Existence of Solutions of Nonlocal Perturbations of Dirichlet Discrete Nonlinear Problems *

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Abstract

This paper is devoted to the study of second order nonlinear difference equations. A Nonlocal Perturbation of a Dirichlet Boundary Value Problem is considered. An exhaustive study of the related Green's function to the linear part is done. The exact expression of the function is given, moreover the range of parameter for which it has constant sign is obtained. Using this, some existence results for the nonlinear problem are deduced from monotone iterative techniques, the classical Krasnoselski fixed point theorem or by application of recent fixed point theorems that combine both theories.

1 Introduction and preliminaries

The study of difference equations represents a useful model for several mathematical models in different areas as economy or population dynamics. Moreover they are a fundamental tool in the discretization of a differential equation. The reader can found the classical theory on the monographs of Goldberg [18] R. P. Agarwal [1], V. Laksmikantham and D. Trigiante [24] and S. Elaydi [15].

The existence of solution for nonlinear difference problems have been treated from different approach, as, among others, the method of upper and lower solutions [3, 4, 5, 11, 16, 17, 26, 27], monotone iterative techniques [9, 10, 34, 31], variational methods [2, 12, 13, 14, 25, 29, 30] or fixed point theorems [7, 28].

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Our aim is to obtain the existence result for the following parameter dependence nonlinear problem

$$(P)\quad \begin{cases} -\Delta^2 u(k-1) = f(k,u(k)),\ k=1,\dots,N-1,\ N\geq 2, \\ \\ u(0)=0,\ u(N)=\mu\sum_{k=a}^{b} u(k),\ \mu>0,\ 1\leq a\leq b\leq N-1, \end{cases}$$

which is a nonlocal perturbation of the Dirichlet problem.

In section 2 we study the Green's function related to the linear problem to be considered. The exact expression of such function is obtained. Moreover, we found the values of the parameter μ for which this function is strictly positive on its square of definition. Section 3 is devoted to the study of the existence of solution for the considered nonlinear boundary value problem by using the classical expansion/contraction Krasnoselskii's fixed point theorem. In section 4 we use some fixed point theorems proved recently in [6], where we combine the existence of a lower solution of (P) and some monotonicity assumptions on function f.

2 Study of the Green's function

Denote $I=\{0,\dots,N\}$, $J=\{1,\dots,N-1\}$ and $J_1=\{1,\dots,N\}$.

The Green's function to the related linear problem

$$\begin{cases} -\Delta^2 u(k-1) = 0,\ k\in J,\ N\geq 2, \\ \\ u(0)=0,\ u(N)=\mu\sum_{k=a}^{b} u(k),\ \mu>0,\ 1\leq a\leq b\leq N-1, \end{cases}$$

is given by a two variables function

$$G:(k,s)\in I\times J\to G(k,s)\in\mathbb{R}.$$

The main result of this section is the following

Theorem 1. *If* $\mu\in\left(0,\frac{2N}{(a+b)(b-a+1)}\right)$ *then* $G(k,s)>0$ *for all* $k\in J_1$ *and* $s\in J$.

Moreover, for such values of μ*, there are two positive constants* $0<m<M$ *for which*

$$m\,G(N,s)\leq G(k,s)\leq M\,G(N,s) \tag{2.1}$$

for all $k\in J_1$ *and* $s\in J$.

Proof. Following the approach in [23], we know that function G follows a form as

$$G(k,s)=\begin{cases} G_1(k,s)\equiv C_1(s)+k\,C_2(s), & \text{if } k\leq s, \\ G_2(k,s)\equiv C_3(s)+k\,C_4(s), & \text{if } k>s. \end{cases}$$

To obtain the expression of the functions C_i, $i \in \{1, \dots, 4\}$, we must use the fact that the expression of the Green's function G is characterized by the following identities:

$$-\Delta^2 G(k-1,k) = 1,$$
$$\Delta^2 G(k-1,s) = 0, \;\; s \neq k,$$
$$G(0,s) = 0, \;\; G(N,s) = \mu \sum_{k=a}^{b} G(k,s).$$

The diagonal equalities give us that

$$\begin{aligned} -1 &= G(k+1,k) - 2G(k,k) + G(k-1,k) \\ &= -(k+1)C_2(k) + C_3(k) + (k+1)C_4(k) \end{aligned}$$

and

$$\begin{aligned} 0 &= G(k+2,k) - 2G(k+1,k) + G(k,k) \\ &= kC_2(k) - C_3(k) - kC_4(k). \end{aligned}$$

Moreover, the condition $G(0,s) = 0$ implies that $C_1(s) = 0$.

We must distinguish three cases, depending on the relative position of s with respect to a and b.

Case I. $1 \leq a \leq s \leq b \leq N-1$.

In this situation, we have that

$$G(N,s) = \mu \sum_{k=a}^{b} G(k,s) = \mu \sum_{k=a}^{s} G_1(k,s) + \mu \sum_{k=s+1}^{b} G_2(k,s).$$

Thus the last equality about Green's function is as follows

$$\begin{aligned} G(N,s) &= \mu \sum_{k=a}^{b} G(k,s) = \mu \sum_{k=a}^{s} kC_2(s) + \mu \sum_{k=s+1}^{b} (C_3(s) + kC_4(s)) \\ &= \mu \frac{(s-a+1)(s+a)}{2} C_2(s) \\ &\quad + \mu(b-s)C_3(s) + \mu \frac{(b+s+1)(b-s)}{2} C_4(s) \\ &= C_3(s) + NC_4(s). \end{aligned}$$

After solving the system we obtain that

$$C_2(s) = \frac{-\mu(b-s)(1+b-s) + 2(N-s)}{(-1+a-b)(a+b)\mu + 2N},$$

$$C_3(s) = s$$

and

$$C_4(s) = \frac{a\mu - a^2\mu + s(-2+\mu+2b\mu-\mu s)}{(-1+a-b)(a+b)\mu+2N}.$$

Finally for $\mu \neq \frac{2N}{(a+b)(b-a+1)}$ the related Green's function to our problem is given by the following expression:

$$G(k,s) = \begin{cases} \dfrac{k\left(-\mu(b-s)(1+b-s)+2(N-s)\right)}{(-1+a-b)(a+b)\mu+2N}, & k \leq s, \\ s + \dfrac{k\left(a\mu - a^2\mu + s(-2+\mu+2b\mu-\mu s)\right)}{(-1+a-b)(a+b)\mu+2N}, & k > s. \end{cases}$$

First we will show that $G_1(k,s) > 0$ when $\mu \in \left(0, \frac{2N}{(a+b)(b-a+1)}\right)$.

It is clear that $G_1(k,s) > 0$ if and only if

$$\left(-\mu(b-s)(1+b-s)+2(N-s)\right)\left((-1+a-b)(a+b)\mu+2N\right) > 0.$$

Let us denote $\mu_1 = \frac{2N}{(a+b)(b-a+1)}$ and $\mu_2 = \frac{2(N-s)}{(b-s)(b-s+1)}$.

Now $\mu_1 < \mu_2$ if and only if

$$(N-s)(a+b)(b-a+1) > N(b-s)(b-s+1).$$

But since $a \leq s$ we have that $(b-a+1) \geq (b-s+1)$ and it is easy to check that

$$(N-s)(a+b) > N(b-s),$$

and we conclude that $\mu_1 < \mu_2$, which gives us that $G_1(k,s) > 0$ when $\mu \in (0, \mu_1)$.

Now we will show that $G_2(k,s) > 0$ for all $\mu \in (0, \mu_1)$.

Let $\mu \in (0, \mu_1)$. From the expression of Green's function we have that $G_2(k,s) > 0$ whenever

$$s\left((-1+a-b)(a+b)\mu+2N\right) + k\left(a\mu - a^2\mu + s(-2+\mu+2b\mu-\mu s)\right) > 0,$$

which is the same as

$$2s(N-k) > \mu\left[s(a+b)(b-a+1) + a^2k - ak - ks + ks^2 - 2bsk\right].$$

Since $\mu < \frac{2N}{(a+b)(b-a+1)}$ it is enough to show that

$$2s(N-k) > \frac{2N(s(a+b)(b-a+1) + a^2k - ak - ks + ks^2 - 2bsk)}{(a+b)(b-a+1)},$$

or, which is the same

$$s(N-k)(a+b)(b-a+1) > N(s(a+b)(b-a+1) + a^2k - ak - ks + ks^2 - 2bsk).$$

After some simplifications, we have to check that

$$Na + Ns + 2Nbs + sa^2 > Na^2 + Ns^2 + sa + sb + sb^2. \tag{2.2}$$

We will prove this inequality by induction on N:

For $N = b + 1$ we have that

$$(b+1)a + (b+1)s + 2(b+1)bs + sa^2 > (b+1)a^2 + (b+1)s^2 + sa + sb + sb^2,$$

which is equivalent to

$$ba + a + s + b^2 s + 2bs + sa^2 > ba^2 + a^2 + bs^2 + s^2 + sa. \tag{2.3}$$

We will prove the last one by induction on b.

If $b = s$ we have

$$sa + a + s + s^3 + 2s^2 + sa^2 > sa^2 + a^2 + s^3 + s^2 + sa.$$

This is the same as

$$s^2 + a + s > a^2,$$

which is true for all $s \geq a$.

Now let us suppose that (2.3) is true for $b = b_0 \geq s$, i.e.

$$b_0 a + a + s + b_0^2 s + 2b_0 s + sa^2 > b_0 a^2 + a^2 + b_0 s^2 + s^2 + sa.$$

For $b = b_0 + 1$ we have to prove that

$$b_0 a + 2a + s + (b_0+1)^2 s + 2(b_0+1)s + sa^2 > (b_0+1)a^2 + a^2 + b_0 s^2 + 2s^2 + sa.$$

But using our inductive assumption, the above inequality is equivalent to

$$a + 3s + 2b_0 s > a^2 + s^2,$$

which is true.

Thus inequality (2.2) holds for $N = b + 1$.

Now let us suppose that (2.2) is true and for $N = N_0 \geq b + 1$, i.e.

$$N_0 a + N_0 s + 2N_0 bs + sa^2 > N_0 a^2 + N_0 s^2 + sa + sb + sb^2.$$

For $N = N_0 + 1$ we have to prove that

$$(N_0+1)a + (N_0+1)s + 2(N_0+1)bs + sa^2 > (N_0+1)a^2 + (N_0+1)s^2 + sa + sb + sb^2.$$

Using our inductive assumption, the previous inequality is equivalent to

$$a + s + 2bs > a^2 + s^2,$$

which is trivially fulfilled.

Thus we have showed that (2.2) holds for every N.

As a consequence the last result gives us that $G_2(k,s) > 0$ for all $\mu \in (0, \mu_1)$.

In the sequel, we are interested in to find two positive constants m and M, satisfying inequalities (2.1) for $a \le s \le b$.

Let us consider the case when $k \le s$. Then

$$\frac{G_1(k,s)}{G_2(N,s)} = \frac{k}{\mu} \frac{2(N-s) - \mu(b-s)(b-s+1)}{N\left(a - a^2 + s\left(1 + 2b - s\right)\right) - s(a+b)(b-a+1)}.$$

Let us denote $q(s) = 2(N-s) - \mu(b-s)(b-s+1)$, where $1 \le a \le s \le b \le N-1$. Clearly $q'(s_0) = 0$ if and only if $s_0 = \frac{\mu(2b+1)-2}{2\mu}$.

Obviously, $s_0 \ge 1$ if and only if $\mu \ge \frac{2}{2b-1}$ and $s_0 \le b$ if and only if $\mu \le 2$.

Since $\frac{2}{2b-1} \le 2$ for every $b \ge 1$, we have the following possibilities:

Case 1: $\mu \in \left(0, \frac{2}{2b-1}\right)$, so $s_0 < 1$ and $q(b) \le q(s) \le q(1)$ for all $1 \le s \le b$.

Case 2: $\mu \in \left[\frac{2}{2b-1}, \frac{2N}{(a+b)(b-a+1)}\right)$ and $N < (a+b)(b-a+1)$. So we have that $1 \le s_0 < b$. Thus $\min\{q(1), q(b)\} \le q(s) \le q(s_0)$ for all $1 \le s \le b$.

Case 3: $\mu \in \left[\frac{2}{2b-1}, \frac{2N}{(a+b)(b-a+1)}\right)$ and $N > (a+b)(b-a+1)$. So we have $b < s_0$, which gives us that $q(1) \le q(s) \le q(b)$ for all $1 \le s \le b$.

Let us now denote $g(s) = -Ns^2 + s\left(2Nb + N + a^2 - b^2 - a - b\right) + Na(1-a)$.

We have $g'(s_1) = 0$ if and only if $s_1 = \frac{2Nb+N+a^2-b^2-a-b}{2N} > 1$, and we distinguish the two following cases:

Case 1: If $N < (a+b)(b-a+1)$, then $s_1 \le b$. Thus we obtain that $\min\{g(1), g(b)\} \le g(s) \le g(s_1)$ for every $1 \le s \le b$.

Case 2: If $N \ge (a+b)(b-a+1)$, then $s_1 > b$ and we have that $g(1) \le g(s) \le g(b)$ for every $1 \le s \le b$.

As consequence, we arrive at the following consequences:

1) if $\mu \in \left(0, \frac{2}{2b-1}\right)$ and $N < (a+b)(b-a+1)$ then

$$\frac{k}{\mu}\frac{q(b)}{g(s_1)} G(N,s) \le G(k,s) \le \frac{k}{\mu}\frac{q(1)}{\min\{g(1), g(b)\}} G(N,s),$$

2) if $\mu \in \left(0, \frac{2}{2b-1}\right)$ and $N \ge (a+b)(b-a+1)$ then

$$\frac{k}{\mu}\frac{q(b)}{g(b)} G(N,s) \le G(k,s) \le \frac{k}{\mu}\frac{q(1)}{g(1)} G(N,s),$$

3) if $\mu \in \left[\frac{2}{2b-1}, \frac{2N}{(a+b)(b-a+1)}\right)$ and $N < (a+b)(b-a+1)$ then

$$\frac{k}{\mu}\frac{\min\{q(1),q(b)\}}{g(s_1)}G(N,s) \le G(k,s) \le \frac{k}{\mu}\frac{q(s_0)}{\min\{g(1),g(b)\}}G(N,s),$$

4) if $\mu \in \left[\frac{2}{2b-1}, \frac{2N}{(a+b)(b-a+1)}\right)$ and $N \ge (a+b)(b-a+1)$ then

$$\frac{k}{\mu}\frac{q(1)}{g(b)}G(N,s) \le G(k,s) \le \frac{k}{\mu}\frac{q(b)}{g(1)}G(N,s).$$

Now, let us consider the case when $k > s$:

$$\frac{G_2(k,s)}{G_2(N,s)} = \frac{1}{\mu}\frac{s\left((-1+a-b)(a+b)\mu+2N\right)+k\left(a\mu-a^2\mu+s(-2+\mu+2b\mu-\mu s)\right)}{N\left(a-a^2+s\left(1+2b-s\right)\right)-s(a+b)(b-a+1)}.$$

Let us denote

$$p(s) = -k\mu s^2 + s\left(2N - a\mu - \mu b^2 - b\mu + a^2\mu - 2k + k\mu + 2kb\mu\right) + ka\mu - ka^2\mu,$$

clearly, $p'(s_2) = 0$ if and only if $s_2 = \frac{1}{2}\frac{-a\mu-\mu b^2+2N-b\mu+a^2\mu-2k+k\mu+2kb\mu}{k\mu} > 1$.

Since

$$s_2 \le b \text{ if and only if } \mu \ge \frac{2(N-k)}{b^2-a^2+a+b-k}$$

and

$$\frac{2(N-k)}{b^2-a^2+a+b-k} < \frac{2N}{b^2-a^2+a+b} \text{ if and only if } N < (a+b)(b-a+1)$$

we arrive to the following cases

Case 1: If $N < (a+b)(b-a+1)$ and

1.1) $\mu \in \left(0, \frac{2(N-k)}{b^2-a^2+a+b-k}\right)$ then $s_2 > b$, so we have $p(1) \le p(s) \le p(b)$ for all $1 \le s \le b$.

1.2) $\mu \in \left[\frac{2(N-k)}{b^2-a^2+a+b-k}, \frac{2N}{b^2-a^2+a+b}\right)$ then $s_2 \ge b$, so $\min\{p(1),p(b)\} \le p(s) \le p(s_2)$ for all $1 \le s \le b$.

Case 2: If $N \ge (a+b)(b-a+1)$ then $s_2 > b$ and we have $p(1) \le p(s) \le p(b)$ for all $1 \le s \le b$.

As consequence, we deduce the following inequalities

1) if $\mu \in \left(0, \frac{2(N-k)}{b^2-a^2+a+b-k}\right)$ and $N < (a+b)(b-a+1)$ then

$$\frac{1}{\mu}\frac{p(1)}{g(s_1)}G(N,s) \le G(k,s) \le \frac{1}{\mu}\frac{p(b)}{\min\{g(1),g(b)\}}G(N,s),$$

2) if $\mu \in \left[\frac{2(N-k)}{b^2-a^2+a+b-k}, \frac{2N}{b^2-a^2+a+b}\right)$ and $N < (a+b)(b-a+1)$ then

$$\frac{1}{\mu}\frac{\min\{p(1),p(b)\}}{g(s_1)}G(N,s) \le G(k,s) \le \frac{1}{\mu}\frac{p(s_2)}{\min\{g(1),g(b)\}}G(N,s),$$

3) if $\mu \in \left(0, \frac{2N}{b^2-a^2+a+b}\right)$ and $N \ge (a+b)(b-a+1)$ then

$$\frac{1}{\mu}\frac{p(1)}{g(b)}G(N,s) \le G(k,s) \le \frac{1}{\mu}\frac{p(b)}{g(1)}G(N,s).$$

Finally, if we denote

$$m_* = \frac{1}{\mu}\frac{\min\{kq(1),kq(b),p(1),p(b)\}}{\max\{g(b),g(s_1)\}}$$

and

$$M^* = \frac{1}{\mu}\frac{\max\{kq(1),kq(s_0),kq(b),p(b),p(s_2)\}}{\min\{g(1),g(b)\}},$$

then we deduce that for all $\mu \in \left(0, \frac{2N}{(a+b)\,(b-a+1)}\right)$ the following inequalities hold:

$$m_*G(N,s) \le G(k,s) \le M^*G(N,s) \quad \text{for all} \quad 1 \le k \le N \;\; \text{and } a \le s \le b.$$

Case II. We study now the situation of $s < a$. In this case, the last equality about Green's function is

$$\begin{aligned} G(N,s) &= \mu\sum_{k=a}^{b} G(k,s) = \mu\sum_{k=a}^{b}(C_3(s)+kC_4(s)) \\ &= \mu(b-a+1)C_3(s) + \mu\frac{(b+a)(b-a+1)}{2}C_4(s) \\ &= C_3(s) + NC_4(s). \end{aligned}$$

After some calculations, we obtain that

$$C_2(s) = 1 - \frac{2s(\mu(b-a+1)-1)}{(-1+a-b)(a+b)\mu+2N},$$

$$C_3(s) = s \text{ and } C_4(s) = \frac{2s(\mu(b-a+1)-1)}{(-1+a-b)(a+b)\mu+2N}.$$

Finally, we conclude that

$$G(k,s) = \begin{cases} k\left(1+\dfrac{2s\,(\mu(b-a+1)-1)}{(-1+a-b)(a+b)\mu+2N}\right), & k \le s, \\ s\left(1+\dfrac{2k\,(\mu(b-a+1)-1)}{(-1+a-b)(a+b)\mu+2N}\right), & k > s. \end{cases}$$

Now for $\mu \in \left(0, \frac{2N}{(a+b)(b-a+1)}\right)$ both $G_1(k,s)$ and $G_2(k,s)$ are positive.

In order to prove the inequalities (2.1), let us first consider the case $k \leq s$. Then

$$\begin{aligned}
\frac{G_1(k,s)}{G_2(N,s)} &= \frac{k\left(2(N-s)-\mu(b-a+1)(a+b-2s)\right)}{\mu s(b-a+1)(2N-a-b)} \\
&= \frac{k\left(2s\left(\mu(b-a+1)-1\right)+2N-\mu(a+b)(b-a+1)\right)}{\mu s(b-a+1)(2N-a-b)}.
\end{aligned}$$

Case 1: If $\mu(b-a+1)-1 \geq 0$ then $\mu \in \left[\frac{1}{b-a+1}, \frac{2N}{(a+b)(b-a+1)}\right)$ and we deduce

$$\begin{aligned}
\frac{G_1(k,s)}{G_2(N,s)} &\geq \frac{k\left(2\left(\mu(b-a+1)-1\right)+2N-\mu(a+b)(b-a+1)\right)}{\mu\,(N-1)\,(b-a+1)(2N-a-b)} \\
&= \frac{k\left(2N-2-\mu(a+b-2)(b-a+1)\right)}{\mu\,(N-1)\,(b-a+1)(2N-a-b)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{G_1(k,s)}{G_2(N,s)} &\leq \frac{k\left(2(N-1)\left(\mu(b-a+1)-1\right)+2N-\mu(a+b)(b-a+1)\right)}{\mu(b-a+1)(2N-a-b)} \\
&= \frac{k\left(\mu(b-a+1)(2N-a-b-2)+2\right)}{\mu(b-a+1)(2N-a-b)}.
\end{aligned}$$

Case 2: If $\mu(b-a+1)-1 < 0$ then $\mu \in \left(0, \frac{1}{b-a+1}\right)$ and we have

$$\frac{k\left(\mu(b-a+1)(2N-a-b-2)+2\right)}{\mu\,(N-1)\,(b-a+1)(2N-a-b)} \leq \frac{G_1(k,s)}{G_2(N,s)}$$

and

$$\frac{G_1(k,s)}{G_2(N,s)} \leq \frac{k\left(2N-2-\mu(a+b-2)(b-a+1)\right)}{\mu(b-a+1)(2N-a-b)}.$$

In the other case, i.e. when $k > s$ we have that

$$\frac{G_2(k,s)}{G_2(N,s)} = \frac{2(N-k)+\mu(b-a+1)(2N-a-b)}{\mu(b-a+1)(2N-a-b)},$$

which does not depend on s.

Finally if we denote

$$m_{a_1}(k) = \frac{2(N-k)+\mu(b-a+1)(2N-a-b)}{\mu(b-a+1)(2N-a-b)},$$

$$m_{a_2}(k) = \frac{k\left(\mu(b-a+1)(2N-a-b-2)+2\right)}{\mu(N-1)(b-a+1)(2N-a-b)},$$

$$m_{a_3}(k) = \frac{k\left(2N-2-\mu(a+b-2)(b-a+1)\right)}{\mu(b-a+1)(2N-a-b)},$$

$$m_a = \min\{m_{a_1}, m_{a_2}, m_{a_3}\} \text{ for } k = 1, ..., N$$

and

$$M_a = \max\{m_{a_1}, m_{a_2}, m_{a_3}\} \text{ for } k = 1, ..., N,$$

then we obtain that for all $\mu \in \left(0, \frac{2N}{(a+b)(b-a+1)}\right)$ the following inequalities hold:

$$m_a G(N,s) \le G(k,s) \le M_a G(N,s) \quad \text{for all} \quad 1 \le k \le N \ \text{ and } 1 \le s < a.$$

<u>Case III.</u> If $s > b$ then last equality about Green's function gives us that

$$\begin{aligned} G(N,s) &= \mu \sum_{k=a}^{b} G(k,s) = \mu \sum_{k=a}^{b} k C_2(s) \\ &= \mu \frac{(b+a)(b-a+1)}{2} C_2(s) = C_3(s) + N C_4(s). \end{aligned}$$

After some calculations, we obtain that

$$C_2(s) = \frac{2(N-s)}{(-1+a-b)(a+b)\mu + 2N},$$

$$C_3(s) = s \text{ and } C_4(s) = \frac{\mu(a+b)(b-a+1) - 2s}{(-1+a-b)(a+b)\mu + 2N}.$$

So finally we have

$$G(k,s) = \begin{cases} \dfrac{2k\,(N-s)}{(-1+a-b)(a+b)\mu + 2N}, & k \le s, \\ s + \dfrac{k\,(\mu(a+b)(b-a+1) - 2s)}{(-1+a-b)(a+b)\mu + 2N}, & k > s. \end{cases}$$

Again we have that for $\mu \in \left(0, \frac{2N}{(a+b)(b-a+1)}\right)$ both $G_1(k,s)$ and $G_2(k,s)$ are positive.

Finally, to prove inequalities (2.1) we consider first the case $k \le s$. Then

$$\begin{aligned} \frac{G_1(k,s)}{G_2(N,s)} &= \frac{2k\,(N-s)}{s\,(2N - \mu(a+b)(b-a+1)) + N\,(\mu(a+b)(b-a+1) - 2s)} \\ &= \frac{2k}{\mu(a+b)(b-a+1)} \end{aligned}$$

which does not depend on s.

In the second case when $k > s$ we obtain

$$\begin{aligned} \frac{G_2(k,s)}{G_2(N,s)} &= \frac{s\,(2N - \mu(a+b)(b-a+1)) + k\,(\mu(a+b)(b-a+1) - 2s)}{s\,(2N - \mu(a+b)(b-a+1)) + N\,(\mu(a+b)(b-a+1) - 2s)} \\ &= \frac{s\,(2\,(N-k) - \mu(a+b)(b-a+1)) + \mu k(a+b)(b-a+1)}{\mu(a+b)(b-a+1)\,(N-s)}. \end{aligned}$$

We have different possibilities:

Case 1: If $2\,(N-k)-\mu(a+b)(b-a+1)\geq 0$ then $\mu\in\left(0,\frac{2(N-k)}{(a+b)(b-a+1)}\right)$

$$\frac{G_2(k,s)}{G_2(N,s)}\leq\frac{(N-1)\,(2\,(N-k)-\mu(a+b)(b-a+1))+\mu k(a+b)(b-a+1)}{\mu(a+b)(b-a+1)}$$

and

$$\frac{2\,(N-k)+\mu(k-1)(a+b)(b-a+1)}{\mu(a+b)(b-a+1)\,(N-1)}\leq\frac{G_2(k,s)}{G_2(N,s)}$$

Case 2: If $2\,(N-k)-\mu(a+b)(b-a+1)<0$ then $\mu\in\left[\frac{2(N-k)}{(a+b)(b-a+1)},\frac{2N}{(a+b)(b-a+1)}\right)$

$$\frac{G_2(k,s)}{G_2(N,s)}\leq\frac{2\,(N-k)-\mu(a+b)(b-a+1)+\mu k(a+b)(b-a+1)}{\mu(a+b)(b-a+1)}$$

and

$$\frac{2(N-1)\,(N-k)-\mu(N-k-1)(a+b)(b-a+1)}{\mu(a+b)(b-a+1)\,(N-1)}\leq\frac{G_2(k,s)}{G_2(N,s)}.$$

Finally if we denote

$$m_{b_1}(k)=\frac{2k}{\mu(a+b)(b-a+1)}$$

$$m_{b_2}(k)=\frac{2\,(N-k)+\mu(k-1)(a+b)(b-a+1)}{\mu(a+b)(b-a+1)\,(N-1)}$$

$$m_{b_3}(k)=\frac{2(N-1)\,(N-k)-\mu(N-k-1)(a+b)(b-a+1)}{\mu(a+b)(b-a+1)\,(N-1)}$$

$$M_{b_2}(k)=\frac{(N-1)\,(2\,(N-k)-\mu(a+b)(b-a+1))+\mu k(a+b)(b-a+1)}{\mu(a+b)(b-a+1)}$$

$$M_{b_3}(k)=\frac{2\,(N-k)-\mu(a+b)(b-a+1)+\mu k(a+b)(b-a+1)}{\mu(a+b)(b-a+1)}$$

$$m_b=\min\{m_{b_1},m_{b_2},m_{b_3}\}\text{ for }k=1,...,N$$

and

$$M_b=\max\{m_{b_1},M_{b_2},M_{b_3}\}\text{ for }k=1,...,N,$$

we obtain that for all $\mu\in\left(0,\frac{2N}{(a+b)(b-a+1)}\right)$ the following inequalities hold:

$$m_bG(N,s)\leq G(k,s)\leq M_bG(N,s),\quad\text{for all}\quad 1\leq k\leq N\ \text{ and } b<s\leq N-1.$$

Finally, we conclude that, by defining $m=\min\{m_*,m_a,m_b\}$ and $M=\max\{M_*,M_a,M_b\}$, then for all $\mu\in\left(0,\frac{2N}{(a+b)(b-a+1)}\right)$ the inequalities (2.1) hold for our problem. □

3 Krasnoselskii's fixed point theorem

In this chapter we will prove the existence of solutions of problem (P), using the results proved in the previous section about the related Green's function and the classical expansion/contraction Krasnoselskii's fixed point theorem.

We now recall some definitions that will be useful in the sequel: a subset K of a real Banach space N is a *cone* if and only if it is closed, $K+K \subset K$, $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A cone K defines the partial ordering in N given by $x \preceq y$ if and only if $y - x \in K$. The notation $x \prec y$ means $x \preceq y$ and $y \neq x$. The cone K is called *normal* with normal constant $c \geq 1$ if and only if $\|x\| \leq c\|y\|$ for all $x, y \in N$ with $0 \preceq x \preceq y$. Whenever $\operatorname{int}(K) \neq \emptyset$ the symbol $x \ll y$ means $y - x \in \operatorname{int}(K)$ and the cone is said *solid*. ∂K denotes the boundary of K and $d(x, \partial K)$ is the distance of x to the boundary of K.

The closed ball of center $x_0 \in N$ and radius $r > 0$ is denoted by

$$B[x_0, r] = \{x \in N : \|x - x_0\| \leq r\},$$

and for $x,\ y \in N$, with $x \preceq y$, we define the interval

$$[x, y] = \{z \in N : x \preceq z \preceq y\}.$$

In the sequel we enunciate the classical expansion/contraction Krasnoselskii's fixed point Theorem (see [33, Theorem 13.D]).

Theorem 2. *Let $T : K \to K$ be a completely continuous operator and $0 < r < R$. Moreover, if one of the following conditions are fulfilled:*

(i) $\|Tu\| \leq \|u\|$ for any $u \in K$ with $\|u\| = r$ and $\|Tu\| \geq \|u\|$ for any $u \in K$ with $\|u\| = R$, or

(ii) $\|Tu\| \geq \|u\|$ for any $u \in K$ with $\|u\| = r$ and $\|Tu\| \leq \|u\|$ for any $u \in K$ with $\|u\| = R$,

then operator T has a fixed point in K such that $r < \|x\| < R$.

Let us define the operator

$$Tu(k) := \sum_{s=1}^{N-1} G(k,s) f(s, u(s)), \quad k \in I, \tag{3.4}$$

and a cone

$$K = \left\{ u : I \to [0, \infty),\ u(k) \geq \frac{m}{M} \|u\|, k \in J_1 \right\}, \tag{3.5}$$

where

$$\|u\| := \max\{|u(k)|, \quad k \in I\}.$$

To the end of this chapter we assume the following condition:

(F) $f : J \times [0, \infty) \to [0, \infty)$ is a continuous function.

Let us introduce the following notations that will be used along the paper:

$$f_0^- = \lim_{u\to 0^+} \min_{s\in J} \frac{f(s,u)}{u},$$

$$f_0^+ = \lim_{u\to 0^+} \max_{s\in J} \frac{f(s,u)}{u},$$

$$f_\infty^- = \lim_{u\to \infty} \min_{s\in J} \frac{f(s,u)}{u}$$

and

$$f_\infty^+ = \lim_{u\to \infty} \max_{s\in J} \frac{f(s,u)}{u}.$$

Now, in order to deduce existence results for problem (P), we introduce the following conditions:

(H1) $\mu \in \left(0, \frac{2N}{(a+b)(b-a+1)}\right)$.

(H2) $f_0^- = \infty$ and $f_\infty^+ = 0$.

(H3) $f_0^+ = 0$ and $f_\infty^- = \infty$.

Let $u \in K$ be arbitrarily chosen, then, by assuming conditions (F) and $(H1)$ we have that $T\,u \geq 0$ on I. Moreover, from (2.1), we deduce that the following inequalities are fulfilled for all $k \in J_1$:

$$\begin{aligned} Tu(k) &= \sum_{s=1}^{N-1} G(k,s) f(s,u(s)) \geq m \sum_{s=1}^{N-1} G(N,s) f(s,u(s)) \\ &\geq \frac{m}{M} \sum_{s=1}^{N-1} \max_{k\in I} \{G(k,s)\}\, f(s,u(s)) \geq \frac{m}{M} \max_{k\in I} \left\{ \sum_{s=1}^{N-1} G(k,s) f(s,u(s)) \right\} \\ &= \frac{m}{M} \|Tu\|. \end{aligned}$$

In other words, $T : K \to K$.

Moreover, since f is continuous, it is clear that T is completely continuous. So we are in the hypothesis of Krasnoselskii's fixed point Theorem 2.

Theorem 3. *Suppose that conditions (F), (H1) and either (H2) or (H3) are satisfied. Then problem (P) has a positive solution on* J_1.

Proof. First, notice that the solutions of problem (P) coincide with the fixed points of operator T.

Assuming, at the beginning, that condition $(H2)$ is fulfilled.

Since $f_0^- = \infty$, for $\delta_1 \geq \frac{1}{\frac{m}{M} \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s)\right\}} > 0$ there exists $r > 0$, such that $f(s,u) \geq \delta_1 u$ for all $0 < u \leq r$ and $s \in J$.

Let $u \in K, \|u\| = r$ then

$$\begin{aligned} \|Tu\| &= \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s) f(s,u(s))\right\} \geq \delta_1 \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s)u(s)\right\} \\ &\geq \delta_1 \|u\| \frac{m}{M} \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s)\right\} \geq \|u\| . \end{aligned}$$

Since $f_\infty^+ = 0$, for $0 < \delta_2 \leq \frac{1}{\max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s)\right\}}$, we can choose $R > r > 0$, such that $f(s,u) \leq \delta_2 u$ for all $u \geq R$ and $s \in J$.

Let $u \in K, \|u\| = \frac{M}{m} R$ then $u(k) \geq \frac{m}{M}\|u\| = R$, for all $k \in J$ and

$$\begin{aligned} \|Tu\| &= \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s) f(s,u(s))\right\} \leq \delta_2 \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s)u(s)\right\} \\ &\leq \delta_2 \|u\| \max_{k\in I}\left\{\sum_{s=1}^{N-1} G(k,s)\right\} \leq \|u\| . \end{aligned}$$

If condition $(H3)$ is fulfilled, the proof is analogous.

In both situations, the existence of a positive solution on J_1 holds from Theorem 2 and the fact that the fixed points of operator T coincide with the solutions of problem (P). □

Example 4. *Consider the following problem*

$$\begin{cases} -\Delta^2 u(k-1) + g(k)\, u^{\beta}(k) = 0, \ k \in J, \ N \geq 2, \ \beta > 0 \\ \\ u(0) = 0, \ u(N) = \frac{\alpha}{N-1}\sum_{k=1}^{N-1} u(k), \ 0 < \alpha < 2, \end{cases}$$

with $g(k) > 0$ for all $k \in J$.

It is obvious that conditions (F) and $(H1)$ are satisfied.

Furthermore, condition $(H3)$ is fulfilled for all $\beta > 1$, and condition $(H2)$ holds for any $\beta \in [0,1)$.

So in both situations we are in the conditions of Theorem 3 and we can ensure that this problem has a positive solution on J_1.

When $\beta = 1$, our problem is linear and, in the particular case of $g(k) = 1$ for all $k \in J$, has as unique solution $u \equiv 0$.

We point out that in this case neither $(H2)$ nor $(H3)$ are fulfilled. So this problem shows us that the conditions imposed in Theorem 3 are, in some sense optimal.

4 Monotonicity assumption

In this section we will prove an existence results under the assumption of existence of a lower or an upper solution in a solid cone and using some monotonicity assumption about f.

The result we will use is the following (see [6, Theorem 2.1]):

Theorem 5. *Let N be a real Banach space, K a solid cone and $T\colon K \to K$ a completely continuous operator. Assume that*

(1) there exist $\alpha_1 \in K$, with $\alpha_1 \preceq T\alpha_1$, and $R_1 > 0$ such that $B[\alpha_1, R_1] \subset K$,

(2) the map T is monotone non-increasing in the set

$$K_2 = \{x \in K : R_1 \le \|x\| \le \|\alpha_1\|\},$$

(3) there exists $r_1 > 0$, with $r_1 \neq R_1$, such that $Tx \not\succeq x$ for all $x \in K$ with $\|x\| = r_1$.

Then the map T has at least one non-zero fixed point $x_1 \in K\backslash\{x \in K; \|x\| = r_1\}$, such that

$$\min\{r_1, R_1\} \le \|x_1\| \le \max\{r_1, R_1\}.$$

To apply the previous theorem we define the following cone

$$K_1 := \left\{u : J_1 \to [0, \infty),\ u(k) \ge \frac{m}{M}\|u\|, k \in J_1\right\}.$$

The existence result is the following:

Theorem 6. *Assume that (F) and $(H1)$ are fulfilled together with the following hypotheses:*

(H4) (lower solution) There exists $\alpha : I \to (0, \infty)$, such that

$$R_1 := \min_{k \in J_1}\left\{\alpha(k) - \frac{m}{M}\|\alpha\|\right\} > 0,$$

and

$$-\Delta^2\alpha(k-1) \le f(k, \alpha(k)),\ k \in J,$$

$$\alpha(0) = 0, \alpha(N) = \mu\sum_{k=a}^{b}\alpha(k),\ 1 \le a \le b \le N-1.$$

(H5) For any $s \in J$, $f(s, x)$ is non-increasing in $x \in \left[\frac{m}{m+M}R_1, \max_{k\in J}\{|\alpha(k)|\}\right]$.

(H6) $f_0^+ = 0$ or $f_\infty^+ = 0$.

Then problem (P) has at least one positive solution on J_1.

Proof. From the continuity of f, we deduce that operator T defined in (3.4) for, in this case, functions u defined on J_1, is completely continuous. Arguing as in the previous section, we can ensure that operator T applies K_1 in K_1.

Denote α_1 as the restriction of α to J_1. By definition it is obvious that $\alpha_1 \in \operatorname{int}(K_1)$.

Let's see now that $\alpha_1 \preceq T\alpha_1$.

From $(H4)$, we have that there is $h \le 0$ on J, such that

$$-\Delta^2\alpha(k-1) = f(k, \alpha(k)) + h(k),\ k \in J,$$

So, using Theorem 1 and $(H1)$, we have that for all $k \in J_1$ it is satisfied that

$$\begin{aligned} \alpha_1(k) &= \sum_{s=1}^{N-1} G(k,s) f(s,\alpha_1(s)) + \sum_{s=1}^{N-1} G(k,s) h(s) \\ &= T\alpha_1(k) + \sum_{s=1}^{N-1} G(k,s) h(s) \le T\alpha_1(k). \end{aligned}$$

Moreover, arguing as in the previous section, since $h \le 0$ on J_1, we have the following inequalities are fulfilled for all $k \in J_1$:

$$\begin{aligned} T\alpha_1(k) - \alpha_1(k) &= -\sum_{s=1}^{N-1} G(k,s) h(s) \\ &\ge \frac{m}{M} \left\| -\sum_{s=1}^{N-1} G(k,s) h(s) \right\| = \frac{m}{M} \left\| T\alpha_1 - \alpha_1 \right\|. \end{aligned}$$

As consequence $\alpha_1 \preceq T\alpha_1$, as we want to prove.

By definition of α, we know that there exists $\bar{R}_1 > 0$ such that for all $k \in J_1$ it is fulfilled

$$\alpha_1(k) - \frac{m}{M} \|\alpha_1\| \ge R_1 =: \frac{m+M}{M} \bar{R}_1 > 0.$$

Lets us show now that $B[\alpha_1, \bar{R}_1] \subset K_1$.

If $u \in B[\alpha_1, \bar{R}_1]$ we have that $\alpha_1(k) - \bar{R}_1 \le u(k) \le \alpha_1(k) + \bar{R}_1$ for all $k \in J_1$.

Since $\alpha_1(k) > \bar{R}_1$, then $\alpha_1(k) - \bar{R}_1 \le u(k)$ gives us that $u \ge 0$ on J_1.

Now using that $\alpha_1(k) - \frac{m}{M} \|\alpha_1\| \ge \frac{m+M}{M} \bar{R}_1$ and $u(k) \le \alpha_1(k) + \bar{R}_1$ we obtain

$$u(k) \ge \alpha_1(k) - \bar{R}_1 \ge \frac{m}{M} \|\alpha_1\| + \frac{m}{M} \bar{R}_1 \ge \frac{m}{M} \|u\| \text{ for all } k \in J_1.$$

So, condition (1) in Theorem 5 holds.

Notice that if $u \in K_1$ is such that $\bar{R}_1 \le \|u\|$, then for all $s \in J_1$ the following inequalities hold:

$$u(s) \ge \frac{m}{M} \|u\| \ge \frac{m}{M} \bar{R}_1 = \frac{m}{m+M} R_1.$$

From $(H1)$ and $(H5)$ it follows that operator T is non increasing on

$$\{x \in K_1 : \bar{R}_1 \le \|x\| \le \|\alpha_1\|\}$$

and condition (2) in Theorem 5 is fulfilled.

(H6) Gives us that there exists $r_1 < \bar{R}_1$ (or $r_1 > \bar{R}_1$) and δ_2 as in the proof of Theorem 3, such that $f(s,u) \le \delta_2 u$ for all $0 < u \le r_1$ (for all $u \ge r_1$) and $s \in J$. As in that case, one can show that if $u \in K_1, \|u\| = r_1$ then

$$\|Tu\| \le \|u\|.$$

Thus we have showed that all the conditions in Theorem 5 are fulfilled and we can ensure the existence of a non trivial fixed point u of operator T such that $u \in K_1 \backslash \{x \in K; \|x\| = r_1\}$, and

$$\min\{r_1, \bar{R}_1\} \leq \|x_1\| \leq \max\{r_1, \bar{R}_1\}.$$

It is important to note that such fixed point is a solution $u : J \to (0, \infty)$ of the following difference equation:

$$(P_1) \quad \left\{ \begin{array}{l} -\Delta^2 u(k-1) = f(k, u(k)),\ k = 2, \dots, N-1,\ N \geq 2, \\ \\ u(N) = \mu \sum_{k=a}^{b} u(k),\ \mu > 0,\ 1 \leq a \leq b \leq N-1. \end{array} \right.$$

Formally, we have no information about the value of this function at $k = 0$. However, by using the properties of the Green's function G, we can extend the function u to a solution of our problem (P) as follows:

Since we know that

$$-u(2) + 2\,u(1) \quad = \quad \sum_{s=1}^{N-1} (-G(2,s) + 2\,G(1,s))\, f(s, u(s)),$$

using that $G(0,s) = 0$, $-\Delta^2 G(s-1, s) = 1$ and $-\Delta^2 G(k,s) = 0$ for all $s \in J$ and $k \neq s$, we have that the previous expression can be rewritten as

$$-u(2) + 2\,u(1) \quad = \quad \sum_{s=1}^{N-1} (-\Delta^2 G(0,s))\, f(s, u(s)) = f(1, u(1)).$$

In particular, if we define $v : I \to [0, \infty)$ as

$$v(k) = \left\{ \begin{array}{ll} 0, & \text{if } k = 0, \\ u(k), & \text{if } k \in J_1, \end{array} \right.$$

one can easily check that v is a solution of problem (P), which is strictly positive on J_1, and the proof is concluded. □

Corollary 7. *Under the assumptions of previous Theorem, if, instead of condition (H6), the following property holds:*
(H6)* $f_0^+ = 0$ *and* $f_\infty^+ = 0$.
Then problem (P) *has two positive solutions on* J_1.

Example 8. *Consider problem* (P) *for* $\mu = \frac{1}{b-a+1}$ *and*

$$f(k,x) = \left\{ \begin{array}{ll} g(k)\, x^\beta, & \text{if } x \in (0, \varepsilon]\,, \\ g(k)\, \frac{\varepsilon^{\beta+1}}{x}, & \text{if } x \in [\varepsilon, \infty), \end{array} \right.$$

with $\beta > 1$, $g(k) > 0$ *for all* $k \in J$, *and* $\varepsilon < \frac{m(M-m)}{M(M+m)} C$.

Obviously, since $a + b < 2N$, *condition* $(H1)$ *holds.*

We may choose $\alpha(0) = 0, \alpha(k) = C > 0$, where C is a constant for $k \in J_1$.
Thus $-\Delta^2\alpha(0) = C$ and $\Delta^2\alpha(k) = 0$ for $k \in J_1$.
Moreover $\alpha(N) = \mu \sum_{k=a}^{b} \alpha(k)$ and $R_1 = \frac{M-m}{M} C > 0$.
Notice that, by construction, $C \geq \varepsilon$. So condition $(H4)$ is fulfilled for all

$$C \geq g(1)^{\frac{1}{1-\beta}} \left(\frac{m\,(M-m)}{M(M+m)} \right)^{\frac{\beta+1}{1-\beta}}.$$

Thus $f(k,x)$ is continuous, satisfies (F) and is non-increasing in $\left[\frac{m(M-m)}{M(M+m)}C, C\right]$, which gives us that $(H5)$ holds and we also have that $f_0^+ = f_\infty^+ = 0$, so $(H6^*)$ is satisfied, and, form Corollary 7, we can ensure the existence of two positive solutions on J_1.

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