1(k), \\ 0, & \text{otherwise.} \end{cases}$$

Case 2.2 In the other case when $p < 1$ ($M \in (0,1)$), using similar arguments as before, we obtain that $G_1(k,s) \equiv 0$ and

$$G(k,s) = \begin{cases} (-1)^{\frac{k-s}{n}+1}\, p^{k-s-n}, & \text{if } s \in A_2(k), \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to verify that both in the positive and negative cases, we have that $|G(k,s)| < p^{-n}$.

Arguing in a similar manner with $M = 0$, we have that only the trivial Green's function has sense for this situation.

We can summarize the above results as follows:

Theorem 1. The Green's function related to problem

$$G(k,s) = \begin{cases} -(-M)^{\frac{k-s}{n}-1}, & \text{if } |M| > 1 \text{ and } s \in A_1(k), \\ (-M)^{\frac{k-s}{n}-1}, & \text{if } |M| < 1, M \neq 0 \text{ and } s \in A_2(k), \\ 0, & \text{otherwise.} \end{cases}$$

3 | POSITIVE GREEN'S FUNCTION

To the end of this section, let us assume that $M \in (-1,0)$ and also the following conditions hold:

$(H1)$ $g \geq 0$ on $\mathbb{Z}$ and $0 < \sum_{s=-\infty}^{\infty} g(s) < \infty$.

$(H2)$ $f : [0,\infty) \to [0,\infty)$ is a continuous function such that $f(x) > 0$ for all $x > 0$.

$(H3)$ f is decreasing in $(0,\delta)$ for some $\delta > 0$.

Let us introduce the notation

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} \text{ and } f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Remark 1. Note that, unless $f \equiv 0$ on $(0,\delta)$, we have that $f_0 = \infty$.

Theorem 2. Assume that $(H1) - (H3)$ hold. Moreover, if $f_0 = \infty$ and $f_\infty = 0$, then (1) has a nontrivial solution.

The following lemma is needed to prove our main result.

Lemma 1. ([11]) Let B be a Banach space and let $K \subset B$ be a cone and $\preceq$ be the order induced by K on B. Assume Ω_1, Ω_2 are bounded open subsets of B with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $F : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to K$ be a completely continuous operator such that either

(i) $Fu \npreceq u$ for any $u \in K \cap \partial\Omega_1$ and $Fu \nsucceq u$ for any $u \in K \cap \partial\Omega_2$,

or

(ii) $Fu \nsucceq u$ for any $u \in K \cap \partial\Omega_1$ and $Fu \npreceq u$ for any $u \in K \cap \partial\Omega_2$.

Then F has a fixed point in $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

Proof of Theorem 2.. From Theorem 1, we have that

$$G(k,s) = \begin{cases} (-M)^{d-1}\,, \text{ if } s = k - dn, d = 1, 2, ... \\ 0, \text{ otherwise.} \end{cases}$$

Moreover,

$$\sum_{k=-\infty}^{\infty} G(k,s) = \sum_{d=0}^{\infty} (-M)^d = \frac{1}{1+M} > 0$$

and $0 \le G(k,s) \le 1$ for every $k, s \in \mathbb{Z}$.

Define the cone K_1 in X by

$$K_1 = \left\{ u \in X,\ u(k) \ge 0,\ k \in \mathbb{Z},\ \sum_{k=-\infty}^{\infty} u(k) \ge \frac{1}{1+M} \|u\|_X \right\}.$$

It is clear that, in this case, for all $u,\ v \in X$, the induced order by K_1 in X is given by

$$u \preceq v \text{ if and only if } v(k) \ge u(k) \text{ for all } \ k \in \mathbb{Z} \ \text{ and } \ \sum_{k=-\infty}^{\infty} (v-u)(k) \ge \frac{1}{1+M} \|v-u\|_X\,.$$

Now, we introduce the operator

$$Tu(k) = \sum_{s=-\infty}^{\infty} G(k,s) g(s) f(u(s)).$$

Notice that, if $u \in K_1$, by means of the discrete Lebesque dominated convergence Theorem, it is not difficult to verify that $T\,u \in X$.

From the definition of the Green's function G, the solutions of problem (1) – (2) coincide with the fixed points of operator T.

From $(H1)$ and $(H2)$ it follows that $Tu(k) \geq 0$ for all $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} Tu(k) &= \sum_{k=-\infty}^{\infty}\left(\sum_{s=-\infty}^{\infty} G(k,s)g(s)f(u(s))\right)\\ &= \sum_{s=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} G(k,s)\right)g(s)f(u(s))\\ &= \frac{1}{1+M}\sum_{s=-\infty}^{\infty} g(s)f(u(s))\end{aligned}$$

and, since $0 \leq G(k,s) \leq 1$,

$$Tu(k) \leq \sum_{s=-\infty}^{\infty} g(s)f(u(s)) \text{ for every } k \in \mathbb{Z}.$$

Thus

$$\sum_{k=-\infty}^{\infty} Tu(k) \geq \frac{1}{1+M}\|Tu\|_X ,$$

i.e., $T(K_1) \subset K_1$.

For any $J \subset \mathbb{Z}$, we denote $|J| = \sum_{k\in J} 1$. So, we define

$$\Gamma := \sup_I \sum_{s=-\infty}^{\infty}\left(\frac{1}{|I|}\sum_{k\in I} G(k,s)\right)g(s) > 0,$$

where the supremum is taken over all finite subsets I of $\mathbb{Z}$.

By using $(H3)$, we have that there is $r_1 \in (0,\delta)$ such that $f(x) \geq \theta x$ for every $x \in (0, r_1]$, where $\theta\Gamma > 1$.

Let $u_1 \in K_1$ be such that $\|u_1\|_X = r_1$ and $(0 \preceq) Tu_1 \preceq u_1$. Then

$$\begin{aligned}\|u_1\|_X &\geq \|Tu_1\|_X \geq \frac{1}{|I|}\sum_{k\in I} Tu_1(k)\\ &= \sum_{s=-\infty}^{\infty}\left(\frac{1}{|I|}\sum_{k\in I} G(k,s)\right)g(s)f(u_1(s))\\ &\geq f\left(\|u_1\|_X\right)\sum_{s=-\infty}^{\infty}\left(\frac{1}{|I|}\sum_{k\in I} G(k,s)\right)g(s),\end{aligned}$$

where I is an arbitrary nonempty finite subset of $\mathbb{Z}$.

Let $\{I_n\}$ be a sequence of nonempty finite subsets of $\mathbb{Z}$, such that

$$\lim_{n\to\infty}\sum_{s=-\infty}^{\infty}\left(\frac{1}{|I_n|}\sum_{k\in I_n} G(k,s)\right)g(s) = \Gamma.$$

Then, taking the limit in the inequality above we obtain that

$$\|u_1\|_X \geq \|u_1\|_X\,\theta\Gamma > \|u_1\|_X ,$$

which is a contradiction. So if $u_1 \in K_1$ is such that $\|u_1\|_X = r_1$, then $Tu_1 \npreceq u_1$.

Suppose that $f_\infty = 0$, then from ([8], Theorem 2.1) we have that

$$\lim_{u\to\infty} \frac{\max\limits_{z\in[0,u]} f(z)}{u} = 0.$$

Then there exists $r_2 > r_1$, such that $\max\limits_{z\in[0,r_2]} f(z) < \frac{r_2}{\sum\limits_{s=-\infty}^{\infty} g(s)}$.

If there exists $u_2 \in K_1$, such that $\|u_2\| = r_2$ and $Tu_2 \succeq u_2$ ($\succeq 0$), then

$$\begin{aligned} r_2 &= \|u_2\|_X \le \|Tu_2\|_X = \max_{k\in\mathbb{Z}} \sum_{s=-\infty}^{\infty} G(k,s)g(s)f(u_2(s)) \\ &\le \sum_{s=-\infty}^{\infty} g(s)f(u_2(s)) < r_2 \sum_{s=-\infty}^{\infty} g(s) \left(\frac{1}{\sum\limits_{s=-\infty}^{\infty} g(s)} \right) = r_2, \end{aligned}$$

which is a contradiction. So $Tu_2 \not\succeq u_2$ for every $u_2 \in K_1$ with $\|u_2\|_X = r_2$.

Since T is a compact operator, then it has a fixed point u such that $r_1 < \|u\|_X < r_2$.

From the expression of G and conditions $(H1)$ and $(H2)$, we conclude that u is a solution of problem (1)-(2). ☐

Remark 2. The case $M \in (-\infty, -1)$ can be applied to the problem

$$u(k+n) + Mu(k) + g(k)\, f(u(k)) = 0, \ k \in \mathbb{Z}, \tag{7}$$

coupled to condition (2).

From Theorem 1, it follows that

$$G(k,s) = \begin{cases} -(-M)^{-d-1}, \text{ if } s = k + dn, d = 0, 1, 2, ... \\ 0, \text{ otherwise.} \end{cases}$$

Moreover,

$$\sum_{k=-\infty}^{\infty} G(k,s) = \frac{1}{M} \sum_{d=0}^{\infty} (-M)^{-d} = \frac{1}{1+M} < 0$$

and $0 \le -G(k,s) \le -1/M$ for every $k, s \in \mathbb{Z}$.

One can define the cone K_1^* in X by

$$K_1^* = \left\{ u \in X, \ u(k) \ge 0, \ k \in \mathbb{Z}, \ \sum_{k=-\infty}^{\infty} u(k) \ge \frac{M}{1+M} \|u\|_X \right\},$$

the operator

$$Tu(k) = -\sum_{s=-\infty}^{\infty} G(k,s)g(s)f(u(s))$$

and

$$\Gamma := \sup_I \sum_{s=-\infty}^{\infty} \left(\frac{1}{|I|} \sum_{k\in I} (-G(k,s)) \right) g(s) > 0,$$

where the supremum is taken over all finite subsets I of $\mathbb{Z}$.

Then, using similar arguments as before, it can be deduced that there exists $r_1 \in (0,\delta)$ such that $f(x) \geq \theta x$ for every $x \in (0,r_1]$, where $\theta\,\Gamma > 1$ and there exists $r_2 > r_1$, such that $\max\limits_{z\in[0,r_2]} f(z) < -\frac{M\,r_2}{\sum\limits_{s=-\infty}^{\infty} g(s)}$.

Since T is a compact operator, then there exists a solution u of problem (7), (2) such that $r_1 < \|u\|_X < r_2$.

4 | SIGN-CHANGING GREEN'S FUNCTION

Now, along this section, let us suppose that $M > 1$.

We can study the problem with even more general right hand side:

$$u(k+n) + Mu(k) = F(k,u(k)),\ n \in \mathbb{N},\ k \in \mathbb{Z}. \tag{8}$$

From Theorem 1 the related Green's function changes its sign on $\mathbb{Z}\times\mathbb{Z}$.

To deduce the existence results, we follow the approach given in [3] for second order ordinary differential equations. Let us denote G^+ and G^- as the positive and negative parts of the Green's function $G(k,s)$.

In order to ensure the existence results, we make the following assumptions:

$(H1^*)$ $F : \mathbb{Z}\times[0,\infty) \to [0,\infty)$ is continuous.

$(H2^*)$ There exists a function $v : \mathbb{Z} \to (0,\infty)$ such that

$$\sum_{s=-\infty}^{\infty} G(k,s)v(s) > 0 \text{ and } \sum_{s=-\infty}^{\infty} v(s) < \infty$$

for any $k \in \mathbb{Z}$.

Also, there exist positive constants m_* and m^* such that

$$m_* v(s) \leq F(s,x) \leq m^* v(s) \text{ for every } s \in \mathbb{Z} \text{ and } x \geq 0.$$

Moreover, these constants satisfy that $\frac{m^*}{m_*} < \gamma$, where

$$\gamma = \min_{k\in\mathbb{Z}} \frac{\sum_{s=-\infty}^{\infty} G^+(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^-(k,s)v(s)} > 1.$$

Lemma 2. For any $c \in \mathbb{Z}$ the following inequalities hold:

$$\sum_{k=c}^{\infty} G(k,s) \geq 0 \text{ for all } s \in \mathbb{Z} \text{ and } \sum_{k=c}^{\infty} G(k,s) > 0 \text{ for all } s \geq c.$$

In addition,

$$\min_{s\geq c}\left\{\sum_{k=c}^{\infty} G(k,s)\right\} = \frac{1}{M} - \frac{1}{M^2}.$$

Proof. One can check that

$$\sum_{k=c}^{\infty} G(k,s) = \begin{cases} 0, \text{ if } s < c, \\ \frac{1}{M}, \text{ if } s \in [c, c+n-1], \\ \frac{1}{M} - \frac{1}{M^2}, \text{ if } s \in [c+n, c+2n-1], \\ \frac{1}{M} - \frac{1}{M^2} + \frac{1}{M^3}, \text{ if } s \in [c+2n, c+3n-1], \\ \vdots \end{cases}$$

Thus

$$\sum_{k=c}^{\infty} G(k,s) = \begin{cases} 0, \text{ if } s < c, \\ \frac{1+\frac{(-1)^{t-1}}{M^t}}{M+1}, \text{ if } s \in [c+(t-1)n, c+tn-1], \ t = 1,2,3,\ldots \end{cases}$$

From the above expression it is obvious that

$$\min_{s \geq c} \left\{ \sum_{k=c}^{\infty} G(k,s) \right\} = \frac{1}{M} - \frac{1}{M^2}.$$

□

Denote

$$\sigma = M \min_{c \in \mathbb{Z}} \left\{ \min_{s \geq c} \left\{ \sum_{k=c}^{\infty} G(k,s) \right\} \right\} = 1 - \frac{1}{M}$$

and let us define the cone

$$K_2 = \{u \in X, \quad u(k) \geq 0 \text{ for all } k \in \mathbb{Z}\}.$$

It is clear that u is a solution of problem (8), (2) if and only if it is a fixed point of the operator

$$Tu(k) = \sum_{s=-\infty}^{\infty} G(k,s) F(s, u(s)).$$

Lemma 3. Assume that conditions $(H1^*)$ and $(H2^*)$ hold. Then T is a completely continuous operator which maps the cone K_2 to itself.

Proof. Let $u \in K_2$. Since $|G| \leq 1$, it is clear that $\|T\|_X < \infty$. This, together $(H2^*)$ and the discrete Lebesque dominated convergence theorem, implies that $T\,u \in X$.

Moreover, the following inequalities hold for all $k \in \mathbb{Z}$:

$$\begin{aligned} Tu(k) &= \sum_{s=-\infty}^{\infty} G(k,s)F(s,u(s)) = \sum_{s=-\infty}^{\infty} \left(G^+(k,s) - G^-(k,s) \right) F(s,u(s)) \\ &\geq \sum_{s=-\infty}^{\infty} m_* v(s) G^+(k,s) - m^* v(s) G^-(k,s) \\ &> m_* \left(\sum_{s=-\infty}^{\infty} v(s) G^+(k,s) - \gamma v(s) G^-(k,s) \right) \geq 0. \end{aligned}$$

□

Now, from condition $(H2^*)$ one can deduce that

$$F_0 = \lim_{x\to 0^+}\left\{\min_{t\in\mathbb{Z}} \frac{F(t,x)}{x}\right\} = \infty \quad \text{and} \quad F_\infty = \lim_{x\to\infty}\left\{\max_{t\in\mathbb{Z}} \frac{F(t,x)}{x}\right\} = 0.$$

We will use some classical results regarding the fixed point index. We compile these results in the following lemma. Let Ω be an open bounded subset of a cone K and let us denote $\overline{\Omega}$ and $\partial\Omega$ its closure and boundary. Moreover, let us denote $\Omega_k = \Omega \cap K$.

Lemma 4 ([1], Lemma 12.1)**.** Let Ω_K be an open bounded set with $0 \in \Omega_K$ and $\overline{\Omega_K} \neq K$. Assume that $F : \overline{\Omega_K} \to K$ is a completely continuous map such that $x \neq Fx$ for all $x \in \partial\Omega_k$. Then the fixed point index $i_K(F, \Omega_K)$ has the following properties:

1. If there exists $e \in K \setminus \{0\}$ such that $x \neq Fx + \lambda e$ for all $x \in \partial\Omega_K$ and all $\lambda > 0$, then $i_K(F, \Omega_K) = 0$.
2. If $x \neq \mu Fx$ for all $x \in \partial\Omega_K$ and every $\mu \leq 1$, then $i_K(F, \Omega_K) = 1$.
3. If $i_K(F, \Omega_K) \neq 0$, then F has a fixed point in Ω_K.
4. Let Ω_K^1 be an open set with $\overline{\Omega_K^1} \subset \Omega_k$. If $i_K(F, \Omega_K) = 1$ and $i_K(F, \Omega_K^1) = 0$, then F has a fixed point in $\Omega_K \setminus \overline{\Omega_K^1}$. The same result holds if $i_K(F, \Omega_K) = 0$ and $i_K(F, \Omega_K^1) = 1$.

Theorem 3. Assume that conditions $(H1^*)$ and $(H2^*)$ hold. Then there exists at least one positive solution of problem (8), (2).

Proof. From the definition of F_0 it follows that there exists $\delta_1 > 0$ such that for all $\|u\|_X \leq \delta_1$ we have that

$$F(k, u(k)) \cdot \min_{c\in\mathbb{Z}}\left\{\min_{s\geq c}\left\{\sum_{k=c}^{\infty} G(k,s)\right\}\right\} > u(k), \quad \text{for all } k \in \mathbb{Z}.$$

Let

$$\Omega_1 = \left\{u \in K_2;\, \|u\|_X < \delta_1\right\}$$

and choose $u \in \partial\Omega_1$ and $e \in K_2 \setminus \{0\}$. We will prove that $u \neq Tu + \lambda e$ for every $\lambda > 0$.

Assume, on the contrary, that there exists some $\lambda > 0$ such that $u = Tu + \lambda e$, i.e.

$$u(k) = Tu(k) + \lambda e(k) \geq Tu(k) \text{ for every } k \in \mathbb{Z}.$$

Then

$$\begin{aligned}
\sum_{k=c}^{\infty} u(k) \geq \sum_{k=c}^{\infty} Tu(k) &= \sum_{k=c}^{\infty}\sum_{s=-\infty}^{\infty} G(k,s)F(s,u(s)) \\
&= \sum_{s=-\infty}^{\infty}\left(\sum_{k=c}^{\infty} G(k,s)\right) F(s,u(s)) \\
&\geq \sum_{s=c}^{\infty}\left(\sum_{k=c}^{\infty} G(k,s)\right) F(s,u(s)) > \sum_{s=c}^{\infty} u(s),
\end{aligned}$$

which is a contradiction.

Therefore we deduce that $i_{K_2}(T, \Omega_1) = 0$.

Now let us define $\widetilde{F}(t,x) = \max_{0 \le y \le x} F(t,y)$. Clearly

$$\lim_{x \to +\infty} \frac{\sum_{s=-\infty}^{\infty} \widetilde{F}(s,x)}{x} \le \lim_{x \to +\infty} \frac{\sum_{s=-\infty}^{\infty} m^* v(s)}{x} = 0,$$

so there exists $\delta_2 > \delta_1$ such that if $\|u\|_X \ge \delta_2$ then

$$\sum_{s=-\infty}^{\infty} \widetilde{F}(s, \|u\|_X) < \sigma(M+1)\, \|u\|_X \,.$$

Let

$$\Omega_2 = \left\{ u \in K_2;\ \ \|u\|_X < \delta_2 \right\}$$

and $u \in \partial\Omega_2$.

We will prove that $u \neq \mu T u$ for every $\mu \le 1$. Assume on the contrary, that there exists some $\mu \le 1$ such that $u(k) = \mu T u(k)$ for all $k \in \mathbb{Z}$.

Then, by using that $\sigma \in (0,1)$, we have that

$$\begin{aligned} \|u\|_X &\le \sum_{k=-\infty}^{\infty} u(k) = \mu \sum_{k=-\infty}^{\infty} Tu(k) = \mu \sum_{k=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} G(k,s) F(s,u(s)) \\ &= \mu \sum_{s=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} G(k,s) \right) F(s,u(s)) \le \frac{\mu}{M+1} \sum_{s=-\infty}^{\infty} \widetilde{F}(s, \|u\|_X) \\ &< \mu\sigma \|u\|_X \le \sigma \|u\|_X < \|u\|_X \,, \end{aligned}$$

which is a contradiction.

As a consequence we obtain that $i_{K_2}(T, \Omega_2) = 1$.

So, from Lemma 4, we deduce that operator T has at least a fixed point u on K_2 such that $\delta_1 < \|u\|_X < \delta_2$. From Lemma 3, we know that u is a solution of problem (8), (2). □

Remark 3. Following the steps above, one can obtain similar results in the case when $M \in (0,1)$, using the fact that

$$\min_{c \in \mathbb{Z}} \left\{ \min_{s \ge c} \left\{ \sum_{k=c}^{\infty} G(k,s) \right\} \right\} = 1 - M.$$

Example 1. Let us consider the following problem

$$u(k+n) + Mu(k) = F(k, u(k)),\ n \in \mathbb{N},\ k \in \mathbb{Z},$$

where

$$F(s,x) = \begin{cases} \frac{2+x^2}{1+x^2} \left(\frac{M}{2} \right)^{\frac{s}{n}}, & \text{if } s < 0, \\ \frac{2+x^2}{1+x^2} M^{-\frac{s}{n}}, & \text{if } s \ge 0. \end{cases}$$

We claim that the problem has at least one positive solution if $M = \sqrt{2}$.

Clearly condition $(H1^*)$ holds. Moreover, if

$$v(s) = \begin{cases} \left(\frac{M}{2}\right)^{\frac{s}{n}}, & \text{if } s < 0, \\ M^{-\frac{s}{n}}, & \text{if } s \geq 0, \end{cases}$$

then one can easily check that $\sum_{s=-\infty}^{\infty} G(k,s)v(s) > 0$ and $\sum_{s=-\infty}^{\infty} v(s) < \infty$ for any $k \in \mathbb{Z}$.

If $k < 0$ we have that

$$\begin{aligned} \sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s) &= \sum_{d=0}^{\infty} \frac{1}{M^{2d+1}} v(k+2dn) \\ &= \sum_{d=0}^{-\left[\frac{k}{2n}\right]-1} \frac{1}{M^{2d+1}} \left(\frac{M}{2}\right)^{\frac{k+2dn}{n}} + \sum_{d=-\left[\frac{k}{2n}\right]}^{\infty} \frac{1}{M^{2d+1}} M^{\frac{-k-2dn}{n}} \\ &= \frac{M^{\frac{k}{n}-1}}{2^{\frac{k}{n}}} \sum_{d=0}^{-\left[\frac{k}{2n}\right]-1} \left(\frac{1}{2}\right)^{2d} + \frac{1}{M^{\frac{k}{n}+1}} \sum_{d=-\left[\frac{k}{2n}\right]}^{\infty} \left(\frac{1}{M}\right)^{4d} \\ &= \frac{M^{\frac{k}{n}-1}}{3 \cdot 2^{\frac{k}{n}-2}} \left(1 - 4^{\left[\frac{k}{2n}\right]}\right) + \frac{M^{4\left[\frac{k}{2n}\right]-\frac{k}{n}+3}}{M^4 - 1} \end{aligned}$$

and

$$\begin{aligned} \sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s) &= \sum_{d=0}^{\infty} \frac{1}{M^{2d+2}} v\left(k + (2d+1)\,n\right) \\ &= \sum_{d=0}^{-\left[\frac{k+n}{2n}\right]-1} \frac{1}{M^{2d+2}} \left(\frac{M}{2}\right)^{\frac{k+(2d+1)n}{n}} \\ &\quad + \sum_{d=-\left[\frac{k+n}{2n}\right]}^{\infty} \frac{1}{M^{2d+2}} M^{\frac{-k-(2d+1)n}{n}} \\ &= \frac{M^{\frac{k}{n}-1}}{2^{\frac{k}{n}+1}} \sum_{d=0}^{-\left[\frac{k+n}{2n}\right]-1} \left(\frac{1}{2}\right)^{2d} + \frac{1}{M^{\frac{k}{n}+3}} \sum_{d=-\left[\frac{k+n}{2n}\right]}^{\infty} \left(\frac{1}{M}\right)^{4d} \\ &= \frac{M^{\frac{k}{n}-1}}{3 \cdot 2^{\frac{k}{n}-1}} \left(1 - 4^{\left[\frac{k+n}{2n}\right]}\right) + \frac{M^{4\left[\frac{k+n}{2n}\right]-\frac{k}{n}+1}}{M^4 - 1}. \end{aligned}$$

If $k \in \left[-2rn - 1, -\left(2r+1\right)n\right]$, $r = 0, 1, 2, \dots$, then $\left[\frac{k}{2n}\right] = -r - 1$ and $\left[\frac{k+n}{2n}\right] = -r$. One can check that

$$\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s) = \frac{M^{\frac{k}{n}-1}\left(M^4 - 1\right)\left(1 - \frac{1}{4^{r+1}}\right) + 3 \cdot 2^{\frac{k}{n}-2} M^{-4r-\frac{k}{n}-1}}{3 \cdot 2^{\frac{k}{n}-2}\left(M^4 - 1\right)}$$

and

$$\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s) = \frac{M^{\frac{k}{n}-1}\left(M^4 - 1\right)\left(1 - \frac{1}{4^{r}}\right) + 3 \cdot 2^{\frac{k}{n}-1} M^{-4r-\frac{k}{n}+1}}{3 \cdot 2^{\frac{k}{n}-1}\left(M^4 - 1\right)}.$$

Moreover

$$\frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} = 2\frac{M^{\frac{k}{n}-1}\left(M^4-1\right)\left(1-\frac{1}{4^{r+1}}\right)+3\cdot 2^{\frac{k}{n}-2}M^{-4r-\frac{k}{n}-1}}{M^{\frac{k}{n}-1}\left(M^4-1\right)\left(1-\frac{1}{4^{r}}\right)+3\cdot 2^{\frac{k}{n}-1}M^{-4r-\frac{k}{n}+1}}.$$

As a result we obtain that $\frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} \geq 2$ if and only if $M \geq \sqrt{2}$ as the equality holds for $M = \sqrt{2}$.

If $k \in [-(2r+1)n-1, -(2r+2)n]$, then $\left[\frac{k}{2n}\right] = \left[\frac{k+n}{2n}\right] = -r-1$. In this situation we have that

$$\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s) = \frac{M^{\frac{k}{n}-1}\left(M^4-1\right)\left(1-\frac{1}{4^{r+1}}\right)+3\cdot 2^{\frac{k}{n}-2}M^{-4r-\frac{k}{n}-1}}{3\cdot 2^{\frac{k}{n}-2}\left(M^4-1\right)}$$

and

$$\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s) = \frac{M^{\frac{k}{n}-1}\left(M^4-1\right)\left(1-\frac{1}{4^{r+1}}\right)+3\cdot 2^{\frac{k}{n}-1}M^{-4r-\frac{k}{n}-3}}{3\cdot 2^{\frac{k}{n}-1}\left(M^4-1\right)}.$$

Obviously, we have

$$\frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} = 2\frac{M^{\frac{k}{n}-1}\left(M^4-1\right)\left(1-\frac{1}{4^{r+1}}\right)+3\cdot 2^{\frac{k}{n}-2}M^{-4r-\frac{k}{n}-1}}{M^{\frac{k}{n}-1}\left(M^4-1\right)\left(1-\frac{1}{4^{r+1}}\right)+3\cdot 2^{\frac{k}{n}-1}M^{-4r-\frac{k}{n}-3}}.$$

Again we deduce that $\frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} \geq 2$ if and only if $M \geq \sqrt{2}$. The equality stands for $M = \sqrt{2}$.

If $k \geq 0$ then

$$\frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} = \frac{\sum_{d=0}^{\infty} \frac{1}{M^{2d+1}} M^{-\frac{k}{n}-2d}}{\sum_{d=0}^{\infty} \frac{1}{M^{2d+2}} M^{-\frac{k}{n}-2d-1}} = M^2.$$

It is obvious that $\frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} \geq 2$ if and only if $M \geq \sqrt{2}$.

The above results gives us that $\gamma = \min_{k\in\mathbb{Z}} \frac{\sum_{s=-\infty}^{\infty} G^{+}(k,s)v(s)}{\sum_{s=-\infty}^{\infty} G^{-}(k,s)v(s)} \geq 2$ if and only if $M \geq \sqrt{2}$.

As a consequence, if $M = \sqrt{2}$ then $\gamma = 2$ and there exist $m_* = 1$ and $m^* = 2$, such that condition $(H2^*)$ holds. Now from Theorem 3 we obtain the existence result.

Author contributions

Both authors have contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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