Positive solutions for fractional boundary value problems with integral boundary conditions and parameter dependence

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Abstract

We study the existence of positive solutions for a Riemann fractional boundary value problem with integral boundary conditions and parameter dependence. To state our results, we use Guo-Krasnoselskii fixed point theorem. Some examples are showed to point out the applicability of the obtained results.

Key words. Fractional differential equation, Integral boundary conditions, Positive solutions, Green's function, Fixed point theorem.
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1 Introduction

Fractional calculus appears in many fields of engineering and sciences as rheology, viscoelasticity, electrochemistry, electromagnetism, and so forth. Many different books and monographs are devoted to the development of fractional calculus. See for instance [5, 9, 10, 11, 14, 15, 16, 18]. The interest of the study of fractional order differential equations lies in the fact that there are more degrees of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Recent results on fractional differential equations can be seen in [6, 7, 8, 12].

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, population dynamics and so forth, for more details see [1, 4, 6, 17].

As in dynamic of populations, many fields of engineering and sciences focus their interest on existence of positive solutions. We mention the works [2, 3, 4, 13].

Motivated by this and the above cited works, in this paper we investigate the existence of positive solutions of the following fractional differential equation with integral boundary conditions.

$$D^{\delta}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 1 < \delta \leq 2, \tag{1}$$

$$u(0) = 0,\ u(1) = \lambda \int_0^1 h(r)u(r)dr. \tag{2}$$

Where D^δ is the Riemann-Liouville fractional derivative and f is a given function.

The boundary conditions (2) can be thought as a mechanism putted at the end point of an oscillator, which is characterized by the weighted function h and the parameter λ, that controls its displacement according to the feedback from devices measuring the displacements along different parts of the oscillator.

This paper is organized as follows. In section 2, we recall some definitions concerning the fractional integral and derivative, and related basic properties which will be used in the sequel. We consider an auxiliary problem to derive the Green's function. Our main existence results are given in Section 3. Some examples are introduced in the last section.

2 Preliminaries

Here we present some basic knowledge and definitions for fractional calculus which will be used in the sequel.

Definition 2.1 *([15, 18]). The Riemann-Liouville fractional primitive of order $\delta > 0$ of a function $f : (0,1] \to \mathbb{R}$ is given by*

$$I_0^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau)d\tau,$$

provided that the right side is pointwise defined on $(0,1]$, and where Γ is the gamma function.

Definition 2.2 *([15, 18]). For a continuous function $f : (0,1] \to \mathbb{R}$, The Riemann-Liouville derivative of fractional order $\delta > 0$ is given by*

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \left(\int_0^t (t-\tau)^{n-\delta-1} f(\tau)d\tau \right),\ n = [\delta] + 1,$$

where $[\delta]$ denotes the integer part of the real number δ.

Lemma 2.1 *([15, 18]). Let $\delta > 0$, then the solutions of the fractional differential equation*

$$D^\delta u(t) = 0$$

are given by the following expression

$$u(t) = c_1 t^{\delta-1} + c_2 t^{\delta-2} + ... + c_n t^{\delta-n},\ c_i \in \mathbb{R}, i = 1, ..., n,\ n = [\delta] + 1.$$

From Lemma 2.1 we deduce the following result.

Lemma 2.2 *([15, 18]). Let $\delta > 0$, then*

$$I^\delta\left(D^\delta u(t)\right) = u(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + ... + c_n t^{\delta-n},\ c_i \in \mathbb{R}, i = 1, ..., n,\ n = [\delta] + 1.$$

In order to get the expression for the Green's function of boundary value problem $(1) - (2)$, we start by solving the following auxiliary problem:

$$D^\delta u(t) + \sigma(t) = 0, \quad 0 < t < 1,\ 1 < \delta \leq 2, \tag{3}$$

$$u(0) = 0,\ u(1) = \lambda \int_0^1 h(r)u(r)dr. \tag{4}$$

Lemma 2.3 *Let $1 < \delta \leq 2$. Suppose that $1 - \lambda \int_0^1 h(r)r^{\delta-1}dr \neq 0$. A function $u \in C[0,1]$ is a solution of the linear boundary value problem (3)-(4) if and only if it satisfies the integral equation*

$$u(t) = \int_0^1 G(t,s)\sigma(s)ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = G_1(t,s) + G_2(t,s)$$

with

$$G_1(t,s) = \begin{cases} \frac{t^{\delta-1}(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \leq s \leq t \leq 1; \\ \frac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5}$$

and

$$G_2(t,s) = \frac{\lambda t^{\delta-1}}{1 - \lambda \int_0^1 h(r)r^{\delta-1}dr} \int_0^1 h(r)G_1(r,s)dr \tag{6}$$

Proof. By Lemma 2.2 we have that the u is a solution of the linear equation (3) if and only if it satisfies

$$u(t) = -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds + c_1 t^{\delta-1} + c_2 t^{\delta-2}.$$

Condition $u(0) = 0$ implies necessarily that $c_2 = 0$.
Since $u(1) = \lambda \int_0^1 h(r)u(r)dr$, we deduce that

$$c_1 = \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds + \lambda c_1 \int_0^1 h(s)s^{\delta-1}ds - \frac{\lambda}{\Gamma(\delta)} \int_0^1 h(r) \int_0^r (r-s)^{\delta-1}\sigma(s)dsdr.$$

Now, since $1 - \lambda \int_0^1 h(r)r^{\delta-1}dr \neq 0$, we have

$$c_1 = \frac{1}{\Gamma(\delta)(1 - \lambda \int_0^1 h(r)r^{\delta-1}dr)} \left(\int_0^1 (1-s)^{\delta-1}\sigma(s)ds - \lambda \int_0^1 h(r) \int_0^r (r-s)^{\delta-1}\sigma(s)dsdr\right)$$

Finally, we have the expression

$$\begin{aligned}
u(t) &= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds \\
&\quad +\frac{t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds \\
&\quad +\frac{t^{\delta-1}(1-\lambda\int_0^1 h(r)r^{\delta-1}dr+\lambda\int_0^1 h(r)r^{\delta-1}dr)}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad +\frac{\lambda t^{\delta-1}\int_0^1 h(r)r^{\delta-1}dr}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t (1-s)^{\delta-1}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad +\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)r^{\delta-1}dr\cdot\int_0^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t (1-s)^{\delta-1}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad +\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^1 r^{\delta-1}(1-s)^{\delta-1}\sigma(s)dsdr \\
&\quad -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr
\end{aligned}$$

$$\begin{aligned}
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t (1-s)^{\delta-1}\sigma(s)ds + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1 (1-s)^{\delta-1}\sigma(s)ds \\
&\quad + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r r^{\delta-1}(1-s)^{\delta-1}\sigma(s)dsdr \\
&\quad - \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
&\quad + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_r^1 r^{\delta-1}(1-s)^{\delta-1}\sigma(s)dsdr \\
&= \int_0^1 G_1(t,s)\sigma(s)ds + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^1 G_1(r,s)\sigma(s)dsdr \\
&= \int_0^1 G_1(t,s)\sigma(s)ds + \int_0^1 \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)G_1(r,s)dr\sigma(s)ds \\
&= \int_0^1 G_1(t,s)\sigma(s)ds + \int_0^1 G_2(t,s)\sigma(s)ds.
\end{aligned}$$

□

As a direct consequence of the previous result, we deduce the following properties.

Lemma 2.4 *The function $G_1(t,s)$ defined in Lemma 2.3 has the following properties:*

1. $G_1(t,s) \in \mathcal{C}\ ([0,1]\times[0,1])$.
2. $G_1(t,s) > 0$ *for* $(t,s)\in(0,1)\times(0,1)$ *and* $G_1(0,s) = 0 = G_1(1,s)$ *for* $s\in[0,1]$.
3. $G_1(t,s) = G_1(1-s,1-t),\ \forall t,s\in[0,1]$.

In next result, we deduce two inequalities that, as we will see, will be fundamental to ensure the existence of the solutions of the nonlinear problem (1)-(2).

Lemma 2.5 *Let the function $G_1(t,s)$ be defined in Lemma 2.3 and fix $t_0\in(0,1)$, then G_1 satisfies the following inequalities:*

$$G_1(t,s) \le \frac{s^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, \quad \forall t\in[0,1],\ s\in[0,1] \tag{7}$$

and

$$s^{\delta-1}(1-s)^{\delta-1}k(t,t_0) \le G_1(t,s), \quad \forall t\in[0,1],\ s\in[t_0,1], \tag{8}$$

with

$$k(t,t_0) := \begin{cases} \frac{t^{\delta-1}}{\Gamma(\delta)} & \text{if } 0\le t\le t_0<1 \\ \min\left\{\frac{t^{\delta-1}}{\Gamma(\delta)}, \frac{t^{\delta-1}(1-t_0)^{\delta-1}-(t-t_0)^{\delta-1}}{\Gamma(\delta)\, t_0^{\delta-1}\,(1-t_0)^{\delta-1}}\right\} & \text{if } 0<t_0<t\le 1 \end{cases}.$$

Proof.
For $s > t$,

$$\frac{\partial G_1}{\partial t}(t,s) = \frac{\delta - 1}{\Gamma(\delta)}(1-s)^{\delta-1}t^{\delta-2} > 0.$$

For $s < t$, since $1 < \delta \leq 2$, we have

$$\frac{\partial G_1}{\partial t}(t,s) = \frac{\delta - 1}{\Gamma(\delta)}\left((1-s)^{\delta-1}t^{\delta-2} - (t-s)^{\delta-2}\right) \leq \frac{\delta - 1}{\Gamma(\delta)}\left(t^{\delta-2} - (t-s)^{\delta-2}\right) < 0.$$

As a consequence, it is fulfilled that

$$G_1(t,s) \leq G_1(s,s) = \frac{s^{\delta-1}\,(1-s)^{\delta-1}}{\Gamma(\delta)} \quad \forall t,s \in [0,1]$$

and inequality (7) holds.

By using the third property on Lemma 2.4, we deduce that

$$\frac{\partial G_1}{\partial s}(t,s) > 0 \quad \text{for } 0 \leq s < t \leq 1$$

and

$$\frac{\partial G_1}{\partial s}(t,s) < 0 \quad \text{if } 0 \leq t < s \leq 1.$$

Now, we introduce the following function:

$$F_1(t,s) = \frac{G_1(t,s)}{s^{\delta-1}\,(1-s)^{\delta-1}}, \quad (t,s) \in [0,1] \times (0,1),$$

as a direct consequence of previous arguments, we deduce that

$$\frac{\partial F_1}{\partial t}(t,s) < 0 \quad \text{for } 0 \leq s < t \leq 1$$

and

$$\frac{\partial F_1}{\partial t}(t,s) > 0 \quad \text{if } 0 \leq t < s \leq 1.$$

As a consequence, we have that

$$\frac{G_1(t,s)}{s^{\delta-1}\,(1-s)^{\delta-1}} \leq \frac{G_1(s,s)}{s^{\delta-1}\,(1-s)^{\delta-1}} = \frac{1}{\Gamma(\delta)}.$$

By the other hand,

$$\frac{\partial F_1}{\partial s}(t,s) = \begin{cases} -\frac{t^{\delta-1}}{\Gamma(\delta-1)\,s^{\delta}} & 0 \leq t < s \leq 1, \\ \\ \frac{(\delta-1)\left(t\left(s^2-2st+t\right)(t-s)^{\delta}-(t-s)^2 t^{\delta}(1-s)^{\delta}\right)}{t\,\Gamma(\delta)(t-s)^2\,(1-s)^{\delta}s^{\delta}} & 0 \leq s < t \leq 1. \end{cases}$$

As a direct consequence, we deduce that

$$\frac{\partial F_1}{\partial s}(t,s) < 0 \quad \text{for } 0 \le t < s \le 1.$$

On the other hand, for the case $0 \le s < t \le 1$ we have that $\frac{\partial F_1}{\partial s}(t,s) > 0$ if and only if

$$h_1(t,s,\delta) := (1-s)^\delta t^{\delta-1}(t-s)^{2-\delta} < s^2 - 2\,s\,t + t =: h_2(t,s).$$

Now, since

$$\frac{\partial h_1}{\partial \delta}(t,s,\delta) = (1-s)^\delta t^{\delta-1}(t-s)^{2-\delta}\,\log\left(\frac{t-t\,s}{t-s}\right),$$

we have that h_1 is strictly increasing on the δ interval $[1,2]$ for any $0 \le s < t \le 1$ given.

Thus, since $h_2(t,s) - h_1(t,s,2) = (1-t)\,s^2 > 0$, we conclude that $\frac{\partial F_1}{\partial s}(t,s) > 0$ for all $0 < s < t < 1$.

So, for any $t_0 \in (0,1)$ fixed, we have that

$$\begin{aligned}\frac{G_1(t,s)}{s^{\delta-1}\,(1-s)^{\delta-1}} &\ge \min\left\{\lim_{s\to 1^-}\frac{G_1(t,s)}{s\,(1-s)^{\delta-1}}, \frac{G_1(t,t_0)}{t_0^{\delta-1}\,(1-t_0)^{\delta-1}}\right\}\\ &= \min\left\{\frac{t^{\delta-1}}{\Gamma(\delta)}, \frac{G_1(t,t_0)}{t_0^{\delta-1}\,(1-t_0)^{\delta-1}}\right\} =: k(t,t_0), \quad \forall t\in[0,1],\ s\in[t_0,1],\end{aligned}$$

and the result is concluded. □

By virtue of this lemma, we can give now the main result of this section.

Lemma 2.6 *Let $t_0 \in (0,1)$ be fixed and h introduced at the boundary conditions* (2)*. Denote by $A = \int_0^1 h(r) r^{\delta-1} dr$, $B = \int_0^1 h(r)dr$ and $C_0 = \int_{t_0}^1 k(r,t_0)\,h(r)dr$. Assume that $h \ge 0$ on $[0,1]$ and $1 - \lambda\,A > 0$. Then the Green's function $G(t,s)$ defined in Lemma 2.3 satisfies the inequalities*

$$\frac{\lambda\,C_0\,t^{\delta-1}}{1-\lambda A}\,s^{\delta-1}(1-s)^{\delta-1} \le G(t,s) \le \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)s^{\delta-1}(1-s)^{\delta-1},\ \forall t,s\in[0,1]. \tag{9}$$

Proof. From the definition of G, the inequality (7) and the fact that $1 < \delta \le 2$, we have the following inequalities for all t, $s \in [0,1]$:

$$\begin{aligned}G(t,s) &\le \frac{1}{\Gamma(\delta)}s^{\delta-1}(1-s)^{\delta-1} + \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 \frac{1}{\Gamma(\delta)}s^{\delta-1}(1-s)^{\delta-1}h(r)dr\\ &\le \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)s^{\delta-1}(1-s)^{\delta-1}.\end{aligned}$$

On the other hand, by Lemma 2.4 (2) and (8), we have for all $t,\ s \in [0,1]$:

$$\begin{aligned} G(t,s) &= G_1(t,s) + G_2(t,s) \\ &\geq \frac{\lambda t^{\delta-1}}{1-\lambda \int_0^1 h(r) r^{\delta-1} dr} \int_{t_0}^1 h(r) G_1(r,s) dr \\ &\geq \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0\, s^{\delta-1}(1-s)^{\delta-1}, \end{aligned}$$

as we want to prove. □

As a direct consequence, we deduce the following Corollary:

Corollary 2.1 *If $h \geq 0$ on $[0,1]$ and $1-\lambda\, A > 0$ then the Green's function $G(t,s)$ defined in Lemma 2.3 satisfies the inequalities*

$$\frac{\lambda\, t}{1-\lambda A} C_0\, s^{\delta-1}(1-s)^{\delta-1} \leq t^{2-\delta} G(t,s) \leq \frac{1}{\Gamma(\delta)} \left(1 + \frac{\lambda B}{1-\lambda A}\right) s^{\delta-1}(1-s)^{\delta-1},\ \forall t,s \in [0,1]$$

3 Main Results

Now for any $u : (0,1] \to \mathbb{R}$, we define function $\overline{u} : [0,1] \to \mathbb{R}$ as follows:

$$\overline{u}(t) = \begin{cases} t^{2-\delta} u(t) & \text{if} \quad t \in (0,1], \\ \lim_{t\to 0^+} t^{2-\delta} u(t) & \text{if} \quad t = 0, \end{cases}$$

provided that such limit exists.

Consider the Banach space

$$E = C_\delta[0,1] := \{\bar{u} : [0,1] \to \mathbb{R}, \text{is a continuous function in } [0,1]\}$$

endowed with the maximum norm $\|u\| = \max\limits_{0\leq t\leq 1} |\bar{u}(t)|$ and define the cone $P_0 \subset E$ by

$$P_0 = \{u \in E,\ \bar{u}(t) \geq t^{2-\delta} p(t,t_0) \|u\|,\ for\ all\ t \in [0,1]\},$$

where

$$p(t,t_0) = \Gamma(\delta) \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0 \bigg/ \left(1 + \frac{\lambda B}{1-\lambda A}\right),\ t \in [0,1],$$

with $t_0 \in (0,1)$ fixed, and A, B and C_0 introduced in Lemma 2.6.

Notice that, provided that $h \geq 0$ on $[0,1]$ and $1-\lambda\, A > 0$, we deduce from (9) that $0 \leq p(t,t_0) \leq 1$ for all $t \in [0,1]$ and $t_0 \in (0,1)$.

Now, we assume the following hypothesis on the nonlinear part of the equation:

(H_1) Function $f : [0,1] \times \mathbb{R} \to [0,\infty)$ is continuous.

So, we define the operator $T : P_0 \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s, \bar{u}(s)) ds, \ \ t \in [0,1] \tag{10}$$

Lemma 3.1 *$T : P_0 \to P_0$ is completely continuous.*

Proof:
Let us prove in first that $T(P_0) \subset P_0$. Notice from the definition of T and Corollary 2.1 that for $u \in P_0$, $Tu(t) \geq 0$ for all $t \in [0,1]$ and

$$\begin{aligned} t^{2-\delta}(Tu)(t) &= \int_0^1 t^{2-\delta} G(t,s) f(s,\bar{u}(s)) ds \\ &\geq \int_0^1 t^{2-\delta} \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0\, s^{\delta-1}(1-s)^{\delta-1} f(s,\bar{u}(s)) ds \\ &= t^{2-\delta} \frac{\Gamma(\delta) \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0}{\left(1+\frac{\lambda B}{1-\lambda A}\right)} \int_0^1 \frac{\left(1+\frac{\lambda B}{1-\lambda A}\right)}{\Gamma(\delta)} s^{\delta-1}(1-s)^{\delta-1} f(s,\bar{u}(s)) ds \\ &\geq t^{2-\delta} p(t,t_0) \int_0^1 \max_{0\leq t\leq 1} \left\{ t^{2-\delta} G(t,s) \right\} f(s,\bar{u}(s)) ds \\ &\geq t^{2-\delta} p(t,t_0) \max_{0\leq t\leq 1} \left\{ \int_0^1 t^{2-\delta} G(t,s) f(s,\bar{u}(s)) ds \right\} \\ &= t^{2-\delta} p(t,t_0) \|Tu\|. \end{aligned}$$

Thus, $T(P_0) \subset P_0$.

In addition, since f is a continuous function it follows that T is a continuous operator.
Next, we show that T is uniformly bounded.
Let $D \subset P$ be a bounded set, i.e. there exists a constant $L > 0$ such that $\|u\| \leq L$, for all $u \in D$. Set

$$M = \max_{0\leq s\leq 1, 0\leq u\leq L} \{f(s,\bar{u}(s))\}.$$

Then, from Lemma 2.6, and for all $u \in D$, we have

$$\begin{aligned} |t^{2-\delta} Tu(t)| &= \left| \int_0^1 t^{2-\delta} G(t,s) f(s,\bar{u}(s)) ds \right| \\ &\leq \frac{M}{\Gamma(\delta)} \left(1 + \frac{\lambda B}{1-\lambda A}\right) \int_0^1 s^{\delta-1}(1-s)^{\delta-1} ds \\ &= M \left(1 + \frac{\lambda B}{1-\lambda A}\right) \frac{\Gamma(\delta)}{\Gamma(2\delta)}. \end{aligned}$$

Hence, $T(D)$ is bounded.
Finally, we show that T is equicontinuous, as follows.
For all $\epsilon > 0$ and for each $u \in P$, let $t_1, \ t_2 \in [0,1]$, be such that $t_1 < t_2$.

We have to prove that there is $\eta > 0$ valid for all $u \in D$, such that $|t_2^{2-\delta} Tu(t_2) - t_1^{2-\delta} Tu(t_1)| < \epsilon$, when $t_2 - t_1 < \eta$.

One has

$$\begin{aligned}
|t_2^{2-\delta} Tu(t_2) - t_1^{2-\delta} Tu(t_1)| &= \left| \int_0^1 [t_2^{2-\delta} G(t_2, s) ds - t_1^{2-\delta} G(t_1, s)] f(s, \bar{u}(s)) ds \right| \\
&\leq \int_0^1 |t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)| f(s, \bar{u}(s)) ds \\
&\leq M \int_0^1 |t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)| ds
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_0^1 |t_2^{2-\delta} G(t_2, s) - t_1^{\delta-2} G(t_1, s)| ds &\leq \int_0^1 |t_2^{2-\delta} G_1(t_2, s) - t_1^{\delta-2} G_1(t_1, s)| ds \\
&\quad + \int_0^1 |t_2^{2-\delta} G_2(t_2, s) - t_1^{\delta-2} G_2(t_1, s)| ds.
\end{aligned}$$

From the expression of G_1, we get

$$\begin{aligned}
\int_0^1 |t_2^{2-\delta} G_1(t_2, s) - t_1^{2-\delta} G_1(t_1, s)| ds &= \int_0^{t_1} |t_2^{2-\delta} G_1(t_2, s) - t_1^{2-\delta} G_1(t_1, s)| ds \\
&\quad + \int_{t_1}^{t_2} |t_2^{2-\delta} G_1(t_2, s) - t_1^{2-\delta} G_1(t_1, s)| ds \\
&\quad + \int_{t_2}^{1} |t_2^{2-\delta} G_1(t_2, s) - t_1^{2-\delta-2} G_1(t_1, s)| ds \\
&= \int_0^{t_1} \left| \frac{t_2(1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)} - \frac{t_1(1-s)^{\delta-1} - t_1^{2-\delta}(t_1-s)^{\delta-1}}{\Gamma(\delta)} \right| ds \\
&\quad + \int_{t_1}^{t_2} \left| \frac{t_2(1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)} - \frac{t_1(1-s)^{\delta-1}}{\Gamma(\delta)} \right| ds \\
&\quad + \int_{t_2}^{1} \left| \frac{t_2(1-s)^{\delta-1}}{\Gamma(\delta)} - \frac{t_1(1-s)^{\delta-1}}{\Gamma(\delta)} \right| ds \\
&= \int_0^{t_1} \left| \frac{(t_2-t_1)(1-s)^{\delta-1} + t_1^{2-\delta}(t_1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)} \right| ds \\
&\quad + \int_{t_1}^{t_2} \left| \frac{(t_2-t_1)(1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)} \right| ds \\
&\quad + \int_{t_2}^{1} \left| \frac{(t_2-t_1)(1-s)^{\delta-1}}{\Gamma(\delta)} \right| ds \\
&= \left| \frac{t_2^{2-\delta} (t_2-t_1)^{\delta} + (t_1-t_2) \Big((1-t_1)^{\delta} + t_1 + t_2 - 1 \Big)}{\Gamma(\delta+1)} \right| \\
&\quad + \left| \frac{(t_2-t_1) \left((1-t_1)^{\delta} - (1-t_2)^{\delta} \right) - t_2^{2-\delta} (t_2-t_1)^{\delta}}{\Gamma(\delta+1)} \right| \\
&\quad + \frac{(t_2-t_1)(1-t_2^{\delta})}{\Gamma(\delta+1)}.
\end{aligned}$$

So, we have that there is $\eta > 0$ valid for all $u \in D$, such that $|t_2^{2-\delta} Tu(t_2) - t_1^{2-\delta} Tu(t_1)| < \epsilon$, when $t_2 - t_1 < \eta$.

Now, denote by $H(s) = \int_0^1 h(r) G_1(r,s) dr$ and $h^* = \max_{t\in[0,1]}\{h(t)\}$. Then, from the expression of $G_2(t,s)$ and the inequality (7), using that

$$\int_0^1 H(s) ds \le h^* \int_0^1 \int_0^1 G_1(r,s) dr ds \le h^* \int_0^1 \frac{s^{\delta-1}\,(1-s)^{\delta-1}}{\Gamma(\delta)} ds = h^* \frac{\Gamma(\delta)}{\Gamma(2\,\delta)},$$

we get

$$\begin{aligned} \int_0^1 |t_2^{2-\delta} G_2(t_2,s) - t_1^{2-\delta} G_2(t_1,s))| ds &= \int_0^1 \frac{\lambda(t_2-t_1)}{1-\lambda A} H(s) ds \\ &\le \frac{\Gamma(\delta)}{\Gamma(2\,\delta)} \frac{\lambda\, h^*}{1-\lambda A}(t_2-t_1). \end{aligned}$$

Thus, we obtain theat the set $T(D)$ is equicontinuous in E. $\square$

Now, we are in position to prove the existence of positive solutions of the nonlinear boundary value problem (BVP). For this we use the known Guo-Krasnoselskii fixed point theorem.

Theorem 3.1 *Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ be a completely continuous operator such that one of following assertions is fulfilled:*

(i) $\|Tu\| \ge \|u\|,\ u \in P \cap \partial\Omega_1,$ *and* $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_2,$

(ii) $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_1,$ *and* $\|Tu\| \ge |u\|, u \in P \cap \partial\Omega_2$

Then operator T has at least one fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Let introduce some notations,

$$f_0 = \lim_{\bar{u}\to 0^+} \left\{ \min_{t\in[0,1]} \left\{ \frac{f\,(t,\bar{u})}{\bar{u}} \right\} \right\} \quad \text{and } f_\infty = \lim_{\bar{u}\to\infty} \left\{ \min_{t\in[0,1]} \left\{ \frac{f\,(t,\bar{u})}{\bar{u}} \right\} \right\},$$

$$f^0 = \lim_{\bar{u}\to 0^+} \left\{ \max_{t\in[0,1]} \left\{ \frac{f\,(t,\bar{u})}{\bar{u}} \right\} \right\} \quad \text{and } f^\infty = \lim_{\bar{u}\to\infty} \left\{ \max_{t\in[0,1]} \left\{ \frac{f\,(t,\bar{u})}{\bar{u}} \right\} \right\}.$$

Theorem 3.2 *Assume that $h \ge 0$ on $[0,1]$, $1 - \lambda\, A > 0$ and (H_1) hold coupled with one of the following conditions*

1. *Sublinear case: $f_0 = \infty$ and $f^\infty = 0$.*
2. *Superlinear case: $f^0 = 0$ and $f_\infty = \infty$.*

Then Problem (1)-(2) has at least one positive solution.

Proof: Consider the first situation

1. Since $f_0 = \infty$, then there exists a constant $R_1 > 0$ such that $f(t, \bar{u}) \geq r_1 \bar{u}$ for all $0 < \bar{u} \leq R_1$ and $t \in [0, 1]$, where $r_1 > 0$ is defined as

   $$r_1 := \frac{(1 - \lambda A)\,(1 - \lambda A + \lambda B)\,\Gamma(1 + 2\delta)}{\lambda^2\, C_0^2\, \delta\, \Gamma^3(\delta)} \tag{11}$$

   Take $u \in P_0$ such that $\|u\| = R_1$. Then from expression (11), we get

   $$\begin{aligned}
   \|Tu\| &:= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) f(s, \bar{u}(s)) ds \right\} \\
   &\geq r_1 \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) \bar{u}(s) ds \right\} \\
   &\geq r_1 \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) s^{2-\delta} p(s, t_0) \|u\| ds \right\} \\
   &\geq r_1\, \|u\| \max_{t\in[0,1]} \left\{ t \int_0^1 \frac{\lambda}{1 - \lambda A}\, C_0\, s\, (1-s)^{\delta-1} p(s, t_0)\, ds \right\} \\
   &= r_1 \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1 - \lambda A)\,(1 - \lambda A + \lambda B)} \int_0^1 s^{\delta}\, (1-s)^{\delta-1}\, ds \\
   &= r_1 \|u\| \frac{\lambda^2\, C_0^2\, \delta\, \Gamma^3(\delta)}{(1 - \lambda A)\,(1 - \lambda A + \lambda B)\,\Gamma(1 + 2\delta)} \\
   &= \|u\|.
   \end{aligned}$$

   By the other hand, since $f(t, \cdot)$ is a continuous function on $[0, \infty)$, we define a new function:

   $$\hat{f}(t, \bar{u}) := \max_{y\in[0,\bar{u}]} \{f(t, y)\}.$$

   Clearly $\hat{f}(t, \cdot)$ is nondecreasing on $[0, \infty)$. Moreover, since $f^\infty = 0$ it is obvious that

   $$\lim_{\bar{u}\to\infty} \left\{ \max_{t\in[0,1]} \frac{\hat{f}\,(t, \bar{u})}{\bar{u}} \right\} = 0.$$

   Choose now $r_2 > 0$ defined as the following constant:

   $$r_2 = \frac{(1 - \lambda A)\,\Gamma(2\delta)}{(1 - \lambda A + \lambda B)\,\Gamma(\delta)}. \tag{12}$$

Therefore there exists a constant $R_2 > R_1 > 0$ such that $\hat{f}(t, \bar{u}) \leq r_2 \bar{u}$ for all $\bar{u} \geq R_2$ and $t \in [0, 1]$.
Consider $u \in P_0$ such that $\|u\| = R_2$. Then from the definition of $\hat{f}$, inequality (12) and lemma (2.5), we attain at the following inequalities:

$$\begin{aligned}
\|Tu\| &:= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) f(s, \bar{u}(s)) ds \right\} \\
&\leq \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) \hat{f}(s, \|u\|) ds \right\} \\
&\leq r_2 \|u\| \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) ds \right\} \\
&\leq r_2 \|u\| \frac{(1 - \lambda A + \lambda B)}{\Gamma(\delta)(1 - \lambda A)} \int_0^1 s^{\delta-1}(1-s)^{\delta-1} ds \\
&= r_2 \|u\| \frac{(1 - \lambda A + \lambda B)\, \Gamma(\delta)}{(1 - \lambda A)\, \Gamma(2\delta)} \\
&= \|u\|.
\end{aligned}$$

Thus, by the first part of Guo-Krasnoselskii fixed point theorem, we conclude that the problem $(1) - (2)$ has at least one positive solution u such that

$$R_1 \leq \|u\| \leq R_2.$$

2. Consider now the second case (ii)

Let $r_2 > 0$ be chosen as in equation (12). Since $f^0 = 0$, there exists a constant $\tau_1 > 0$ such that $f(t, \bar{u}) \leq r_2 \bar{u}$ for $0 \leq \bar{u} \leq \tau_1$ and $t \in [0, 1]$.

Take $u \in P_0$ such that $\|u\| = \tau_1$. Then, arguing as in the previous case, we have

$$\begin{aligned}
\|Tu\| &:= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) f(s, \bar{u}(s)) ds \right\} \leq r_2 \|u\| \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) ds \right\} \\
&\leq r_2 \|u\| \frac{(1 - \lambda A + \lambda B)\, \Gamma(\delta)}{(1 - \lambda A)\, \Gamma(2\delta)} = \|u\|.
\end{aligned}$$

Now, by denoting the incomplete beta function as

$$\mathcal{B}_z(a, b) := \int_0^z t^{a-1}\, (1-t)^{b-1}\, dt.$$

For any fixed $t_1 \in (0, 1)$, we define $r_3 > 0$ as follows:

$$r_3 = \frac{(1 - \lambda A)\,(1 - \lambda A + \lambda B)}{\lambda^2\, C_0^2\, \Gamma(\delta)} \left(\frac{\sqrt{\pi}\, \Gamma(\delta)}{\Gamma\left(\delta + \frac{1}{2}\right) 4^{\delta}} - \mathcal{B}_{t_1}(\delta + 1, \delta) \right)^{-1}. \tag{13}$$

The fact that $f_\infty = \infty$ ensures that there exists a constant $\tau_2 > \tau_1 > 0$ such that $f(t, \bar{u}) \geq r_3 \bar{u}$ for all $\bar{u} \geq \tau_2$ and $t \in [0, 1]$.

By the definition of $p(t, t_0)$ is clear that

$$p_1 := \min_{t \in [t_1, 1]} \{t^{2-\delta}\, p(t, t_0)\} > 0.$$

Let now $u \in P_0$ be such that $\|u\| = \tau_2 / p_1$. As consequence, since $u \in P_0$, the following inequality holds :

$$\bar{u}(t) \geq t^{2-\delta}\, p(t, t_0) \|u\| \geq p_1 \, \|u\| = \tau_2 \qquad \text{for all } \ t \in [t_1, 1].$$

So, condition (ii) gives us the following properties:

$$\begin{aligned}
\|Tu\| &:= \max_{t \in [0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) f(s, \bar{u}(s)) ds \right\} \\
&\geq \max_{t \in [0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s) f(s, \bar{u}(s)) ds \right\} \\
&\geq r_3 \max_{t \in [0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s) \bar{u}(s) ds \right\} \\
&\geq r_3 \|u\| \max_{t \in [0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s) s^{2-\delta} p(s, t_0) ds \right\} \\
&\geq r_3 \, \|u\| \max_{t \in [0,1]} \left\{ t \int_{t_1}^1 \frac{\lambda}{1 - \lambda A}\, C_0 \, s \, (1-s)^{\delta - 1} p(s, t_0) \, ds \right\} \\
&= r_3 \|u\| \, \frac{\lambda^2 \, C_0^2 \, \Gamma(\delta)}{(1 - \lambda A)\, (1 - \lambda A + \lambda B)} \int_{t_1}^1 s^\delta \, (1-s)^{\delta - 1} \, ds \\
&= r_3 \|u\| \, \frac{\lambda^2 \, C_0^2 \, \Gamma(\delta)}{(1 - \lambda A)\, (1 - \lambda A + \lambda B)} \left(\frac{\sqrt{\pi}\, \Gamma(\delta)}{\Gamma\left(\delta + \frac{1}{2}\right) 4^\delta} - \mathcal{B}_{t_1}(\delta + 1, \delta) \right) \\
&= \|u\|.
\end{aligned}$$

Therefore, by the second part of Guo-Krasnoselskii fixed point theorem, we conclude that the problem (1)-(2) has at least one positive solution.

4 Examples

Example 4.1 *The problem*

$$\left\{ \begin{array}{l} D^{\frac{3}{2}} u(t) + f(t, u(t) = 0 \\ \\ u(0) = 0, \quad u(1) = \lambda \int_0^1 s^{\frac{1}{2}} u(s) ds, \end{array} \right. \tag{14}$$

with

$$f(t,x)=\begin{cases} t+\sqrt{x}\,\arctan\left(\frac{1}{x}\right) & \textit{if} \quad x>0,\ t\in[0,1] \\ t & \textit{if} \quad x=0,\ t\in[0,1], \end{cases}$$

has at least a positive solution for any $0<\lambda<2$.
Here, $\delta=\frac{3}{2}$ *and* $h(t)=t^{\frac{1}{2}}$. *Then,* $A=\int_0^1 t^{\frac{1}{2}}t^{\frac{3}{2}-1}dt=\frac{1}{2}$, *and* $1-\lambda A>0$ *for any* $\lambda<2$.
It is clear that $f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous and,*

$$\lim_{u\to 0^+}\left\{\min_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}=+\infty,$$

and

$$\lim_{u\to +\infty}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}=0.$$

Then by the first part of Theorem 3.2, the problem (14) has at least one positive solution.

Example 4.2 *The problem*

$$\begin{cases} D^{\frac{3}{2}}u(t)+u^{\beta}(t)+(t-1)u(t)=0, \quad t\in(0,1) \\ \\ u(0)=0, \quad u(1)=\lambda\int_0^1 e^{s}u(s)ds, \end{cases} \tag{15}$$

has at least a positive solution for any $\beta>1$ *and* $0<\lambda<0.796413$.
Here, $\delta=\frac{3}{2}$, $h(t)=e^{t}$ *and* $f(t,u)=u^{\beta}+(t-1)\,u$.
A numerical calculation leads to $A=\int_0^1 e^{t}t^{\frac{1}{2}}dt\approx 1.25563$ *and* $1-\lambda A>0$ *for any* $\lambda\in(0,0.796413)$.
$f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous and,*

$$\lim_{u\to 0^+}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}=0,$$

and

$$\lim_{u\to +\infty}\left\{\min_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}=+\infty.$$

Then by the second part of Theorem 3.2, *the problem* (15) *has at least one positive solution.*

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