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On the gamma-logistic map and applications to a delayed neoclassical model of economic growth

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Abstract In this work, we study the stability properties of a delay-differential neoclassical model of economic growth, based on the original model proposed by Solow in 1956. We consider a logistic-type production function, which comes from combining a Cobb-Douglas function and a linear pollution effect caused by increasing concentrations of capital. The difference between the production function and the classical logistic map comes from the presence of a parameter $\gamma \in (0,1)$ in the exponent of one factor. We call this new function the gamma-logistic map. Our main purpose is to obtain sharp global stability conditions for the positive equilibrium of the model and to study how the stability properties of such equilibrium depend on the relevant model parameters. This study is developed by using some properties of the gamma-logistic map and some well-known results connecting stability in delay differential equations and discrete dynamical systems. Finally, we also compare the obtained results with the ones written in related articles.

Keywords Delay differential equation · Neoclassical growth model · Global stability · Gamma-logistic map

Mathematics Subject Classification (2000) 34K20 · 91B62

1 Introduction

In 1956, R. M. Solow [21] proposed a model of economic growth in terms of capital accumulation and labor. He supposed there is a unique commodity and a production function $P \equiv P(K,L)$ depending on the capital stock K and the labor force L. In his model, the evolution of the economic system is studied via the introduction of a new variable: the capital-labor ratio $x = K/L$. The principal equation in the Solow's work is

$$x'(t) = -\alpha x(t) + s(x(t))p_1(x(t)), \tag{1}$$

where $\alpha > 0$ is the rate of labor growth, $s(x(t))$ is the ratio of saving in each instant and $p_1(x(t)) := P(x(t),1)$, with P assumed to be homogeneous. If $s(x)$ is constant, an appropriate choice of P for an economic growth model leads to the existence of a unique equilibrium, which is a global attractor [2].

Day [4] showed how rich the dynamics of a neoclassical model can be by studying the difference equation version of the model

$$x_{n+1} = \frac{sp_1(x_n)p_2(x_n)}{1-\alpha}, \quad n \in \mathbb{N}, \tag{2}$$

where p_2 is a pollution function that appears due to the accumulation of capital and $\mathbb{N}$ denotes the set of nonnegative integer numbers.

The discrete model (2) includes a dependence on past states. For a continuous model, the consideration of time delays leads to a functional delay differential equation of the type

$$x'(t) = -\alpha x(t) + sP_1(x_t)P_2(x_t), \tag{3}$$

where $x_t : [-h,0] \to \mathbb{R}$ is defined as $x_t(\theta) = x(t+\theta)$, for some $h \geq 0$, and $P_1, P_2 : \mathcal{C}_+ \to \mathbb{R}$ are certain functionals

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defined in the Banach space

$$\mathcal{C}_+ = \{\phi\colon [-h,0] \to [0,\infty) : \phi \text{ is continuous}\},$$

endowed with the supremum norm. In this way, (3) becomes a "continuous" version of (2).

In addition to the former assumptions and according to [15], production takes time to take place after decisions are made, and such decisions also need time because they depend on the information about the market, which is not obtained instanteneously. Following this direction (taking $P_1(\phi) = \frac{\beta}{s}\phi(-h)$, $\beta > 0$, and $P_2(\phi) = 1 - \phi(-h)$), the authors of [15] proposed a model with a logistic term and fixed delay, i.e.,

$$x'(t) = -\alpha x(t) + \beta x(t-h)(1 - x(t-h)), \tag{4}$$

and they analyzed the local stability of its unique positive equilibrium.

As recognized in [15], equation (4) is a simplification of a more general model

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t-h)(1 - x(t-h)), \tag{5}$$

where the term $\beta x^\gamma(1-x)$, $\gamma \in (0,1]$, comes from considering the Cobb-Douglas production function $p_1(x) = \beta x^\gamma$, as in [4,21], and multiplying it by a factor $(1-x)$ which reflects a linear influence of pollution on per-capita output. As highlighted in [15,16], such production function may lead to solutions that take negative values. Then, the equation

$$x'(t) = -\alpha x(t) + \max\{0, \beta x^\gamma(t-h)(1 - x(t-h))\}, \tag{6}$$

can be used to avoid such problem. We remark that every solution of (6) starting above 1 will clearly enter in $(0,1)$ for a time interval of length greater than or equal to h. Hence we can focus on the study of the dynamics of (6) in $(0,1)$, which is just the case of (5).

Following the approach used in the recent paper [3] for some related neoclassical models, we study the stability properties of the positive equilibrium of (5). We also consider the more general case of variable delay, which may be used for the cases of temporal variation in production lags.

The proof of our main results are based on the interplay between delay differential equations and scalar maps (see, e.g., [9,13,14] and references therein). In particular, we need to study some dynamical properties of the discrete dynamical system

$$x_{n+1} = g(x_n) := \delta x_n^\gamma (1 - x_n), \quad n \in \mathbb{N}. \tag{7}$$

where the function $g : (0,1) \to \mathbb{R}$ is defined by $g(x) := \delta x^\gamma(1-x)$ and $\delta := \frac{\beta}{\alpha}$. We refer to g as the gamma-logistic map, in analogy with other related models recently considered in [11,12]. This map has been used by Avilés [1] in the framework of cooperative interaction in a group of individuals (for $1 \le \gamma \le 2$), and by Eskola and Parvinen [6] in the context of populations with Allee effects (for $\gamma = 2$). We analyze the case $\gamma \in (0,1)$, which represents a generalization of the famous quadratic map ($\gamma = 1$), and has an independent interest to improve its flexibility to fit population data, in the line of [11].

The paper is organized as follows: in Section 2, we analyze the discrete dynamical system (7) generated by g. In Section 3, we study the delay-differential model using a result from [3] (stated as Theorem 2), which provides a relationship between the stability properties of the equilibria of both dynamical systems (5) and (7).

2 Discrete dynamical system

In this section we discuss some important properties of the dynamical system

$$x_{n+1} = g(x_n) = \delta x_n^\gamma (1 - x_n), \quad n \in \mathbb{N}. \tag{8}$$

In other words, we are interested in the behaviour of the sequence $x_n = g^n(x_0)$, for any $x_0 \in (0,1)$, where

$$g^n = \overbrace{g \circ \cdots \circ g}^{n}$$

(see Figure 1 for a sketch of g with different values of the parameters). This notation will be employed in the context of any discrete dynamical system.

In Proposition 1, we show some basic features of the function g such as the existence of a unique fixed point. In Theorem 1, a global stability condition is provided. Finally, the influence of the parameter γ on the stability and the size of the equilibrium is respectively analyzed in Proposition 3 and Theorem 4.

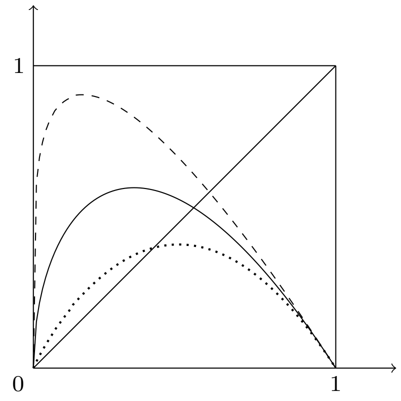

Fig. 1 The function g for $\delta = 1.55$ and $\gamma = 0.2$ (dashed), $\gamma = 0.5$ (solid) and $\gamma = 0.925$ (dotted). The figure was made by using the LaTeX package tikz.

We recall an important tool for the global stability analysis of scalar maps: the Schwarzian derivative. For a given $\mathcal{C}^3$ map $f : (a, b) \to (a, b)$, the Schwarzian derivative of f is defined for every $x \in (a, b)$, $f'(x) \neq 0$, by the expression:

$$Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

Proposition 1 *The function g has the following properties:*

1. *g is $\mathcal{C}^\infty$, nonnegative and $g(0^+) = g(1^-) = 0$.*
2. *g is concave and has a unique critical point $c := \frac{\gamma}{\gamma+1}$, which is a global maximum.*
3. *Provided the condition*
$$\delta < (\gamma + 1)\left(\frac{\gamma + 1}{\gamma}\right)^\gamma , \tag{9}$$
then $g((0, 1)) \subset (0, 1)$.
4. *g has a unique fixed point p in $(0, 1)$ and*
$$(g(x) - x)(x - p) \leq 0$$
for any $x \in (0, 1)$.
5. *$Sg(x) < 0$, for all $x > c$.*

Proof Assertion 1 is trivial by the definition of g.

It can be checked by an inductive argument that the n-th derivative of g is

$$g^{(n)}(x) = \delta x^{\gamma-n}[\gamma - (n - 1) - (\gamma + 1)x] \prod_{i=0}^{n-2}(\gamma - i).$$

The first, second and third derivatives of g are, respectively,

$$g'(x) = \delta x^{\gamma-1}[\gamma - (\gamma + 1)x],$$
$$g''(x) = \delta \gamma x^{\gamma-2}[\gamma - 1 - (\gamma + 1)x],$$
$$g'''(x) = \delta \gamma(\gamma - 1)x^{\gamma-3}[\gamma - 2 - (\gamma + 1)x].$$

It is easy to check that g'' is negative using that $0 < \gamma < 1$. It is also trivial that $g'(c) = 0$ is equivalent to $c = \frac{\gamma}{\gamma+1}$. By using that $g'(x) > 0$ while $x < c$ and $g'(x) < 0$ while $x > c$, one can prove that c is a global maximum. This proves Assertion 2.

As c is a global maximum and g is nonnegative, $g((0, 1)) \subset (0, 1)$ is equivalent to

$$1 > g(c) = \delta \frac{1}{\gamma + 1}\left(\frac{\gamma}{\gamma + 1}\right)^\gamma ,$$

which proves Assertion 3.

Let $z(x) := g(x) - x$. Then, $z(0) = 0$, $z'(0^+) = \infty$ and $z(1) = -1$, so there exists at least one fixed point of g in $(0, 1)$. Take p as the lowest of the positive fixed points of g, which exists because $z'(0^+) = \infty$, which also implies that $g(x) - x > 0$, for $0 < x < p$. Moreover $z'(p) \leq 0$ or, equivalently, $g'(p) \leq 1$. As $g''(x) < 0$, p is the unique fixed point of g in $(0, 1)$ and the proof of Assertion 4 finishes.

By doing some calculations, one can reach

$$Sg(x) = \frac{\gamma(\gamma + 1)q(x)}{2x^2[\gamma - (\gamma + 1)x]^2}, \quad x \neq c,$$

where q is a polynomial of degree two defined by

$$q(x) := -(\gamma+1)(\gamma+2)x^2 + 2(\gamma+2)(\gamma-1)x - \gamma(\gamma-1).$$

As $q''(x) = -2(\gamma + 1)(\gamma + 2) < 0$, for all $x \in (0, 1)$, $q(0^+) > 0$ and $q'(0^+) < 0$, then there is at most one root of q in $(0, 1)$. By checking $q(c) = -\frac{3\gamma}{\gamma+1} < 0$, we conclude that there is a unique positive root x^* of q in $(0, 1)$ and it satisfies $x^* < c < 1$. Then $q(x) < 0$, while $x > c$. This implies $Sg(x) < 0$, for all $x > c$, and proves Assertion 5.

Remark 1 Condition (9) becomes relevant to continue the study because it ensures that the solutions of the difference equation (8) are well defined for all $x_0 \in (0, 1)$ and $n \geq 0$. Hence, in the following, we will assume that (9) holds. Although we have only supposed that $\delta > 0$, condition (9) implies that $\delta < 4$.

We will say that p is a global attractor of (8) if

$$\lim_{n\to\infty} g^n(x) = p,$$

for every $x \in (0, 1)$. By using Corollaries 2.9 and 2.10 in [5], we can assure that p is a global attractor for (8) if $-1 \leq g'(p) < 1$ and $Sg(x) < 0$, for any $x > c$. We can apply those results due to Assertion 4 in Proposition 1.

Theorem 1 *Let p be the unique fixed point of g in $(0, 1)$. This point is a global attractor for (8) if and only if*

$$\delta \leq (\gamma + 1)\left(\frac{\gamma + 2}{\gamma + 1}\right)^\gamma . \tag{10}$$

In the former case, for (8), convergence to p is eventually monotone (monotone after some $k \in \mathbb{N}$) if and only if

$$\delta \leq \gamma\left(\frac{\gamma + 1}{\gamma}\right)^\gamma . \tag{11}$$

Proof First we will check the sufficient condition for p to be a global attractor. By Corollary 2.10 in [5], checking that $-1 \leq g'(p) < 1$ and $Sg(x) < 0$ for all $x > c$, is sufficient to prove that p is a global attractor. As the negativity condition of the Schwarzian derivative was checked in Assertion 5 of Proposition 1, it only remains to obtain a condition equivalent to $-1 \leq g'(p) < 1$.

On the one hand, if p is an equilibrium of (8), then it satisfies

$$p = g(p) = \delta p^{\gamma}(1-p)$$

or, equivalently,

$$\delta p^{\gamma-1}(1-p) = 1. \tag{12}$$

On the other hand,

$$g'(p) = \delta p^{\gamma-1}[\gamma(1-p)-p] = \gamma - \delta p^{\gamma}, \tag{13}$$

where the last identity can be deduced by using (12).

Now, the required condition is equivalent to

$$-1 \leq \gamma - \delta p^{\gamma} < 1. \tag{14}$$

The second inequality of (14) is always fulfilled because $\gamma < 1$. The first inequality of (14) is equivalent to

$$p \leq \left(\frac{\gamma+1}{\delta}\right)^{\frac{1}{\gamma}} =: \eta. \tag{15}$$

By using Assertion 4 of Proposition 1, one can write (15) as

$$\begin{aligned}\delta\left(\frac{\gamma+1}{\delta}\right)\left[1-\left(\frac{\gamma+1}{\delta}\right)^{\frac{1}{\gamma}}\right] &= g(\eta)\\ &\leq \eta = \left(\frac{\gamma+1}{\delta}\right)^{\frac{1}{\gamma}}.\end{aligned}$$

By doing some calculations one can reach that (15) is equivalent to (10).

In a scalar discrete dynamical system generated by a continuous function, global attraction implies local asymptotic stability (see [18]), that is, the condition $|g'(p)| \leq 1$ must hold. We have seen that $g'(p) < 1$ for any value of the parameters. Hence, $g'(p) \geq -1$, which is equivalent to (10), is a necessary condition for the local asymptotic stability of p.

Now, we proceed with the second part of the Theorem. It is easy to see that (11) is equivalent to $g'(p) \geq 0$. In particular, (11) implies (10). If $x_0 \in (0,p)$, then $x_0 < x_1 = g(x_0) < p$ and therefore (x_n) is strictly increasing. Analogously, if $x_0 \in (p,c]$, then $p < g(x_0) < x_0$ and (x_n) is strictly decreasing. Finally, if $x_0 \in (c,1)$, then $g(x_0) \in (0,c)$ and the subsequence $(x_n)_{n\geq 2}$ is included in the former cases. If (11) does not hold, then one can take the solution of starting at the critical point c to prove that there is no monotonicity.

Remark 2 For (8), asymptotic stability and global attraction of p are equivalent due to the negativity of Sg in $(c,1)$ and Corollary 2.10 of [5].

Another question about (8) is if extinction, that is,

$$\lim_{n\to\infty} x_n = 0,$$

is possible for some values of the parameters and initial conditions. The next proposition shows that this cannot occur.

Proposition 2 *The discrete dynamical system (8) is uniformly persistent (in the sense of [20]), that is, there exists* $\alpha > 0$ *such that*

$$\liminf_{n\to\infty} x_n \geq \alpha, \tag{16}$$

for any $x_0 \in (0,1)$.

Proof First, we will suppose that $c < p$. In such case, $I := [g^2(c), g(c)]$ is an invariant interval for g. This can be checked by the properties of g in Proposition 1. It is obvious that $g(x) \leq g(c)$, for any $x \in (0,1)$. We also have that $g(x) \geq x$ for any $x \in [g^2(c), p]$ and $g(x) \geq g^2(c)$ if $x \in (p, g(c)]$. Then, $g(I) \subset I$. Moreover, I is also an attractor for (8). For each $x \in (0, g^2(c))$, there exists $n \in \mathbb{N}$ such that $g^n(x) \in I$. This last assertion can be checked by taking into account that

$$\lim_{n\to\infty} {g_{|(0,c]}}^{-n}(c) = 0.$$

We finish this case by noting that $g(x) \in [0, g^2(c))$ if $x \in (g(c), 1)$. Therefore, (16) holds with $\alpha = g^2(c) > 0$.

If $c \geq p$, then $g'(p) \geq 0$ due to Assertion 2 of Proposition 1. An application of the second part of Theorem 1 leads us to conclude that every sequence $x_n = g^n(x_0)$ converges to p, and therefore (16) holds with $\alpha = p > 0$.

By using Theorem 1 and its main expression (10), we can write a result about the influence of γ on the stability of p, for each fixed $\delta \in (0,4)$, and obtain the values of δ for which γ produces a stability switch in p.

Proposition 3 *The equilibrium* p *is stable if* $\delta \leq 1$, *unstable if* $\delta \geq 3$ *and increasing* γ *stabilizes* p *if* $\delta \in (1,3)$.

Proof Let $T : (0,1) \to \mathbb{R}$ be defined as

$$T(\gamma) := (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma}.$$

T is trivially positive. By taking logarithms in its definition, one can check that

$$T'(\gamma) = T(\gamma)\left(\frac{2}{(\gamma+1)(\gamma+2)} + \ln\left(\frac{\gamma+2}{\gamma+1}\right)\right) > 0.$$

Then,

$$\lim_{\gamma\to 0^+} T(\gamma) = 1, \quad \lim_{\gamma\to 1^-} T(\gamma) = 3.$$

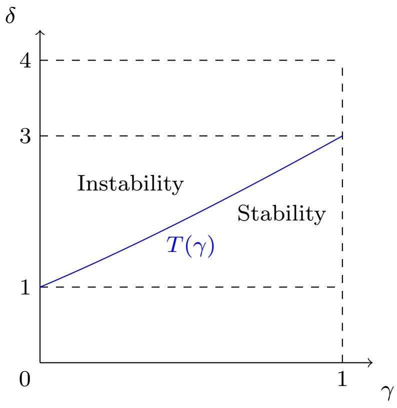

Fig. 2 The function T provides the curve of stability switches for each $\gamma \in (0,1)$. The figure was made by using the LaTeX package tikz.

Since $T(\gamma)$ is the right-hand side of (10), which provides the stability condition for p, the assertions in the statement of the theorem will hold (see Figure 2).

We have also obtained that the capital-labor ratio decreases as the production function tends to be logistic ($\gamma \to 1^-$).

Proposition 4 *For any $\gamma \in (0,1)$, let $p := p(\gamma)$ be the unique fixed point of g in $(0,1)$. Then p is decreasing in γ.*

Proof We recall that p is an equilibrium of g if (12) is satisfied. Let $F : D := (0,1) \times (0,1) \to \mathbb{R}$ be defined as

$$F(p,\gamma) = \delta p^{\gamma-1}(1-p) - 1.$$

Then, the fixed points p of g are characterized by the equation $F(p,\gamma) = 0$. As F is a $\mathcal{C}^1$ map, if we guarantee that $\frac{\partial F}{\partial p} \neq 0$ in D, we can apply the Implicit Function Theorem and $p(\gamma)$ would be $\mathcal{C}^1$ and

$$p'(\gamma) = \frac{-\dfrac{\partial F}{\partial \gamma}(p(\gamma),\gamma)}{\dfrac{\partial F}{\partial p}(p(\gamma),\gamma)}. \tag{17}$$

It is easy to check that

$$\frac{\partial F}{\partial p}(p,\gamma) = \delta p^{\gamma-2}[(1-p)\gamma - 1] < 0,$$

for any $(p,\gamma) \in D$. Then, (17) becomes

$$p'(\gamma) = \frac{-\delta p^{\gamma-1}(1-p)\ln p}{\delta p^{\gamma-2}[(1-p)\gamma - 1]} = \frac{-(1-p)p\ln p}{(1-p)\gamma - 1} < 0$$

and p is decreasing.

3 Stability properties of the delay differential model

Let h be a nonnegative real number and τ be a continuous function such that $\tau(t) \in [0,h]$. The delay differential equation

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t-\tau(t))(1 - x(t-\tau(t))). \tag{18}$$

belongs to the class of equations

$$x'(t) = -g_1(x(t))f_2(x(t-\tau(t))) + f_1(x(t-\tau(t)))g_2(x(t)),$$

considered in [3, Section 2].

Indeed, take $f_2 = g_2 \equiv 1$, $g_1(x) = \alpha x$ and $f_1 = \beta x^\gamma(1-x)$ for $(a,b) = (0,1)$. All of these functions are continuous and g_1 is strictly increasing. Assertions 3 and 4 of Proposition 1, respectively, ensure that $g = g_1^{-1} \circ f_1$ maps $(0,1)$ into $(0,1)$ and there is only one solution of $g(x) = x$, named p, in such interval. As g_1 and g_2 are Lipschitz continuous functions, for any $\phi \in \mathcal{C}_{(0,1)} := \{\varphi \in \mathcal{C}_+ : \varphi(t) \in (0,1),\ t \in [-h,0]\}$, there is a unique solution $x(t,\phi)$ of (18) with initial condition ϕ and it is defined for $t \geq 0$ (see [8]). Finally, due to the thesis of Theorem 1, p is a global attractor for (8) if we suppose that condition (10) holds.

Theorem 3.2 of [3] connects the global stability of the discrete dynamical system generated by (8) and the global attraction of the positive equilibrium of (18). That result uses the notion of strong attractor given by Liz and Ruiz-Herrera [14], which is equivalent to the concept of global attractor in the scalar case. Such theorem is stated as follows.

Theorem 2 *If p is a global attractor of (8) on $(0,1)$, then p is a global attractor of (18), that is, if $x(t) = x(t,\phi)$ is the solution of (18) with initial condition $\phi \in \mathcal{C}_{(0,1)}$, then*

$$\lim_{t\to\infty} x(t,\phi) = p.$$

The previous result allows us to write another one about the global stability of p in (18).

Theorem 3 *Let p be the unique fixed point of g in $(0,1)$. If*

$$\delta \leq (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^\gamma, \tag{19}$$

then p is a global attractor for the delay differential equation (18).

3.1 Constant delay

In the particular case of $\tau(t) = h$, for every t, we can provide more information about the stability of p for (18). Then, the following results will be related to the study of the equation

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t-h)(1 - x(t-h)). \tag{20}$$

First, we will show a local stability result (Theorem 4) and then a sharper global stability result (Theorem 5) is given.

Theorem 4 *For a given $\gamma \in (0,1)$, let p be the unique fixed point of g in $(0,1)$. Then, p is an asymptotically stable equilibrium for (20) if either (19) holds or (19) does not hold and*

$$h < \frac{\arccos\left(\frac{1}{\gamma - \delta p^\gamma}\right)}{\alpha\sqrt{-1 + (\gamma - \delta p^\gamma)^2}}. \tag{21}$$

Proof We recall that the equilibria of (20) are given by the equilibria of (8). Consider the linearized equation of (20) about the positive equilibrium p:

$$x'(t) = -\alpha x(t) + \alpha g'(p)x(t-h).$$

Applying Theorem 4.7 in [19] with $A = -\alpha$ and $B = \alpha g'(p)$, it follows that p is asymptotically stable if either $|g'(p)| \le 1$ or both $g'(p) < -1$ and

$$\arccos\left(\frac{1}{g'(p)}\right) > h\alpha\sqrt{-1 + (g'(p))^2}$$

hold. By (13), we can replace $g'(p)$ by $\gamma - \delta p^\gamma$.

Remark 3 When $\gamma = 1$, the equilibrium of (20) is $p = 1 - \alpha/\beta$. As a limit case of Theorem 4 as $\gamma \to 1^-$, we recover the results for $\gamma = 1$ established in Theorems 2 and 3 of [15].

The invariance principle [9, Theorem 2.3] establishes that an invariant and attracting interval $[a,b]$ for g is also invariant and attracting for (20) for all values of the delay h, that is,

$$a \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le b,$$

for any solution of (20) starting at $\phi \in \mathcal{C}_{(0,1)}$. In view of Proposition 2, we can choose $a = \min\{p, g^2(c)\}$, $b = g(c)$.

Using this invariance principle, we can derive some other dynamical properties of (20) from the results of [7]. Although in the framework of [7] the analogous g (φ with their notation) is defined in $(0,\infty)$ and satisfies $g(x) > 0$, it is clear that for large t any solution of (20) coincides with one solution of

$$x'(t) = -\alpha x(t) + \beta f(x(t-h)),$$

for some continuous function $f : (0,\infty) \to (0,\infty)$, with $f(x) = \beta x^\gamma(1-x)$, $x \in (0,M)$, $g(c) < M < 1$ and

$$\lim_{x\to\infty} f(x) = 0,$$

which is in the framework of [7].

Another consequence of the invariance principle is that (20) is uniformly persistent.

In Theorem 5, we show a condition for p to become a global attractor. As in Theorem 3, we can write an absolute stability condition, but the difference is that we can also give a delay-dependent stability condition. Theorem 3 and Corollary 17 in [7] are used for proving it.

Theorem 5 *Let p be the unique fixed point of g in $(0,1)$. If either (19) holds or (19) does not hold and*

$$h \le \frac{1}{\alpha}\ln\left(\frac{\gamma - \delta p^\gamma}{1 + \gamma - \delta p^\gamma}\right), \tag{22}$$

then p is a global attractor for the delay differential equation (20).

Proof The first part comes from a direct application of Theorem 3. The second part comes from [7, Corollary 17]: if p is a global attractor for the discrete dynamical system

$$x_{n+1} = \zeta(x_n) := e^{-\alpha h}p + (1 - e^{-\alpha h})g(x_n),\ n \in \mathbb{N}, \tag{23}$$

then p is a global attractor of (20). The unique equilibrium of (23) is also p. Then, by an application of Corollary 2.10 in [5], it is sufficient for ζ to satisfy $S\zeta < 0$ in $(c,1)$ and $-1 \le \zeta'(p) < 1$. The first condition is trivially satisfied because $S\zeta = Sg$. The second one is equivalent to (22).

In Figure 3 we present a stability diagram for (20) in the plane (γ, h). For each $\delta = \beta/\alpha$, we recall that these diagrams only make sense if (9) holds. Moreover, provided (19) holds, p is a global attractor for (20) for every h.

4 Discussion

In the last decades, Solow's model of economic growth [21] has achieved a large recognition by the economists [2]. It is a model that tries to explain the long-run behaviour of a market for which it is supposed there

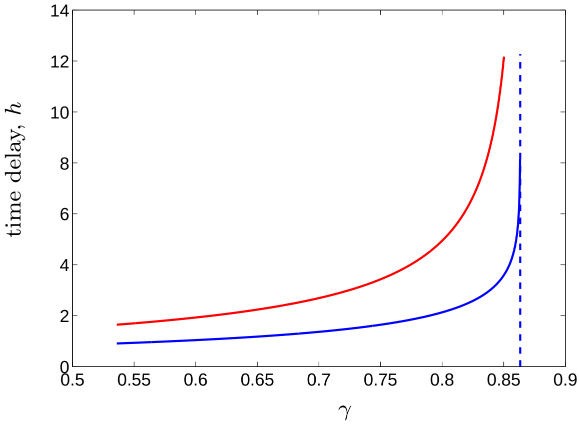

Fig. 3 Diagram of stability for (20) in the plane (γ, h) for $\alpha = 1$ and $\beta = 2.7$. The dashed line represents the threshold value for γ to ensure that p is a global attractor (absolute stability). Below the lower solid curve, the equilibrium p is a global attractor (delay-dependent condition). The upper solid curve represents the local stability switch of p. Both solid curves were numerically calculated and are shown for the parameter values for which (9) holds. The graphics were made by using MATLAB.

is only one commodity. Through the study of Solow's model, one can show how the relation between capital and labor changes in time. The evolution of the capital-labor ratio depends inversely on labor growth and directly on a production function. The production function picks up the information about how increasing relation capital-labor ratio affects to the commodity. By taking into account the Cobb-Douglas function, x^γ can be chosen as the production function. Another remarkable feature about Solow's neoclassical model is the consideration of a pollution function because of capital accumulation. This function reduces the positive effect of the production on the above-mentioned ratio. An example for the pollution function is $1-x$, which is decreasing in x, but not positive all over $(0, \infty)$. Hence, we have needed to consider a normalized variable x in order to work in the set $(0, 1)$, where the pollution function is positive. Then, our differential model includes a logistic-type term. Finally, although Sollow's original model was proposed in terms of an ordinary differential equation, the presence of delays in the logistic-type term is justified by delays in production [15]. We state here again the model we have worked with:

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t-\tau(t))(1 - x(t-\tau(t))). \qquad (24)$$

It is known that, in the ordinary case, a unique equilibrium appears if we take reasonable hypotheses on the production function and this equilibrium becomes a global attractor [2]. A relevant question is if the presence of delays affects the stability of our model. The authors of [15] have dealt with the study of local stability of some particular cases of the delay-differential model (24). In this paper, following the lines of [3], we have extended the study in [15] by providing global stability results for the delay-differential model (24). We have seen that, for some values of the parameters, the global attraction properties of the positive equilibrium are preserved for (24). The key idea is studying auxiliar discrete dynamical systems due to its connection to continuous dynamical systems [7,9]. The particular conditions for (24) to have a global attractor can be found in Theorem 3 and, in the case of a constant τ, in Theorem 5, where a sharper condition is provided. The parameter γ provides more flexibility [1,3,11,12] for data fitting, and allows changes in the discrete dynamical behaviour of $x^\gamma(1-x)$ as γ is varied. Then, the behaviour of the delay-differential model (24) can change accordingly as well.

For example, Proposition 4 explains how γ affects the equilibrium size, and it applies to the delay differential equation (24) because the positive equilibrium is the same as in equation (8). Roughly speaking, in terms of economics, it says that the capital-labor ratio decreases as the marginal productivity of capital rises, which is a typical characteristic of economic models based on the Cobb-Douglas function. Another interesting fact is that, in view of Proposition 3, an increasing value of the marginal productivity of capital tends to stabilize the model, thus preventing growth cycles in the capital-labor ratio.

Our results can also be applied in population dynamics. Indeed, for constant delay and $\gamma = 1$, equation (24) is known as the blowfly logistic equation, and it was introduced by Maynard Smith [17] to model an age-structured population with two stages; see [10, Section 4] for more references. Thus, equation (24) can be viewed as a generalization of the blowfly equation, which allows a greater flexibility to fit the recruitment function to field data. In particular, the global stability conditions given in Theorems 3 and 5 provide in the limit case $\gamma = 1$ the stability results for the blowfly equation given in Section 4.2 of [10].

Future work should be done in order to investigate if (20) satisfies "L.A.S. implies G.A.S." property. Other interesting problem would be to study equation (18) with state-dependent delay.

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Compliance with Ethical Standards

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