INTERNATIONAL DOCTORAL
SCHOOL OF THE USC

Andrés
Pérez Rodríguez

PhD Thesis

Genetic algebras and associated evolution operators

Santiago de Compostela, 2026

Doctoral Programme in Mathematics

ESCOLA DE DOUTORAMENTO
INTERNACIONAL DA USC

DOCTORAL THESIS

Genetic algebras and associated evolution operators

Author

Andrés Pérez Rodríguez

Supervisor: Manuel Ladra

Tutor: Manuel Ladra

PHD PROGRAMME IN MATHEMATICS

SANTIAGO DE COMPOSTELA – 2026

The results presented in this dissertation were obtained within the framework of an *Ayuda para la formación del profesorado universitario*, grant FPU21/05685, from the Ministerio de Ciencia, Innovación y Universidades.

The author also received funding from the Ministerio de Ciencia e Innovación and the Agencia Estatal de Investigación under the projects PID2020-115155GB-I00 and PID2024-155502NB-I00, and from the Consellería de Cultura, Educación e Ordenación Universitaria da Xunta de Galicia, within the Competitive Reference Group scheme through the project ED431C 2023/31.

The author declares no conflicts of interest related to this thesis.

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

Cofinanciado por la Unión Europea

AGENCIA ESTATAL DE INVESTIGACIÓN

XUNTA DE GALICIA | CONSELLERÍA DE CULTURA, EDUCACIÓN E UNIVERSIDADE

Á miña irmá,
por facerme sentir sempre como un adulto «premium»;

á miña nai,
por coller sempre o teléfono, mesmo cando chamo cinco veces ao día;

e á miña avoa Celsa,
que, aínda dende o ceo, segue a chegarme o recendo das súas filloas.

Agradecementos

Non podo comezar estes agradecementos doutra maneira que non sexa mencionando ao meu director de tese. Manolo, estouche fondamente agradecido por todo o que fixeches por min, por confiar en min desde o primeiro momento para realizar o Traballo Fin de Grao, continuar co Traballo Fin de Máster e, finalmente, chegar ata esta tese. Grazas por todas as conversacións, matemáticas ou non, por todos os teus consellos e, sobre todo, pola túa preocupación constante por min, moitas veces máis propia dun avó ca dun director de tese. Oxalá lembres esta tese con tanto cariño e ledicia como o farei eu. Simplemente, grazas.

Por outro lado, tamén lle teño moito que agradecer a todos os meus colaboradores, sen os cales esta tese non tería sido posible. Pilar, ti, xunto con Manolo, fuches quen me iniciou no mundo da investigación. Considérote, cando menos, como unha directora de tese máis, polo teu apoio e preocupación por min ao comezo desta etapa, así como por levarme ao meu primeiro congreso, axudándome a dar os primeiros pasos neste camiño. Só me queda desexarche o mellor, tanto no persoal como no profesional. Xabi, moitas grazas por estar sempre aí, por ofrecerme a túa axuda cando máis a precisaba e por dicirme sempre que as miñas charlas están súper ben, algo que nunca deixa de dar ánimo. Yolanda, moitas grazas por chegar a Santiago no momento perfecto, por estar sempre dispoñible e por axudarme con esta investigación de corte máis biolóxico, da que tiña tantas gañas. Ademais, vaiche tocar ter a Andrés para rato, porque aínda nos faltan moitas cousas por facer. Por último, tamén quero mencionar a Nacer. Grazas de corazón por acollerme tan ben en Mulhouse durante as miñas dúas estadías, e por facerme sentir como na casa desde o primeiro día. Verdadeiramente, sentinme moi acollido por todo o departamento.

Por suposto, tamén quero agradecer á miña familia toda a confianza depositada en min ao longo deste camiño, especialmente aos meus pais, Rosi e Salva, e á miña irmá, Tamara, polo seu apoio constante. Gustaríame tamén facer unha mención moi especial aos meus avós. Cando comecei esta tese aínda estaban os catro; hoxe, por desgraza, só un segue comigo, pero estou seguro de que os demais me ollan dende o ceo con orgullo e me acompañaron dalgún xeito ao longo de todo este proceso.

Sen dúbida, algo fundamental no desenvolvemento dunha tese é contar cun ambiente de traballo propicio, non só desde o punto de vista académico, senón tamén humano e social. Neste sentido, síntome profundamente agradecido ao Departamento, e moi especialmente á área de Álxebra, pola magnífica acollida que recibín desde o primeiro día. Grazas a todos aqueles profesores que confiaron en min para impartir as súas clases e me fixeron sentir parte do equipo. Un agradecemento moi especial merece Óscar, que foi verdadeiramente clave nesta etapa: ademais das longas e divertidas conversas sobre temas banais, e tamén dalgunha que outra xornada de sufrimento compartido impartindo Álxebra Linear e Multilinear, confiou en min para codirixir un TFG, algo que significou moito para min tanto a nivel profesional como persoal. Quero tamén expresar o meu máis sincero agradecemento a todo o meu alumnado. Aínda lembro o primeiro día que entrei na aula para dar Espazos, completamente aterrado, sen imaxinar que aquela experiencia acabaría marcando a miña vida e axudándome a descubrir que, se hai algo ao que realmente quero dedicarme, é á docencia. Aínda que lles estou profundamente agradecido a todos, de maneira especial quero mencionar a Estela por ser a miña alumna de TFG, pola súa implicación e polo camiño compartido. Xente coma ti, faino todo moito máis sinxelo.

A maiores, os meus compañeiros de doutoramento e do Máster en Matemáticas fixeron que esta experiencia fose moito máis doada e, nalgúns días, mesmo que apetecese vir traballar. Moitas grazas a Ángel, Víctor, Raúl, Samuel, Isaac, Paula e Juanma por facer desta unha etapa tan especial. Quero mencionar tamén aquelas persoas que coñecín durante a carreira e que seguen da miña man a día de hoxe: ao meu ministro de economía de confianza, Manu; á miña $\sqrt{34}$ favorita, Sara; e á *Miss Biblioteca* máis merecida, Iria. E, por suposto, estes dous grupos non son disxuntos. A súa intersección ten nome propio: o meu compi de despacho, de risas e choros, de festas e de moito traballo, de éxitos compartidos e dixomedíxomes constantes. Brais, esta aventura tería sido infinitamente máis aburrida (e seguramente máis curta) sen ti. Quérovos moito a todos, e oxalá sigamos compartindo experiencias (quen sabe se tamén traballo) durante moitos anos máis.

Por último, aínda que non menos importante, están todos os meus amigos de Celanova. Pouca xente hai que me queira tanto e que me faga sentir tan ben comigo mesmo coma eles. Compañeiros de casas rurais, de viaxes, de concertos, de aventuras, de festas e de tantas outras cousas que xa forman parte de quen son. Porque, coas nosas loucuras, simplemente somos nós. Álvaro, Ana, Edu, María, Iván, Sou, Roi, Diego, Vera, Ser, Ru... quérovos infinito. Quero unha vida ao voso carón.

Para rematar, está a persoa coa que todo cobra sentido. A miña compañeira de vida, en Celanova e en Santiago, nos días bos e nos días difíciles. A persoa que máis me quere, que máis confía en min e que sempre está cando fai falta. Con ela comparto choros, risas, silencios, e tamén esas batallas a base de plumas de bádminton cheas de raza. Con ela todo flúe, todo pesa menos, todo é mellor. Albiña, gordi, amiga da alma... grazas por ser casa, por ser calma e por ser forza. Contigo pecho estes agradecementos, sabendo que, pase o que pase, quero camiñar sempre da túa man.

Andrés Pérez Rodríguez
2 de febreiro de 2026

Contents

Abstract

The study of populations and the mechanisms that regulate them is essential for understanding ecosystems. In particular, analysing how a population evolves over time has long been regarded as a central mathematical challenge. Among the many existing mathematical frameworks for modelling population dynamics, this dissertation adopts an algebraic viewpoint, examining the role of certain nonassociative algebras, collectively known as genetic algebras, that provide a powerful tool for describing inheritance patterns in genetics.

Although several classes of genetic algebras have been introduced in the literature, this thesis addresses two of them, each treated in a separate part: evolution algebras and gonosomal algebras. The first part adopts a purely algebraic perspective, setting aside biological interpretations in order to investigate the intrinsic structure of evolution algebras as mathematical objects, including the structure of their subalgebra lattices and their capacity for deformation. The second part adopts a biological viewpoint, combining the algebraic framework of gonosomal algebras with the analysis of their associated evolution operators, whose trajectories and limit points are computed to derive the corresponding biological interpretations.

Resumo en galego

Nas últimas décadas, o aumento das presións ambientais como o cambio climático, a perda de hábitats e o acelerado declive da biodiversidade, intensificou tanto a preocupación social como científica por comprender como se comportan os ecosistemas e como responden ás perturbacións. Entre as catro leis da ecoloxía formuladas por Barry Commoner, a primeira, *Todo está conectado con todo*, subliña que os ecosistemas son complexos e profundamente interdependentes (véxase [35]). Nas propias palabras de Commoner,

> *o sistema está estabilizado polas súas propiedades dinámicas de autocompensación; estas mesmas propiedades, se se someten a unha tensión excesiva, poden conducir a un colapso dramático... [e]... o sistema ecolóxico é un amplificador, de xeito que unha pequena perturbación nunha rede pode ter efectos grandes, afastados e retardados no tempo.*

Neste contexto, comprender como evolucionan as poboacións biolóxicas adquiriu unha relevancia particular, xa que a súa estabilidade ou declive a longo prazo adoita reflectir procesos ecolóxicos máis profundos.

Unha poboación defínese como un conxunto de individuos da mesma especie que habitan unha rexión xeográfica específica nun momento dado e que teñen capacidade para reproducirse entre si. A rama da bioloxía adicada á súa análise é a chamada *dinámica de poboacións*, a cal non só abrangue o estudo de parámetros demográficos como a supervivencia ou o éxito reprodutivo, senón tamén a transmisión xenética ao longo das xeracións. En efecto, a herdanza desempeña un papel fundamental na configuración do comportamento a longo prazo e da capacidade de adaptación das especies.

O estudo matemático da herdanza comezou en 1856 cos experimentos de Gregor Mendel, quen empregou plantas de chícharos e razoamentos numéricos sinxelos para formular as súas leis da herdanza. Desde as ideas pioneiras de Mendel ata a xenética matemática contemporánea, persiste unha cuestión central: como evolucionan as poboacións ao longo do tempo baixo a influencia dos mecanismos hereditarios? Para dar resposta a esta pregunta, as matemáticas apoiáronse na teoría da probabilidade, nos procesos estocásticos, na álxebra, nos sistemas dinámicos, na análise non lineal e nas ecuacións diferenciais e en diferenzas. Unha visión xeral completa do desenvolvemento histórico da dinámica de poboacións pode atoparse en [6].

Neste manuscrito, porén, a énfase ponse na aproximación alxébrica, en particular no papel das álxebras non asociativas como marco para a modelaxe e comprensión da dinámica de poboacións. En 1940, Etherington introduciu a linguaxe formal da álxebra abstracta no estudo da xenética a través dos seus traballos [43–45], nos que reformulou as leis mendelianas en termos alxébricos. Desde entón, a investigación nesta área seguiu a medrar, e xurdiron diversas familias de álxebras orientadas á modelaxe da herdanza, entre as que se atopan as álxebras *báricas*, *gaméticas*, *cigóticas*, *de evolución*, *de Bernstein*, *train* e *gonosomais*, entre outras. Este conxunto de estruturas, maioritariamente non asociativas, coñécese de forma xeral baixo o nome de *álxebras xenéticas*, e constitúen a día de hoxe unha ferramenta fundamental na xenética teórica de poboacións. As referencias máis completas sobre o desenvolvemento matemático destas álxebras poden atoparse en [72, 88, 90, 108].

Nesta tese centrámonos en dúas clases particulares de álxebras xenéticas: as *álxebras de evolución* e as *álxebras gonosomais*, que se estudan en dúas partes diferenciadas. A primeira parte adopta unha perspectiva puramente alxébrica, abstraéndose da interpretación xenética para analizar a súa estrutura e propiedades intrínsecas como obxectos matemáticos. A segunda parte adopta un enfoque máis próximo á bioloxía, combinando o marco alxébrico cos sistemas dinámicos discretos para estudar a evolución temporal de determinadas poboacións.

Parte I: Álxebras de evolución

As álxebras de evolución son estruturas conmutativas pero non asociativas introducidas por J. P. Tian e P. Vojtěchovský en 2006 (véxase [100]) como un marco matemático para modelar a herdanza non mendeliana, considerada hoxe a linguaxe básica da bioloxía molecular. Dous anos máis tarde, Tian publicou a monografía [99], na que se estudan con maior detalle as súas propiedades e aplicacións biolóxicas. En particular, motívase a construción destas álxebras como unha maneira sinxela de describir matematicamente a autorreplicación característica dos organismos que se reproducen de forma asexual, como son moitos procariotas.

Sexan $e_1, e_2, \dots, e_n$ os posibles tipos xenéticos dunha poboación, e supoñamos que as condicións ambientais permanecen constantes ao longo das xeracións. Interpretando o produto como reprodución, o produto de dous xenotipos distintos $e_i e_j$ non ten significado biolóxico, mentres que $e_i e_i$ pode interpretarse como un proceso de autorreplicación. Deste xeito, a distribución de frecuencias dos tipos na seguinte xeración pode describirse mediante unha regra da forma

$$e_i e_i = \sum_{k=1}^{n} c_{ik} e_k, \qquad e_i e_j = 0,\ i \neq j,$$

onde os coeficientes c_{ik} representan a probabilidade de que un individuo do tipo e_i dea lugar a un individuo do tipo e_k ao reproducirse. A extensión bilinear deste produto ao espazo vectorial xerado polos tipos xenéticos define unha estrutura alxébrica chamada *álxebra de evolución*. Así, entenderemos unha álxebra de evolución (de dimensión finita) $\mathcal{E}$ como unha álxebra sobre un corpo $\mathbb{K}$ que admite unha base $B = \{e_1, \dots, e_n\}$ tal que $e_i e_j = 0$ para todo $i \neq j$.

Na actualidade, a teoría das álxebras de evolución constitúe un campo moi activo da investigación matemática. Un dos problemas centrais é o da súa clasificación, que foi abordado en dimensións baixas (véxanse [26, 32]) e no caso nilpotente (véxanse [41, 57]), aínda que as clasificacións completas resultan en xeral difíciles de obter. Por este motivo, gran parte da literatura céntrase no estudo das súas propiedades estruturais, incluíndo os seus ideais, condicións de simplicidade e semisimplicidade, así como as súas derivacións e automorfismos. Ademais, exploráronse diversas conexións con outras áreas das matemáticas, destacando o uso da teoría de grafos como ferramenta fundamental para o seu estudo (véxanse [40, 42]), así como as relacións con cadeas de Markov (véxase [86]) ou a teoría de grupos (véxase [39]).

Capítulo 1: Álxebras de evolución: preliminares e novas contribucións estruturais

Neste capítulo repásanse os conceptos fundamentais relacionados coas álxebras de evolución, revisándose as súas definicións clásicas, propiedades básicas e conexións coa teoría de grafos. Analízanse tamén aspectos estruturais, como subálxebras e ideais, e preséntase a clasificación das álxebras de evolución de dimensión dous. Posteriormente, trátanse as álxebras de evolución nilpotentes, caracterízanse e clasifícanse sobre o corpo dos complexos, establecendo a base para estudar familias máis xerais de álxebras solubles. En particular, recibirán unha atención especial as álxebras solubles con subálxebra derivada de dimensión un (denotadas por $\mathcal{T}_{\mathbb{K}}$) e as álxebras solubles con índice de solubilidade máximo, familias que serán fundamentais para os desenvolvementos posteriores. Finalmente, introdúcense tamén as álxebras de evolución supersolubles, caracterízanse dentro do contexto das álxebras solubles e defínense por primeira vez neste ámbito os ideais $\mathcal{E}$-supersolubles, detallándose as súas propiedades principais. Non obstante, este capítulo non é unha simple compilación de resultados xa coñecidos sobre álxebras de evolución, senón que tamén contén numerosas contribucións orixinais respecto das súas subálxebras.

En primeiro lugar, cómpre salientar os resultados que deron lugar ao artigo [70], no cal se estudan subálxebras no caso das álxebras de evolución regulares. Nótese que, en xeral, as álxebras de evolución non son cerradas baixo subálxebras, é dicir, unha subálxebra dunha álxebra de evolución non admite necesariamente unha base natural. Non obstante, o seguinte resultado amosa o contrario no caso regular.

Teorema 1.3.2. *Sexa $\mathcal{E}$ unha álxebra de evolución regular sobre calquera corpo. Entón, toda subálxebra de $\mathcal{E}$ admite unha base natural.*

Este resultado permítenos asumir, sen perda de xeneralidade, que toda subálxebra admite unha base na que todos os elementos teñen soportes disxuntos. Isto, fixo posible caracterizar as subálxebras de codimensión un en calquera álxebra regular.

Corolario 1.3.7. *Sexa $\mathcal{E}$ unha álxebra de evolución regular de dimensión maior que dous sobre un corpo calquera $\mathbb{K}$, con base natural $B = \{e_1, \ldots, e_n\}$ e matriz de estrutura $M_B(\mathcal{E}) = (\omega_{ij})$. Se o subespazo $\mathsf{span}\{e_i, v\colon i \neq p, q\}$ é unha subálxebra para algún $v \in \mathsf{span}\{e_p, e_q\}$, entón as constantes de estrutura satisfán*

$$\omega_{ip}^2\omega_{iq}\omega_{pp} + \omega_{iq}^3\omega_{qp} = \omega_{ip}^3\omega_{pq} + \omega_{ip}\omega_{iq}^2\omega_{qq}$$

para todo $i \neq p, q$. En particular, se $\mathcal{E}$ é de dimensión tres e $\mathbb{K}$ é alxebricamente pechado ou $\mathbb{K} = \mathbb{R}$, entón tamén se cumpre o recíproco.

Por outra banda, tamén presentamos os resultados que deron lugar ao artigo [51], no cal se tratan as álxebras de evolución completas, unha propiedade relacionada coa estrutura das súas subálxebras. Este concepto foi introducido e estudado en [29], onde se formulan dúas conxecturas relativas á súa estrutura e clasificación. A proba destas dúas conxecturas reduciuse ao seguinte resultado, cuxa proba se basea nalgunhas ferramentas básicas da xeometría alxébrica.

Teorema 1.4.1. *Sexa $A = (a_{ij})_{1\leq i,j\leq n}$ unha matriz complexa e invertible. Entón, o sistema de ecuacións*

$$\begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

sempre admite unha solución non trivial. En particular, existe unha solución na que polo menos dúas coordenadas son non nulas.

En consecuencia, os dous resultados probados relativos ás álxebras de evolución completas, foron os seguintes.

Teorema 1.4.4 [29, Conjecture 5.2]. *Sexa $\mathcal{E}$ unha álxebra de evolución regular de dimensión maior que un. Entón, $\mathcal{E}$ non é completa.*

Teorema 1.4.5 [29, Conjecture 5.3]. *Sexa $\mathcal{E}$ unha álxebra de evolución completa de dimensión n non nilpotente. Entón, $\mathcal{E}$ é isomorfa a unha das seguintes álxebras:*

$$\{e_1^2 = e_1\} \oplus \mathbb{C}^{n-1} \quad ou \quad \{e_1^2 = e_1\} \oplus \widetilde{\mathcal{E}} \oplus \mathbb{C}^{n-s-1},$$

onde $\widetilde{\mathcal{E}}$ é unha álxebra de evolución de dimensión s con índice de nilpotencia máximo e $\mathbb{C}^k$ denota a álxebra de evolución abeliana de dimensión k.

Capítulo 2: Sobre o retículo de subálxebras de álxebras de evolución

O estudo da teoría de retículos desenvolveuse no marco de diversas estruturas alxébricas, centrándose principalmente en dúas propiedades clásicas definidas por identidades: a *distributividade* e a *modularidade*. O caso dos grupos é especialmente relevante, pois a relación entre as súas propiedades e a estrutura do retículo de subgrupos constitúe un dos primeiros resultados neste ámbito. Por exemplo, o teorema de Ore establece que o retículo de subgrupos dun grupo G é distributivo se e só se G é localmente cíclico. Ademais, é ben coñecido que o retículo de subgrupos normais dun grupo (e, polo tanto, o retículo de subgrupos de calquera grupo abeliano) é modular. Estudos análogos realizáronse tamén en estruturas non asociativas, como as álxebras de Lie (véxanse [52, 64]), de Leibniz (véxase [98]) ou as álxebras de Lie restrinxidas (véxanse [76, 85]). Porén, esta relación é pouco coñecida no caso das álxebras xenéticas (véxase [79]) e resulta completamente descoñecida para as álxebras de evolución. Así, o principal obxectivo deste capítulo é o estudo da distributividade e da modularidade neste ámbito, que deu lugar ao artigo [68].

En primeiro lugar, empregando a caracterización das álxebras de evolución nilpotentes como aquelas que admiten unha matriz de estrutura estritamente triangular superior, fomos quen de describir a distributividade, afirmando ademais que, neste caso, o retículo de subálxebras é precisamente unha cadea.

Teorema 2.2.1. *Sexa $\mathcal{E}$ unha álxebra de evolución nilpotente de dimensión n sobre calquera corpo $\mathbb{K}$ cunha base natural $B = \{e_1, \dots, e_n\}$ tal que a súa matriz de estrutura $M_B(\mathcal{E}) = (\omega_{ij})$ é estritamente triangular superior. Entón, as seguintes afirmacións son equivalentes:*

(i) *$\mathcal{E}$ ten índice de nilpotencia máximo;*

(ii) *o seu retículo de subálxebras é unha cadea de lonxitude n;*

(iii) *$\mathcal{E}$ é distributiva;*

A maiores, como primeiro paso para o estudo da modularidade, conseguimos unha condición necesaria que restrinxe a existencia de elementos absolutamente nilpotentes.

Proposición 2.2.4. *Sexa $\mathcal{E}$ unha álxebra de evolución nilpotente sobre un corpo de característica distinta de dous. Se $\mathcal{E}$ é modular, entón non existen elementos absolutamente nilpotentes fóra do anulador.*

Finalmente, vemos como esta condición necesaria se converte nunha caracterización cando traballamos sobre corpos cuadraticamente pechados e o anulador ten dimensión un.

Corolario 2.2.9. *Sexa $\mathcal{E}$ unha álxebra de evolución sobre un corpo cuadraticamente pechado de característica distinta de dous e cuxo anulador ten dimensión un. Entón, as seguintes afirmacións son equivalentes:*

(i) *$\mathcal{E}$ é distributiva;*

(ii) *$\mathcal{E}$ é modular; e*

(iii) *$\mathcal{E}$ ten índice de nilpotencia máximo.*

Rematado o caso nilpotente, a segunda parte do capítulo adícase ao contexto máis xeral das álxebras de evolución solubles. En particular, estúdanse a distributividade e a modularidade nas dúas familias de álxebras solubles ben desenvolvidas no capítulo un: as álxebras de evolución cuxa subálxebra derivada ten dimensión un ($\mathcal{T}_{\mathbb{K}}$) e as álxebras de evolución con índice de solubilidade máximo.

Corolario 2.3.2. *Sexa $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ cun corpo $\mathbb{K}$ cuadraticamente pechado de característica distinta de dous ou $\mathbb{K} = \mathbb{R}$. Entón, $\mathcal{E}$ é distributiva se e só se é modular e se e só se $\dim \mathcal{E} = 2$, isto é, $\mathcal{E}$ é isomorfa á álxebra de evolución de dimensión dous con produto $e_1^2 = -e_2^2 = e_1 + e_2$.*

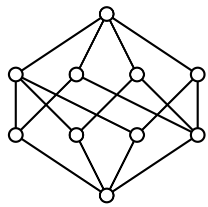
Corolario 2.3.9. *Sexa $\mathcal{E}$ unha álxebra de evolución soluble con índice de solubilidade máximo sobre un corpo de característica distinta de dous. Se $\mathcal{E}$ é modular, entón a serie derivada non ten dous termos consecutivos que non sexan ideais básicos.*

Cómpre salientar que a condición necesaria recollida no resultado anterior é equivalente á supersolubilidade da álxebra. Aínda que esta propiedade non caracteriza por si soa a modularidade, resulta estar moi próxima a facelo, pois abonda con engadir unha condición técnica adicional que (aínda que non se inclúe aquí por non proceder tal nivel de profundidade) se describe e analiza con detalle na memoria.

A análise da modularidade remata coa súa caracterización en álxebras supersolubles regulares sobre corpos alxebricamente pechados.

Teorema 2.4.5. *Sexa $\mathcal{E}$ unha álxebra de evolución regular supersoluble sobre un corpo alxebricamente pechado $\mathbb{K}$. Entón, $\mathcal{E}$ é modular se e só se ten dimensión menor ou igual que dous ou $\mathcal{E}$ é isomorfa á álxebra de evolución con base natural $\{e_1, e_2, e_3\}$ e produto $e_1^2 = e_1$, $e_2^2 = e_2$, $e_3^2 = \frac{1}{4}e_1 + \frac{1}{4}e_2 + e_3$, cuxo retículo de subálxebras é xustamente o seguinte:*

Subalg. of dim. 1	**Subalg. of dim.** 2
$\mathsf{span}\{e_1\}$	$\mathsf{span}\{e_1, e_2\}$
$\mathsf{span}\{e_2\}$	$\mathsf{span}\{e_1, e_2 + 2e_3\}$
$\mathsf{span}\{e_1 + e_2\}$	$\mathsf{span}\{e_2, e_1 + 2e_3\}$
$\mathsf{span}\{e_1 + e_2 + 2e_3\}$	$\mathsf{span}\{e_3, e_1 + e_2\}$



Capítulo 3: Unha teoría de Frattini para álxebras de evolución

A teoría de Frattini orixinouse na teoría de grupos, onde Giovanni Frattini introduciu en 1885 o subgrupo de Frattini (véxase [49]), definido como a intersección de todos os subgrupos maximais dun grupo dado. A súa demostrada versatilidade e gran cantidade de propiedades salientables fixeron que, co tempo, se establecese un marco análogo para álxebras. Así, dada unha álxebra $\mathcal{A}$, a súa *subálxebra de Frattini*, $F(\mathcal{A})$, definiuse como a intersección de todas as súas subálxebras maximais. Tamén se introduciu o seu *ideal de Frattini*, $\phi(\mathcal{A})$, como o maior ideal contido na subálxebra de Frattini. Esta extensión motivou numerosos estudos en álxebras non asociativas (véxase [102]) e, en específico, nos marcos das álxebras de Lie (véxanse [9, 10, 78]) e Leibniz (véxase [11]). Non obstante, ata agora, non se explorou este enfoque no contexto das álxebras xenéticas ou de evolución. Por iso, este capítulo pretende desenvolver unha teoría de Frattini no marco das álxebras de evolución, cuxos resultados se presentan na publicación [69].

Comezamos analizando a noción de nilradical (tradicionalmente definido como o maior ideal nilpotente) no ámbito das álxebras de evolución. Porén, tal e como amosa o seguinte exemplo, pode existir máis dun ideal nilpotente maximal.

Exemplo 3.2.1. Sexa $\mathcal{E}$ a álxebra de evolución cunha base natural $\{e_1, e_2, e_3, e_4\}$ e produto dado por $e_1^2 = -e_2^2 = e_3 + e_4$ e $e_3^2 = -e_4^2 = e_1 + e_2$. Os subespazos $\mathcal{N}_1 = \mathsf{span}\{e_1, e_2, e_3 + e_4\}$ e $\mathcal{N}_2 = \mathsf{span}\{e_3, e_4, e_1 + e_2\}$ son dous ideais nilpotentes maximais distintos. Non obstante, $\mathcal{N}_1 + \mathcal{N}_2 = \mathcal{E}$ non é nilpotente.

Non obstante, amosamos que no caso de álxebras solubles con subálxebra derivada de dimensión un ($\mathcal{T}_{\mathbb{K}}$), entón ese único ideal nilpotente maximal existe.

Teorema 3.2.4. *Sexa* $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$. *Entón, existe o seu nilradical,* $\mathrm{Nil}(\mathcal{E})$. *De feito, asumindo, sen perda de xeneralidade, que* $\mathcal{E} = \mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$ *con* $k \geq 2$ *e* $\lambda_1, \ldots, \lambda_k \neq 0$, *tense*

$$\mathrm{Nil}(\mathcal{E}) = \mathsf{span}\{\lambda_2 e_1 - \lambda_1 e_2, \lambda_3 e_1 - \lambda_1 e_3, \ldots, \lambda_k e_1 - \lambda_1 e_k, e_{k+1}, \ldots, e_n\}.$$

Empregando este resultado, definiuse o chamado *nilradical supersoluble*, denotado por $\mathrm{SNil}(\mathcal{E})$, que se presenta como a alternativa máis adecuada ao nilradical no contexto das álxebras de evolución. Ademais, amosamos un exemplo que ilustra claramente como se constrúe este nilradical supersoluble. Este ten propiedades moi favorables, entre as que destaca que é o maior ideal nilpotente $\mathcal{E}$-supersoluble da álxebra de evolución correspondente. Ademais, se resulta ser un ideal nilpotente maximal, entón o nilradical existe e ambos conceptos coinciden necesariamente.

Este nilradical supersoluble, permitiunos estudar a subálxebra e o ideal de Frattini nalgúns casos particulares, e incluso obter condicións necesarias e suficientes para os casos nos que son cero. Os principais resultados obtidos son os seguintes.

Teorema 3.3.1. *Sexa* $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ *con* $\mathbb{K}$ *un corpo de característica distinta de dous. Entón,*

$$F(\mathcal{E}) = \phi(\mathcal{E}) = \begin{cases} 0, & \textit{se } \operatorname{codim}(\operatorname{ann}(\mathcal{E})) = 2; \\ \mathcal{E}^2, & \textit{noutro caso.} \end{cases}$$

Teorema 3.3.3. *Sexa* $\mathcal{E}$ *unha álxebra de evolución. Se* $\mathcal{E}$ *é* ϕ*-libre e* $\mathrm{SNil}(\mathcal{E})^2$ *é un ideal, entón* $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$.

Teorema 3.3.6. *Sexa* $\mathcal{E}$ *unha álxebra de evolución sobre un corpo* $\mathbb{K}$ *de característica distinta de dous, cunha base natural* $B = \{e_1, \ldots, e_n\}$ *tal que* $\operatorname{supp}\big(\mathrm{SNil}(\mathcal{E})\big) = \operatorname{supp}(\mathcal{E})$. *Entón,* $\mathcal{E}$ *é* ϕ*-libre se e só se* $\mathcal{E} = K \oplus \operatorname{ann}(\mathcal{E})$, *onde* K *é unha suma directa de* $m \leq \lfloor \frac{n}{2} \rfloor$ *copias da álxebra de evolución de dimensión dous con produto* $e_1^2 = -e_2^2 = e_1 + e_2$.

Por último, como aplicación ao estudo da teoría de Frattini, analizamos tamén as álxebras de evolución dualmente atomísticas. Unha álxebra de evolución dise *dualmente atomística* se toda subálxebra pode expresarse como intersección de subálxebras maximais. Obsérvese que, en particular, a subálxebra nula 0 debe ser tamén unha intersección de subálxebras maximais, polo que necesariamente tanto a subálxebra como o ideal de Frattini son triviais. Este feito motiva de maneira natural o emprego dos resultados anteriores para obter o noso resultado final de clasificación.

Teorema 3.4.3. *Sexa* $\mathcal{E}$ *unha álxebra de evolución que é ou ben case abeliana ou ben satisfai que* $\operatorname{supp}\big(\mathrm{SNil}(\mathcal{E})\big) = \operatorname{supp}(\mathcal{E})$. *Se* $\mathcal{E}$ *é dualmente atomística, entón é isomorfa a unha das seguintes álxebras de evolución:*

- $e_1^2 = -e_2^2 = e_1 + e_2$.
- $e_1^2 = e_1,\ e_2^2 = \cdots = e_n^2 = 0$, *con* $n \in \mathbb{N}$.

Capítulo 4: Deformacións e dexeneracións de álxebras de evolución

Este capítulo adícase ao estudo das deformacións e dexeneracións no ámbito das álxebras de evolución presentado no artigo [75]. Mentres que o estudo de ambos conceptos foi moi desenvolvido no contexto dalgunhas variedades clásicas, como son as álxebras asociativas ou as de Lie, este non é o caso das álxebras de evolución.

As *deformacións formais* foron introducidas por Gerstenhaber en [53] para álxebras asociativas, e logo xeneralizadas por Nijenhuis e Richardson en [83, 84] para álxebras de Lie. De maneira informal, unha deformación dunha estrutura alxébrica $\mathcal{A}$ con produto μ consiste en construír un novo produto μ_t sobre o espazo de series formais $\mathcal{A}[[t]]$ da forma $\mu_t = \mu + \sum_{k \geq 1} \mu_i t^i$, onde cada μ_i é unha nova aplicación bilinear. Seguindo esta idea, o que primeiro fixemos foi introducir o concepto de deformación formal no ámbito das álxebras de evolución, impoñendo que o produto de elementos diferentes da base natural da correspondente álxebra siga a ser cero. Isto, de maneira directa, da lugar a unha álxebra de evolución sobre o anel $\mathbb{K}[[t]]$.

O noso seguinte obxectivo foi estudar cando dúas deformacións dunha álxebra de evolución son, en esencia, a mesma. Para iso, introducimos o concepto de deformacións equivalentes e probamos que, neste caso, a diferenza dos termos de primeira orde ten *forma de derivación*. En consecuencia, tamén puidemos estudar as deformacións triviais dunha álxebra de evolución, é dicir, aquelas que son equivalentes ao produto orixinal.

Teorema 4.1.10. *Sexan ν_t e λ_t dúas deformacións dunha álxebra de evolución $\mathcal{E} = (V, \mu)$ sobre un corpo calquera $\mathbb{K}$. Se son equivalentes, entón existe un morfismo lineal $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$ tal que, para todo $u, v \in V$,*

$$\lambda_1(u, v) - \nu_1(u, v) = \varphi(uv) - u\varphi(v) - \varphi(u)v.$$

Corolario 4.1.12. *Sexa ν_t unha deformación dunha álxebra de evolución $\mathcal{E} = (V, \mu)$ sobre un corpo calquera $\mathbb{K}$. Se ν_t é unha deformación trivial, entón existe un morfismo lineal $\varphi \in \mathrm{End}(V)$ tal que*

$$\nu_1(u, v) = \varphi(uv) - u\varphi(v) - \varphi(u)v$$

para todo $u, v \in V$.

Por último, demostramos tamén que toda álxebra de evolución admite deformacións, o que evidencia un claro contraste coa rixidez que adoitan presentar as álxebras semisimples.

Teorema 4.1.15. *Toda álxebra de evolución admite unha deformación non trivial de primeira orde.*

Por outra banda, dexenerar é un concepto que, dalgunha maneira, é oposto a deformar, e que foi moi estudado no caso das álxebras de Lie (véxanse, por exemplo, as referencias [17, 31, 97]). Para o estudo das dexeneracións adoptamos o enfoque formal presentado en [74, Subsection 5.2] para álxebras asociativas. Sexa $\{g_t\}_{t\neq 0}$ unha familia continua de aplicacións lineais invertibles nun espazo vectorial V de dimensión n sobre $\mathbb{K}$ e sexa $\mathcal{A}_1$ unha álxebra sobre $\mathbb{K}$ cun espazo vectorial subxacente V e produto μ_1. Cando o límite

$$\mu_0(x,y) = \lim_{t\to 0} g_t \cdot \mu_1(x,y) := \lim_{t\to 0} g_t\big(\mu_1(g_t^{-1}x, g_t^{-1}y)\big)$$

existe para todo $x, y \in V$, dicimos que a álxebra $\mathcal{A}_0$, con espazo vectorial subxacente V e produto μ_0, é unha *dexeneración formal* de $\mathcal{A}_1$. Estas definicións poñen de manifesto o carácter dual destes dous conceptos: as dexeneracións formais tenden a simplificar a estrutura alxébrica, achegándoa ao caso abeliano, mentres que as deformacións formais xeran, en xeral, leis de multiplicación máis complexas.

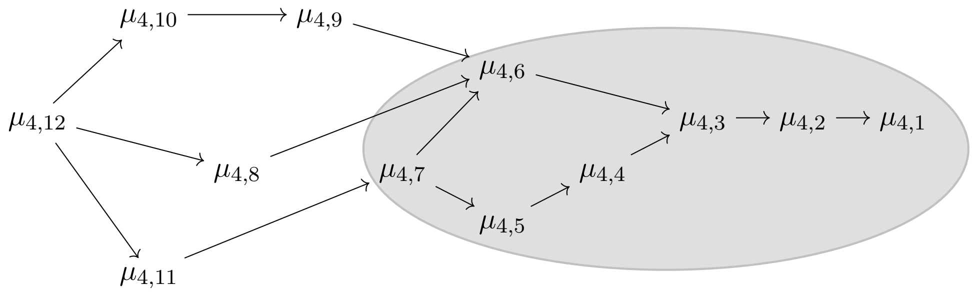
O primeiro que fixemos foi introducir as dexeneracións no ámbito das álxebras de evolución coa restrición de que as aplicacións g_t sexan cambios de base que levan bases naturais en bases naturais e, logo de amosar que en xeral a transitividade non se ten, establecemos numerosos criterios que nos permitisen determinar se unha dexeneración pode existir ou non. Finalmente, isto permitiunos, empregando as ben coñecidas clasificacións das álxebras de evolución nilpotentes, construír uns diagramas amosando numerosas relacións de dexeneración en dimensión dous, tres e catro. Nótese que, como as únicas clases de isomorfía en dimensión dous son $\mu_{2,1}\colon e_1^2 = e_2^2 = 0$ e $\mu_{2,2}\colon e_1^2 = e_2, e_2^2 = 0$, e ademais toda álxebra dexenera na abeliana da mesma dimensión, o correspondente diagrama é simplemente $\mu_{2,2} \to \mu_{2,1}$.

Teorema 4.3.13. *Todas as relacións de dexeneración transitiva entre as álxebras de nilpotentes de dimensión tres sobre os complexos quedan recollidas no diagrama*

$$\mu_{3,4} \to \mu_{3,3} \to \mu_{3,2} \to \mu_{3,1}.$$

Ademais, $\mu_{3,2}$ é tamén unha dexeneración de $\mu_{3,4}$.

Proposición 4.3.16. *O seguinte diagrama recolle varias relacións de dexeneración transitiva entre as álxebras de evolución nilpotentes de dimensión catro sobre os complexos. En particular, o diagrama representa todas as dexeneracións transitivas entre as álxebras $\{\mu_{4,i}\}_{i=1}^{7}$.*



Aínda que poderían existir máis relacións de dexeneración, unha das principais consecuencias do diagrama anterior é que $\mu_{4,12}$ é a única álxebra ríxida dentro das nilpotentes de dimensión catro.

Parte II: Álxebras e operadores gonosomais

As álxebras gonosomais foron introducidas por Richard Varro en 2016 en [105] co obxectivo de fornecer un marco alxébrico para modelar a herdanza ligada ao sexo en poboacións bisexuais. Unha das principais dificultades á hora de construír modelos deste tipo de herdanza reside en que a transmisión da información xenética depende non só do material hereditario dos individuos, senón tamén do mecanismo de determinación do sexo subxacente. Neste contexto, as álxebras gonosomais destacan pola súa versatilidade, xa que a súa estrutura alxébrica pode adaptarse para representar unha ampla variedade de patróns de herdanza e sistemas de determinación sexual.

Sexan f_i, con $i = 1, \ldots, n$, os posibles tipos xenéticos femininos, e h_p, con $p = 1, \ldots, m$, os posibles tipos xenéticos masculinos dunha poboación bisexual. A reprodución entre individuos do mesmo sexo, $f_i f_j$ e $h_p h_q$, carece de significado biolóxico, polo que a reprodución da poboación descríbese mediante unha regra bilinear na que

$$f_i f_j = 0, \qquad h_p h_q = 0,$$

e

$$f_i h_p = h_p f_i = \sum_{k=1}^{n} \gamma_{ipk} f_k + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} h_r,$$

onde os coeficientes γ_{ipk} (resp. $\widetilde{\gamma}_{ipr}$) representan a probabilidade de obter unha descendente feminina do tipo f_k (resp. masculina do tipo h_r) a partir da reprodución dunha femia do tipo f_i cun macho do tipo h_p. Como estes coeficientes representan probabilidades, satisfán as condicións

$$0 \leq \gamma_{ipk} \leq 1, \quad 0 \leq \widetilde{\gamma}_{ipr} \leq 1, \quad \sum_{k=1}^{n} \gamma_{ipk} + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} = 1.$$

A extensión bilinear deste produto ao espazo vectorial xerado por todos os tipos xenéticos define unha estrutura alxébrica coñecida como *álxebra gonosomal*.

A diferenza do que ocorre coas álxebras de evolución, a maioría dos traballos sobre álxebras gonosomais non as estudan desde unha perspectiva puramente alxébrica. Isto débese en parte a que estas álxebras adoitan ter dimensións elevadas, o que dificulta considerablemente o seu estudo estrutural. Pola contra, a investigación neste ámbito céntrase habitualmente na modelaxe de poboacións concretas mediante sistemas dinámicos en tempo discreto asociados aos coeficientes da álxebra. Estes sistemas describen a evolución temporal das frecuencias xenéticas masculinas e femininas e permítennos analizar o seu comportamento dinámico (véxanse, por exemplo, [3, 4, 92, 94]).

Capítulo 5: Revisión das álxebras xenéticas para poboacións bisexuais

Este capítulo ten un carácter fundamentalmente expositivo e está adicado á revisión das *álxebras gonosomais*, sen achegar novos resultados de investigación. O seu propósito é afianzar e sistematizar o marco teórico destas estruturas alxébricas, que xogan un papel esencial na modelaxe da herdanza ligada ao sexo. A exposición artéllase arredor do caso da hemofilia, que serve como fío condutor para introducir de maneira progresiva as distintas definicións, construcións e ferramentas asociadas.

As álxebras gonosomais foron concibidas para describir poboacións bisexuais, nas que a reprodución require a interacción entre dous tipos sexuais diferenciados. Neste contexto, a transmisión hereditaria vese condicionada non só polo material xenético transmitido, senón tamén polos sistemas de determinación sexual, o que limita a aplicabilidade de modelos máis clásicos, como as *álxebras dibáricas* ou as *álxebras de evolución de poboacións bisexuais*. A imposibilidade de tratar adecuadamente exemplos como a hemofilia dentro destes marcos motiva a aparición das álxebras gonosomais como unha extensión natural e máis flexible.

Ao longo do capítulo revísanse distintas construcións destas álxebras e preséntanse exemplos biolóxicos representativos, poñendo de manifesto como poden combinarse para describir situacións xenéticas de maior complexidade. Así mesmo, introdúcense os *operadores gonosomais* asociados, que permiten interpretar estas estruturas desde un punto de vista dinámico. As traxectorias destes operadores definen sistemas en tempo discreto cuxa análise proporciona información sobre a evolución das proporcións de individuos ao longo das xeracións.

Capítulo 6: Dinámicas de sistemas xenéticos cun só xenotipo masculino

Neste capítulo analízase a dinámica de distintos sistemas xenéticos modelados mediante álxebras e operadores gonosomais, co obxectivo de describir a evolución de poboacións concretas e extraer as correspondentes interpretacións biolóxicas. En todos os casos considéranse sistemas de determinación sexual con varios xenotipos femininos e un único xenotipo masculino, unha situación frecuente en diversos contextos biolóxicos. O estudo céntrase na análise das traxectorias e dos puntos límite de certos sistemas dinámicos discretos, que permiten caracterizar o comportamento asintótico das poboacións. Neste resumo desenvólvese con maior detalle o primeiro modelo, no que se introduce explicitamente o denominado *operador gonosomal normalizado* co obxectivo de ilustrar con detalle a metodoloxía empregada, mentres que nos exemplos restantes preséntanse directamente os cruzamentos e as conclusións biolóxicas obtidas.

En primeiro lugar, analízase o sistema de determinación sexual ZW seguido polas cochinillas *Armadillidium vulgare* (véxase [36]), no que intervén a bacteria intracelular *Wolbachia*, herdada por vía materna e responsable da feminización de machos xenéticos. Cando *Wolbachia* infecta un macho ZZ, que denotamos por ZZ+w, este convértese nunha femia fértil. Así, considéranse tres xenotipos femininos, ZZ+w, ZW e ZW+w, e un único xenotipo masculino, ZZ. Os posibles cruzamentos veñen dados por

$$\begin{aligned} (\text{ZZ+w}) \times \text{ZZ} \quad &\rightarrowtail \quad \eta\,(\text{ZZ+w}),\ (1-\eta)\,\text{ZZ}; \\ \text{ZW} \times \text{ZZ} \quad &\rightarrowtail \quad \tfrac{1}{2}\,\text{ZW},\ \tfrac{1}{2}\,\text{ZZ}; \\ (\text{ZW+w}) \times \text{ZZ} \quad &\rightarrowtail \quad \tfrac{\eta}{2}\,\text{ZW+w},\ \tfrac{\eta}{2}\,\text{ZZ+w},\ \tfrac{1-\eta}{2}\,\text{ZW},\ \tfrac{1-\eta}{2}\,\text{ZZ}; \end{aligned}$$

onde η ($\frac{1}{2} < \eta < 1$) representa a taxa de transmisión de *Wolbachia*. A partir destes cruzamentos constrúese a álxebra gonosomal correspondente e o operador gonosomal normalizado asociado:

$$\widetilde{V}_\eta : \begin{cases} x_1' = \frac{\eta u(2x_1+x_3)}{2u(x_1+x_2+x_3)}, & x_3' = \frac{\eta u x_3}{2u(x_1+x_2+x_3)}, \\ x_2' = \frac{u(x_2+(1-\eta)x_3)}{2u(x_1+x_2+x_3)}, & u' = \frac{u((1-\eta)(2x_1+x_3)+x_2)}{2u(x_1+x_2+x_3)}; \end{cases}$$

cuxo comportamento dinámico se estuda mediante acoutamentos das súas iteracións. En particular, obtense o seguinte resultado límite.

Teorema 6.2.4. *Considérese o operador* $\widetilde{V}_\eta$ *con* $\frac{1}{2} < \eta < 1$. *Entón, para un punto inicial* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$, *cúmprese que*

$$\lim_{k\to\infty} \widetilde{V}_\eta^k(s) = \begin{cases} (0, \frac{1}{2}, 0, \frac{1}{2}), & \textit{se } x_1^{(0)} = x_3^{(0)} = 0, \\ (\eta, 0, 0, 1-\eta), & \textit{noutro caso.} \end{cases}$$

Este resultado permite concluír que para calquera estado inicial $s \in S^{3,1}$ (é dicir, calquera distribución de probabilidade sobre o conxunto de xenotipos posibles

$\{$ZZ+w, ZW, ZW+w, ZZ$\}$), a evolución da poboación é sempre estable. Se no estado inicial non hai individuos dos tipos ZZ+w nin ZW+w, a poboación tende ao estado de equilibrio $(0, \frac{1}{2}, 0, \frac{1}{2})$, no que os xenotipos ZW e ZZ aparecen en igual proporción. Pola contra, se no estado inicial hai individuos infectados por *Wolbachia*, a poboación converxe ao estado de equilibrio $(\eta, 0, 0, 1-\eta)$.

A continuación, tamén estudamos a dinámica de poboacións de roedores con femias XY fértiles atípicas, como *Myopus schisticolor* e *Mus minutoides* (véxanse [77, 107]). Neste caso, temos tres xenotipos femininos, XX, XX* e X*Y, e un único xenotipo masculino, XY, onde o cromosoma X* elimina o gonosoma Y durante a gametoxénese. Os cruzamentos veñen dados por

$$\begin{aligned}
\mathrm{XX} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX}, \tfrac{1}{2}\mathrm{XY};\\
\mathrm{XX}^* \times \mathrm{XY} &\rightarrowtail \tfrac{1}{4}\mathrm{XX}, \tfrac{1}{4}\mathrm{XX}^*, \tfrac{1}{4}\mathrm{X}^*\mathrm{Y}, \tfrac{1}{4}\mathrm{XY};\\
\mathrm{X}^*\mathrm{Y} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX}^*, \tfrac{1}{2}\mathrm{X}^*\mathrm{Y}.
\end{aligned}$$

Neste caso, puidemos concluír que, se no estado inicial non hai individuos dos tipos XX* nin X*Y, a poboación tende ao estado de equilibrio $\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, no que os xenotipos XX e XY se distribúen de maneira equitativa. Pola contra, se existen individuos XX* ou X*Y, a poboación tende a $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, onde todos os xenotipos posibles aparecen na mesma proporción.

Unha análise similar realízase para unha poboación de *Dicrostonyx torquatus* (lemming ártico), na que os xenotipos son os mesmos, pero cunha regra de herdanza distinta (véxase [105, Exemplo 16]), dada polos cruzamentos

$$\begin{aligned}
\mathrm{XX} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX}, \tfrac{1}{2}\mathrm{XY};\\
\mathrm{XX}^* \times \mathrm{XY} &\rightarrowtail \tfrac{1}{4}\mathrm{XX}, \tfrac{1}{4}\mathrm{XX}^*, \tfrac{1}{4}\mathrm{X}^*\mathrm{Y}, \tfrac{1}{4}\mathrm{XY};\\
\mathrm{X}^*\mathrm{Y} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{3}\mathrm{XX}^*, \tfrac{1}{3}\mathrm{X}^*\mathrm{Y}, \tfrac{1}{3}\mathrm{XY}.
\end{aligned}$$

Tamén neste caso a poboación presenta un comportamento estable, con estados de equilibrio explicitamente descritos. Se no estado inicial non hai individuos dos tipos XX* nin X*Y, a poboación tende ao estado de equilibrio $\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, no que os xenotipos XX e XY están distribuídos de maneira igualitaria. Noutro caso, a poboación tende ao estado de equilibrio $\left(\frac{7}{20}, \frac{7}{60}, \frac{7}{60}, \frac{5}{12}\right)$.

Por último, analízanse poboacións de peixes cíclidos africanos con determinación sexual polixénica (véxanse [81, 87]). Estes sistemas presentan dous loci independentes, XY e ZW, sendo dominante o alelo W sobre Y. Considerando unha versión simplificada do modelo biolóxico e asumindo a eliminación do cromosoma Y durante a gametoxénese en presenza de W, obtense o seguinte conxunto de cruzamentos:

$$\begin{aligned}
\mathrm{ZWXX} \times \mathrm{ZZXY} &\rightarrowtail \tfrac{1}{4}\mathrm{ZZXX}, \tfrac{1}{4}\mathrm{ZZXY}, \tfrac{1}{4}\mathrm{ZWXX}, \tfrac{1}{4}\mathrm{ZWXY};\\
\mathrm{ZZXX} \times \mathrm{ZZXY} &\rightarrowtail \tfrac{1}{2}\mathrm{ZZXX}, \tfrac{1}{2}\mathrm{ZZXY};\\
\mathrm{ZWXY} \times \mathrm{ZZXY} &\rightarrowtail \tfrac{1}{4}\mathrm{ZZXX}, \tfrac{1}{4}\mathrm{ZZXY}, \tfrac{1}{4}\mathrm{ZWXX}, \tfrac{1}{4}\mathrm{ZWXY}.
\end{aligned}$$

O estudo do operador gonosomal correspondente mostra que, para calquera estado inicial $s \in S^{3,1}$ (a distribución de probabilidade sobre o conxunto de xenotipos posibles $\{\mathrm{ZZXX}, \mathrm{ZWXX}, \mathrm{ZWXY}, \mathrm{ZZXY}\}$), a evolución futura da poboación tende sempre ao estado de equilibrio $\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$. É dicir, os individuos ZZXX e ZZXY sobreviven na mesma proporción, mentres que os individuos ZWXX e ZWXY desaparecen a longo prazo.

Publications included in this thesis

The contents of this thesis appear in the following publications. We would like to acknowledge the corresponding publishers for allowing the reproduction of the results obtained.

[68] M. Ladra, P. Páez-Guillán, and A. Pérez-Rodríguez, *On the subalgebra lattice of solvable evolution algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **119** (2025), no. 3, Paper No. 83, 18 pp.

Title: On the subalgebra lattice of solvable evolution algebras.

Year: 2025.

Author 1: Manuel Ladra. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Author 2: Pilar Páez-Guillán.

Author 3: Andrés Pérez-Rodríguez. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Journal: Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas. RACSAM.

DOI: 10.1007/s13398-025-01752-x

Publisher: Springer Italia.

ISSN: 1578-7303 (Print), 1579-1505 (Online).

Journal permission: Open access, see page 168.

Quality indexes: JCR Mathematics, Impact factor: 1.6, 53/492 (Q1), 2024.

Corresponding chapter or section: Chapter 2.

Ph.D. student contribution: Essential. The candidate participated in the design of the research, the development of the proofs, the subsequent analysis of the results, the writing of the work, as well as in the dissemination of the results through presentations at scientific conferences.

[69] M. Ladra and A. Pérez-Rodríguez, *A Frattini theory for evolution algebras*, Ric. Mat. (2025), DOI:10.1007/s11587-025-01027-y.

Title: A Frattini theory for evolution algebras.

Year: 2025.

Author 1: Manuel Ladra. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Author 2: Andrés Pérez-Rodríguez. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Journal: Ricerche di Matematica. A Journal of Pure and Applied Mathematics

DOI: 10.1007/s11587-025-01027-y

Publisher: Springer Italia.

ISSN: 0035-5038 (Print), 1827-3491 (Online).

Journal permission: See page 169.

Quality indexes: JCR Mathematics, Impact factor: 1.1, 108/492 (Q1), 2024.

Corresponding chapter or section: Chapter 3.

Ph.D. student contribution: Essential. The candidate participated in the design of the research, the development of the proofs, the subsequent analysis of the results, the writing of the work, as well as in the dissemination of the results through presentations at scientific conferences.

[70] M. Ladra and A. Pérez-Rodríguez, *Regular evolution algebras are closed under subalgebras*, C. R. Math. Acad. Sci. Paris **363** (2025), 1461–1465.

Title: Regular evolution algebras are closed under subalgebras.

Year: 2025.

Author 1: Manuel Ladra. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Author 2: Andrés Pérez-Rodríguez. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Journal: Comptes Rendus Mathématique. Académie des Sciences. Paris

DOI: 10.5802/crmath.804

Publisher: Académie des sciences. Institut de France.

ISSN: 1631-073X (Print), 1778-3569 (Online).

Journal permission: Open access, see page 170.

Quality indexes: JCR Mathematics, Impact factor: 0.7, 242/492 (Q2), 2024.

Corresponding chapter or section: Section 1.3.

Ph.D. student contribution: Essential. The candidate participated in the design of the research, the development of the proofs, the subsequent analysis of the results, the writing of the work, as well as in the dissemination of the results through presentations at scientific conferences.

Moreover, we include information about an article which, although accepted, it has not been published yet.

[22] Y. Cabrera Casado, M. Ladra, and A. Pérez-Rodríguez, *Gonosomal algebras and operators associated to genetic systems with a single male genotype*, arXiv: 2502.06299 (2025). To appear in Publicacions Matemàtiques.

Title: Gonosomal algebras and operators associated to genetic systems with a single male genotype.

Preprint: arXiv:2502.06299

Year: 2025.

Author 1: Yolanda Cabrera Casado. Departamento de Matemática Aplicada, E.T.S. Ingeniería Informática, Universidad de Málaga.

Author 2: Manuel Ladra. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Author 3: Andrés Pérez-Rodríguez. CITMAga – Departamento de Matemáticas, Universidade de Santiago de Compostela.

Journal: Publicacions Matemàtiques.

Provisional arXiv DOI: 10.48550/arXiv.2502.06299

Publisher: Universitat Autònoma de Barcelona

ISSN: 0214-1493 (Print), 2014-4350 (Online).

Journal permission: Open access, see page 171.

Quality indexes: JCR Mathematics, Impact factor: 1.0, 130/492 (Q2), 2024.

Corresponding chapter or section: Chapter 6.

Ph.D. student contribution: Essential. The candidate participated in the design of the research, the development of the proofs, the subsequent analysis of the results, the writing of the work, as well as in the dissemination of the results through presentations at scientific conferences.

Introduction

Over the last decades, growing environmental pressures such as climate change, habitat loss, and the accelerating decline of biodiversity have intensified both social and scientific concern for understanding how ecosystems behave and respond to disturbances. Among the four laws of ecology formulated by Barry Commoner, the first one, *Everything is connected to everything else*, emphasises that ecosystems are complex and deeply interdependent (see [35]). In Commoner's own words,

> *the system is stabilised by its dynamic self-compensating properties; these same properties, if overstressed, can lead to a dramatic collapse... [and]... the ecological system is an amplifier, so that a small perturbation in one network may have large, distant, long-delayed effects.*

In this context, understanding how biological populations evolve has become particularly relevant, since their long-term stability or decline often reflects deeper ecological processes.

A population is defined as a group of individuals of the same species that inhabit a specific geographic region at a given time and have the ability to interbreed. The analysis of populations, along with the factors and mechanisms that regulate them, is undertaken by *population dynamics*, historically recognised as one of the most prominent branches of mathematical biology. By analysing demographic parameters such as survival, reproductive success, and distribution, one can predict species' future and identify key points where intervention may be necessary to ensure its preservation. Understanding these processes, however, requires not only analysing population sizes and interactions but also examining how genetic information is transmitted across generations, since inheritance plays a fundamental role in shaping the long-term behaviour and adaptability of species.

The mathematical study of heredity began in 1856 with the experiments of Gregor Mendel, who used pea plants and simple numerical reasoning to formulate his laws of inheritance. From Mendel's pioneering ideas to contemporary mathematical genetics, a central question persists: how do populations evolve over time under the influence of hereditary mechanisms? To answer it, mathematics has drawn upon probability theory, stochastic processes, algebra, dynamical systems, nonlinear analysis, and differential and difference equations. A comprehensive overview of the historical development of population dynamics can be found in [6], which includes models ranging from the Fibonacci sequence introduced by Leonardo of Pisa to the Lotka-Volterra predator-prey system, as well as more recent formulations inspired by demographic policies such as China's one-child rule.

In this manuscript, however, the emphasis lies on the algebraic approach, particularly on the role of nonassociative algebras as a framework for modelling and understanding population dynamics. In 1940, I. M. H. Etherington introduced the formal language of abstract algebra into the study of genetics through his seminal papers [43–45], where he reformulated Mendelian laws in algebraic terms. Since then, research in this area has continued to grow, and several families of nonassociative algebras, such as baric, gametic, zygotic, evolution, Bernstein, and train algebras, have provided important contributions to theoretical population genetics. This is the reason why the term *genetic algebra* was coined to refer to all those algebras, most of them nonassociative, that make it possible to model inheritance in the field of genetics.

The theory of nonassociative algebras experienced a major development throughout the twentieth century, and although genetic algebras arise within this context, their study follows a rather different path. To better appreciate this distinction, let us briefly recall the general framework of nonassociative algebras and the way they are usually classified. In the broadest sense, an algebra $\mathcal{A}$ is a vector space equipped with a bilinear product. In particular, given a basis $\{e_i\}_{i\in\Lambda}$ of $\mathcal{A}$, the product is determined by the so-called structure constants c_{ij}^k, via the rule $e_ie_j = \sum_{k\in\Lambda} c_{ij}^k e_k$. The algebra $\mathcal{A}$ is said to be nonassociative if the product is not assumed to be nonassociative, that is, the product may or may not be associative. This general notion of an algebra is too broad to yield meaningful structural results on its own. Therefore, it is customary to restrict attention to particular classes, or varieties, obtained by imposing polynomial identities on the bilinear operation. Some of the most extensively studied varieties over the past century include associative algebras, Lie algebras, Leibniz algebras, Jordan algebras, alternative algebras, and Malcev algebras, among others. These varieties share certain structural properties that allow similar types of questions to be posed about them; however, the answers to such questions may vary significantly depending on the distinct behaviour of each variety under similar conditions.

Genetic algebras, however, do not fit into these varieties. Although they are defined through several conditions on their structure constants, usually motivated by their biological interpretation, these conditions are not polynomial identities in the usual sense, and thus their algebraic study requires different tools and approaches. Therefore, beyond their fundamental role in modelling inheritance in genetics, their algebraic study is also of intrinsic interest in pure mathematics, since it raises the question of whether these algebras can, in some sense, be treated analogously to varieties. In particular, it is natural to ask to what extent the classical structural problems considered for varieties, such as their classification, can be meaningfully formulated and solved in the framework of genetic algebras. The most comprehensive references for the mathematical research in this area are [72, 88, 90, 108].

In this thesis, we focus on two classes of genetic algebras, each treated in a separate part: *evolution algebras* and *gonosomal algebras*. The first part adopts a purely algebraic perspective, abstracting away the genetic interpretation to analyse their intrinsic structure and behaviour as mathematical objects. The second part takes a more biological viewpoint, combining the algebraic framework with discrete dynamical systems to study the evolution of certain populations. So, we now turn to motivate and introduce both structures, outlining their background, current state of the art, and describing the investigations carried out in this thesis through a brief overview of the chapters.

Evolution algebras

Evolution algebras are commutative but nonassociative structures introduced in 2006 by J. P. Tian and P. Vojtěchovský (see [100]) as the mathematical framework of modelling non-Mendelian inheritance, which is actually considered the basic language of molecular biology. Two years later, Tian published a new monograph [99] in which their properties and biological applications are studied in more detail. In particular, he motivates the construction of evolution algebras as a simple way to mathematically describe the inheritance of asexually reproducing organisms, such as many prokaryotes. In these species, reproduction does not involve the combination of genetic material from two parents; instead, each individual gives rise to new individuals whose genetic type depends solely on that of the parent, possibly affected by small random changes or mutations.

Let us denote by $e_1, e_2, \ldots, e_n$ the possible genetic types in the population. We assume that environmental conditions remain constant from generation to generation, so that the reproductive behaviour of each type does not change over time. The construction of evolution algebras is inspired by the self-replication rule observed in non-Mendelian inheritance. Interpreting the product as a reproduction, the product of two distinct genotypes, e_ie_j, has no biological meaning, whereas e_ie_i can be viewed as self-replication (binary fission, for instance). Hence, the frequency distribution of types in the next generation can be described through a bilinear rule of the form

$$e_ie_i = \sum_{k=1}^{n} c_{ik}e_k \quad \text{and} \quad e_ie_j = 0, i \neq j,$$

where the coefficients c_{ik} represent the probability that an individual of type e_i produces an individual of type e_k.

The bilinear extension of this product on the vector space spanned by the set of genetic types defines an algebraic structure called an *evolution algebra*. From a biological viewpoint, the coefficients c_{ik} represent probabilities, and hence satisfy $0 \leq c_{ik} \leq 1$ and $\sum_{k=1}^{n} c_{ik} = 1$ for every i. However, in general, these coefficients are usually regarded as arbitrary elements of a given field. Therefore, we will understand a (finite-dimensional) *evolution algebra* $\mathcal{E}$ as an algebra over a field $\mathbb{K}$ which admits a basis $B = \{e_1, \dots, e_n\}$ such that $e_i e_j = 0$ for all $i \neq j$.

The theory of evolution algebras is currently a very active field of mathematical research. In particular, one of the main challenges is their classification problem: to give a list of algebras or of (parametric) families of algebras, pairwise nonisomorphic, such that any algebra in the class is isomorphic to one of them. This problem has been addressed in low dimensions (see [26, 32]), and later for the nilpotent case (see [41, 57]). Complete classifications are, as expected, difficult to obtain, so most publications on evolution algebras focus on analysing their intrinsic and structural properties. Researchers have studied, for instance, their ideals (see [13, 21]), conditions for simplicity and semisimplicity (see [24]), as well as their derivations (see [20, 89]) and automorphisms (see [38, 39]). Moreover, numerous connections between evolution algebras and other areas of mathematics have been explored. In particular, graph theory has proved to be a powerful tool in studying their structural properties (see [40, 42]). The relationship between evolution algebras and Markov chains has also been investigated (see [86]), and in [93] evolution algebras are associated with function spaces defined by Gibbs measures on graphs, providing a natural link to thermodynamics and its applications in biology, physics, and mathematics.

Motivated by the developments described above, Part I is devoted to the study of evolution algebras. Our general philosophy throughout this part is to investigate to what extent classical questions and structural ideas that have been successfully explored in well-established varieties of algebras can be meaningfully formulated and solved in the framework of evolution algebras. With this in mind, Chapter 1 provides a comprehensive overview of this algebraic structure, recalling the necessary concepts, notations, and results, along with original contributions that will be essential for the development of subsequent chapters.

Although several works have examined ideals of evolution algebras, studies devoted to their subalgebras are comparatively scarce (see [29]). This motivates Chapters 2 and 3, which focus on understanding the behaviour of subalgebras in this setting. Chapter 2 analyses the structure of the set of subalgebras as a lattice, with special attention to two classical lattice-theoretical properties: distributivity and modularity. Chapter 3, in turn, centres on maximal subalgebras and investigates the so-called Frattini subalgebra, defined as their intersection. It is worth noting that analogous studies have a long history in group theory (see [49, 96]) and have also been developed for other nonassociative structures, such as Lie algebras (see [10, 52, 64, 78]).

Finally, Chapter 4 deals with deformations and degenerations of evolution algebras, a topic that has received little attention in this context despite being extensively investigated for associative algebras (see [53]) and Lie algebras (see [83, 84]).

Gonosomal algebras

Gonosomal algebras were introduced by Richard Varro in 2016 in [105] with the aim of providing an algebraic framework to model sex-linked inheritance in bisexual populations. One of the main difficulties in constructing models for such inheritance lies in the fact that the transmission of genetic information depends not only on the hereditary material of the individuals but also on the underlying sex-determination mechanism. Gonosomal algebras offer remarkable versatility in this regard, as their algebraic structure can be adapted to represent a wide range of inheritance patterns and sex-determination systems.

Let us denote by f_i with $i = 1, \ldots, n$, the possible female genetic types and by h_p with $p = 1, \ldots, m$, the possible male genetic types in a bisexual population. Note that the reproduction of two individuals of the same sex, $f_i f_j$ and $h_p h_q$, does not make biological sense. Therefore, we can represent the reproduction of the population through a bilinear rule such that for all $1 \leq i, j \leq n$ and $1 \leq p, q \leq m$, we have that

$$f_i f_j = 0, \quad h_p h_q = 0 \quad \text{and} \quad f_i h_p = h_p f_i = \sum_{k=1}^{n} \gamma_{ipk} f_k + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} h_r,$$

where the coefficients γ_{ipk} (resp. $\widetilde{\gamma}_{ipr}$) represent the probability of getting a female of type f_k (resp. a male of type h_r) in the offspring of a female of type f_i with a male of type h_p. Consequently, as the coefficients γ_{ipk} and $\widetilde{\gamma}_{ipr}$ represent probabilities, they satisfy that $0 \leq \gamma_{ipk} \leq 1$, $0 \leq \widetilde{\gamma}_{ipr} \leq 1$ and $\sum_{k=1}^{n} \gamma_{ipk} + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} = 1$ for all $i, k = 1, \ldots, n$ and $p, r = 1, \ldots, m$. The bilinear extension of this product, together with the restrictions on the coefficients, on the vector space spanned by the set of genetic types, defines an algebraic structure known as *gonosomal algebra*.

Unlike in the case of evolution algebras, most works on gonosomal algebras do not approach them from a purely algebraic perspective. This is partly because these algebras often have very high dimensions, making their structural study considerably difficult. Instead, research in this area typically focuses on modelling specific types of populations. In such studies, one constructs the so-called associated gonosomal operator

$$\begin{cases} x'_k &= \sum_{i,p=1}^{n,m} \gamma_{ipk} x_i y_p, \quad k = 1, \ldots, n; \\ y'_r &= \sum_{i,p=1}^{n,m} \widetilde{\gamma}_{ipr} x_i y_p, \quad r = 1, \ldots, m; \end{cases}$$

and its normalised version

$$\begin{cases} x'_k &= \dfrac{\sum_{i,p=1}^{n,m} \gamma_{ipk} x_i y_p}{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{p=1}^{m} y_p\right)}, \quad k = 1, \ldots, n; \\ y'_r &= \dfrac{\sum_{i,p=1}^{n,m} \widetilde{\gamma}_{ipr} x_i y_p}{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{p=1}^{m} y_p\right)}, \quad r = 1, \ldots, m. \end{cases}$$

They are discrete-time dynamical systems derived from the coefficients of the gonosomal algebra and which describe the temporal evolution of the corresponding population. The analysis then centres on the dynamical behaviour of these systems, usually by computing trajectories and investigating the existence of equilibrium points or limit states for an arbitrary initial point (see, for instance, [3, 4, 92, 94]).

Motivated by these considerations, Part II is devoted to the study of gonosomal algebras and their associated operators. The general spirit throughout this part is to explore how algebraic methods can be effectively combined with discrete-time dynamical systems to model and analyse the evolution of populations under various sex-determination mechanisms. To this end, Chapter 5 introduces gonosomal algebras and motivates their necessity. It presents the main definitions and properties related to these algebras, recalls some of the most relevant existing results, and explains the construction of the associated gonosomal operators.

Finally, Chapter 6 applies this framework to the study of populations with sex-determination systems in which a single male genotype interacts with several female genotypes. Using gonosomal algebras and their associated operators, we analyse the long-term behaviour of the corresponding populations, determine equilibrium states, and interpret the results from a biological perspective.

Hypotheses and objectives

The main purposes of this thesis are to study evolution and gonosomal algebras, as well as the evolution operators associated with the latter. In what follows, we briefly describe the hypotheses (**H**) and specific objectives (**O**) of this work.

H1 The lattice of subalgebras of a Lie algebra was extensively studied during the last decades of the past century (see [52, 64]). Two key features underlying this theory are that every one-dimensional subspace is automatically a subalgebra, due to skew-symmetry, and that every subalgebra is itself a Lie algebra, since Lie algebras form a variety. Although evolution algebras are also nonassociative, neither of these properties holds in their setting. Combined with the fact that subalgebras of evolution algebras (see [29]) have received comparatively little attention in contrast with ideals (see, for instance, [13, 21, 24]), the study of subalgebra lattices in evolution algebras is potentially interesting. To analyse this hypothesis, we formulate the following objectives.

- **O1** Owing to the lack of studies on subalgebras in evolution algebras, an essential objective is to characterise subalgebras in several key classes of evolution algebras, notably the regular and the nilpotent cases.
- **O2** As already said in the introduction, evolution algebras do not form a variety. One of the main consequences of this lack of variety structure is that subalgebras of evolution algebras are not themselves evolution algebras in general. So we aim to find sufficient and necessary conditions for an evolution algebra to be closed under subalgebras.
- **O3** To study evolution algebras whose subalgebra lattices satisfy classical lattice-theoretic properties, namely distributivity, modularity and upper and lower semimodularity.

H2 Closely related to subalgebra lattices, we also need to consider Frattini theory. Although originated in group theory (see [49]), Frattini theory has also been extensively studied in the framework of nonassociative varieties of algebras, most notably in the context of Lie (see [78, 102]) or Leibniz algebras (see [11]). Traditionally, the Frattini subalgebra is defined as the intersection of maximal subalgebras, and, as before, although there exist several publications concerning maximal ideals, that is not the case for maximal subalgebras. Therefore, motivated by the need to study them and to obtain results concerning the Frattini subalgebra, we establish the following objectives.

O4 In the case of Lie and Leibniz algebras, the nilradical, defined as their maximal nilpotent ideal, plays a fundamental role in determining whether their Frattini subalgebras are trivial or not. Motivated by this, we aim to investigate to what extent this approach can be applied in the setting of evolution algebras and, if this is not the case, to introduce an analogous concept that can effectively support the development of a Frattini theory in this context.

O5 To develop, or at least initiate, a Frattini theory for evolution algebras by establishing sufficient and necessary conditions for the Frattini subalgebra of an evolution algebra to be trivial.

O6 To explore applications of Frattini theory in the context of evolution algebras, such as the characterisation of dually atomistic algebras.

H3 Formal deformations and degenerations have been extensively studied for associative (see [53]), Lie (see [83, 84]) and Leibniz algebras (see [7, 62]). In particular, much attention has been devoted to the analysis of equivalent deformations (typically governed by the second cohomology space) as well as to the classification of rigid algebras. Furthermore, numerous works have constructed diagrams that describe the degeneration relations within specific families of algebras (see [15, 17, 48]). To the best of our knowledge, only one publication addresses these questions in the context of evolution algebras (see [27]), which provides a clear motivation for developing this theory in that setting.

O7 To define formal deformations for evolution algebras and to obtain sufficient and necessary conditions for two such deformations to be equivalent.

O8 To define formal degenerations for evolution algebras, to derive necessary conditions for one evolution algebra to degenerate into another, and to attempt the construction of diagrams representing these degeneration relations.

O9 To characterise (formally) rigid evolution algebras, that is, to determine whether an evolution algebra admits any nontrivial degenerations.

H4 Gonosomal algebras constitute the most recent algebraic framework for modelling the evolution of bisexual populations (see [105]). Their flexible structure allows one to represent a wide range of sex-related genetic phenomena. Every gonosomal algebra naturally determines a quadratic operator, its associated gonosomal evolution operator, which is a discrete-time dynamical system linking the genetic states of two successive generations (see [91]). A central challenge in this context is to determine the limit points of the trajectories generated by these operators from an arbitrary initial state. Although several works analyse the dynamical behaviour of particular operators (see [1, 3]), the lack of a general method for studying nonlinear discrete dynamical systems significantly complicates the problem, leaving this as a difficult and largely open area of research.

O10 To study gonosomal algebras and their associated (normalised) gonosomal operators of some specific sex-determination systems in bisexual populations.

O11 To analyse the trajectories and limit points of these gonosomal operators.

O12 To identify potential applications of the results in biology, ecology, and medicine.

Methodology

This thesis has followed the classic methodology in basic research in mathematics. The general procedure could be summarised in the identification of open problems through the comprehensive study of the existing literature in a certain topic (in the case of this thesis, genetic algebras and other nonassociative structures), with the aim of proposing definitions, conjectures of results that generalise others already known, or which can be compared with them, together with the searching for techniques and new examples that are significant enough or have important applications in other areas of mathematics. Occasionally, the exploratory work has relied on computer-assisted calculations, which were used to test conjectures and to examine the behaviour of specific examples. Note that the described stages of the aforementioned process need not be linear, as often mathematicians will find themselves back in the stage of reviewing the literature for various reasons, such as the announcement of new relevant results.

An essential part of this methodology is the continuous interaction with other researchers. This includes attending conferences, participating in seminars, and carrying out research stays that encourage the exchange of ideas and collaborations with other specialists in the field. In this regard, we highlight the visit of Professor Yolanda Cabrera Casado (Universidad de Málaga) to the Universidade de Santiago de Compostela during the development of this thesis. In addition, two research stays, totaling four months, were completed at the Université de Haute-Alsace (Mulhouse, France) under the supervision of Professor Abdenacer Makhlouf. The first stay was from 3 June to 5 July 2024, and the second was from 1 March to 31 May 2025.

Results and discussion

Part I

Evolution algebras

Chapter 1

Evolution algebras: background and new structural contributions

This chapter provides a comprehensive review of evolution algebras, recalling fundamental definitions and results while also incorporating original contributions that will be essential to the first part of this thesis.

Section 1.1 revisits well-established definitions and properties of evolution algebras, with particular emphasis on their connections to their associated digraphs. It also reviews the basic theory of ideals and subalgebras in this setting, and concludes with the classification of two-dimensional evolution algebras over the complex numbers. In Section 1.2, we begin with the characterisation of nilpotent evolution algebras and their classification over the complex numbers, and then turn to two families of solvable evolution algebras, the first of which will play a key role in our subsequent developments. Sections 1.3 and 1.4 are devoted to the study of subalgebras of two well-known classes of evolution algebras, namely regular and complete evolution algebras, respectively. The results obtained in these sections have appeared in our publications [70] and [51]. Finally, Section 1.5 introduces supersolvable evolution algebras, provides their characterisation within the solvable setting, and briefly defines $\mathcal{E}$-supersolvable ideals, outlining their main properties.

We include a short comment on notation here. $\mathbb{K}$ will denote an arbitrary field, and $\mathbb{K}^*$ will stand for $\mathbb{K}\setminus\{0\}$. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra and $S \subseteq \mathcal{A}$ a subset. We will use $\mathsf{span}\{S\}$ to denote the $\mathbb{K}$-linear span of S, and we use $+$ and $\oplus$ to denote sums and direct sums of vector spaces, respectively.

1.1 Basic definitions and facts about evolution algebras

An *evolution algebra* over a field $\mathbb{K}$ is a $\mathbb{K}$-algebra $\mathcal{E}$ which admits a distinguished basis $B = \{e_1, \dots, e_n, \dots\}$ such that $e_ie_j = 0$ for any $i \neq j$. Such a basis is called a *natural basis*. Throughout this thesis, all evolution algebras will be assumed to be finite-dimensional, that is, B will be a finite set. Fixed a natural basis $B = \{e_1, \dots, e_n\}$ in $\mathcal{E}$, the scalars $\omega_{ik} \in \mathbb{K}$ such that $e_i^2 = \sum_{k=1}^{n} \omega_{ik} e_k$ are called the *structure constants of $\mathcal{E}$ relative to B*, and the matrix

$$M_B(\mathcal{E}) := \begin{pmatrix} \omega_{11} & \dots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \dots & \omega_{nn} \end{pmatrix}$$

is called the *structure matrix of $\mathcal{E}$ relative to B*. Note that, given a natural basis, an evolution algebra is completely determined by its structure matrix, since the multiplication on the whole algebra is obtained by bilinear extension from the squares of the basis elements. Indeed, for any $u, v \in \mathcal{E}$, say $u = \sum_{i=1}^n \mu_i e_i$ and $v = \sum_{i=1}^n \nu_i e_i$ with $\mu_i, \nu_i \in \mathbb{K}$, we have

$$uv = \left(\sum_{i=1}^n \mu_i e_i\right)\left(\sum_{i=1}^n \nu_i e_i\right) = \sum_{i=1}^n \mu_i \nu_i e_i^2 = \sum_{i=1}^n \mu_i \nu_i \sum_{k=1}^n \omega_{ik} e_k.$$

It is worth noting that a finite-dimensional algebra is an evolution algebra if and only if there exists a basis $B = \{e_1, \dots, e_n\}$ with respect to which the multiplication table is diagonal:

	e_1	$\dots$	e_i	$\dots$	e_n
e_1	$\sum_{k=1}^n \omega_{1k} e_k$	$\dots$	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
e_i	0	$\dots$	$\sum_{k=1}^n \omega_{ik} e_k$	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
e_n	0	$\dots$	0	$\dots$	$\sum_{k=1}^n \omega_{nk} e_k$

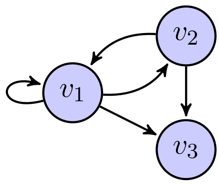
The problem of deciding whether a given algebra is an evolution algebra has been addressed in [18, Theorem 1], where it is shown that this occurs precisely when a certain set of matrices is simultaneously diagonalisable via congruence.

By definition, every evolution algebra is commutative and thus flexible. However, in general, they are neither associative nor power-associative. In fact, a complete classification of associative evolution algebras is given in [47], while a characterization of power-associativity appears in [19]. Furthermore, evolution algebras do not, in general, belong to any of the classical varieties of nonassociative algebras, such as Lie, Leibniz, alternative, or Jordan algebras.

Remark 1.1.1. The concepts of *homomorphism*, *endomorphism*, *isomorphism*, *automorphism*, and *derivation* in evolution algebras are analogous to those used in the general context of algebras. In particular, every isomorphism between evolution algebras maps natural bases to natural bases. Moreover, it is worth mentioning that numerous works have been devoted to the study of automorphisms (see [38, 39]) and derivations (see [20, 89]) in the framework of evolution algebras.

A survey of the existing literature shows that graph theory is a powerful tool for studying evolution algebras (see, for instance, [24, 37, 40–42]). While there are several ways to associate a graph with an evolution algebra, in this work, we focus on directed graphs (digraphs). This construction assumes a basic knowledge of graph theory; the reader may consult [34, Chapter 7] for further background if necessary.

Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{e_1, \dots, e_n\}$ and structure matrix $M_B(\mathcal{E}) = (\omega_{ij})$. The digraph $\Gamma(\mathcal{E}, B) = (V, E)$, with $V = \{v_1, \dots, v_n\}$ the set of vertices and $E = \{(v_i, v_j) \in V \times V : \omega_{ij} \neq 0\}$ the set of edges, is called the *digraph attached to $\mathcal{E}$ relative to B*. For instance, consider the evolution algebra $\mathcal{E}_1$ with natural basis $B_1 = \{e_1, e_2, e_3, e_4\}$ and product given by $e_1^2 = e_2 + e_3$, $e_2^2 = 0$, $e_3^2 = -2e_4$ and $e_4^2 = 5e_3$; and the evolution algebra $\mathcal{E}_2$ with natural basis $B_2 = \{e_1, e_2, e_3\}$ and product given by $e_1^2 = -2e_1 + 2e_2 + e_3$, $e_2^2 = 3e_1 - e_3$ and $e_3^2 = 0$. Then, the attached digraphs $\Gamma(\mathcal{E}_1, B_1)$ and $\Gamma(\mathcal{E}_2, B_2)$ are:



Remark 1.1.2. It should be noted that, in general, the digraph associated with an evolution algebra depends on the choice of natural basis. That is, a single evolution algebra may give rise to different digraphs depending on the natural basis considered (see [40, Examples 2.4 & 2.5]). Despite this drawback, we shall see throughout this preliminary chapter that this graph-theoretic approach is useful for studying several properties of evolution algebras.

We now turn our attention to the notions of subalgebra and ideal in the setting of evolution algebras, which are, in fact, analogous to those used in the general theory of algebras. A subspace U of an evolution algebra $\mathcal{E}$ is called a *subalgebra* of $\mathcal{E}$ if it is closed under the product. Moreover, a subalgebra $I \subseteq \mathcal{E}$ is called an *ideal* of $\mathcal{E}$ if $I\mathcal{E} \subseteq I$. Note that evolution algebras are not closed under subalgebras (see [19, Example 1.4.1]) nor under ideals (see [19, Example 1.4.6]). Hence, it is natural to introduce the following two notions. A subalgebra (resp. ideal) of an evolution algebra is said to be an *evolution subalgebra* (resp. *evolution ideal*) if it is an evolution algebra, that is, it admits a natural basis.

Recall that for an (evolution) algebra $\mathcal{E}$, a *flag of subalgebras* (or simply a *flag*) is an increasing sequence of subalgebras, each properly contained in the next: $0 = U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_k = \mathcal{E}$. In particular, such a sequence is said to be a *complete flag of subalgebras* if each subalgebra has one dimension larger than the preceding, that is, $0 = U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_n = \mathcal{E}$ where $\dim U_i = i$ for all $i = 0, \ldots, n$. Still in the context of subalgebras, we recall that, unlike the case of ideals, the sum $U + V$ of two subalgebras U and V is not necessarily a subalgebra. Hence, we introduce the following notation.

Notation 1.1.3. Given a subset S of an evolution algebra $\mathcal{E}$, we denote by $\langle S \rangle$ the subalgebra of $\mathcal{E}$ generated by S, that is, the smallest subalgebra of $\mathcal{E}$ containing S. In particular, given two subalgebras U and V, $\langle U, V \rangle$ denotes the smallest subalgebra of $\mathcal{E}$ which contains both U and V.

Concerning ideals, we also introduce the following concepts. An ideal I of $\mathcal{E}$ is called a *basic ideal* of $\mathcal{E}$ relative to B if it admits a natural basis consisting of vectors from B. Moreover, adopting the standard terminology from algebras, an ideal I of $\mathcal{E}$ (or a subalgebra, in general) is said to be *abelian* if $I^2 = 0$; and an evolution algebra $\mathcal{E}$ is said to be *abelian* if $\mathcal{E}^2 = 0$. Moreover, an evolution algebra is said to be *almost abelian* if it is nonabelian, but it has an abelian ideal of codimension one. Following [25], an evolution algebra $\mathcal{E}$ is called *semiprime* if it has no nonzero abelian ideals. An element $u \in \mathcal{E}$ is said to be *idempotent* if $u^2 = u$ and *absolute nilpotent* if $u^2 = 0$.

Remark 1.1.4. Every one-dimensional subalgebra, say $\mathsf{span}\{u\}$, is spanned by either an absolute nilpotent element or an idempotent. Indeed, if $u^2 = 0$ then u is absolute nilpotent; if $u^2 = ku$ with $k \in \mathbb{K}^*$, then $u' = \frac{1}{k}u$ is an idempotent, since $(u')^2 = u'$.

One of the most frequently used ideals in the theory of evolution algebras is the annihilator. Recall that the *annihilator* of an evolution algebra $\mathcal{E}$ with natural basis $B = \{e_1, \ldots, e_n\}$ is characterised by [19, Proposition 1.5.3],

$$\operatorname{ann}(\mathcal{E}) \coloneqq \{u \in \mathcal{E} \colon u\mathcal{E} = 0\} = \mathsf{span}\{e \in B \colon e^2 = 0\}.$$

If $\operatorname{ann}(\mathcal{E}) = 0$, we say that $\mathcal{E}$ is *nondegenerate*. Otherwise, we say that $\mathcal{E}$ is *degenerate*. Moreover, following [41, Definition 3.3], we can also define the chain of ideals $\operatorname{ann}^i(\mathcal{E})$, $i \geq 1$, where:

- $\operatorname{ann}^1(\mathcal{E}) \coloneqq \operatorname{ann}(\mathcal{E})$,
- $\operatorname{ann}^i(\mathcal{E})$ with $i \geq 2$ is defined by $\operatorname{ann}^i(\mathcal{E}) / \operatorname{ann}^{i-1}(\mathcal{E}) \coloneqq \operatorname{ann}(\mathcal{E} / \operatorname{ann}^{i-1}(\mathcal{E}))$.

Equivalently, $\operatorname{ann}^i(\mathcal{E}) := \mathsf{span}\{e \in B \colon e^2 \in \operatorname{ann}^{i-1}(\mathcal{E})\}$ for all $i \geq 2$. The chain of ideals

$$0 \subseteq \operatorname{ann}^1(\mathcal{E}) \subseteq \cdots \subseteq \operatorname{ann}^r(\mathcal{E}) \subseteq \ldots$$

is called the *upper annihilating series* of $\mathcal{E}$. Note that the annihilator of an evolution algebra is an example of a basic abelian ideal, whereas the subsequent terms in the upper annihilating series correspond to basic nonabelian ideals.

Remark 1.1.5. The upper annihilating series of an evolution algebra $\mathcal{E}$ with basis B can be read directly from its digraph $\Gamma(\mathcal{E}, B)$. The first term $\operatorname{ann}(\mathcal{E})$ is spanned by the basis elements corresponding to sink vertices (vertices with no outgoing edges). Iteratively, $\operatorname{ann}^{k+1}(\mathcal{E})$ is obtained by removing the sinks and all attached edges from the subdigraph corresponding to $\operatorname{ann}^k(\mathcal{E})$, and taking the span of the new sinks together with all sinks found in previous steps, until no sinks remain.

It is also known that evolution algebras are closed with respect to quotients by ideals (see [19, Lemma 1.4.11]). Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{e_1, \ldots, e_n\}$ and I an ideal of $\mathcal{E}$. Then, although

$$B_{\mathcal{E}/I} := \{\overline{e} = e + I \colon e \in B,\ e \notin I\}$$

is not necessarily a natural basis of $\mathcal{E}/I$, it always contains a natural basis of the quotient algebra (see [19, Remark 1.4.12]). Particularly, if we have a basic ideal $I = \mathsf{span}\{e_i \colon i \in \Lambda\}$, with $\Lambda \subset \{1, \ldots, n\}$, we will consider the set $B_{\mathcal{E}/I} = \{\overline{e_i} \colon i \notin \Lambda\}$ as the natural basis of the quotient algebra $\mathcal{E}/I$.

Remark 1.1.6. Let U be a subalgebra of an evolution algebra $\mathcal{E}$ and I a basic ideal of $\mathcal{E}$ relative to a natural basis B such that $I \subset U$. Then, it is easy to check that U/I is a basic ideal of $\mathcal{E}/I$ relative to $B_{\mathcal{E}/I}$ if and only if U is a basic ideal of $\mathcal{E}$.

Now, we recall the concept of the support of an element. Given an evolution algebra $\mathcal{E}$ with a natural basis $B = \{e_1, \ldots, e_n\}$ and an element $u = \sum_{i=1}^n \mu_i e_i \in \mathcal{E}$, we define its *support* relative to B as $\operatorname{supp}_B(u) := \{i : \mu_i \neq 0\}$. In a similar way, we define the *support of a subspace* $U \subset \mathcal{E}$ as $\operatorname{supp}_B(U) := \cup_{u \in U} \operatorname{supp}_B(u)$. Clearly, if B_U is a basis of U, then $\operatorname{supp}_B(U) = \cup_{u \in B_U} \operatorname{supp}_B(u)$. When the choice of the natural basis is clear, we simply write supp instead of supp_B.

Remark 1.1.7. Let $\mathcal{E}$ be an evolution algebra with a natural basis $B = \{e_1, \ldots, e_n\}$. If I is an ideal of $\mathcal{E}$, then $\mathsf{span}\{e_i \colon i \in \operatorname{supp}(I)\}$ is also an ideal.

Finally, Table 1.1.1 presents the classification of nonabelian two-dimensional evolution algebras over $\mathbb{C}$ established in [32, Theorem 4.1]. Besides individual algebras, the table also includes parametric families, which naturally give rise to certain isomorphism relations. In fact, all algebras listed are pairwise nonisomorphic except for the following cases: $\mathcal{E}_5(a_2, a_3) \cong \mathcal{E}_5(a_3, a_2)$; and, if $a_4 \neq 0$, then $\mathcal{E}_6(a_4) \cong \mathcal{E}_6(a_4')$ if and only if $\frac{a_4}{a_4'} = \cos\left(\frac{2\pi k}{3}\right) + i \sin\left(\frac{2\pi k}{3}\right)$ for some $k = 1, 2, 3$.

$\mathcal{E}$	**Product**
$\mathcal{E}_1$	$e_1^2 = e_1,\ e_2^2 = 0$
$\mathcal{E}_2$	$e_1^2 = e_2^2 = e_1$
$\mathcal{E}_3$	$e_1^2 = -e_2^2 = e_1 + e_2$
$\mathcal{E}_4$	$e_1^2 = e_2,\ e_2^2 = 0$
$\mathcal{E}_5(a_2, a_3)$	$e_1^2 = e_1 + a_2 e_2,\ e_2^2 = a_3 e_1 + e_2$ with $1 - a_2 a_3 \neq 0$
$\mathcal{E}_5(a_4)$	$e_1^2 = e_2,\ e_2^2 = e_1 + a_4 e_2$

Table 1.1.1: Classification of two-dimensional evolution algebras over $\mathbb{C}$.

1.2 Nilpotent and solvable evolution algebras

Throughout this manuscript, we will work extensively with nilpotent and solvable evolution algebras. Given an evolution algebra $\mathcal{E}$, we define the following sequences of subalgebras:

$$\mathcal{E}^{\langle 1\rangle} = \mathcal{E}, \qquad \mathcal{E}^{\langle k+1\rangle} = \mathcal{E}^{\langle k\rangle}\mathcal{E};$$
$$\mathcal{E}^{1} = \mathcal{E}, \qquad \mathcal{E}^{k+1} = \sum_{i=1}^{k} \mathcal{E}^{i}\mathcal{E}^{k+1-i};$$
$$\mathcal{E}^{(1)} = \mathcal{E}, \qquad \mathcal{E}^{(k+1)} = \mathcal{E}^{(k)}\mathcal{E}^{(k)}.$$

An evolution algebra $\mathcal{E}$ is called *right nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathcal{E}^{\langle n\rangle} = 0$ (and the minimal such number is called *index of right nilpotency*), *nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathcal{E}^{n} = 0$ (and the minimal such number is called *index of nilpotency*), and *solvable* if there exists $n \in \mathbb{N}$ such that $\mathcal{E}^{(n)} = 0$ (and the minimal such number is called *index of solvability*). As in Lie algebras, the sequence $0 = \mathcal{E}^{(n)} \subsetneq \mathcal{E}^{(n-1)} \subsetneq \cdots \subsetneq \mathcal{E}^{(2)} \subsetneq \mathcal{E}$ will be called the *derived series of* $\mathcal{E}$ and, particularly, $\mathcal{E}^{\langle 2\rangle} = \mathcal{E}^{2} = \mathcal{E}^{(2)}$ will be called the *derived subalgebra of* $\mathcal{E}$. Moreover, as introduced in [46], we define the *principal powers* of an element $u \in \mathcal{E}$, recursively, as $u^1 = u$, $u^n = u^{n-1}u$; and we define its *plenary powers* as $u^{(0)} = u$, $u^{(n)} = u^{(n-1)}u^{(n-1)}$.

Nilpotent evolution algebras have received considerable attention in the literature. First, recall that a commutative algebra is right nilpotent if and only if it is nilpotent (see [109, Chapter 4, Proposition 1]); in particular, this applies to evolution algebras. A key feature of nilpotent evolution algebras is that their structure matrix can be assumed to be strictly (upper or lower) triangular by [32, Theorem 2.7]. Consequently, we also have that $\mathcal{E}$ is nilpotent if and only if the upper annihilating series reaches $\mathcal{E}$.

That is, there exists an integer $r \geq 1$ such that $\operatorname{ann}^r(\mathcal{E}) = \mathcal{E}$. Moreover, as presented in [41, Definition 3.4], the *type* of a nilpotent evolution algebra $\mathcal{E}$ is defined as the sequence $[n_1, \ldots, n_r]$ such that

$$n_i = \dim\big(\operatorname{ann}(\mathcal{E}/\operatorname{ann}^{i-1}(\mathcal{E}))\big) = \dim\big(\operatorname{ann}^i(\mathcal{E})\big) - \dim\big(\operatorname{ann}^{i-1}(\mathcal{E})\big),$$

for all $i = 1, \ldots, r$. Furthermore, recall that nilpotency can be read from the associated digraph: an evolution algebra is nilpotent if and only if its associated digraph contains no oriented cycles (see [40, Theorem 3.4]).

In what follows, nilpotent evolution algebras of dimension n over a field $\mathbb{K}$ will be denoted by $\mathcal{N}_n(\mathbb{K})$. In particular, Table 1.2.1 displays the classification of nilpotent evolution algebras up to dimension four over $\mathbb{C}$, originally established in [57, Theorems 5.1, 5.2, 5.3 & 6.1] and later refined in [41, Theorem 5.1], together with their type and the dimension of their square.

$\mathcal{E}$	**Product**	**Type of** $\mathcal{E}$	**Dimension of** $\mathcal{E}^2$
$\mathcal{E}_{1,1}$	$e_1^2 = 0$	$[1]$	0
$\mathcal{E}_{2,1}$	$e_1^2 = e_2^2 = 0$	$[2]$	0
$\mathcal{E}_{2,2}$	$e_1^2 = e_2,\ e_2^2 = 0$	$[1,1]$	1
$\mathcal{E}_{3,1}$	$e_1^2 = e_2^2 = e_3^2 = 0$	$[3]$	0
$\mathcal{E}_{3,2}$	$e_1^2 = e_2,\ e_2^2 = e_3^2 = 0$	$[2,1]$	1
$\mathcal{E}_{3,3}$	$e_1^2 = e_2^2 = e_3,\ e_3^2 = 0$	$[1,2]$	1
$\mathcal{E}_{3,4}$	$e_1^2 = e_2,\ e_2^2 = e_3,\ e_3^2 = 0$	$[1,1,1]$	2
$\mathcal{E}_{4,1}$	$e_1^2 = e_2^2 = e_3^2 = e_4^2 = 0$	$[4]$	0
$\mathcal{E}_{4,2}$	$e_1^2 = e_2,\ e_2^2 = e_3^2 = e_4^2 = 0$	$[3,1]$	1
$\mathcal{E}_{4,3}$	$e_1^2 = e_2^2 = e_3,\ e_3^2 = e_4^2 = 0$	$[2,2]$	1
$\mathcal{E}_{4,4}$	$e_1^2 = e_3,\ e_2^2 = e_4,\ e_3^2 = e_4^2 = 0$	$[2,2]$	2
$\mathcal{E}_{4,5}$	$e_1^2 = e_2,\ e_2^2 = e_4,\ e_3^2 = e_4^2 = 0$	$[2,1,1]$	2
$\mathcal{E}_{4,6}$	$e_1^2 = e_2^2 = e_3^2 = e_4,\ e_4^2 = 0$	$[1,3]$	1
$\mathcal{E}_{4,7}$	$e_1^2 = e_2,\ e_2^2 = e_3^2 = e_4,\ e_4^2 = 0$	$[1,2,1]$	2
$\mathcal{E}_{4,8}$	$e_1^2 = e_2 + ie_3,\ e_2^2 = e_3^2 = e_4,\ e_4^2 = 0$	$[1,2,1]$	2
$\mathcal{E}_{4,9}$	$e_1^2 = e_2^2 = e_3,\ e_3^2 = e_4,\ e_4^2 = 0$	$[1,1,2]$	2
$\mathcal{E}_{4,10}$	$e_1^2 = e_3,\ e_2^2 = e_3 + e_4,\ e_3^2 = e_4,\ e_4^2 = 0$	$[1,1,2]$	2
$\mathcal{E}_{4,11}$	$e_1^2 = e_2,\ e_2^2 = e_3,\ e_3^2 = e_4,\ e_4^2 = 0$	$[1,1,1,1]$	3
$\mathcal{E}_{4,12}$	$e_1^2 = e_2 + e_3,\ e_2^2 = e_3,\ e_3^2 = e_4,\ e_4^2 = 0$	$[1,1,1,1]$	3

Table 1.2.1: Nilpotent evolution algebras up to dimension four over $\mathbb{C}$.

Although solvable evolution algebras, unlike the nilpotent ones, lack both a classification and any characterisation in terms of their structure matrix or associated digraphs, the following two remarks provide some basic structural information.

Remark 1.2.1. Every one-dimensional subalgebra of a solvable evolution algebra is abelian and, consequently, it is spanned by an absolute nilpotent element. To see this, suppose to the contrary that $\mathsf{span}\{u\}$ is a one-dimensional ideal of a solvable evolution algebra $\mathcal{E}$ such that $u^2 = ku$ for some $k \in \mathbb{K}^*$. Then, $u \in \mathcal{E}^{(n)}$ for all $n \in \mathbb{N}$, which contradicts the solvability of $\mathcal{E}$.

Remark 1.2.2. Every solvable evolution algebra $\mathcal{E}$ admits a complete flag of subalgebras. The proof proceeds by induction on $\dim \mathcal{E}$. Since $\mathcal{E}$ is solvable, we have $\mathcal{E}^2 \subsetneq \mathcal{E}$, so by the inductive hypothesis $\mathcal{E}^2$ admits a complete flag of subalgebras. Moreover, as $\mathcal{E}/\mathcal{E}^2$ is abelian, the remaining terms of the complete flag follow.

Recall that every nilpotent evolution algebra is solvable, but the converse does not hold in general. In the next subsection, we describe the family of complex solvable evolution algebras with one-dimensional derived subalgebras, which includes examples of solvable but nonnilpotent algebras.

1.2.1 Solvable evolution algebras with one-dimensional derived subalgebras

This family has already been introduced and characterised in [28]. Given a vector space V over $\mathbb{C}$, with basis $B = \{e_1, \ldots, e_n\}$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, not all of them zero, and $\sum_{j=1}^{k} \lambda_j = 0$ for a $k \in \{1, \ldots, n\}$, we define $\mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$ as the evolution algebra with natural basis B and product given by

$$\mathcal{E}_k(\lambda_1, \ldots, \lambda_n)\colon e_i^2 = \lambda_i(e_1 + \cdots + e_k), \tag{1.2.1}$$

for all $i = 1, \ldots, n$. The importance of this type of solvable algebras lies in the fact that every complex solvable evolution algebra whose derived subalgebra is one-dimensional is isomorphic to one of them [28, Proposition 2.4]. Moreover, as explained in [28, Remark 2.6], the evolution algebras described in (1.2.1) can be divided in two disjoint classes. First, when $\lambda_i = 0$ for all $1 \leq i \leq k$, this evolution algebra is nilpotent. The second one is when $\lambda_i \neq 0$ for some $1 \leq i \leq k$. Then, by a natural basis transformation, one can assume that $e_1^2 = e_1 + \cdots + e_k$ and hence, these evolution algebras are not nilpotent. Although this family was originally defined over $\mathbb{C}$, it can be similarly defined over any field, and both the above-mentioned proposition and remark extend readily to this more general setting.

Notation 1.2.3. We will denote by $\mathcal{T}_{\mathbb{K}}$ the set of all solvable but nonnilpotent evolution algebras with one-dimensional derived subalgebra over a field $\mathbb{K}$. Accordingly, we will say that an evolution algebra $\mathcal{E}$ of this type is an element of $\mathcal{T}_{\mathbb{K}}$, $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$. For simplicity in exposition, we slightly abuse notation by also considering a square matrix A to be in $\mathcal{T}_{\mathbb{K}}$, $A \in \mathcal{T}_{\mathbb{K}}$, if it can serve as the structure matrix of an evolution algebra in $\mathcal{T}_{\mathbb{K}}$.

We now present a slightly stronger characterisation of the family $\mathcal{T}_{\mathbb{K}}$.

Proposition 1.2.4. *Every evolution algebra of $\mathcal{T}_{\mathbb{K}}$ is isomorphic to an evolution algebra $\mathcal{E}_k(\lambda_1, \dots, \lambda_n)$ with $k \geq 2$ and $\lambda_1, \dots, \lambda_k \neq 0$.*

Proof. Consider an evolution algebra $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$. By [28, Proposition 2.4 & Remark 2.6], $\mathcal{E}$ is isomorphic to $\mathcal{E}_k(\lambda_1, \dots, \lambda_n)$ with $k \geq 2$ and $\lambda_i \neq 0$ for some $1 \leq i \leq k$ (actually, for at least two indices). Then, simply reordering the natural basis, there exists a natural number $2 \leq m \leq k$ such that $\lambda_1, \dots, \lambda_m \neq 0$ and $\lambda_{m+1}, \dots, \lambda_k = 0$. If $m = k$, we are done. Otherwise, we can consider the following natural basis transformation:

$$f_1 = \frac{1}{\lambda_1}(e_1 + e_{m+1} + \cdots + e_k) \qquad \text{and} \qquad f_i = \frac{1}{\lambda_1} e_i,$$

for all $i = 2, \dots, n$. Then, we get that

$$f_1^2 = f_1 + \cdots + f_m \qquad \text{and} \qquad f_i^2 = \frac{\lambda_i}{\lambda_1}(f_1 + \cdots + f_m),$$

for all $i = 2, \dots, n$. Consequently, it holds that $\mathcal{E}$ is isomorphic to

$$\mathcal{E}_m\left(1, \frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_m}{\lambda_1}, 0, \dots, 0, \frac{\lambda_{k+1}}{\lambda_1}, \dots, \frac{\lambda_n}{\lambda_1}\right),$$

where $1, \frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_m}{\lambda_1} \neq 0$, what yields the claim. □

Remark 1.2.5. Because of Proposition 1.2.4, we may assume without loss of generality that every evolution algebra in $\mathcal{T}_{\mathbb{K}}$ is of the form $\mathcal{E}_k(\lambda_1, \dots, \lambda_n)$ with $k \geq 2$ and $\lambda_1, \dots, \lambda_k \neq 0$.

Remark 1.2.6. Every evolution algebra in $\mathcal{T}_{\mathbb{K}}$ splits over its annihilator with respect to a certain natural basis. Given an evolution algebra $\mathcal{E}_k(\lambda_1, \dots, \lambda_n)$ with $k \geq 2$ and $\lambda_1, \dots, \lambda_k \neq 0$, the result follows from considering the ideal $\mathsf{span}\{e_i \colon \lambda_i \neq 0\}$, which is complemented by $\operatorname{ann}(\mathcal{E}) = \mathsf{span}\{e_i \colon \lambda_i = 0\}$.

Finally, a new family of solvable evolution algebras is introduced. Although these algebras may not possess one-dimensional derived subalgebras, they generalise (1.2.1).

Definition 1.2.7. Let $\mathcal{E}_1, \ldots, \mathcal{E}_r \in \mathcal{T}_{\mathbb{K}}$ with natural bases $B_1, \ldots, B_r$, respectively. Suppose that $r \geq 1$. Then, it will be said that an evolution algebra $\mathcal{E}$ belongs to the family $\mathcal{F}(\mathcal{E}_1, \ldots, \mathcal{E}_r)$ if there exists a natural basis B such that $\cup_{i=1}^r B_i \subseteq B$ and

$$M_B(\mathcal{E}) = \begin{pmatrix} A_{m\times m} & 0_{m\times(n-m)} \\ C_{(n-m)\times m} & L_{(n-m)\times(n-m)} \end{pmatrix}, \tag{1.2.2}$$

where $m = |\cup_{i=1}^r B_i|$, A is block diagonal (in which each block is $M_{B_i}(\mathcal{E}_i)$ for $1 \leq i \leq r$), L is strictly lower triangular and C can be any matrix. Notice that $\oplus_{i=1}^r \mathcal{E}_i \subseteq \mathcal{E}$ is an ideal, and the quotient $\mathcal{E}/(\oplus_{i=1}^r \mathcal{E}_i)$ is a nilpotent evolution algebra.

It is worth noting that Definition 1.2.7 describes a quite varied family of solvable evolution algebras since we can find solvable evolution algebras of arbitrary dimension and any index of solvability. More precisely, let $n \geq 2$ and let k be an integer such that $2 \leq k \leq n$. Consider the evolution algebra $\mathcal{E}$ with natural basis $\{e_1, \ldots, e_n\}$ and multiplication given by:

$$e_1^2 = -e_2^2 = e_1 + e_2, \qquad e_i^2 = e_{i-1}, \text{ for all } 3 \leq i \leq k,$$

and $e_i^2 = 0$ for all $k < i \leq n$. Then, $\mathcal{E}$ is a solvable but nonnilpotent evolution algebra of dimension n with index of solvability equal to $k+1$.

1.2.2 Evolution algebras with maximum index of solvability

In contrast to solvable evolution algebras with one-dimensional derived subalgebra, this subsection is devoted to establishing some results about solvable evolution algebras with maximum index of solvability. Recall that if a solvable evolution algebra $\mathcal{E}$ of dimension n has maximum index of solvability, then its index of solvability is $n+1$ (see [28, Proposition 4.2]). First, we characterise these solvable evolution algebras.

Proposition 1.2.8. *Let $\mathcal{E}$ be a solvable evolution algebra. Then, $\mathcal{E}$ has maximum index of solvability if and only if* $\operatorname{codim} \mathcal{E}^2 = 1$.

Proof. Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{e_1, \ldots, e_n\}$ which has maximum index of solvability. Then, the derived series has n nonzero terms, that is,

$$n = \dim \mathcal{E} > \dim \mathcal{E}^{(2)} > \ldots > \dim \mathcal{E}^{(n)} > \dim \mathcal{E}^{(n+1)} = 0. \tag{1.2.3}$$

From (1.2.3), it holds that $\dim \mathcal{E}^{(2)} = \dim \mathcal{E}^2 = \dim\left(\mathsf{span}\{e_1^2, \ldots, e_n^2\}\right) = n-1$. Conversely, if $\operatorname{codim} \mathcal{E}^2 = 1$, then there exists a reordering of the natural basis such that $\{e_1^2, \ldots, e_{n-1}^2\}$ is a linearly independent subset. Now, consider $\mathcal{E}^{(s)} = \mathsf{span}\{u_1, \ldots, u_l\}$, where $u_i = \sum_{j=1}^n \mu_{ij} e_j$ with $\mu_{ij} \in \mathbb{K}$. Without loss of generality, assume that the matrix $(\mu_{ij})_{i,j=1}^{l,n}$ is in row echelon form. Moreover, notice that n belongs to the support of at least one element, say u_l, meaning that $\mu_{ln} \neq 0$. Otherwise, since $\{e_1^2, \ldots, e_{n-1}^2\}$ is linearly independent, then

$$\dim \mathcal{E}^{(s+1)} = \dim \left(\mathsf{span}\{u_1^2, \ldots, u_l^2\} \right) = l = \dim \mathcal{E}^{(s)},$$

contradicting the solvability of $\mathcal{E}$. Then, use the nonzero scalar μ_{ln} to perform elementary row operations and make zeros in the last column. Finally, after all these changes we have that $n \in \operatorname{supp}(u_l)$ but $n \notin \operatorname{supp}(u_i)$ for all $i = 1, \ldots, l-1$. Therefore, applying again the linear independency of $\{e_1^2, \ldots, e_{n-1}^2\}$, we get that

$$\dim \mathcal{E}^{(s+1)} \geq \dim \left(\mathsf{span}\{u_1^2, \ldots, u_{l-1}^2\} \right) = l - 1 = \dim \mathcal{E}^{(s)} - 1.$$

As s was taken arbitrarily and $\mathcal{E}^{(s+1)} \subsetneq \mathcal{E}^{(s)}$, it holds that $\dim \mathcal{E}^{(s+1)} = \dim \mathcal{E}^{(s)} - 1$ for any $s = 1, \ldots, n$, which yields the claim. □

The following result determines subalgebras of solvable evolution algebras with maximum index of solvability.

Proposition 1.2.9. *Let $\mathcal{E}$ be a solvable evolution algebra with maximum index of solvability over a field $\mathbb{K}$ of characteristic different from 2. Then, there exists a reordering of the natural basis such that $e_n^2 = -\lambda_1^2 e_1^2 - \cdots - \lambda_m^2 e_m^2$ for some $0 \leq m \leq n-1$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{K}^*$. In this case, $\mathsf{span}\{\pm\lambda_1 e_1 \pm \cdots \pm \lambda_m e_m \pm e_n\}$, with the signs chosen arbitrarily, are precisely all the one-dimensional subalgebras of $\mathcal{E}$. Consequently, $\mathcal{E}$ has 2^m one-dimensional subalgebras.*

Proof. As a consequence of Proposition 1.2.8, there exists a reordering of the natural basis such that $\sigma_1 e_1^2 + \cdots + \sigma_m e_m^2 + e_n^2 = 0$ for some $0 \leq m \leq n-1$ and $\sigma_1, \ldots, \sigma_m \in \mathbb{K}^*$. Since all one-dimensional subalgebras of a solvable algebra are necessarily abelian (Remark 1.2.1), they are spanned by an absolute nilpotent element. So, we have that $\mathsf{span}\{\pm\sqrt{\sigma_1} e_1 \pm \cdots \pm \sqrt{\sigma_m} e_m \pm e_n\}$, with the signs chosen arbitrarily, are precisely all those subalgebras. Finally, notice that $\sqrt{\sigma_1}, \ldots, \sqrt{\sigma_m}$ necessarily belong to $\mathbb{K}^*$; otherwise, there would exist no one-dimensional subalgebras, which contradicts the existence of a complete flag of subalgebras in every solvable algebra (Remark 1.2.2). Hence, taking $\lambda_i = \sqrt{\sigma_i}$ for all $i = 1, \ldots, m$, the result follows. □

Remark 1.2.10. As $\lambda_1, \ldots, \lambda_m \in \mathbb{K}^*$ in the previous proposition, with the suitable natural basis change ($f_i = \lambda_i e_i$ for all $i = 1, \ldots, m$) one can assume that $e_n^2 = -e_1^2 - \cdots - e_m^2$ for some $m \leq n-1$. Particularly, the one-dimensional subalgebras are $\mathsf{span}\{\pm e_1 \pm \cdots \pm e_m \pm e_n\}$.

Remark 1.2.11. In the case of evolution algebras over fields of characteristic two, the alternating signs disappear, and therefore there is only one one-dimensional subalgebra.

In the context of complex evolution algebras, [28, Proposition 4.2] establishes an equivalence between maximum index of nilpotency and maximum index of solvability in the nilpotent case. Moreover, [28, Theorem 4.5] states that a nilpotent evolution algebra reaches its maximum index of nilpotency if and only if the first (upper or lower) diagonal of its structure matrix has no zero elements. These proofs extend without difficulty to any field. Consequently, we are able to characterise all subalgebras in this particular case.

Corollary 1.2.12. *Let $\mathcal{E}$ be a nilpotent evolution algebra with maximum index of nilpotency over a field $\mathbb{K}$, $B = \{e_1, \ldots, e_n\}$ a natural basis of $\mathcal{E}$ such that $M_B(\mathcal{E}) = (\omega_{ij})$ is strictly upper triangular and U a subalgebra. Then $U = \mathsf{span}\{e_k, \ldots, e_n\}$ for some $k = 1, \ldots, n$.*

Proof. First, by Proposition 1.2.9, since $e_n^2 = 0$, the only one-dimensional subalgebra is $\mathsf{span}\{e_n\}$. To study the other subalgebras, consider an arbitrary element $u = \sum_{i=1}^n \mu_i e_i \in \mathcal{E}$ and define $k = \min\{\mathrm{supp}(u)\}$. Hence, we have that

$$\begin{aligned}
u^{(0)} &= u = \mu_k e_k + v_{k+1}, \\
&\vdots \\
u^{(i)} &= \mu_k^{2^i} \omega_{k(k+1)}^{2^{i-1}} \cdots \omega_{(k+i-1)(k+i)} e_{k+i} + v_{k+i+1}, \\
&\vdots \\
u^{(n-k)} &= \mu_k^{2^{n-k}} \omega_{k(k+1)}^{2^{n-k-1}} \cdots \omega_{(n-1)n} e_n;
\end{aligned}$$

with $v_{k+i+1} \in \mathsf{span}\{e_{k+i+1}, \ldots, e_n\}$ and $\mu_k^{2^i} \omega_{k(k+1)}^{2^{i-1}} \cdots \omega_{(k+i-1)(k+i)} \neq 0$ for all $i = 0, \ldots, n-k$. As all of the previous elements belong to $\langle u \rangle$, it holds that $e_n \in \langle u \rangle$. In the same way, as $u^{(n-k-1)} = \lambda_1 e_{n-1} + \lambda_2 e_n \in \langle u \rangle$ with $\lambda_1, \lambda_2 \in \mathbb{K}$, $\lambda_1 \neq 0$, and $e_n \in \langle u \rangle$, it also holds that $e_{n-1} \in \langle u \rangle$. Inductively, it comes that $\langle u \rangle = \mathsf{span}\{e_k, \ldots, e_n\}$. □

Example 1.2.13. For instance, the unique proper nonzero subalgebras of the evolution algebra $\mathcal{E}$ with natural basis $\{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_2$, $e_2^2 = e_3$ and $e_3^2 = 0$ are $\mathsf{span}\{e_3\}$ and $\mathsf{span}\{e_2, e_3\}$.

Finally, the following two lemmas about solvable evolution algebras with maximum index of solvability will also be instrumental throughout our study.

Lemma 1.2.14. *Let $\mathcal{E}$ be a solvable evolution algebra of dimension n with maximum index of solvability over a field $\mathbb{K}$ of characteristic different from 2. Then $\mathcal{E}^{(n-1)}$ has at most two subalgebras.*

Proof. By (1.2.3), we can assume that $\mathcal{E}^{(n)} = \mathsf{span}\{v\}$ and $\mathcal{E}^{(n-1)} = \mathsf{span}\{v, u\}$, where $v^2 = 0$, $u^2 = k_1 v$ and $vu = uv = k_2 v$ with $k_1, k_2 \in \mathbb{K}$, at least one of them nonzero. Hence, any proper nonzero subalgebra of $\mathcal{E}^{(n-1)}$ will be spanned by an absolute nilpotent element $\lambda_1 v + \lambda_2 u$. Therefore, we have

$$0 = (\lambda_1 v + \lambda_2 u)^2 = \lambda_1^2 v^2 + \lambda_2^2 u^2 + 2\lambda_1\lambda_2 vu = \lambda_2(\lambda_2 k_1 + 2\lambda_1 k_2)v. \quad (1.2.4)$$

Then, it is easy to deduce that, if $k_1 \neq 0$, $\mathcal{E}^{(n-1)}$ has at most two different subalgebras: $\mathsf{span}\{v\}$, and $\mathsf{span}\{v - \frac{2k_2}{k_1}u\}$ when $k_2 \neq 0$; and if $k_1 = 0$ it has two different subalgebras: $\mathsf{span}\{v\}$ and $\mathsf{span}\{u\}$. □

Lemma 1.2.15. *Any two-dimensional solvable evolution algebra over a field $\mathbb{K}$ is abelian or isomorphic to one of the following nonisomorphic algebras:*

(i) $e_1^2 = -e_2^2 = e_1 + e_2$;

(ii) $e_1^2 = e_2$, $e_2^2 = 0$

Proof. Note that if $\mathcal{E}$ is a two-dimensional solvable evolution algebra, then its product is given by $e_1^2 = \omega_{11}e_1 + \omega_{12}e_2$ and $e_2^2 = k(\omega_{11}e_1 + \omega_{12}e_2)$, with $\omega_{11}, \omega_{12}, k \in \mathbb{K}$, and such that $(\omega_{11}e_1 + \omega_{12}e_2)^2 = 0$. From this last condition, if $\omega_{12} \neq 0$ we have that $k = -\frac{w_{11}^2}{\omega_{12}^2}$. Now, consider the following possibilities:

1. If $\omega_{11}, \omega_{12} \neq 0$, the natural basis $\left\{f_1 = \frac{1}{\omega_{11}}e_1, f_2 = \frac{\omega_{12}}{\omega_{11}^2}e_2\right\}$ gives $f_1^2 = -f_2^2 = f_1 + f_2$.
2. If $\omega_{11} = 0$ and $\omega_{12} \neq 0$, then $k = 0$. Consequently, taking the natural basis $\{f_1 = e_1, f_2 = \omega_{12}e_2\}$ we get that $f_1^2 = f_2$ and $f_2^2 = 0$.
3. If $\omega_{11}, \omega_{12} = 0$, the algebra is abelian.

Since the case with $\omega_{11} \neq 0$ and $\omega_{12} = 0$ does not yield a solvable evolution algebra, the result follows. □

1.3 Regular evolution algebras

An evolution algebra $\mathcal{E}$ is said to be *regular* (or *perfect*) if $\mathcal{E} = \mathcal{E}^2$, meaning that it is generated by the squares of the elements of the natural basis. Since we are only considering finite-dimensional evolution algebras, this condition is equivalent to the structure matrix being nonsingular. Recall that an evolution algebra is said to be *simple* if it is nonabelian and it has no nonzero proper ideals, and *semisimple* if it can be written as a direct sum of simple evolution algebras. Since an evolution algebra is simple if and only if it is regular and its associated digraph is strongly connected (see [19, Corollary 2.2.10]), both simple and semisimple algebras are natural examples of regular evolution algebras.

Regular evolution algebras have been thoroughly investigated and possess particularly desirable properties. For instance, they have a unique natural basis (see [40, Theorem 4.4]) and, consequently, their associated digraphs do not depend on the considered natural basis (see [40, Corollary 4.5]). Their automorphism groups are finite (see [40, Theorem 4.8]), they are universally finite (see [37]), and their algebras of derivations have also been described (see [42, Theorem 4.1]). Furthermore, unlike the case of subalgebras, ideals in regular evolution algebras have also been investigated. In fact, every ideal in this setting is basic (see [13, Proposition 4.2]), so the class of regular evolution algebras is closed under taking ideals.

The following two subsections address subalgebras in the regular case. First, we show that regular evolution algebras are closed not only under ideals, but also under subalgebras, which allows us to assume, without loss of generality, that every subalgebra in this setting has a basis consisting of vectors with disjoint supports. In the second, we use this result to characterise the existence of codimension-one subalgebras. All these results have given rise to the paper [70].

1.3.1 Regular evolution algebras are closed under subalgebras

First, note that every one-dimensional subalgebra is trivially an evolution subalgebra, so we first show that they are given by the nontrivial solutions of a nonlinear polynomial system of equations.

Lemma 1.3.1. *Let $\mathcal{E}$ be a regular evolution algebra with basis $B = \{e_1, \dots, e_n\}$ and structure matrix $M_B(\mathcal{E})$. Then, the one-dimensional subspace $\operatorname{span}\{\lambda_1 e_1 + \cdots + \lambda_n e_n\}$ is a proper nonzero subalgebra of $\mathcal{E}$ if and only if the vector $(\lambda_1, \dots, \lambda_n)$ is proportional to a nontrivial solution of*

$$\begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} = \left(M_B(\mathcal{E})^t\right)^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{1.3.1}$$

Proof. A subspace $\mathsf{span}\{\lambda_1 e_1 + \cdots + \lambda_n e_n\}$ is a subalgebra if and only if there exists a scalar $k \neq 0$ such that

$$\left(\sum_{i=1}^{n} \lambda_i e_i\right)^2 = \sum_{i=1}^{n} \lambda_i^2 \sum_{j=1}^{n} \omega_{ij} e_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \omega_{ji} \lambda_j^2\right) e_i = k \left(\sum_{i=1}^{n} \lambda_i e_i\right),$$

where the second equality simply follows from interchanging the indices i and j. For this condition to hold, it is necessary that $\sum_{j=1}^{n} \omega_{ji}\lambda_j^2 = k\lambda_i$ for all $i = 1, \ldots, n$. Moreover, note that $k \neq 0$; otherwise, $(\lambda_1 e_1 + \cdots + \lambda_n e_n)^2 = \lambda_1^2 e_1^2 + \cdots + \lambda_n^2 e_n^2 = 0$, contradicting the regularity of $\mathcal{E}$. In fact, without loss of generality, it can be assumed that $k = 1$ (by setting $\lambda_i' = \frac{\lambda_i}{k}$), so we obtain (1.3.1). □

Next, we present our main theorem, which shows that regular evolution algebras are closed under subalgebras. This property not only offers a more precise and structured approach to understanding subalgebras but also significantly simplifies their study in higher dimensions.

Theorem 1.3.2. *Let $\mathcal{E}$ be a regular evolution algebra over any field $\mathbb{K}$. Then, every subalgebra of $\mathcal{E}$ admits a natural basis.*

Proof. Let $\mathcal{E}$ be a regular evolution algebra with natural basis $\{e_1, \ldots, e_n\}$, and consider a proper subalgebra $U = \mathsf{span}\{u_i \colon 1 \leq i \leq m\}$ of dimension $m < n$, where each $u_i = \sum_{j=1}^{n} \mu_{ij} e_j$ with $\mu_{ij} \in \mathbb{K}$. Without loss of generality, we assume that the matrix $(\mu_{ij})_{i,j=1}^{m}, n$ is in reduced row echelon form. Furthermore, by appropriately reordering the natural basis, we can assume that the leading coefficients form the identity matrix, i.e.

$$u_1 = e_1 + v_1,\ u_2 = e_2 + v_2,\ \ldots,\ u_m = e_m + v_m,$$

where $v_1, \ldots, v_m \in \mathsf{span}\{e_{m+1}, \ldots, e_n\}$.

For the sake of contradiction, assume that U does not admit a natural basis. Then, there exist distinct indices $k, l \in \{1, \ldots, m\}$ such that $u_k u_l \neq 0$. Moreover, as $\mathcal{E}$ is regular, we have $\operatorname{rank}\{e_1^2, \ldots, e_n^2\} = n$, which implies that $\operatorname{rank}\{u_1, \ldots, u_m\} = \operatorname{rank}\{u_1^2, \ldots, u_m^2\} = m$. Then, since $\{u_1^2, \ldots, u_m^2\}$ also spans the subalgebra U, there exist scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ such that

$$\lambda_1 u_1^2 + \cdots + \lambda_m u_m^2 + u_k u_l = 0.$$

Expanding each term,

$$\lambda_1(e_1^2 + v_1^2) + \cdots + \lambda_m(e_m^2 + v_m^2) + u_k u_l = 0.$$

Since $u_k u_l \in \mathsf{span}\{e_{m+1}^2, \ldots, e_n^2\}$, the linear independence of $\{e_1^2, \ldots, e_n^2\}$ forces $\lambda_1 = \cdots = \lambda_m = 0$, contradicting that $u_k u_l \neq 0$. □

Remark 1.3.3. The converse of the previous result does not hold in general. As shown in Corollary 1.2.12, all subalgebras of a nilpotent evolution algebra with the maximum index of nilpotency are basic ideals.

Corollary 1.3.4. *Let $\mathcal{E}$ be a regular evolution algebra over any field $\mathbb{K}$ with natural basis $B = \{e_1, \ldots, e_n\}$. Then, every subalgebra U of $\mathcal{E}$ admits a basis $\{u_1, \ldots, u_m\}$ where the supports of its elements are disjoint.*

Proof. The result follows from Theorem 1.3.2 and from the fact that if we have two elements u and v in a regular evolution algebra, then $uv = 0$ if and only if their supports are disjoint (see [40, Remark 4.3]). □

1.3.2 Conditions for the existence of codimension-one subalgebras

Corollary 1.3.4 significantly simplifies the study of subalgebras in the regular case. For example, given a regular evolution algebra $\mathcal{E}$ with natural basis $B = \{e_1, \ldots, e_n\}$, any subalgebra of codimension one can be written as

$$\mathsf{span}\{e_i, v \colon i \neq p, q\}, \tag{1.3.2}$$

where $v \in \mathsf{span}\{e_p, e_q\}$ for some distinct indices $p < q$. Using this description, we establish precise conditions for the existence of subalgebras of codimension one in regular evolution algebras.

Proposition 1.3.5. *Let $\mathcal{E}$ be a regular evolution algebra of dimension greater than two over any field $\mathbb{K}$ with natural basis $B = \{e_1, \ldots, e_n\}$ and $M_B(\mathcal{E}) = (\omega_{ij})$.*

Suppose that $\mathcal{E}$ has a subalgebra of codimension one. In this case, there necessarily exist distinct indices $p < q$ such that

$$\operatorname{rank} M_{p,q} \leq 1,$$

where $M_{p,q}$ denotes the submatrix of $M_B(\mathcal{E})$ formed by the p-th and q-th columns after removing the p-th and q-th rows.

Conversely, if $\operatorname{rank} M_{p,q} \leq 1$*, the existence of subalgebras of codimension one can be characterised as follows:*

1. *If* $\operatorname{rank} M_{p,q} = 1$*, then at least one row of $M_{p,q}$ is nonzero; let (α, β) be such a row. In this case, $\mathcal{E}$ has a subalgebra of codimension one if and only if*

$$\alpha^2\beta\omega_{pp} + \beta^3\omega_{qp} = \alpha^3\omega_{pq} + \alpha\beta^2\omega_{qq}, \tag{1.3.3}$$

and the corresponding subalgebra is given by (1.3.2) *with $v = \alpha e_p + \beta e_q$.*

2. *If* $\operatorname{rank} M_{p,q} = 0$*, then the subalgebras of codimension one are* (1.3.2) *where* $v = e_p + \lambda e_q$ *for any* $\lambda \in \mathbb{K}^*$ *satisfying the equation*

$$\omega_{qp}\lambda^3 - \omega_{qq}\lambda^2 + \omega_{pp}\lambda - \omega_{pq} = 0, \tag{1.3.4}$$

and additionally, $\mathsf{span}\{e_i \colon i \neq q\}$ *when* $\omega_{pq} = 0$ *and* $\mathsf{span}\{e_i \colon i \neq p\}$ *when* $\omega_{qp} = 0$*.*

Proof. Let $\pi_{p,q}$ be the linear projection of $\mathcal{E}$ onto $\mathsf{span}\{e_p, e_q\}$ along $\mathsf{span}\{e_i \colon i \neq p, q\}$. Observe that (1.3.2) with $v = \alpha e_p + \beta e_q$ defines a subalgebra of $\mathcal{E}$ if and only if the following conditions hold:

$$\pi_{p,q}(e_i^2) = \omega_{ip}e_p + \omega_{iq}e_q \in \mathsf{span}\{\alpha e_p + \beta e_q\}, \text{ for all } i \neq p, q; \text{ and} \tag{1.3.5}$$

$$\begin{aligned}\pi_{p,q}\big((\alpha e_p + \beta e_q)^2\big) &= (\alpha^2\omega_{pp} + \beta^2\omega_{qp})e_p + (\alpha^2\omega_{pq} + \beta^2\omega_{qq})e_q \\ &\in \mathsf{span}\{\alpha e_p + \beta e_q\}.\end{aligned} \tag{1.3.6}$$

From (1.3.5), it follows that $\operatorname{rank} M_{p,q} \leq 1$ is a necessary condition. Conversely, now assume that $\operatorname{rank} M_{p,q} \leq 1$ and study the following two cases separately:

1. If $\operatorname{rank} M_{p,q} = 1$, then (α, β) must correspond with a nonzero row of $M_{p,q}$ for (1.3.2) to be a subalgebra of $\mathcal{E}$. Now, we consider the following scenarios for (1.3.6) to be satisfied:

   (a) if $\alpha = 0$ and $\beta \neq 0$, then (1.3.6) holds if and only if $\omega_{qp} = 0$;

   (b) if $\beta = 0$ and $\alpha \neq 0$, then (1.3.6) holds if and only if $\omega_{pq} = 0$;

   (c) if $\alpha, \beta \neq 0$, then (1.3.6) holds if and only if there exists a scalar $k \in \mathbb{K}^*$ such that $\alpha^2\omega_{pp} + \beta^2\omega_{qp} = k\alpha$ and $\alpha^2\omega_{pq} + \beta^2\omega_{qq} = k\beta$, or equivalently, if (1.3.3) holds.

   Notice that if $\alpha = 0$ and $\beta \neq 0$ (resp. $\beta = 0$ and $\alpha \neq 0$), then (1.3.3) holds if and only if $\omega_{qp} = 0$ (resp. $\omega_{pq} = 0$). Thus, we conclude that, in this case, (1.3.2) is a subalgebra if and only if (1.3.3).

2. If $\operatorname{rank} M_{p,q} = 0$, then it is evident that $\mathsf{span}\{e_i \colon i \neq q\}$ and $\mathsf{span}\{e_i \colon i \neq p\}$ are subalgebras if and only if $\omega_{pq} = 0$ and $\omega_{qp} = 0$, respectively. Moreover, (1.3.2) with $v = \alpha e_p + \beta e_q$ for some $\alpha, \beta \neq 0$, or equivalently, with $v = e_p + \lambda e_q$ for $\lambda \neq 0$, is a subalgebra if and only if there exists a scalar $k \in \mathbb{K}^*$ such that $\omega_{pp} + \lambda^2\omega_{qp} = k$ and $\omega_{pq} + \lambda^2\omega_{qq} = k\lambda$, which is equivalent to λ being a nonzero solution of (1.3.4).

Thus, the result follows. □

Remark 1.3.6. Every nonzero proper subalgebra of a two-dimensional regular evolution algebra is one-dimensional and therefore trivially admits a natural basis. Nevertheless, Proposition 1.3.5 also plays a key role in fully characterising all subalgebras in this case. Specifically, given a regular evolution algebra $\mathcal{E}$ over any field $\mathbb{K}$ with natural basis $B = \{e_1, e_2\}$ and structure matrix $M_B(\mathcal{E}) = (\omega_{ij})$, the subalgebras are given by $\mathsf{span}\{e_1 + \lambda e_2\}$ for any $\lambda \in \mathbb{K}^*$ satisfying the equation

$$\omega_{21}\lambda^3 - \omega_{22}\lambda^2 + \omega_{11}\lambda - \omega_{12} = 0;$$

and additionally, $\mathsf{span}\{e_1\}$ when $\omega_{12} = 0$ and $\mathsf{span}\{e_2\}$ when $\omega_{21} = 0$.

Corollary 1.3.7. *Let $\mathcal{E}$ be a regular evolution algebra of dimension greater than two over any field $\mathbb{K}$ with natural basis $B = \{e_1, \dots, e_n\}$ and $M_B(\mathcal{E}) = (\omega_{ij})$. If the subspace* (1.3.2) *is a subalgebra for some $v \in \mathsf{span}\{e_p, e_q\}$, then the structure constants satisfy*

$$\omega_{ip}^2\omega_{iq}\omega_{pp} + \omega_{iq}^3\omega_{qp} = \omega_{ip}^3\omega_{pq} + \omega_{ip}\omega_{iq}^2\omega_{qq} \tag{1.3.7}$$

for all $i \neq p, q$. Particularly, if $\mathcal{E}$ is three-dimensional and $\mathbb{K}$ is algebraically closed or $\mathbb{K} = \mathbb{R}$, then the converse also holds.

Proof. The necessity follows straightforwardly from Proposition 1.3.5.

For sufficiency in dimension three, note first that for any distinct indices $p, q \in \{1, 2, 3\}$ we always have $\operatorname{rank} M_{p,q} \leq 1$. If $\operatorname{rank} M_{p,q} = 1$, the claim follows immediately from hypothesis (1.3.7) together with condition (1.3.3). On the other hand, if $\operatorname{rank} M_{p,q} = 0$, since $\mathbb{K}$ is algebraically closed or $\mathbb{K} = \mathbb{R}$, then equation (1.3.4) with $\omega_{pq}, \omega_{qp} \neq 0$ always admits a solution in $\mathbb{K}$. Therefore, the result follows. □

We conclude this note with two examples which show how the hypothesis for equivalence cannot be relaxed. Example 1.3.8 illustrates the importance of the choice of the field, while Example 1.3.9 highlights the necessity of the evolution algebra to have dimension three.

Example 1.3.8. Let $\mathcal{E}$ be the regular evolution algebra over $\mathbb{Q}$ with natural basis $\{e_1, e_2, e_3\}$ and structure matrix given by

$$M_B(\mathcal{E}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

First, note that $M_{2,3} = (0, 0)$ and that the structure constants verify the relation

$$\omega_{12}^2\omega_{13}\omega_{22} + \omega_{13}^3\omega_{32} = 0 = \omega_{12}^3\omega_{23} + \omega_{12}\omega_{13}^2\omega_{33}.$$

However, we have that $\omega_{23}, \omega_{32} \neq 0$, and the polynomial $\omega_{32}x^3 - \omega_{33}x^2 + \omega_{22}x - \omega_{23} = x^3 - x - 1$ is irreducible over $\mathbb{Q}$. Consequently, according to Proposition 1.3.5, there are no subalgebras of dimension two of the form $\mathsf{span}\{e_1, v\}$ with $v \in \mathsf{span}\{e_2, e_3\}$. Moreover, since

$$\omega_{21}^2\omega_{23}\omega_{11} + \omega_{23}^3\omega_{31} \neq \omega_{21}^3\omega_{13} + \omega_{21}\omega_{23}^2\omega_{33},$$
$$\omega_{31}^2\omega_{32}\omega_{11} + \omega_{32}^3\omega_{21} \neq \omega_{31}^3\omega_{12} + \omega_{31}\omega_{32}^2\omega_{22},$$

$\mathcal{E}$ does not have subalgebras of dimension two.

Example 1.3.9. Let $\mathcal{E}$ be a regular evolution algebra over any field $\mathbb{K}$ with natural basis $\{e_1, e_2, e_3, e_4\}$ and the two last columns of the structure matrix $M_B(\mathcal{E})$ given by

$$\begin{pmatrix} \omega_{13} & \omega_{14} \\ \omega_{23} & \omega_{24} \\ \omega_{33} & \omega_{34} \\ \omega_{43} & \omega_{44} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ -3 & 2 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that (1.3.7) holds for $i \neq 3, 4$:

$$\omega_{13}^2\omega_{14}\omega_{33} + \omega_{14}^3\omega_{43} = 2 = \omega_{13}^3\omega_{34} + \omega_{13}\omega_{14}^2\omega_{44},$$
$$\omega_{23}^2\omega_{24}\omega_{33} + \omega_{24}^3\omega_{43} = 2 = \omega_{23}^3\omega_{34} + \omega_{23}\omega_{24}^2\omega_{44}.$$

However,

$$\operatorname{rank} M_{3,4} = \operatorname{rank} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = 2,$$

what yields, by Proposition 1.3.5, that there are no subalgebras of dimension three of the form $\mathsf{span}\{e_1, e_2, v\}$ with $v \in \mathsf{span}\{e_3, e_4\}$.

1.4 Complete evolution algebras

As already mentioned, evolution algebras are not, in general, closed under taking subalgebras. Consequently, besides ordinary subalgebras and subalgebras admitting a natural basis (evolution subalgebras), it is natural to consider subalgebras admitting a natural basis that extends to a natural basis of the entire algebra. With respect to this notion, an evolution algebra $\mathcal{E}$ is said to be *complete* if every subalgebra of $\mathcal{E}$ admits a natural basis extendable to a natural basis of $\mathcal{E}$. This concept was introduced and studied in [29], where two conjectures concerning the structure and classification of complete evolution algebras were proposed.

The purpose of this section is to prove these two conjectures by reducing them to a result on the existence and form of solutions of a certain nonlinear polynomial system, established using elementary tools from algebraic geometry, and to derive several further consequences. These results gave rise to the paper [51].

1.4.1 The key result

The main goal of this section is to prove a relaxed version of [29, Conjecture 5.1], which will be sufficient for the statements about evolution algebras that we aim to establish later. In its original form, the conjecture says that given a complex invertible matrix $A = (a_{ij})_{1\leq i,j\leq n}$, the system of equations

$$\begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{1.4.1}$$

admits a solution $(x_1, x_2, \ldots, x_n)$ such that $x_i \neq 0$ for all i. Nevertheless, we now show that there exists a solution with at least two nonzero components, without requiring all components to be nonzero.

Theorem 1.4.1. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a complex invertible matrix. Then, the system of equations* (1.4.1) *always admits a nontrivial solution. In particular, there exists a solution in which at least two coordinates are nonzero.*

Proof. For convenience, define the polynomials $F_i(x_1, \ldots, x_n) = x_i^2 - \sum_{j=1}^n a_{ij}x_j$ for all $i = 1, \ldots, n$. Note that the system $\{F_i = 0\}_{i=1}^n$ coincides with (1.4.1). Moreover, we homogenise them to obtain homogeneous polynomials defining hypersurfaces in $\mathbb{P}^n(\mathbb{C})$:

$$\widetilde{F}_i(x_1, \ldots, x_n, t) = x_i^2 - t\sum_{j=1}^n a_{ij}x_j.$$

We first show that the hypersurfaces $\widetilde{F}_1, \ldots, \widetilde{F}_n$ share no common component. Let us assume that the homogeneous polynomials $\widetilde{F}_1, \ldots, \widetilde{F}_n$ admit a common nonconstant divisor. Let $g(x, t) = g(x_1, \ldots, x_n, t)$ be an irreducible homogeneous polynomial of degree $d \geq 1$ dividing all $\widetilde{F}_i$. If we consider the specialisation $t = 0$, then $\widetilde{F}_i(x, 0) = x_i^2$, so the polynomial $g(x, 0)$ must divide x_i^2 for every i. Since the variables $x_1, \ldots, x_n$ are algebraically independent, we have $\gcd(x_1^2, \ldots, x_n^2) = 1$, and therefore $g(x, 0)$ must be a nonzero constant, which yields a contradiction with the fact that g was homogeneous of degree $d \geq 1$ because such a polynomial necessarily vanishes at the origin, whereas a nonzero constant does not.

Since $\widetilde{F}_1, \dots, \widetilde{F}_n$ share no common components, Bézout's theorem can be applied. Hence, these n hypersurfaces intersect in a finite set, in particular, in 2^n points of $\mathbb{P}^n(\mathbb{C})$, counting their multiplicities. Moreover, it is easy to check that there are no intersection points at infinity. Indeed, if $t = 0$, then it forces $x_1 = \cdots = x_n = 0$. This does not represent any projective point, and therefore all intersection points lie in the affine space $\mathbb{A}^n(\mathbb{C})$ (after rescaling $t = 1$).

We now claim that points with less than two nonzero coordinates have multiplicity one. A point has multiplicity one precisely when the hypersurfaces intersect transversely at that point. This happens exactly when the Jacobian matrix has full rank at that point. The Jacobian matrix of the affine system is

$$J(x_1, \dots, x_n) = \begin{pmatrix} 2x_1 - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & 2x_2 - a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & 2x_3 - a_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & 2x_n - a_{nn} \end{pmatrix}.$$

At the origin, $J(0, \dots, 0) = -A$. Since A is invertible by hypothesis, the origin has multiplicity one. Let us consider a possible solution of the form $(x_1, 0, \dots, 0)$ with $x_1 \neq 0$. Substituting into the system forces that necessarily $x_1 = a_{11}$ and $a_{21} = \cdots = a_{n1} = 0$. Plugging this into the Jacobian shows that

$$\det J(x_1, 0, \dots, 0) = a_{11} \begin{vmatrix} -a_{22} & \cdots & -a_{2n} \\ \vdots & \ddots & \vdots \\ -a_{n2} & \cdots & -a_{nn} \end{vmatrix} = (-1)^{n-1} \det(A) \neq 0.$$

Hence, this point also has multiplicity one. The same reasoning applies to any point with only one nonzero coordinate.

Finally, note that there are at most $n+1$ points with less than two nonzero entries, and all of them have multiplicity one. Since $n + 1 < 2^n$ for all $n \geq 2$, not all the intersection points can be of that form. Therefore, at least one solution must have at least two nonzero components. □

Remark 1.4.2. The previous result does not necessarily hold over fields that are not algebraically closed, such as $\mathbb{R}$. For instance, consider the matrix

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

A computer-assisted calculation shows that the only real solution is $(1, 0, 0)$, which does not have at least two nonzero components.

1.4.2 Consequences on evolution algebra structures

The previous theorem yields several noteworthy implications for the structure theory of complex evolution algebras, which are described in detail below. Note that all algebras in this section are over $\mathbb{C}$.

Existence of nonzero proper subalgebras

The system (1.4.1) does not only arise in [29], it appears naturally in Lemma 1.3.1, where a one-to-one correspondence is established between all one-dimensional subalgebras of a regular evolution algebra $\mathcal{E}$ and the nontrivial solutions of the system (1.3.1). Since $\mathcal{E}$ is regular, the structure matrix $M_B(\mathcal{E})$ is invertible, and therefore Theorem 1.4.1 applies directly. We thus have the following straightforward consequence.

Theorem 1.4.3. *Every complex evolution algebra $\mathcal{E}$ admits a nonzero proper subalgebra.*

Proof. If an evolution algebra $\mathcal{E}$ is regular, then the result follows straightforwardly from Theorem 1.4.1. If $\mathcal{E}$ is abelian, i.e., $\mathcal{E}^2 = 0$, then the result follows automatically from the fact that every subspace of an abelian algebra is a subalgebra. Otherwise, if $\mathcal{E}$ is neither regular nor abelian, then $0 \subsetneq \mathcal{E}^2 \subsetneq \mathcal{E}$, so $\mathcal{E}^2$ is a proper nonzero subalgebra. □

It is worth recalling that simple evolution algebras (those with no nonzero proper ideals) have already been characterised: they are precisely the regular algebras whose associated digraph is strongly connected. Thus, the relevance of the previous result lies in the fact that no analogous characterisation for subalgebras is possible, since every evolution algebra admits at least one nonzero proper subalgebra.

Characterising complete evolution algebras

We show that the two conjectures concerning complete evolution algebras formulated in [29], and stated below for completeness, are correct.

Theorem 1.4.4 [29, Conjecture 5.2]**.** *Let $\mathcal{E}$ be a regular evolution algebra of dimension greater than one. Then, $\mathcal{E}$ is not complete.*

Theorem 1.4.5 [29, Conjecture 5.3]. *Let $\mathcal{E}$ be an n-dimensional nonnilpotent complete evolution algebra. Then, $\mathcal{E}$ is isomorphic to one of the following pairwise nonisomorphic algebras:*

$$\{e_1^2 = e_1\} \oplus \mathbb{C}^{n-1} \quad \textit{or} \quad \{e_1^2 = e_1\} \oplus \widetilde{\mathcal{E}} \oplus \mathbb{C}^{n-s-1},$$

where $\widetilde{\mathcal{E}}$ is an s-dimensional evolution algebra with maximal index of nilpotency and $\mathbb{C}^k$ denotes the k-dimensional abelian evolution algebra over $\mathbb{C}$.

Outline of both proofs. The proofs of both results in [29] rely on the assumed validity of [29, Conjecture 5.1]. The arguments of both proofs require the existence of a solution $(x_1, \ldots, x_n)$ of the system (1.4.1) with all coordinates nonzero, which then yields a one-dimensional subalgebra $\mathsf{span}\{x_1e_1 + \cdots + x_ne_n\}$ that cannot be extended to a natural basis of the whole algebra.

However, we do not actually need all coordinates to be nonzero: having just two nonzero coordinates already suffices. To explain this, we recall the notion of *natural vector* introduced in [13], meaning a vector that can be extended to a natural basis of the whole algebra. As shown in [13, Theorem 2.4], if $u = x_1e_1 + \cdots + x_ne_n$ is an idempotent, $u^2 = u \neq 0$, then u is a natural vector if and only if $\operatorname{rank}\left(\{e_i^2 \colon x_i \neq 0\}\right) = 1$. In our situation, this condition holds only when exactly one x_i is nonzero. Note that the system (1.4.1) involves an invertible matrix, which comes from the structure matrix of a regular evolution algebra, and therefore $\{e_1^2, \ldots, e_n^2\}$ is linearly independent. Hence, any idempotent supported on at least two basis elements whose squares are linearly independent fails to be a natural vector, giving exactly the obstruction required in the proofs. □

Note that Theorems 1.4.4 and 1.4.5 together with [29, Theorem 4.2] entirely classify complete evolution algebras over the complex numbers.

Corollary 1.4.6. *Let $\mathcal{E}$ be an n-dimensional complete evolution algebra. Then, $\mathcal{E}$ is isomorphic to one of the following pairwise nonisomorphic algebras:*

$$\{e_1^2 = e_1\}, \quad \widetilde{\mathcal{E}} \oplus \mathbb{C}^{n-s}, \quad \{e_1^2 = e_1\} \oplus \mathbb{C}^{n-1} \quad \textit{or} \quad \{e_1^2 = e_1\} \oplus \widetilde{\mathcal{E}} \oplus \mathbb{C}^{n-s-1},$$

where $\widetilde{\mathcal{E}}$ is an s-dimensional evolution algebra with maximal index of nilpotency and $\mathbb{C}^k$ denotes the k-dimensional abelian evolution algebra over $\mathbb{C}$.

Idempotents

As stated in [82], the existence of idempotent elements in an arbitrary evolution algebra is still an open problem. Given an evolution algebra $\mathcal{E}$ with natural basis

$\{e_1, \ldots, e_n\}$ and structure matrix $M_B(\mathcal{E}) = (\omega_{ij})$, an element $u = x_1e_1 + \cdots + x_ne_n$ is an idempotent if we have

$$u^2 = \sum_{i=1}^{n} x_i^2 e_i^2 = \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} \omega_{ij} e_j = \sum_{i=1}^{n} x_i e_i = u,$$

which yields the condition $\sum_{j=1}^{n} x_j^2 \omega_{ji} = x_i$ for all $i = 1, \ldots, n$, or, equivalently,

$$M_B(\mathcal{E})^t \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{1.4.2}$$

Note that (1.4.2) is equivalent to (1.3.1) when $M_B(\mathcal{E})$ is invertible, i.e., when $\mathcal{E}$ is regular, and in this case, idempotents correspond exactly to one-dimensional subalgebras. Thus, we have the following immediate consequence of Theorem 1.4.1.

Theorem 1.4.7. *Every complex regular evolution algebra admits an idempotent.*

We conclude this section by proposing a new conjecture concerning evolution algebras. For this, observe first that the absence of idempotent elements is a necessary condition for an evolution algebra to be solvable. Indeed, if an idempotent $u \in \mathcal{E}$ exists, then $u \in \mathcal{E}^{(n)}$ for all $n \in \mathbb{N}$, which contradicts the assumption that $\mathcal{E}$ is solvable.

Conjecture 1.4.8. *Let $\mathcal{E}$ be a complex evolution algebra. Then, the following assertions are equivalent:*

(i) *$\mathcal{E}$ is solvable;*

(ii) *$\mathcal{E}$ admits no idempotents; and*

(iii) *the system* (1.4.2) *only admits the trivial solution.*

Let us provide some evidence in support of Conjecture 1.4.8.

- If $n = 1$, then the conjecture is obviously true.
- If $n = 2$, we may rely on the classification given in Table 1.1.1. If $\mathcal{E}$ is regular, Theorem 1.4.7 already guarantees the existence of an idempotent. Hence, it remains to check that every nonregular and nonsolvable isomorphism class also contains an idempotent. According to the classification, such algebras are

$$\mathcal{E}_1 \colon e_1^2 = e_1, e_2^2 = 0 \quad \text{and} \quad \mathcal{E}_2 \colon e_1^2 = e_2^2 = e_1,$$

and both clearly contain the idempotent e_1. Thus, Conjecture 1.4.8 holds for $n = 2$.

If true, this conjecture would provide a structural characterization of solvable evolution algebras, which, to the best of our knowledge, does not yet exist.

1.5 Supersolvable evolution algebras

Throughout our studies, we will naturally find supersolvable evolution algebras. An evolution algebra $\mathcal{E}$ is said to be *supersolvable* if there exists a complete flag made up of ideals, that is, there exists a chain of ideals $0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = \mathcal{E}$ where $\dim I_i = i$ for all $i = 0, \ldots, n$. For example, notice that every nilpotent evolution algebra is supersolvable. If we consider, without loss of generality, that its structure matrix is strictly upper triangular, then

$$0 \subsetneq \mathsf{span}\{e_n\} \subsetneq \mathsf{span}\{e_{n-1,}e_n\} \subsetneq \cdots \subsetneq \mathsf{span}\{e_2, \ldots, e_n\} \subsetneq \mathcal{E}$$

is trivially a complete flag made up of ideals.

Remark 1.5.1. Unlike the case of groups or Lie algebras, there exist supersolvable evolution algebras which are not solvable. For instance, consider the regular evolution algebra $\mathcal{E}$ with natural basis $\{e_1, \ldots, e_n\}$ and product given by $e_i^2 = e_i$ for any $i = 1, \ldots, n$.

Remark 1.5.2. Let $\mathcal{E}$ be a supersolvable evolution algebra, $U \subseteq \mathcal{E}$ a subalgebra and $I \subseteq \mathcal{E}$ an ideal. Then, U and $\mathcal{E}/I$ are supersolvable. Since $\mathcal{E}$ is supersolvable, there exits a complete flag made up of ideals of $\mathcal{E}$, say $0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = \mathcal{E}$. Then, it is easy to check that $\{I_i \cap U\}_{i=1}^n$ and $\{(I_i + I)/I\}_{i=1}^n$ (keeping only the different terms) give rise to complete flag of ideals of U and $\mathcal{E}/I$, respectively.

Remark 1.5.3. If an evolution algebra $\mathcal{E}$ is supersolvable, then every maximal subalgebra of $\mathcal{E}$ has codimension one. This result for Lie algebras can be found in [9, Theorem 7], but the proof works in general. Let M be a maximal subalgebra of $\mathcal{E}$ and I a one-dimensional ideal of $\mathcal{E}$. We proceed by induction on $\dim \mathcal{E}$. If $I \subseteq M$, then M/I is a maximal subalgebra of $\mathcal{E}/I$ with codimension one; hence $\operatorname{codim} M = 1$. If instead, $I \nsubseteq M$, then $\mathcal{E} = M + I$, $M \cap I = 0$, which implies again that $\operatorname{codim} M = \dim I = 1$.

Example 1.5.4. In addition to nilpotent evolution algebras, it is easy to check that evolution algebras in $\mathcal{T}_{\mathbb{K}}$ and those which have a structure matrix like (1.2.2) are also supersolvable.

As shown in Remark 1.2.2, every solvable evolution algebra admits a complete flag of subalgebras. However, this fact does not extend to the case of ideals, that is, not every solvable evolution algebra is supersolvable. Hence, we now characterise supersolvability within solvable evolution algebras by using the family $\mathcal{T}_{\mathbb{K}}$.

Proposition 1.5.5. *Let $\mathcal{E}$ be an evolution algebra over $\mathbb{K}$. Every one-dimensional abelian ideal is either spanned by an element of the annihilator or is the derived subalgebra of a basic ideal isomorphic to an evolution algebra in $\mathcal{T}_{\mathbb{K}}$.*

Proof. Let I be a one-dimensional abelian ideal of $\mathcal{E}$, spanned by an element $u = \sum_{i=1}^{n} \mu_i e_i$. If $u \notin \operatorname{ann}(\mathcal{E})$, there exists at least one index $k \in \operatorname{supp}(u)$ such that $e_k^2 \neq 0$. Since I is a one-dimensional abelian ideal, it follows that e_i^2 is collinear with u for all $i \in \operatorname{supp}(u)$, and that $u^2 = (\sum_{i=1}^{n} \mu_i e_i)^2 = 0$. Now, define $J = \operatorname{span}\{e_i \colon i \in \operatorname{supp}(u)\}$. Clearly, J is a solvable but nonnilpotent basic ideal of $\mathcal{E}$ with $J^2 = I$, so $J \in \mathcal{T}_{\mathbb{K}}$, completing the proof. □

Theorem 1.5.6. *Let $\mathcal{E}$ be a solvable evolution algebra over a field $\mathbb{K}$. Then, $\mathcal{E}$ is supersolvable if and only if every quotient by an ideal is degenerate or has a basic ideal isomorphic to an evolution algebra of $\mathcal{T}_{\mathbb{K}}$.*

Proof. Since supersolvability is closed under quotients (Remark 1.5.2) and all one-dimensional ideals of a solvable algebra are abelian (Remark 1.2.1), the result follows straightforwardly from Proposition 1.5.5. □

Remark 1.5.7. We cannot remove the condition of the ideal isomorphic to an evolution algebra of $\mathcal{T}_{\mathbb{K}}$ being basic. Indeed, let $\mathcal{E}$ be an evolution algebra with natural basis $\{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_1 + e_2$, $e_2^2 = e_3$ and $e_3^2 = -e_1 - e_2 - e_3$. Notice that $I = \operatorname{span}\{f_1 = e_1 + e_2, f_2 = e_3\}$ is an evolution ideal such that $f_1^2 = f_1 + f_2$ and $f_2^2 = -f_1 - f_2$, then $I \cong \mathcal{E}_2(1, -1)$. Nevertheless, $\mathcal{E}$ does not have one-dimensional ideals, so it is not supersolvable.

1.5.1 $\mathcal{E}$-supersolvable ideals

Note that, in general, it is not true that if an ideal $I \subseteq \mathcal{E}$ and the quotient algebra $\mathcal{E}/I$ are both supersolvable, then $\mathcal{E}$ must also be supersolvable. For instance, consider the example given in Remark 1.5.7. The ideal $I = \operatorname{span}\{e_1 + e_2, e_3\}$ is supersolvable since $0 \subsetneq \operatorname{span}\{e_1 + e_2 + e_3\} \subsetneq I$ is a complete flag of ideals of I, and $\mathcal{E}/I$ is trivially supersolvable. However, as stated before, $\mathcal{E}$ itself is not supersolvable since it has no one-dimensional ideals. This motivates the introduction of $\mathcal{E}$-supersolvable ideals, which are actually inspired by the notion of G-supersolvable normal subgroups.

Definition 1.5.8. Let $\mathcal{E}$ be an evolution algebra and $I \subseteq \mathcal{E}$ an ideal. We say that I is an *$\mathcal{E}$-supersolvable ideal* if it admits a complete flag made up of ideals of $\mathcal{E}$. That is, there exists a chain

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = I$$

such that each I_i is an ideal of $\mathcal{E}$ and $\dim I_i = i$ for every $0 \leq i \leq r$.

The notion of $\mathcal{E}$-supersolvability enjoys several properties, which mostly follow directly from its definition, using arguments analogous to those in Remark 1.5.2. Let $\mathcal{E}$ be an evolution algebra and let I, J be ideals of $\mathcal{E}$ such that $J \subset I$. Then, the following assertions hold:

1. If $\mathcal{E}$ is supersolvable, then every ideal of $\mathcal{E}$ is clearly $\mathcal{E}$-supersolvable.
2. If I is $\mathcal{E}$-supersolvable, then J is $\mathcal{E}$-supersolvable and I/J is $\mathcal{E}/J$-supersolvable.
3. If J is $\mathcal{E}$-supersolvable and I/J is $\mathcal{E}/J$-supersolvable, then I is $\mathcal{E}$-supersolvable.
4. The sum of $\mathcal{E}$-supersolvable ideals is again $\mathcal{E}$-supersolvable.
5. If I is basic, then I is supersolvable if and only if it is $\mathcal{E}$-supersolvable.

We conclude this chapter by observing that the obstruction described at the beginning of this subsection disappears when working with the notion of $\mathcal{E}$-supersolvability. In fact, as a consequence of the third property, if I is an $\mathcal{E}$-supersolvable ideal and $\mathcal{E}/I$ is supersolvable, then $\mathcal{E}$ itself is supersolvable.

Chapter 2

On the subalgebra lattice of evolution algebras

Lattice theories have been developed in several algebraic structures by mainly working with two classical lattice-theoretic properties defined by identities: *distributivity* and *modularity*. In particular, the relationship between the properties of a group and the structure of its subgroup lattice has been deeply studied and found interesting in group theory (see [96]). For instance, Ore's Theorem establishes that the subgroup lattice of a group G is distributive if and only if G is locally cyclic. Moreover, it is also well known that the lattice of normal subgroups (and consequently the subgroup lattice of an abelian group) is modular. Both results highlight such a strong connection and make the case for groups especially representative.

Along the same line, the lattice of right ideals of a ring (see [14]) and the lattice of submodules of a module (see [30]) have also been investigated. Moreover, the lattice of ideals of an associative algebra has been addressed in [60] due to its importance in representation theory. Last but not least, this kind of studies has also been carried out in nonassociative structures such as Lie (see [52, 64]) or Leibniz (see [98]) algebras, and even in the context of restricted Lie algebras (see [76, 85]). However, this relationship is not well known in genetic algebras (see [79]) and is completely unsettled in the particular case of evolution algebras.

Hence, the motivation for this chapter is to establish such relations between evolution algebras and *lattice theory*, particularly to investigate how far the properties of an evolution algebra determine those of its subalgebra lattice, and vice versa. Mainly, we focus on solvable evolution algebras, in which distributivity and modularity are studied using, inter alia, the notions of quasi-ideal, supersolvability, and the terms of the derived series.

The text is structured into four sections. After this brief introduction, Section 2.1 is devoted to reviewing the lattice-theoretical framework required for the development of the chapter. We first present the necessary concepts from lattice theory, focusing mainly on the definitions of and relations between distributivity, modularity, and upper and lower semimodularity. Since the subalgebras of an algebra, ordered by inclusion, form a lattice, we then particularise these notions to this setting. Although already considered in other contexts, we highlight the equivalence between modularity and the property that every subalgebra is a quasi-ideal (Proposition 2.1.5), which will play a key role in our study.

Sections 2.2 and 2.3 are devoted to the study of the subalgebra lattice of solvable evolution algebras, illustrated with several examples that capture most of the obtained results. Section 2.2 focuses on the nilpotent case. Since the structure matrix of a nilpotent evolution algebra is strictly triangular, we characterise distributivity in terms of its index of nilpotency (see Theorem 2.2.1). Moreover, we deduce a necessary condition for modularity (see Proposition 2.2.4), which becomes a characterisation under an easily checkable additional hypothesis related to the annihilator of the algebra (see Corollary 2.2.9). Section 2.3, in turn, addresses solvable but not nilpotent evolution algebras. In particular, we characterise distributivity and modularity within the family $\mathcal{T}_{\mathbb{K}}$ (see Corollary 2.3.2) and subsequently determine the modularity of solvable evolution algebras with maximum index of solvability by computing the terms of the derived series and using supersolvability (see Proposition 2.3.7, Theorem 2.3.11 and Corollary 2.3.12). It is worth noting that these two sections have given rise to the paper [68].

Recall that, as stated in Remark 1.5.1, there exist supersolvable evolution algebras that are not solvable and, in particular, are regular. Consequently, Section 2.4 is devoted to studying the subalgebra lattice of such algebras over algebraically closed fields. In fact, we provide a complete characterisation of when these lattices are modular (see Theorem 2.4.5).

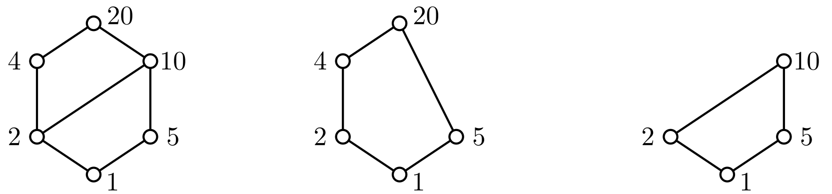
2.1 Preliminaries on lattice theory

Let $(X, \leq)$ be a partially ordered set. For a subset $S \subseteq X$, an element $x \in X$ is called an *upper bound* of S if $y \leq x$ for all $y \in S$. If there exists a least element among the upper bounds of S, it is called the *supremum* (or *least upper bound*) of S. Dually, an element $x \in X$ is a *lower bound* of S if $x \leq y$ for all $y \in S$, and the greatest among the lower bounds, when it exists, is called the *infimum* (or *greatest lower bound*) of S. Note that both the supremum and the infimum, whenever they exist, are unique.

A partially ordered set $\mathcal{L}$ is called a *lattice* if every pair of elements admits both a supremum and an infimum. For lattices, given two elements $x, y \in \mathcal{L}$, their supremum is usually denoted by $x \vee y$ (the *join*) and their infimum by $x \wedge y$ (the *meet*). Thus, a partially ordered set $\mathcal{L}$ is a lattice if and only if $x \vee y$ and $x \wedge y$ exist for all $x, y \in \mathcal{L}$. Moreover, a nonempty subset $S \subseteq \mathcal{L}$ is a *sublattice* of $\mathcal{L}$ if it is closed under both operations, i.e., $x \vee y, x \wedge y \in S$ for all $x, y \in S$.

Lattices can be conveniently visualised through *Hasse diagrams*, which provide a graphical representation of the partial order. Recall that, given two elements x, y in a lattice $\mathcal{L}$, we say that y *covers* x (or that x is *covered* by y) if $x < y$ and there is no $z \in \mathcal{L}$ such that $x < z < y$, and it is denoted by $x \prec y$ or $y \succ x$. The Hasse diagram of $\mathcal{L}$ is then obtained by representing each element as a small circle and drawing a straight line between x and y whenever y covers x, placing greater elements above smaller ones.

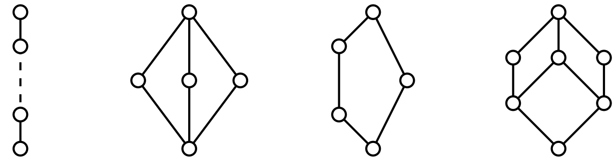
Example 2.1.1. A classic example of a lattice is given by the set of positive divisors of a fixed integer, ordered by divisibility. In this lattice, the supremum of two elements is their least common multiple, while the infimum is their greatest common divisor. For instance, consider the lattice $(D_{20}, |)$, where $D_{20} = \{1, 2, 4, 5, 10, 20\}$ is the set of divisors of 20 and $|$ denotes the relation "divides". We now consider the following Hasse diagrams:



$(D_{20}, |)$ $(A = \{1, 2, 4, 5, 20\}, |)$ $(B = \{1, 2, 5, 10\}, |)$

This example clearly illustrates the concept of a sublattice. Indeed, $(B, |)$ is a sublattice of $(D_{20}, |)$, whereas $(A, |)$ is not, since $5 \vee 2 = 10 \notin A$.

A lattice $\mathcal{L}$ is called *distributive* if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (or equivalently, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) for all $x, y, z \in \mathcal{L}$. Similarly, $\mathcal{L}$ is said to be modular if $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in \mathcal{L}$ with $x \leq z$. Moreover, as shown in [54, Lemma 1.4.12], this condition is equivalent to requiring that $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$, so modularity can also be defined by an identity.



A_n M_3 N_5 D_2

Figure 2.1.1: Some important lattices

Regarding Figure 2.1.1, it is easy to check that a chain A_n is both distributive and modular; the diamond M_3 is modular but not distributive; while the pentagon N_5 is neither distributive nor modular. In fact, the diamond and the pentagon play a key role characterising these two properties. By a well-known theorem of Birkhoff (see [54, Theorem 2.1.1]), a lattice is distributive if and only if it does not contain a diamond or a pentagon as a sublattice. Similarly, Dedekind characterised modular lattices as those that do not contain a pentagon as a sublattice (see [54, Theorem 2.1.2]). Consequently, every distributive lattice is also modular, but the converse does not necessarily hold.

Remark 2.1.2. Since both properties are defined by identities, sublattices of distributive and modular lattices are also distributive and modular, respectively.

A lattice $\mathcal{L}$ is said to be *semimodular* (or *upper semimodular*) if $x \wedge y \prec x$ implies $y \prec x \vee y$ for all $x, y \in \mathcal{L}$. Dually, a lattice $\mathcal{L}$ is said to be *lower semimodular* if $y \prec x \vee y$ implies $x \wedge y \prec x$ for all $x, y \in \mathcal{L}$. Note that modular lattices are both upper and lower semimodular. However, the centred hexagon D_2 of Figure 2.1.1 is semimodular but not modular.

The classical *Jordan-Hölder Chain Condition* (see [54, Theorem 4.2.1]) states that if $\mathcal{L}$ is a semimodular lattice of finite length (that is, every maximal chain has finite length), then any two maximal chains have the same length. Recall that, given two elements $x, y \in \mathcal{L}$, $x < y$, the set $[x, y] := \{z \in \mathcal{L} \colon x \leq z \leq y\}$ endowed with the inherited partial order, is a sublattice of $\mathcal{L}$. Thus, the result can be reformulated in terms of maximal chains between any two elements $x < y$ of $\mathcal{L}$. This allows us to define a *height function* δ on $\mathcal{L}$ by letting $\delta(x)$ be the length of a maximal chain from 0 to x:

$$\delta(x) = n \quad \text{if} \quad 0 = a_0 \prec a_1 \prec \cdots \prec a_n = x.$$

Note that, in the semimodular case, δ is well defined. An important property of the height function in semimodular lattices (see [54, Theorem 4.2.2]), which will play a key role in what follows, is that for all $x, y \in \mathcal{L}$,

$$\delta(x \vee y) + \delta(x \wedge y) \leq \delta(x) + \delta(y). \tag{2.1.1}$$

2.1.1 Already in the framework of evolution algebras

The set of subalgebras of an algebra, ordered by inclusion, forms a lattice where the supremum of two subalgebras is the subalgebra generated by the union, and the infimum is their intersection. In this chapter, we focus on the subalgebra lattice of evolution algebras; thus, from now on, we use the notation $\vee = \langle \cdot \rangle$ and $\wedge = \cap$.

Remark 2.1.3. It is worth noting that, throughout this work, we only consider subalgebra lattices in the standard sense, namely lattices whose elements are subspaces closed under the algebra product. We do not consider, for instance, lattices of evolution subalgebras. A drawback of working with evolution subalgebras, rather than with subalgebras in the usual sense, is that this class is not closed under intersection. Let $\mathcal{E}$ be the evolution algebra with natural basis $B = \{e_1, e_2, e_3, e_4\}$ and multiplication given by $e_1^2 = -e_2^2 = e_1 + e_2$, $e_3^2 = -e_2 + e_3$, and $e_4^2 = 0$. It is easy to check that $U = \mathsf{span}\{e_1, e_2, e_3\}$ and $V = \mathsf{span}\{e_1 + e_4, e_2 - e_4, e_3 - e_4\}$ are two distinct evolution subalgebras. However, $U \cap V = \mathsf{span}\{e_1 + e_2, e_1 + e_3\}$ does not admit a natural basis (see [19, Example 1.4.1]), and thus $U \cap V$ is not an evolution subalgebra.

We will say that an evolution algebra $\mathcal{E}$ is distributive, modular, semimodular or lower semimodular if its subalgebra lattice has the corresponding property. For the reader's convenience, we recall all these definitions in the context of subalgebras. We say that $\mathcal{E}$ is *distributive* if, for all subalgebras U, V, W of $\mathcal{E}$, one has $\langle U, V \cap W\rangle = \langle U, V\rangle \cap \langle U, W\rangle$, and that $\mathcal{E}$ is *modular* if $\langle U, V \cap W\rangle = \langle U, V\rangle \cap W$ for all subalgebras U, V, W of $\mathcal{E}$ with $U \subseteq W$. Note that in the setting of subalgebras, the fact that U covers V means that V is a maximal subalgebra of U. Hence, using the notions used in [103], we say that a subalgebra U of $\mathcal{E}$ is *upper semimodular* if, whenever $U \cap V$ is maximal in V, then U is maximal in $\langle U, V\rangle$; and that U is *lower semimodular* in $\mathcal{E}$ if, whenever V is maximal in $\langle U, V\rangle$, then $U \cap V$ is maximal in U. Note that the algebra $\mathcal{E}$ is *upper semimodular* (resp. *lower semimodular*) if all of its subalgebras are upper semimodular (resp. lower semimodular) in $\mathcal{E}$.

Remark 2.1.4. Let U be a subalgebra of an evolution algebra $\mathcal{E}$, and let I be an ideal of $\mathcal{E}$ contained in U. Then, U is upper semimodular (resp. lower semimodular) in $\mathcal{E}$ if and only if U/I is upper semimodular (resp. lower semimodular) in $\mathcal{E}/I$ (see [103, Lemmas 1.2 & 2.2]). Consequently, if $\mathcal{E}$ is upper semimodular (resp. lower semimodular), then every quotient by an ideal enjoys the same property; that is, upper (and lower) semimodularity is preserved under quotients by ideals.

In order to study modularity in evolution algebras, we conclude this section by stating two results in the context of solvable evolution algebras. These results have already been established in other algebraic structures, and their proofs readily extend to the present setting; we include them here adapted to evolution algebras for completeness (see, for instance, [5, Theorem 2.2] and [98, Theorem 5.1]). Furthermore, recall that a subalgebra U of $\mathcal{E}$ is called a *quasi-ideal of* $\mathcal{E}$ if $\langle U, V\rangle = U + V$ for all subalgebras V of $\mathcal{E}$. Note that every ideal is a quasi-ideal, but as we will see on several occasions throughout this chapter, the converse does not necessarily hold.

Proposition 2.1.5 [5]. *Let $\mathcal{E}$ be a solvable evolution algebra. Then, the following assertions are equivalent:*

(i) *$\mathcal{E}$ is modular;*

(ii) *$\mathcal{E}$ is upper semimodular; and*

(iii) *every subalgebra is a quasi-ideal.*

Proof.
(i)⇒(ii): Straightforward.
(ii)⇒(iii): Let U and V be two subalgebras of $\mathcal{E}$. So, we must prove that $\langle U, V\rangle = U + V$. Since $\mathcal{E}$ is upper semimodular, by (2.1.1) we have $\delta(\langle U, V\rangle) + \delta(U \cap V) \leq \delta(U) + \delta(V)$. By Remark 1.2.2, we can put dimensions instead of heights to obtain

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V) \geqslant \dim(\langle U, V\rangle).$$

Since $\dim(\langle U, V\rangle) \leqslant \dim(U+V)$ but, in general, $U+V \subseteq \langle U, V\rangle$, we can conclude that $\langle U, V\rangle = U + V$, as desired.
(iii)⇒(i): The proof is modelled on [85, Lemma 6.11]. Let U, V and W subalgebras of $\mathcal{E}$ such that $U \subseteq W$. Take $w \in \langle U, V\rangle \cap W = (U + V) \cap W$, and write $w = u + v$ for some $u \in U$ and $v \in V$. Then, $u \in U \subseteq W$, yielding that $v \in V \cap W$. Therefore, $w \in U + (V \cap W) = \langle U, V \cap W\rangle$. Then, $\langle U, V\rangle \cap W \subseteq \langle U, V \cap W\rangle$. The other inclusion is analogous. Thus, $\mathcal{E}$ is modular. □

Proposition 2.1.6 [98]**.** *Let $\mathcal{E}$ be a solvable evolution algebra. If $\mathcal{E}$ is supersolvable, then it is lower semimodular.*

Proof. Let U and V be subalgebras of $\mathcal{E}$ such that V is maximal in $\langle U, V\rangle$. Since $\mathcal{E}$ is supersolvable, by Remarks 1.5.2 and 1.5.3 (which hold for nonassociative algebras in general), it holds that V has codimension one in $\langle U, V\rangle$, so $\langle U, V\rangle = U + V$. Therefore,

$$\dim U - \dim(U \cap V) = \dim(\langle U, V\rangle) - \dim V = 1,$$

whence $U \cap V$ is maximal in U. □

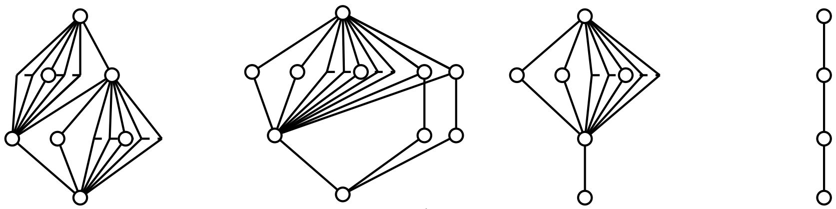
2.2 The subalgebra lattice of nilpotent evolution algebras

As the starting point for our analysis of distributivity and modularity, we describe the subalgebra lattices of all three-dimensional nilpotent evolution algebras both over $\mathbb{C}$ and $\mathbb{R}$, as classified in [57, Theorem 5.3]. In particular, Table 2.2.1 lists all the proper subalgebras of each isomorphism class, while Figure 2.2.1 shows the corresponding Hasse diagrams.

The abelian case is omitted, since every subspace is then a subalgebra and the lattice is trivially distributive and modular. We write $\mathcal{E}_{3,i}(\mathbb{C})$ or $\mathcal{E}_{3,i}(\mathbb{R})$ to indicate whether the multiplications are taken over $\mathbb{C}$ or $\mathbb{R}$. Note in particular that $\mathcal{E}_{3,3}^{-1}$ is considered only over $\mathbb{R}$, as over $\mathbb{C}$ one has $\mathcal{E}_{3,3}(\mathbb{C}) \cong \mathcal{E}_{3,3}^{-1}$.

$\mathcal{E}$	**Product**	**Field**	**Subalg. of dim.** 1	**Subalg. of dim.** 2
$\mathcal{E}_{3,2}$	$e_1^2 = e_2$ $e_2^2 = 0$ $e_3^2 = 0$	$\mathbb{C}$	$\mathsf{span}\{e_2\}$ $\mathsf{span}\{e_3\}$ $\mathsf{span}\{e_2 + \alpha e_3\}, \alpha \in \mathbb{C}^*$	$\mathsf{span}\{e_1 + \alpha e_3, e_2\}, \alpha \in \mathbb{C}$ $\mathsf{span}\{e_2, e_3\}$
		$\mathbb{R}$	$\mathsf{span}\{e_2\}$ $\mathsf{span}\{e_3\}$ $\mathsf{span}\{e_2 + \alpha e_3\}, \alpha \in \mathbb{R}^*$	$\mathsf{span}\{e_1 + \alpha e_3, e_2\}, \alpha \in \mathbb{R}$ $\mathsf{span}\{e_2, e_3\}$
$\mathcal{E}_{3,3}$	$e_1^2 = e_3$ $e_2^2 = e_3$ $e_3^2 = 0$	$\mathbb{C}$	$\mathsf{span}\{e_3\}$ $\mathsf{span}\{e_1 + ie_2\}$ $\mathsf{span}\{e_1 - ie_2\}$	$\mathsf{span}\{e_1, e_3\}$ $\mathsf{span}\{e_2, e_3\}$ $\mathsf{span}\{e_1 + \alpha e_2, e_3\}, \alpha \in \mathbb{C}^* \backslash \{\pm i\}$ $\mathsf{span}\{e_1 + ie_2, e_3\}$ $\mathsf{span}\{e_1 - ie_2, e_3\}$
		$\mathbb{R}$	$\mathsf{span}\{e_3\}$	$\mathsf{span}\{e_1, e_3\}$ $\mathsf{span}\{e_2, e_3\}$ $\mathsf{span}\{e_1 + \alpha e_2, e_3\}, \alpha \in \mathbb{R}^*$
$\mathcal{E}_{3,3}^{-1}$	$e_1^2 = e_3$ $e_2^2 = -e_3$ $e_3^2 = 0$	$\mathbb{R}$	$\mathsf{span}\{e_3\}$ $\mathsf{span}\{e_1 + e_2\}$ $\mathsf{span}\{e_1 - e_2\}$	$\mathsf{span}\{e_1, e_3\}$ $\mathsf{span}\{e_2, e_3\}$ $\mathsf{span}\{e_1 + \alpha e_2, e_3\}, \alpha \in \mathbb{R}^* \backslash \{\pm 1\}$ $\mathsf{span}\{e_1 + e_2, e_3\}$ $\mathsf{span}\{e_1 - e_2, e_3\}$
$\mathcal{E}_{3,4}$	$e_1^2 = e_2$ $e_2^2 = e_3$ $e_3^2 = 0$	$\mathbb{C}$ and $\mathbb{R}$	$\mathsf{span}\{e_3\}$	$\mathsf{span}\{e_2, e_3\}$

Table 2.2.1: All proper subalgebras of three-dimensional nilpotent evolution algebras over $\mathbb{C}$ and $\mathbb{R}$



$\mathcal{E}_{3,2}(\mathbb{C})$ and $\mathcal{E}_{3,2}(\mathbb{R})$ $\quad$ $\mathcal{E}_{3,3}(\mathbb{C})$ and $\mathcal{E}_{3,3}^{-1}$ $\quad$ $\mathcal{E}_{3,3}(\mathbb{R})$ $\quad$ $\mathcal{E}_{3,4}(\mathbb{C})$ and $\mathcal{E}_{3,4}(\mathbb{R})$

Figure 2.2.1: Subalgebra lattices of three-dimensional nilpotent evolution algebras over $\mathbb{C}$ and $\mathbb{R}$

To analyse their distributivity and modularity, we examine the presence of sublattices isomorphic to the diamond and the pentagon. From the Hasse diagrams, one can see that $\mathcal{E}_{3,4}(\mathbb{C})$ and $\mathcal{E}_{3,4}(\mathbb{R})$ contain neither diamonds nor pentagons, hence they are distributive. On the other hand, $\mathcal{E}_{3,2}(\mathbb{C})$, $\mathcal{E}_{3,2}(\mathbb{R})$ and $\mathcal{E}_{3,3}(\mathbb{R})$ contain diamonds but no pentagons, and therefore they are modular but not distributive. Finally, the lattices of $\mathcal{E}_{3,3}(\mathbb{C})$ and $\mathcal{E}_{3,3}^{-1}$ contain pentagons, which shows that they are not modular.

Although in the case of three-dimensional nilpotent evolution algebras distributivity and modularity can be characterised by explicitly computing all proper subalgebras and drawing their subalgebra lattices, this approach is not feasible in higher dimensions. We therefore establish some general results that will allow us to study these properties in higher dimensions, and even over other ground fields.

2.2.1 Distributivity

This subsection provides a characterisation of distributivity in terms of the index of nilpotency.

Theorem 2.2.1. *Let $\mathcal{E}$ be a nilpotent evolution algebra of dimension n over a field $\mathbb{K}$ with a natural basis $B = \{e_1, \ldots, e_n\}$ such that the structure matrix $M_B(\mathcal{E}) = (\omega_{ij})$ is strictly upper triangular. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ has maximum index of nilpotency;*

(ii) *its subalgebra lattice is a chain of length n;*

(iii) *$\mathcal{E}$ is distributive; and*

(iv) *$\mathcal{E}$ is spanned by the principal powers of an element; precisely, the support of such element contains $\{1, \ldots, n-1\}$.*

Proof.

(i)⇒(ii): If $\mathcal{E}$ is a nilpotent evolution algebra with maximum index of nilpotency, by Corollary 1.2.12, all subalgebras are $\mathsf{span}\{e_k, \ldots, e_n\}$, $k = 1, \ldots, n$. Consequently, its subalgebra lattice is a chain of length n.

(ii)⇒(iii): Simply note that chains are distributive lattices.

(iii)⇒(i): Suppose, for the sake of contradiction, that the structure matrix has a zero entry on its first upper diagonal, say $\omega_{k(k+1)}$. Since $\omega_{k(k+1)} = 0$, we have that $e_k^2 \in \mathsf{span}\{e_{k+2}, \ldots, e_n\}$. Therefore, the subspaces $U = \mathsf{span}\{e_k, e_{k+2}, \ldots, e_n\}$, $V = \mathsf{span}\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ and $W = \mathsf{span}\{e_k + e_{k+1}, e_{k+2}, \ldots, e_n\}$ are subalgebras. However, a direct computation shows

$$U = \langle U, V \cap W \rangle \neq \langle U, V \rangle \cap \langle U, W \rangle = \mathsf{span}\{e_k, \ldots, e_n\},$$

and thus its subalgebra lattice is not distributive, a contradiction.

(iv)⇔(i): Consider an arbitrary element $u = \sum_{i=1}^{n} \mu_i e_i \in \mathcal{E}$. Then, we have

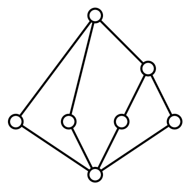
$$
\begin{aligned}
u^2 &= \mu_1^2 \omega_{12} e_2 + v_3, \text{ with } v_3 \in \mathsf{span}\{e_3, \dots, e_n\};\\
&\vdots\\
u^i &= \mu_1^2 \mu_2 \dots \mu_{i-1} \omega_{12}\omega_{23} \dots \omega_{(i-1)i} e_i + v_{i+1}, \text{ with } v_{i+1} \in \mathsf{span}\{e_{i+1}, \dots, e_n\};\\
&\vdots\\
u^n &= \mu_1^2 \mu_2 \dots \mu_{n-1} \omega_{12}\omega_{23} \dots \omega_{(n-1)n} e_n \in \mathsf{span}\{e_n\}.
\end{aligned}
$$

Consequently, $\mathcal{E} = \mathsf{span}\{u, u^2, \dots, u^n\}$ if and only if $\mu_i, \omega_{i(i+1)} \neq 0$ for all $i = 1, \dots, n-1$, what yields the claim. □

The following two examples illustrate how the hypothesis of nilpotency cannot be relaxed to solvability. Example 2.2.2 shows that solvable evolution algebras with maximum index of solvability are not necessarily distributive, while Example 2.2.3 shows that even when they are distributive, their subalgebra lattice is not necessarily a chain.

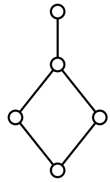
Example 2.2.2. Let $\mathcal{E}$ be an evolution algebra over a field $\mathbb{K}$ of characteristic not two with natural basis $\{e_1, e_2, e_3\}$ and multiplication given by $e_1^2 = 2e_1 + 2e_2 + 4e_3$, $e_2^2 = 2e_1 + 2e_2$, and $e_3^2 = -e_1^2 - e_2^2 = -4e_1 - 4e_2 - 4e_3$. In fact, it is solvable with maximum index of solvability since $\mathcal{E}^{(2)} = \mathsf{span}\{e_1 + e_2, e_3\}$, $\mathcal{E}^{(3)} = \mathsf{span}\{e_1 + e_2 + e_3\}$ and $\mathcal{E}^{(4)} = 0$. Moreover, applying Proposition 1.5.5 along with some routine computations, we obtain all its subalgebras. Consequently, the subalgebra lattice contains a diamond, and thus it is not distributive.

Subalg. of dim. 1	**Subalg. of dim.** 2
$\mathsf{span}\{e_1 + e_2 + e_3\}$	$\mathsf{span}\{e_1 + e_2, e_3\}$
$\mathsf{span}\{e_1 - e_2 + e_3\}$	
$\mathsf{span}\{e_1 + e_2 - e_3\}$	
$\mathsf{span}\{e_1 - e_2 - e_3\}$	



Example 2.2.3. Let $\mathcal{E}$ be an evolution algebra over a field $\mathbb{K}$ of characteristic not two with natural basis $\{e_1, e_2, e_3\}$ and multiplication given by $e_1^2 = -e_2^2 = e_1 + e_2$ and $e_3^2 = e_2$. In fact, it is solvable with maximum index of solvability since $\mathcal{E}^{(2)} = \mathsf{span}\{e_1, e_2\}$, $\mathcal{E}^{(3)} = \mathsf{span}\{e_1 + e_2\}$ and $\mathcal{E}^{(4)} = 0$. Again, it is easy to show that its subalgebras are the following, and although its subalgebra lattice is distributive, it is not a chain.

Subalg. of dim. 1	Subalg. of dim. 2
$\mathsf{span}\{e_1+e_2\}$	$\mathsf{span}\{e_1,e_2\}$
$\mathsf{span}\{e_1-e_2\}$	



2.2.2 Modularity

Now, we use the notion of quasi-ideal and apply Theorem 2.1.5 to study modularity in nilpotent evolution algebras. We begin by establishing a necessary condition, and subsequently obtain a characterisation over quadratically closed fields.

Proposition 2.2.4. *Let $\mathcal{E}$ be a nilpotent evolution algebra over a field $\mathbb{K}$ of characteristic not two. If $\mathcal{E}$ is modular, then there are no absolute nilpotent elements outside the annihilator.*

Proof. Consider a natural basis $B=\{e_1,\dots,e_n\}$ of $\mathcal{E}$ such that the structure matrix is strictly upper triangular. Suppose, for the sake of contradiction, that an absolute nilpotent element outside the annihilator exists. Note that we can assume, without loss of generality, that such an element u verifies that $\operatorname{supp}(u)\cap\operatorname{supp}(\operatorname{ann}(\mathcal{E}))=\emptyset$. Then, take $u=\mu_1e_1+\cdots+\mu_ke_k$ with $\mu_k\neq 0$ and $e_1,\dots,e_k\notin\operatorname{ann}(\mathcal{E})$, such that $u^2=0$. In fact, since u is not an element of the annihilator, at least two scalars μ_i must be nonzero. Moreover, the element $v=\mu_1e_1+\cdots-\mu_ke_k$ is also absolute nilpotent, since $v^2=u^2=0$. Then, $U=\mathsf{span}\{u\}$ and $V=\mathsf{span}\{v\}$ are one-dimensional subalgebras. However, we now show that U and V are not quasi-ideals. First, we claim that $uv\neq 0$. Suppose that $uv=0$, then

$$0=u^2-uv=(\mu_1^2e_1^2+\cdots+\mu_k^2e_k^2)-(\mu_1^2e_1^2+\cdots-\mu_k^2e_k^2)=2\mu_k^2e_k^2.$$

Since the characteristic of the field is not two and $\mu_k\neq 0$ we conclude that $e_k^2=0$, a contradiction with $e_k\notin\operatorname{ann}(\mathcal{E})$. Next, we claim that $\operatorname{supp}(uv)\cap\operatorname{supp}(u)=\operatorname{supp}(uv)\cap\operatorname{supp}(v)=\emptyset$. By hypothesis, we have that

$$\mu_1^2e_1^2+\cdots+\mu_{k-1}^2e_{k-1}^2=-\mu_k^2e_k^2\in\mathsf{span}\{e_l\mid l>k\}.$$

Therefore,

$$uv=(\mu_1^2e_1^2+\cdots+\mu_{k-1}^2e_{k-1}^2)-\mu_k^2e_k^2\in\mathsf{span}\{e_l\mid l>k\}.$$

Hence, since $\operatorname{supp}(u)=\operatorname{supp}(v)\subset\{1,\dots,k\}$ and $\operatorname{supp}(uv)\subset\{k+1,\dots,n\}$, their intersection is the empty set, and therefore $uv\notin\mathsf{span}\{u,v\}$. Then, we conclude that $\dim(\langle U,V\rangle)\geqslant 3$, so U and V are not quasi-ideals. By Proposition 2.1.5, we are done. □

Remark 2.2.5. The existence of absolute nilpotent elements depends on the ground field. For example, in the evolution algebra $\mathcal{E}_{3,3}(\mathbb{C})$, the elements $e_1 \pm ie_2$ are absolute nilpotents lying outside the annihilator. By contrast, in $\mathcal{E}_{3,3}(\mathbb{R})$ there are no absolute nilpotent elements outside the annihilator.

Remark 2.2.6. Assume, without loss of generality, that the multiplication of a nilpotent evolution algebra $\mathcal{E}$ with natural basis $B = \{e_1, \dots, e_n\}$ satisfies

$$e_i^2 \neq 0, \quad 1 \leq i \leq k, \qquad e_i^2 = 0, \quad k+1 \leq i \leq n; \tag{2.2.1}$$

with $k \leq n$. Over $\mathbb{C}$ (or, more generally, over a quadratically closed field), Proposition 2.4.3 can be equivalently formulated as follows: if $\mathcal{E}$ is modular then, necessarily, $\operatorname{rank}\big(M_B(\mathcal{E})\big) = k$. Indeed, $\operatorname{rank}\big(M_B(\mathcal{E})\big) < k$ if and only if there exists a nontrivial linear relation $\alpha_1 e_1^2 + \cdots + \alpha_k e_k^2 = 0$, which yields an absolute nilpotent element $\sqrt{\alpha_1}e_1 + \cdots + \sqrt{\alpha_k}e_k$.

In what follows we present two examples. The first one shows that the converse of Proposition 2.3.7 does not hold in general, whereas the second one shows that the result is not necessarily true over fields of characteristic two.

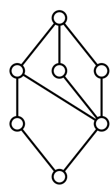
Example 2.2.7. Let $\mathcal{E}$ be a nilpotent evolution algebra over a field $\mathbb{K}$ of characteristic not two with natural basis $B = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and product given by $e_1^2 = e_4 + e_5 + e_6$, $e_2^2 = -e_5$, $e_3^2 = -e_6$ and $e_4^2 = e_5^2 = e_6^2 = 0$. Consider the subalgebras $U = \mathsf{span}\{e_1 + e_2, e_4 + e_6\}$ and $V = \mathsf{span}\{e_1 + e_3, e_4 + e_5\}$. Note that there are no absolute nilpotent elements outside the annihilator, since $\{e_1^2, e_2^3, e_3^2\}$ is a linearly independent subset. However,

$$\langle U, V\rangle = \mathsf{span}\{e_1 + e_2, e_1 + e_3, e_4, e_5, e_6\} \neq U + V.$$

Therefore, U and V are not quasi-ideals and, consequently, $\mathcal{E}$ is neither modular nor upper semimodular.

Example 2.2.8. Let $\mathcal{E}$ be the evolution algebra over the finite field with two elements, $\mathbb{Z}/2\mathbb{Z}$, with natural basis $B = \{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_2^2 = e_3$ and $e_3^2 = 0$. In fact, $e_1 + e_2$ is an absolute nilpotent element outside the annihilator. However, $\mathcal{E}$ is modular.

Subalg. of dim. 1	**Subalg. of dim.** 2
$\mathsf{span}\{e_3\}$	$\mathsf{span}\{e_1, e_3\}$
$\mathsf{span}\{e_1 + e_2\}$	$\mathsf{span}\{e_2, e_3\}$
	$\mathsf{span}\{e_1 + e_2, e_3\}$



Corollary 2.2.9. *Let $\mathcal{E}$ be a nilpotent evolution algebra over a quadratically closed field of characteristic different from two such that its annihilator is one-dimensional. Then, the following assertions are equivalent:*

(i) *$\mathcal{E}$ is distributive;*

(ii) *$\mathcal{E}$ is modular; and*

(iii) *$\mathcal{E}$ has maximum index of nilpotency.*

Proof. The implication (i)⇒(ii) follows from the fact that all distributive lattices are also modular; and (iii)⇒(i) is a straightforward consequence of Theorem 2.2.1. Therefore, it only remains to show that (ii)⇒(iii). Assume, by contrary, that $\mathcal{E}$ does not have maximum index of nilpotency. Hence, $\mathrm{rank}\,(M_B(\mathcal{E})) < n-1$ and, applying Remark 2.2.6, we get that $\mathcal{E}$ is not modular. □

The following example shows that the equivalence between distributivity and modularity given by Corollary 2.2.9 does not necessarily hold if the dimension of the annihilator is greater than one.

Example 2.2.10. The evolution algebra $\mathcal{E}_{3,2}(\mathbb{C})$ has a two-dimensional annihilator, $\mathrm{ann}(\mathcal{E}_{3,2}(\mathbb{C})) = \mathsf{span}\{e_2, e_3\}$. As illustrated in Figure 2.2.1, its subalgebra lattice is modular but not distributive.

Although not included in [68], we conclude this section by formulating the following conjecture, which would provide a characterization of modularity in the nilpotent setting over quadratically closed fields.

Conjecture 2.2.11. *Let $\mathcal{E}$ be a nilpotent evolution algebra of dimension n over a quadratically closed field $\mathbb{K}$. Then, the following statements are equivalent:*

(i) *$\mathcal{E}$ is modular;*

(ii) *$\mathcal{E}$ is isomorphic to $\widetilde{\mathcal{E}} \oplus \mathbb{K}^{n-k}$, where $\widetilde{\mathcal{E}}$ is a k-dimensional evolution algebra with maximum index of nilpotency and $\mathbb{K}^{n-k}$ denotes the $(n-k)$-dimensional zero evolution algebra; and*

If this conjecture holds, then over quadratically closed fields modular nilpotent evolution algebras would coincide precisely with complete nilpotent evolution algebras (see [29, Theorem 4.2]). This suggests a strong connection between modularity and completeness, and consequently between quasi-ideals and evolution subalgebras that admit a natural basis extendable to a natural basis of the whole algebra.

2.3 The subalgebra lattice of solvable evolution algebras

This section is devoted to the study of distributivity and modularity in solvable but nonnilpotent evolution algebras. Our first goal is to characterise these properties within the family of solvable nonnilpotent evolution algebras with one-dimensional derived subalgebra. As stated in Notation 1.2.3, all such algebras over a specific field $\mathbb{K}$ are denoted by $\mathcal{T}_{\mathbb{K}}$. For what follows, it is also important to keep in mind the definition of the evolution algebras $\mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$, given in (1.2.1).

Proposition 2.3.1. *Let $\mathcal{E} = \mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$ with $k \geq 2$ and $\lambda_1, \ldots, \lambda_k \neq 0$ over a field $\mathbb{K}$ of characteristic not two. If $k \geq 3$ or $\mathcal{E}$ is degenerate, then $\mathcal{E}$ is not modular. Moreover, if $\mathbb{K}$ is a quadratically closed field or $\mathbb{K} = \mathbb{R}$, when $k = 2$ and $\mathcal{E}$ is nondegenerate then $\mathcal{E}$ is neither modular.*

Proof. First, we show that if $k \geq 3$ then $\mathcal{E}$ is not modular. Consider the elements $u = -e_1 + e_2 + \cdots + e_k$ and $v = e_1 + \cdots + e_{k-1} - e_k$, and note that $U = \mathrm{span}\{u\}$ and $V = \mathrm{span}\{v\}$ are one-dimensional subalgebras since both u and v are absolute nilpotent. Indeed, $u^2 = v^2 = (e_1 + \cdots + e_k) \sum_{i=1}^{k} \lambda_i = 0$. Now, we claim that U is never a quasi-ideal. To prove it, we consider the following two cases:

1. If $uv \neq 0$, then $\langle U, V \rangle = \mathrm{span}\{u, v, e_1 + \cdots + e_k\} \neq U + V$. Consequently, U and V are not quasi-ideals.

2. If $uv = 0$, then we have that $u^2 - uv = 2(e_1^2 + e_k^2) = 0$, so $e_1 + e_k$ is an absolute nilpotent element and $\mathrm{span}\{e_1 + e_k\}$ is a subalgebra. Moreover, note that $u(e_1 + e_k) = -e_1^2 + e_k^2 \neq 0$; otherwise, since $e_1^2 + e_k^2 = 0$, we would have that $e_1, e_k \in \mathrm{ann}(\mathcal{E})$, a contradiction with $\lambda_1, \lambda_k \neq 0$. Consequently, we have that $\langle U, e_1 + e_k \rangle = \mathrm{span}\{u, e_1 + e_k, e_1 + \cdots + e_k\} \neq U + \mathrm{span}\{e_1 + e_k\}$, yielding that U is not a quasi-ideal.

Finally, we focus on the case $k = 2$. Note that, in this setting, $\mathrm{span}\{e_1, e_2\}$ is a subalgebra and we can suppose, without loss of generality, that $e_1^2 = -e_2^2 = e_1 + e_2$. If $\mathcal{E}$ is degenerate, then there exists an element of the natural basis, e_i with $i \geq 3$, such that its square is zero, say e_3. Then it is enough to consider the subalgebras $\mathrm{span}\{e_1 - e_2\}$ and $\mathrm{span}\{e_1 + e_2 + e_3\}$, which are not quasi-ideals since $\langle e_1 - e_2, e_1 + e_2 + e_3 \rangle = \mathrm{span}\{e_1, e_2, e_3\}$. So, it only remains to check the case when $\mathcal{E}$ is nondegenerate. As distinguished in the statement of the result, we consider the following two cases:

1. If $\mathbb{K}$ is quadratically closed of characteristic not two, simply consider the one-dimensional subalgebra $\mathrm{span}\{\sqrt{\lambda_3} e_2 + e_3\}$. In fact, it is not a quasi-ideal since $\langle \sqrt{\lambda_3} e_2 + e_3, e_1 - e_2 \rangle = \mathrm{span}\{e_1, e_2, e_3\}$.

2. If $\mathbb{K} = \mathbb{R}$, then it is easy to check that, depending on the sign of λ_3, either $\mathsf{span}\{\sqrt{|\lambda_3|}e_1 + e_3\}$ or $\mathsf{span}\{\sqrt{|\lambda_3|}e_2 + e_3\}$ is a subalgebra. In both cases, it is not a quasi-ideal since $\langle\sqrt{|\lambda_3|}e_i + e_3, e_1 - e_2\rangle = \mathsf{span}\{e_1, e_2, e_3\}$.

In conclusion, if $k = 2$, $\mathcal{E}$ is nondegenerate and $\mathbb{K}$ is as stated, there is always a subalgebra that is not a quasi-ideal. So, by Proposition 2.1.5, $\mathcal{E}$ is not modular neither in this case. □

Corollary 2.3.2. *Let $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ with $\mathbb{K}$ a quadratically closed field of characteristic not two or $\mathbb{K} = \mathbb{R}$. Then, it is distributive if and only if it is modular and if and only if $n = 2$, i.e. $\mathcal{E} \cong \mathcal{E}_2(1, -1)$.*

Proof. By Lemma 1.2.15, if $n = 2$ then $\mathcal{E} \cong \mathcal{E}_2(1, -1)$, whose subalgebras are $\mathsf{span}\{e_1 - e_2\}$ and $\mathsf{span}\{e_1 + e_2\}$ and give rise to a distributive lattice. The converse follows straightforwardly from Propositions 1.2.4 and 2.3.1. □

Corollary 2.3.3. *The direct sum of evolution algebras of $\mathcal{T}_{\mathbb{K}}$ with $\mathbb{K}$ a quadratically closed field of characteristic not two or $\mathbb{K} = \mathbb{R}$ is not modular.*

Proof. If any of the addends has dimension greater than two then, by Corollary 2.3.2, it is not modular. Therefore, the only possibility is that all the addends have dimension two. Nevertheless, we claim that it is not modular either. Let $\mathcal{E}_1 = \mathsf{span}\{e_1, e_2\}$ and $\mathcal{E}_2 = \mathsf{span}\{e_3, e_4\}$ be the first two addends of the direct sum. Again, by Lemma 1.2.15, their multiplication is given by $e_1^2 = -e_2^2 = e_1 + e_2$ and $e_3^2 = -e_4^2 = e_3 + e_4$, respectively. Finally, it is enough to consider the subalgebras $U = \mathsf{span}\{e_1 - e_2 + e_3 + e_4\}$ and $V = \mathsf{span}\{e_1 + e_2 + e_3 - e_4\}$ since

$$\begin{aligned}\langle U, V\rangle &= \mathsf{span}\{e_1 - e_2 + e_3 + e_4, e_1 + e_2 + e_3 - e_4, e_1 + e_2 + e_3 + e_4\}\\ &= \mathsf{span}\{e_1 + e_3, e_2, e_4\} \neq U + V,\end{aligned}$$

so they are not quasi-ideals. The result follows from Proposition 2.1.5. □

Finally, over quadratically closed fields, we generalise the previous two cases by characterising distributivity and modularity in the evolution algebras described in Definition 1.2.7.

Theorem 2.3.4. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_r \in \mathcal{T}_{\mathbb{K}}$ with $r \geq 1$ and $\mathbb{K}$ a quadratically closed field of characteristic not two, and $\mathcal{E}$ an evolution algebra of the family $\mathcal{F}(\mathcal{E}_1, \ldots, \mathcal{E}_r)$. Then, the following statements are equivalent:*

(i) *$\mathcal{E}$ is distributive;*

(ii) *$\mathcal{E}$ is modular and, consequently, upper semimodular;*

(iii) $r = 1$, $\mathcal{E}_1 \cong \mathcal{E}_2(1, -1)$ *and the quotient* $\mathcal{E}/\mathcal{E}_1^2$ *is a nilpotent evolution algebra with maximum index of nilpotency; and*

(iv) $\mathcal{E}$ *has maximum index of solvability.*

In this case, there exists a natural basis such that the structure matrix is

$$M_B(\mathcal{E}) = \begin{pmatrix} A_{2\times 2} & 0_{2\times(n-2)} \\ C_{(n-2)\times 2} & L_{(n-2)\times(n-2)} \end{pmatrix},$$

where $A = \left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, L *is strictly lower triangular with no zeros in its first lower diagonal and* $M_B(\mathcal{E})$ *has rank* $n-1$.

Proof.

(i)⇒(ii). Obvious.

(ii)⇒(iii). Suppose that the structure of $\mathcal{E}$ is not the one described in (iii). Firstly, if $\dim \mathcal{E}_1 > 2$ or $r > 1$, $\mathcal{E}$ is not modular by Corollaries 2.3.2 and 2.3.3, respectively. Secondly, we claim that $\mathcal{E}$ is not modular if $\mathcal{E}/\mathcal{E}_1^2$ does not have maximum index of nilpotency. We consider the following two cases:

1. If the annihilator of $\mathcal{E}/\mathcal{E}_1^2$ is one-dimensional, it is not upper semimodular (or modular) by Corollary 2.2.9. Then, there exists a subalgebra $U \subset \mathcal{E}/\mathcal{E}_1^2$ which is not upper semimodular. Denote by π the usual projection to the quotient. Since $\mathcal{E}_1^2$ is ideal, by [103, Lemma 1.2], $V = \pi^{-1}(U) \subset \mathcal{E}$ is not upper semimodular either. Then, $\mathcal{E}$ is not upper semimodular.

2. If the annihilator of $\mathcal{E}/\mathcal{E}_1^2$ has dimension greater than 1, then there exists an index $i \in \{3, \ldots, n\}$ such that $e_i^2 \in \mathrm{span}\{e_1 + e_2\} = \mathcal{E}_1^2$. Assume, without loss of generality, that $e_3^2 = \lambda(e_1 + e_2)$ with $\lambda \in \mathbb{K}$. Next, we show that $\mathrm{span}\{e_1 - e_2\}$ is not a quasi-ideal.

   (a) If $\lambda = 0$, then $\mathrm{span}\{e_1 + e_2 + e_3\}$ is a subalgebra. However, $\langle e_1 - e_2, e_1 + e_2 + e_3\rangle = \mathrm{span}\{e_1, e_2, e_3\}$.

   (b) If $\lambda \neq 0$, then $\mathrm{span}\{\sqrt{\lambda} e_2 + e_3\}$ is a subalgebra. However, $\langle e_1 - e_2, \sqrt{\lambda} e_2 + e_3\rangle = \mathrm{span}\{e_1, e_2, e_3\}$.

   Consequently, we are done by Proposition 2.1.5.

(iii)⇒(i). If $e_3^2 \notin \mathrm{span}\{e_1 - e_2\}$, following the argument used in Corollary 1.2.12, it is easy to prove that all its subalgebras are exactly $\mathrm{span}\{e_1 + e_2\}$, $\mathrm{span}\{e_1 - e_2\}$ and those of the form $\mathrm{span}\{e_1, \ldots, e_k\}$ with $k = 2, \ldots, n$. Consequently, its subalgebra lattice is almost a chain, actually, a chain with a rhombus at the bottom, which

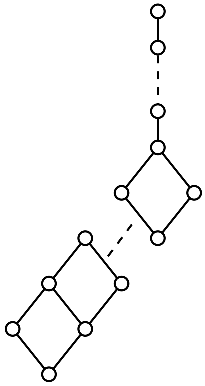


Figure 2.3.1: Helpful Hasse diagram for the proof of Theorem 2.3.4

is distributive. Nevertheless, if $e_3^2 \in \mathsf{span}\{e_1 - e_2\}$, there exist more subalgebras than in the previous case. In fact, $\mathsf{span}\{e_1 + e_2\}$, $\mathsf{span}\{e_1 - e_2\}$, $\mathsf{span}\{e_1, e_2\}$ and $\mathsf{span}\{e_1 - e_2, e_3\}$ are exactly the subalgebras contained in $\mathsf{span}\{e_1, e_2, e_3\}$. Analogously, if $e_4^2 \in \mathsf{span}\{e_1 - e_2, e_3\}$, then $\mathsf{span}\{e_1 - e_2, e_3, e_4\}$ and $\mathsf{span}\{e_1, e_2, e_3\}$ are (together with the previous five) the subalgebras contained in $\mathsf{span}\{e_1, e_2, e_3, e_4\}$. Following this argument, we deduce that its subalgebra lattice is a chain with a succession of rhombuses linked by an edge at the bottom as shown in Figure 2.3.1, which is also distributive.

(iii)⇔(iv). It is straightforward from Proposition 1.2.8.

□

2.3.1 The case of maximum index of solvability

As we have seen in Theorems 2.2.1 and 2.3.4, evolution algebras with maximum index of solvability have the best properties. We now turn our attention to this case, focusing on their modularity and its connection with supersolvability. Before proceeding, however, we first state some technical lemmas that will be used in the subsequent results.

Lemma 2.3.5. *Let $\mathcal{E}$ be a solvable evolution algebra with index of solvability n such that $\mathcal{E}^{(m)}$, with $m < n$, is an ideal. Then $\mathcal{E}/\mathcal{E}^{(m)}$ is an evolution algebra with index of solvability m.*

Proof. It is enough to realise that the terms of the derived series of $\mathcal{E}/\mathcal{E}^{(m)}$ are exactly

$$\left(\mathcal{E}/\mathcal{E}^{(m)}\right)^{(l)} = \mathcal{E}^{(l)}/\mathcal{E}^{(m)}, \quad l = 1, \ldots, m.$$

□

Lemma 2.3.6. *Let $\mathcal{E}$ be an n-dimensional solvable evolution algebra with maximum index of solvability over a field $\mathbb{K}$ of characteristic not two such that $e_n^2 = -e_1^2 - \cdots - e_m^2$, with $m < n$, and let $u \in \mathcal{E}$ be an absolute nilpotent element. Then,*

$$\operatorname{ann}_{\mathcal{E}}(u) := \{v \in \mathcal{E} \colon vu = 0\} = \mathsf{span}\{u, e_{m+1}, \ldots, e_{n-1}\}.$$

Proof. By Proposition 1.2.9 and Remark 1.2.10, every absolute nilpotent element is a multiple of $\pm e_1 \pm \cdots \pm e_m \pm e_n$. Then, if we consider an arbitrary element $v = \mu_1 e_1 + \cdots + \mu_n e_n$ we have that

$$\begin{aligned} v \in \operatorname{ann}_{\mathcal{E}}(u) \iff vu &= (\mu_1 e_1 + \cdots + \mu_n e_n)u \\ &= (\mu_1 e_1 + \cdots + \mu_m e_m + \mu_n e_n)u = 0 \\ \iff \mu_1 e_1 + \cdots &+ \mu_m e_m + \mu_n e_n \in \mathsf{span}\{u\} \text{ or } \mu_1, \ldots, \mu_m, \mu_n = 0, \end{aligned}$$

what yields the claim. Just notice that the second equivalence is a consequence of the fact that $e_n^2 + e_1^2 + \cdots + e_m^2 = 0$ (an its multiples, obviously) is the only linear dependence relationship in the set $\{e_1^2, \ldots, e_n^2\}$. □

Next, we establish a necessary but not sufficient condition for a solvable evolution algebra with maximum index of solvability to be modular.

Proposition 2.3.7. *Let $\mathcal{E}$ be an n-dimensional solvable evolution algebra with maximum index of solvability over a field of characteristic not two. If $\mathcal{E}$ is modular, then $\mathcal{E}^{(n)}$ or $\mathcal{E}^{(n-1)}$ is a basic ideal.*

Proof. By Proposition 1.2.9 and Remark 1.2.10, there exists a natural basis such that $e_n^2 = -e_1^2 - \cdots - e_m^2$, with $m \in \mathbb{N}$, $m < n$. First, note that $\mathcal{E}^{(n)}$ is basic if and only if $m = 0$, indeed $\mathcal{E}^{(n)} = \mathsf{span}\{e_n\}$. Moreover, if $\mathcal{E}^{(n)}$ is not basic, note that $\mathcal{E}^{(n-1)}$ is basic if and only if $m = 1$ and it has two different subalgebras. Indeed, such subalgebras are $\mathsf{span}\{e_1 + e_n\}$ and $\mathsf{span}\{e_1 - e_n\}$, which yield that $\mathcal{E}^{(n-1)} = \mathsf{span}\{e_1, e_n\}$ is basic. We will now show that any other different situation (which implies that neither $\mathcal{E}^{(n)}$ nor $\mathcal{E}^{(n-1)}$ are basic ideals) is not compatible with modularity.

1. Assume that $m > 1$ and $\mathcal{E}^{(n-1)}$ has two different subalgebras, say $V = \mathsf{span}\{v\}$ and $W = \mathsf{span}\{w\}$. Hence, by Proposition 1.2.9 and Lemma 1.2.14, there exists a one-dimensional subalgebra, $U = \mathsf{span}\{u\}$, which is not contained in $\mathcal{E}^{(n-1)}$. Take U so that the number $l \in \mathbb{N}$ is such that $U \subset \mathcal{E}^{(l)}$ but $U \not\subseteq \mathcal{E}^{(l+1)}$ is maximum. Since $\mathcal{E}$ is modular, by Proposition 2.1.5 every subalgebra is a quasi-ideal, then $U + V = \mathsf{span}\{u, v\}$ and $U + W = \mathsf{span}\{u, w\}$ are subalgebras. On the one hand, as $(U + V)^2, (U + W)^2 \subset \mathcal{E}^{(l+1)}$, and

$(U+V)^{(3)}, (U+W)^{(3)} = 0$, and the multiples of v and w are the only absolute nilpotent elements of $\mathcal{E}^{(l+1)}$, it holds that $0 \neq uv \in V$ and $0 \neq uw \in W$. On the other hand, it holds that either $0 \neq vw \in V$ or $0 \neq vw \in W$. Consider the first case; the other is analogue. As $0 \neq uv \in V$ and $0 \neq vw \in V$, there exist scalars $\lambda_1, \lambda_2 \in \mathbb{K}^*$ such that $v(\lambda_1 u + \lambda_2 w) = 0$. Then, by Lemma 2.3.6, it holds that $\lambda_1 u + \lambda_2 w \in V$, which contradicts the linear independence of $\{u, v, w\}$.

2. Assume now that $m \geq 1$ but $\mathcal{E}^{(n-1)}$ only has one subalgebra, say $\mathcal{E}^{(n)} = \mathsf{span}\{v\}$. Looking at the proof of Lemma 1.2.14, there exists an element $w \in \mathcal{E}$ such that $\mathcal{E}^{(n-1)} = \mathsf{span}\{v, w\}$ and $v^2 = 0, w^2 = v$ and $vw = 0$. Again, we take U as in the previous case, and we consider the subalgebra $\mathsf{span}\{u, v\}$, where $0 \neq uv \in \mathcal{E}^{(n)}$. As a consequence of Lemma 2.3.6, it holds that $\mathrm{supp}(w) \subset \{m+1, \ldots, n-1\}$ and, since $0 \neq uv, w^2 \in \mathcal{E}^{(n)}$, there exists a contradiction with the linear independence of $\{e_1^2, \ldots, e_{n-1}^2\}$.

As in both cases we get a contradiction, the result follows. □

Remark 2.3.8. Notice that the fact that $\mathcal{E}^{(n-1)}$ is a basic ideal is not equivalent to the existence of a reordering of the natural basis such that $e_n^2 = -e_1^2$. For instance, consider an evolution algebra $\mathcal{E}$ over a field of characteristic not two with natural basis $\{e_1, e_2, e_3\}$ and product given by $e_1^2 = -e_3^2 = e_2$ and $e_2^2 = e_1 + e_3$. Note that $e_3^2 = -e_1^2$ but $\mathcal{E}^2 = \mathsf{span}\{e_2, e_1 + e_3\}$, which is not a basic ideal. Actually, it is easy to check that its only proper subalgebras are $\mathsf{span}\{e_1 + e_3\}$, $\mathsf{span}\{e_1 - e_3\}$ and $\mathsf{span}\{e_2, e_1 + e_3\}$, so its subalgebra lattice is a pentagon, and consequently $\mathcal{E}$ is not modular.

Corollary 2.3.9. *Let $\mathcal{E}$ be a solvable evolution algebra with maximum index of solvability over any field of characteristic not two. If $\mathcal{E}$ is modular, then the derived series does not have two consecutive terms that are not basic ideals.*

Proof. Since modularity is closed under quotients, the result follows straightforward from Proposition 2.3.7, Lemma 2.3.5 and Remark 1.1.6. □

Remark 2.3.10. The converse of Corollary 2.3.9 is not necessarily true. Let $\mathcal{E}$ be an evolution algebra over a field of characteristic not two with natural basis $\{e_1, e_2, e_3, e_4\}$ and product given by $e_1^2 = -e_2^2 = e_1 + e_2$, $e_3^2 = e_1 + e_3 + e_4$ and $e_4^2 = -e_2 + e_3 + e_4$. In fact, it is solvable with maximum index of solvability since $\mathcal{E}^2 = \mathsf{span}\{e_1, e_2, e_3 + e_4\}$, $\mathcal{E}^{(3)} = \mathsf{span}\{e_1, e_2\}$, $\mathcal{E}^{(4)} = \mathsf{span}\{e_1 + e_2\}$ and $\mathcal{E}^{(5)} = 0$. Moreover, notice that every other term of the derived series is a basic ideal. However, $U = \mathsf{span}\{e_1 - e_2, e_3 + e_4\}$ and $V = \mathsf{span}\{e_1 - e_2, e_3 - e_4\}$ are not quasi-ideals since

$$\langle U, V \rangle = \mathsf{span}\{e_1, e_2, e_3, e_4\} \neq \mathsf{span}\{e_1 - e_2, e_3, e_4\} = U + V.$$

The next result proves that this necessary condition is equivalent to supersolvability. Then, although it is not a sufficient condition for modularity, by Proposition 2.1.6, it is indeed sufficient for lower semimodularity.

Theorem 2.3.11. *Let $\mathcal{E}$ be a solvable n-dimensional evolution algebra with maximum index of solvability over any field of characteristic not two. Then, the following statements are equivalent:*

(i) *$\mathcal{E}$ is supersolvable;*

(ii) *there exists a natural basis B such that the structure matrix is block lower triangular with zeros or blocks $\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ in its diagonal and rank $n-1$; and*

(iii) *the derived series does not have two consecutive terms that are not basic ideals.*

Proof. Before starting the proof recall that, by Remark 1.1.6, the structure matrix of a quotient by a basic ideal can be seen as the result of removing the corresponding rows and columns in the original one. Then, by Lemma 2.3.5, item (ii) is preserved under quotients by basic ideals.

(i) $\Rightarrow$ (ii). As $\mathcal{E}$ has maximum index of solvability and is supersolvable, by Theorem 1.5.6, we have that $\mathcal{E}$ is degenerate or there exists a basic ideal isomorphic to $\mathcal{E}_2(1,-1)$. Then, the result follows from the fact that supersolvability is closed under quotients.

(ii) $\Rightarrow$ (iii). We will use induction on $\dim \mathcal{E}$. Let $M_B(\mathcal{E}) = (\omega_{ij})$ be a structure matrix as described. If $\omega_{11} = 0$, then there clearly exists a basic ideal $\mathcal{E}^{(n)} = \mathsf{span}\{e_1\}$. Otherwise, we have that $\omega_{11} = \omega_{12} = -\omega_{21} = -\omega_{22} = 1$ and, reasoning as in the second case of the proof of Proposition 2.3.7, we get that $\mathcal{E}^{(n-1)} = \mathsf{span}\{e_1, e_2\}$. Then, we have just proved that $\mathcal{E}^{(n)}$ or $\mathcal{E}^{(n-1)}$ is a basic ideal. Denote by I this basic ideal. Next, by Lemma 2.3.5, the induction hypothesis can be applied, and the derived series of $\mathcal{E}/I$ does not have two consecutive terms that are not basic ideals. Finally, using Remark 1.1.6, the result follows.

(iii) $\Rightarrow$ (i). We will again use induction on $\dim \mathcal{E}$. By hypothesis, $\mathcal{E}^{(n)}$ or $\mathcal{E}^{(n-1)}$ is a basic ideal. Suppose that $\mathcal{E}^{(n)}$ is basic. Then, consider the quotient $\mathcal{E}/\mathcal{E}^{(n)}$, which is a solvable evolution algebra with maximum index of solvability by Lemma 2.3.5. Then, by the induction hypothesis, $\mathcal{E}/\mathcal{E}^{(n)}$ is supersolvable. As $\mathcal{E}^{(n)}$ is a one-dimensional ideal, $\mathcal{E}$ is clearly supersolvable. Otherwise, reasoning the same $\mathcal{E}/\mathcal{E}^{(n-1)}$ is supersolvable, and since $\mathcal{E}^{(n-1)}$ is $\mathcal{E}$-supersolvable (see Lemma 1.2.15 and the fifth property of $\mathcal{E}$-supersolvability on page 38), the result follows. □

In our last result, we will work with evolution algebras whose structure matrix $(\omega_{ij})_{i,j=1}^{n}$ is as described in Theorem 2.3.11. Moreover, define Λ as the subset of pairs $(i, i+1)$, which corresponds with the blocks $\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ in the structure matrix, that is,

$$\Lambda = \left\{ (i, i+1) \colon \left(\begin{smallmatrix} \omega_{ii} & \omega_{i(i+1)} \\ \omega_{(i+1)i} & \omega_{(i+1)(i+1)} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right) \right\}.$$

Indeed, as reasoned in the proof of Theorem 2.3.4 it is easy to check that all subalgebras can be written as

$$K + \mathsf{span}\{e_{k_1} - e_{k_1+1}, e_{k_2} - e_{k_2+1}, \dots, e_{k_l} \pm e_{k_l+1}\}$$

for certain $k_1 < \ldots < k_l$, $(k_1, k_1+1), (k_2, k_2+1), \dots, (k_l, k_l+1) \in \Lambda$ and K is a subspace spanned by elements of the natural basis. In addition, also denote by π_{ij} the linear projection of $\mathcal{E}$ onto $\mathsf{span}\{e_i, e_j\}$ along $\mathsf{span}\{e_k \colon k \neq i, j\}$.

Corollary 2.3.12. *Let $\mathcal{E}$ be a solvable evolution algebra with maximum index of solvability over any field of characteristic different from two. Then, $\mathcal{E}$ is modular if and only if the derived series does not have two consecutive terms that are not basic ideals and there does not exist a subalgebra $U = K + \mathsf{span}\{e_i - e_{i+1}, e_j \pm e_{j+1}\}$ where $i < j$, $(i, i+1), (j, j+1) \in \Lambda$ and K is a subspace spanned by elements of the natural basis different from e_i, e_{i+1}, e_j and e_{j+1}; and such that $\pi_{i(i+1)}(e_j)$ or $\pi_{i(i+1)}(e_{j+1}) \notin \mathsf{span}\{e_i - e_{i+1}\}$.*

Proof. First, we prove the sufficiency. By hypothesis, every subalgebra of $\mathcal{E}$ can be written as

$$K + \mathsf{span}\{e_{k_1} - e_{k_1+1}, e_{k_2} - e_{k_2+1}, \dots, e_{k_l} \pm e_{k_l+1}\},$$

with $\pi_{(k_i, k_i+1)}(e_{k_j})$ and $\pi_{(k_i, k_i+1)}(e_{k_j+1}) \in \mathsf{span}\{e_{k_i} - e_{k_i+1}\}$ for all $1 \leq i < j \leq l$. It is easy to see that, owing to the hypothesis on the projections, all subalgebras of this form are quasi-ideals. Hence, by Proposition 2.1.5, the modularity of $\mathcal{E}$ follows.

For the necessity, as a consequence of Proposition 2.3.7, we just need to prove that if there exists a subalgebra, as stated, then $\mathcal{E}$ is not modular. Without loss of generality, suppose that $U = K + \mathsf{span}\{e_i - e_{i+1}, e_j + e_{j+1}\}$, with $i < j$, is a subalgebra. Then, $V = K + \mathsf{span}\{e_i - e_{i+1}, e_j - e_{j+1}\}$ is also a subalgebra. Nevertheless, as $\pi_{i(i+1)}(e_j)$ or $\pi_{i(i+1)}(e_{j+1}) \notin \mathsf{span}\{e_i - e_{i+1}\}$, we have that

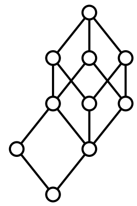
$$\begin{aligned} \langle U, V \rangle &= K + \mathsf{span}\{e_i, e_{i+1}, e_j, e_{j+1}\} \\ &\neq K + \mathsf{span}\{e_i - e_{i+1}, e_j, e_{j+1}\} = U + V. \end{aligned}$$

Since U is not a quasi-ideal, by Proposition 2.1.5, $\mathcal{E}$ is not modular. □

Remark 2.3.13. We cannot omit that $\pi_{i(i+1)}(e_j)$ or $\pi_{i(i+1)}(e_{j+1}) \notin \mathsf{span}\{e_i - e_{i+1}\}$ in the previous corollary. Let $\mathcal{E}$ be an evolution algebra with natural basis

$\{e_1, e_2, e_3, e_4\}$ and product given by $e_1^2 = -e_2^2 = e_1 + e_2$, $e_3^2 = e_1 - e_2 + e_3 + e_4$ and $e_4^2 = e_1 - e_2 - e_3 - e_4$. It is easy to check that $\mathcal{E}$ is solvable with maximum index of solvability and that its subalgebras are all quasi-ideals. Consequently, $\mathcal{E}$ is modular.

Subalg. dim. 1	**Subalg. dim.** 2	**Subalg. dim.** 3
$\text{span}\{e_1 + e_2\}$	$\text{span}\{e_1, e_2\}$	$\text{span}\{e_1, e_2, e_3 + e_4\}$
$\text{span}\{e_1 - e_2\}$	$\text{span}\{e_1 - e_2, e_3 + e_4\}$	$\text{span}\{e_1, e_2, e_3 - e_4\}$
	$\text{span}\{e_1 - e_2, e_3 - e_4\}$	$\text{span}\{e_1 - e_2, e_3, e_4\}$



2.4 Modularity in a specific case of regular evolution algebras

As stated in Remark 1.5.1, there exist supersolvable evolution algebras which are not solvable. In fact, we now state the following elementary result, which characterises supersolvability in the regular setting.

Proposition 2.4.1. *Let $\mathcal{E}$ be a regular evolution algebra over an arbitrary field $\mathbb{K}$. Then, $\mathcal{E}$ is supersolvable if and only if there exists a natural basis such that the structure matrix is lower triangular with ones on the diagonal.*

Proof. The sufficiency is immediate: such a structure matrix directly yields a complete flag of ideals: $0 \subsetneq \text{span}\{e_1\} \subsetneq \cdots \subsetneq \text{span}\{e_1, \ldots, e_{n-1}\} \subsetneq \mathcal{E}$.

For the necessity, note that in the regular setting all ideals are basic, which implies that a triangular matrix can be obtained by simply reordering the elements of the natural basis; let (ω_{ij}) denote such a structure matrix. Finally, to normalise the diagonal entries to ones, it suffices to consider the natural basis $\{f_i = \frac{1}{\omega_{ii}} e_i\}_{i=1}^n$. □

In particular, our purpose is to characterise modularity within three-dimensional supersolvable regular evolution algebras (and, in fact, to go beyond this case). Note that, by the previous proposition, whenever working with this type of evolution algebra we may assume that their structure matrices are

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \mu & \rho & 1 \end{pmatrix}, \text{ with } \lambda, \mu, \rho \in \mathbb{K}. \tag{2.4.1}$$

Consequently, an evolution algebra over a field $\mathbb{K}$ which admits a natural basis such that the structure matrix is (2.4.1) will be denoted by $\mathcal{E}_{\mathbb{K}}^{\text{reg}}(\lambda, \mu, \rho)$. Moreover, by

Lemma 1.3.1, their one-dimensional subalgebras are given by the nontrivial solutions $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{K}^3$ of the following nonlinear system of equations:

$$\begin{cases} x^2 = x - \lambda y + (\lambda\rho - \mu)z, \\ y^2 = y - \rho z, \\ z^2 = z. \end{cases} \tag{2.4.2}$$

Remark 2.4.2. Note that although the algebras considered in this section are not solvable, Proposition 2.1.5 still applies in this setting. In fact, the arguments used in its proof (particularly, in (ii)⇒(iii)) remain valid, since these algebras also admit a complete flag of subalgebras.

Proposition 2.4.3. *Let $\mathcal{E} = \mathcal{E}^{\text{reg}}_{\mathbb{K}}(\lambda, \mu, \rho)$ with $\mathbb{K}$ an algebraically closed field of characteristic not two. If $\mathcal{E}$ is modular, then the following assertions hold:*

(i) $\rho \neq 0$;

(ii) $\rho^2\lambda - \rho\mu + \mu^2 = 0$; *and*

(iii) $\lambda = 0$.

Proof. Since we are working over an algebraically closed field, the system (2.4.2) always admits a solution with all components nonzero. Consequently, a subalgebra of the form $\mathsf{span}\{\alpha e_1 + \beta e_2 + e_3\}$ with $\alpha, \beta \neq 0$ always exists. Denote by U one of them. Similarly, it is easy to check that $\mathsf{span}\{\gamma e_1 + e_2\}$ is a subalgebra if and only if γ is a solution of the quadratic equation $x^2 - x + \lambda = 0$. Since this equation has at least one nonzero solution, we denote by V a subalgebra of the form $\mathsf{span}\{\gamma e_1 + e_2\}$ with $\gamma \neq 0$.

(i) For the sake of contradiction, assume that $\rho = 0$. In this case, observe that $\mathsf{span}\{\delta e_1 + e_3\}$ is a subalgebra if and only if δ is a solution of $x^2 - x + \mu = 0$. As before, this equation has at least one nonzero solution, so denote by W a subalgebra of the form $\mathsf{span}\{\delta e_1 + e_3\}$ with $\delta \neq 0$. By Theorem 1.3.2, we then obtain that $V + W$ is not a subalgebra, since it does not admit a natural basis. Therefore, they are not quasi-ideals and the claim follows from Remark 2.4.2.

(ii) Since $\mathcal{E}$ is modular then, by Remark 2.4.2, $U + V$ is a subalgebra. Hence, again by Theorem 1.3.2, for $U + V$ to be a subalgebra it must first admit a natural basis, but this happens if and only if $\alpha e_1 + \beta e_2 = k(\gamma e_1 + e_2)$ for some $k \in \mathbb{K}^*$. Consequently, it results in a two-dimensional subalgebra which can be written as $\mathsf{span}\{v, e_3\}$ with $v \in \mathsf{span}\{e_1, e_2\}$, which exists, following Corollary 1.3.7, when the following identity is satisfied:

$$\mu^2\rho + \rho^3\lambda = \rho^2\mu \implies \rho(\rho^2\lambda - \rho\mu + \mu^2) = 0.$$

Since $\rho \neq 0$ by item (i), the claim follows.

(iii) Note that, since $\rho \neq 0$ by item (i), the only two-dimensional subalgebra of the form $\mathsf{span}\{v, e_3\}$, with $v \in \mathsf{span}\{e_1, e_2\}$, that may exist is $\mathsf{span}\{\mu e_1 + \rho e_2, e_3\}$. Moreover, consider subalgebras $V_1 = \mathsf{span}\{\gamma_1 e_1 + e_2\}$ and $V_2 = \mathsf{span}\{\gamma_2 e_1 + e_2\}$ with γ_1, γ_2 solutions of $x^2 - x + \lambda = 0$. We now claim that if $\gamma_1, \gamma_2 \neq 0$, then $\gamma_1 = \gamma_2$, that is, $x^2 - x + \lambda = 0$ has only one nonzero solution. Indeed, if $\gamma_1, \gamma_2 \neq 0$, as all subalgebras are quasi-ideals, we have

$$\langle U, V_1 \rangle = \langle U, V_2 \rangle = \mathsf{span}\{\mu e_1 + \rho e_2, e_3\},$$

which implies that $\mu e_1 + \rho e_2 = k_1(\gamma_1 e_1 + e_2) = k_2(\gamma_2 e_1 + e_2)$ with $k_1, k_2 \in \mathbb{K}$, what yields that $\gamma_1 = \gamma_2$. Consequently, we have that either $\lambda = 0$ or $\lambda = \frac{1}{4}$.

We now show that the case $\lambda = \frac{1}{4}$ is not valid. For the sake of contradiction, if $\lambda = \frac{1}{4}$ then, by item (ii), we have that $\mu = \frac{1}{2}\rho$. It is easy to check that $\mathsf{span}\{\frac{1}{2}e_1 + e_2\}$ is a subalgebra. Moreover, looking at (2.4.2), all one-dimensional evolution algebras of the form $\mathsf{span}\{\alpha e_1 + \beta e_2 + e_3\}$, with $\alpha \neq 0$ or $\beta \neq 0$, are given by the nontrivial solutions of

$$\begin{cases} x^2 = x - \frac{1}{4}y - \frac{1}{4}\rho, \\ y^2 = y - \rho. \end{cases} \tag{2.4.3}$$

In fact, all these subalgebras must satisfy

$$\left\langle \frac{1}{2}e_1 + e_2, \alpha e_1 + \beta e_2 + e_3 \right\rangle = \mathsf{span}\left\{ \frac{1}{2}e_1 + e_2, \alpha e_1 + \beta e_2 + e_3 \right\}.$$

Consequently, it must hold that $\beta = 2\alpha$ for all (α, β) solution of (2.4.3). Then, we get that $\alpha^2 = \frac{1}{2}\alpha - \frac{1}{4}\rho$ and $4\alpha^2 = \alpha - \rho$, where we easily obtain that $\alpha = \rho = 0$, a contradiction with item (i). □

Theorem 2.4.4. *Let $\mathcal{E}$ be a three-dimensional supersolvable regular evolution algebra over an algebraically closed field $\mathbb{K}$ of characteristic not two. Then, $\mathcal{E}$ is modular if and only if $\mathcal{E} \cong \mathcal{E}_{\mathbb{K}}^{\mathrm{reg}}(0, \frac{1}{4}, \frac{1}{4})$.*

Proof. If $\mathcal{E}$ is modular, by Proposition 2.4.3 we have that $\lambda = 0$, so $\mathsf{span}\{e_1 + e_2\}$ is a subalgebra. Moreover, any one-dimensional subalgebra $\mathsf{span}\{\alpha e_1 + \beta e_2 + e_3\}$ is given by

$$\begin{cases} x^2 = x - \mu, \\ y^2 = y - \rho. \end{cases} \tag{2.4.4}$$

Since $\mathcal{E}$ is modular, then $\langle e_1 + e_2, \alpha e_1 + \beta e_2 + e_3 \rangle = \mathsf{span}\{e_1 + e_2, \alpha e_1 + \beta e_2 + e_3\}$ for any α and β solutions of $x^2 - x + \mu = 0$ and $y^2 - y + \rho = 0$, respectively. Notice that this holds if and only if $\alpha = \beta$ in any case, i.e, if and only if $\rho = \mu = \frac{1}{4}$.

Conversely, by an straightforward computation we show that the one-dimensional subalgebras of $\mathcal{E}_{\mathbb{K}}^{\text{reg}}(0, \frac{1}{4}, \frac{1}{4})$ are $\mathsf{span}\{e_1\}$, $\mathsf{span}\{e_2\}$, $\mathsf{span}\{e_1+e_2\}$ and $\mathsf{span}\{e_1+e_2+2e_3\}$, which are all quasi-ideals. As every subalgebra of codimension one is a quasi-ideal, then all its subalgebras are quasi ideals and consequently $\mathcal{E}_{\mathbb{K}}^{\text{reg}}(0, \frac{1}{4}, \frac{1}{4})$ is modular.

Subalg. of dim. 1	**Subalg. of dim.** 2
$\mathsf{span}\{e_1\}$	$\mathsf{span}\{e_1, e_2\}$
$\mathsf{span}\{e_2\}$	$\mathsf{span}\{e_1, e_2 + 2e_3\}$
$\mathsf{span}\{e_1 + e_2\}$	$\mathsf{span}\{e_2, e_1 + 2e_3\}$
$\mathsf{span}\{e_1 + e_2 + 2e_3\}$	$\mathsf{span}\{e_3, e_1 + e_2\}$

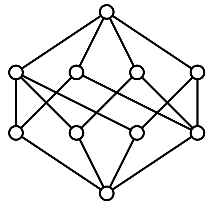


Figure 2.4.1: Proper subalgebras and subalgebra lattice of $\mathcal{E}_{\mathbb{K}}^{\text{reg}}(0, \frac{1}{4}, \frac{1}{4})$ with $\mathbb{K}$ a field of characteristic not 2.

□

Moreover, it has been possible to characterise modularity in supersolvable regular evolution algebras in general, as shown in the next theorem.

Theorem 2.4.5. *Let $\mathcal{E}$ be a supersolvable regular evolution algebra over an algebraically closed field $\mathbb{K}$ of characteristic not two. Then, $\mathcal{E}$ is modular if and only if $n \leq 2$ or $\mathcal{E} \cong \mathcal{E}_{\mathbb{K}}^{\text{reg}}(0, \frac{1}{4}, \frac{1}{4})$.*

Proof. The fact that every one or two-dimensional algebra is modular together with Theorem 2.4.4 reduce our problem to the case of dimension greater than three.

Let $\mathcal{E}$ be a n-dimensional evolution algebra ($n > 3$) with natural basis $B = \{e_1, \ldots, e_n\}$ and with product given by a structure matrix $M_B(\mathcal{E}) = (\omega_{ij})$ of the form described in Proposition 2.4.1. Then, by Theorem 2.4.4 it holds that $\omega_{21} = 0$ and $\omega_{31} = \omega_{32} = \frac{1}{4}$. Otherwise, there would exist a subalgebra which is not upper semimodular.

Next, consider the quotient algebra $\mathcal{E}/\,\mathsf{span}\{e_1\}$. By Remark 1.1.6, its structure matrix relative to the natural basis $\{\overline{e_2}, \ldots, \overline{e_n}\}$ is the result of deleting the first row and the first column of $M_B(\mathcal{E})$. However, the entry $(2, 1)$ of this new matrix is $\frac{1}{4}$, and thus the subalgebra $\mathsf{span}\{\overline{e_2}, \overline{e_3}, \overline{e_4}\}$ is not modular by Proposition 2.4.3. Since $\mathcal{E}/\,\mathsf{span}\{e_1\}$ is not modular, finally Remark 2.1.4 yields the claim. □

Chapter 3

A Frattini theory for evolution algebras

Frattini theory originates in group theory, where the *Frattini subgroup*, defined as the intersection of all maximal subgroups of a given group, was first introduced by Giovanni Frattini in 1885 (see [49]). The Frattini subgroup captures the notion of the group's nongenerators, as it consists precisely of those elements that can be removed from any generating set without losing the generation of the group. In addition, this subgroup has several other remarkable properties; particularly, it is always a characteristic subgroup and, consequently, it is normal. Furthermore, the Frattini subgroup of a finite group is nilpotent. Last but not least, it also serves as a criterion for nilpotency, as it contains the derived subgroup if and only if the group itself is nilpotent (see, for instance, [56, pp. 156–158]).

Over time, Frattini theory has been extended beyond group theory to establish a parallel framework for algebras. The similarities between Lie algebras and groups led to the analogous definition of the Frattini subalgebra as the intersection of all maximal subalgebras. This extension motivated numerous investigations, notably by Barnes (see [9]), Barnes and Gastineau-Hills (see [10]), and Marshall (see [78]), among many others. Although less explored, related studies have also emerged in the broader context of nonassociative algebras (see [102]), and more specifically, in the areas of Leibniz algebras and restricted Lie algebras (see [71]). However, to the best of our knowledge, no similar study has been conducted in the setting of genetic algebras and, in particular, evolution algebras.

Motivated by these developments and others, this chapter aims to establish a Frattini theory for evolution algebras, defining their Frattini subalgebras as the intersection of all maximal subalgebras and their Frattini ideals as the largest ideals contained within the Frattini subalgebras. We now outline how this investigation was conducted by describing the structure of the chapter in detail and presenting the purposes and results of each of the four sections. Following this introduction, Section 3.1 presents a review of the essential background on Frattini theory needed for the subsequent developments.

As a first step in our study, Section 3.2 revisits the concept of the nilradical in the context of evolution algebras. We begin by showing that, unlike in other nonassociative structures such as Lie or Leibniz algebras, an evolution algebra may admit several maximal nilpotent ideals (see Example 3.2.1). Hence, its nilradical cannot be defined as its largest nilpotent ideal. The main goal of this section is therefore to propose a suitable definition of the nilradical in this setting. To this end, we first focus on solvable evolution algebras with one-dimensional derived subalgebras, where the maximal nilpotent ideal is unique (see Theorem 3.2.4). Using this result, we inductively define a series of nilpotent ideals leading to the notion of the supersolvable nilradical (Definition 3.2.11), which captures the essence of the nilradical while enjoying several desirable properties (see Proposition 3.2.13 and Corollary 3.2.17).

Section 3.3 develops the Frattini theory for evolution algebras, establishing necessary and sufficient conditions for the Frattini subalgebra and ideal to be trivial. First, we characterise these objects in the context of solvable evolution algebras with one-dimensional derived subalgebras (see Theorem 3.3.1). We then establish a necessary condition for an evolution algebra to be ϕ-free in terms of its supersolvable nilradical (see Theorem 3.3.3). Finally, we also prove the sufficiency of these conditions in a particular case related to the support of the supersolvable nilradical (see Theorem 3.3.6).

Finally, Section 3.4 explores the role of the Frattini subalgebra in the study of dually atomistic evolution algebras. In particular, we characterise this property within specific families of evolution algebras (Theorem 3.4.3) by introducing almost (basic) abelian evolution algebras.

The results and developments presented in this chapter can be found in [69].

3.1 Preliminaries on Frattini theory

Given an evolution algebra $\mathcal{E}$, its *Frattini subalgebra*, $F(\mathcal{E})$, is defined as the intersection of all maximal subalgebras of $\mathcal{E}$; and its *Frattini ideal*, $\phi(\mathcal{E})$, as the largest ideal contained in $F(\mathcal{E})$. Moreover, $\mathcal{E}$ is said to be *ϕ-free* if $\phi(\mathcal{E}) = 0$. The main elementary property of the Frattini subalgebra is that if U is a subalgebra of $\mathcal{E}$ such that $U + F(\mathcal{E}) = \mathcal{E}$, then necessarily $U = \mathcal{E}$. Actually, as stated in [101, Theorem 1], $F(\mathcal{E})$ can be characterised as the set of nongenerators of $\mathcal{E}$.

We now state some further deep results concerning the Frattini subalgebra that will be essential for our investigation. Although these results were already established in a general nonassociative framework, we include them here for completeness and reproduce their proofs in the specific setting of evolution algebras due to their importance and frequent use in the subsequent arguments.

Lemma 3.1.1 [78, Lemma 1] & [101, Theorem 6]**.** *The Frattini subalgebra of an evolution algebra $\mathcal{E}$, $F(\mathcal{E})$, is contained in its derived subalgebra $\mathcal{E}^2$. Moreover, if $\mathcal{E}$ is nilpotent, then $F(\mathcal{E}) = \mathcal{E}^2$.*

Proof. Suppose, for the sake of contradiction, that there exists an element $x \in F(\mathcal{E})$ with $x \notin \mathcal{E}^2$. Then there exists a subalgebra of $\mathcal{E}$ of codimension one that contains $\mathcal{E}^2$ but not x. This contradicts the fact that x belongs to all maximal subalgebras.

To prove the second assertion, let M be a maximal subalgebra of $\mathcal{E}$. Then, there exists an integer $r \geq 2$ such all all products of r elements of $\mathcal{E}$ belong to M, whereas some product of $r-1$ elements of $\mathcal{E}$, namely x, does not belong to M. Consequently, $\langle x, M\rangle = \mathcal{E}$. Hence,

$$\mathcal{E}^2 = \langle x^2, xM, M^2\rangle \subseteq M,$$

which implies that $\mathcal{E}^2 \subseteq F(\mathcal{E})$. As a consequence of the previous inclusion, the result follows. □

Remark 3.1.2. The case of Lie algebras is specially representative since the converse of the previous result also holds: a Lie algebra $\mathcal{L}$ satisfies $F(\mathcal{L}) = \mathcal{L}^2$ if and only if $\mathcal{L}$ is nilpotent (see [8]). However, we will see later that this is not necessarily true for evolution algebras.

Lemma 3.1.3 [102, Lemma 4.1]**.** *Let $\mathcal{E}$ be an evolution algebra. If U is a subalgebra of $\mathcal{E}$, and I is an ideal of $\mathcal{E}$ contained in $F(U)$, then I is contained in $F(\mathcal{E})$.*

Proof. Suppose that I is not contained in $F(\mathcal{E})$. Then, there is a maximal subalgebra M of $\mathcal{E}$ such that $\mathcal{E} = I + M$. Then, we have

$$U = \mathcal{E} \cap U = (I + M) \cap U = I + M \cap U = F(U) + M \cap U = M \cap U.$$

Consequently, U is contained in M, whence I is contained in M, a contradiction. □

Lemma 3.1.4 [102, Lemma 7.1]**.** *Let I be an ideal of an evolution algebra $\mathcal{E}$, and let U be a subalgebra of $\mathcal{E}$ which is minimal with respect to the property that $\mathcal{E} = I + U$. Then, $I \cap U \subseteq \phi(U)$.*

Proof. For the sake of contradiction, suppose that $I \cap U \nsubseteq \phi(U)$. Then, since $I \cap U$ is an ideal of U, $I \cap U \nsubseteq F(U)$. Then, it follows that there exits a maximal subalgebra M of U such that $I \cap U \nsubseteq M$. Clearly, $U = I \cap U + M$, and so $\mathcal{E} = I + (I \cap U + M) = I + M$, which contradicts the minimality of U. The result follows □

Lemma 3.1.5 [102, Lemma 7.2]**.** *Let $\mathcal{E}$ be an (evolution) algebra. If I is an abelian ideal of $\mathcal{E}$ such that $\phi(\mathcal{E}) \cap I = 0$, then there exists a subalgebra U of $\mathcal{E}$ such that $\mathcal{E} = U \oplus I$.*

Proof. Take U a subalgebra of $\mathcal{E}$ which is minimal with respect to the property that $\mathcal{E} = I + U$. Then, by Lemma 3.1.4, $I \cap U \subseteq \phi(U)$. Now, $I \cap U$ is an ideal of U and $(I \cap U)I \subseteq I^2 = 0$, so $I \cap U$ is an ideal of $\mathcal{E}$. Hence, using Lemma 3.1.3, $I \cap U \subseteq \phi(\mathcal{E}) \cap I = 0$ and $\mathcal{E} = I \oplus U$. □

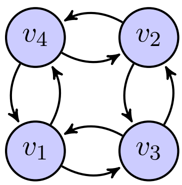
3.2 Defining the nilradical of an evolution algebra

Traditionally, the nilradical of a commutative ring is the ideal consisting of all nilpotent elements. Similarly, the nilradical of a Lie algebra $\mathcal{L}$, $\mathrm{Nil}(\mathcal{L})$, is its maximal nilpotent ideal, which exists since the sum of any two nilpotent ideals is also nilpotent. This notion also extends to Leibniz algebras (see [12, Corollary 4]). The nilradical is closely connected to the Frattini subalgebra and ideal: for Lie and Leibniz algebras one has

$$\mathrm{Nil}(\mathcal{L})^2 = F(\mathrm{Nil}(\mathcal{L})) \subset \phi(\mathcal{L}) \subset F(\mathcal{L}),$$

which follows from Lemma 3.1.3 and the fact that $\mathrm{Nil}(\mathcal{L})^2$ is also an ideal of $\mathcal{L}$ (a consequence of the Jacobi identity). Therefore, whenever $\mathrm{Nil}(\mathcal{L})^2 \neq 0$, both the Frattini subalgebra and ideal are automatically nontrivial. However, as shown in the next example, more than one maximal nilpotent ideal may exist in the context of evolution algebras, so that the previous approach cannot be applied directly in this context.

Example 3.2.1. Let $\mathcal{E}$ be the evolution algebra with natural basis $\{e_1, e_2, e_3, e_4\}$ and product given by $e_1^2 = -e_2^2 = e_3 + e_4$ and $e_3^2 = -e_4^2 = e_1 + e_2$. The subspaces $\mathcal{N}_1 = \mathsf{span}\{e_1, e_2, e_3 + e_4\}$ and $\mathcal{N}_2 = \mathsf{span}\{e_3, e_4, e_1 + e_2\}$ are two different maximal nilpotent ideals. However, $\mathcal{N}_1 + \mathcal{N}_2 = \mathcal{E}$ is not nilpotent since its associated digraph contains oriented cycles.



Remark 3.2.2. In contrast to the previous example, an evolution algebra may admit a unique maximal nilpotent ideal. In this case, we call this ideal the *nilradical* of $\mathcal{E}$ and denote it by $\mathrm{Nil}(\mathcal{E})$. Moreover, when $\mathcal{E}$ is nilpotent, we shall say that $\mathcal{E}$ itself is its nilradical, $\mathrm{Nil}(\mathcal{E}) = \mathcal{E}$.

We next show that the nilradical of an evolution algebra in $\mathcal{T}_{\mathbb{K}}$, say $\mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$ with $k \geq 2$ and $\lambda_1, \ldots, \lambda_k \neq 0$ by Remark 1.2.5, exists and can be perfectly characterised in terms of the natural number k and the scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$.

Lemma 3.2.3. *Let $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$. Then, its derived subalgebra, $\mathcal{E}^2$, is contained in every maximal nilpotent ideal.*

Proof. Let $\mathcal{N}$ be a maximal nilpotent ideal of $\mathcal{E}$. Assume, for the sake of contradiction, that $\mathcal{E}^2 \nsubseteq \mathcal{N}$. Since $\mathcal{N}$ is nilpotent, if $i \in \mathrm{supp}(\mathcal{N})$ then it necessarily holds that

$e_i^2 = 0$. Consequently, $\mathcal{N} = \mathrm{ann}(\mathcal{E})$. Nevertheless, note that $\mathcal{E}^2 + \mathcal{N}$ is also nilpotent since $(\mathcal{E}^2 + \mathcal{N})^2 = 0$, which contradicts the maximality of $\mathcal{N}$ or the nonnilpotency of $\mathcal{E}$. □

Theorem 3.2.4. *Let $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$. Then, its nilradical,* $\mathrm{Nil}(\mathcal{E})$*, exists. In fact, assuming, without loss of generality, that $\mathcal{E} = \mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$ with $k \geq 2$ and $\lambda_1, \ldots, \lambda_k \neq 0$, it holds that*

$$\mathrm{Nil}(\mathcal{E}) = \mathsf{span}\{\lambda_2 e_1 - \lambda_1 e_2, \lambda_3 e_1 - \lambda_1 e_3, \ldots, \lambda_k e_1 - \lambda_1 e_k, e_{k+1}, \ldots, e_n\}.$$

Proof. For simplicity, we denote by $\mathcal{N}$ the subspace described just above and prove that it is the only maximal nilpotent ideal of $\mathcal{E}$. First, $\mathcal{N}$ is an ideal because $\mathcal{E}^2 = \mathsf{span}\{e_1 + \cdots + e_k\} \subset \mathcal{N}$. In fact, we have that

$$\begin{aligned}(\lambda_2 e_1 - \lambda_1 e_2) &+ (\lambda_3 e_1 - \lambda_1 e_2) + \cdots + (\lambda_k e_1 - \lambda_1 e_k) \\ &= (\lambda_2 + \lambda_3 + \cdots + \lambda_k) e_1 - \lambda_1 e_2 - \lambda_1 e_3 - \cdots - \lambda_1 e_k \\ &= -\lambda_1 e_1 - \lambda_1 e_2 - \lambda_1 e_3 - \cdots - \lambda_1 e_k = -\lambda_1 (e_1 + \cdots + e_k);\end{aligned}$$

and, since $\lambda_1 \neq 0$ by hypothesis, we conclude that $e_1 + \cdots + e_k \in \mathcal{N}$. Secondly, $\mathcal{N}$ is a maximal ideal since it has codimension one. Thirdly, $\mathcal{N}$ is nilpotent since $\mathcal{N}^3 = \mathcal{N}^2 \mathcal{N} = 0$. In fact, it holds that

$$(\lambda_i e_1 - \lambda_1 e_i)(e_1 + \cdots + e_k) = (\lambda_i \lambda_1 - \lambda_1 \lambda_i)(e_1 + \cdots + e_k) = 0, \qquad (3.2.1)$$

for all $i = 2, \ldots, k$ and $e_i(e_1 + \cdots + e_k) = 0$ for all $i = k+1, \ldots, n$.

Next, for the sake of contradiction, suppose that there exists another maximal nilpotent ideal $\mathcal{M}$ and consider a nonzero element $u = \sum_{i=1}^n \mu_i e_i$ such that $u \in \mathcal{M}$ but $u \notin \mathcal{N}$. Then, we have that

$$\begin{aligned}u &+ \frac{\mu_2}{\lambda_1}(\lambda_2 e_1 - \lambda_1 e_2) + \cdots + \frac{\mu_k}{\lambda_1}(\lambda_k e_1 - \lambda_1 e_k) \\ &- \mu_{k+1} e_{k+1} - \cdots - \mu_n e_n = \left(\mu_1 + \frac{\mu_2 \lambda_2}{\lambda_1} + \cdots + \frac{\mu_k \lambda_k}{\lambda_1}\right) e_1 \neq 0.\end{aligned} \qquad (3.2.2)$$

Note that if (3.2.2) were equal to zero, there would be a contradiction with the fact that $u \notin \mathcal{N}$. Consequently, set

$$\Lambda := \mu_1 + \frac{\mu_2 \lambda_2}{\lambda_1} + \cdots + \frac{\mu_k \lambda_k}{\lambda_1},$$

and note that $\Lambda \neq 0$. Moreover, by Lemma 3.2.3, it holds that $e_1 + \cdots + e_k \in \mathcal{M}$. Additionally, it holds that $u(e_1 + \cdots + e_k) = 0$. Otherwise, $u(e_1 + \cdots + e_k) =$

$\mathbb{K}^*(e_1 + \cdots + e_k)$ and, consequently, $\mathcal{M}^{\langle n \rangle} \neq 0$ for any $n \in \mathbb{N}$, which contradicts the nilpotency of $\mathcal{M}$. Finally, putting all this together, we get that

$$\begin{aligned} 0 = & \, u\,(e_1 + \cdots + e_k) \\ & + \frac{\mu_2}{\lambda_1}(\lambda_2 e_1 - \lambda_1 e_2)(e_1 + \cdots + e_k) + \cdots + \frac{\mu_k}{\lambda_1}(\lambda_k e_1 - \lambda_1 e_k)(e_1 + \cdots + e_k) \\ & - (\mu_{k+1} e_{k+1} + \cdots + \mu_n e_n)\,(e_1 + \cdots + e_k) \\ = & \, \Lambda e_1(e_1 + \cdots + e_k) = \Lambda e_1^2 = \Lambda \lambda_1 (e_1 + \cdots + e_k), \end{aligned}$$

which is impossible because $\Lambda, \lambda_1 \neq 0$, thus leading to a contradiction with the assumption that $\mathcal{M}$ is another maximal nilpotent ideal. □

The following property will also be instrumental throughout our study.

Corollary 3.2.5. *Let $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$. Then,* $\operatorname{ann}_{\mathcal{E}}(\mathcal{E}^2) := \{x \in \mathcal{E} : x\mathcal{E}^2 = 0\} = \operatorname{Nil}(\mathcal{E})$.

Proof. As shown in (3.2.1), $\mathcal{E}^2 \operatorname{Nil}(\mathcal{E}) = 0$. Consequently, $\operatorname{Nil}(\mathcal{E}) \subset \operatorname{ann}_{\mathcal{E}}(\mathcal{E}^2)$. Now, assume that $\operatorname{Nil}(\mathcal{E}) \subsetneq \operatorname{ann}_{\mathcal{E}}(\mathcal{E}^2)$. As $\operatorname{Nil}(\mathcal{E})$ has codimension one and $\operatorname{ann}_{\mathcal{E}}(\mathcal{E}^2)$ is a subspace, then $\operatorname{ann}_{\mathcal{E}}(\mathcal{E}^2) = \mathcal{E}$, thus a contradiction with the nonnilpotency of $\mathcal{E}$. □

After this point, our main aim is to establish a good definition for the nilradical of an evolution algebra.

3.2.1 The basic nilradical of an evolution algebra

Given an evolution algebra $\mathcal{E}$ with natural basis B, recall the construction of the upper annihilating series of $\mathcal{E}$ presented on page 16. Note that, as we are only considering finite-dimensional algebras, there exists an integer $r \geq 1$ such that $\operatorname{ann}^r(\mathcal{E}) = \operatorname{ann}^{r+1}(\mathcal{E}) = \operatorname{ann}^{r+2}(\mathcal{E}) = \cdots$, that is, the upper annihilating series stabilises for some $r \geq 1$.

Proposition 3.2.6. *Let $\mathcal{E}$ be an evolution algebra, and $r \geq 1$ the number of steps until the upper annihilating series stabilises. Then, $\operatorname{ann}^r(\mathcal{E})$ is the largest basic nilpotent ideal of $\mathcal{E}$.*

Proof. First, notice that there exists a unique maximal basic nilpotent ideal since the sum of basic nilpotent ideals is clearly nilpotent due to their strictly triangular structure matrix. So, since $\operatorname{ann}^r(\mathcal{E})$ is clearly a basic nilpotent ideal, say $\operatorname{ann}^r(\mathcal{E}) = \mathsf{span}\{e_k, \dots, e_n\}$, it is clearly contained in the largest one. For the sake of contradiction, assume that $I = \mathsf{span}\{e_l, \dots, e_n\}$ with $l < k$ is a larger basic nilpotent ideal. As I is a nilpotent evolution algebra by itself, its structure matrix can be supposed to be strictly upper triangular. This implies that $e_{k-1}^2 \in \mathsf{span}\{e_k, \dots, e_n\} = \operatorname{ann}^r(\mathcal{E})$, contradicting the definition of r. □

Because of the previous result, we introduce the following definition.

Definition 3.2.7. The *basic nilradical* of an evolution algebra $\mathcal{E}$, denoted by $\mathrm{BNil}(\mathcal{E})$, is defined as the largest nilpotent basic ideal of $\mathcal{E}$. Equivalently, $\mathrm{BNil}(\mathcal{E}) = \mathrm{ann}^r(\mathcal{E})$, where r is the number of steps until the upper annihilating series of $\mathcal{E}$ stabilises.

However, the basic nilradical does not coincide with the nilradical in the particular case considered in Theorem 3.2.4. In fact, if $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ then $\mathrm{BNil}(\mathcal{E}) = \mathrm{ann}(\mathcal{E}) \subsetneq \mathrm{Nil}(\mathcal{E})$. This makes us realise that the basic nilradical is far from being a suitable definition for the nilradical of an evolution algebra.

3.2.2 The supersolvable nilradical of an evolution algebra

Let $\mathcal{E}$ be an evolution algebra with a natural basis B. We introduce the sequence of ideals $\mathcal{N}^i(\mathcal{E})$, $i \geq 1$. To define $\mathcal{N}^1(\mathcal{E})$, we consider the sum of all one-dimensional abelian ideals, which is actually the sum of a finite number of them, namely $I_{1,j} = \mathsf{span}\{w_{1,j}\}$ for $j \in \Lambda_1$. By Proposition 1.5.5, we consider a partition of the set $\Lambda_1 = \Gamma_1 \sqcup \overline{\Gamma_1}$ such that $I_{1,j} \subset \mathrm{ann}(\mathcal{E})$ for any $j \in \Gamma_1$ (in fact, $\sum_{j \in \Gamma_1} I_{1,j} = \mathrm{ann}(\mathcal{E})$) and such that every $I_{1,j}$ with $j \in \overline{\Gamma_1}$ is the derived subalgebra of the basic ideal $\mathcal{E}_{1,j} = \mathsf{span}\{e_k \colon k \in \mathrm{supp}(w_{1,j})\} \in \mathcal{T}_{\mathbb{K}}$ of $\mathcal{E}$. Since the nilradical of each $\mathcal{E}_{1,j}$ with $j \in \overline{\Gamma_1}$ is characterised by Theorem 3.2.4, we define the ideal

$$\mathcal{N}^1(\mathcal{E}) := \sum_{j \in \overline{\Gamma_1}} \mathrm{Nil}(\mathcal{E}_{1,j}) + \sum_{j \in \Gamma_1} I_{1,j} = \sum_{I \subset \mathcal{E} \text{ basic ideal, } I \in \mathcal{T}_{\mathbb{K}}} \mathrm{Nil}(I) + \mathrm{ann}(\mathcal{E}). \quad (3.2.3)$$

Inductively, to define $\mathcal{N}^i(\mathcal{E})$ with $i \geq 2$, assume that $\mathcal{N}^{i-1}(\mathcal{E})$ is an ideal of $\mathcal{E}$ and consider the sum of all $(1 + \dim \mathcal{N}^{i-1}(\mathcal{E}))$-dimensional ideals which can be written as $\mathsf{span}\{w\} + \mathcal{N}^{i-1}(\mathcal{E})$ with $w^2 \in \mathcal{N}^{i-1}(\mathcal{E})$ and $\mathrm{supp}(w) \cap \mathrm{supp}(\mathcal{N}^{i-1}(\mathcal{E})) = \emptyset$. Again, this sum can be supposed to be the sum of a finite number of them, namely $I_{i,j} = \mathsf{span}\{w_{i,j}\} + \mathcal{N}^{i-1}(\mathcal{E})$ for $j \in \Lambda_i$, such that

$$w_{i,j}^2 \in \mathcal{N}^{i-1}(\mathcal{E}) \quad \text{and} \quad \mathrm{supp}(w_{i,j}) \cap \mathrm{supp}\left(\mathcal{N}^{i-1}(\mathcal{E})\right) = \emptyset \text{ for any } j \in \Lambda_i. \quad (3.2.4)$$

By construction, every $\overline{I_{i,j}} = I_{i,j}/\mathcal{N}^{i-1}(\mathcal{E}) = \mathsf{span}\{\overline{w_{i,j}}\}$ for $j \in \Lambda_i$ is a one-dimensional abelian ideal of the quotient evolution algebra $\mathcal{E}/\mathcal{N}^{i-1}(\mathcal{E})$. Then, again applying Proposition 1.5.5, consider a partition $\Lambda_i = \Gamma_i \sqcup \overline{\Gamma_i}$ such that every $\overline{I_{i,j}} \subset \mathrm{ann}\left(\mathcal{E}/\mathcal{N}^{i-1}(\mathcal{E})\right)$ with $j \in \Gamma_i$ and such that every $\overline{I_{i,j}}$, with $j \in \overline{\Gamma_i}$, is the derived subalgebra of the basic ideal $\mathcal{E}_{i,j} = \mathsf{span}\{\overline{e_k} \colon k \in \mathrm{supp}(w_{i,j})\} \in \mathcal{T}_{\mathbb{K}}$ of $\mathcal{E}/\mathcal{N}^{i-1}(\mathcal{E})$. Since the nilradical of every $\mathcal{E}_{i,j}$ with $j \in \overline{\Gamma_1}$ is characterised, we define $\mathcal{N}^i(\mathcal{E})$ by

$$\mathcal{N}^i(\mathcal{E})/\mathcal{N}^{i-1}(\mathcal{E}) := \sum_{j \in \overline{\Gamma_i}} \mathrm{Nil}(\mathcal{E}_{i,j}) + \sum_{j \in \Gamma_i} \overline{I_{i,j}}, \quad (3.2.5)$$

which is clearly an ideal of $\mathcal{E}/\mathcal{N}^{i-1}(\mathcal{E})$ since it is the sum of ideals of $\mathcal{E}/\mathcal{N}^{i-1}(\mathcal{E})$. Consequently, $\mathcal{N}^i(\mathcal{E})$ is an ideal of $\mathcal{E}$.

Remark 3.2.8. Notice that the sum of $\sum_{j\in\overline{\Gamma_1}} \mathrm{Nil}(\mathcal{E}_{1,j})$ and $\sum_{j\in\Gamma_1} I_{1,j}$ in (3.2.3) is not necessarily direct. For instance, consider the evolution algebra $\mathcal{E} = \mathcal{E}_3(1,-1,0)$. Its one-dimensional abelian ideals are $I_{1,1} = \mathsf{span}\{e_3\}$ and $I_{1,2} = \mathsf{span}\{e_1 + e_2 + e_3\}$. Moreover, $I_{1,1} = \mathrm{ann}(\mathcal{E})$ and $\mathrm{Nil}(\mathcal{E}_{1,2}) = \mathrm{Nil}(\mathcal{E}) = \mathsf{span}\{e_1 + e_2, e_3\}$. Nevertheless, $\mathrm{Nil}(\mathcal{E}_{1,2}) \cap I_{1,1} = \mathsf{span}\{e_3\} \neq 0$. However, due to Remark 1.2.6, such a sum may be assumed to be direct without loss of generality, by considering that all the basic ideals of $\mathcal{T}_{\mathbb{K}}$ are evolution algebras of the form $\mathcal{E}_k(\lambda_1,\dots,\lambda_n)$ with $k \geq 2$ and $\lambda_1,\dots,\lambda_k \neq 0$.

Remark 3.2.9. Notice that, unlike what happens in (3.2.3), $\sum_{j\in\Gamma_i} \overline{I_{i,j}}$ does not necessarily coincides with $\mathrm{ann}\left(\mathcal{E}/\mathcal{N}^{i-1}(\mathcal{E})\right)$ in (3.2.5). For instance, consider the evolution algebra $\mathcal{E}$ with natural basis $\{e_1,e_2,e_3,e_4\}$ and product given by $e_1^2 = -e_2^2 = e_1 + e_2$, $e_3^2 = e_1 + e_2 + e_3 + e_4$ and $e_4^2 = -e_3 - e_4$. It is easy to check that $\mathcal{N}^1(\mathcal{E}) = \mathsf{span}\{e_1 + e_2\}$, hence $\mathcal{E}/\mathcal{N}^1(\mathcal{E})$ can be seen as an evolution algebra with natural basis $\{\overline{e_1},\overline{e_3},\overline{e_4}\}$ and product given by $\overline{e_1}^2 = 0$ and $\overline{e_3}^2 = -\overline{e_4}^2 = \overline{e_3} + \overline{e_4}$. However, although $\overline{e_1} \in \mathrm{ann}(\mathcal{E}/\mathcal{N}^1(\mathcal{E}))$, we have $\overline{e_1} \notin \mathcal{N}^2(\mathcal{E})/\mathcal{N}^1(\mathcal{E}) = \mathsf{span}\{\overline{e_3} + \overline{e_4}\}$.

Subsequently, we prove that every term in the chain of ideals

$$0 \subseteq \mathcal{N}^1(\mathcal{E}) \subseteq \cdots \subseteq \mathcal{N}^r(\mathcal{E}) \subseteq \cdots$$

is nilpotent and $\mathcal{E}$-supersolvable.

Proposition 3.2.10. *$\mathcal{N}^i(\mathcal{E})$ is an $\mathcal{E}$-supersolvable nilpotent ideal of $\mathcal{E}$ for every $i \geq 1$.*

Proof. We use induction on $i \geq 1$. First, notice that every $\mathcal{E}_{1,j}$ is $\mathcal{E}$-supersolvable. Then, $\mathcal{N}^1(\mathcal{E})$ is $\mathcal{E}$-supersolvable since it is the sum of $\mathcal{E}$-supersolvable ideals. Moreover, it holds that $\mathcal{N}^1(\mathcal{E})^{\langle 2\rangle} = \mathsf{span}\{w_{1,j}\colon j \in \overline{\Gamma_1}\}$ and $\mathcal{N}^1(\mathcal{E})^{\langle 3\rangle} = 0$ by Corollary 3.2.5, what yields the nilpotency of $\mathcal{N}^1(\mathcal{E})$. Then, assume the assertion is true for i, that is, $\mathcal{N}^i(\mathcal{E})$ is an $\mathcal{E}$-supersolvable nilpotent ideal. Hence, $\mathcal{E}$-supersolvability follows straightforwardly from the fact that $\mathcal{N}^{i+1}(\mathcal{E})/\mathcal{N}^i(\mathcal{E})$ is clearly $(\mathcal{E}/\mathcal{N}^i(\mathcal{E}))$-supersolvable. For nilpotency, just notice that $\mathcal{N}^{i+1}(\mathcal{E})^{\langle 2\rangle} \subset \mathsf{span}\{w_{i+1,j}\colon j \in \overline{\Gamma_{i+1}}\} + \mathcal{N}^i(\mathcal{E})$ and, by (3.2.4), $\mathcal{N}^{i+1}(\mathcal{E})^{\langle k\rangle} \subset \mathcal{N}^i(\mathcal{E})^{\langle k-2\rangle}$ for any $k \geq 3$. The result follows. □

Hence, we introduce the following definition.

Definition 3.2.11. Let $\mathcal{E}$ be an evolution algebra. The chain of $\mathcal{E}$-supersolvable nilpotent ideals defined by (3.2.3) and (3.2.5),

$$0 \subseteq \mathcal{N}^1(\mathcal{E}) \subseteq \cdots \subseteq \mathcal{N}^r(\mathcal{E}) \subseteq \cdots,$$

will be called the *$\mathcal{E}$-supersolvable nilpotent series of $\mathcal{E}$*.

Our main objective is now to prove that, given an evolution algebra $\mathcal{E}$, the term $\mathcal{N}^r(\mathcal{E})$ in the $\mathcal{E}$-supersolvable nilpotent series, where $r \geq 1$ is the number of steps until the series stabilises, is the largest $\mathcal{E}$-supersolvable nilpotent ideal of $\mathcal{E}$.

Before proceeding, we first characterise the nilpotent ideals of an evolution algebra $\mathcal{E}$ in terms of its $\mathcal{E}$-supersolvable nilpotent series, which will be instrumental in the following steps. To fix notation, given a subspace U of an evolution algebra $\mathcal{E}$, we denote by π_U the linear projection $\pi_U : \mathcal{E} \longrightarrow \mathsf{span}\{e_i : i \in \operatorname{supp}(U)\}$ along $\mathsf{span}\{e_i : i \notin \operatorname{supp}(U)\}$.

Proposition 3.2.12. *Let $\mathcal{E}$ be an evolution algebra, I an ideal and $\mathcal{N}^k(\mathcal{E})$ the largest term of the $\mathcal{E}$-supersolvable nilpotent series contained in I. Then, I is nilpotent if and only if there exists $l \in \mathbb{N}$ such that $I^{\langle l \rangle} \subset \mathcal{N}^k(\mathcal{E})$ and $\pi_{\mathcal{N}^k(\mathcal{E})}(w) \in \mathcal{N}^k(\mathcal{E})$ for any $w \in I$.*

Proof. First, we prove the sufficiency. By hypothesis, $\pi_{\mathcal{N}^k(\mathcal{E})}(w) \in \mathcal{N}^k(\mathcal{E})$ for all $w \in I$. Hence, just performing the corresponding elementary operations, I can be written as $K + \mathcal{N}^k(\mathcal{E})$, where $K = \mathsf{span}\{w_1, \ldots, w_r\}$ is a subspace (not necessarily a subalgebra) such that $\operatorname{supp}(K) \cap \operatorname{supp}(\mathcal{N}^k(\mathcal{E})) = \emptyset$. Consequently, $K \cdot \mathcal{N}^k(\mathcal{E}) = 0$. Then, as there exist $l, s \in \mathbb{N}$ such that $I^{\langle l \rangle} \subset \mathcal{N}^k(\mathcal{E})$ and $\mathcal{N}^k(\mathcal{E})^{\langle s \rangle} = 0$, it holds that

$$\begin{aligned} I^{\langle l+s \rangle} &= (\cdots (I^{\langle l \rangle} \cdot I) \overset{s)}{\cdots}) \cdot I \subseteq (\cdots (\mathcal{N}^k(\mathcal{E}) \cdot I) \overset{s)}{\cdots}) \cdot I \\ &= (\cdots (\mathcal{N}^k(\mathcal{E}) \cdot (K + \mathcal{N}^k(\mathcal{E}))) \overset{s)}{\cdots}) \cdot (K + \mathcal{N}^k(\mathcal{E})) = \mathcal{N}^k(\mathcal{E})^{\langle s \rangle} = 0. \end{aligned}$$

Next, we prove the necessity of both conditions. On the one hand, if $I^{\langle l \rangle} \not\subseteq \mathcal{N}^k(\mathcal{E})$ for any $l \in \mathbb{N}$, then $I^{\langle l \rangle} \neq 0$ for any $l \in \mathbb{N}$, which contradicts the nilpotency of I. On the other hand, if there exists an element $w \in I$ such that $\pi_{\mathcal{N}^k(\mathcal{E})}(w) \notin \mathcal{N}^k(\mathcal{E})$, then, by Corollary 3.2.5, there exists an index $i \leq k$ and an element $w_{i,j}$ with $j \in \overline{\Gamma_i}$ such that $ww_{i,j} \in \mathbb{K}^* w_{i,j} + \mathcal{N}^{i-1}(\mathcal{E})$. Consequently, $(\cdots((ww_{i,j})w)\cdots)w \neq 0$, which again contradicts the nilpotency of I. $\square$

As shown in Example 1.3.8, the sum of two nilpotent ideals in an evolution algebra is not necessarily nilpotent. However, we next prove that this holds when one of the summands is the largest term of the $\mathcal{E}$-supersolvable nilpotent series, $\mathcal{N}^r(\mathcal{E})$.

Proposition 3.2.13. *Let $\mathcal{E}$ be an evolution algebra. If I is a nilpotent ideal, then $I + \mathcal{N}^r(\mathcal{E})$ is also a nilpotent ideal.*

Proof. We first show that $(I + \mathcal{N}^r(\mathcal{E}))^{\langle k \rangle} \subseteq I^{\langle k \rangle} + \mathcal{N}^r(\mathcal{E})$ for any $k \geq 1$ by induction on k. When $k = 1$, the result is trivially true. Then, suppose the assertion is true for k. Hence, it follows that

$$(I + \mathcal{N}^r(\mathcal{E}))^{\langle k+1 \rangle} \subseteq (I^{\langle k \rangle} + \mathcal{N}^r(\mathcal{E}))(I + \mathcal{N}^r(\mathcal{E}))$$

$$= I^{\langle k+1\rangle} + \mathcal{N}^r(\mathcal{E})(I^{\langle k\rangle} + I + \mathcal{N}^r(\mathcal{E})) \subseteq I^{\langle k+1\rangle} + \mathcal{N}^r(\mathcal{E}),$$

and the claim is established. Moreover, as I is nilpotent by hypothesis, there exists a number $l \in \mathbb{N}$ such that $I^{\langle l\rangle} = 0$ and, consequently, $(I + \mathcal{N}^r(\mathcal{E}))^{\langle l\rangle} \subseteq \mathcal{N}^r(\mathcal{E})$.

Next, we claim that $\pi_{\mathcal{N}^r(\mathcal{E})}(w) \in \mathcal{N}^r(\mathcal{E})$ for any $w \in I + \mathcal{N}^r(\mathcal{E})$. Otherwise, by Corollary 3.2.5, there would exist an element $w \in I$, an index $i \leq r$ and an element $w_{i,j}$ with $j \in \overline{\Gamma_i}$ such that $ww_{i,j} \in \mathbb{K}^* w_{i,j} + \mathcal{N}^{i-1}(\mathcal{E})$; and moreover, $ww_{i,j} \in I$ since I is an ideal. Consequently, $(\cdots(((ww_{i,j})w)w)\cdots)w \neq 0$, which contradicts the nilpotency of I.

Since $(I+\mathcal{N}^r(\mathcal{E}))^{\langle l\rangle} \subseteq \mathcal{N}^r(\mathcal{E})$ and $\pi_{\mathcal{N}^r(\mathcal{E})}(w) \in \mathcal{N}^r(\mathcal{E})$ for any $w \in I+\mathcal{N}^r(\mathcal{E})$, the result follows from Proposition 3.2.12. □

Theorem 3.2.14. *Let $\mathcal{E}$ be an evolution algebra, and $r \geq 1$ the number of steps until the $\mathcal{E}$-supersolvable nilpotent series stabilises. Then, $\mathcal{N}^r(\mathcal{E})$ is the largest $\mathcal{E}$-supersolvable nilpotent ideal of $\mathcal{E}$.*

Proof. Notice that $\mathcal{N}^r(\mathcal{E})$ is an $\mathcal{E}$-supersolvable nilpotent ideal by Proposition 3.2.10. First, we show that $\mathcal{N}^r(\mathcal{E})$ is a maximal $\mathcal{E}$-supersolvable nilpotent ideal of $\mathcal{E}$. Assume, for the sake of contradiction, that there exists an $\mathcal{E}$-supersolvable nilpotent ideal I such that $\mathcal{N}^r(\mathcal{E}) \subsetneq I$. Then, as $\mathcal{N}^r(\mathcal{E})$ and I are $\mathcal{E}$-supersolvable, there clearly exists a $(1 + \dim \mathcal{N}^r(\mathcal{E}))$-dimensional nilpotent ideal J such that $\mathcal{N}^r(\mathcal{E}) \subsetneq J \subseteq I$. As a consequence of Proposition 3.2.12, J can be written as $\mathsf{span}\{w\} + \mathcal{N}^r(\mathcal{E})$ with $w^2 \in \mathcal{N}^r(\mathcal{E})$ and $\mathrm{supp}(w) \cap \mathrm{supp}(\mathcal{N}^r(\mathcal{E})) = \emptyset$, a contradiction with the fact that the series stabilises for r.

Now, assume that another maximal $\mathcal{E}$-supersolvable nilpotent ideal exists, say I, different from $\mathcal{N}^r(\mathcal{E})$. By Proposition 3.2.13, the sum $I + \mathcal{N}^r(\mathcal{E})$ is also an $\mathcal{E}$-supersolvable nilpotent ideal, which strictly contains $\mathcal{N}^r(\mathcal{E})$. However, this contradicts the maximality of $\mathcal{N}^r(\mathcal{E})$ previously shown, completing the proof. □

In view of the previous result, we introduce the following definition.

Definition 3.2.15. The *supersolvable nilradical* of an evolution algebra $\mathcal{E}$, denoted by $\mathrm{SNil}(\mathcal{E})$, is defined as the largest $\mathcal{E}$-supersolvable nilpotent ideal of $\mathcal{E}$. Equivalently, it holds that $\mathrm{SNil}(\mathcal{E}) = \mathcal{N}^r(\mathcal{E})$, where r is the number of steps until the $\mathcal{E}$-supersolvable nilpotent series of $\mathcal{E}$ stabilises.

For the reader's convenience, we now present an example of how to compute the supersolvable nilradical of an evolution algebra.

Example 3.2.16. Let $\mathcal{E}$ be an evolution algebra with natural basis $\{e_1, \ldots, e_8\}$ and structure matrix

$$\left(\begin{array}{ccc|c|cc|cc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
4 & 0 & 2 & 0 & -1 & -1 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1
\end{array}\right)$$

The one-dimensional abelian ideals of $\mathcal{E}$ are $I_{1,1} = \mathsf{span}\{e_4\}$ and $I_{1,2} = \mathsf{span}\{e_1 + e_2 + e_3\}$. In fact, $I_{1,1} = \operatorname{ann}(\mathcal{E})$ and $I_{1,2}$ is the derived subalgebra of the basic ideal $\mathcal{E}_{1,2} = \mathsf{span}\{e_1, e_2, e_3\} = \mathcal{E}_3(1, 1, -2)$. Consequently, following (3.2.3), we have that

$$\mathcal{N}^1(\mathcal{E}) = \mathrm{Nil}(\mathcal{E}_{1,2}) + I_{1,1} = \mathsf{span}\{e_2 - e_1, 2e_1 + e_3, e_4\}.$$

Next, following condition (3.2.4), we consider the ideals $I_{2,1} = \mathsf{span}\{e_5 + e_6\} + \mathcal{N}^1(\mathcal{E})$ and $I_{2,2} = \mathsf{span}\{e_7 + e_8\} + \mathcal{N}^1(\mathcal{E})$. Notice that $\overline{I_{2,1}} = \mathsf{span}\{\overline{e_5} + \overline{e_6}\}$ and $\overline{I_{2,2}} = \mathsf{span}\{\overline{e_7} + \overline{e_8}\}$ are one-dimensional abelian ideals of $\mathcal{E}/\mathcal{N}^1(\mathcal{E})$, which are the derived subalgebras of the basic ideals $\mathcal{E}_{2,1} = \mathsf{span}\{\overline{e_5}, \overline{e_6}\} \cong \mathcal{E}_2(1, -1)$ and $\mathcal{E}_{2,2} = \mathsf{span}\{\overline{e_7}, \overline{e_8}\} \cong \mathcal{E}_2(1, -1)$, respectively. Consequently, following (3.2.5), we have that

$$\mathcal{N}^2(\mathcal{E})/\mathcal{N}^1(\mathcal{E}) = \mathrm{Nil}(\mathcal{E}_{2,1}) + \mathrm{Nil}(\mathcal{E}_{2,2}) = \mathsf{span}\{\overline{e_5} + \overline{e_6}, \overline{e_7} + \overline{e_8}\},$$

which implies that $\mathcal{N}^2(\mathcal{E}) = \mathsf{span}\{e_2 - e_1, 2e_1 + e_3, e_4, e_5 + e_6, e_7 + e_8\}$. Moreover, as $\operatorname{supp}\left(\mathcal{N}^2(\mathcal{E})\right) = \{1, \ldots, 8\}$, the series clearly stabilises in this second step. Hence,

$$\mathrm{SNil}(\mathcal{E}) = \mathcal{N}^2(\mathcal{E}) = \mathsf{span}\{e_2 - e_1, 2e_1 + e_3, e_4, e_5 + e_6, e_7 + e_8\}.$$

Notice that the construction of the supersolvable nilradical is clearly inspired by Theorem 3.2.4. Consequently, unlike the basic nilradical, if $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$, then $\mathrm{SNil}(\mathcal{E}) = \mathrm{Nil}(\mathcal{E})$. In what follows, we characterise the nilradical when the supersolvable nilradical is a maximal nilpotent ideal.

Corollary 3.2.17. *Let $\mathcal{E}$ be an evolution algebra. Then, $\mathrm{Nil}(\mathcal{E}) = \mathrm{SNil}(\mathcal{E})$ if and only if $\mathrm{SNil}(\mathcal{E})$ is a maximal nilpotent ideal.*

Proof. The necessity is straightforward. For the sufficiency, observe that if $\mathrm{SNil}(\mathcal{E})$ is a maximal nilpotent ideal, it must be unique. Indeed, if there existed another maximal nilpotent ideal I, then $\mathrm{SNil}(\mathcal{E}) + I$ would also be nilpotent by Proposition 3.2.13, contradicting the maximality of $\mathrm{SNil}(\mathcal{E})$. □

Finally, the following example shows that evolution algebras whose nilradicals are well-defined but do not coincide with their supersolvable nilradicals also exist.

Example 3.2.18. Let $\mathcal{E}$ be the evolution algebra with natural basis $\{e_1, e_2, e_3, e_4\}$ and product given by $e_1^2 = -e_2^2 = e_1 + e_2 + e_3 + e_4$ and $e_3^2 = -e_4^2 = e_1 + e_2$. It is easy to check that $\mathcal{N} = \mathsf{span}\{e_1 + e_2, e_3, e_4\}$ is the unique maximal nilpotent ideal of $\mathcal{E}$, and consequently, $\mathrm{Nil}(\mathcal{E}) = \mathcal{N}$. However, $\mathcal{E}$ has no one-dimensional abelian ideals, so $\mathrm{SNil}(\mathcal{E}) = 0$.

3.3 The Frattini subalgebra and ideal via the supersolvable nilradical

Regarding its counterpart in both group theory and Lie algebras, the *abelian socle* of an evolution algebra $\mathcal{E}$, $\mathrm{Asoc}(\mathcal{E})$, is defined as the sum of all minimal abelian ideals of $\mathcal{E}$. Nevertheless, during this section we will mainly work with the sum of all one-dimensional abelian ideals of $\mathcal{E}$, which will be denoted by $\mathrm{Asoc}_1(\mathcal{E})$. In fact, as a consequence of Proposition 1.5.5, given an evolution algebra $\mathcal{E}$, it holds that

$$\mathrm{Asoc}_1(\mathcal{E}) = \sum_{I \subset \mathcal{E} \text{ basic ideal, } I \in \mathcal{T}_{\mathbb{K}}} I^2 + \mathrm{ann}(\mathcal{E}). \tag{3.3.1}$$

In particular, when all basic ideals $I \in \mathcal{T}_{\mathbb{K}}$ are assumed to split over its annihilator as stated in Remark 1.2.6, the previous sums are direct. Clearly, $\mathrm{Asoc}_1(\mathcal{E})$ is $\mathcal{E}$-supersolvable, and $\mathrm{SNil}(\mathcal{E}) \cap \mathrm{Asoc}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$. Moreover, if $\mathcal{E}$ is supersolvable (if $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$, for instance) then, we obtain that $\mathrm{Asoc}_1(\mathcal{E}) = \mathrm{Asoc}(\mathcal{E})$ since every ideal in this case is $\mathcal{E}$-supersolvable.

Our first result of this section characterises the Frattini subalgebra and the Frattini ideal of solvable but nonnilpotent evolution algebras with one-dimensional derived subalgebras.

Theorem 3.3.1. *Let $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ with $\mathbb{K}$ a field of characteristic not two. Then, the following assertions are equivalent:*

(i) *the annihilator of $\mathcal{E}$ is of codimension two;*

(ii) *$\mathcal{E}$ is isomorphic to $\mathcal{E}_2(1, -1, 0, \ldots, 0)$;*

(iii) *$\mathcal{E}$ splits over its abelian socle;*

(iv) $F(\mathcal{E}) = \phi(\mathcal{E}) = 0$*; and*

(v) $\mathrm{Nil}(\mathcal{E})$ *is abelian.*

Otherwise, $F(\mathcal{E}) = \phi(\mathcal{E}) = \mathrm{Nil}(\mathcal{E})^2 = \mathcal{E}^2$.

Proof.

(i)⇒(ii): Remark 1.2.5 implies that $\mathcal{E}$ is isomorphic to $\mathcal{E}_2(\lambda, -\lambda, 0, \ldots, 0)$ for some $\lambda \in \mathbb{K}^*$. Thus, by performing the natural basis transformation $f_1 = \frac{1}{\lambda}e_1$, $f_2 = \frac{1}{\lambda}e_2$ and $f_i = e_i$ for all $i = 3, \ldots, n$, the result follows.

(ii)⇒(iii): By (3.3.1) and the fact that $\mathcal{E}$ is supersolvable, we have that the subalgebra $\mathsf{span}\{e_1 - e_2\}$ clearly complements

$$\begin{aligned}\mathrm{Asoc}\left(\mathcal{E}_2(1, -1, 0, \ldots, 0)\right) &= \mathrm{Asoc}_1\left(\mathcal{E}_2(1, -1, 0, \ldots, 0)\right) \\ &= \mathsf{span}\{e_1 + e_2, e_3, \ldots, e_n\}.\end{aligned}$$

(iii)⇒(iv): If $\mathcal{E}$ splits over its abelian socle, there exists a subalgebra $U \subset \mathcal{E}$ such that $\mathcal{E} = \mathrm{Asoc}(\mathcal{E}) \oplus U$. Since $\mathcal{E}$ is supersolvable, $\mathrm{Asoc}(\mathcal{E})$ can be written as the direct sum of some one-dimensional abelian ideals, say $\mathrm{Asoc}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E}) = I_1 \oplus \cdots \oplus I_m$. Then, the subalgebra $M_i = (I_1 \oplus \cdots \oplus \widehat{I_i} \oplus \cdots \oplus I_m) + U$ is a maximal subalgebra of $\mathcal{E}$ for all $i = 1, \ldots, m$. Since $F(\mathcal{E}) \subseteq \mathcal{E}^2$ in general, we have that

$$\phi(\mathcal{E}) \subseteq F(\mathcal{E}) \subseteq (\cap_{i=1}^m M_i) \cap \mathcal{E}^2 = U \cap \mathcal{E}^2 = 0,$$

where the last equality follows from the fact that $\mathcal{E}^2$ is a one-dimensional abelian ideal and, consequently, is contained in $\mathrm{Asoc}(\mathcal{E})$.

(iv)⇒(v): By Theorem 3.2.4, the square of the nilradical, $\mathrm{Nil}(\mathcal{E})^2$, of an evolution algebra $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ is either 0 or $\mathcal{E}^2$. Since $\mathrm{Nil}(\mathcal{E})^2$ is an ideal and $\mathrm{Nil}(\mathcal{E})$ is nilpotent, Lemma 3.1.3 implies that $\mathrm{Nil}(\mathcal{E})^2 = F(\mathrm{Nil}(\mathcal{E})) \subset F(\mathcal{E}) = 0$.

(v)⇒(i): For the sake of contradiction, consider an evolution algebra $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$ such that $\mathrm{codim}\left(\mathrm{ann}(\mathcal{E})\right) > 2$, that is, an evolution algebra $\mathcal{E} = \mathcal{E}_k(\lambda_1, \ldots, \lambda_n)$ with $k \geq 3$ and $\lambda_1, \ldots, \lambda_k \neq 0$. Then, by Theorem 3.2.4, we have $\lambda_2 e_1 - \lambda_1 e_2, \lambda_3 e_1 - \lambda_1 e_3 \in \mathrm{Nil}(\mathcal{E})$. However, $(\lambda_2 e_1 - \lambda_1 e_2)(\lambda_3 e_1 - \lambda_1 e_3) = \lambda_2\lambda_3 e_1^2 \neq 0$, what yields that $\mathrm{Nil}(\mathcal{E})$ is not abelian, a contradiction. Furthermore, in this case, we have that $\mathcal{E}^2 = \mathrm{Nil}(\mathcal{E})^2 = F(\mathrm{Nil}(\mathcal{E})) \subset F(\mathcal{E}) \subset \mathcal{E}^2$. Since $\mathcal{E}^2$ is an ideal, then $F(\mathcal{E}) = \phi(\mathcal{E}) = \mathcal{E}^2$, and the final assertion follows. □

Next, our goal is to develop a necessary condition for an evolution algebra to be ϕ-free, using the basic and supersolvable nilradicals. However, we should be aware of one of the main weaknesses of this supersolvable nilradical compared to the nilradical of a Lie or Leibniz algebra: in general, its square is not guaranteed to be an ideal, as shown in the following example.

Example 3.3.2. Let $\mathcal{E}$ be the evolution algebra with natural basis $\{e_1, e_2, e_3, e_4, e_5\}$ and product given by $e_1^2 = -e_2^2 = e_1+e_2+e_3$, $e_3^2 = 0$, $e_4^2 = e_1+e_2+e_3+e_4+e_5$ and $e_5^2 = -e_3 - e_4 - e_5$. Actually, its supersolvable nilradical is $\mathrm{SNil}(\mathcal{E}) = \mathsf{span}\{e_1 + e_2, e_3, e_4 + e_5\}$. However, its square $\mathrm{SNil}(\mathcal{E})^2 = \mathsf{span}\{e_1 + e_2\}$ is not an ideal.

Theorem 3.3.3. *Let $\mathcal{E}$ be an evolution algebra. If $\mathcal{E}$ is ϕ-free, then the following hold:*

(i) $\mathrm{BNil}(\mathcal{E}) = \mathrm{ann}(\mathcal{E})$*; and*

(ii) *if* $\mathrm{SNil}(\mathcal{E})^2$ *is an ideal, then* $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$.

Proof. For (i), note that $\mathrm{BNil}(\mathcal{E})^2$ is always an ideal. In fact, $\mathcal{E} \cdot \mathrm{BNil}(\mathcal{E})^2 \subset \mathcal{E} \cdot \mathrm{BNil}(\mathcal{E}) = \mathrm{BNil}(\mathcal{E})^2$. Hence, by Lemmas 3.1.1 and 3.1.3, we have that $\phi(\mathrm{BNil}(\mathcal{E})) = \mathrm{BNil}(\mathcal{E})^2 \subset \phi(\mathcal{E}) = 0$. Consequently, as the annihilator is the largest basic abelian ideal, the result follows.

The proof of (ii) is modelled on [102, Theorem 7.4]. As $\mathrm{SNil}(\mathcal{E})^2$ is an ideal by hypothesis, by Lemmas 3.1.1 and 3.1.3, we have that $\phi(\mathrm{SNil}(\mathcal{E})) = \mathrm{SNil}(\mathcal{E})^2 \subset \phi(\mathcal{E}) = 0$. Moreover, by Lemma 3.1.5, there exists a subalgebra $U \subset \mathcal{E}$ such that $\mathcal{E} = \mathrm{Asoc}_1(\mathcal{E}) \oplus U$. Then, as $\mathrm{Asoc}_1(\mathcal{E}) \subset \mathcal{N}^1(\mathcal{E}) \subset \mathrm{SNil}(\mathcal{E})$ in general, it holds that

$$\mathrm{SNil}(\mathcal{E}) = \mathcal{E} \cap \mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E}) \oplus (U \cap \mathrm{SNil}(\mathcal{E})). \tag{3.3.2}$$

Next, we claim that $U \cap \mathrm{SNil}(\mathcal{E})$ is an abelian ideal of $\mathcal{E}$. As $U \cap \mathrm{SNil}(\mathcal{E})$ is an ideal of U, it holds that

$$\begin{aligned} \mathcal{E} \cdot (U \cap \mathrm{SNil}(\mathcal{E})) &= \mathrm{Asoc}_1(\mathcal{E}) \cdot (U \cap \mathrm{SNil}(\mathcal{E})) + U \cdot (U \cap \mathrm{SNil}(\mathcal{E})) \\ &\subset \mathrm{Asoc}_1(\mathcal{E})\, \mathrm{SNil}(\mathcal{E}) + U \cap \mathrm{SNil}(\mathcal{E}) \\ &\subset \mathrm{SNil}(\mathcal{E})^2 + U \cap \mathrm{SNil}(\mathcal{E}) = U \cap \mathrm{SNil}(\mathcal{E}). \end{aligned}$$

Moreover, $\left(U \cap \mathrm{SNil}(\mathcal{E})\right)^2 \subset \mathrm{SNil}(\mathcal{E})^2 = 0$. Therefore, $U \cap \mathrm{SNil}(\mathcal{E}) = 0$ or $U \cap \mathrm{SNil}(\mathcal{E})$ contains a one-dimensional abelian ideal of $\mathcal{E}$ (which could be itself). Otherwise, we would contradict the $\mathcal{E}$-supersolvability of $\mathrm{SNil}(\mathcal{E})$. The latter implies that

$$\mathrm{Asoc}_1(\mathcal{E}) \cap (U \cap \mathrm{SNil}(\mathcal{E})) \neq 0,$$

which contradicts (3.3.2). Hence, we conclude that $U \cap \mathrm{SNil}(\mathcal{E}) = 0$, and thus $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$. □

The following result makes easier the validation of the condition $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$.

Proposition 3.3.4. *Let $\mathcal{E}$ be an evolution algebra. Then,* $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$ *if and only if* $\mathrm{SNil}(\mathcal{E})^2 = 0$ *and* $\mathrm{SNil}(\mathcal{E}) = \mathcal{N}^1(\mathcal{E})$.

Proof. First, assume that $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$. Since $\mathrm{Asoc}_1(\mathcal{E})$ is abelian and, in general, $\mathrm{Asoc}_1(\mathcal{E}) \subset \mathcal{N}^1(\mathcal{E})$, we are done.

Conversely, since $\mathrm{SNil}(\mathcal{E}) = \mathcal{N}^1(\mathcal{E})$ and $\mathrm{SNil}(\mathcal{E})$ is abelian, the nilradical of every basic ideal $I \in \mathcal{T}_{\mathbb{K}}$ of $\mathcal{E}$ must be abelian. Equivalently, by Theorem 3.3.1, each I must be isomorphic to $\mathcal{E}_2(1, -1, 0, \dots, 0)$. Notice that the nilradical of these algebras can be written as the sum of one-dimensional abelian ideals. Indeed,

$$\mathrm{Nil}\left(\mathcal{E}_2(1, -1, 0, \dots, 0)\right) = \mathsf{span}\{e_1 + e_2\} + \mathsf{span}\{e_3\} + \cdots + \mathsf{span}\{e_n\}.$$

Thus, the result follows. □

The following example shows that the converse of Theorem 3.3.3 is not true in general.

Example 3.3.5. Let $\mathcal{E}$ be the evolution algebra with natural basis $\{e_1, e_2, e_3\}$ and product $e_1^2 = -e_2^2 = e_1 + e_2$ and $e_3^2 = e_2$. As shown in Example 2.2.3, we have $\mathrm{Asoc}_1(\mathcal{E}) = \mathrm{SNil}(\mathcal{E}) = \mathsf{span}\{e_1 + e_2\}$ but $\phi(\mathcal{E}) = \mathsf{span}\{e_1, e_2\} \neq 0$.

We now focus on a specific case in which the equivalence holds even when we omit the hypothesis of the square of the supersolvable nilradical being ideal, that is, when $\mathrm{supp}\left(\mathrm{SNil}(\mathcal{E})\right) = \mathrm{supp}(\mathcal{E})$ as in Example 3.2.16.

Theorem 3.3.6. *Let $\mathcal{E}$ be an evolution algebra over a field $\mathbb{K}$ of characteristic not two with natural basis $B = \{e_1, \dots, e_n\}$ such that* $\mathrm{supp}\left(\mathrm{SNil}(\mathcal{E})\right) = \mathrm{supp}(\mathcal{E})$. *Then, the following assertions are equivalent:*

(i) *$\mathcal{E}$ is ϕ-free;*

(ii) *$\mathcal{E}$ splits over* $\mathrm{Asoc}_1(\mathcal{E})$*;*

(iii) *$\mathcal{E}$ splits over its annihilator, and its complement can be seen as the direct sum of copies of $\mathcal{E}_2(1, -1)$; that is, $\mathcal{E} = K \oplus \mathrm{ann}(\mathcal{E})$ where $K \cong \bigoplus_{i=1}^{m} \mathcal{E}_2(1, -1)$ with $m \leq \lfloor \frac{n}{2} \rfloor$;*

(iv) $\mathrm{SNil}(\mathcal{E})^2 = 0$ *and* $\mathrm{SNil}(\mathcal{E}) = \mathcal{N}^1(\mathcal{E})$.

Proof.

(i)⇒(ii): This follows directly from Lemma 3.1.5.

(ii)⇒(iii): Assume that $\mathcal{E}$ splits over $\mathrm{Asoc}_1(\mathcal{E})$, meaning that there exists a subalgebra U of $\mathcal{E}$ such that $\mathcal{E} = \mathrm{Asoc}_1(\mathcal{E}) \oplus U$. By Remark 1.1.7, we define the basic ideal $I = \mathsf{span}\{e_i \colon i \in \operatorname{supp}(\operatorname{Asoc}_1(\mathcal{E}))\}$. Now, by a combination of Remarks 1.2.5 and 1.2.6, Proposition 1.5.5 and Theorem 3.3.1 (iii), it follows that

$$I \cong \left(\bigoplus_{i=1}^{m} \mathcal{E}_2(1,-1) \right) \oplus \operatorname{ann}(\mathcal{E}).$$

Thus, without loss of generality, if we set $\dim(\operatorname{ann}(\mathcal{E})) = r$, we can write that

$$\begin{aligned}\operatorname{Asoc}_1(\mathcal{E}) &= \mathcal{N}^1(\mathcal{E}) \\ &= \mathsf{span}\{e_1 + e_2, e_3 + e_4, \ldots, e_{2m} + e_{2m+1}, e_{2m+2}, \ldots, e_{2m+r+1}\}.\end{aligned}$$

Furthermore, since $\operatorname{Asoc}_1(\mathcal{E}) \subset I$, it holds that $I = \mathcal{E} \cap I = \operatorname{Asoc}_1(\mathcal{E}) \oplus (U \cap I)$, where we can write that $U \cap I = \mathsf{span}\{u_1, \ldots, u_m\}$ with $u_i \in \mathbb{K}^*(e_{2i-1} - e_{2i}) + \operatorname{Asoc}_1(\mathcal{E})$ in such way that $u_i^2 = 0$ for all $1 \leq i \leq m$.

Now, we claim that $\mathcal{E} = I$. Denote $k = 2m + r + 2$ and assume, for the sake of contradiction, that $k \leq n$. Then, since $\mathcal{E} = \operatorname{Asoc}_1(\mathcal{E}) \oplus U$, notice that there must also exist elements $u_k = e_k + w_k, \ldots, u_n = e_n + w_n \in U$, where $w_j \in \operatorname{Asoc}_1(\mathcal{E})$ for all $k \leq j \leq n$. Now, distinguish the following two cases, both of which will lead to a contradiction:

1. If there exists at least one $w_j \notin \operatorname{ann}(\mathcal{E})$ then, there clearly exist one u_i with $1 \leq i \leq m$ such that $u_i u_j \in \mathbb{K}^*(e_{2i-1} + e_{2i}) \subset U$, which contradicts the fact that $e_{2i-1} + e_{2i} \in \operatorname{Asoc}_1(\mathcal{E})$.

2. If every $w_j \in \operatorname{ann}(\mathcal{E})$ then, by the construction of the supersolvable nilradical, there exists at least one element in the set $\{e_k, \ldots, e_n\}$, say e_k without loss of generality, such that $e_k^2 \in \mathcal{N}^2(\mathcal{E})$ and

$$\operatorname{supp}(e_k^2) \cap \operatorname{supp}(\mathcal{N}^1(\mathcal{E})) \neq \emptyset.$$

Now, consider the quotient evolution algebra $\mathcal{E}/\mathcal{N}^1(\mathcal{E})$ and the following two subcases:

   (a) If $\overline{e_k^2} = 0$, then $u_k^2 = e_k^2 + w_k^2 = e_k^2 \in \mathcal{N}^1(\mathcal{E})$, a contradiction with the fact that U complements $\operatorname{Asoc}_1(\mathcal{E})$.

   (b) If $\overline{e_k^2} \neq 0$, then there exist elements $e_k, \ldots, e_s$, with $k < s \leq n$, such that $\mathsf{span}\{\overline{e_k}, \ldots, \overline{e_s}\} \in \mathcal{T}_{\mathbb{K}}$ is a basic ideal of $\mathcal{E}/\mathcal{N}^1(\mathcal{E})$. Assume, without loss of generality, that $\mathsf{span}\{\overline{e_k}, \ldots, \overline{e_s}\} \cong \mathcal{E}_p(\lambda_1, \ldots, \lambda_{s-k+1})$ with $2 \leq p \leq s-k+1$ and $\lambda_1, \ldots, \lambda_p \neq 0$. Then, by Theorem 3.2.4, its nilradical is

$$\mathsf{span}\{\lambda_2\overline{e_k} - \lambda_1\overline{e_{k+1}}, \ldots, \lambda_p\overline{e_k} - \lambda_1\overline{e_{k+p-1}}, \overline{e_{k+p}}, \ldots, \overline{e_s}\}.$$

Then, we have

$$(u_k + \cdots + u_s)(\lambda_l u_k - \lambda_1 u_{k+l-1}) \\ = (e_k + \cdots + e_{k+p-1})(\lambda_2 e_k - \lambda_1 e_{k+1}) \in \mathcal{N}^1(\mathcal{E}),$$

for all $2 \leq l \leq p$. Since U complements $\mathrm{Asoc}_1(\mathcal{E}) = \mathcal{N}^1(\mathcal{E})$, all the previous products must be zero. Consequently, $\mathrm{rank}\,\{e_k^2, \ldots, e_{k+p-1}^2\} = 1$ and $e_k^2 + \cdots + e_{k+p-1}^2 = 0$. Moreover, as every $w_i \in \mathrm{ann}(\mathcal{E})$, we necessarily have that $e_k^2 \in \mathbb{K}^*(e_k + \cdots + e_{k+p-1}) + \mathrm{ann}(\mathcal{E})$. Thus, we have that $e_k^2 e_k^2 = 0$ and we conclude that $e_k^2 \in \mathrm{Asoc}_1(\mathcal{E})$, contradicting the definition of I.

As in both cases we get a contradiction, the result follows.

(iii)⇔(iv): Straightforward from the construction of the supersolvable nilradical.

(iii)⇒(i): By [102, Theorem 4.8], we have that $\phi(\mathcal{E}) = \phi(\mathcal{E}_2(1,-1)) \oplus \cdots \oplus \phi(\mathcal{E}_2(1,-1)) \oplus \phi(\mathrm{ann}(\mathcal{E})) = 0$. Thus, $\mathcal{E}$ is ϕ-free, completing the proof. □

3.4 Dually atomistic evolution algebras

An evolution algebra $\mathcal{E}$ will be called *dually atomistic* if every proper subalgebra of $\mathcal{E}$ is an intersection of maximal subalgebras of $\mathcal{E}$. It is easy to see that if $\mathcal{E}$ is dually atomistic, then so is every quotient algebra of $\mathcal{E}$ and, moreover, it is ϕ-free.

In the context of nonassociative algebras, Scheiderer proved in [95] that every dually atomistic Lie algebra is either abelian, almost abelian or simple over a field of characteristic zero. A slightly weaker version of this result, which holds over any field, was established in [85]. Specifically, if $\mathcal{L}$ is a dually atomistic Lie algebra over an arbitrary field, then L is either abelian, almost abelian or semisimple. However, an analogous result cannot be established in the context of evolution algebras.

Example 3.4.1. Let $\mathcal{E}$ be the complex evolution algebra with natural basis $\{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_3^2 = \frac{1}{4}e_1 + \frac{1}{4}e_2 + e_3$. Although $\mathcal{E}$ is neither abelian, almost abelian, nor semisimple, Figure 2.4.1 shows that $\mathcal{E}$ is dually atomistic.

Our goal in this section is to apply the concepts previously developed to study the dually atomistic property within specific families of evolution algebras. We exclude the semisimple case from our analysis since semisimple evolution algebras are semiprime and thus have a trivial nilradical. Additionally, abelian evolution algebras are clearly dually atomistic.

Remark 3.4.2. Since every dually atomistic evolution algebra $\mathcal{E}$ is ϕ-free, Theorem 3.3.3 provides two straightforward necessary conditions: $\mathrm{BNil}(\mathcal{E}) = \mathrm{ann}(\mathcal{E})$; and if $\mathrm{SNil}(\mathcal{E})^2$ is an ideal, then $\mathrm{SNil}(\mathcal{E}) = \mathrm{Asoc}_1(\mathcal{E})$.

In particular, this section aims to prove the following classification result.

Theorem 3.4.3. *Let $\mathcal{E}$ be an evolution algebra that is either almost abelian or satisfies* $\mathrm{supp}\left(\mathrm{SNil}(\mathcal{E})\right) = \mathrm{supp}(\mathcal{E})$*. If $\mathcal{E}$ is dually atomistic, then it is isomorphic to one of the following pairwise nonisomorphic almost abelian evolution algebras:*

- $\mathcal{E}_2(1,-1)\colon e_1^2 = -e_2^2 = e_1 + e_2$.
- $\mathcal{E}_{n,1}\colon e_1^2 = e_1, e_2^2 = \cdots = e_n^2 = 0$*, with* $n \in \mathbb{N}$.

The proof of this theorem will be a consequence of the results which follow.

Almost abelian evolution algebras had not been previously considered. So, we now provide their characterisation. To do so, we first introduce the concept of almost basic abelian evolution algebras, which will be essential for our purpose.

Definition 3.4.4. An evolution algebra $\mathcal{E}$ will be called *almost basic abelian* if it has an abelian basic ideal of codimension one, that is, its annihilator is of codimension one.

Proposition 3.4.5. *Let $\mathcal{E}$ be an almost abelian evolution algebra. Then, $\mathcal{E}$ is almost basic abelian, nilpotent, or lies in $\mathcal{T}_{\mathbb{K}}$ and its annihilator is of codimension two.*

Proof. Assume that $\mathcal{E}$ is almost abelian but not almost basic abelian and let $I = \mathsf{span}\{u_i \colon 1 \leq i \leq n-1\}$ with $u_i = \sum_{j=1}^{n} \mu_{ij} e_j$, $\mu_{ij} \in \mathbb{K}$ be an abelian ideal of codimension one. Without loss of generality, suppose that the matrix $(\mu_{ij})_{i,j=1}^{n-1,n}$ is in reduced row echelon form. As $\mathcal{E}$ is not almost basic abelian then I is not basic. Consequently, there exists $e_k \in B$ such that $I = \mathsf{span}\{e_1 + \lambda_1 e_k, \ldots, e_{k-1} + \lambda_{k-1} e_k, e_{k+1}, \ldots, e_n\}$, where at least one of $\lambda_1, \ldots, \lambda_{k-1} \in \mathbb{K}$ is nonzero.

Now, we claim that there is only one nonzero scalar among $\lambda_1, \ldots, \lambda_{k-1} \in \mathbb{K}$. Indeed, if at least two of them were nonzero, say $\lambda_p, \lambda_q \neq 0$ with $1 \leq p, q \leq k-1$, then, since I is abelian, we would have

$$(e_p + \lambda_p e_k)(e_q + \lambda_q e_k) = \lambda_p \lambda_q e_k^2 = 0,$$

which implies that $e_k \in \mathrm{ann}(\mathcal{E})$. Moreover, in this case, we have that $(e_i + \lambda_i e_k)^2 = e_i^2 = 0$ for all $i = 1, \ldots, k-1$, meaning that $\mathcal{E}$ is abelian, a contradiction. Hence, there must be exactly one nonzero scalar, say $\lambda_p \neq 0$. Consequently, we necessarily have $e_p^2 = -\lambda_p^2 e_k^2 \neq 0$ and $e_i^2 = 0$ for any $i \neq p, k$. Therefore, if $e_p^2, e_k^2 \in \mathsf{span}\{e_i \in B \colon i \neq p, k\}$, then $\mathcal{E}$ is nilpotent but not almost basic abelian. Otherwise, if $e_p^2, e_k^2 \in \mathbb{K}^*(e_p + \lambda_p e_k) + \mathsf{span}\{e_i \in B\colon i \neq p, k\}$, then, via a suitable natural basis transformation, it is easy to see that $\mathcal{E}$ is isomorphic to $\mathcal{E}_2(1, -1, 0 \ldots, 0)$, what yields the claim by Theorem 3.3.1. □

As a consequence of the previous result, the study of the dually atomistic property in almost abelian evolution algebras reduces to three specific cases: the nilpotent case, the class of almost basic abelian evolution algebras, and the family $\mathcal{T}_{\mathbb{K}}$. Before proceeding, note that any nonabelian nilpotent evolution algebra $\mathcal{E}$ has a nontrivial Frattini subalgebra, $F(\mathcal{E}) = \mathcal{E}^2 \neq 0$, which ensures that it is not dually atomistic. Next, we study this property in the almost basic abelian case.

Proposition 3.4.6. *Let $\mathcal{E}$ be an almost basic abelian evolution algebra of dimension n. Then, $\mathcal{E}$ is isomorphic to one of the following pairwise nonisomorphic evolution algebras:*

- $\mathcal{E}_{n,1}\colon e_1^2 = e_1, e_2^2 = \cdots = e_n^2 = 0$;
- $\mathcal{E}_{n,2}\colon e_1^2 = e_2, e_2^2 = \cdots = e_n^2 = 0$.

Moreover, if $\mathcal{E}$ is dually atomistic, then it is necessarily isomorphic to $\mathcal{E}_{n,1}$.

Proof. Without loss of generality, assume that the product of $\mathcal{E}$ is given by $e_1^2 = \sum_{i=1}^n \alpha_i e_i$ and $e_2^2 = \cdots = e_n^2 = 0$. If $\alpha_1 \neq 0$, then we can define the element $x = \sum_{i=1}^n \frac{\alpha_i}{\alpha_1^2} e_i$, which is clearly idempotent. Thus, by considering the natural basis $\{x, e_2, \ldots, e_n\}$, it follows that $\mathcal{E} \cong \mathcal{E}_{n,1}$. Otherwise, suppose that $\alpha_1 = 0$ and consider the lowest index k such that $\alpha_k \neq 0$. Then, by considering the natural basis $\{e_1, e_1^2, e_2, \ldots, \widehat{e_k}, \ldots, e_n\}$, it follows that $\mathcal{E} \cong \mathcal{E}_{n,2}$. Moreover, notice that $\mathcal{E}_{n,2}$ is not dually atomistic. Since it is nilpotent, we have that $F(\mathcal{E}_{n,2}) = \phi(\mathcal{E}_{n,2}) = \mathcal{E}_{n,2}^2 = \mathsf{span}\{e_2\} \neq 0$.

Next, we check that $\mathcal{E}_{n,1}$ is dually atomistic. Its subalgebras are of one of the following types: $U = \mathsf{span}\{u_1, \ldots, u_m\}$ or $V = \mathsf{span}\{e_1\} + U$, where $u_1, \ldots, u_m \in \mathsf{span}\{e_2, \ldots, e_n\}$. Both types can be easily expressed as the intersection of maximal subalgebras. First, consider a linear independent subset $\{u_{m+1}, \ldots, u_{n-1}\}$ such that $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_{n-1}\}$ is a basis of $\mathsf{span}\{e_2, \ldots, e_n\}$. Then, we have that

$$V = \bigcap_{i=m+1}^{n-1} \mathsf{span}\{e_1, u_1, \ldots, u_m, u_{m+1}, \ldots, \widehat{u_i}, \ldots, u_{n-1}\} \quad \text{and}$$
$$U = V \cap \mathsf{span}\{e_2, \ldots, e_n\}.$$

The result follows. □

Finally, we fully characterise dually atomistic evolution algebras which satisfy $\operatorname{supp}\big(\operatorname{SNil}(\mathcal{E})\big) = \operatorname{supp}(\mathcal{E})$ (a family that includes $\mathcal{T}_{\mathbb{K}}$) through the following two technical lemmas.

Lemma 3.4.7. *Let $\mathcal{E} = \mathcal{E}_2(1,-1,0,\dots,0)$. Then, $M = \mathsf{span}\{e_1 - e_2, e_3, \dots, e_n\}$ is the only maximal subalgebra such that $e_1 + e_2 \notin M$.*

Proof. First, note that $\mathcal{E}_2(1,-1,0,\dots,0)$ is clearly supersolvable, then all maximal subalgebras have codimension one. Thus, consider a maximal subalgebra $M = \mathsf{span}\{u_i \colon 1 \leq i \leq n-1\}$ with $u_i = \sum_{j=1}^{n} \mu_{ij} e_j$, $\mu_{ij} \in \mathbb{K}$ such that $e_1 + e_2 \notin M$. Assume that the matrix $(\mu_{ij})_{i,j=1}^{n-1,n}$ is in reduced row echelon form. Now, we claim that $\mu_{22} = 0$. Otherwise, $\mu_{22} = 1$ and there would exist an element e_k of the natural basis with $k > 2$ such that $M = \mathsf{span}\{e_1 + \lambda_1 e_k, \dots, e_{k-1} + \lambda_{k-1} e_k, e_{k+1}, \dots, e_n\}$ with $\lambda_1, \dots, \lambda_{k-1} \in \mathbb{K}$. However, in this case, $(e_1 + \lambda_1 e_k)^2 = e_1^2 \in M$, which contradicts the fact that $e_1 + e_2 \notin M$. Then, $M = \mathsf{span}\{e_1 + \lambda e_2, e_3, \dots, e_n\}$ with $\lambda \in \mathbb{K}$. Moreover, it is easy to check that M is a subalgebra if and only if $\lambda = \pm 1$ but, as $e_1 + e_2 \notin M$, $\lambda = -1$ necessarily. □

Lemma 3.4.8. *Let $\mathcal{E} = \mathcal{E}_2(1,-1) \oplus \mathcal{E}_2(1,-1)$ be the evolution algebra with natural basis $\{e_1, e_2, e_3, e_4\}$ and multiplication given by $e_1^2 = -e_2^2 = e_1 + e_2$, $e_3^2 = -e_4^2 = e_3 + e_4$. Then, every maximal subalgebra of $\mathcal{E}$ that contains the subalgebra $\mathsf{span}\{e_1 + e_2 + e_3 + e_4\}$ also contains $\mathcal{E}^2$.*

Proof. First, note that $\mathcal{E}$ is supersolvable, and so all maximal subalgebras have codimension one. Additionally, observe that $\mathsf{span}\{e_1 + e_2, e_3, e_4\}$ and $\mathsf{span}\{e_1, e_2, e_3 + e_4\}$ are maximal subalgebras that contain $\mathsf{span}\{e_1 + e_2 + e_3 + e_4\}$, and both contain $\mathcal{E}^2 = \mathsf{span}\{e_1 + e_2, e_3 + e_4\}$. Now let $U = \mathsf{span}\{u_1, u_2, u_3\}$ be a maximal subalgebra containing $\mathsf{span}\{e_1 + e_2 + e_3 + e_4\}$ and assume U is different from the two subalgebras above. Without loss of generality, we can write:

$$u_1 = e_1 + e_2 + e_3 + e_4, \quad u_2 = e_2 + \alpha e_3 + \beta e_4, \quad u_3 = e_3 + \gamma e_4,$$

for some scalars $\alpha, \beta, \gamma \in \mathbb{K}$. Since U is closed under multiplication, we must have that

$$u_1 u_3 = e_3^2 + \gamma e_4^2 = (1-\gamma)(e_3 + e_4) \in U,$$

which holds if and only if $\gamma = 1$. Consequently, as $e_1 + e_2 = u_1 - u_3$, U contains both $e_1 + e_2$ and $e_3 + e_4$, so $\mathcal{E}^2 \subseteq U$, as claimed. □

Proposition 3.4.9. *Let $\mathcal{E}$ be an evolution algebra which satisfies $\operatorname{supp}\big(\operatorname{SNil}(\mathcal{E})\big) = \operatorname{supp}(\mathcal{E})$. Then, $\mathcal{E}$ is dually atomistic if and only if $\mathcal{E} \cong \mathcal{E}_2(1,-1)$.*

Proof. If $\mathcal{E} \cong \mathcal{E}_2(1,-1)$, its only nonzero subalgebras are $\mathsf{span}\{e_1+e_2\}$ and $\mathsf{span}\{e_1-e_2\}$. Then, $\mathcal{E}$ is clearly dually atomistic.

Conversely, assume that $\mathcal{E} \ncong \mathcal{E}_2(1,-1)$ and distinguish the following two cases.

1. If $\mathcal{E} \in \mathcal{T}_{\mathbb{K}}$, then, by Proposition 3.3.4, $\mathcal{E}$ could only be dually atomistic if the codimension of the annihilator is two. In this case, by Lemma 3.4.7, the element e_1+e_2 is contained in all maximal subalgebras except for $M = \mathsf{span}\{e_1-e_2, e_3, \dots, e_n\}$. Then, consider the subalgebra $\mathsf{span}\{e_1+e_2+e_3\}$. In fact, $e_1+e_2+e_3 \notin \mathsf{span}\{e_1+e_2\}$ but $\mathsf{span}\{e_1+e_2+e_3\} \nsubseteq M$, so it cannot be expressed as the intersection of maximal subalgebras, and consequently $\mathcal{E}$ is not dually atomistic.

2. If $\mathcal{E} \notin \mathcal{T}_{\mathbb{K}}$, then, by Theorem 3.3.6, $\mathcal{E}$ could only be dually atomistic if it can be written as $K \oplus \operatorname{ann}(\mathcal{E})$ where $K \cong \bigoplus_{i=1}^{m} \mathcal{E}_2(1,-1)$ with $m \geq 2$. However, in this case, $\mathcal{E}_2(1,-1) \oplus \mathcal{E}_2(1,-1)$ is a quotient algebra of $\mathcal{E}$, which is not dually atomistic by Lemma 3.4.8.

Since in both cases we get a contradiction, the result follows. □

Proof of Theorem 3.4.3. It follows from the combination of Propositions 3.4.5, 3.4.6 and 3.4.9. □

Chapter 4

Deformations and degenerations of evolution algebras

This chapter is devoted to the study of deformations and degenerations of evolution algebras. While these concepts have been extensively investigated in classical settings such as associative or Lie algebras, they remain largely unexplored in the framework of evolution algebras. To the best of our knowledge, the only related study in this direction is [27], which deals with degenerations of evolution algebras with one-dimensional squares.

Although there are numerous definitions and perspectives for both concepts, we adopt a more formal point of view here. *Formal deformations*, introduced by Gerstenhaber [53] for associative algebras and later generalized to Lie algebras by Nijenhuis and Richardson [83, 84], provide a way to study how the multiplication of an algebra can be perturbed while remaining inside the same class. Roughly speaking, a deformation of an algebraic structure $\mathcal{A}$ with product μ consists in constructing a new product μ_t over the formal power series space $\mathcal{A}[[t]]$ given by $\mu_t = \mu + \sum_{k\geq 1} t^k \mu_k$, where each μ_k is a bilinear map on $\mathcal{A}$. In general, the goal of deformation theory is to understand how new multiplications μ_t enrich or modify the original structure, by determining their existence and classifying them up to the so-called *equivalence*, a process typically controlled by the second cohomology space.

On the other hand, *degeneration* is a concept that is somewhat opposite to deformation, and it has been extensively studied in the classical case of Lie algebras (see, for example, references [17, 31, 97]). Indeed, since Lie algebras are defined by polynomial identities, the set of n-dimensional Lie algebras over a field $\mathbb{K}$, $\mathcal{L}_n(\mathbb{K})$, forms an affine algebraic variety in the n^3-dimensional affine space $\mathbb{K}^{n^3}$. Moreover, the general linear group $\mathrm{GL}(n, \mathbb{K})$ acts naturally on $\mathcal{L}_n(\mathbb{K})$ by change of basis (see [63]). In this framework, given two n-dimensional Lie algebras $\mathcal{L}_1$ and $\mathcal{L}_0$ over a field $\mathbb{K}$, we say that $\mathcal{L}_1$ degenerates to $\mathcal{L}_0$, and write $\mathcal{L}_1 \longrightarrow_{\text{deg}} \mathcal{L}_0$, if $\mathcal{L}_0$ lies in the Zariski closure of the orbit of $\mathcal{L}_1$ under this action. In the same vein, degenerations have also been studied in several other varieties of (not necessarily associative) algebras (see the survey [61]), with complete descriptions available for low-dimensional complex associative algebras (see, for example, [50, 59, 73, 80]).

However, since evolution algebras over a field $\mathbb{K}$ are not defined by identities, the n-dimensional ones do not form an affine algebraic variety of $\mathbb{K}^{n^3}$ and, consequently, the Zariski topology cannot be considered. Hence, we will adopt the formal viewpoint presented in [74, Subsection 5.2] for associative algebras. Given a continuous family $\{g_t\}_{t\neq 0}$ of invertible linear maps on an n-dimensional vector space V over $\mathbb{K}$ and an algebra $\mathcal{A}_1$ over $\mathbb{K}$ with underlying vector space V and product μ_1, when the limit

$$\mu_0(x, y) = \lim_{t\to 0} g_t \cdot \mu_1(x, y) := \lim_{t\to 0} g_t(\mu_1(g_t^{-1}x, g_t^{-1}y)) \tag{4.0.1}$$

exists for all $x, y \in V$, we say that the algebra $\mathcal{A}_0$, with the same underlying vector space V and product μ_0, is a *formal degeneration* of $\mathcal{A}_1$. As stated in [74, Proposition 5.1], when working with algebraic varieties like associative or Lie algebras, every formal degeneration is also a degeneration in the usual sense.

The above definitions highlight the dual nature of the two concepts: formal degenerations tend to simplify the algebraic structure, often producing algebras that are closer to the abelian case, while formal deformations typically generate more intricate multiplication laws. In this work, we study both notions in the setting of evolution algebras, adapting the classical approaches and making the necessary changes due to the lack of variety structure but the existence of a natural basis.

This chapter, whose results are presented in detail in the preprint [75], is organized into three sections. In Section 4.1, we introduce formal deformations of evolution algebras (see Definition 4.1.1) by requiring that the product of distinct elements of the natural basis remains zero. This condition naturally leads to an evolution algebra structure over the power series ring $\mathbb{K}[[t]]$. We also define the notion of equivalence of deformations and prove that, if two deformations are equivalent, the difference of their first-order terms takes a derivation-like expression (see Theorem 4.1.10). Finally, we show that every evolution algebra admits a nontrivial deformation (see Theorem 4.1.15), in sharp contrast to the rigidity typically observed in semisimple algebras.

Since the second cohomology group traditionally governs infinitesimal deformations, Section 4.2 introduces a definition of the second cohomology space for evolution algebras (see Definition 4.2.1), which likewise controls such deformations. As an illustrative example, we compute the second cohomology space of all two-dimensional complex evolution algebras, thereby obtaining all their infinitesimal deformations up to equivalence (see Theorem 4.2.5).

Finally, Section 4.3 is devoted to the study of formal degenerations, understood as a sort of dual procedure to formal deformations. In particular, we establish several criteria to determine whether a degeneration exists (see Proposition 4.3.7). On the other hand, we show that degenerations lack transitivity (see Example 4.3.8 and

Remark 4.3.9), which can be seen as their main limitation. The section concludes with Hasse diagrams illustrating the degeneration relation among three and four-dimensional evolution algebras (see Theorem 4.3.13 and Proposition 4.3.16).

At the end of this chapter, we also include Appendices 4.A, 4.B, and 4.C, which contain the detailed computations supporting several results in the chapter. They are collected at the end to avoid interrupting the flow of the exposition with extensive calculations, while still providing the necessary details for completeness and verification.

Notation 4.0.1. Throughout this chapter, we regard an evolution algebra $\mathcal{E}$ as a pair (V, μ), where V is its underlying vector space and μ is a bilinear map $\mu\colon V \times V \to V$ defining its product. For simplicity of notation, we will often refer to an evolution algebra $\mathcal{E}$ directly by its product μ, and we will write $e_i e_j$ instead of $\mu(e_i, e_j)$ when there is no risk of confusion.

4.1 Formal evolution deformations

Let $\mathcal{E} = (V, \mu)$ be a finite dimensional evolution algebra over a field $\mathbb{K}$, and let $\mathbb{K}[[t]]$ denote the formal power series ring in one variable t. We define a formal space $V[[t]] := V \otimes \mathbb{K}[[t]]$, which is the result of extending the coefficient domain of $\mathcal{E}$ from $\mathbb{K}$ to $\mathbb{K}[[t]]$. Observe that each element $u \in V[[t]]$ can thus be written as a power series $u = \sum_{k \geq 0} u_k t^k$, where $u_k \in V$. Moreover, note that any $\mathbb{K}$-bilinear map $\nu\colon V \times V \longrightarrow V$ extends naturally to a $\mathbb{K}[[t]]$-bilinear map from $V[[t]] \times V[[t]]$ to $V[[t]]$. In this setting, the study of deformations aims to define new multiplication laws on the space $V[[t]]$ that yield new $\mathbb{K}[[t]]$-evolution algebra structures. This leads to the following definition.

Definition 4.1.1. Let $\mathcal{E} = (V, \mu)$ be an evolution algebra with natural basis $B = \{e_1, \ldots, e_n\}$. A *formal evolution deformation* of $\mathcal{E}$ is given by a $\mathbb{K}[[t]]$-bilinear map $\nu_t\colon V[[t]] \times V[[t]] \longrightarrow V[[t]]$ of the form $\nu_t = \mu + \sum_{k \geq 1} \nu_k t^k$, where each $\nu_k\colon V \times V \longrightarrow V$ is a $\mathbb{K}$-bilinear map (extended to be $\mathbb{K}[[t]]$-bilinear) and the following "condition of naturalness" is satisfied:

$$\nu_t(e_i, e_j) = 0, \quad \text{for all } i \neq j. \tag{4.1.1}$$

Remark 4.1.2.

1. Setting $t = 0$ in the previous definition recovers the original algebra $\mathcal{E}$.

2. By bilinearity, a formal evolution deformation of an evolution algebra $\mathcal{E}$ is uniquely determined by how it acts on the elements of the natural basis of $\mathcal{E}$.

3. Note that condition (4.1.1) holds if and only if $\nu_k(e_i, e_j)$ for all $i \neq j$ and for all $k \geq 0$. Consequently, each map ν_k can be seen as an evolution algebra on its own. If we denote by ρ_{ij}^k the structure constants of ν_k with respect to B, then the deformation ν_t expands as follows:

$$\begin{aligned}\nu_t(e_i, e_i) &= \mu(e_i, e_i) + \sum_{k\geq 1} t^k \nu_k(e_i, e_i)\\ &= \sum_{j=1}^{n} \omega_{ij} e_j + \sum_{k\geq 1} t^k \left(\sum_{j=1}^{n} \rho_{ij}^k e_j\right) = \sum_{j=1}^{n} \left(\omega_{ij} + \sum_{k\geq 1} \rho_{ij}^k t^k\right) e_j.\end{aligned}$$

4. As a consequence of condition (4.1.1), a formal evolution deformation ν_t of an evolution algebra $\mathcal{E} = (V, \mu)$ defines an evolution $\mathbb{K}[[t]]$-algebra structure on the vector space $V[[t]]$, with the same natural basis as the original. It is worth mentioning that evolution algebras over rings (in particular, integral domains) have already been considered in [23].

5. From now on, and for the sake of simplicity, we will use the term deformation to refer to a formal evolution deformation.

Although addressed from a different perspective, [33] provides several examples of chains of evolution algebras that, somehow, align with the notion of formal evolution deformations. Next, we present one of these examples.

Example 4.1.3 [33, Example 1]**.** This example models a time-homogeneous Markov process described in [65]. In fact, using the Taylor series expansion of the functions $\sin(t)$, $\cos(t)$, and e^t, we obtain the following structure matrix:

$$\begin{aligned}\omega_{ii}^{[t]} &= \frac{2}{3} e^{-\frac{3}{2}At} \cos(\alpha t) + \frac{1}{3}\\ &= \frac{2}{3}\left(\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{3}{2}A)^n}{n!} t^n\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} t^{2n}\right) + \frac{1}{3}, \quad i = 1, 2, 3;\\ \omega_{12}^{[t]} = \omega_{23}^{[t]} = \omega_{31}^{[t]} &= e^{-\frac{3}{2}At}\left(\frac{1}{\sqrt{3}} \sin(\alpha t) - \frac{1}{3}\cos(\alpha t)\right) + \frac{1}{3}\\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{3}{2}A)^n}{n!} t^n\right)\left(\frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} t^{2n+1} - \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} t^{2n}\right) + \frac{1}{3};\\ \omega_{21}^{[t]} = \omega_{32}^{[t]} = \omega_{13}^{[t]} &= -e^{-\frac{3}{2}At}\left(\frac{1}{\sqrt{3}} \sin(\alpha t) + \frac{1}{3}\cos(\alpha t)\right) + \frac{1}{3}\\ &= -\left(\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{3}{2}A)^n}{n!} t^n\right)\left(\frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} t^{2n+1} + \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} t^{2n}\right) + \frac{1}{3};\end{aligned}$$

with $A > 0$ and $\alpha = \frac{\sqrt{3}}{2}A$. It is easy to see that each entry of this matrix belongs to $\mathbb{K}[[t]]$, thus yielding an structure that could be treated as a deformation.

Definition 4.1.4. Let $\nu_t = \mu + \sum_{k\geq 1} \nu_k t^k$ be a deformation of an evolution algebra μ. Each coefficient ν_k is called the *coefficient of order* k. Particularly, the first-order coefficient ν_1 is called the *infinitesimal* of ν_t. Moreover, if ν_t is a truncated deformation, that is, there exists an integer m such that $\nu_t = \mu + \sum_{k=1}^{m} \nu_k t^k$ with $\nu_m \neq 0$, then we say that the deformation ν_t is of *order* m.

Analogously to Definition 4.1.1, one can also consider evolution algebra structures on $V \otimes \mathbb{K}[t]/(t^{m+1})$, which correspond to considering deformations up to order m. The next definition introduces a particularly important example of this.

Definition 4.1.5. Let $\mathcal{E} = (V, \mu)$ be an evolution algebra. Deformations of $\mathcal{E}$ over the vector space $V \otimes \mathbb{K}[t]/(t^2)$ are called *infinitesimal deformations* of $\mathcal{E}$. The set of all such deformations will be denoted by $\operatorname{InfDef}(\mathcal{E})$.

Remark 4.1.6. Although first-order and infinitesimal deformations are often treated interchangeably in the literature, we consider them as distinct notions. First-order deformations are deformations of the form $\nu = \mu + t\nu_1$ over the ring of formal power series $\mathbb{K}[[t]]$, whereas infinitesimal deformations are deformations of the same form but defined over the truncated ring $\mathbb{K}[t]/(t^2)$. While both notions agree at the level of first-order terms, they differ in the underlying base ring.

4.1.1 Equivalence of deformations

The next issue is to determine when two deformations should be regarded as essentially the same. This is addressed by introducing the notion of equivalence between deformations.

Definition 4.1.7. Let ν_t and λ_t be two formal evolution deformations of an evolution algebra $\mathcal{E} = (V, \mu)$. We say that ν_t and λ_t are *equivalent* if there exists a $\mathbb{K}[[t]]$-linear map $\phi_t \colon V[[t]] \longrightarrow V[[t]]$ of the form $\phi_t = \operatorname{Id} + t\phi_1 + t^2\phi_2 + \cdots$, where each $\phi_k \colon V \longrightarrow V$ is $\mathbb{K}$-linear and such that

$$\phi_t(\nu_t(u, v)) = \lambda_t(\phi_t(u), \phi_t(v)) \tag{4.1.2}$$

for all $u, v \in \mathcal{E}$. Notice that given a natural basis $\{e_1, \ldots, e_n\}$ of $\mathcal{E}$, since ν_t and λ_t are $\mathbb{K}[[t]]$-bilinear and ϕ_t is $\mathbb{K}[[t]]$-linear, condition (4.1.2) is equivalent to requiring that

$$\phi_t(\nu_t(e_i, e_j)) = \lambda_t(\phi_t(e_i), \phi_t(e_j)) \tag{4.1.3}$$

for all $i, j = 1, \ldots, n$. If ν_t and λ_t are equivalent through a formal isomorphism ϕ_t, we write $\nu_t \cong_{\phi_t} \lambda_t$.

Remark 4.1.8. Given a ring R, it is known that an element $r = \sum_{k\geq 0} r_k t^k$ is invertible in the ring of formal power series $R[[t]]$ if and only if r_0 is invertible in R. Consequently, any linear map $\phi \in \operatorname{End}_{\mathbb{K}}(V,V)[[t]]$ of the form $\phi = \operatorname{Id} + t\phi_1 + t^2\phi_2 + \cdots$, as the one considered in the previous definition, is invertible and yields a $\mathbb{K}[[t]]$-automorphism of $V[[t]]$.

Let $\mathcal{E} = (V, \mu)$ be an evolution algebra with natural basis $B = \{e_1, \ldots, e_n\}$ and structure matrix $M_B(\mathcal{E}) = (\omega_{ij})$. Then, given two formal evolution deformations ν_t and λ_t of μ, we can express their products as

$$\begin{aligned}
\nu_t(e_i, e_i) &= \sum_{j=1}^{n} \left(\omega_{ij} + \rho_{ij}^1 t + \rho_{ij}^2 t^2 + \cdots\right) e_j = \sum_{j=1}^{n} \left(\omega_{ij} + \sum_{k\geq 1} \rho_{ij}^k t^k\right) e_j,\\
\lambda_t(e_i, e_i) &= \sum_{j=1}^{n} \left(\omega_{ij} + \sigma_{ij}^1 t + \sigma_{ij}^2 t^2 + \cdots\right) e_j = \sum_{j=1}^{n} \left(\omega_{ij} + \sum_{k\geq 1} \sigma_{ij}^k t^k\right) e_j,
\end{aligned} \tag{4.1.4}$$

for all $i = 1, \ldots, n$, where ρ_{ij}^k and σ_{ij}^k are scalars in $\mathbb{K}$ representing the structure constants of the coefficients of each order k.

Proposition 4.1.9. *Let ν_t and λ_t be two deformations of an evolution algebra $\mathcal{E} = (V, \mu)$ over any field $\mathbb{K}$, with expansions as in* (4.1.4)*. If they are equivalent, then there exists a matrix $(\xi_{ij})_{i,j=1}^n$ such that all the following conditions are satisfied:*

$$\xi_{ji}\omega_{jk} + \xi_{ij}\omega_{ik} = 0, \quad \textit{for all } 1 \leq i,j,k \leq n \textit{ such that } i \neq j; \tag{4.1.5}$$

$$\rho_{ik}^1 + \sum_{p=1}^{n} \omega_{ip}\xi_{kp} = \sigma_{ik}^1 + 2\xi_{ii}\omega_{ik}, \quad \textit{for all } 1 \leq i,k \leq n. \tag{4.1.6}$$

Proof. Let $\phi_t = \operatorname{Id} + t\phi_1 + t^2\phi_2 + \cdots$ be a formal isomorphism such that $\nu_t \cong_{\phi_t} \lambda_t$, and denote by $(\xi_{ij})_{i,j=1}^n$ the matrix of ϕ_1 with respect to the basis B. All the following computations are made modulo t^2. In view of (4.1.3), when $i \neq j$ we have

$$\begin{aligned}
0 = \phi_t(\mu_t(e_i, e_j)) &= \lambda_t(\phi_t(e_i), \phi_t(e_j))\\
&= \lambda_t(e_i + t\phi_1(e_i),\ e_j + t\phi_1(e_j)) && \operatorname{mod}\ (t^2)\\
&= \lambda_t(e_i, e_j) + t\big(\lambda_t(\phi_1(e_i), e_j) + \lambda_t(\phi_1(e_j), e_i)\big) && \operatorname{mod}\ (t^2)\\
&= t\left(\lambda_t\left(\sum_{p=1}^{n} \xi_{pi} e_p,\ e_j\right) + \lambda_t\left(\sum_{p=1}^{n} \xi_{pj} e_p,\ e_i\right)\right) && \operatorname{mod}\ (t^2)\\
&= t\big(\xi_{ji}\lambda_t(e_j, e_j) + \xi_{ij}\lambda_t(e_i, e_i)\big) && \operatorname{mod}\ (t^2)
\end{aligned}$$

$$
\begin{aligned}
&= t\left(\xi_{ji}\sum_{k=1}^{n}\omega_{jk}e_k + \xi_{ij}\sum_{k=1}^{n}\omega_{ik}e_k\right) && \mod (t^2)\\
&= t\sum_{k=1}^{n}\left(\xi_{ji}\omega_{jk} + \xi_{ij}\omega_{ik}\right)e_k && \mod (t^2).
\end{aligned}
$$

Therefore, we obtain $\xi_{ji}\omega_{jk}+\xi_{ij}\omega_{ik} = 0$ for all $1 \leq i \neq j \leq n$ and for all $1 \leq k \leq n$, which correspond exactly to the conditions presented in (4.1.5). Again, in view of (4.1.3), when $i = j$ we have

$$
\begin{aligned}
\phi_t(\nu_t(e_i, e_i)) &= \sum_{k=1}^{n}\left(\omega_{ik} + \rho_{ik}^1 t\right)\phi_t(e_k) && \mod (t^2)\\
&= \sum_{k=1}^{n}\left(\omega_{ik} + \rho_{ik}^1 t\right)(e_k + t\phi_1(e_k)) && \mod (t^2)\\
&= \sum_{k=1}^{n}\omega_{ij}e_k + t\left(\sum_{k=1}^{n}\rho_{ik}^1 e_k + \sum_{k=1}^{n}\omega_{ik}\phi_1(e_k)\right) && \mod (t^2)\\
&= \sum_{k=1}^{n}\omega_{ik}e_k + t\left(\sum_{k=1}^{n}\rho_{ik}^1 e_k + \sum_{k=1}^{n}\omega_{ik}\sum_{p=1}^{n}\xi_{pk}e_p\right) && \mod (t^2)\\
&= \sum_{k=1}^{n}\omega_{ik}e_k + t\sum_{k=1}^{n}\left(\rho_{ik}^1 + \sum_{p=1}^{n}\omega_{ip}\xi_{kp}\right)e_k && \mod (t^2);
\end{aligned}
\tag{4.1.7}
$$

and

$$
\begin{aligned}
\lambda_t(\phi_t(e_i), \phi_t(e_i)) &= \lambda_t\left(e_i + t\phi_1(e_i), e_i + t\phi_1(e_i)\right) && \mod (t^2)\\
&= \lambda_t(e_i, e_i) + 2t\lambda_t(e_i, \phi_1(e_i)) && \mod (t^2)\\
&= \lambda_t(e_i, e_i) + 2t\lambda_t\left(e_i, \sum_{p=1}^{n}\xi_{pi}e_p\right) && \mod (t^2)\\
&= \sum_{k=1}^{n}\left(\omega_{ik} + \sigma_{ik}^1 t\right)e_k \\
&\quad + 2t\left(\xi_{ii}\sum_{k=1}^{n}\left(\omega_{ik} + \sigma_{ik}^1 t\right)e_k\right) && \mod (t^2)\\
&= \sum_{k=1}^{n}\omega_{ik}e_k + t\sum_{k=1}^{n}\left(\sigma_{ik}^1 + 2\xi_{ii}\omega_{ik}\right)e_k && \mod (t^2).
\end{aligned}
\tag{4.1.8}
$$

Therefore, collecting the coefficients of first order in (4.1.7) and (4.1.8), we obtain $\rho_{ik}^1 + \sum_{p=1}^{n}\omega_{ip}\xi_{kp} = \sigma_{ik}^1 + 2\xi_{ii}\omega_{ik}$ for all $1 \leq i, k \leq n$, getting the desired conditions (4.1.6). $\square$

Recall that a derivation of $\mathcal{E}$ is a linear map $d\colon \mathcal{E} \longrightarrow \mathcal{E}$ such that $d(uv) = d(u)v + ud(v)$, for all $u, v \in \mathcal{E}$. In particular, in [99, Section 3.2.6], it was proved that, if $\mathcal{E}$ is an evolution $\mathbb{K}$-algebra with natural basis $B = \{e_1, \ldots, e_n\}$, then a linear map d such that $d(e_i) = \sum_{k=1}^n \xi_{ki} e_k$ is a derivation if and only if it satisfies the following conditions:

$$\xi_{ji}\omega_{jk} + \xi_{ij}\omega_{ik} = 0, \text{ for all } 1 \leq i, j, k \leq n \text{ such that } i \neq j; \tag{4.1.9}$$

$$\sum_{p=1}^{n} \omega_{ip}\xi_{kp} = 2\xi_{ii}\omega_{ik}, \text{ for all } 1 \leq i, k \leq n. \tag{4.1.10}$$

Notice that conditions (4.1.5)–(4.1.6) already mirror those in (4.1.9)–(4.1.10). This resemblance suggests that derivations underlie the equivalence of deformations, as made precise in the following theorem.

Theorem 4.1.10. *Let ν_t and λ_t be two deformations of an evolution algebra $\mathcal{E} = (V, \mu)$ over any field $\mathbb{K}$. If they are equivalent, then there exists a linear morphism $\varphi \in \operatorname{End}_{\mathbb{K}}(V)$ such that, for all $u, v \in V$,*

$$\lambda_1(u, v) - \nu_1(u, v) = \varphi(uv) - u\varphi(v) - \varphi(u)v. \tag{4.1.11}$$

Proof. Given a formal isomorphism $\phi_t = \mathrm{Id} + t\phi_1 + t^2\phi_2 + \cdots$ such that $\nu_t \cong_{\phi_t} \lambda_t$, let us set $\varphi := \phi_1$, and denote its matrix with respect to the natural basis B by $(\xi_{ij})_{i,j=1}^n$. We will verify (4.1.11) on the elements of the natural basis of $\mathcal{E}$, that is,

$$\lambda_1(e_i, e_j) - \nu_1(e_i, e_j) = \phi_1(e_i e_j) - e_i\phi_1(e_j) - \phi_1(e_i)e_j, \tag{4.1.12}$$

for all $1 \leq i, j \leq n$. The result will then follow by the bilinearity of ν_1 and λ_1 and the linearity of ϕ_1. On the one hand, when $i \neq j$, we have that $\nu_1(e_i, e_j) = \lambda_1(e_i, e_j) = 0$ and, by Proposition 4.1.9, it holds

$$\begin{aligned}
\phi_1(e_i e_j) - e_i\phi_1(e_j) - \phi_1(e_i)e_j &= -e_i\left(\sum_{p=1}^{n} \xi_{pj} e_p\right) - e_j\left(\sum_{p=1}^{n} \xi_{pi} e_p\right) \\
&= -\xi_{ij} e_i^2 - \xi_{ji} e_j^2 \\
&= -\sum_{k=1}^{n} \left(\xi_{ij}\omega_{ik} + \xi_{ji}\omega_{jk}\right) e_k = 0.
\end{aligned}$$

Consequently, (4.1.12) is satisfied in this first case. On the other hand, when $i = j$, also as a consequence of Proposition 4.1.9, we have that

$$\phi_1(e_i^2) - 2e_i\phi_1(e_i) = \phi_1\left(\sum_{k=1}^{n} \omega_{ik} e_k\right) - 2e_i\left(\sum_{p=1}^{n} \xi_{pi} e_p\right)$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \omega_{ik} \sum_{p=1}^{n} \xi_{pk} e_p - 2\xi_{ii} e_i^2 \\
&= \sum_{k=1}^{n} \omega_{ik} \sum_{p=1}^{n} \xi_{pk} e_p - 2\xi_{ii} \sum_{k=1}^{n} \omega_{ik} e_k \\
&= \sum_{k=1}^{n} \left(\sum_{p=1}^{n} \omega_{ip} \xi_{kp} - 2\xi_{ii} \omega_{ik} \right) e_k \\
&= \sum_{k=1}^{n} (\sigma_{ik}^1 - \rho_{ik}^1) e_k = \lambda_1(e_i, e_i) - \nu_1(e_i, e_i),
\end{aligned}
$$

what completes the proof of (4.1.12). □

Definition 4.1.11. A deformation ν_t of an evolution algebra $\mathcal{E} = (V, \mu)$ is called *trivial* if ν_t is equivalent to μ.

Hence, as a consequence of Theorem 4.1.10, we get the following result.

Corollary 4.1.12. *Let ν_t be a deformation of an evolution algebra $\mathcal{E} = (V, \mu)$ over any field $\mathbb{K}$. If ν_t is a trivial deformation, then there exists a linear morphism $\varphi \in \operatorname{End}(V)$ such that $\nu_1(u, v) = \varphi(uv) - u\varphi(v) - \varphi(u)v$ for all $u, v \in V$.*

Remark 4.1.13. Note that all the above statements remain valid in characteristic two. The only difference is that the terms of the form $2\xi_{ii}\omega_{ik}$ appearing in (4.1.6) and (4.1.10) vanish automatically, so the identities are accordingly simplified.

4.1.2 Every evolution algebra admits a nontrivial deformation

In many classical varieties of algebras, such as associative and Lie algebras, an algebra is called *(formally) rigid* if all its formal deformations are trivial, a property often ensured by the vanishing of its second cohomology group. In contrast, we next show that every evolution algebra admits a nontrivial first-order deformation.

Lemma 4.1.14. *Let ν_t and λ_t be two equivalent deformations, $\nu_t \cong_{\phi_t} \lambda_t$, of an evolution algebra $\mathcal{E} = (V, \mu)$ with natural basis $\{e_1, \ldots, e_n\}$. If $\sum_{i=1}^{n} \alpha_i \nu_t(e_i, e_i) = 0$ for some scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$, then $\sum_{i=1}^{n} \alpha_i \lambda_t(\phi_t(e_i), \phi_t(e_i)) = 0$.*

Proof. It follows straightforwardly from (4.1.3). In fact,

$$
0 = \phi_t \left(\sum_{i=1}^{n} \alpha_i \nu_t(e_i, e_i) \right) = \sum_{i=1}^{n} \alpha_i \phi_t(\nu_t(e_i, e_i)) = \sum_{i=1}^{n} \alpha_i \lambda_t(\phi_t(e_i), \phi_t(e_i)).
$$

□

Theorem 4.1.15. *Every evolution algebra admits a nontrivial first-order deformation.*

Proof. Let $\mathcal{E} = (V, \mu)$ be an evolution algebra with natural basis $\{e_1, \ldots, e_n\}$ and structure matrix $(\omega_{ij})_{i,j=1}^n$. We show that there always exists a nontrivial first-order deformation $\nu_t = \mu + \nu_1 t$. For the computations that follow, denote by $(\rho_{ij}^1)_{i,j=1}^n$ the structure matrix of ν_1. We consider the regular and the nonregular cases separately:

1. If $\mathcal{E}$ is regular then, from condition (4.1.5), it is deduced that

$$\begin{pmatrix} \omega_{i1} & \ldots & \omega_{in} \\ \omega_{j1} & \ldots & \omega_{jn} \end{pmatrix}^t \begin{pmatrix} \xi_{ij} \\ \xi_{ji} \end{pmatrix} = 0 \tag{4.1.13}$$

for all $1 \leq i \neq j \leq n$. As $\mathcal{E}$ is regular, (4.1.13) is a homogeneous system with a unique solution, and such solution is the trivial one. Then, $\xi_{ij} = 0$ for all $1 \leq i \neq j \leq n$. Next, from (4.1.6) it is deduced that $\rho_{ii}^1 = -\omega_{ii}\xi_{ii}$ for all $1 \leq i \leq n$. If $\omega_{ii} = 0$ for some $1 \leq i \leq n$, then just consider a scalar $\rho_{ii}^1 \neq 0$, what yields that ν_t is a nontrivial deformation. Otherwise, if $\omega_{ii} \neq 0$ for all $1 \leq i \leq n$, we have that $\xi_{ii} = -\frac{\rho_{ii}^1}{\omega_{ii}}$ for all $1 \leq i \leq n$. Consequently, also from (4.1.6), we get that necessarily $\rho_{ik}^1 = \omega_{ik}(\xi_{kk} - 2\xi_{ii})$. Then, taking a ρ_{ik}^1 which does not satisfy the previous relation yields again that ν_t is a nontrivial deformation, which completes the proof.

2. If $\mathcal{E}$ is nonregular, consider the matrix $(\rho_{ij}^1)_{i,j=1}^n = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ with each $\beta_i = \pm 1$. For the sake of contradiction, assume the existence of a formal isomorphism ϕ_t such that $\phi_t(\mu(e_i, e_i)) = \nu_t(\phi_t(e_i), \phi_t(e_i))$ for all $1 \leq i \leq n$ and for every $(\beta_1, \ldots, \beta_n) \in \{\pm 1\}^n$. In fact,

$$\begin{aligned} \nu_t(\phi_t(e_i), \phi_t(e_i)) &= \nu_t(e_i, e_i) + 2t\nu_t(e_i, \phi_1(e_i)) && \mod (t^2) \\ &= \mu(e_i, e_i) + t\beta_i e_i + 2t\xi_{ii}(\mu(e_i, e_i) + t\beta_i e_i) && \mod (t^2) \\ &= \mu(e_i, e_i) + t(2\xi_{ii}\mu(e_i, e_i) + \beta_i e_i) && \mod (t^2). \end{aligned}$$

At the same time, as $\mathcal{E}$ is not regular, there exists a nonempty subset $\Lambda \subset \{1, \ldots, n\}$ such that $\sum_{i \in \Lambda} \alpha_i \mu(e_i, e_i) = 0$ for some scalars $\alpha_i \in \mathbb{K}^*$, $i \in \Lambda$. Consequently, by Lemma 4.1.14, it also holds that $\sum_{i \in \Lambda} \alpha_i \nu_t(\phi_t(e_i), \phi_t(e_i)) = 0$. Hence, looking at the coefficient of t, we have that

$$0 = \sum_{i \in \Lambda} \alpha_i (2\xi_{ii}\mu(e_i, e_i) + \beta_i e_i) \implies \sum_{i \in \Lambda} 2\alpha_i \xi_{ii} \mu(e_i, e_i) = -\sum_{i \in \Lambda} \alpha_i \beta_i e_i. \tag{4.1.14}$$

We now distinguish two subcases according to the characteristic of the base field:

(a) If the characteristic is two, it suffices to take $\beta_i = 1$ for all $i = 1, \ldots, n$. In this case, condition (4.1.14) yields $\sum_{i\in\Lambda} \alpha_i e_i = 0$, a contradiction.

(b) If the characteristic is not two, we claim that (4.1.14) is not satisfied for some $(\beta_1, \ldots, \beta_n) \in \{\pm 1\}^n$. Otherwise, if we changed the sign of β_{i_0} for any index $i_0 \in \Lambda$, there would also exist another formal isomorphism $\overline{\phi_t}$ and, consequently, other scalars $(\overline{\xi_{ii}})_{i\in\Lambda}$ such that

$$\sum_{i\in\Lambda} 2\alpha_i \overline{\xi_{ii}} \mu(e_i, e_i) = \beta_{i_0}\alpha_{i_0} e_{i_0} - \sum_{i\in\Lambda\setminus\{i_0\}} \alpha_i \beta_i e_i. \tag{4.1.15}$$

By subtracting (4.1.15) from (4.1.14), we get that

$$\sum_{i\in\Lambda} 2\alpha_i (\xi_{ii} - \overline{\xi_{ii}}) \mu(e_i, e_i) = -2\beta_{i_0}\alpha_{i_0} e_{i_0},$$

and consequently $e_{i_0} \in \mathsf{span}\{\mu(e_i, e_i), i \in \Lambda\}$. Repeating this process for every $i_0 \in \Lambda$, we get that $\mathsf{span}\{e_i \colon i \in \Lambda\} = \mathsf{span}\{\mu(e_i, e_i) \colon i \in \Lambda\}$, a contradiction with the fact that $\sum_{i\in\Lambda} \alpha_i \mu(e_i, e_i) = 0$ for some $\alpha_i \in \mathbb{K}^*$, $i \in \Lambda$.

In both cases we get a contradiction, thus the result follows in the nonregular setting.

Therefore, it is always possible to construct a first-order nontrivial deformation, what yields the claim. □

Since all computations in the previous proof are done modulo t^2, the same conclusion applies in the context of infinitesimal deformations.

Corollary 4.1.16. *Every evolution algebra admits a nontrivial infinitesimal deformation.*

4.2 Formal evolution deformations from a cohomological perspective

Note that Theorem 4.1.10 expresses the difference of the infinitesimals of two equivalent deformations as a derivation-like expression. This result can be viewed as the natural analogue of a classical fact in deformation theory of associative and Lie algebras, where such difference is a 2-coboundary in the Hochschild or Chevalley-Eilenberg cohomology, respectively (see [53] for associative algebras and see [84] for Lie algebras). Moreover, in these two classical cases, the elements of the second cohomology group can be seen as infinitesimal deformations (up to equivalence). Although evolution algebras do not possess a standard cohomology theory, it is possible to construct a cohomological framework that captures, in an analogous way, the behaviour of infinitesimal deformations.

We first introduce the following notation. Given a vector space V over a field $\mathbb{K}$ with a fixed basis $\{e_1, \ldots, e_n\}$, we denote by $\mathcal{Z}^2(V)$ the set of all bilinear maps on V that vanish on distinct basis elements:

$$\mathcal{Z}^2(V) := \{\theta \in \mathrm{Bil}_{\mathbb{K}}(V \times V, V) \mid \theta(e_i, e_j) = 0 \text{ for all } i \neq j\}.$$

Following the terminology commonly used in cohomology theory, we will refer to $\mathcal{Z}^2(V)$ as the *space of* 2*-cocycles*. Note that $\mathcal{Z}^2(V)$ can be naturally identified with the space of $n \times n$ matrices over $\mathbb{K}$, where $n = \dim(V)$, by associating each $\theta \in \mathcal{Z}^2(V)$ with its structure matrix relative to the fixed basis.

Let $\mathcal{E} = (V, \mu)$ be an evolution algebra. Inspired by the associative case, define the differential operator δ such that

$$\delta_\mu \varphi(u, v) := \varphi(\mu(u, v)) - \mu(u, \varphi(v)) - \mu(\varphi(u), v)$$

for all $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$ and for all $u, v \in V$. Then, considering the image of δ_μ restricted to the space of 2-cocycles $\mathcal{Z}^2(V)$, we define the *space of* 2*-coboundaries*:

$$\mathcal{B}^2(\mathcal{E}) := \{\theta \in \mathcal{Z}^2(V) \mid \theta = \delta_\mu \varphi \text{ for some } \varphi \in \mathrm{End}_{\mathbb{K}}(V)\}.$$

Note that $\mathcal{B}^2(\mathcal{E})$ is a vector subspace of $\mathcal{Z}^2(V)$. Indeed, given two elements $\theta_1 = \delta_\mu \varphi_1$ and $\theta_2 = \delta_\mu \varphi_2$ of $\mathcal{B}^2(\mathcal{E})$, and a scalar $\lambda \in \mathbb{K}$, it is straightforward to check that $\theta_1 + \theta_2 = \delta_\mu(\varphi_1 + \varphi_2)$ and $\lambda\theta_1 = \delta_\mu(\lambda\varphi_1)$.

Hence, we present the following definition.

Definition 4.2.1. Let $\mathcal{E} = (V, \mu)$ be an evolution algebra. The *second cohomology space* of $\mathcal{E}$ is defined as the quotient $\mathcal{H}^2(\mathcal{E}) := \mathcal{Z}^2(V)/\mathcal{B}^2(\mathcal{E})$.

We can now state the following results in view of Theorem 4.1.10 and Corollary 4.1.12.

Corollary 4.2.2. *If ν_t and λ_t are two equivalent formal evolution deformations of an evolution algebra $\mathcal{E}$, then the difference of their infinitesimals is equal to zero up to a* 2*-coboundary.*

Corollary 4.2.3. *Given an evolution algebra $\mathcal{E}$, there exists a bijection between its infinitesimal deformations (up to equivalence) and the elements of $\mathcal{H}^2(\mathcal{E})$.*

Moreover, as a consequence of Theorem 4.1.15 and Remark 4.1.16, we also have the following result.

Corollary 4.2.4. *$\mathcal{H}^2(\mathcal{E})$ is nontrivial for any finite-dimensional evolution algebra $\mathcal{E}$.*

4.2.1 Explicit computation of the second cohomology group

The computation of the second cohomology space (particularly, of the space of 2-coboundaries) of an evolution algebra is feasible in low dimensions and becomes particularly accessible in cases with sparse structure matrices, such as the nilpotent setting. For the sake of completeness, we include here the explicit computation of this space for all two-dimensional evolution algebras over $\mathbb{C}$, listed in Table 1.1.1. As a consequence, we also obtain a characterisation (up to equivalence) of the infinitesimal deformations corresponding to each isomorphism class.

Theorem 4.2.5. *The spaces of* 2*-coboundaries and all the infinitesimal deformations (up to equivalence) for all two-dimensional evolution algebras over* $\mathbb{C}$ *are presented in Table 4.2.1.*

$\mathcal{E}$	$\mathcal{B}^2(\mathcal{E})$	$\mathrm{InfDef}(\mathcal{E})$
$\mathcal{E}_1$	$\mathrm{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix}t,\ \alpha, \beta \in \mathbb{C}$
$\mathcal{E}_2$	$\mathrm{span}\left\{\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}\right\}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}t,\ \alpha \in \mathbb{C}$
$\mathcal{E}_3$	$\mathrm{span}\left\{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\right\}$	$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix}t,\ \alpha, \beta \in \mathbb{C}$
$\mathcal{E}_4$	$\mathrm{span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}t,\ \alpha, \beta, \gamma \in \mathbb{C}$
$\mathcal{E}_5(a_2, a_3)$	$\mathrm{span}\left\{\begin{pmatrix} -1 & -2a_2 \\ a_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ -2a_3 & -1 \end{pmatrix}\right\}$	**Case 1:** $a_2 \neq 0$ $\begin{pmatrix} 1 & a_2 \\ a_3 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix}t,\ \alpha, \beta \in \mathbb{C}$ **Case 2:** $a_2 = a_3 = 0$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}t,\ \alpha, \beta \in \mathbb{C}$ **Case 3:** $a_2 = 0,\ a_3 \neq 0$ $\begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix}t,\ \alpha, \beta \in \mathbb{C}$
$\mathcal{E}_6(a_4)$	$\mathrm{span}\left\{\begin{pmatrix} 0 & 1 \\ -2 & -a_4 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}\right\}$	$\begin{pmatrix} 0 & 1 \\ 1 & a_4 \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}t,\ \alpha, \beta \in \mathbb{C}$

Table 4.2.1: Infinitesimal deformations and spaces $\mathcal{B}^2(\mathcal{E})$ for all 2-dimensional evolution algebras over $\mathbb{C}$.

Proof. We prove the result in detail for the first evolution algebra in Table 4.2.1. The remaining cases are analogous and can be found in Appendix 4.A.

Let $\mathcal{E}_1$ be the evolution algebra with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1$ and $e_2^2 = 0$. If $\theta \in \mathcal{B}^2(\mathcal{E}_1)$, then there must exist a linear morphism

$(\varphi)_{BB} = (\xi_{ij})$ such that the following conditions hold:

$$\begin{aligned}\theta(e_1, e_2) &= \varphi(e_1e_2) - e_1\varphi(e_2) - \varphi(e_1)e_2\\ &= -e_1(\xi_{12}e_1 + \xi_{22}e_2) - e_2(\xi_{11}e_1 + \xi_{21}e_2)\\ &= \xi_{12}e_1^2 + \xi_{21}e_2^2 = \xi_{12}e_1 = 0 \implies \xi_{12} = 0;\\ \theta(e_1, e_1) &= \varphi(e_1^2) - 2e_1\varphi(e_1) = \xi_{11}e_1 + \xi_{21}e_2 - 2\xi_{11}e_1 = -\xi_{11}e_1 + \xi_{21}e_2;\\ \theta(e_2, e_2) &= \varphi(e_2^2) - 2e_2\varphi(e_2) = 0.\end{aligned}$$

Hence, we have that

$$\mathcal{B}^2(\mathcal{E}_1) = \left\{ \left(\begin{smallmatrix} -\xi_{11} & \xi_{21} \\ 0 & 0 \end{smallmatrix}\right) \mid \xi_{11}, \xi_{21} \in \mathbb{C} \right\} = \mathsf{span}\left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \right\},$$

and

$$\mathcal{H}^2(\mathcal{E}_1) = \mathsf{span}\left\{ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_1), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_1) \right\}.$$

Consequently, the set of all infinitesimal formal evolution deformations of $\mathcal{E}_1$ (up to equivalence) is given by

$$\begin{aligned}\mathrm{InfDef}(\mathcal{E}_1) &= \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + (\alpha\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \beta\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right))\, t \mid \alpha, \beta \in \mathbb{C} \}\\ &= \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \alpha & \beta \end{smallmatrix}\right) t \mid \alpha, \beta \in \mathbb{C} \right\}.\end{aligned}$$

□

4.3 Formal evolution degenerations

Since the defining feature of evolution algebras is the existence of a natural basis, the families of invertible linear maps $\{g_t\}_{t\neq 0}$ considered in the study of formal degenerations (4.0.1) in the context of evolution algebras will be required to map natural bases to natural bases. This observation motivates the following definition.

Definition 4.3.1. Let $\mathcal{E} = (V, \mu)$ be an evolution algebra with natural basis $B = \{e_1, \dots, e_n\}$. A nonsingular matrix g defines a *natural basis change* if it represents the change of basis from B to another natural basis $B' = \{f_1, \dots, f_n\}$. In this case, the product μ' of $\mathcal{E}$ with respect to B' is given by

$$\mu'(f_i, f_j) = (g \cdot \mu)(f_i, f_j) = g\big(\mu(g^{-1}f_i, g^{-1}f_j)\big),$$

for all $i, j = 1, \dots, n$, so in particular $\mu'(f_i, f_j) = 0$ for all $i \neq j$.

Remark 4.3.2. The relation between two structure matrices of the same evolution algebra relative to different natural bases, together with the corresponding change of basis matrices, is described in [99, Section 3.2.2] and in more detail in [19, Section 1.3]. Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{e_1, \ldots, e_n\}$. If $B' = \{f_1, \ldots, f_n\}$ is another natural basis and $g = P_{BB'} = (p_{ij})$ and $g^{-1} = P_{B'B} = (q_{ij})$ are the change of basis matrices, then

$$M_{B'}(\mathcal{E})^t = g\, M_B(\mathcal{E})^t\, (g^{-1})^{(2)},$$

where $(g^{-1})^{(2)} = (q_{ij}^2)$.

We now introduce the notion of formal degenerations in the setting of evolution algebras. The following definition is inspired by the concept of a *contraction* (see, for instance, references [16, 48]), a special case of degeneration in the classical context of varieties.

Definition 4.3.3. Let μ and λ be two evolution algebras. We say that λ is a *formal evolution degeneration* of μ, or that μ *formally degenerates* to λ, if there exists a continuous map $g\colon (0,1] \longrightarrow \mathrm{GL}(n, \mathbb{K})$, $t \mapsto g_t$, with each linear isomorphism g_t defining a natural basis change of μ, such that

$$\lambda = \lim_{t \to 0} g_t \cdot \mu.$$

We denote this by $\mu \to \lambda$.

Remark 4.3.4.

1. Since each g_t is an isomorphism for $t \in (0, 1]$, all the algebras $g_t \cdot \mu$ are isomorphic to μ. Hence, to obtain a new evolution algebra via formal evolution degeneration one needs $\det(g_0) = 0$. This is a necessary condition but not a sufficient one.

2. Throughout this paper, we only consider matrices g_t with entries of the form αt^m, with $\alpha \in \mathbb{K}$, $m \in \mathbb{Z}$. Then, the structure constants of $g_t \cdot \mu$ will be elements of $\mathbb{K}[t, t^{-1}]$. However, if a degeneration exists, the structure constants necessarily lie on $\mathbb{K}[t]$ to ensure that the limit exists when $t \to 0$.

3. In what follows, we slightly abuse notation and denote by $B' = \{f_1, \ldots, f_n\}$ both the natural basis of $g_t \cdot \mu$ induced by g_t and that of the limiting algebra λ.

4. As in the case of deformations, we will use the term degeneration to mean a formal evolution degeneration.

Example 4.3.5. Any evolution algebra μ degenerates to the abelian evolution algebra of the same dimension. Simply consider the matrix $g_t = t^{-1}I_n$. Then, we have

$$g_t \cdot \mu(f_i, f_i) = g_t\big(\mu(g_t^{-1}f_i, g_t^{-1}f_i)\big) = t^{-1}\big(\mu(te_i, te_i)\big) = t\mu(e_i, e_i) \xrightarrow{t \to 0} 0.$$

The following proposition gives a connection between degeneration and deformation.

Proposition 4.3.6. *If λ is a degeneration of μ (in the setting of the second item of Remark 4.3.4), then μ is a deformation of λ.*

Proof. By hypothesis, there exists a matrix g_t which is nonsingular for all $t \in (0, 1]$ and whose entries are elements of the form αt^m such that $\lambda = \lim_{t\to 0} g_t \cdot \mu$. Then, the structure constants of $g_t \cdot \mu$ are elements of $\mathbb{K}[t]$. Since $g_t \cdot \mu \cong \mu$ for all $t \in (0, 1]$, it follows that μ is a deformation of λ. □

4.3.1 Main properties and weaknesses

Motivated by the necessary conditions commonly used to establish the existence of a degeneration in the classical setting of Lie algebras (see [15, Proposition 1.8]), we state the following result.

Proposition 4.3.7. *Let μ and λ be two n-dimensional evolution algebras such that λ is a degeneration of μ. Then, the following assertions hold:*

(i) $\dim \operatorname{ann}(\mu) \leq \dim \operatorname{ann}(\lambda)$;

(ii) *if there exists an integer $k > 0$ such that $\dim \operatorname{ann}^i(\mu) = \dim \operatorname{ann}^i(\lambda)$ for all $i \leq k$, then $\dim \operatorname{ann}^{k+1}(\mu) \leq \dim \operatorname{ann}^{k+1}(\lambda)$, that is, the type of μ is less (with the lexicographic order) than the type of λ;*

(iii) $\dim \lambda^2 \leq \dim \mu^2$.

(iv) $\dim \mathcal{B}^2(\mu) \geq \dim \mathcal{B}^2(\lambda)$.

(v) $\dim \mathcal{H}^2(\mu) \leq \dim \mathcal{H}^2(\lambda)$.

Proof. By hypothesis, there exists a continuous map $g\colon (0, 1] \longrightarrow \mathrm{GL}(n, \mathbb{K})$, $t \mapsto g_t$, with each linear isomorphism g_t defining a natural basis change of μ such that $\lambda = \lim_{t\to 0} g_t \cdot \mu$.

(i) Assume that $\dim \operatorname{ann}(\mu) = r$. Then, since $g_t \cdot \mu \cong \mu$ for all $t \in (0, 1]$, we have that $\dim \operatorname{ann}(g_t \cdot \mu) = r$. Say, without loss of generality, that $\operatorname{ann}(g_t \cdot \mu) =$

$\mathsf{span}\{f_1,\dots,f_r\}$. Since $g_t\cdot\mu(f_1,f_1)=\cdots=g_t\cdot\mu(f_r,f_r)=0$, we also have that

$$\lambda(f_i,f_i)=\lim_{t\to 0} g_t\cdot\mu(f_i,f_i)=0,$$

for all $1\le i\le r$. Consequently, $\mathsf{span}\{f_1,\dots,f_r\}\subset \operatorname{ann}(\lambda)$, and the inequality follows.

(ii) Assume that $\dim\operatorname{ann}^k(\mu)=m$ and $\dim\operatorname{ann}^{k+1}(\mu)=r$, with $m\le r$. Reasoning as before, we can say $\operatorname{ann}^k(g_t\cdot\mu)=\mathsf{span}\{f_1,\dots,f_m\}$ and $\operatorname{ann}^{k+1}(g_t\cdot\mu)=\mathsf{span}\{f_1,\dots,f_r\}$. Since $g_t\cdot\mu(f_i,f_i)\in\operatorname{ann}^k=\mathsf{span}\{f_1,\dots,f_m\}$ for all $1\le i\le r$, we also have

$$\lambda(f_i,f_i)=\lim_{t\to 0} g_t\cdot\mu(f_i,f_i)\in\mathsf{span}\{f_1,\dots,f_m\}=\operatorname{ann}^k(\lambda),$$

for all $1\le i\le r$, where the last equality follows from the hypothesis that $\dim\operatorname{ann}^i(\mu)=\dim\operatorname{ann}^i(\lambda)$ for all $i\le k$. Consequently, $\mathsf{span}\{f_1,\dots,f_r\}\subset\operatorname{ann}^{k+1}(\lambda)$, and the inequality follows.

(iii) Since $g_t\cdot\mu\cong\mu$ for all $t\in(0,1]$, we have

$$\begin{aligned}\dim\mu^2&=\dim\big(\mathsf{span}\{\mu(e_i,e_i)\colon i=1,\dots,n\}\big)\\&=\dim\big(\mathsf{span}\{g_t\cdot\mu(f_i,f_i)\colon i=1,\dots,n\}\big)=\dim\,(g_t\cdot\mu)^2.\end{aligned}$$

Moreover, it is easy to check that if $\sum_{i=1}^n\alpha_i\big(g_t\cdot\mu(e_i,e_i)\big)=0$ for some $\alpha_i\in\mathbb{K}$, then $\sum_{i=1}^n\alpha_i\lambda(f_i,f_i)=0$. Indeed,

$$\sum_{i=1}^n\alpha_i\lambda(f_i,f_i)=\sum_{i=1}^n\alpha_i\lim_{t\to 0}g_t\cdot\mu(f_i,f_i)=\lim_{t\to 0}\sum_{i=1}^n\alpha_i\big(g_t\cdot\mu(f_i,f_i)\big)=0.$$

Since the linear dependence relations are preserved, then we get $\dim\mu^2=\dim\,(g_t\cdot\mu)^2\ge\dim\lambda^2$.

(iv) We show that every element in $\mathcal{B}^2(\lambda)$ can be obtained as the limit when $t\to 0$ of an element in $\mathcal{B}^2(g_t\cdot\mu)$. Let $\theta\in\mathcal{B}^2(\lambda)$. Then, for all $i,j=1,\dots,n$ we have that

$$\begin{aligned}\theta(f_i,f_j)&=\varphi(\lambda(f_i,f_j))-\lambda(f_i,\varphi(f_j))-\lambda(\varphi(f_i),f_j)\\&\overset{(*)}{=}\lim_{t\to 0}\varphi(g_t\cdot\mu(f_i,f_j))-\lim_{t\to 0}g_t\cdot\mu(f_i,\varphi(f_j))-\lim_{t\to 0}g_t\cdot\mu(\varphi(f_i),f_j)\\&=\lim_{t\to 0}\Big[\varphi(g_t\cdot\mu(f_i,f_j))-g_t\cdot\mu(f_i,\varphi(f_j))-g_t\cdot\mu(\varphi(f_i),f_j)\Big]\\&=\lim_{t\to 0}\theta_t,\end{aligned}$$

for some $\theta_t \in \mathcal{B}^2(g_t \cdot \mu)$. The equality $(*)$ follows from the fact that the limit and the map φ commute. Indeed, when $i \neq j$ it is trivial; and when $i = j$, if we denote by $\mu_{ik}(t)$ the structure constants of $g_t \cdot \mu$ and by λ_{ik} the structure constants of λ, we have

$$\begin{aligned}
\varphi(\lambda(f_i, f_i)) &= \varphi\left(\sum_{k=1}^{n} \lambda_{ik} f_k\right) = \sum_{k=1}^{n} \lambda_{ik}\varphi(f_k) = \sum_{k=1}^{n} \big(\lim_{t\to 0} \mu_{ik}(t)\big)\varphi(f_k) \\
&= \sum_{k=1}^{n} \lim_{t\to 0} \big(\mu_{ik}(t)\varphi(f_k)\big) = \lim_{t\to 0} \sum_{k=1}^{n} \mu_{ik}(t)\varphi(f_k) \\
&= \lim_{t\to 0} \varphi\left(\sum_{k=1}^{n} \mu_{ik}(t) f_k\right) = \lim_{t\to 0} \varphi\big(g_t \cdot \mu(f_i, f_i)\big).
\end{aligned}$$

Finally, since $g_t \cdot \mu \cong \mu$ for all $t \in (0, 1]$, we conclude that $\dim \mathcal{B}^2(\mu) = \dim \mathcal{B}^2(g_t \cdot \mu) \geq \dim \mathcal{B}^2(\lambda)$.

(v) This follows straightforwardly from the fact that for any given evolution algebra $\mathcal{E}$ it holds that $\dim \mathcal{H}^2(\mathcal{E}) = n^2 - \dim \mathcal{B}^2(\mathcal{E})$.

□

Unlike in the general setting of algebra varieties, degenerations in evolution algebras are not necessarily transitive. As we show in the next two examples, each step in a degeneration chain must be examined with particular care. In the first, we provide full computations to explicitly illustrate how such degenerations are constructed. These details will be mostly omitted in the rest of the chapter but present in the final appendices.

Example 4.3.8. Consider the following two-dimensional evolution algebras:

$$\mu_1 : e_1^2 = e_1,\ e_2^2 = e_2; \qquad \mu_2 : e_1^2 = e_1,\ e_2^2 = 0; \qquad \mu_3 : e_1^2 = e_2,\ e_2^2 = 0.$$

First, it is not difficult to see that

$$\lim_{t\to 0} g_t \cdot \mu_1 = \mu_2, \quad \text{with } g_t = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \left(\text{and } g_t^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\right).$$

Explicitly, we have

$$\begin{aligned}
g_t \cdot \mu_1(f_1, f_1) &= g_t\big(\mu_1(g_t^{-1} f_1, g_t^{-1} f_1)\big) \\
&= g_t\big(\mu_1(e_1, e_1)\big) = g_t(e_1) = f_1 \xrightarrow{t \to 0} f_1, \\
g_t \cdot \mu_1(f_2, f_2) &= g_t\big(\mu_1(g_t^{-1} f_2, g_t^{-1} f_2)\big)
\end{aligned}$$

$$= t^2 g_t(\mu_1(e_2,e_2)) = t^2 g_t(e_2) = t f_2 \xrightarrow{t\to 0} 0;$$

which converges to the product of μ_2. In the same way, we can also see that

$$\lim_{t\to 0} h_t \cdot \mu_2 \neq \mu_3, \quad \text{with } h_t = \begin{pmatrix} t^{-1} & 0 \\ t^{-2} & t^{-1} \end{pmatrix} \quad \left(\text{and } h_t^{-1} = \begin{pmatrix} t & 0 \\ -1 & t \end{pmatrix}\right).$$

Explicitly, we have

$$\begin{aligned} h_t \cdot \mu_2(f_1,f_1) &= h_t\big(\mu_2(h_t^{-1}f_1, h_t^{-1}f_1)\big) = h_t\big(\mu_2(te_1-e_2, te_1-e_2)\big) \\ &= h_t\big(t^2\mu_2(e_1,e_1)+\mu_2(e_2,e_2)\big) = t^2 h_t(e_1) = tf_1 + f_2 \xrightarrow{t\to 0} f_2, \\ h_t \cdot \mu_2(f_2,f_2) &= h_t\big(\mu_2(g_t^{-1}f_2, g_t^{-1}f_2)\big) = t^2 h_t\big(\mu_2(e_2,e_2)\big) = 0 \xrightarrow{t\to 0} 0; \end{aligned}$$

which converges to the product of μ_3. However, there is no degeneration from μ_1 to μ_3. Since μ_1 is regular, it has a unique natural basis (see [40, Theorem 4.4]). Hence, the only possibilities are given by

$$g_t = \begin{pmatrix} g_1(t) & 0 \\ 0 & g_2(t) \end{pmatrix} \left(\text{and } g_t^{-1} = \begin{pmatrix} (g_1(t))^{-1} & 0 \\ 0 & (g_2(t))^{-1} \end{pmatrix}\right) \quad \text{or}$$

$$h_t = \begin{pmatrix} 0 & h_1(t) \\ h_2(t) & 0 \end{pmatrix} \left(\text{and } h_t^{-1} = \begin{pmatrix} 0 & (h_2(t))^{-1} \\ (h_1(t))^{-1} & 0 \end{pmatrix}\right),$$

for some continuous maps $g_1(t)$, $g_1(t)$, $h_1(t)$ and $h_2(t)$ which are different from zero for any $t \in (0,1]$, but it is easy to check that $\lim_{t\to 0} g_t \cdot \mu_1 \neq \mu_3$ and $\lim_{t\to 0} h_t \cdot \mu_1 \neq \mu_3$. We check the first case, the other is analogue. In fact,

$$\begin{aligned} g_t \cdot \mu_1(f_1,f_1) &= (g_1(t))^{-2} g_t\big(\mu_1(e_1,e_1)\big) = (g_1(t))^{-2} g_t(e_1) = (g_1(t))^{-1} f_1, \\ g_t \cdot \mu_1(f_2,f_2) &= (g_2(t))^{-2} g_t\big(\mu_1(e_2,e_2)\big) = (g_2(t))^{-2} g_t(e_2) = (g_2(t))^{-1} f_2; \end{aligned}$$

which does not converge to μ_3 when $t \to 0$ for any continuous maps $g_1(t)$ and $g_2(t)$.

Remark 4.3.9. One might think that the previous example fails because the matrix

$$h_t g_t = \begin{pmatrix} t^{-1} & 0 \\ t^{-2} & t^{-2} \end{pmatrix} \quad \left(\text{and } (h_t g_t)^{-1} = g_t^{-1} h_t^{-1} = \begin{pmatrix} t & 0 \\ -t & t^2 \end{pmatrix}\right)$$

does not define a natural basis change of μ_1. However, in general, even when we have a chain of degenerations $\mu_1 \to \mu_2 \to \mu_3$ given by g_t and h_t, respectively, such that $h_t g_t$ is a natural basis change of the first evolution algebra, we may still have

$$\lim_{t\to 0}(h_t g_t) \cdot \mu_1 \neq \lim_{t\to 0} h_t \cdot \Big(\lim_{t\to 0} g_t \cdot \mu_1\Big).$$

For instance, consider the three-dimensional evolution algebras $\mathcal{E}_{3,4} \equiv \mu_{3,4}$, $\mathcal{E}_{3,2} \equiv \mu_{3,2}$ and $\mathcal{E}_{3,1} \equiv \mu_{3,1}$ in Table 1.2.1. We can easily check that $\mu_{3,2} = \lim_{t\to 0} g_t \cdot \mu_{3,4}$ with $g_t = \operatorname{diag}(1, t, t^2)$ and that $\mu_{3,1} = \lim_{t\to 0} h_t \cdot \mu_{3,2}$ with $h_t = \operatorname{diag}(1, t, 1)$. However, although $h_t g_t = \operatorname{diag}(1, t^2, t^2)$ is a natural basis change of μ_1 for all $t \neq 0$, we have that $\lim_{t\to 0}(h_t g_t) \cdot \mu_1 \neq \mu_3$. Specifically,

$$(h_t g_t) \cdot \mu_1(f_2, f_2) = t^{-4} h_t(g_t(e_3)) = t^{-2} e_2 \xrightarrow{t \to 0} \infty.$$

As a consequence of the lack of transitivity discussed above, we introduce the following definition considering the transitive closure of the relation "being a degeneration ($\to$)".

Definition 4.3.10. Let $\mathcal{F}$ be a family of evolution algebras, and let $\mu, \lambda \in \mathcal{F}$. We say that λ is a *transitive degeneration* of μ, or that μ *transitively degenerates* to λ, among $\mathcal{F}$ if there exists a finite sequence of degenerations

$$\mu = \mu_0 \to \mu_1 \to \cdots \to \mu_k = \lambda,$$

with each $\mu_i \in \mathcal{F}$. We denote this by $\mu \rightsquigarrow \lambda$.

4.3.2 Degenerations of nilpotent evolution algebras up to dimension four

The study of degeneration relations among nilpotent Lie algebras has been extensively developed (see, for instance, [15, 55, 97]). Motivated by this, and taking into account that complex nilpotent evolution algebras up to dimension four have been completely classified (see Table 1.2.1), we devote this final part to the study of transitive degenerations within these families of evolution algebras, explicitly constructing the corresponding maps g_t and the associated Hasse diagrams, which collectively describe all such relations.

Notation 4.3.11. For convenience in what follows, we shall denote the evolution algebras listed in Table 1.2.1 by $\mu_{i,j}$ instead of $\mathcal{E}_{i,j}$. That is, each symbol $\mu_{i,j}$ will refer to the corresponding algebra $\mathcal{E}_{i,j}$ in the classification. Moreover, we denote by E_{ij} the matrix having a 1 in the (i, j)-entry and zeros elsewhere.

Remark 4.3.12. In what follows, we will only use the first three conditions in Proposition 4.3.7. Although the last two are not considered here, we believe that they could provide additional information and lead to stronger results.

Hasse diagrams of nilpotent evolution algebras of dimensions two and three

The unique isomorphism classes in $\mathcal{N}_2(\mathbb{C})$ are $\mu_{2,1} : e_1^2 = e_2^2 = 0$ and $\mu_{2,2} : e_1^2 = e_2$, $e_2^2 = 0$. Then, since every evolution algebra degenerates to the abelian one of the same dimension (Example 4.3.5), the corresponding Hasse diagram is $\mu_{2,2} \to \mu_{2,1}$.

Theorem 4.3.13. *All transitive degenerations among $\mathcal{N}_3(\mathbb{C})$ are captured by the Hasse diagram $\mu_{3,4} \to \mu_{3,3} \to \mu_{3,2} \to \mu_{3,1}$. Moreover, $\mu_{3,2}$ is also a degeneration of $\mu_{3,4}$.*

Proof. From the type and the dimension of the derived subalgebra of each evolution algebra in $\mathcal{N}_3(\mathbb{C})$ (see Table 1.2.1) and Proposition 4.3.7, it follows that the previous Hasse diagram realises the maximal number of transitive degenerations. Therefore, since every evolution algebra degenerates to the abelian one, it remains only to prove that $\mu_{3,4} \to \mu_{3,3}$, $\mu_{3,3} \to \mu_{3,2}$ and $\mu_{3,4} \to \mu_{3,2}$. Although we omit the detailed computations, we explicitly list the suitable transformations in Table 4.3.1.

Degeneration	g_t
$\mu_{3,4} \to \mu_{3,3}$	$\mathrm{diag}(1, t, t^2) + E_{32}$
$\mu_{3,3} \to \mu_{3,2}$	$\mathrm{diag}(t, 1, t^2)$
$\mu_{3,4} \to \mu_{3,2}$	$\mathrm{diag}(1, t, t^2)$

Table 4.3.1: All degenerations in $\mathcal{N}_3(\mathbb{C})$ and the corresponding transformations.

□

Finally, we state the following straightforward results.

Corollary 4.3.14. *$\mu_{3,4}$ is the unique rigid evolution algebra in $\mathcal{N}_3(\mathbb{C})$.*

Corollary 4.3.15. *Let $\mu, \lambda \in \mathcal{N}_3(\mathbb{C})$. Then, $\mu \rightsquigarrow \lambda$ if and only if $\mu \to \lambda$.*

Hasse diagrams of nilpotent evolution algebras of dimension four

We devote this final section to the study of (transitive) degenerations among evolution algebras of dimension four over $\mathbb{C}$. In particular, the next result presents all transitive degeneration relations that we have explicitly established within $\mathcal{N}_4(\mathbb{C})$.

Proposition 4.3.16. *The Hasse diagram in Figure 4.3.1 captures several transitive degeneration relations among $\mathcal{N}_4(\mathbb{C})$. Particularly, the diagram represents all transitive degenerations among the algebras $\{\mu_{4,i}\}_{i=1}^{7}$.*

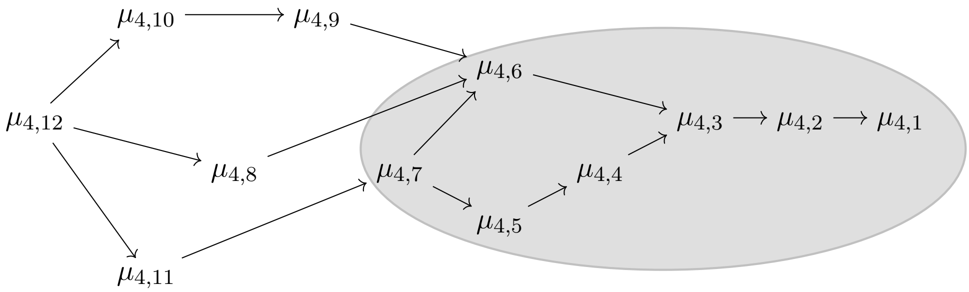


Figure 4.3.1: Several transitive degeneration relations among $\mathcal{N}_4(\mathbb{C})$

Proof. Although we omit the detailed computations, all these degenerations were explicitly constructed using suitable transformations g_t listed in Table 4.3.2. In particular, the grey ellipse highlights all transitive degenerations among the algebras $\mu_{4,1}$ through $\mu_{4,7}$. Looking at the type and the dimension of the derived subalgebras of $\mu_{4,4}$, $\mu_{4,5}$ and $\mu_{4,6}$, and applying Proposition 4.3.7, it is immediate that neither $\mu_{4,4}$ nor $\mu_{4,5}$ degenerate to $\mu_{4,6}$, and conversely, $\mu_{4,6}$ does not degenerate to $\mu_{4,4}$ or $\mu_{4,5}$. Therefore, the Hasse diagram inside the grey ellipse above captures the maximal number of transitive degeneration relations within this family of seven algebras.

Degeneration	g_t
$\mu_{4,12} \longrightarrow \mu_{4,11}$	$\operatorname{diag}(t, t^2, t^4, t^8)$
$\mu_{4,12} \longrightarrow \mu_{4,10}$	$\operatorname{diag}(t^{-1}, t^{-1}, t^{-2}, t^{-4}) - t^{-2}E_{42} + t^{-2}E_{43}$
$\mu_{4,12} \longrightarrow \mu_{4,8}$	$\operatorname{diag}(\sqrt{i}t^{-1}, it^{-2}, -t^{-2}, t^{-4}) + t^{-4}E_{42} - t^{-4}E_{43}$
$\mu_{4,11} \longrightarrow \mu_{4,7}$	$\operatorname{diag}(1, 1, t, t^2) + E_{43}$
$\mu_{4,10} \longrightarrow \mu_{4,9}$	$\operatorname{diag}(t, t, t^2, t^4)$
$\mu_{4,9} \longrightarrow \mu_{4,6}$	$\operatorname{diag}(1, 1, t, t^2) + E_{43}$
$\mu_{4,8} \longrightarrow \mu_{4,6}$	$\operatorname{diag}(t, t^3, t^3, t^6) - it^2E_{43}$
$\mu_{4,7} \longrightarrow \mu_{4,6}$	$\operatorname{diag}(1, t, t, t^2) + E_{42}$
$\mu_{4,7} \longrightarrow \mu_{4,5}$	$\operatorname{diag}(t, t^2, 1, t^4)$
$\mu_{4,6} \longrightarrow \mu_{4,3}$	$\operatorname{diag}(t^2, t^2, 1, t^4)$
$\mu_{4,5} \longrightarrow \mu_{4,4}$	$\operatorname{diag}(1, t, 1, t^2) + E_{32}$
$\mu_{4,4} \longrightarrow \mu_{4,3}$	$\operatorname{diag}(t^{-3}, t^{-1}, t^{-4}, t^{-2}) + t^{-6}E_{43}$
$\mu_{4,3} \longrightarrow \mu_{4,2}$	$\operatorname{diag}(1, t^{-1}, 1, 1)$

Table 4.3.2: All degenerations in $\mathcal{N}_4(\mathbb{C})$ and the corresponding transformations.

$\square$

Remark 4.3.17. Although many of the transformations in Table 4.3.2 (and even earlier ones, such as those in Theorem 4.3.13) may appear difficult to obtain, Proposition 4.3.6 often provides a powerful and practical tool. We illustrate its utility through the case $\mu_{4,12} \to \mu_{4,10}$. If $\mu_{4,10}$ is a degeneration of $\mu_{4,12}$, then, by Proposition 4.3.6, $\mu_{4,12}$ is a deformation of $\mu_{4,10}$. In particular, the structure matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

defines a nilpotent first-order deformation of $\mu_{4,10}$, which is in fact isomorphic to $\mu_{4,12}$ for all $t \in \mathbb{C}^*$. A straightforward computation shows that the product with respect to the natural basis $B' = \{f_1 = t^{-1}e_1, f_2 = t^{-1}e_2 - t^{-2}e_4, f_3 = t^{-2}e_3 + t^{-2}e_4, f_4 = t^{-4}e_4\}$ coincides with the product of $\mu_{4,12}$. Indeed,

$$\begin{aligned} f_1^2 &= t^{-2}e_1^2 = t^{-2}(te_2 + e_3) = t^{-1}e_2 + t^{-2}e_3 = f_2 + f_3, \\ f_2^2 &= t^{-2}e_2^2 + t^{-4}e_4^2 = t^{-2}(e_2 + e_3) = f_3, \\ f_3^2 &= t^{-4}e_3^2 + t^{-4}e_4^2 = t^{-4}e_4 = f_4, \\ f_4^2 &= t^{-8}e_4^2 = 0. \end{aligned}$$

Therefore, a possible transformation g_t is obtained as the change of basis matrix from B' to the original basis B:

$$g_t = \operatorname{diag}(t^{-1}, t^{-1}, t^{-2}, t^{-4}) - t^{-2}E_{42} + t^{-2}E_{43}.$$

Although additional relations may exist beyond those shown, the Hasse diagram of Proposition 4.3.16 provides a coherent and informative global picture of the known degeneration structure and, even if it may not capture all possible transitive degenerations, it still allows us to draw the following conclusion.

Corollary 4.3.18. *$\mu_{4,12}$ is the unique rigid evolution algebra in $\mathcal{N}_4(\mathbb{C})$.*

Appendices of Chapter 4

4.A Calculations for Theorem 4.2.5

This first appendix is devoted to the explicit computations supporting Theorem 4.2.5. These yield Table 4.2.1, which fully characterises the infinitesimal deformations of all isomorphism classes of two-dimensional evolution algebras over $\mathbb{C}$.

Infinitesimal deformations of $\mathcal{E}_2$

Let $\mathcal{E}_2$ be the evolution algebra over $\mathbb{C}$ with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_2^2 = e_1$. If $\theta \in \mathcal{B}^2(\mathcal{E}_2)$, then there must exist a linear morphism $(\varphi)_{BB} = (\xi_{ij})$ such that the following conditions hold:

$$\begin{aligned}
\theta(e_1, e_2) &= \varphi(e_1e_2) - e_1\varphi(e_2) - \varphi(e_1)e_2 = -(\xi_{12} + \xi_{21})e_1 = 0\\
&\implies \xi_{12} = -\xi_{21};\\
\theta(e_1, e_1) &= \varphi(e_1^2) - 2e_1\varphi(e_1) = -\xi_{11}e_1 + \xi_{21}e_2;\\
\theta(e_2, e_2) &= \varphi(e_2^2) - 2e_2\varphi(e_2) = (\xi_{11} - 2\xi_{22})e_1 + \xi_{21}e_2.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\mathcal{B}^2(\mathcal{E}_2) &= \left\{ \left(\begin{smallmatrix} -\xi_{11} & \xi_{21} \\ \xi_{11}-2\xi_{22} & \xi_{21} \end{smallmatrix}\right) \mid \xi_{11}, \xi_{21}, \xi_{22} \in \mathbb{K} \right\}\\
&= \operatorname{span}\left\{ \left(\begin{smallmatrix} -1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ -2 & 0 \end{smallmatrix}\right) \right\},
\end{aligned}$$

and

$$\mathcal{H}^2(\mathcal{E}_2) = \operatorname{span}\left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_2) \right\}.$$

Consequently, the set of all infinitesimal formal evolution deformations of $\mathcal{E}_2$ (up to equivalence) is given by

$$\operatorname{InfDef}(\mathcal{E}_2) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right)t \mid \alpha \in \mathbb{C} \right\}.$$

Infinitesimal deformations of $\mathcal{E}_3$

Let $\mathcal{E}_3$ be the evolution algebra over $\mathbb{C}$ with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = -e_2^2 = e_1 + e_2$. If $\theta \in \mathcal{B}^2(\mathcal{E}_3)$, then there must exist a linear morphism $(\varphi)_{BB} = (\xi_{ij})$ such that the following conditions hold:

$$\begin{aligned}
\theta(e_1, e_2) &= \varphi(e_1 e_2) - e_1\varphi(e_2) - \varphi(e_1)e_2 = -(\xi_{12} - \xi_{21})(e_1 + e_2) = 0 \\
&\implies \xi_{12} = \xi_{21}; \\
\theta(e_1, e_1) &= \varphi(e_1^2) - 2e_1\varphi(e_1) = (\xi_{12} - \xi_{11})e_1 + (\xi_{12} + \xi_{22} - 2\xi_{11})e_2; \\
\theta(e_2, e_2) &= \varphi(e_2^2) - 2e_2\varphi(e_2) = (2\xi_{22} - \xi_{11} - \xi_{12})e_1 + (\xi_{22} - \xi_{12})e_2.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\mathcal{B}^2(\mathcal{E}_3) &= \left\{ \begin{pmatrix} \xi_{12}-\xi_{11} & \xi_{12}+\xi_{22}-2\xi_{11} \\ 2\xi_{22}-\xi_{11}-\xi_{12} & \xi_{22}-\xi_{12} \end{pmatrix} \mid \xi_{11}, \xi_{21}, \xi_{22} \in \mathbb{K} \right\} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \right\},
\end{aligned}$$

and

$$\mathcal{H}^2(\mathcal{E}_3) = \operatorname{span}\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathcal{B}^2(\mathcal{E}_3), \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{B}^2(\mathcal{E}_3) \right\}.$$

Consequently, the set of all infinitesimal formal evolution deformations of $\mathcal{E}_3$ (up to equivalence) is given by

$$\operatorname{InfDef}(\mathcal{E}_3) = \left\{ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix} t \mid \alpha \in \mathbb{K} \right\}.$$

Infinitesimal deformations of $\mathcal{E}_4$

Let $\mathcal{E}_4$ be the evolution algebra over $\mathbb{C}$ with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_2$ and $e_2^2 = 0$. If $\theta \in \mathcal{B}^2(\mathcal{E}_4)$, then there must exist a linear morphism $(\varphi)_{BB} = (\xi_{ij})$ such that the following conditions hold:

$$\begin{aligned}
\theta(e_1, e_2) &= \varphi(e_1 e_2) - e_1\varphi(e_2) - \varphi(e_1)e_2 = -\xi_{12}e_2 = 0 \implies \xi_{12} = 0; \\
\theta(e_1, e_1) &= \varphi(e_1^2) - 2e_1\varphi(e_1) = (\xi_{22} - 2\xi_{11})e_2; \\
\theta(e_2, e_2) &= \varphi(e_2^2) - 2e_2\varphi(e_2) = 0.
\end{aligned}$$

Hence, we have that

$$\mathcal{B}^2(\mathcal{E}_4) = \left\{ \begin{pmatrix} 0 & \xi_{22}-2\xi_{11} \\ 0 & 0 \end{pmatrix} \mid \xi_{11}, \xi_{22} \in \mathbb{K} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\},$$

and

$$\mathcal{H}^2(\mathcal{E}_4) = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{B}^2(\mathcal{E}_4), \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathcal{B}^2(\mathcal{E}_4), \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{B}^2(\mathcal{E}_4) \right\}.$$

Consequently, the set of all infinitesimal formal evolution deformations of $\mathcal{E}_4$ (up to equivalence) is given by

$$\mathrm{InfDef}(\mathcal{E}_4) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right) t \mid \alpha, \beta, \gamma \in \mathbb{K} \right\}.$$

Infinitesimal deformations of $\mathcal{E}_5(a_2, a_3)$

Let $\mathcal{E}_5(a_2, a_3)$ be the evolution algebra over $\mathbb{C}$ with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1 + a_2 e_2$ and $e_2^2 = a_3 e_1 + e_2$ with $1 - a_2 e_3 \neq 0$. If $\theta \in \mathcal{B}^2(\mathcal{E}_5(a_2, a_3))$, then there must exist a linear morphism $(\varphi)_{BB} = (\xi_{ij})$ such that the following conditions hold:

$$\begin{aligned} \theta(e_1, e_2) &= \varphi(e_1 e_2) - e_1 \varphi(e_2) - \varphi(e_1) e_2 \\ &= -\xi_{12}(e_1 + a_2 e_2) - \xi_{21}(a_3 e_1 + e_2) = 0 \implies \xi_{12} = \xi_{21} = 0; \\ \theta(e_1, e_1) &= \varphi(e_1^2) - 2e_1 \varphi(e_1) = -\xi_{11} e_1 + a_2(\xi_{22} - 2\xi_{11}) e_2; \\ \theta(e_2, e_2) &= \varphi(e_2^2) - 2e_2 \varphi(e_2) = a_3(\xi_{11} - 2\xi_{22}) e_1 - \xi_{22} e_2. \end{aligned}$$

Hence, we have that

$$\begin{aligned} \mathcal{B}^2(\mathcal{E}_5) &= \left\{ \left(\begin{smallmatrix} -\xi_{11} & a_2(\xi_{22} - 2\xi_{11}) \\ a_3(\xi_{11} - 2\xi_{22}) & -\xi_{22} \end{smallmatrix}\right) \mid \xi_{11}, \xi_{22} \in \mathbb{K} \right\} \\ &= \mathsf{span} \left\{ \left(\begin{smallmatrix} -1 & -2a_2 \\ a_3 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & a_2 \\ -2a_3 & -1 \end{smallmatrix}\right) \right\}. \end{aligned}$$

Consequently, we now distinguish the set of infinitesimal deformations depending on the values of a_2 and a_3:

1. If $a_2 \neq 0$, then $\mathcal{H}^2(\mathcal{E}_5) = \mathsf{span}\{ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_5), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_5)\}$ and

$$\mathrm{InfDef}(\mathcal{E}_5) = \left\{ \left(\begin{smallmatrix} 1 & a_2 \\ a_3 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \alpha & \beta \end{smallmatrix}\right) t \mid \alpha, \beta \in \mathbb{K} \right\}.$$

2. If $a_2 = a_3 = 0$, then $\mathcal{H}^2(\mathcal{E}_5) = \mathsf{span}\{ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_5), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_5)\}$ and

$$\mathrm{InfDef}(\mathcal{E}_5) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right) t \mid \alpha, \beta \in \mathbb{K} \right\}.$$

3. If $a_2 = 0 \neq a_3$, then $\mathcal{H}^2(\mathcal{E}_5) = \mathsf{span}\{ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_5), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_5)\}$ and

$$\mathrm{InfDef}(\mathcal{E}_5) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ a_3 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & \alpha \\ 0 & \beta \end{smallmatrix}\right) t \mid \alpha, \beta \in \mathbb{K} \right\}.$$

Infinitesimal deformations of $\mathcal{E}_6(a_4)$

Let $\mathcal{E}_6(a_4)$ be the evolution algebra over $\mathbb{C}$ with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_2$ and $e_2^2 = e_1 + a_4 e_2$. If $\theta \in \mathcal{B}^2(\mathcal{E}_6)$, then there must exist a linear morphism $(\varphi)_{BB} = (\xi_{ij})$ such that the following conditions hold:

$$\begin{aligned}
\theta(e_1, e_2) &= \varphi(e_1 e_2) - e_1\varphi(e_2) - \varphi(e_1)e_2 \\
&= -\xi_{21}e_1 - (\xi_{12} + a_4\xi_{21})e_2 = 0 \implies \xi_{12} = \xi_{21} = 0; \\
\theta(e_1, e_1) &= \varphi(e_1^2) - 2e_1\varphi(e_1) = (\xi_{22} - 2\xi_{11})e_2; \\
\theta(e_2, e_2) &= \varphi(e_2^2) - 2e_2\varphi(e_2) = (\xi_{11} - 2\xi_{22})e_1 - a_4\xi_{22}e_2.
\end{aligned}$$

Hence, we have that

$$\mathcal{B}^2(\mathcal{E}_6) = \left\{\left(\begin{smallmatrix} 0 & \xi_{22}-2\xi_{11} \\ \xi_{11}-2\xi_{22} & -a_4\xi_{22} \end{smallmatrix}\right) \mid \xi_{11}, \xi_{22} \in \mathbb{K}\right\} = \mathsf{span}\left\{\left(\begin{smallmatrix} 0 & 1 \\ -2 & -a_4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -2 \\ 1 & 0 \end{smallmatrix}\right)\right\}.$$

and

$$\mathcal{H}^2(\mathcal{E}_6) = \mathsf{span}\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_6), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \mathcal{B}^2(\mathcal{E}_6)\right\}.$$

Consequently, the set of all infinitesimal formal evolution deformations of $\mathcal{E}_4$ (up to equivalence) is given by

$$\operatorname{InfDef}(\mathcal{E}_6) = \left\{\left(\begin{smallmatrix} 0 & 1 \\ 1 & a_4 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)t \mid \alpha, \beta \in \mathbb{K}\right\}.$$

4.B Calculations for Theorem 4.3.13

This appendix is devoted to the explicit computations supporting Theorem 4.3.13 and, particularly, Table 4.3.1, which fully characterise the transitive degeneration relations within $\mathcal{N}_3(\mathbb{C})$.

From $\mu_{3,4}$ to $\mu_{3,3}$

$\mu_{3,4} \longrightarrow \mu_{3,3}$	$g_t = \operatorname{diag}(1, t, t^2) + E_{32}$

$g_t \cdot \mu_{3,4}(f_1, f_1) = g_t(\mu_{3,4}(e_1, e_1)) = g_t(e_2) = tf_2 + f_3 \xrightarrow{t \to 0} f_3,$
$g_t \cdot \mu_{3,4}(f_2, f_2) = t^{-2}g_t(\mu_{3,4}(e_2, e_2)) = t^{-2}g_t(e_3) = f_3 \xrightarrow{t \to 0} f_3,$
$g_t \cdot \mu_{3,4}(f_3, f_3) = t^{-4}g_t(\mu_{3,4}(e_3, e_3)) = 0 \xrightarrow{t \to 0} 0$

From $\mu_{3,3}$ to $\mu_{3,2}$

$\mu_{3,3} \longrightarrow \mu_{3,2}$	$g_t = \mathrm{diag}(t, 1, t^2)$

$$g_t \cdot \mu_{3,3}(f_1, f_1) = t^{-2} g_t(\mu_{3,3}(e_1, e_1)) = t^{-2} g_t(e_3) = f_3 \xrightarrow{t \to 0} f_3;$$
$$g_t \cdot \mu_{3,3}(f_2, f_2) = g_t(\mu_{3,3}(e_2, e_2)) = g_t(e_3) = t^2 f_3 \xrightarrow{t \to 0} 0;$$
$$g_t \cdot \mu_{3,3}(f_3, f_3) = t^{-4} g_t(\mu_{3,3}(e_3, e_3)) = 0 \xrightarrow{t \to 0} 0;$$

From $\mu_{3,4}$ to $\mu_{3,2}$

$\mu_{3,4} \longrightarrow \mu_{3,2}$	$g_t = \mathrm{diag}(1, t, t^2)$

$$g_t \cdot \mu_{3,4}(f_1, f_1) = g_t(\mu_{3,4}(e_1, e_1)) = g_t(e_2) = t f_2 \xrightarrow{t \to 0} 0;$$
$$g_t \cdot \mu_{3,4}(f_2, f_2) = t^{-2} g_t(\mu_{3,4}(e_2, e_2)) = t^{-2} g_t(e_3) = f_3 \xrightarrow{t \to 0} f_3;$$
$$g_t \cdot \mu_{3,3}(f_3, f_3) = 0 \xrightarrow{t \to 0} 0;$$

4.C Calculations for Proposition 4.3.16

This appendix is devoted to the explicit computations supporting Proposition 4.3.16 and, particularly, Table 4.3.2, which yield several transitive degeneration relations within $\mathcal{N}_3(\mathbb{C})$.

From $\mu_{4,12}$ to $\mu_{4,11}$

$\mu_{4,12} \longrightarrow \mu_{4,11}$	$g_t = \mathrm{diag}(t, t^2, t^4, t^8)$

$$g_t \cdot \mu_{4,12}(f_1, f_1) = t^{-2} g_t(\mu_{4,12}(e_1, e_1)) = t^{-2} g_t(e_2 + e_3) = f_2 + t^2 f_3 \xrightarrow{t \to 0} f_2;$$
$$g_t \cdot \mu_{4,12}(f_2, f_2) = t^{-4} g_t(\mu_{4,12}(e_2, e_2)) = t^{-4} g_t(e_3) = f_3 \xrightarrow{t \to 0} f_3;$$
$$g_t \cdot \mu_{4,12}(f_3, f_3) = t^{-8} g_t(\mu_{4,12}(e_3, e_3)) = t^{-8} g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,12}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.$$

From $\mu_{4,12}$ to $\mu_{4,10}$

$\mu_{4,12} \longrightarrow \mu_{4,10}$	$g_t = \mathrm{diag}(t^{-1}, t^{-1}, t^{-2}, t^{-4}) - t^{-2}E_{42} + t^{-2}E_{43}$

$$\begin{aligned}
&g_t \cdot \mu_{4,12}(f_1, f_1) = t^2 g_t(\mu_{4,12}(e_1, e_1)) = t^2 g_t(e_2 + e_3) = t f_2 + f_3 \xrightarrow{t \to 0} f_3;\\
&g_t \cdot \mu_{4,12}(f_2, f_2) = t^2 g_t(\mu_{4,12}(e_2, e_2)) = t^2 g_t(e_3) = f_3 + f_4 \xrightarrow{t \to 0} f_3 + f_4;\\
&g_t \cdot \mu_{4,12}(f_3, f_3) = t^4 g_t(\mu_{4,12}(e_3, e_3)) = t^4 g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;\\
&g_t \cdot \mu_{4,12}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.
\end{aligned}$$

From $\mu_{4,12}$ to $\mu_{4,8}$

$\mu_{4,12} \longrightarrow \mu_{4,8}$	$g_t = \mathrm{diag}(\sqrt{i}t^{-1}, it^{-2}, -t^{-2}, t^{-4}) + t^{-4}E_{42} - t^{-4}E_{43}$

$$\begin{aligned}
&g_t \cdot \mu_{4,12}(f_1, f_1) = \tfrac{t^2}{i} g_t(\mu_{4,12}(e_1, e_1)) = \tfrac{t^2}{i} g_t(e_2 + e_3) = f_2 + i f_3 \xrightarrow{t \to 0} f_2 + i f_3;\\
&g_t \cdot \mu_{4,12}(f_2, f_2) = -t^4 g_t(\mu_{4,12}(e_2, e_2)) = -t^4 g_t(e_3) = t^2 f_3 + f_4 \xrightarrow{t \to 0} f_4;\\
&g_t \cdot \mu_{4,12}(f_3, f_3) = t^4 g_t(\mu_{4,12}(e_3, e_3)) = t^4 g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;\\
&g_t \cdot \mu_{4,12}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.
\end{aligned}$$

From $\mu_{4,11}$ to $\mu_{4,7}$

$\mu_{4,11} \longrightarrow \mu_{4,7}$	$g_t = \mathrm{diag}(1, 1, t, t^2) + E_{43}$

$$\begin{aligned}
&g_t \cdot \mu_{4,11}(f_1, f_1) = g_t(\mu_{4,11}(e_1, e_1)) = g_t(e_2) = f_2 \xrightarrow{t \to 0} f_2;\\
&g_t \cdot \mu_{4,11}(f_2, f_2) = g_t(\mu_{4,11}(e_2, e_2)) = g_t(e_3) = t f_3 + f_4 \xrightarrow{t \to 0} f_4;\\
&g_t \cdot \mu_{4,11}(f_3, f_3) = t^{-2} g_t(\mu_{4,11}(e_3, e_3)) = t^4 g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;\\
&g_t \cdot \mu_{4,11}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.
\end{aligned}$$

From $\mu_{4,10}$ to $\mu_{4,9}$

$\mu_{4,10} \longrightarrow \mu_{4,9}$	$g_t = \mathrm{diag}(t, t, t^2, t^4)$

$$\begin{aligned}
&g_t \cdot \mu_{4,10}(f_1, f_1) = t^{-2} g_t(\mu_{4,10}(e_1, e_1)) = t^{-2} g_t(e_3) = f_3 \xrightarrow{t \to 0} f_3;\\
&g_t \cdot \mu_{4,10}(f_2, f_2) = t^{-2} g_t(\mu_{4,10}(e_2, e_2)) = t^{-2} g_t(e_3 + e_4) = f_3 + t^2 f_4 \xrightarrow{t \to 0} f_3;\\
&g_t \cdot \mu_{4,10}(f_3, f_3) = t^{-4} g_t(\mu_{4,10}(e_3, e_3)) = t^{-4} g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;\\
&g_t \cdot \mu_{4,10}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.
\end{aligned}$$

From $\mu_{4,9}$ to $\mu_{4,6}$

$\mu_{4,9} \longrightarrow \mu_{4,6}$	$g_t = \mathrm{diag}(1, 1, t, t^2) + E_{43}$

$$g_t \cdot \mu_{4,9}(f_1, f_1) = g_t(\mu_{4,9}(e_1, e_1)) = g_t(e_3) = tf_3 + f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,9}(f_2, f_2) = g_t(\mu_{4,9}(e_2, e_2)) = g_t(e_3) = tf_3 + f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,9}(f_3, f_3) = t^{-2}g_t(\mu_{4,9}(e_3, e_3)) = t^{-2}g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,9}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.$$

From $\mu_{4,8}$ to $\mu_{4,6}$

$\mu_{4,8} \longrightarrow \mu_{4,6}$	$g_t = \mathrm{diag}(t, t^3, t^3, t^6) - it^2 E_{43}$

$$g_t \cdot \mu_{4,8}(f_1, f_1) = t^{-2}g_t(\mu_{4,8}(e_1, e_1)) = t^{-2}g_t(e_2 + ie_3) = tf_2 + itf_3 + f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,8}(f_2, f_2) = t^{-6}g_t(\mu_{4,8}(e_2, e_2)) = t^{-6}g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,8}(f_3, f_3) = t^{-6}g_t(\mu_{4,8}(e_3, e_3)) = t^{-6}g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,8}(f_4, f_4) = 0 \xrightarrow{t \to 0} 0.$$

From $\mu_{4,7}$ to $\mu_{4,6}$

$\mu_{4,7} \longrightarrow \mu_{4,6}$	$g_t = \mathrm{diag}(1, t, t, t^2) + E_{42}$

$$g_t \cdot \mu_{4,7}(f_1, f_1) = g_t(\mu_{4,7}(e_1, e_1)) = g_t(e_2) = tf_2 + f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,7}(f_2, f_2) = t^{-4}g_t(\mu_{4,7}(e_2, e_2)) = t^{-4}g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,7}(f_3, f_3) = t^{-2}g_t(\mu_{4,7}(e_3, e_3)) = t^{-2}g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,7}(f_4, f_4) = g_t(\mu_{4,7}(e_4, e_4)) = 0 \xrightarrow{t \to 0} 0.$$

From $\mu_{4,7}$ to $\mu_{4,5}$

$\mu_{4,7} \longrightarrow \mu_{4,5}$	$g_t = \mathrm{diag}(t, t^2, 1, t^4)$

$$g_t \cdot \mu_{4,7}(f_1, f_1) = t^{-2}g_t(\mu_{4,7}(e_1, e_1)) = t^{-2}g_t(e_2) = f_2 \xrightarrow{t \to 0} f_2;$$
$$g_t \cdot \mu_{4,7}(f_2, f_2) = t^{-4}g_t(\mu_{4,7}(e_2, e_2)) = t^{-4}g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,7}(f_3, f_3) = g_t(\mu_{4,7}(e_3, e_3)) = g_t(e_4) = t^4 f_4 \xrightarrow{t \to 0} 0;$$
$$g_t \cdot \mu_{4,7}(f_4, f_4) = g_t(\mu_{4,7}(e_4, e_4)) = 0 \xrightarrow{t \to 0} 0.$$

From $\mu_{4,6}$ to $\mu_{4,3}$

$\mu_{4,6} \longrightarrow \mu_{4,3}$	$g_t = \mathrm{diag}(t^2, t^2, 1, t^4)$

$$g_t \cdot \mu_{4,6}(f_1, f_1) = t^{-4} g_t(\mu_{4,6}(e_1, e_1)) = t^{-4} g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4$$
$$g_t \cdot \mu_{4,6}(f_2, f_2) = t^{-4} g_t(\mu_{4,6}(e_2, e_2)) = t^{-4} g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,6}(f_3, f_3) = g_t(\mu_{4,6}(e_3, e_3)) = g_t(e_4) = t^4 f_4 \xrightarrow{t \to 0} 0;$$
$$g_t \cdot \mu_{4,6}(f_4, f_4) = g_t(\mu_{4,6}(e_4, e_4)) = 0 \xrightarrow{t \to 0} 0.$$

From $\mu_{4,5}$ to $\mu_{4,4}$

$\mu_{4,5} \longrightarrow \mu_{4,4}$	$g_t = \mathrm{diag}(1, t, 1, t^2) + E_{32}$

$$g_t \cdot \mu_{4,5}(f_1, f_1) = g_t(\mu_{4,5}(e_1, e_1)) = g_t(e_2) = t f_2 + f_3 \xrightarrow{t \to 0} f_3;$$
$$g_t \cdot \mu_{4,5}(f_2, f_2) = t^{-2} g_t(\mu_{4,5}(e_2, e_2)) = t^{-2} g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,5}(f_3, f_3) = g_t \cdot \mu_{4,5}(f_4, f_4) = 0.$$

From $\mu_{4,4}$ to $\mu_{4,3}$

$\mu_{4,4} \longrightarrow \mu_{4,3}$	$g_t = \mathrm{diag}(t^{-3}, t^{-1}, t^{-4}, t^{-2}) + t^{-6} E_{43}$

$$g_t \cdot \mu_{4,4}(f_1, f_1) = t^6 g_t(\mu_{4,4}(e_1, e_1)) = t^6 g_t(e_3) = t^2 f_3 + f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,4}(f_2, f_2) = t^2 g_t(\mu_{4,4}(e_2, e_2)) = t^2 g_t(e_4) = f_4 \xrightarrow{t \to 0} f_4;$$
$$g_t \cdot \mu_{4,4}(f_3, f_3) = g_t \cdot \mu_{4,4}(f_4, f_4) = 0.$$

From $\mu_{4,3}$ to $\mu_{4,2}$

$\mu_{4,3} \longrightarrow \mu_{4,2}$	$g_t = \mathrm{diag}(1, t^{-1}, 1, 1)$

$$g_t \cdot \mu_{4,3}(f_1, f_1) = g_t(\mu_{4,3}(e_1, e_1)) = g_t(e_3) = f_3 \xrightarrow{t \to 0} f_3;$$
$$g_t \cdot \mu_{4,3}(f_2, f_2) = t^2 g_t(\mu_{4,3}(e_2, e_2)) = t^2 g_t(e_3) = t^2 f_3 \xrightarrow{t \to 0} 0;$$
$$g_t \cdot \mu_{4,3}(f_3, f_3) = g_t \cdot \mu_{4,3}(f_4, f_4) = 0.$$

Part II

Gonosomal algebras and operators

Chapter 5

A review of genetic algebras modelling bisexual populations

Bisexual populations are those in which sexual differentiation exists, and reproduction requires the interaction between two sexually differentiated types, typically females and males. One of the main challenges when constructing algebraic models of sex-linked inheritance in such populations is that inheritance depends not only on the genetic material transmitted by individuals but also on the sex-determination systems, which are the biological mechanisms that determine the development of sexual characteristics in organisms. In this context, gonosomal algebras represent the most recent and comprehensive framework for algebraically capturing the dynamics of sex-linked inheritance.

It is worth noting that a recent preprint by R. Varro extends the notion of gonosomal algebras by introducing the so-called *gonosomic algebras* [106]; however, this approach will not be considered in this thesis.

Hence, the motivation for this preliminary chapter of the second part of the thesis is to explain the emergence of gonosomal algebras and to provide a review of their main properties and related results, using the case of haemophilia as a guiding example. In particular, this chapter is organised into four sections. Section 5.1 briefly recalls the notions of baric and dibaric algebras, which play a fundamental role in the study of genetic algebras. Next, Section 5.2 is devoted to motivating the introduction of gonosomal algebras. We first recall the framework of evolution algebras of bisexual populations (EABPs) and show that the case of haemophilia cannot be modelled within this setting due to the lack of dibaricity. This impediment naturally leads to the generalisation provided by gonosomal algebras. In Section 5.3, we describe two distinct constructions of gonosomal algebras and illustrate them through the gonosomal algebra associated with haemophilia, whose construction involves both procedures. Finally, Section 5.4 introduces (normalised) gonosomal operators, discrete-time dynamical systems associated with gonosomal algebras, whose trajectories yield the corresponding biological interpretations.

5.1 Baric and dibaric algebras

In 1940, in connection with the formalism of genetics, Etherington introduced the notion of a *baric algebra* in [43]. Given a field $\mathbb{K}$, a $\mathbb{K}$-algebra $\mathcal{A}$ is said to be *baric* (or a *weighted algebra*) if there exists a nontrivial homomorphism $\omega\colon \mathcal{A} \longrightarrow \mathbb{K}$, called the *weight function* of $\mathcal{A}$. Moreover, by [108, Lemma 1.10], a finite-dimensional $\mathbb{K}$-algebra $\mathcal{A}$ is baric if and only if it admits a basis $B = \{e_1, \ldots, e_n\}$ such that $e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k$ with $\sum_{k=1}^{n} \gamma_{ijk} = 1$ for all $1 \leq i, j \leq n$. Consequently, as detailed in [72], evolution algebras associated to free populations (populations with unrestricted random mating) are baric structures.

However, algebraic structures modelling bisexual populations are not baric in general. To overcome this limitation, Etherington also suggested in [43] that the male and female components of a population could be treated separately. This idea was later formalised by Holgate in [58] through the introduction of the so-called *sex differentiation algebra* and the concept of *dibaric algebras*. Let $\mathcal{U} = \langle f, m \rangle$ denote the two-dimensional commutative algebra over $\mathbb{R}$ with multiplication $f^2 = m^2 = 0$ and $fm = mf = \frac{1}{2}(f + m)$. This algebra $\mathcal{U}$ is called the *sex differentiation algebra*. Now, an algebra $\mathcal{A}$ is called *dibaric* if it admits a homomorphism onto the sex differentiation algebra.

Note that every baric algebra is trivially dibaric, since the ground field $\mathbb{K}$ can be identified with the subalgebra of $\mathcal{U}$ generated by $f + m$. Indeed, if and algebra $\mathcal{A}$ admits a weight function $\omega\colon \mathcal{A} \to \mathbb{K}$, then the map $\phi\colon \mathcal{A} \to \mathcal{U}$ defined by $\phi(x) = \omega(x)(f + m)$ is clearly an homomorphism since

$$\phi(x)\phi(y) = \omega(x)\omega(y)(f + m)^2 = \omega(xy)(f + m) = \phi(xy).$$

Hence, dibaric algebras naturally generalise the notion of baric ones. However, as we will see in the following section, the converse does not hold in general: not every dibaric algebra is baric.

5.2 From EABPs to gonosomal algebras

The algebraic modelling of populations with sexual differentiation was first developed by Ladra and Rozikov in [67], who introduced an algebraic structure to describe the inheritance dynamics in populations where reproduction occurs between male and female individuals. The set of females is assumed to consist of finitely many types indexed by $\{1, \ldots, n\}$, while the set of males is partitioned into finitely many types indexed by $\{1, \ldots, \nu\}$. The total number $n + \nu$ is referred to as the dimension of the population. Consider the canonical basis $\{e_1, \ldots, e_{n+\nu}\}$ of the vector space $\mathbb{R}^{n+\nu}$, and split it into female and male basis elements, respectively:

$$e_i^{(f)} = e_i, \quad i = 1, \ldots, n; \qquad e_p^{(m)} = e_{n+p}, \quad p = 1, \ldots, \nu;$$

and introduce the following product rules:

$$\begin{gathered} e_i^{(f)} e_p^{(m)} = e_p^{(m)} e_i^{(f)} = \frac{1}{2} \left(\sum_{k=1}^{n} P_{ip,k}^{(f)} e_k^{(f)} + \sum_{l=1}^{\nu} P_{ip,l}^{(m)} e_l^{(m)} \right), \\ e_i^{(f)} e_j^{(f)} = 0, \quad i, j = 1, \ldots, n; \qquad e_p^{(m)} e_q^{(m)} = 0, \quad p, q = 1, \ldots, \nu; \end{gathered} \tag{5.2.1}$$

where the scalars $P_{ip,k}^{(f)}$ and $P_{ip,l}^{(m)}$ are the corresponding inheritance coefficients of the population, that is, the probability that a female offspring is type k and a male offspring is type l, respectively, when the parental pair is (i, p). Consequently, for all $i, k = 1 \ldots, n$ and $p, l = 1, \ldots, \nu$, we have that

$$P_{ip,k}^{(f)} \geq 0, \quad \sum_{k=1}^{n} P_{ip,k}^{(f)} = 1; \quad \text{and} \quad P_{ip,l}^{(m)} \geq 0, \quad \sum_{l=1}^{\nu} P_{ip,l}^{(m)} = 1. \tag{5.2.2}$$

The algebra defined on the vector space $\mathbb{R}^{n+\nu}$ as the bilinear extension of the multiplication (5.2.1), with the inheritance coefficients satisfying (5.2.2), is referred to by the authors as the *evolution algebra of a bisexual population* (EABP).

By definition, every EABP is commutative and thus flexible. However, in general, they are neither associative nor power-associative. It is also worth noting that EABPs are the paradigmatic example of dibaric algebras that fail to be baric (see [67, Theorems 5.1 and 6.3]). Given an EABP $\mathcal{B}$, its dibaricity follows by considering the map $\phi\colon \mathcal{B} \to \mathcal{U}$ defined by

$$\phi(e_i^{(f)}) = f, \quad i = 1, \ldots, n; \qquad \phi(e_p^{(m)}) = m, \quad p = 1, \ldots, \nu.$$

Although EABPs provide a rigorous framework to model and study the inheritance dynamics of several bisexual populations, the symmetry imposed on the inheritance coefficients in (5.2.2) implies that certain sex-determination systems, such as that associated with haemophilia (see [105, Example 1]), cannot be accurately modelled using an EABP. Haemophilia is a lethal recessive genetic disorder linked to the X chromosome. If $\mathrm{X^h}$ denotes the X chromosome carrying haemophilia, then genotype $\mathrm{X^hX^h}$ is lethal. Consequently, there exist only two female genotypes, XX and $\mathrm{XX^h}$, and two male genotypes, XY and $\mathrm{X^hY}$. Specifically, the results of the four possible crosses are:

$$\begin{aligned} \mathrm{XX} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX}, \tfrac{1}{2}\mathrm{XY}; \\ \mathrm{XX^h} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{4}\mathrm{XX}, \tfrac{1}{4}\mathrm{XX^h}, \tfrac{1}{4}\mathrm{XY}, \tfrac{1}{4}\mathrm{X^hY}; \\ \mathrm{XX} \times \mathrm{X^hY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX^h}, \tfrac{1}{2}\mathrm{XY}; \\ \mathrm{XX^h} \times \mathrm{X^hY} &\rightarrowtail \tfrac{1}{3}\mathrm{XX^h}, \tfrac{1}{3}\mathrm{XY}, \tfrac{1}{3}\mathrm{X^hY}. \end{aligned} \tag{5.2.3}$$

Algebraically, by identifying $f_1 \leftrightarrow$ XX, $f_2 \leftrightarrow$ XX$^{\mathrm{h}}$, $h_1 \leftrightarrow$ XY, and $h_2 \leftrightarrow$ X$^{\mathrm{h}}$Y, these crosses can be encoded by a four-dimensional commutative algebra with basis $\{f_1, f_2, h_1, h_2\}$ and multiplication given by

$$\begin{aligned} f_1h_1 &= \tfrac{1}{2}f_1 + \tfrac{1}{2}h_1, & f_1h_2 &= \tfrac{1}{2}f_2 + \tfrac{1}{2}h_1, \\ f_2h_1 &= \tfrac{1}{4}f_1 + \tfrac{1}{4}f_2 + \tfrac{1}{4}h_1 + \tfrac{1}{4}h_2, & f_2h_2 &= \tfrac{1}{3}f_2 + \tfrac{1}{3}h_1 + \tfrac{1}{3}h_2, \end{aligned} \tag{5.2.4}$$

and $f_if_j = h_ih_j = 0$ for all $i, j = 1, 2$. At first glance, note that the inheritance coefficients do not verify (5.2.2). In fact, in f_2h_2 we have

$$P^{(f)}_{22,1} + P^{(f)}_{22,2} = \frac{2}{3} \neq 1 \quad \text{and} \quad P^{(m)}_{22,1} + P^{(m)}_{22,2} = \frac{2}{3} + \frac{2}{3} \neq 1.$$

More rigorously, one can verify that this algebra is not dibaric [105, Remark 37]. Since EABPs are dibaric and dibaricity is preserved under algebra isomorphisms, it follows that this algebra cannot be isomorphic to an EABP.

To address this limitation, in [105], Varro extends the definition of EABP by introducing gonosomal algebras. Given a field $\mathbb{K}$ of characteristic not two, a $\mathbb{K}$-algebra $\mathcal{A}$ is *gonosomal* of type (n, m) if it admits a basis $(f_i)_{1\leq i\leq n} \cup (h_p)_{1\leq p\leq m}$ such that for all $1 \leq i, j \leq n$ and $1 \leq p, q \leq m$ we have that

$$f_if_j = 0, \quad h_ph_q = 0 \quad \text{and} \quad f_ih_p = h_pf_i = \sum_{k=1}^{n} \gamma_{ipk} f_k + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} h_r,$$

where the structure constants satisfy

$$\sum_{k=1}^{n} \gamma_{ipk} + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} = 1.$$

The basis $(f_i)_{1\leq i\leq n} \cup (h_p)_{1\leq p\leq n}$ is called a *gonosomal basis* of $\mathcal{A}$. Although gonosomal algebras were originally defined over arbitrary fields, all gonosomal algebras considered throughout this thesis will be assumed to be *stochastic*, that is, defined over $\mathbb{R}$ with all inheritance coefficients being nonnegative.

In this definition, the vectors $(f_i)_{1\leq i\leq n}$ (resp. $(h_i)_{1\leq i\leq m}$) are interpreted as genetic types observed in females (resp. in males), and the structure constant γ_{ipk} (resp. $\widetilde{\gamma}_{ipr}$) represents the female (resp. male) proportion of type f_k (resp. h_r) in the progeny of a female of type f_i with a male of type h_p.

Remark 5.2.1. Note that if

$$\sum_{k=1}^{n} \gamma_{ipk} = \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} \quad \text{for all } 1 \leq i \leq n,\ 1 \leq p \leq m,$$

then, setting $P^{(f)}_{ip,k} = 2\gamma_{ipk}$ and $P^{(m)}_{ip,r} = 2\widetilde{\gamma}_{ipr}$, we obtain

$$e_i\widetilde{e}_p = \frac{1}{2}\sum_{k=1}^{n} P^{(f)}_{ip,k}\, e_k + \frac{1}{2}\sum_{r=1}^{m} P^{(m)}_{ip,r}\, \widetilde{e}_r, \quad \text{where } \sum_{k=1}^{n} P^{(f)}_{ip,k} = \sum_{r=1}^{m} P^{(m)}_{ip,r} = 1,$$

which coincides with the product of an EABP, given by (5.2.1) and (5.2.2).

Gonosomal algebras are commutative by definition and thus flexible. However, in general, they are neither associative nor power-associative. Moreover, since gonosomal algebras are a generalisation of EABPs, they are not baric algebras in general. Finally, they are not dibaric either. In fact, dibaricity in gonosomal algebras has been characterised in [105, Proposition 39], and it is equivalent to the existence of a nonzero solution of a system of quadratic equations. Moreover, note that, although the algebra structure (5.2.4) does not lie in the framework of EABPs, it lies in the framework of gonosomal algebras.

5.3 Some Varro's constructions of gonosomal algebras

As a generalisation of EABPs, gonosomal algebras provide an algebraic representation of a wide variety of sex-related genetic phenomena. In particular, in [105], Varro illustrates this versatility by describing several constructions of gonosomal algebras and presenting nearly twenty genetic examples of sex-linked inheritance mechanisms observed in bisexual populations, most of which cannot be modelled within the framework of EABPs. Although each construction has its own intrinsic interest, we now focus on two of them, those that will play a central role in the investigation developed in the next chapter.

The first method we describe is how to construct gonosomal algebras *from the commutative duplicate of a baric algebra*. Although inspired by [105, Proposition 12], the presentation below has been slightly adapted to make the construction more transparent and better suited to our framework. For that, we first recall that given a commutative $\mathbb{K}$-algebra $\mathcal{A}$, the quotient space $D(\mathcal{A}) = (\mathcal{A}\otimes\mathcal{A})/I$, where $I = \mathsf{span}\{x\otimes y - y\otimes x\colon x, y\in\mathcal{A}\}$ endowed with the component-wise multiplication is called the *commutative duplicate* of $\mathcal{A}$. We will denote the elements of $D(\mathcal{A})$ by $x\otimes y$. Moreover, the surjective morphism $\mu\colon D(\mathcal{A}) \longrightarrow \mathcal{A}^2$, $x\otimes y\mapsto xy$ is called the *Etherington morphism*. Finally, we also denote $\Lambda := \{1,\dots,n\}$ and $\Gamma := \Lambda\times\Lambda$, for convenience in what follows.

Construction 5.3.1. *Let $\mathcal{A}$ be a baric algebra with basis $\{e_i\}_{i\in\Lambda}$ and multiplication given by $e_je_i = e_ie_j = \sum_{k\in\Lambda}\gamma_{ijk}e_k$ such that $\sum_{k\in\Lambda}\gamma_{ijk} = 1$ for any $i, j\in\Lambda$, and $D(\mathcal{A})$ the commutative duplicate of $\mathcal{A}$. Consider two subspaces of $D(\mathcal{A})$, say*

$$F = \mathsf{span}\{f_{rs} = e_r\otimes e_s\colon (r,s)\in\Omega\}$$

$$M = \mathsf{span}\{h_{kl} = e_k \otimes e_l \colon (k,l) \in \Psi\},$$

where $\Omega, \Psi \subsetneq \Gamma$ *with* $\Omega \cap \Psi = \emptyset$*, such that*

$$\mu(F) \otimes \mu(M) = \mathsf{span}\{e_r e_s \otimes e_k e_l \colon (r,s) \in \Omega, (k,l) \in \Psi\} \subset F \oplus M.$$

Then, the subspace $F \oplus M \subset D(\mathcal{A})$ *with the product given by*

$$\begin{aligned} f_{rs}h_{kl} = (e_r e_s) \otimes (e_k e_l) &= \left(\sum_{p\in\Lambda} \gamma_{rsp} e_p\right) \otimes \left(\sum_{q\in\Lambda} \gamma_{klq} e_q\right) \\ &= \sum_{(p,q)\in\Omega} \gamma_{rsp}\gamma_{klq} f_{pq} + \sum_{(p,q)\in\Psi} \gamma_{rsp}\gamma_{klq} h_{pq}, \end{aligned}$$

for all $(r,s) \in \Omega$ *and* $(k,l) \in \Psi$*, and zero in another case, is a gonosomal algebra with basis* $(f_{rs})_{(r,s)\in\Lambda} \cup (h_{kl})_{(k,l)\in\Psi}$.

Proof. It suffices to show that $\sum_{(p,q)\in\Omega} \gamma_{rsp}\gamma_{klq} + \sum_{(p,q)\in\Psi} \gamma_{rsp}\gamma_{klq} = 1$. First, note that, since $\mathcal{A}$ is a baric algebra, then $\sum_{p\in\Lambda} \gamma_{rsp} = \sum_{q\in\Lambda} \gamma_{klq} = 1$ for all $(r,s) \in \Omega$ and $(k,l) \in \Psi$. Consequently,

$$\left(\sum_{p\in\Lambda} \gamma_{rsp}\right)\left(\sum_{q\in\Lambda} \gamma_{klq}\right) = \sum_{(p,q)\in\Gamma} \gamma_{rsp}\gamma_{klq} = 1.$$

Moreover, since $\mu(F) \otimes \mu(M) \subset F \oplus M$ by hypothesis, we can write

$$1 = \sum_{(p,q)\in\Gamma} \gamma_{rsp}\gamma_{klq} = \sum_{(p,q)\in\Omega} \gamma_{rsp}\gamma_{klq} + \sum_{(p,q)\in\Psi} \gamma_{rsp}\gamma_{klq},$$

what yields the claim. □

As shown in [105, Proposition 10], given a gonosomal algebra, we can construct others by reducing the elements of its gonosomal basis. In the result below, we denote $[\![1,k]\!] = \{1,\ldots,k\}$ for any integer $k \geq 1$.

Construction 5.3.2. *Let* $\mathcal{A}$ *be a gonosomal algebra,* $(f_i)_{i\in[\![1,n]\!]} \cup (h_p)_{p\in[\![1,m]\!]}$ *its gonosomal basis and product given by* $f_i h_p = \sum_{k=1}^n \gamma_{ipk} f_k + \sum_{r=1}^m \widetilde{\gamma}_{ipr} h_r$*. If there exist two subsets* $I \subsetneq [\![1,n]\!]$ *and* $J \subsetneq [\![1,m]\!]$ *such that for all* $i \in [\![1,n]\!]\backslash I$ *and* $p \in [\![1,m]\!]\backslash J$ *we have* $\sigma_{ip} = 1 - (\sum_{k\in I} \gamma_{ipk} + \sum_{r\in J} \widetilde{\gamma}_{ipr}) \neq 0$*, then the subspace spanned by* $(f_i)_{i\in[\![1,n]\!]\backslash I} \cup (h_p)_{p\in[\![1,m]\!]\backslash J}$ *with multiplication*

$$f_i * h_p = \sigma_{ip}^{-1}\left(\sum_{k\in[\![1,n]\!]\backslash I} \gamma_{ipk} f_k + \sum_{r\in[\![1,m]\!]\backslash J} \widetilde{\gamma}_{ipr} h_r\right)$$

and $f_i * f_j = h_p * h_q = 0$ *for all* $i,j \in [\![1,n]\!]\backslash I$ *and* $p,q \in [\![1,m]\!]\backslash J$*, is a gonosomal algebra.*

Proof. The result follows immediately from the definition of σ_{ip}. Indeed,

$$\sigma_{ip} = 1 - \left(\sum_{k \in I} \gamma_{ipk} + \sum_{r \in J} \widetilde{\gamma}_{ipr} \right) = \sum_{k \in [\![1,n]\!] \setminus I} \gamma_{ipk} + \sum_{r \in [\![1,m]\!] \setminus J} \widetilde{\gamma}_{ipr}.$$

Consequently, $\sigma_{ip}^{-1}(\sum_{k \in [\![1,n]\!] \setminus I} \gamma_{ipk} + \sum_{r \in [\![1,m]\!] \setminus J} \widetilde{\gamma}_{ipr}) = 1$, what yields a gonosomal algebra. $\square$

Although both constructions may appear rather abstract at first, we now illustrate their applicability by showing how the gonosomal algebra modelling haemophilia can be obtained through a combination of the two procedures. Consider the baric algebra $\mathcal{A}$ with basis $\{e_1, e_2, e_3\}$ and multiplication given by $e_i e_j = \frac{1}{2}(e_i + e_j)$ for all $i, j = 1, 2, 3$. Applying Construction 5.3.1, define the following subspaces of $D(\mathcal{A})$:

$$F = \mathsf{span}\{f_1 = e_1 \otimes e_1, f_2 = e_1 \otimes e_2, f_3 = e_2 \otimes e_2\},$$
$$M = \mathsf{span}\{h_1 = e_1 \otimes e_3, h_2 = e_2 \otimes e_3\}.$$

As required, it is straightforward to verify that $\mu(F) \oplus \mu(M) \subset F \oplus M$. Indeed,

$$f_1 h_1 = e_1^2 \otimes e_1 e_3 = e_1 \otimes \frac{1}{2}(e_1 + e_3) = \frac{1}{2} f_1 + \frac{1}{2} h_1,$$
$$f_1 h_2 = e_1^2 \otimes e_2 e_3 = e_1 \otimes \frac{1}{2}(e_2 + e_3) = \frac{1}{2} f_2 + \frac{1}{2} h_1,$$
$$f_2 h_1 = e_1 e_2 \otimes e_1 e_3 = \frac{1}{2}(e_1 + e_2) \otimes \frac{1}{2}(e_1 + e_3) = \frac{1}{4} f_1 + \frac{1}{4} h_1 + \frac{1}{4} f_2 + \frac{1}{4} h_2,$$
$$f_2 h_2 = e_1 e_2 \otimes e_2 e_3 = \frac{1}{2}(e_1 + e_2) \otimes \frac{1}{2}(e_2 + e_3) = \frac{1}{4} f_2 + \frac{1}{4} h_1 + \frac{1}{4} f_3 + \frac{1}{4} h_2,$$
$$f_3 h_1 = e_2^2 \otimes e_1 e_3 = e_2 \otimes \frac{1}{2}(e_1 + e_3) = \frac{1}{2} f_2 + \frac{1}{2} h_2,$$
$$f_1 h_2 = e_2^2 \otimes e_2 e_3 = e_1 \otimes \frac{1}{2}(e_2 + e_3) = \frac{1}{2} f_3 + \frac{1}{2} h_2.$$

Finally, we just need to reduce the gonosomal basis $\{f_1, f_2, f_3, h_1, h_2\}$ by taking $I = \{3\}$ in Construction 5.3.2. In this case, it is easy to check that $\sigma_{11} = \sigma_{12} = \sigma_{21} = 1$ and $\sigma_{22} = \frac{3}{4}$. Hence, we obtain the gonosomal algebra with basis $\{f_1, f_2, h_1, h_2\}$ and multiplication

$$f_1 h_1 = \frac{1}{2} f_1 + \frac{1}{2} h_1, \qquad f_1 h_2 = \frac{1}{2} f_2 + \frac{1}{2} h_1,$$
$$f_2 h_1 = \frac{1}{4} f_1 + \frac{1}{4} h_1 + \frac{1}{4} f_2 + \frac{1}{4} h_2, \qquad f_2 h_2 = \frac{1}{3} f_2 + \frac{1}{3} h_1 + \frac{1}{3} h_2.$$

Therefore, identifying $e_1 \leftrightarrow$ X, $e_2 \leftrightarrow$ $\mathrm{X^h}$ and $e_3 \leftrightarrow$ Y, the resulting gonosomal algebra precisely realises the crosses in (5.2.3).

5.4 Gonosomal operators

This section is devoted to the construction of discrete-time dynamical systems that model the sex-linked inheritance mechanisms already represented by gonosomal algebras, the so-called *gonosomal evolution operators*. As a preliminary example, we start by considering the case of haemophilia. Let $s = (x, y, u, v) \in \mathbb{R}^4$ denote an initial state of the population, where the components correspond to the proportions of the genotypes $\{\mathrm{XX}, \mathrm{XX^h}, \mathrm{XY}, \mathrm{X^hY}\}$, respectively. The state in the next generation $s' = (x', y', u', v')$ is determined by the evolution operator $V\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ constructed according to the crossing rules in (5.2.3):

$$W : \begin{cases} x' &= \frac{1}{2}xu + \frac{1}{4}yu, \\ y' &= \frac{1}{2}xv + \frac{1}{4}yu + \frac{1}{3}yv, \\ u' &= \frac{1}{2}xu + \frac{1}{2}xv + \frac{1}{4}yu + \frac{1}{3}yv, \\ v' &= \frac{1}{4}yu + \frac{1}{3}yv. \end{cases} \tag{5.4.1}$$

This example can be generalised. In fact, every gonosomal algebra gives rise to a quadratic operator, called a gonosomal operator, which connects the genetic states of two successive generations. As described in [91, Section 10.1], given a gonosomal algebra with inheritance real coefficients γ_{ipk} and $\widetilde{\gamma}_{ipr}$, the corresponding gonosomal evolution operator is defined in coordinate form by $V\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, $(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x'_1, \ldots, x'_n, y'_1, \ldots, y'_m)$, where

$$V : \begin{cases} x'_k &= \sum_{i,p=1}^{n,m} \gamma_{ipk} x_i y_p, \quad k = 1, \ldots, n; \\ y'_r &= \sum_{i,p=1}^{n,m} \widetilde{\gamma}_{ipr} x_i y_p, \quad r = 1, \ldots, m. \end{cases} \tag{5.4.2}$$

The population evolves by starting from an arbitrary state s, then passing to the state $s' = V(s)$ in the next generation, then to the state $s'' = V(s') = V(V(s))$, and so on. Thus, starting from an initial point $s^{(0)} \in \mathbb{R}^{n+m}$, the evolution of the population is described by the following discrete-time dynamical system:

$$s^{(0)}, \quad s^{(1)} = V\big(s^{(0)}\big), \quad s^{(2)} = V^2\big(s^{(0)}\big), \quad s^{(3)} = V^3\big(s^{(0)}\big), \quad \ldots$$

The main problem for a given dynamical system is to describe the limit points of the trajectory $\{s^{(n)}\}_{n=0}^{\infty}$ for an arbitrary initial state $s^{(0)}$. However, this study in the case of dynamical systems generated by gonosomal algebras, is generally, complicated. Numerous works have focused on analysing the dynamical behaviour of specific operators (see [2, 4, 92, 94], among others), typically by determining their fixed points (since the possible limits of trajectories must be fixed points) and studying the eigenvalues of the associated Jacobian matrix to characterise their stability (attractive, repelling, or saddle points). In addition, some of these studies investigate the invariance of certain subsets of $\mathbb{R}^4$, which, in particular cases, allows for a complete description of the asymptotic behaviour for specific initial conditions.

5.4.1 Normalised gonosomal operators

Although gonosomal operators (5.4.2) connect, in some sense, the genetic states of two successive generations, these states do not, in general, correspond to probability distributions of genotypes within the population. In other words, the main limitation of such operators is that they do not necessarily map the simplex

$$S^{n+m-1} = \left\{ s = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}_{\geq 0} \colon \sum_{k=1}^{n} x_k + \sum_{r=1}^{m} y_r = 1 \right\}$$

to itself, since

$$\begin{aligned} \sum_{k=1}^{n} x'_k + \sum_{r=1}^{m} y'_r &= \sum_{k=1}^{n} \sum_{i,p=1}^{n,m} \gamma_{ipk} x_i y_p + \sum_{r=1}^{m} \sum_{i,p=1}^{n,m} \widetilde{\gamma}_{ipr} x_i y_p \\ &= \sum_{i,p=1}^{n,m} x_i y_p \left(\sum_{k=1}^{n} \gamma_{ipk} + \sum_{r=1}^{m} \widetilde{\gamma}_{ipr} \right) = \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{p=1}^{m} y_p \right), \end{aligned}$$

which is generally different from one. This significantly complicates the biological interpretation of their dynamics.

To construct an operator that maps the simplex to itself, it is important to ensure that the resulting values remain within the simplex. A natural approach is to normalise each component by $(\sum_{i=1}^{n} x_i)(\sum_{p=1}^{m} y_p)$. This normalisation preserves the probabilistic structure of the simplex by adjusting proportions consistently. To properly define this normalisation, we restrict our attention to the subset

$$S^{n,m} = \left\{ s \in S^{n+m-1} \colon (x_1, \ldots, x_n) \neq (0, \ldots, 0) \text{ and } (y_1, \ldots, y_m) \neq (0, \ldots, 0) \right\}.$$

Indeed, the excluded points, those for which $(x_1, \ldots, x_n)$ or $(y_1, \ldots, y_m)$ is 0, verify $V(s) = (0, \ldots, 0)$ under (5.4.2), and therefore do not contribute to their dynamics. Hence, we now introduce the *normalised gonosomal operator* $\widetilde{V}$ with coefficients

γ_{ipk} and $\widetilde{\gamma}_{ipr}$ on the set $S^{n,m}$ in coordinate form as

$$\widetilde{V}:\begin{cases} x'_k &= \dfrac{\sum_{i,p=1}^{n,m}\gamma_{ipk}x_iy_p}{\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{p=1}^{m}y_p\right)}, \quad k=1,\dots,n; \\ y'_r &= \dfrac{\sum_{i,p=1}^{n,m}\widetilde{\gamma}_{ipr}x_iy_p}{\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{p=1}^{m}y_p\right)}, \quad r=1,\dots,m. \end{cases} \tag{5.4.3}$$

As stated in [91, Proposition 10.3], the operator $\widetilde{V}$ maps $S^{n,m}$ to itself if and only if the condition

$$(\gamma_{ip1},\dots,\gamma_{ipn},\widetilde{\gamma}_{ip1},\dots,\widetilde{\gamma}_{ipm})\in S^{n,m}$$

holds for all $i=1,\dots,n$ and $p=1,\dots,m$. Furthermore, the fixed points of a given gonosomal operator V and the fixed points of its normalised version $\widetilde{V}$ are closely related. Recall that a fixed point $s=(x_1,\dots,x_n,y_1,\dots,y_m)$ of (5.4.2) is called *nonnegative* and *normalisable* if all coordinates are nonnegative $Z=\sum_{i=1}^{n}x_i+\sum_{p=1}^{m}y_p>0$. As shown in [91, Proposition 10.4], there exists a one-to-one correspondence between nonnegative and normalisable fixed points of (5.4.2) and all the fixed points of (5.4.3). Concretely, $s=(x_1,\dots,x_n,y_1,\dots,y_m)$ is a nonnegative and normalisable fixed point of (5.4.2) if and only if

$$\widetilde{s}=(x_1/Z,\dots,x_n/Z,y_1/Z,\dots,y_m/Z)$$

is a fixed point of (5.4.3).

Remark 5.4.1. In view of all these properties and their biological consistency, in the next chapter, we will be mainly interested in determining the limit points of trajectories of normalised gonosomal operators, while using the (non-normalised) gonosomal operators only for the computation of the fixed points.

We conclude this section by presenting the normalised version of the gonosomal operator modelling haemophilia (5.4.1) and deriving the corresponding biological interpretations as an illustrative example:

$$\widetilde{W}:\begin{cases} x' &= \dfrac{\frac{1}{2}xu+\frac{1}{4}yu}{(x+y)(u+v)}=\dfrac{2xu+yu}{4(x+y)(u+v)}, \\ y' &= \dfrac{\frac{1}{2}xv+\frac{1}{4}yu+\frac{1}{3}yv}{(x+y)(u+v)}=\dfrac{6xv+3yu+4yv}{12(x+y)(u+v)}, \\ u' &= \dfrac{\frac{1}{2}xu+\frac{1}{2}xv+\frac{1}{4}yu+\frac{1}{3}yv}{(x+y)(u+v)}=\dfrac{6xu+6xv+3yu+4yv}{12(x+y)(u+v)}, \\ v' &= \dfrac{\frac{1}{4}yu+\frac{1}{3}yv}{(x+y)(u+v)}=\dfrac{3yu+4yv}{12(x+y)(u+v)}. \end{cases} \tag{5.4.4}$$

In this particular case, $\widetilde{W}$ maps the set

$$S^{2,2} = \Big\{(x,y,u,v) \in \mathbb{R}^4_{\geq 0} \colon x+y>0, u+v>0, x+y+u+v=1\Big\} \subset S^3$$

to itself, since

$$\begin{aligned}
(\gamma_{111}, \gamma_{112}, \widetilde{\gamma}_{111}, \widetilde{\gamma}_{112}) &= (\tfrac{1}{2}, 0, \tfrac{1}{2}, 0) \in S^{2,2},\\
(\gamma_{121}, \gamma_{122}, \widetilde{\gamma}_{121}, \widetilde{\gamma}_{122}) &= (0, \tfrac{1}{2}, \tfrac{1}{2}, 0) \in S^{2,2},\\
(\gamma_{211}, \gamma_{212}, \widetilde{\gamma}_{211}, \widetilde{\gamma}_{212}) &= (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \in S^{2,2},\\
(\gamma_{221}, \gamma_{222}, \widetilde{\gamma}_{221}, \widetilde{\gamma}_{222}) &= (0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \in S^{2,2}.
\end{aligned}$$

Moreover, since $(2,0,2,0)$ is the only nonnegative and normalisable fixed point of (5.4.1), the normalised operator (5.4.4) admits a unique fixed point $(\frac{1}{2}, 0, \frac{1}{2}, 0)$. In fact, as proved in [3], for any initial point $s \in S^{2,2}$,

$$\lim_{n\to\infty} \widetilde{W}^n(s) = (\tfrac{1}{2}, 0, \tfrac{1}{2}, 0),$$

showing that this point is globally attractive within $S^{2,2}$. This means that, as time goes to infinity, the genotypes XX and XY always persist in the population, whereas XX^h and X^hY tend to disappear. Consequently, haemophilia can be maintained in the population only through the occurrence of new mutations in the genes encoding coagulation factors.

Chapter 6

Gonosomal algebras and operators associated to genetic systems with a single male genotype

As already stated in the previous chapter, the study of gonosomal algebras arises from the algebraic modelling of genetic systems in which sex plays a determining role. Typically, sex is controlled by a pair of chromosomes known as *gonosomes*. The most widely known sex-determination system is the male heterogametic XX/XY system found in most mammals, in which XY individuals develop as males and XX individuals as females. Other organisms, such as birds, have a ZZ/ZW chromosomal system in which heterogametic ZW individuals develop as females and homogametic ZZ individuals as males.

Although the previous sex-determination systems are relatively straightforward to model, in general, these mechanisms exhibit great diversity across species. For instance, sex can be determined by environmental factors (such as temperature [104]), and there even exist polygenic sex-determination systems (see [81]), in which the sex of an organism is influenced by multiple genes rather than by a single gene or chromosome.

In particular, this chapter focuses on studying gonosomal algebras and operators that model sex-determination systems with a finite number of female genotypes but only one male genotype. Although several examples of such genetic systems can be found in the literature (see [66, 105]), it is worthwhile to explain from a genetic perspective why and how these situations naturally arise. It is natural to ask: *how common are genetic systems in which only a single male genotype occurs?* The answer depends on the specific *sex-determination mechanism* and on the *position of the loci on the gonosomes*. In what follows, we outline some representative scenarios:

1. **ZW system.** Two distinct cases can occur depending on the location of the locus:

   (a) *Locus in a pseudoautosomal region.* Suppose the Z chromosome carries a single allele a, while the W chromosome carries several alleles, namely

$\{a, a_1, \ldots, a_n\}$, of the same gene. Then the possible female genotypes are $\{aa, aa_1, \ldots, aa_n\}$, whereas the male genotype is uniquely aa.

   (b) *Locus in the gonosomal (non-recombining) part of the Z chromosome.* Assume several alleles $\{a, a_1, \ldots, a_n\}$ exist, with $\{a_1, \ldots, a_n\}$ being lethal but their lethality is neutralised in females by a gene on the W chromosome. The allele a is non-lethal and recessive with respect to the others. Thus, the viable female genotypes are $\{aW, a_1W, \ldots, a_nW\}$, and the only viable male genotype is aa, since all other combinations are lethal.

2. **XY system.** Again, two main cases can be distinguished according to the locus position:

   (a) *Locus on the gonosomal part of the X chromosome.* Let the gene be multiallelic, with alleles $\{a_1, \ldots, a_n\}$, and assume the non-lethal allele a_1 is dominant over the lethal alleles $\{a_2, \ldots, a_n\}$. Then, the female genotypes are a_1a_k for all $1 \le k \le n$, and the male genotype is uniquely a_1Y.

   (b) *Locus in a pseudoautosomal region of the X and Y chromosomes.* Consider again several alleles $\{a_1, \ldots, a_n\}$, where a_1 is non-lethal, and the remaining alleles are lethal in males but non-lethal in females. In this case, females exhibit the genotypes a_ia_j, while males have the genotype a_1a_1.

These biological scenarios highlight the relevance of modelling genetic systems with a single male genotype, as they capture situations that actually occur in nature. Motivated by this, in the present work, we study the corresponding gonosomal algebras and their normalised gonosomal operators in several particular cases, and we determine the limit points of their trajectories, illustrating the development of specific populations and deriving the associated biological interpretations.

This chapter is structured into five sections. Following this introduction, Section 6.1 covers the preliminaries, reviewing the basic language and the fundamental concepts of gonosomal algebras and (normalised) gonosomal operators within the context of genetic systems with a single male genotype. Subsequently, Sections 6.2 and 6.3 build upon such genetic examples given in [105]. In particular, in Section 6.2 we explore a ZW sex-determination system in which infection by the bacterium *Wolbachia* induces ZZ individuals to develop as females, and we analyse its development in terms of Wolbachia's transmission rate to the offspring. Section 6.3 is devoted to the study of XY sex-determination systems that model the population of some rodents, such as the African pygmy mouse or the Arctic lemming. Notably, in both sections, we pioneer the restriction of the domain of certain normalised gonosomal operators to obtain meaningful biological interpretations. Finally, in Section 6.4, we provide the first complete mathematical model of African cichlid fish populations, addressing a case that had not been previously explored in the literature. All these results are presented in detail in the preprint [22].

6.1 Preliminaries and notations

Throughout this chapter, we denote $\Lambda := \{1, \dots, n\}$, and we adopt the convention that the set of natural numbers $\mathbb{N}$ contains zero, $\mathbb{N} = \{0, 1, 2, \dots\}$. Moreover, we denote by $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ the set of positive integers. Due to the existence of a single male genotype, all the considered gonosomal algebras will be assumed to be $\mathbb{R}$-algebras which admit a gonosomal basis $(f_i)_{i\in\Lambda} \cup h$ such that for all $i, j \in \Lambda$ we have that $f_i f_j = hh = 0$ and $f_i h = h f_i = \sum_{k\in\Lambda} \gamma_{ik} f_k + \widetilde{\gamma}_i h$, where $\gamma_{ij}, \widetilde{\gamma}_i \geq 0$ and $\sum_{k\in\Lambda} \gamma_{ik} + \widetilde{\gamma}_i = 1$. Note that, in this case, the structure constant γ_{ik} (resp. $\widetilde{\gamma}_i$) represents the proportion of individuals f_k (resp. of individuals h) in the offspring of a female f_i and a male h.

Next, we recall Constructions 5.3.1 and 5.3.2 in the framework of genetic systems with a single male genotype.

Construction 6.1.1 (Particular case of Construction 5.3.1)**.** *Let $\mathcal{A}$ be a baric $\mathbb{R}$-algebra with basis $\{e_i\}_{i\in\Lambda}$ and multiplication given by $e_j e_i = e_i e_j = \sum_{k\in\Lambda} \gamma_{ijk} e_k$ such that $\sum_{k\in\Lambda} \gamma_{ijk} = 1$ for any $i, j \in \Lambda$, and $D(\mathcal{A})$ the commutative duplicate of $\mathcal{A}$. Let $\Gamma = \Lambda \times \Lambda$ and consider a subset $\Omega \subsetneq \Gamma$ and a pair $(k, l) \in \Gamma\backslash\Omega$. Let $F = \mathsf{span}\{f_{rs} = e_r \otimes e_s \colon (r, s) \in \Omega\}$ and $M = \mathsf{span}\{h = e_k \otimes e_l\}$ be two subspaces of $D(\mathcal{A})$ such that*

$$\mu(F) \otimes \mu(M) = \mathsf{span}\{e_r e_s \otimes e_k e_l \colon (r, s) \in \Omega\} \subset F \oplus M.$$

Then, the subspace $F \oplus M \subset D(\mathcal{A})$ with the product given by

$$f_{rs} h := \sum_{(p,q)\in\Omega} \gamma_{rsp}\gamma_{klq} f_{pq} + \gamma_{rsk}\gamma_{kll} h,$$

for any $(r, s) \in \Omega$ and zero in another case, is a gonosomal algebra with basis $(f_{rs})_{(r,s)\in\Omega} \cup h$.

Construction 6.1.2 (Particular case of Construction 5.3.2)**.** *Let $\mathcal{A}$ be a gonosomal $\mathbb{K}$-algebra with gonosomal basis $(f_i)_{i\in\Lambda} \cup h$ and product given by $f_i h = \sum_{k\in\Lambda} \gamma_{ik} f_k + \widetilde{\gamma}_i h$. If there is a subset $I \subsetneq \Lambda$ such that for all $i \in \Lambda\backslash I$ we have $\sigma_i := 1 - \sum_{k\in I} \gamma_{ik} \neq$*

0*, then the subspace spanned by* $(f_i)_{i\in\Lambda\setminus I} \cup h$ *with the product given by*

$$f_i * h = \sigma_i^{-1}\left(\sum_{k\in\Lambda\setminus I} \gamma_{ik} f_k + \widetilde{\gamma}_i h\right)$$

and $f_i * f_j = h * h = 0$ *for all* $i, j \in \Lambda\setminus I$*, is a gonosomal algebra.*

Next, we recall the construction of the associated (normalised) gonosomal operator. Given the inheritance coefficients $\{\gamma_{ij}\}_{i,j\in\Lambda}$ and $\{\widetilde{\gamma}_i\}_{i\in\Lambda}$ of a gonosomal algebra with a single male genotype, the corresponding *gonosomal evolution operator* and its *normalised* version are defined in coordinate form by

$$V : \begin{cases} x_j' &= u\sum_{i\in\Lambda}\gamma_{ij}x_i; \\ u' &= u\sum_{i\in\Lambda}\widetilde{\gamma}_i x_i. \end{cases} \quad \text{and} \quad \widetilde{V} : \begin{cases} x_j' &= \dfrac{u\sum_{i\in\Lambda}\gamma_{ij}x_i}{u\sum_{i\in\Lambda}x_i}; \\ u' &= \dfrac{u\sum_{i\in\Lambda}\widetilde{\gamma}_i x_i}{u\sum_{i\in\Lambda}x_i}. \end{cases} \tag{6.1.1}$$

for all $j \in \Lambda$. Notice that the operator $\widetilde{V}$ maps the simplex S^n to itself. Furthermore, for applications in genetics, we work with the subset

$$S^{n,1} = \left\{(x_1, \ldots, x_n, u) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i\in\Lambda} x_i > 0, u > 0, \sum_{i\in\Lambda} x_i + u = 1\right\} \subset S^n$$

whose elements can be interpreted as the frequency distributions of the genotypes f_i and h. We know that $\widetilde{V}$ also maps $S^{n,1}$ to itself if and only if $(\gamma_{i1}, \ldots, \gamma_{in}, \widetilde{\gamma}_i) \in S^{n,1}$ for any $i \in \Lambda$ (see [91, Proposition 10.3]). Moreover, we have that if $\widetilde{V}$ maps $S^{n,1}$ to itself then, given an arbitrary initial point $s = (x_1, \ldots, x_n, u) \in S^{n,1}$, it holds that $u^{(k)} > 0$ for any $k \in \mathbb{N}$. Consequently, the study of the trajectories of the normalised gonosomal evolution operator $\widetilde{V}$ is equivalent to the study of those of the following operator:

$$\begin{cases} x_j' = \dfrac{\sum_{i\in\Lambda}\gamma_{ij}x_i}{\sum_{i\in\Lambda}x_i}, \quad j \in \Lambda \\ u' = \dfrac{\sum_{i\in\Lambda}\widetilde{\gamma}_i x_i}{\sum_{i\in\Lambda}x_i}. \end{cases} \tag{6.1.2}$$

From now on, if $\widetilde{V}$ maps $S^{n,1}$ to itself, we will use both operators $\widetilde{V}$ and (6.1.2) interchangeably. Moreover, recall that there exists a one-to-one correspondence between the nonnegative and normalisable fixed points of V and the fixed points of $\widetilde{V}$. Concretely, $(x_1, \ldots, x_n, u)$ is a nonnegative and normalisable fixed point of V if and only if $(\sum_{i\in\Lambda} x_i + u)^{-1}(x_1, \ldots, x_n, u)$ is a fixed point of $\widetilde{V}$.

6.2 ZW systems with male feminization

This section is dedicated to studying the ZW sex-determination system that Woodlice follows (see [36]). *Wolbachia* is an intracellular maternally inherited bacteria affecting a wide range of arthropods. In the case of woodlice of the *Armadillium vulgare* species, *Wolbachia* is responsible for the feminisation of genetic males. When Wolbachia infects a male with genotype ZZ, denoted by ZZ+w, it becomes a female, which can cross with a male ZZ. So, in this population, there are three female genotypes: ZZ+w, ZW and ZW+w; and a single male genotype ZZ. The results of the three possible crosses are:

$$\begin{aligned} (\text{ZZ+w}) \times \text{ZZ} \quad &\rightarrowtail \quad \eta(\text{ZZ+w}), (1-\eta)\text{ZZ}; \\ \text{ZW} \times \text{ZZ} \quad &\rightarrowtail \quad \tfrac{1}{2}\text{ZW}, \tfrac{1}{2}\text{ZZ}; \\ (\text{ZW+w}) \times \text{ZZ} \quad &\rightarrowtail \quad \tfrac{\eta}{2}\text{ZW+w}, \tfrac{\eta}{2}\text{ZZ+w}, \tfrac{1-\eta}{2}\text{ZW}, \tfrac{1-\eta}{2}\text{ZZ}; \end{aligned} \tag{6.2.1}$$

where η ($\frac{1}{2} < \eta < 1$) denotes the transmission rate of Wolbachia in the offspring. Although empirical evidence indicates that the transmission rate is typically very high (close to 100%), for completeness we carry out this study for all $0 < \eta \leq 1$. As explained in detail in [105, Example 15], the results of crosses can be retrieved by a gonosomal algebra obtained by Construction 6.1.1. Consider the baric algebra $\mathcal{A}$ with basis $\{e_1, e_2, e_3, e_4\}$ and product defined by

$$\begin{aligned} e_i^2 &= e_i, \\ e_1e_2 &= (1-\eta)e_1 + \eta e_2, \\ e_1e_3 &= \frac{1}{2}e_1 + \frac{1}{2}e_3, \\ e_1e_4 = e_2e_3 = e_2e_4 &= \frac{1-\eta}{2}(e_1+e_3) + \frac{\eta}{2}(e_2+e_4). \end{aligned}$$

for all $i = 1, 2, 3, 4$. Hence, considering the following subspaces of $D(\mathcal{A})$:

$$\begin{aligned} F &= \mathsf{span}\{f_1 = e_1 \otimes e_2, f_2 = e_1 \otimes e_3, f_3 = e_1 \otimes e_4\}, \\ M &= \mathsf{span}\{h = e_1 \otimes e_1\}, \end{aligned}$$

we get the gonosomal algebra with basis $\{f_1, f_2, f_3, h\}$ and multiplication given by

$$\begin{aligned} f_1h &= (e_1e_2) \otimes e_1^2 = ((1-\eta)e_1 + \eta e_2) \otimes e_1 = \eta f_1 + (1-\eta)h, \\ f_2h &= (e_1e_3) \otimes e_1^2 = \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) \otimes e_1 = \frac{1}{2}f_2 + \frac{1}{2}h, \\ f_3h &= (e_1e_4) \otimes e_1^2 \\ &= \left(\frac{1-\eta}{2}(e_1+e_3) + \frac{\eta}{2}(e_2+e_4)\right) \otimes e_1 = \frac{\eta}{2}f_1 + \frac{1-\eta}{2}f_2 + \frac{\eta}{2}f_3 + \frac{1-\eta}{2}h. \end{aligned}$$

So, making the correspondences $e_1 \leftrightarrow$ Z, $e_2 \leftrightarrow$ Z+w, $e_3 \leftrightarrow$ W and $e_4 \leftrightarrow$ W+w, we retrieve the results of crosses (6.2.1). Next, consider the associated gonosomal evolution operator V_η, with $0 < \eta \leq 1$, given by

$$V_\eta : \begin{cases} x_1' = \eta x_1 u + \frac{\eta}{2} x_3 u; & x_3' = \frac{\eta}{2} x_3 u; \\ x_2' = \frac{1}{2} x_2 u + \frac{1-\eta}{2} x_3 u; & u' = (1-\eta) x_1 u + \frac{1}{2} x_2 u + \frac{1-\eta}{2} x_3 u. \end{cases} \tag{6.2.2}$$

Proposition 6.2.1. *The operator V_η, with $0 < \eta < \frac{1}{2}$ or $\frac{1}{2} < \eta < 1$, has three nonzero fixed points: $(\frac{2}{2-\eta}, \frac{2}{2-\eta}, \frac{-2}{2-\eta}, \frac{2}{\eta})$, $(0,2,0,2)$ and $(\frac{1}{1-\eta}, 0, 0, \frac{1}{\eta})$.*

Moreover, $V_{\frac{1}{2}}$ and V_1 have infinitely many nonzero fixed points: the first one has $(\frac{4}{3}, \frac{4}{3}, -\frac{4}{3}, 4)$ and the family $(\rho, 2-\rho, 0, 2)$ with $\rho \in \mathbb{R}$ and the second one has the family $(\beta, 2, -\beta, 2)$ with $\beta \in \mathbb{R}$.

Proof. We need to solve the system of equations given by

$$\begin{cases} x_1 = \eta x_1 u + \frac{\eta}{2} x_3 u; & x_3 = \frac{\eta}{2} x_3 u; \\ x_2 = \frac{1}{2} x_2 u + \frac{1-\eta}{2} x_3 u; & u = (1-\eta) x_1 u + \frac{1}{2} x_2 u + \frac{1-\eta}{2} x_3 u. \end{cases} \tag{6.2.3}$$

First, assume that $0 < \eta < \frac{1}{2}$ or $\frac{1}{2} < \eta < 1$. If $x_3 = 0$, it is deduced that $x_1 = 0$ or $u = \frac{1}{\eta}$. On the one hand, if $x_1 = 0$, we get either $u = 0$ or $x_2 = 2$. But if $x_3 = x_1 = u = 0$, then necessarily $x_2 = 0$. However, if $x_2 = 2$, we have the nontrivial solution $(0,2,0,2)$. On the other hand, if $u = \frac{1}{\eta}$ then we obtain that $x_2 = 0$ because $\eta \neq \frac{1}{2}$. Finally, as $\eta \neq 1$, then $x_1 = \frac{1}{1-\eta}$. Therefore, $(\frac{1}{1-\eta}, 0, 0, \frac{1}{\eta})$ is another solution of (6.2.3). Next, assume that $x_3 \neq 0$. Hence, $u = \frac{2}{\eta}$. This implies that $x_1 = x_2 = -x_3$. We conclude that $x_1 = \frac{2}{2-\eta}$. Thus, we obtain the solution $(\frac{2}{2-\eta}, \frac{2}{2-\eta}, \frac{-2}{2-\eta}, \frac{2}{\eta})$.

Now, consider the cases $\eta = \frac{1}{2}$ and $\eta = 1$. It is easy to check that, for $\eta = \frac{1}{2}$, the solutions of (6.2.3) are the points $(\frac{4}{3}, \frac{4}{3}, -\frac{4}{3}, 4)$ and $(\rho, 2-\rho, 0, 2)$ with $\rho \in \mathbb{R}$ and for $\eta = 1$ the solutions are the family of points $(\beta, 2, -\beta, 2)$ with $\beta \in \mathbb{R}$. □

Once the fixed points have been obtained, we consider the normalised version of (6.2.2), that is

$$\widetilde{V}_\eta : \begin{cases} x_1' = \frac{\eta u(2x_1+x_3)}{2u(x_1+x_2+x_3)}; & x_3' = \frac{\eta u x_3}{2u(x_1+x_2+x_3)}; \\ x_2' = \frac{u(x_2+(1-\eta)x_3)}{2u(x_1+x_2+x_3)}; & u' = \frac{u((1-\eta)(2x_1+x_3)+x_2)}{2u(x_1+x_2+x_3)}. \end{cases} \tag{6.2.4}$$

Next, we study the limit points of the different trajectories for arbitrary initial points.

6.2.1 Case 1: $\frac{1}{2} < \eta < 1$

In this case, it is clear that $\widetilde{V}_\eta$ maps $S^{3,1}$ to itself and that its fixed points are $(0, \frac{1}{2}, 0, \frac{1}{2})$ and $(\eta, 0, 0, 1-\eta)$.

Proposition 6.2.2. *Consider the operator $\widetilde{V}_\eta$ defined by* (6.2.4) *with $\frac{1}{2} < \eta < 1$. Then, for any initial point $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$, the following assertions hold:*

(i) $\frac{1}{2} \leq x_1^{(k)} + x_2^{(k)} \leq \eta$ *for all $k \in \mathbb{N}^*$;*

(ii) $x_3^{(k)} \leq \eta$ *and* $x_3^{(k+1)} \leq x_3^{(k)}$ *for all $k \in \mathbb{N}^*$;*

(iii) *there exists a natural number n_0 such that $x_2^{(k)} + (1+\eta)x_3^{(k)} \leq 2\eta(x_2^{(k)} + x_3^{(k)})$ for all $k \geq n_0$; and*

(iv) *there exists a natural number m_0 such that $x_1^{(k+1)} \geq x_1^{(k)}$ for all $k \geq m_0$.*

Proof. Take $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. First, we introduce the notation $\alpha^{(k)} := x_1^{(k)} + x_2^{(k)} + x_3^{(k)}$ for any $k \in \mathbb{N}$. For items (i) and (ii), consider a number $k \in \mathbb{N}^*$. Then, we observe that

$$\frac{1}{2} \leq \frac{\alpha^{(k-1)} + (2\eta-1)x_1^{(k-1)}}{2\alpha^{(k-1)}} \leq \frac{\alpha^{(k-1)} + (2\eta-1)\alpha^{(k-1)}}{2\alpha^{(k-1)}} \leq \eta,$$

where the second term is actually $x_1^{(k)} + x_2^{(k)}$, what yields the claim.

For item (ii), observe that $2\alpha^{(k)} \geq 2(x_1^{(k)} + x_2^{(k)}) \geq 1$ using item (i). Then,

$$x_1^{(k)} + x_3^{(k)} = \frac{\eta(x_1^{(k-1)} + x_3^{(k-1)})}{\alpha^{(k-1)}} \leq \eta \quad \text{and} \quad x_3^{(k+1)} = \frac{\eta x_3^{(k)}}{2\alpha^{(k)}} \leq \eta x_3^{(k)}.$$

From the first one, we get that $x_3^{(k)} \leq x_1^{(k)} + x_3^{(k)} \leq \eta$. From the second one, as $\eta x_3^{(k)} < \eta$ and $\eta x_3^{(k)} < x_3^{(k)}$, we are done.

For item (iii), we first claim that there exists a natural number $n_0 \in \mathbb{N}$ such that $x_2^{(n_0)} + (1+\eta)x_3^{(n_0)} \leq 2\eta(x_2^{(n_0)} + x_3^{(n_0)})$. Indeed, by contrary, assume that $x_2^{(k)} + (1+\eta)x_3^{(k)} > 2\eta(x_2^{(k)} + x_3^{(k)})$ for all $k \in \mathbb{N}$. Equivalently, we have that $x_3^{(k)}(1-\eta) > x_2^{(k)}(2\eta-1)$ for all $k \in \mathbb{N}$. Since k can be chosen arbitrarily large, for all $k \geq 1$ we have

$$(1-\eta)\frac{\eta x_3^{(k-1)}}{2\alpha^{(k-1)}} > (2\eta-1)\frac{x_2^{(k-1)} + (1-\eta)x_3^{(k-1)}}{2\alpha^{(k-1)}}$$

which is equivalent to $(1-\eta)^2 x_3^{(k-1)} > (2\eta-1)x_2^{(k-1)}$. In the same way, for all $k \geq 2$ we get that $(1-\eta)(1-\eta-\eta^2)x_3^{(k-2)} > (2\eta-1)x_2^{(k-2)}$. Inductively, it is easy to check that

$$x_3^{(k-l)}(1-\eta)\left(1-\sum_{j=1}^{l}\eta^j\right) > (2\eta-1)x_2^{(k-l)} \tag{6.2.5}$$

for all $l \in \mathbb{N}$, $l \leq k$. Hence, as $\frac{1}{2} < \eta < 1$, the series $\sum_{j=1}^{\infty}\eta^j = \frac{\eta}{1-\eta} > 1$. Therefore, there exists a number $m \in \mathbb{N}$ large enough such that $\sum_{j=1}^{m}\eta^j > 1$. Then, taking a number $k \in \mathbb{N}$, $k \geq m$, we obtain that $x_3^{(k-m)}(1-\eta)(1-\sum_{j=1}^{m}\eta^j) < 0$, which is a contradiction with the fact that $(2\eta-1)x_2^{(k-m)} \geq 0$ and (6.2.5).

Moreover, looking at the previous calculations, we have just proved that if $x_3^{(k)}(1-\eta)^2 \leq x_2^{(k)}(2\eta-1)$ for certain $k \in \mathbb{N}$, then $x_3^{(k+1)}(1-\eta) \leq x_2^{(k+1)}(2\eta-1)$. But, taking into account that $\frac{1}{2} < \eta < 1$ and using that there exists a number $n_0 \in \mathbb{N}$ such that $x_3^{(n_0)}(1-\eta) \leq x_2^{(n_0)}(2\eta-1)$, we get that $x_3^{(n_0)}(1-\eta)^2 < x_3^{(n_0)}(1-\eta) \leq x_2^{(n_0)}(2\eta-1)$, which completes the proof.

Lastly, for proving item (iv), we know that there exists a $n_0 \in \mathbb{N}$ large enough such that $x_2^{(k)} + (1+\eta)x_3^{(k)} \leq 2\eta(x_2^{(k)} + x_3^{(k)})$ for all $k \geq n_0$ by item (iii). Now, we have that $\alpha^{(k)} \leq \eta$ for all $k \geq n_0 + 1$. Indeed,

$$\alpha^{(k)} = \frac{2\eta x_1^{(k-1)} + (1+\eta)x_3^{(k-1)} + x_2^{(k-1)}}{2\alpha^{(k-1)}} \leq \frac{2\eta(x_1^{(k-1)} + x_2^{(k-1)} + x_3^{(k-1)})}{2\alpha^{(k-1)}} = \eta$$

for all $k \geq n_0+1$. Then, taking $m_0 = n_0+1$ we obtain that $2x_1^{(k)}(\alpha^{(k)}-\eta)-\eta x_3^{(k)} \leq 0$ for all $k \geq m_0$, which implies that $2\alpha^{(k)}x_1^{(k)} \leq \eta(2x_1^{(k)} + x_3^{(k)})$ for all $k \geq m_0$. Therefore,

$$x_1^{(k+1)} = \frac{\eta(2x_1^{(k)} + x_3^{(k)})}{2\alpha^{(k)}} \geq x_1^{(k)}$$

for all $k \geq m_0$, and the result follows. □

Remark 6.2.3. Let be the operator $\widetilde{V}$ defined in (6.2.4) and suppose that $\lim_{k\to\infty} x_1^{(k)}$, $\lim_{k\to\infty} x_2^{(k)}$ and $\lim_{k\to\infty} x_3^{(k)}$ exist for any initial point $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. Then, observe that $\lim_{k\to\infty} u^{(k)} \neq 0$. Indeed, if suppose that $\lim_{k\to\infty} u^{(k)} = 0$ then $\lim_{k\to\infty} x_1^{(k)} + x_2^{(k)} + x_3^{(k)} = 1$ and necessarily $\lim_{k\to\infty}(1-\eta)(2x_1^{(k)} + x_3^{(k)}) + x_2^{(k)} = 0$. As $\eta \neq 1$, it holds that $\lim_{k\to\infty} x_1^{(k)} = \lim_{k\to\infty} x_2^{(k)} = \lim_{k\to\infty} x_3^{(k)} = 0$, a contradiction with the fact that $x_1^{(k)} + x_2^{(k)} + x_3^{(k)} + u^{(k)} = 1$ for any $k \geq 0$.

Theorem 6.2.4. *Consider the operator* $\widetilde{V}_\eta$ *given by* (6.2.4) *with* $\frac{1}{2} < \eta < 1$. *Then, for an initial point* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$, *it holds that*

$$\lim_{k\to\infty} \widetilde{V}_\eta^k(s) = \begin{cases} (0, \frac{1}{2}, 0, \frac{1}{2}), & \text{if } x_1^{(0)} = x_3^{(0)} = 0, \\ (\eta, 0, 0, 1-\eta), & \text{otherwise.} \end{cases}$$

Proof. Consider $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. First, note that if $x_1^{(0)} = x_3^{(0)} = 0$, then clearly $\widetilde{V}_\eta^k(s) = (0, \frac{1}{2}, 0, \frac{1}{2})$ for all $k \in \mathbb{N}^*$ and so

$$\lim_{k\to\infty} \widetilde{V}_\eta^k(s) = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right).$$

Otherwise, we can ensure $x_1^{(1)} \neq 0$. By Proposition 6.2.2 (iv), there exists a number $m_0 \in \mathbb{N}$ such that $\{x_1^{(k)}\}_{k\geq m_0}$ is an increasing, bounded, positive sequence. Consequently, the limit of $\{x_1^{(k)}\}_{k\in\mathbb{N}}$ exists and is positive. Analogously, by Proposition 6.2.2 (ii), we have that $\{x_3^{(k)}\}_{k\in\mathbb{N}^*}$ is a descending bounded positive sequence, so its limit exists as well. Since

$$x_2^{(k)} = \frac{\eta(2x_1^{(k)} + x_3^{(k)})}{2x_1^{(k+1)}} - x_1^{(k)} - x_3^{(k)} \quad \text{and} \quad u^{(k)} = 1 - (x_1^{(k)} + x_2^{(k)} + x_3^{(k)})$$

for all $k \geq 1$, we deduce that $\{x_2^{(k)}\}_{k\in\mathbb{N}}$ and $\{u^{(k)}\}_{k\in\mathbb{N}}$ are also two convergent sequences. As a consequence of Remark 6.2.3, if we take limits on both sides of the expressions which define the dynamical system (6.2.4), we have that the possible limits are exactly its fixed points but, as $\lim_{k\to\infty} x_1^{(k)} > 0$, we conclude that $\lim_{k\to\infty} \widetilde{V}_\eta^k(s) = (\eta, 0, 0, 1-\eta)$. □

6.2.2 Case 2: $\frac{1}{2} < \eta < 1$

The arguments in this case are very similar to those in the previous one, so we omit the analogous details. It is straightforward to see that $\widetilde{V}_\eta$ with $0 < \eta < \frac{1}{2}$ maps $S^{3,1}$ to itself and that its fixed points are again $(0, \frac{1}{2}, 0, \frac{1}{2})$ and $(\eta, 0, 0, 1-\eta)$.

Proposition 6.2.5. *Consider the operator* $\widetilde{V}_\eta$ *defined by* (6.2.4) *with* $0 < \eta < \frac{1}{2}$. *Then, for any initial point* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$, *the following assertions hold:*

(i) $\eta \leq x_1^{(k)} + x_2^{(k)} \leq \frac{1}{2}$ *for all* $k \in \mathbb{N}^*$;

(ii) $x_3^{(k+1)} \leq \frac{1}{2} x_3^{(k+1)}$ *for all* $k \in \mathbb{N}^*$;

(iii) *there exists a natural number* n_0 *such that* $2\eta x_1^{(k)} + (1+\eta)x_3^{(k)} \leq x_1^{(k)} + x_3^{(k)}$ *for all* $k \geq n_0$*; and*

(iv) *there exists a natural number* m_0 *such that* $x_2^{(k+1)} \geq x_2^{(k)}$ *for all* $k \geq m_0$.

Proof. The proofs of items (i) and (ii) are straightforward and analogous to those of Proposition 6.2.2. For item (iii) assume by contradiction that for every natural number n there exists another natural number $k \geq n$, such that $2\eta x_1^{(k)} + (1+\eta)x_3^{(k)} > x_1^{(k)} + x_3^{(k)}$, or equivalently, $(2\eta - 1)x_1^{(k)} + \eta x_3^{(k)} > 0$. This implies that

$$2(2\eta-1)x_1^{(k-1)} + \big[(2\eta-1)+\eta\big]x_3^{(k-1)} > 0.$$

Proceeding similarly, we obtain

$$2^2(2\eta-1)x_1^{(k-2)} + \big[2(2\eta-1)+(2\eta-1)+\eta\big]x_3^{(k-1)} > 0.$$

Inductively, it is straightforward to verify that

$$2^k(2\eta-1)x_1^{(0)} + \big[\eta+(2\eta-1)(2^k-1)\big]x_3^{(0)} > 0. \tag{6.2.6}$$

However, since n can be chosen arbitrarily large, the term $\eta + (2\eta-1)(2^k-1)$ eventually becomes negative for all $k > n$, which contradicts (6.2.6).

Lastly, to prove item (iv), we know that there exists a $n_0 \in \mathbb{N}$ large enough such that $2\eta x_1^{(k)} + (1+\eta)x_3^{(k)} \leq x_1^{(k)} + x_3^{(k)}$ for all $k \geq n_0$ by item (iii). Hence, we have that $\alpha^{(k)} \leq \frac{1}{2}$ for all $k \geq n_0 + 1$. Indeed,

$$\alpha^{(k)} = \frac{2\eta x_1^{(k-1)} + (1+\eta)x_3^{(k-1)} + x_2^{(k-1)}}{2\alpha^{(k-1)}} \leq \frac{x_1^{(k-1)} + x_2^{(k-1)} + x_3^{(k-1)}}{2\alpha^{(k-1)}} = \frac{1}{2}$$

for all $k \geq n_0 + 1$. Then, taking $m_0 = n_0 + 1$ we obtain that $(1-2\alpha^{(k)})x_2^{(k)} + (1-\eta)x_3^{(k)} \geq 0$ for all $k \geq m_0$, which implies that $x_2^{(k)} + (1-\eta)x_3^{(k)} \geq 2x_2^{(k)}\alpha^{(k)}$, for all $k \geq m_0$. Therefore,

$$x_2^{(k+1)} = \frac{x_2^{(k)} + (1-\eta)x_3^{(k)}}{2\alpha^{(k)}} \geq x_2^{(k)}$$

for all $k \geq m_0$, and the result follows. □

Theorem 6.2.6. *Consider the operator* $\widetilde{V}_\eta$ *given by* (6.2.4) *with* $\frac{1}{2} < \eta < 1$. *Then, for an initial point* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$*, it holds that*

$$\lim_{k\to\infty} \widetilde{V}_\eta^k(s) = \begin{cases} (\eta, 0, 0, 1-\eta), & \text{if } x_2^{(0)} = x_3^{(0)} = 0; \\ (0, \frac{1}{2}, 0, \frac{1}{2}), & \text{otherwise.} \end{cases}$$

Proof. The proof is analogous to that of Theorem 6.2.4. □

6.2.3 Case 3: $\eta = \frac{1}{2}$

In this case, we also have that $S^{3,1}$ is invariant under $\widetilde{V}_{\frac{1}{2}}$. We obtain the following result regarding its limit points.

Theorem 6.2.7. *Consider the operator* $\widetilde{V}_{\frac{1}{2}}$ *defined by* (6.2.4)*. Then, for an initial point* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$*, it holds that*

$$\lim_{k\to\infty} \widetilde{V}_{\frac{1}{2}}^k(s) = \begin{cases} \widetilde{V}_{\frac{1}{2}}(s), & \text{if } x_3^{(0)} = 0, \\ \left(\frac{x_1^{(1)}+x_3^{(1)}}{1+4x_3^{(1)}}, \frac{1+2(x_3^{(1)}-x_1^{(1)})}{2(1+4x_3^{(1)})}, 0, \frac{1}{2}\right), & \text{otherwise.} \end{cases}$$

Proof. Let $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. If $x_3^{(0)} = 0$, then it is easy to check that

$$\widetilde{V}_{\frac{1}{2}}^k(s) = \widetilde{V}_{\frac{1}{2}}(s) = \left(\frac{x_1^{(0)}}{2(x_1^{(0)} + x_2^{(0)})}, \frac{x_2^{(0)}}{2(x_1^{(0)} + x_2^{(0)})}, 0, \frac{1}{2}\right)$$

for all $k \in \mathbb{N}^*$. Consequently, $\lim_{k\to\infty} \widetilde{V}_{\frac{1}{2}}^k(s) = \widetilde{V}_{\frac{1}{2}}(s)$.

Next, assume that $x_3^{(0)} \neq 0$. Then, as $x_1^{(k)} + x_2^{(k)} = \frac{1}{2}$ for any $k \in \mathbb{N}^*$, we have that

$$x_1^{(k)} = \frac{x_1^{(k-1)} + \frac{1}{2}x_3^{(k-1)}}{1 + 2x_3^{(k-1)}} = \frac{x_1^{(k-2)} + (\frac{1}{2} + \frac{1}{4})x_3^{(k-2)}}{1 + (2+1)x_3^{(k-2)}} = \cdots = \frac{x_1^{(1)} + x_3^{(1)} \sum_{j=1}^{k-1} \frac{1}{2^j}}{1 + x_3^{(1)} \sum_{j=-1}^{k-3} \frac{1}{2^j}}$$

for all $k \geq 2$. Hence,

$$\lim_{k\to\infty} x_1^{(k)} = \frac{x_1^{(1)} + x_3^{(1)} \sum_{k=1}^{\infty} \frac{1}{2^k}}{1 + x_3^{(1)} \sum_{k=-1}^{\infty} \frac{1}{2^k}} = \frac{x_1^{(1)} + x_3^{(1)}}{1 + 4x_3^{(1)}}$$

and

$$\lim_{k\to\infty} x_2^{(k)} = \frac{1 + 2(x_3^{(1)} - x_1^{(1)})}{2(1 + 4x_3^{(1)})}.$$

Analogously, we have that

$$x_3^{(k)} = \frac{\frac{1}{2}x_3^{(k-1)}}{1 + 2x_3^{(k-1)}} = \frac{\frac{1}{4}x_3^{(k-2)}}{1 + (2+1)x_3^{(k-2)}} = \cdots = \frac{\frac{1}{2^{k-1}}x_3^{(1)}}{1 + x_3^{(1)} \sum_{j=-1}^{k-3} \frac{1}{2^j}}$$

for all $k \geq 2$ and, so $\lim_{k\to\infty} x_3^{(k)} = 0$. Finally, $\lim_{k\to\infty} u^{(k)} = \frac{1}{2}$, and the result follows. □

6.2.4 Case 4: $\eta = 1$

First, notice that, unlike the previous cases, $\widetilde{V}_1$ does not necessarily map $S^{3,1}$ to itself. In fact, we have the following result.

Lemma 6.2.8. *The set* $\{(x_1, x_2, x_3, u) \in S^{3,1} : x_2 > 0\}$ *is the largest invariant subset of* $S^{3,1}$ *with respect to* $\widetilde{V}_1$.

Proof. Clearly, if we take an initial point $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$ such that $x_2^{(0)} > 0$, then $x_2^{(k)} = u^{(k)} > 0$ for any $k \in \mathbb{N}$. To prove that it is the largest invariant contained in $S^{3,1}$, consider an initial point $s \in S^{3,1}$ with $x_2^{(0)} = 0$. So, $u^{(1)} = 0$ and thus $s^{(1)} \notin S^{3,1}$. □

Theorem 6.2.9. *Consider the operator* $\widetilde{V}_1$ *defined by* (6.2.4)*. Then, for an initial point* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$ *with* $x_2^{(0)} > 0$*, it holds that*

$$\lim_{k\to\infty} \widetilde{V}_1^k(s) = \begin{cases} (0, \frac{1}{2}, 0, \frac{1}{2}), & \text{if } x_1^{(0)} = x_3^{(0)} = 0, \\ (1, 0, 0, 0), & \text{otherwise.} \end{cases}$$

Proof. Take an arbitrary initial point $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$ with $x_2^{(0)} > 0$. Firstly, notice that if $x_1^{(0)} = x_3^{(0)} = 0$, then clearly $\widetilde{V}_1^k(s) = (0, \frac{1}{2}, 0, \frac{1}{2})$ for any $k \in \mathbb{N}^*$, and so $\lim_{k\to\infty} \widetilde{V}_1^k(s) = (0, \frac{1}{2}, 0, \frac{1}{2})$.

Otherwise, we can ensure that $x_1^{(1)} \neq 0$. Moreover, we have that

$$x_i^{(k)} = \frac{x_i^{(k-1)}}{2(x_1^{(k-1)} + x_2^{(k-1)} + x_3^{(k-1)})} = \cdots = \frac{x_i^{(1)}}{2^{k-1}(x_1^{(1)} + x_3^{(1)}) + 2x_2^{(1)}},$$

for $i = 2, 3$ and for all $k \geq 2$. This implies that $\lim_{k\to\infty} x_2^{(k)} = \lim_{k\to\infty} x_3^{(k)} = 0$. In addition, since $x_2^{(k)} = u^{(k)}$ for all $k \in \mathbb{N}^*$, we also have that $\lim_{k\to\infty} u^{(k)} = 0$. Thus, $\lim_{k\to\infty} \widetilde{V}_1^k(s) = (1, 0, 0, 0)$.

□

Biological interpretation 6.2.10. For the *Armadillium vulgare* population, we can conclude that for any initial state $s \in S^{3,1}$ (the probability distribution on the set of possible genotypes $\{$ZZ+w, ZW, ZW+w, ZZ$\}$), the future of the population is always stable. If the transmission rate is $\frac{1}{2} < \eta < 1$, then, in the absence of ZZ+w and ZW+w individuals in the initial state, the population tends to the equilibrium state $(0, \frac{1}{2}, 0, \frac{1}{2})$, where ZW and ZZ are equally distributed. Otherwise, the population tends to the equilibrium state $(\eta, 0, 0, 1-\eta)$. Analogously, if $0 < \eta < \frac{1}{2}$, and there are no ZW and ZW+w individuals initially, the population tends to the equilibrium state $(\eta, 0, 0, 1-\eta)$; otherwise, it converges to $(0, \frac{1}{2}, 0, \frac{1}{2})$. Notice that, in any case, the individuals of genotype ZW+w tend to extinction, which precisely reflects the situation (1.b) in the introduction, where only the genotypes ZW and ZZ persist, together with the ZZ+w individuals infected by *Wolbachia*. In the case of $\eta = \frac{1}{2}$, although the limit always exists, it depends on the initial point. Moreover, again, the ZW+w individuals always tend to extinction. Finally, for $\eta = 1$ and if the probability of genotypes ZZ+w and ZW+w is zero, then the population tends to the equilibrium state $(0, \frac{1}{2}, 0, \frac{1}{2})$ where ZW and ZZ are distributed equally. Otherwise, the population tends to $(1, 0, 0, 0)$, meaning that the proportion of males gradually decreases and tends to zero. As a result, the population will eventually face extinction due to the insignificant proportion of males.

6.3 XY systems with fertile XY females

6.3.1 Generalising the wood lemming sex-determination system

This subsection aims to develop the dynamic aspects of a (normalised) gonosomal operator that generalises the modelling of some rodent populations with atypical fertile XY females (see [77, 107]), such as *Myopus schisticolor* (wood lemming) and *Mus minutoides* (African pygmy mouse). In this case, we describe three female genotypes: XX, XX* and X*Y; and only one male genotype: XY. In these genotypes, notation X* refers to a chromosome carrying a gene which inactivates the action of gonosome Y and causes its elimination during gametogenesis, so X*Y female only give rise to ova of type X*. The results of the three possible crosses are:

$$\begin{aligned} \mathrm{XX} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX}, \tfrac{1}{2}\mathrm{XY}; \\ \mathrm{XX^*} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{4}\mathrm{XX}, \tfrac{1}{4}\mathrm{XX^*}, \tfrac{1}{4}\mathrm{X^*Y}, \tfrac{1}{4}\mathrm{XY}; \\ \mathrm{X^*Y} \times \mathrm{XY} &\rightarrowtail \tfrac{1}{2}\mathrm{XX^*}, \tfrac{1}{2}\mathrm{X^*Y}. \end{aligned} \tag{6.3.1}$$

Then, as explained in [105, Example 16], applying Construction 6.1.1, it is possible to obtain a gonosomal algebra which realises the results of crosses. Define the baric algebra $\mathcal{A}$ with basis $\{e_1, e_2, e_3\}$ and multiplication given by

$$e_i^2 = e_i, \quad e_1 e_i = \frac{1}{2}e_1 + \frac{1}{2}e_i, \quad e_2 e_3 = e_2,$$

for all $i = 1, 2, 3$. Hence, considering the following subspaces of $D(\mathcal{A})$:

$$\begin{aligned} F &= \mathsf{span}\{f_1 = e_1 \otimes e_1, f_2 = e_1 \otimes e_2, f_3 = e_2 \otimes e_3\}, \\ M &= \mathsf{span}\{h = e_1 \otimes e_3\}, \end{aligned}$$

we get the gonosomal algebra with basis $\{f_1, f_2, f_3, h\}$ and multiplication given by

$$\begin{aligned} f_1 h &= e_1^2 \otimes (e_1 e_3) = e_1 \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{2}f_1 + \frac{1}{2}h, \\ f_2 h &= (e_1 e_2) \otimes (e_1 e_3) \\ &= \left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right) \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{4}f_1 + \frac{1}{4}f_2 + \frac{1}{4}f_3 + \frac{1}{4}h, \\ f_3 h &= (e_2 e_3) \otimes (e_1 e_3) = e_2 \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{2}f_2 + \frac{1}{2}f_3. \end{aligned} \tag{6.3.2}$$

Motivated by this example, we introduce a generalised gonosomal algebra with gonosomal basis $(f_i)_{i\in\Lambda} \cup h$ and parameters $0 \le \gamma_i \le \frac{1}{2}$ (some nonzero), whose product is given by

$$f_i h = \gamma_i f_1 + \sum_{j=2}^{n} \frac{1-2\gamma_i}{n-1} f_j + \gamma_i h, \tag{6.3.3}$$

for any $i \in \Lambda$. Its associated gonosomal operator is

$$V_{\gamma_1,\ldots,\gamma_n} : \left\{ \begin{array}{lllll} x_1' & = & u' & = & u \sum_{i\in\Lambda} \gamma_i x_i, \\ x_j' & & & = & u \sum_{i\in\Lambda} \frac{1-2\gamma_i}{n-1} x_i, \text{ for any } j = 2, \ldots, n. \end{array} \right. \tag{6.3.4}$$

Remark 6.3.1. Note that if $n = 3$, $\gamma_1 = \frac{1}{2}, \gamma_2 = \frac{1}{4}$ and $\gamma_3 = 0$ in (6.3.3), then we precisely recover the gonosomal algebra (6.3.2).

We first compute the fixed points of the gonosomal operator given by (6.3.4) depending on the parameters $\{\gamma_i\}_{i\in\Lambda}$. For that, as $x_1^{(k)} = u^{(k)}$ and $x_2^{(k)} = \cdots = x_n^{(k)}$ for any $k \ge 1$, we can equivalently compute the fixed points on the simplified two-dimensional discrete-time dynamical system given by

$$SV_{n,\gamma,\lambda} : \left\{ \begin{array}{lll} x' & = & x\left(\gamma x + \lambda y\right), \\ y' & = & x\left(\frac{1-2\gamma}{n-1} x + (1 - \frac{2\lambda}{n-1}) y\right); \end{array} \right. \tag{6.3.5}$$

where $\gamma = \gamma_1$, $\lambda = \sum_{i=2}^{n} \gamma_i$ and $(\gamma, \lambda) \neq (0, 0)$. Clearly, there is a one-to-one correspondence between the fixed points of (6.3.5) and the fixed points of (6.3.4). Moreover, this correspondence also holds if we restrict to nonzero, nonnegative fixed points.

Proposition 6.3.2. *Consider the operator $SV_{n,\gamma,\lambda}$ defined by* (6.3.5)*, where $n \in \mathbb{N}$, $n > 1$, $0 \le \gamma \le \frac{1}{2}$, $0 \le \lambda \le \frac{n-1}{2}$ and $(\gamma, \lambda) \neq (0, 0)$. Then, $SV_{n,\gamma,\lambda}$ has the following nonzero fixed points:*

(i) *if* $\lambda = 0$ *and* $\gamma \neq 0$*, then the only fixed point is* $\left(\frac{1}{\gamma}, \frac{1-2\gamma}{\gamma(\gamma-1)(n-1)}\right)$*; and*

(ii) *if* $\lambda \neq 0$*, then we distinguish two cases:*

(a) *if* $\lambda = (n-1)\gamma$*, then the only fixed point is* $\left(\frac{1}{1-\gamma}, \frac{1-2\gamma}{\gamma(1-\gamma)(n-1)}\right)$*; and*

(b) *if* $\lambda \neq (n-1)\gamma$*, then there exist at most two fixed points. In fact,* $\left(x, \frac{1-\gamma x}{\lambda}\right)$ *is a fixed point if and only if* x *is a solution of the quadratic equation*

$$(\lambda - (n-1)\gamma)\, x^2 + ((n-1)(\gamma+1) - 2\lambda)\, x - n + 1 = 0. \qquad (6.3.6)$$

Proof. We need to solve the system of equations defined by

$$\begin{cases} x &= x\,(\gamma x + \lambda y)\,, \\ y &= x\left(\frac{1-2\gamma}{n-1} x + (1 - \frac{2\lambda}{n-1}) y\right). \end{cases}$$

First, if $x = 0$, then $y = 0$. For $x \neq 0$, it holds that $\gamma x + \lambda y = 1$. If $\lambda = 0$, then $\gamma \neq 0$ and $x = \frac{1}{\gamma}$. So, from the second equation, we get that $y = \frac{1-2\gamma}{\gamma(\gamma-1)(n-1)}$. If $\lambda \neq 0$, then $y = \frac{1-\gamma x}{\lambda}$ and replacing this expression in the second equation, we obtain precisely the equation (6.3.6). Now, we have to consider two cases. For $\lambda = (n-1)\gamma$ we have that $x = \frac{1}{1-\gamma}$ and $y = \frac{1-2\gamma}{\gamma(1-\gamma)(n-1)}$. In the case of $\lambda \neq (n-1)\gamma$, observe that the discriminant of the equation (6.3.6) is $((n-1)\gamma - 2\lambda)^2 + (n-1)^2(1-2\gamma) \geq 0$, and so, both solutions are also real, which yields the claim. □

Next, we study sufficient and necessary conditions for the nonzero fixed points to be nonnegative. We will denote by $\Delta := ((n-1)\gamma - 2\lambda)^2 + (n-1)^2(1-2\gamma)$ the radicand of (6.3.6).

Proposition 6.3.3. *Consider the discrete-time dynamical system* $SV_{n,\gamma,\lambda}$ *defined above by* (6.3.5)*, where* $n \in \mathbb{N}$, $n > 1$, $0 \leq \gamma \leq \frac{1}{2}$, $0 \leq \lambda \leq \frac{n-1}{2}$ *and* $(\gamma, \lambda) \neq (0,0)$*. The nonzero fixed points of* $SV_{n,\gamma,\lambda}$ *satisfy the following assertions:*

(i) *if* $\lambda = 0$*, then the only fixed point is nonnegative if and only if* $\gamma = \frac{1}{2}$*;*

(ii) *if* $\lambda \neq 0$ *and* $\lambda = (n-1)\gamma$*, then the only fixed point is always nonnegative;*

(iii) *if* $\lambda \neq 0$*, then we state the following:*

(a) *if* $\lambda > (n-1)\gamma$*, then we only have one nonnegative fixed point; and*

(b) *if* $\lambda < (n-1)\gamma$*, then every fixed point is nonnegative if and only if* $\gamma = \frac{1}{2}$*. Otherwise, at least one nonnegative fixed point exists.*

Proof. Items (i) and (ii) are consequences of Proposition 6.3.2. To prove item (iii), denote by

$$x_1 = \frac{2\lambda - (n-1)(\gamma+1) - \sqrt{\Delta}}{2(\lambda - (n-1)\gamma)} \quad \text{and} \quad x_2 = \frac{2\lambda - (n-1)(\gamma+1) + \sqrt{\Delta}}{2(\lambda - (n-1)\gamma)}$$

both solutions of (6.3.6). Assume that $\lambda > (n-1)\gamma$. Since $2\lambda \leq n-1$ and $(n-1)(1+\gamma) + \sqrt{\Delta} > n-1$ then $x_1 < 0$. Hence, $(x_1, \frac{1-\gamma x_1}{\lambda})$ is not a nonnegative fixed point. Now, we prove that $x_2 > 0$. We have that:

$$\begin{aligned} x_2 > 0 &\iff \sqrt{\Delta} > (n-1)(\gamma+1) - 2\lambda \\ &\iff (n-1)^2\gamma^2 + 4\lambda^2 - 4\lambda\gamma(n-1) + (n-1)^2(1-2\gamma) \\ &\qquad > (n-1)^2(\gamma+1)^2 + 4\lambda^2 - 4\lambda(\gamma+1)(n-1) \\ &\iff -4(n-1)^2\gamma > -4(n-1)\lambda \iff (n-1)\gamma < \lambda, \end{aligned} \tag{6.3.7}$$

but, this is true by hypothesis. Moreover, if $\gamma \neq 0$, then $x_2 \leq \frac{1}{\gamma}$. Indeed,

$$\begin{aligned} x_2 \leq \frac{1}{\gamma} &\iff \sqrt{\Delta} \leq \frac{1-\gamma}{\gamma}(2\lambda - \gamma(n-1)) \\ &\iff (n-1)^2\gamma^2 + 4\lambda^2 - 4\lambda\gamma(n-1) + (n-1)^2(1-2\gamma) \\ &\qquad \leq \frac{(1-\gamma)^2}{\gamma^2}(4\lambda^2 + \gamma^2(n-1)^2 - 4\lambda\gamma(n-1)) \\ &\iff 0 \leq \frac{4\lambda}{\gamma^2}(1-2\gamma)(\lambda - \gamma(n-1)) \end{aligned} \tag{6.3.8}$$

and this last inequality is true by hypothesis. Therefore, the only nonnegative fixed point is $(x_2, \frac{1-\gamma x_2}{\lambda})$.

Next, suppose that $\lambda < (n-1)\gamma$. It is obvious that $x_2 \leq x_1$. Reasoning similarly as we did in (6.3.7), we get that $x_2 > 0$ (and, consequently, $x_1 > 0$). Moreover, $x_1 \leq \frac{1}{\gamma}$ if and only if $\gamma = \frac{1}{2}$. Indeed, analogously to (6.3.8), we have:

$$\begin{aligned} x_1 \leq \frac{1}{\gamma} &\iff \sqrt{\Delta} \leq \frac{1-\gamma}{\gamma}(\gamma(n-1) - 2\lambda) \\ &\iff 0 \leq \frac{4\lambda}{\gamma^2}(1-2\gamma)(\lambda - \gamma(n-1)) \end{aligned} \tag{6.3.9}$$

and this is verified if and only if $\gamma = \frac{1}{2}$. Therefore, $(x_1, \frac{1-\gamma x_1}{\lambda})$ and $(x_2, \frac{1-\gamma x_2}{\lambda})$ are both nonnegative fixed points for $\gamma = \frac{1}{2}$. In any case, as $\lambda < (n-1)\gamma$, observe that

$$x_2 \leq \frac{1}{\gamma} \iff \sqrt{\Delta} \geq \frac{1-\gamma}{\gamma}(\gamma(n-1) - 2\lambda).$$

Note that if $\frac{1-\gamma}{\gamma}(\gamma(n-1)-2\lambda) \leq 0$, the inequality is true. For $\frac{1-\gamma}{\gamma}(\gamma(n-1)-2\lambda) > 0$, following a reasoning similar to (6.3.9), we obtain that

$$\sqrt{\Delta} \geq \frac{1-\gamma}{\gamma}(\gamma(n-1)-2\lambda) \iff \frac{4\lambda}{\gamma^2}(1-2\gamma)(\lambda-\gamma(n-1)) \leq 0,$$

but this is always true. Hence, the fixed point $(x_2, \frac{1-\gamma x_2}{\lambda})$ is nonnegative. □

Corollary 6.3.4. *Consider the gonosomal operator $V_{\gamma_1,\ldots,\gamma_n}$ defined by* (6.3.4)*, where $n \in \mathbb{N}$, $n > 1$, $0 \leq \gamma_i \leq \frac{1}{2}$ for any $i \in \Lambda$ and some nonzero. Then, every nonzero fixed point is nonnegative if and only if $\gamma_1 = \frac{1}{2}$ or $\sum_{i=2}^{n} \gamma_i = (n-1)\gamma_1$.*

Next, we consider the normalised version of the gonosomal operator defined by (6.3.4), that is,

$$\widetilde{V}_{\gamma_1,\ldots,\gamma_n} : \begin{cases} x_1' = u' = \frac{u\sum_{i\in\Lambda}\gamma_i x_i}{u\sum_{i\in\Lambda} x_i}, \\ x_j' = \frac{u\sum_{i\in\Lambda}(1-2\gamma_i)x_i}{(n-1)u\sum_{i\in\Lambda} x_i}, \text{ for any } j = 2,\ldots,n; \end{cases} \tag{6.3.10}$$

with $n > 1$ and $0 \leq \gamma_i \leq \frac{1}{2}$ for any $i \in \Lambda$ and some nonzero. Notice that, in general, the previous operator does not map the set $S^{n,1}$ to itself, as shown in the next result.

Lemma 6.3.5. *The operator $\widetilde{V}_{\gamma_1,\ldots,\gamma_n}$ defined by* (6.3.10) *maps $S^{n,1}$ to itself if and only if $\gamma_i \neq 0$ for any $i \in \Lambda$.*

Proof. It follows straightforward since $(\gamma_i, \frac{1-2\gamma_i}{n-1}, \ldots, \frac{1-2\gamma_i}{n-1}, \gamma_i) \in S^{n,1}$ if and only if $\gamma_i \neq 0$ for any $i \in \Lambda$.

□

Thanks to the following result, it is not necessary to restrict the values of $\{\gamma_i\}_{i\in\Lambda}$ so much. Instead, it suffices to take an invariant subset of $S^{n,1}$ under $\widetilde{V}_{\gamma_1,\ldots,\gamma_n}$. Actually, we will determine the largest invariant subset. To do this, let $\widetilde{V}_{\gamma_1,\ldots,\gamma_n}$ be the normalised gonosomal operator defined by (6.3.10) and consider the following two subsets of $S^{n,1}$, which depend on the parameters $\gamma_1,\ldots,\gamma_n$:

$$R_{\gamma_1,\ldots,\gamma_n} = \{(x_1,\ldots,x_n,u) \in S^{n,1} \colon \gamma_i x_i > 0 \text{ for some } i \in \Lambda\},$$
$$T_{\gamma_1,\ldots,\gamma_n} = \{(x_1,\ldots,x_n,u) \in S^{n,1} \colon (1-2\gamma_i)x_i > 0 \text{ for some } i \in \Lambda\}.$$

For simplicity, if there is no risk of confusion, we will denote these sets as R and T. We establish the next result.

Proposition 6.3.6. *In the setting above, if $\gamma_1 > 0$, then R is the biggest invariant with respect to $\widetilde{V}_{\gamma_1,\ldots,\gamma_n}$. However, if $\gamma_1 = 0$, then such subset is $R \cap T$.*

Proof. Let $s^{(0)} = (x_1, \ldots, x_n, u) \in S^{n,1}$. First, notice that by the construction of the normalised gonosomal operator, $s^{(1)}$ always belongs to the simplex of $\mathbb{R}^{n+1}$. For the first part, suppose that $s^{(0)} \in R$. It is clear that $s^{(1)} \in R$ since

$$x^{(1)} = u^{(1)} = \frac{u \sum_{i\in\Lambda} \gamma_i x_i}{u \sum_{i\in\Lambda} x_i} > 0.$$

To prove that R is the biggest invariant contained in $S^{n,1}$ just consider an initial point $s^{(0)} \in S^{n,1} \setminus R$. So, $u^{(1)} = 0$ and thus $s^{(1)} \notin S^{n,1}$. For the second part, assume that $\gamma_1 = 0$ and consider an initial point $s^{(0)} = (x_1, \ldots, x_n, u) \in R \cap T$. Then, we know there exist $i, j \in \Lambda$ such that $x_i\gamma_i > 0$ and $(1 - 2\gamma_j)x_j > 0$. Consequently,

$$x^{(1)} = u^{(1)} = \frac{u \sum_{i\in\Lambda} \gamma_i x_i}{u \sum_{i\in\Lambda} x_i} > 0 \quad \text{and} \quad x_k^{(1)} = \frac{u \sum_{i\in\Lambda}(1 - 2\gamma_i)x_i}{(n-1)u \sum_{i\in\Lambda} x_i} > 0$$

for any $k = 2, \ldots, n$. Hence, there exist $i, j \in \Lambda$ such that $\gamma_i x_i^{(1)} > 0$ and $(1 - 2\gamma_j)x_j^{(1)} > 0$, which yields that $s^{(1)} \in T$. Now, we have to see that $R \cap T$ is the largest invariant set contained in $S^{n,1}$. Consider $s^{(0)} \in S^{n,1} \setminus R$, so $x_i\gamma_i = 0$ for any $i \in \Lambda$. Thus, $u^{(1)} = 0$ and accordingly $s^{(1)} \notin S^{n,1}$. Now, we take an element $s^{(0)} \in S^{n,1} \setminus T$, implying that $(1 - 2\gamma_i)x_i = 0$ for all $i \in \Lambda$. Therefore $x_k^{(1)} = 0$ for any $k = 2, \ldots, n$ and thus $u^{(2)} = 0$ which implies that $s^{(2)} \notin S^{n,1}$. □

Next, we study the limits of (6.3.10) in cases where every nonzero fixed point of $V_{\gamma_1,\ldots,\gamma_n}$ is nonnegative. This happens whether $\gamma_1 = \frac{1}{2}$ or $\sum_{i=2}^{n} \gamma_i = (n-1)\gamma_1$ by Corollary 6.3.4. Notice that, necessarily $\gamma_1 \neq 0$, then, by Proposition 6.3.6, R is the largest invariant subset of $S^{n,1}$ with respect to $\widetilde{V}_{\gamma_1,\ldots,\gamma_n}$. Again, to carry out this study, we consider the "normalised" version of (6.3.5), which will be given by

$$\widetilde{SV}_{n,\gamma,\lambda} : \begin{cases} x' &= \frac{x(\gamma x + \lambda y)}{x(x+(n-1)y)}, \\ y' &= \frac{x((1-2\gamma)x+(n-1-2\lambda)y)}{x(n-1)(x+(n-1)y)}, \end{cases} \tag{6.3.11}$$

with $n \in \mathbb{N}$, $n > 1$, $0 < \gamma = \gamma_1 \leq \frac{1}{2}$ and $0 \leq \lambda = \sum_{i=2}^{n} \gamma_i \leq \frac{n-1}{2}$. Observe that the set

$$\widetilde{S^{n,1}} := \{(x, y) \in \mathbb{R}^2 \colon x > 0, y \geq 0, 2x + (n-1)y = 1\}$$

is invariant with respect to $\widetilde{SV}_{n,\gamma,\lambda}$.

Theorem 6.3.7. *In the setting above, if $\gamma = \frac{1}{2}$ or $\lambda = (n-1)\gamma$, then, for any initial point $s = (x^{(0)}, y^{(0)}) \in \widetilde{S^{n,1}}$, we get that $\lim_{m\to\infty} \widetilde{SV}^{m}_{n,\gamma,\lambda}(s)$ exists. In fact, we have:*

(i) *if* $\gamma \neq \frac{1}{2}$, *then* $\lim_{k\to\infty} \widetilde{SV}^k_{n,\gamma,\lambda}(s) = \left(\gamma, \frac{1-2\gamma}{n-1}\right)$.

(ii) *if* $\gamma = \frac{1}{2}$, *then we distinguish the two following cases:*

(a) *if* $\lambda \geq \frac{n-1}{4}$, *then* $\lim_{k\to\infty} \widetilde{SV}^k_{n,\gamma,\lambda}(s) = (\frac{1}{2}, 0)$*;*

(b) *if* $\lambda < \frac{n-1}{4}$, *then*

$$\lim_{k\to\infty} \widetilde{SV}^k_{n,\gamma,\lambda}(s) = \begin{cases} (\frac{1}{2}, 0), & \text{if } s = (\frac{1}{2}, 0), \\ \left(\frac{2\lambda}{n-1}, \frac{n-1-4\lambda}{(n-1)^2}\right), & \text{otherwise.} \end{cases}$$

Proof. Let $s = (x^{(0)}, y^{(0)}) \in \widetilde{S^{n,1}}$. For item (i), notice that $x' = \frac{\gamma x+(n-1)\gamma y}{x+(n-1)y} = \gamma$ and $y' = \frac{(1-2\gamma)x+(n-1)(1-2\gamma)y}{(n-1)(x+(n-1)y)} = \frac{1-2\gamma}{n-1}$, which do not depend on the initial point. The dynamical system converges to this point in the first iteration. For item (ii), as $2x^{(k)} + (n-1)y^{(k)} = 1$ for any $k \in \mathbb{N}$, it is enough to study the limits of the sequence $\{y^{(k)}\}_{k\in\mathbb{N}}$. We can write

$$y^{(k+1)} = \frac{(n-1-2\lambda)y^{(k)}}{(n-1)\big(x^{(k)} + (n-1)y^{(k)}\big)} = \frac{2(n-1-2\lambda)y^{(k)}}{(n-1)\big(1 + (n-1)y^{(k)}\big)} \qquad (6.3.12)$$

for any $k \in \mathbb{N}$. If $y^{(0)} = 0$, clearly $\lim\limits_{k\to\infty} y^{(k)} = 0$. On the other hand, we have that:

$$y^{(k)} \begin{cases} < \frac{n-1-4\lambda}{(n-1)^2} & \Longleftrightarrow \; y^{(0)} < \frac{n-1-4\lambda}{(n-1)^2}, \\ > \frac{n-1-4\lambda}{(n-1)^2} & \Longleftrightarrow \; y^{(0)} > \frac{n-1-4\lambda}{(n-1)^2}, \\ = \frac{n-1-4\lambda}{(n-1)^2} & \Longleftrightarrow \; y^{(0)} = \frac{n-1-4\lambda}{(n-1)^2}; \end{cases} \qquad (6.3.13)$$

because $y^{(k+1)} < \frac{n-1-4\lambda}{(n-1)^2} \Leftrightarrow y^{(k)} < \frac{n-1-4\lambda}{(n-1)^2}$. From this, and the fact that $y^{(k+1)} > y^{(k)} \Leftrightarrow y^{(k)} < \frac{n-1-4\lambda}{(n-1)^2}$, we get that

$$y^{(k+1)} \begin{cases} > y^{(k)} & \Longleftrightarrow \; y^{(0)} < \frac{n-1-4\lambda}{(n-1)^2}, \\ < y^{(k)} & \Longleftrightarrow \; y^{(0)} > \frac{n-1-4\lambda}{(n-1)^2}, \\ = \frac{n-1-4\lambda}{(n-1)^2} & \Longleftrightarrow \; y^{(0)} = \frac{n-1-4\lambda}{(n-1)^2}. \end{cases} \qquad (6.3.14)$$

Therefore, $\{y^{(k)}\}_{k\in\mathbb{N}}$ is an increasing (resp. decreasing) and upper (resp. lower) bounded sequence, which implies that its limit always exists. Now, we note that necessarily $\lim_{k\to\infty} x^{(k)} \neq 0$. Indeed, if $\lim_{k\to\infty} x^{(k)} = 0$ then $\lim_{k\to\infty} x^{(k)} + (n-1)y^{(k)} = 1$. So, $\lim_{k\to\infty} \lambda y^{(k)} = 0$ a contradiction since $2x^{(k)} + (n-1)y^{(k)} = 1$ for any $k \in \mathbb{N}$. Taking this into account and taking limits in both sides of (6.3.12), we get that $L_y\big((n-1)^2 L_y + 4\lambda - (n-1)\big) = 0$, that is, $L_y = 0$ or $L_y = \frac{n-1-4\lambda}{(n-1)^2}$. Observe that if $n-1 = 4\lambda \neq 0$, then $L_y = 0$. Now, since $\widetilde{S^{n,1}}$ is an invariant subset with respect to $\widetilde{SV}_{n,\gamma,\lambda}$, we have that $y^{(k)} \geq 0$ for all $k \in \mathbb{N}$. Hence $L_y \geq 0$. So, if $\lambda > \frac{n-1}{4}$, necessarily $L_y = 0$. For $\lambda < \frac{n-1}{4}$, whether $y^{(0)} < \frac{n-1-4\lambda}{(n-1)^2}$ or $y^{(0)} > \frac{n-1-4\lambda}{(n-1)^2}$ happens, applying (6.3.13) and (6.3.14), we obtain that $L_y = \frac{n-1-4\lambda}{(n-1)^2}$. Finally, the result follows from the fact that $x^{(k)} = \frac{1-(n-1)y^{(k)}}{2}$ for any $k \in \mathbb{N}$. □

Remark 6.3.8. Let $s = (x_1^{(0)}, \ldots, x_n^{(0)}, u^{(0)}) \in R$. For any $k \in \mathbb{N}^*$, it is easy to check that

$$\widetilde{SV}^{k}_{n,\gamma,\lambda}(x_1^{(1)}, x_2^{(1)}) = (x_1^{(k+1)}, x_2^{(k+1)})$$

where $x_1^{(1)}$, $x_2^{(1)}$, $x_1^{(k+1)}$ and $x_2^{(k+1)}$ are given by

$$\widetilde{V}_{\gamma_1,\ldots,\gamma_n}(s) = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}, u^{(1)})$$

and

$$\widetilde{V}^{k+1}_{\gamma_1,\ldots,\gamma_n}(s) = (x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_n^{(k+1)}, u^{(k+1)})$$

for any $k \in \mathbb{N}^*$.

Hence, we establish the following corollary.

Corollary 6.3.9. *Consider the gonosomal operator $V_{\gamma_1,\ldots,\gamma_n}$ given by* (6.3.4)*, whose fixed points are nonnegative and normalisable. Let $\widetilde{V}_{\gamma_1,\ldots,\gamma_n}$ be its corresponding normalised gonosomal operator. Then, for any initial point $s \in R$, we get that $\lim\limits_{k\to\infty} \widetilde{V}^{k}_{\gamma_1,\ldots,\gamma_n}(s)$ exists.*

Proof. It follows from Corollary 6.3.4, Theorem 6.3.7 and Remark 6.3.8. □

Finally, we apply the previous results to study the dynamic behaviour of the population with crosses (6.3.1).

Biological interpretation 6.3.10. As stated in Remark 6.3.1, if $n = 3$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{4}$ and $\gamma_3 = 0$ in (6.3.3), then we precisely recover the gonosomal algebra (6.3.2). Moreover, the associated normalised gonosomal operator (6.3.10) in this particular case is

$$\widetilde{V} : \begin{cases} x_1' = u' = \frac{u(2x_1+x_2)}{4u(x_1+x_2+x_3)}, \\ x_2' = x_3' = \frac{u(x_2+2x_3)}{4u(x_1+x_2+x_3)}. \end{cases}$$

Since $\gamma_1, \gamma_2 \neq 0$ but $\gamma_3 = 0$ then, by Proposition 6.3.6, we have that

$$R = \{(x_1, x_2, x_3, u) \in S^{3,1} \colon x_1 \neq 0 \text{ or } x_2 \neq 0\}$$

is the biggest subset of $S^{3,1}$ which $\widetilde{V}$ maps to itself. Consequently, Theorem 6.3.7 and Corollary 6.3.9 can be applied. Then, since $\lambda = \gamma_2 + \gamma_3 = \frac{1}{4} < \frac{1}{2} = \frac{n-1}{4}$, for any initial point $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in R$ it holds that

$$\lim_{k\to\infty} \widetilde{V}^k_{\frac{1}{2},\frac{1}{4},0}(s) = \begin{cases} (\frac{1}{2}, 0, 0, \frac{1}{2}), & \text{if } x_2^{(0)} = x_3^{(0)} = 0; \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), & \text{otherwise.} \end{cases}$$

Hence, we can conclude that for any initial state $s \in R$ (the probability distribution on the set of possible genotypes{XX, XX*, X*Y, XY}), the future of the population is always stable. If there are no XX* and X*Y individuals in the initial state, the population tends to the equilibrium state $(\frac{1}{2}, 0, 0, \frac{1}{2})$ where XX and XY are distributed equally. Otherwise, the population tends to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, where all possible genotypes appear in the same proportion.

6.3.2 The Arctic lemming sex-determination system

This subsection is devoted to studying the dynamical behaviour of a *Dicrostonyx torquatus* (Artic lemming) population, in which the possible genotypes are the same as in the previous case (6.3.1), except that X*Y females give birth to normal XY males (see [105, Example 16]). In this case the result of crosses are:

$$\begin{aligned} \text{XX} \times \text{XY} &\rightarrowtail \tfrac{1}{2}\text{XX}, \tfrac{1}{2}\text{XY}; \\ \text{XX*} \times \text{XY} &\rightarrowtail \tfrac{1}{4}\text{XX}, \tfrac{1}{4}\text{XX*}, \tfrac{1}{4}\text{X*Y}, \tfrac{1}{4}\text{XY}; \\ \text{X*Y} \times \text{XY} &\rightarrowtail \tfrac{1}{3}\text{XX*}, \tfrac{1}{3}\text{X*Y}, \tfrac{1}{3}\text{XY}. \end{aligned}$$

Remark 6.3.11. Unlike the previous sex-determination systems already studied, the dynamical behaviour of *Dicrostonyx torquatus* cannot be realised as the commutative duplicate of a baric algebra. If we identify $e_1 \leftrightarrow$ X, $e_2 \leftrightarrow$ X* and $e_3 \leftrightarrow$ Y, the result of the third cross is

$$(e_2 \otimes e_3)(e_1 \otimes e_3) = \frac{1}{3} e_1 \otimes e_2 + \frac{1}{3} e_2 \otimes e_3 + \frac{1}{3} e_1 \otimes e_3. \tag{6.3.15}$$

Now, suppose there exists a baric algebra with basis $\{e_1, e_2, e_3\}$ such that gives rise to (6.3.15) by following Construction 6.1.1 where $F = \mathsf{span}\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_3\}$ and $M = \mathsf{span}\{e_1 \otimes e_3\}$. We can write $\mu(e_2 \otimes e_3) = e_2 e_3 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

and $\mu(e_1 \otimes e_3) = e_1e_3 = \beta_1e_1 + \beta_2e_2 + \beta_3e_3$ with $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 1$. Thus, we have that

$$\begin{aligned}(e_2 \otimes e_3)(e_1 \otimes e_3) = \alpha_1\beta_1 e_1 \otimes e_1 + \alpha_2\beta_2 e_2 \otimes e_2 \\ + \alpha_3\beta_3 e_3 \otimes e_3 + (\alpha_1\beta_2 + \alpha_2\beta_1)e_1 \otimes e_2 \\ + (\alpha_1\beta_3 + \alpha_3\beta_1)e_1 \otimes e_3 + (\alpha_2\beta_3 + \alpha_3\beta_2)e_2 \otimes e_3.\end{aligned}$$

From (6.3.15) and the previous expression, we get that $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = 0$ and $\alpha_1\beta_2 + \alpha_2\beta_1 = \alpha_1\beta_3 + \alpha_3\beta_1 = \alpha_2\beta_3 + \alpha_3\beta_2 = \frac{1}{3}$. However, this system of equations does not admit any solution, which yields that such baric algebra does not exist.

Hence, we now combine both Constructions 6.1.1 and 6.1.2 to model this genetic system. Considering the identification $e_1 \leftrightarrow$ X, $e_2 \leftrightarrow$ X* and $e_3 \leftrightarrow$ Y, define the baric algebra $\mathcal{A}$ with basis $\{e_1, e_2, e_3\}$ and product given by

$$e_i^2 = e_i, \quad e_1e_i = \frac{1}{2}(e_1 + e_i), \quad e_2e_3 = \frac{1}{2}(e_2 + e_3),$$

for all $i = 1, 2, 3$. Then, considering the following subspaces of $D(\mathcal{A})$:

$$\begin{aligned}F &= \mathsf{span}\{f_1 = e_1 \otimes e_1, f_2 = e_1 \otimes e_2, f_3 = e_2 \otimes e_3, f_4 = e_3 \otimes e_3\},\\ M &= \mathsf{span}\{h = e_1 \otimes e_3\}.\end{aligned}$$

we define a gonosomal algebra with basis $\{f_1, f_2, f_3, f_4, h\}$ and product given by

$$\begin{aligned}f_1h &= e_1^2 \otimes (e_1e_3) = e_1 \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{2}f_1 + \frac{1}{2}h,\\ f_2h &= (e_1e_2) \otimes (e_1e_3)\\ &= \left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right) \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{4}f_1 + \frac{1}{4}f_2 + \frac{1}{4}f_3 + \frac{1}{4}h,\\ f_3h &= (e_2e_3) \otimes (e_1e_3)\\ &= \left(\frac{1}{2}e_2 + \frac{1}{2}e_3\right) \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{4}f_2 + \frac{1}{4}f_3 + \frac{1}{4}f_4 + \frac{1}{4}h,\\ f_4h &= e_3^2 \otimes (e_1e_3) = e_3 \otimes \left(\frac{1}{2}e_1 + \frac{1}{2}e_3\right) = \frac{1}{2}f_4 + \frac{1}{2}h.\end{aligned}$$

Note that $f_4 = e_3 \otimes e_3 \leftrightarrow$ YY is not a possible genotype, but as the baric algebra $\mathcal{A}$ is defined, we need to include it to $\mu(F) \otimes \mu(M) \subset F \oplus M$. Finally, we just need to reduce the gonosomal basis $\{f_1, f_2, f_3, f_4, h\}$ by taking $I = \{4\}$ in Construction 6.1.2. It is easy to check that $\sigma_{11} = \sigma_{21} = 1$ and $\sigma_{31} = \frac{3}{4}$. Therefore, the resulting gonosomal algebra, which realises the results of crosses, has gonosomal basis $\{f_1, f_2, f_3, h\}$ and its product is given by

$$f_1 h = \frac{1}{2} f_1 + \frac{1}{2} h,$$
$$f_2 h = \frac{1}{4} f_1 + \frac{1}{4} f_2 + \frac{1}{4} f_3 + \frac{1}{4} h,$$
$$f_3 h = \frac{4}{3}\left(\frac{1}{4} f_2 + \frac{1}{4} f_3 + \frac{1}{4} h\right) = \frac{1}{3} f_2 + \frac{1}{3} f_3 + \frac{1}{3} h;$$

and, consequently, its associated gonosomal operator is

$$V\colon \begin{cases} x_1' = \frac{1}{2}x_1 u + \frac{1}{4}x_2 u, & x_3' = \frac{1}{4}x_2 u + \frac{1}{3}x_3 u, \\ x_2' = \frac{1}{4}x_2 u + \frac{1}{3}x_3 u, & u' = \frac{1}{2}x_1 u + \frac{1}{4}x_2 u + \frac{1}{3}x_3 u; \end{cases} \tag{6.3.16}$$

whose fixed points are given by the following result.

Proposition 6.3.12. *The operator* (6.3.16) *has two nonzero fixed points:* $(2, 0, 0, 2)$ *and* $(\frac{36}{25}, \frac{12}{25}, \frac{12}{25}, \frac{12}{7})$.

Proof. As $x_2' = x_3'$, we need to solve the system of equations given by

$$x_1 = u\left(\frac{1}{2}x_1 + \frac{1}{4}x_2\right), \quad x_2 = \frac{7}{12}x_2 u \quad \text{and} \quad u = u\left(\frac{1}{2}x_1 + \frac{7}{12}x_2\right).$$

First, if $x_2 = 0$, it is easy to check that $x_1 = u = 0$ or $x_1 = u = 2$. Otherwise, we get $u = \frac{12}{7}$. Then, from the third equation, we get that $\frac{1}{2}x_1 + \frac{7}{12}x_2 = 1$, or equivalently, $x_1 = 2 - \frac{7}{6}x_2$. Changing this expression of x_1 in the first equation, we get $x_2 = \frac{12}{25}$ and, consequently, $x_1 = \frac{36}{25}$. □

Next, we consider the normalised version of (6.3.16), that is,

$$\widetilde{V}\colon \begin{cases} x_1' = \frac{u(6x_1+3x_2)}{12u(x_1+x_2+x_3)}, & x_3' = \frac{u(3x_2+4x_3)}{12u(x_1+x_2+x_3)}, \\ x_2' = \frac{u(3x_2+4x_3)}{12u(x_1+x_2+x_3)}, & u' = \frac{u(6x_1+3x_2+4x_3)}{12u(x_1+x_2+x_3)}. \end{cases} \tag{6.3.17}$$

Hence, this normalised gonosomal operator has $(\frac{1}{2}, 0, 0, \frac{1}{2})$ and $(\frac{7}{20}, \frac{7}{60}, \frac{7}{60}, \frac{5}{12})$ as fixed points. Moreover, it is easy to check that $\widetilde{V}$ maps $S^{3,1}$ to itself.

Proposition 6.3.13. *Consider the operator* $\widetilde{V}$ *defined by* (6.3.17) *and an initial point* $s^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. *Then, the following assertions are equivalent:*

(i) $6x_1^{(0)} \leq 6x_2^{(0)} + 12x_3^{(0)}$ *(resp.* $6x_1^{(0)} \geq 6x_2^{(0)} + 12x_3^{(0)}$*);*

(ii) $x_1^{(k)} \leq 3x_2^{(k)}$ *(resp.* $x_1^{(k)} \geq 3x_2^{(k)}$*) for any* $k \in \mathbb{N}^*$*;*

(iii) $x_2^{(k)} \geq \frac{7}{60}$ *(resp.* $x_2^{(k)} \leq \frac{7}{60}$*) for any* $k \geq 2$*; and*

(iv) $x_2^{(k)} \geq \frac{7}{24} - \frac{1}{2}x_1^{(k)}$ *(resp.* $x_2^{(k)} \leq \frac{7}{24} - \frac{1}{2}x_1^{(k)}$*) for any* $k \geq 2$.

Proof. Let $s^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. We only prove the result for one inequality; the other is trivially analogue. First, we prove that (i) is equivalent to (ii). Notice that $6x_1^{(0)} < 6x_2^{(0)} + 12x_3^{(0)}$ is equivalent to $x_1^{(1)} < 3x_2^{(1)}$. Indeed, adding $3x_2^{(0)}$ to both sides we have that

$$6x_1^{(0)} \leq 6x_2^{(0)} + 12x_3^{(0)} \iff 6x_1^{(0)} + 3x_2^{(0)} \leq 3(3x_2^{(0)} + 4x_3^{(0)}) \iff x_1^{(1)} \leq 3x_2^{(1)}.$$

Moreover, we claim that $x_1^{(k)} \leq 3x_2^{(k)}$ is equivalent to $x_1^{(k+1)} \leq 3x_2^{(k+1)}$ for any $k \in \mathbb{N}^*$. Indeed, it holds that

$$\begin{aligned} x_1^{(k)} \leq 3x_2^{(k)} &\iff 6x_1^{(k)} \leq 18x_2^{(k)} \\ &\iff 6x_1^{(k)} + 3x_2^{(k)} \leq 3(7x_2^{(k)}) \iff x_1^{(k+1)} \leq 3x_2^{(k+1)}, \end{aligned}$$

which completes the proof.

Next, we prove that both (iii) and (iv) are equivalent to (ii). Just notice that for any $k \in \mathbb{N}^*$, we have that

$$x_2^{(k+1)} = \frac{7x_2^{(k)}}{12(x_1^{(k)} + 2x_2^{(k)})} \geq \frac{7}{60} \iff x_1^{(k)} \leq 3x_2^{(k)}.$$

and

$$\begin{aligned} x_2^{(k+1)} \geq \frac{7}{24} - \frac{1}{2}x_1^{(k+1)} &\iff \frac{7x_2^{(k)}}{12(x_1^{(k)} + 2x_2^{(k)})} \geq \frac{7}{24} - \frac{6x_1^{(k)} + 3x_2^{(k)}}{24(x_1^{(k)} + 2x_2^{(k)})} \\ &\iff x_1^{(k)} \leq 3x_2^{(k)}; \end{aligned}$$

what yields the claim. □

Remark 6.3.14. Let $\widetilde{V}$ be the operator defined in (6.3.17) and assume that the three limits $\lim_{k\to\infty} x_1^{(k)}$, $\lim_{k\to\infty} x_2^{(k)}$ and $\lim_{k\to\infty} x_3^{(k)}$ exist for any initial point $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$. Then, notice that $\lim_{k\to\infty} u^{(k)} \neq 0$. Indeed, by contrary, if $\lim_{k\to\infty} u^{(k)} = 0$ then $\lim_{k\to\infty} 6x_1^{(k)} + 3x_2^{(k)} + 4x_3^{(k)} = 0$. Then, it necessarily holds that $\lim_{k\to\infty} x_1^{(k)} = \lim_{k\to\infty} x_2^{(k)} = \lim_{k\to\infty} x_3^{(k)} = 0$, a contradiction with the fact that $x_1^{(k)} + x_2^{(k)} + x_3^{(k)} + u^{(k)} = 1$ for any $k \geq 0$.

Theorem 6.3.15. *Consider the normalised gonosomal operator* $\widetilde{V}$ *given by* (6.3.17). *Then, for any initial point* $s = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}) \in S^{3,1}$*, it holds that*

$$\lim_{k\to\infty} \widetilde{V}^k(s) = \begin{cases} (\frac{1}{2}, 0, 0, \frac{1}{2}), & \text{if } x_2^{(0)} = x_3^{(0)} = 0, \\ (\frac{7}{20}, \frac{7}{60}, \frac{7}{60}, \frac{5}{12}), & \text{otherwise.} \end{cases}$$

Proof. First, notice that if $x_2^{(0)} = x_3^{(0)} = 0$, then clearly $\widetilde{V}^n(s) = (\frac{1}{2}, 0, 0, \frac{1}{2})$ for all $n \in \mathbb{N}^*$, and so $\lim_{n\to\infty} \widetilde{V}^n(s) = (\frac{1}{2}, 0, 0, \frac{1}{2})$.

Otherwise, we can ensure that $x_2^{(k)} \neq 0$ for any $k \in \mathbb{N}^*$. Moreover, after some computations, it is easy to check that $x_2^{(k+1)} \geq x_2^{(k)}$ (resp. $x_2^{(k+1)} \leq x_2^{(k)}$) for any $k \in \mathbb{N}^*$ if and only if $x_2^{(k)} \leq \frac{7}{24} - \frac{1}{2}x_1^{(k)}$ (resp. $x_2^{(k)} \geq \frac{7}{24} - \frac{1}{2}x_1^{(k)}$) for any $k \in \mathbb{N}^*$. Then, by Proposition 6.3.13, for any $k \in \mathbb{N}^*$, we have that

$$x_2^{(k)} \begin{cases} \geq \frac{7}{60}, & \text{if } 6x_1^{(0)} \leq 6x_2^{(0)} + 12x_3^{(0)}, \\ \leq \frac{7}{60}, & \text{if } 6x_1^{(0)} \geq 6x_2^{(0)} + 12x_3^{(0)}, \end{cases}$$

and

$$x_2^{(k+1)} \begin{cases} \leq x_2^{(k)}, & \text{if } 6x_1^{(0)} \leq 6x_2^{(0)} + 12x_3^{(0)}, \\ \geq x_2^{(k)}, & \text{if } 6x_1^{(0)} \geq 6x_2^{(0)} + 12x_3^{(0)}, \end{cases}$$

Consequently, in both cases, we have that $\{x_2^{(k)}\}_{k\in\mathbb{N}^*}$ is a monotone bounded sequence; its limit exists and, moreover, is a positive number. Furthermore, since

$$x_1^{(k)} = \frac{7x_2^{(k)}}{12x_2^{(k+1)}} - 2x_2^{(k)} \quad \text{and} \quad u^{(k+1)} = \frac{6x_1^{(k)} + 7x_2^{(k)}}{12(x_1^{(k)} + 2x_2^{(k)})}$$

for any $k \in \mathbb{N}^*$, we deduce that $\{x_1^{(k)}\}_{k\in\mathbb{N}^*}$ and $\{u^{(k)}\}_{k\in\mathbb{N}^*}$ converge. As a consequence of Remark 6.3.14, if we take limits on both sides of the expressions which define the operator (6.3.17), we have that the possible limits are exactly its fixed points: $(\frac{1}{2}, 0, 0, \frac{1}{2})$ and $(\frac{7}{20}, \frac{7}{60}, \frac{7}{60}, \frac{5}{12})$. Hence, as the limit of $\{x_2^{(k)}\}_{k\in\mathbb{N}^*}$ is positive, necessarily $\lim_{k\to\infty} \widetilde{V}^k(s) = (\frac{7}{20}, \frac{7}{60}, \frac{7}{60}, \frac{5}{12})$.

□

Biological interpretation 6.3.16. For *Dicrostonyx torquatus* population, given the previous result, we can conclude that for any initial state $s \in S^{3,1}$ (the probability distribution on the set of possible genotypes $\{\text{XX}, \text{XX*}, \text{X*Y}, \text{XY}\}$), the future of the population is always stable. If there are no XX* and X*Y individuals in the initial state, the population tends to the equilibrium state $(\frac{1}{2}, 0, 0, \frac{1}{2})$, where XX and XY are distributed equally. Otherwise, the population tends to the equilibrium state $(\frac{7}{20}, \frac{7}{60}, \frac{7}{60}, \frac{5}{12})$, where the first female genotype and the male genotype are the most frequent.

6.4 A combination of XY systems and ZW systems

In this section, we describe the dynamic behaviour of some African cichlid fish populations (see [81,87]), which has not yet been modelled. These species with polygenic sex determination (see [81]) have a multi-locus system, where alleles at an XY locus on chromosome seven and a ZW locus on chromosome five segregate independently. Most importantly, the W allele overrides the Y male determiner, so ZWXY individuals are females. Hence, when a female with a ZW sex determiner is mated to a male with an XY sex determiner, they produce siblings with four possible sex classes: ZZXX, ZWXX and ZWXY females, and ZZXY males. That is,

$$\mathrm{ZWXX} \times \mathrm{ZZXY} \quad \rightarrowtail \quad \tfrac{1}{4}\mathrm{ZZXX}, \tfrac{1}{4}\mathrm{ZZXY}, \tfrac{1}{4}\mathrm{ZWXX}, \tfrac{1}{4}\mathrm{ZWXY}.$$

As shown in [87], many other genotypes and crosses with different outcomes are possible. However, we will consider a simplified version in which the only female and male genotypes are the previous ones. Moreover, to show such dominance of W over Y, we will assume that W causes the elimination of Y during gametogenesis. Hence, the remaining crosses are:

$$\begin{aligned} \mathrm{ZZXX} \times \mathrm{ZZXY} \quad &\rightarrowtail \quad \tfrac{1}{2}\mathrm{ZZXX}, \tfrac{1}{2}\mathrm{ZZXY};\ \text{and} \\ \mathrm{ZWXY} \times \mathrm{ZZXY} \quad &\rightarrowtail \quad \tfrac{1}{4}\mathrm{ZZXX}, \tfrac{1}{4}\mathrm{ZZXY}, \tfrac{1}{4}\mathrm{ZWXX}, \tfrac{1}{4}\mathrm{ZWXY}. \end{aligned}$$

In order to build the corresponding gonosomal algebra, we consider two spaces A and B, with bases $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively. Then, the space $\mathcal{A} = A \otimes B$ with basis $\{e_{(i,j)} = a_i \otimes b_j\}_{i,j\in\{1,2\}}$ and the multiplication given by

$$e_{(i,j)}e_{(k,l)} = \begin{cases} \frac{1}{4}\big(e_{(i,j)} + e_{(i,l)} + e_{(k,j)} + e_{(k,l)}\big), & \text{if } (i,j),(k,l) \neq (2,2), \\ \frac{1}{2}\big(e_{(i,1)} + e_{(2,1)}\big), & \text{otherwise}; \end{cases}$$

is a baric algebra. Next, we define

$$I = \mathsf{span}\,\{e_{(i,j)}\otimes e_{(k,l)} - e_{(k,l)}\otimes e_{(i,j)}, e_{(i,j)}\otimes e_{(k,l)} - e_{(i,l)}\otimes e_{(k,j)} : i,j,k,l \in \{1,2\}\},$$

and take the subspaces $F = \mathsf{span}\{e_{(1,1)} \otimes e_{(1,1)}, e_{(1,1)} \otimes e_{(2,1)}, e_{(1,1)} \otimes e_{(2,2)}\}$ and $M = \mathsf{span}\{e_{(1,1)} \otimes e_{(1,2)}\}$ of the quotient $(\mathcal{A} \otimes \mathcal{A})/I$. So, analogously to Construction 6.1.1, this allows us to obtain a gonosomal algebra with gonosomal basis

$\{f_1 = e_{(1,1)} \otimes e_{(1,1)}, f_2 = e_{(1,1)} \otimes e_{(2,1)}, f_3 = e_{(1,1)} \otimes e_{(2,2)}, h = e_{(1,1)} \otimes e_{(1,2)}\}$ and product given by

$$\begin{aligned}
f_1 h &= e_{(1,1)}^2 \otimes (e_{(1,1)} e_{(1,2)}) = e_{(1,1)} \otimes \left(\frac{1}{2}e_{(1,1)} + \frac{1}{2}e_{(1,2)}\right) = \frac{1}{2}f_1 + \frac{1}{2}h, \\
f_2 h &= (e_{(1,1)} e_{(2,1)}) \otimes (e_{(1,1)} e_{(1,2)}) \\
&= \left(\frac{1}{2}e_{(1,1)} + \frac{1}{2}e_{(2,1)}\right) \otimes \left(\frac{1}{2}e_{(1,1)} + \frac{1}{2}e_{(1,2)}\right) = \frac{1}{4}f_1 + \frac{1}{4}f_2 + \frac{1}{4}f_3 + \frac{1}{4}h, \\
f_3 h &= (e_{(1,1)} e_{(2,2)}) \otimes (e_{(1,1)} e_{(1,2)}) \\
&= \left(\frac{1}{2}e_{(1,1)} + \frac{1}{2}e_{(2,1)}\right) \otimes \left(\frac{1}{2}e_{(1,1)} + \frac{1}{2}e_{(1,2)}\right) = \frac{1}{4}f_1 + \frac{1}{4}f_2 + \frac{1}{4}f_3 + \frac{1}{4}h.
\end{aligned}$$

Using the coding $a_1 \leftrightarrow$ Z, $a_2 \leftrightarrow$ W, $b_1 \leftrightarrow$ X and $b_2 \leftrightarrow$ Y, we obtain the desired frequency distribution of crosses. Furthermore, the corresponding normalised gonosomal operator is given by

$$\widetilde{V}: \begin{cases} x_1' = u' = \frac{2x_1+x_2+x_3}{4(x_1+x_2+x_3)}, \\ x_2' = x_3' = \frac{x_2+x_3}{4(x_1+x_2+x_3)}. \end{cases} \tag{6.4.1}$$

Note that if $n = 3$, $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = \gamma_3 = \frac{1}{4}$, then the normalised gonosomal operator (6.3.10) correspond exactly with (6.4.1). Now, by Lemma 6.3.5, it is clear that $S^{3,1}$ is invariant with respect to $\widetilde{V}$. Moreover, as $\gamma_1 = \frac{1}{2}$ then, by Corollary 6.3.4, all nonzero fixed points are nonnegative. Consequently, Theorem 6.3.7 and Corollary 6.3.9 can be applied. Then, since $\lambda = \gamma_2 + \gamma_3 = \frac{1}{2} \geq \frac{1}{2} = \frac{n-1}{4}$, for any initial point $s = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, u^{(0)}\right) \in S^{3,1}$ it holds that

$$\lim_{k\to\infty} \widetilde{V}^k_{\frac{1}{2},\frac{1}{4},0}(s) = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right).$$

Biological interpretation 6.4.1. For the African cichlid fish population, we can conclude that for any initial state $s \in S^{3,1}$ (the probability distribution on the set of possible genotypes $\{\text{ZZXX}, \text{ZWXX}, \text{ZWXY}, \text{ZZXY}\}$), the future of the population always tends to the equilibrium state $(\frac{1}{2}, 0, 0, \frac{1}{2})$, that is, ZZXX and ZZXY individuals will survive in the same proportion, but ZWXX and ZWXY individuals will disappear in the future.

Conclusions

The results obtained in this thesis show that all the objectives initially proposed have been fully achieved. Overall, this work contributes to the development of the theory of genetic algebras, with a particular focus on evolution and gonosomal algebras. The results combine algebraic properties and methods with the study of their dynamical behaviour arising from biological models. The main conclusions of the thesis are summarised below.

C1 Although Chapter 1 may at first appear to be a comprehensive review of what is already known about evolution algebras, it actually incorporates several new contributions concerning their subalgebras (**O1**), which are subsequently used throughout the thesis. This demonstrates that the study of evolution algebras' substructures is not restricted to ideals alone, but can be meaningfully extended to subalgebras. Among other results, we highlight the following advances:

- **C1.1** We obtained several results concerning subalgebras in the solvable case. In particular, we characterised all subalgebras of evolution algebras with maximal index of nilpotency (Corollary 1.2.12), as well as all subalgebras of dimension one in the more general setting of maximal index of solvability (Proposition 1.2.9).
- **C1.2** Although evolution algebras are not closed under taking subalgebras in general, in Theorem 1.3.2 we showed that, in the regular case, every subalgebra necessarily admits a natural basis (**O2**). From our perspective, this constitutes one of the most significant results of the thesis, since it allows one to assume, without loss of generality, that every subalgebra in the regular setting has a basis consisting of vectors with disjoint supports (Corollary 1.3.4). Moreover, this result can be used to characterise the existence of codimension-one subalgebras in regular evolution algebras (Proposition 1.3.5). The results presented in this item have led to the short publication [70].
- **C1.3** We proved two conjectures formulated in [29] concerning both the structure and the classification of complete evolution algebras, a property

closely related to the structure of their subalgebras, by using elementary tools from algebraic geometry. These results are collected in the preprint [51].

C2 The relationship between an evolution algebra and the structure of its subalgebra lattice (**O3**) is developed in Chapter 2. We obtained several results on both distributivity and modularity by introducing, among other notions, concepts such as quasi-ideals or supersolvability in the setting of evolution algebras. The nilpotent case is particularly representative, since in this context both properties have been succesfully studied (Theorem 2.2.1 and Corollary 2.2.9), with the index of nilpotency playing a key role. In addition, we obtain further results on modularity in the more general framework of solvable evolution algebras (see, for instance, Corollaries 2.3.9 and 2.3.2). The results obtained in the nilpotent and solvable settings have led to the paper [68].

Finally, it is also worth noting that, beyond the solvable case, we initiate a lattice-theoretical study of regular evolution algebras, characterising modularity among the supersolvable ones over algebraically closed fields (Theorem 2.4.5).

C3 Along the same lines, Chapter 3 continues the study of subalgebras of evolution algebras, focusing in particular on the maximal ones. We show that analysing their intersection, traditionally known as the Frattini subalgebra, yields several structural results that ultimately led to the article [69]. This investigation is carried out in several stages, which we now describe:

C3.1 To facilitate the study of the Frattini subalgebra, we first revisit the notion of the nilradical in the setting of evolution algebras (**O4**). While this concept is traditionally defined as the largest nilpotent ideal, it is not well defined in the context of evolution algebras (Example 3.2.1). After establishing several preliminary results, we introduce the so-called supersolvable nilradical (Definition 3.2.15) and justify that it provides a suitable and natural replacement for the classical notion of nilradical in the theory of evolution algebras.

C3.2 In analogy with the role played by the nilradical in the study of the Frattini subalgebra and ideal for Lie and Leibniz algebras, we employ the supersolvable nilradical to obtain analogous results in the setting of evolution algebras (**O5**). In particular, these results provide criteria for the triviality of the Frattini subalgebra and the Frattini ideal (Theorems 3.3.3 and 3.3.6).

C3.3 Finally, building upon all the previous results and continuing the study of subalgebras of evolution algebras, we investigate the dually atomistic property within certain families of evolution algebras (**O6**). Recall that an evolution algebra is said to be dually atomistic if every subalgebra can be expressed as an intersection of maximal subalgebras, a notion that is closely related to the behaviour of the Frattini subalgebra and the Frattini ideal and is well established in lattice theory. After introducing the class of almost (basic) abelian evolution algebras, which had not previously been considered in the literature, we show how this property plays a key role in this setting (Theorem 3.4.3).

C4 One of the most extensively studied families of evolution algebras throughout this thesis consists of solvable nonnilpotent evolution algebras with one-dimensional derived subalgebras (the family $\mathcal{T}_{\mathbb{K}}$, see Notation 1.2.3). This class enjoys several strong structural properties and, as a consequence, is relatively easy to study. Among the most relevant features of this family, we highlight the following:

- As shown in Corollary 2.3.2, an algebra in this class is modular if and only if it is distributive, and this occurs precisely when it is isomorphic to the evolution algebra $\mathcal{E}_2(1,-1)$ defined by $e_1^2 = -e_2^2 = e_1 + e_2$.
- Corollary 2.3.3 shows that any direct sum of algebras in this family is neither modular nor distributive.
- Theorem 3.2.4 establishes that the nilradical of such algebras is well defined and provides an explicit characterisation of it.
- Theorem 3.3.1 characterises their Frattini subalgebras and ideals. More precisely, one has $F(\mathcal{E}) = \phi(\mathcal{E}) = 0$ when the annihilator has codimension two, whereas $F(\mathcal{E}) = \phi(\mathcal{E}) = \mathcal{E}^2$ otherwise.
- As a consequence of Theorem 3.4.3, an algebra in this family is dually atomistic if and only if it is again isomorphic to $\mathcal{E}_2(1,-1)$.

C5 In the first part of Chapter 4, we introduce formal deformations of evolution algebras (Definition 4.1.1) by requiring that the product of distinct elements of a natural basis remains zero. This naturally leads to an evolution algebra structure on the power series ring $\mathbb{K}[[t]]$. We then define when two deformations are equivalent and prove Theorem 4.1.10, which states that, in this case, the difference of their first-order terms has a derivation-like form (**O7**). Finally, Theorem 4.1.15 shows that every evolution algebra admits a nontrivial deformation (**O9**), in contrast with the rigidity typically found in semisimple algebras. All these results gave rise to the first part of the preprint [75].

C6 In the second part of Chapter 4, we study formal degenerations of evolution algebras (**O8**), which can be regarded as a dual procedure to formal deformations, and we establish several criteria to decide when a degeneration may exist (Proposition 4.3.7). Finally, Theorem 4.3.13 and Proposition 4.3.16 present Hasse diagrams describing several (though not necessarily all) degeneration relations among three- and four-dimensional evolution algebras. These results completed the research presented in the preprint [75].

C7 Following the review of genetic algebras modelling bisexual populations presented in Chapter 5, Chapter 6 examines several genetic systems that model bisexual populations under the common assumption of a single male genotype. More precisely, we consider all such systems appearing in [105] and construct their associated gonosomal algebras and the corresponding (normalised) gonosomal operators (**O10**). Although the study of nonlinear discrete dynamical systems is, in general, an open problem with no universal methods, in this setting, we can analyse each case individually by computing trajectories and describing the limit points of the associated operators (**O11**). This provides a complete description of the long-term behaviour of the populations across successive generations (**O12**). Moreover, the techniques developed in this chapter, mainly based on suitable bounding arguments, allow us to treat all the considered cases and may prove useful for the study of more general genetic systems in future work. Altogether, the results of this last item motivated the article [22], which has been accepted for publication and is currently in press.

Bibliography

[1] A. T. Absalamov, *Asymptotical behavior of trajectories for an evolution operator*, Uzbek Math. J. (2019), no. 4, 4–11.

[2] A. T. Absalamov, *On the eigenvalues of a gonosomal evolution operator*, Uzbek Math. J. (2020), no. 4, 4–10.

[3] A. T. Absalamov, *The global attractiveness of the fixed point of a gonosomal evolution operator*, Discontinuity, Nonlinearity, and Complexity **10** (2021), no. 1, 143–149.

[4] A. T. Absalamov and U. A. Rozikov, *The dynamics of gonosomal evolution operators*, J. Appl. Nonlinear Dyn. **9** (2020), no. 2, 247–257.

[5] J. A. Anquela, *On Herstein's theorems relating modularity in A and $A^{(+)}$*, J. Algebra **163** (1994), no. 1, 207–218.

[6] N. Bacaër, *A short history of mathematical population dynamics*, Springer-Verlag London, Ltd., 2011.

[7] D. Balavoine, *Déformations et rigidité géométrique des algèbres de Leibniz*, Comm. Algebra **24** (1996), no. 3, 1017–1034.

[8] D. W. Barnes, *Nilpotency of Lie algebras*, Math. Z. **79** (1962), 237–238.

[9] D. W. Barnes, *On the cohomology of soluble Lie algebras*, Math. Z. **101** (1967), 343–349.

[10] D. W. Barnes and H. M. Gastineau-Hills, *On the theory of soluble Lie algebras*, Math. Z. **106** (1968), 343–354.

[11] C. Batten, L. Bosko-Dunbar, A. Hedges, J. T. Hird, K. Stagg, and E. Stitzinger, *A Frattini theory for Leibniz algebras*, Comm. Algebra **41** (2013), no. 4, 1547–1557.

[12] L. Bosko, A. Hedges, J. T. Hird, N. Schwartz, and K. Stagg, *Jacobson's refinement of Engel's theorem for Leibniz algebras*, Involve **4** (2011), no. 3, 293–296.

[13] N. Boudi, Y. Cabrera Casado, and M. Siles Molina, *Natural families in evolution algebras*, Publ. Mat. **66** (2022), no. 1, 159–181.

[14] H. H. Brungs, *Rings with a distributive lattice of right ideals*, J. Algebra **40** (1976), no. 2, 392–400.

[15] D. Burde, *Degenerations of 7-dimensional nilpotent Lie algebras*, Comm. Algebra **33** (2005), no. 4, 1259–1277.

[16] D. Burde, *Contractions of Lie algebras and algebraic groups*, Arch. Math. (Brno) **43** (2007), no. 5, 321–332.

[17] D. Burde and C. Steinhoff, *Classification of orbit closures of 4-dimensional complex Lie algebras*, J. Algebra **214** (1999), no. 2, 729–739.

[18] M. D. Bustamante, P. Mellon, and M. V. Velasco, *Determining when an algebra is an evolution algebra*, Mathematics **8** (2020), no. 8, 1349.

[19] Y. Cabrera Casado, *Evolution algebras*, PhD thesis, Universidad de Málaga, 2016.

[20] Y. Cabrera Casado, P. Cadavid, and T. Reis, *Derivations and loops of some evolution algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2023), no. 3, Paper No. 119, 22 pp.

[21] Y. Cabrera Casado, M. Kanuni, and M. Siles Molina, *Basic ideals in evolution algebras*, Linear Algebra Appl. **570** (2019), 148–180.

[22] Y. Cabrera Casado, M. Ladra, and A. Pérez-Rodríguez, *Gonosomal algebras and operators associated to genetic systems with a single male genotype*, arXiv:2502.06299 (2025). To appear in Publicacions Matemàtiques.

[23] Y. Cabrera Casado, D. Martín Barquero, and C. Martín González, *Two-dimensional perfect evolution algebras over domains*, J. Algebraic Combin. **58** (2023), no. 2, 569–587.

[24] Y. Cabrera Casado, D. Martín Barquero, C. Martín González, and A. Tocino, *Connecting ideals in evolution algebras with hereditary subsets of its associated graph*, Collect. Math. **76** (2025), no. 2, 357–372.

[25] Y. Cabrera Casado, M. Siles Molina, and M. V. Velasco, *Evolution algebras of arbitrary dimension and their decompositions*, Linear Algebra Appl. **495** (2016), 122–162.

[26] Y. Cabrera Casado, M. Siles Molina, and M. V. Velasco, *Classification of three-dimensional evolution algebras*, Linear Algebra Appl. **524** (2017), 68–108.

[27] A. J. Calderón Martín, A. Fernández Ouaridi, and I. Kaygorodov, *Non-degenerate evolution algebras*, J. Pure Appl. Algebra **228** (2024), no. 6, Paper No. 107594, 26 pp.

[28] L. M. Camacho, J. R. Gómez, B. A. Omirov, and R. M. Turdibaev, *Some properties of evolution algebras*, Bull. Korean Math. Soc. **50** (2013), no. 5, 1481–1494.

[29] L. M. Camacho, A. K. Khudoyberdiyev, and B. A. Omirov, *On the property of subalgebras of evolution algebras*, Algebr. Represent. Theory **22** (2019), no. 2, 281–296.

[30] V. Camillo, *Distributive modules*, J. Algebra **36** (1975), no. 1, 16–25.

[31] R. Carles, *Rigidité dans les variétés d'algèbres de Lie et composantes irréductibles*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 25, A1223–A1226.

[32] J. M. Casas, M. Ladra, B. A. Omirov, and U. A. Rozikov, *On evolution algebras*, Algebra Colloq. **21** (2014), no. 2, 331–342.

[33] J. M. Casas, M. Ladra, and U. A. Rozikov, *A chain of evolution algebras*, Linear Algebra Appl. **435** (2011), no. 4, 852–870.

[34] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, sixth ed., Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2016.

[35] B. Commoner, *The closing circle: Nature, man, and technology*, Alfred A. Knopf, New York, 1971.

[36] R. Cordaux, A. Michel-Salzat, M. Frelon-Raimond, T. Rigaud, and D. Bouchon, *Evidence for a new feminizing wolbachia strain in the isopod armadillidium vulgare: evolutionary implications*, Heredity **93** (2004), no. 1, 78–84.

[37] C. Costoya, P. Ligouras, A. Tocino, and A. Viruel, *Regular evolution algebras are universally finite*, Proc. Amer. Math. Soc. **150** (2022), no. 3, 919–925.

[38] C. Costoya, P. Mayorga, and A. Viruel, *Permutation representations and automorphisms of evolution algebras*, Mediterr. J. Math. **22** (2025), no. 6, Paper No. 149, 16.

[39] C. Costoya, V. Muñoz, A. Tocino, and A. Viruel, *Automorphism groups of Cayley evolution algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2023), no. 2, Paper No. 82, 11.

[40] A. Elduque and A. Labra, *Evolution algebras and graphs*, J. Algebra Appl. **14** (2015), no. 7, 1550103, 10.

[41] A. Elduque and A. Labra, *On nilpotent evolution algebras*, Linear Algebra Appl. **505** (2016), 11–31.

[42] A. Elduque and A. Labra, *Evolution algebras, automorphisms, and graphs*, Linear Multilinear Algebra **69** (2021), no. 2, 331–342.

[43] I. M. H. Etherington, *Genetic algebras*, Proc. Roy. Soc. Edinburgh **59** (1939), 242–258.

[44] I. M. H. Etherington, *Duplication of linear algebras*, Proc. Edinburgh Math. Soc. (2) **6** (1941), 222–230.

[45] I. M. H. Etherington, *Non-associative algebra and the symbolism of genetics*, Proc. Roy. Soc. Edinburgh Sect. B **61** (1941), 24–42.

[46] D. Fernández-Ternero, V. M. Gómez-Sousa, and J. Núñez Valdés, *Dealing with the resolubility of evolution algebras*, Math. Comput. Sci. **16** (2022), no. 1, Paper No. 4, 8 pp.

[47] D. Fernández Ternero, V. M. Gómez Sousa, and J. Núñez-Valdés, *A characterization of associative evolution algebras*, Contemporary Mathematics **4** (2023), no. 1, 42–48.

[48] A. Fialowski and M. de Montigny, *On deformations and contractions of Lie algebras*, SIGMA Symmetry Integrability Geom. Methods Appl. **2** (2006), Paper 048, 10 pp.

[49] G. Frattini, *Intorno alla generazione dei gruppi di operazioni*, Rom. Acc. L. Rend. (4) **1** (1885), 281–285 (Italian).

[50] P. Gabriel, *Finite representation type is open*, Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Carleton Mathematical Lecture Notes, vol. 9, Carleton Univ., Ottawa, ON, 1974, Paper No. 10, 23 pp.

[51] X. García-Martínez and A. Pérez-Rodríguez, *A note on complete evolution algebras*, arXiv:2512.12418 (2025).

[52] A. G. Geĭn, *Semimodular Lie algebras*, Sib. Math. J. **17** (1976), no. 2, 189–193.

[53] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103.

[54] G. Grätzer, *General lattice theory*, second ed., Birkhäuser Verlag, Basel, 1998.

[55] F. Grunewald and J. O'Halloran, *Varieties of nilpotent Lie algebras of dimension less than six*, J. Algebra **112** (1988), no. 2, 315–325.

[56] M. Hall, Jr., *The theory of groups*, Chelsea Publishing Co., New York, 1976, Reprinting of the 1968 edition.

[57] A. S. Hegazi and H. Abdelwahab, *Nilpotent evolution algebras over arbitrary fields*, Linear Algebra Appl. **486** (2015), 345–360.

[58] P. Holgate, *Genetic algebras associated with sex linkage*, Proc. Edinburgh Math. Soc. (2) **17** (1970), 113–120.

[59] N. M. Ivanova and C. A. Pallikaros, *Degenerations of complex associative algebras of dimension three via Lie and Jordan algebras*, Adv. Group Theory Appl. **18** (2024), 41–79.

[60] J. P. Jans, *On the indecomposable representations of algebras*, Ann. of Math. (2) **66** (1957), 418–429.

[61] I. Kaygorodov, M. Khrypchenko, and P. Páez-Guillán, *The geometric classification of non-associative algebras: a survey*, Commun. Math. **32** (2024), no. 2, 185–284.

[62] A. K. Khudoyberdiyev, M. Ladra, K. K. Masutova, and B. A. Omirov, *Some irreducible components of the variety of complex $(n+1)$-dimensional Leibniz algebras*, J. Geom. Phys. **121** (2017), 228–246.

[63] A. Kirillov and Y. Neretin, *The variety A_n of n-dimensional Lie algebra structures*, Am. Math. Soc. Transl. **137** (1987), 21–30.

[64] B. Kolman, *Semi-modular Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I Math. **29** (1965), 149–163.

[65] A. N. Kolmogorov, *On the analytic methods of probability theory*, Uspekhi matematicheskikh nauk (1938), no. 5, 5–41.

[66] A. Labra, M. Ladra, and U. A. Rozikov, *An evolution algebra in population genetics*, Linear Algebra Appl. **457** (2014), 348–362.

[67] M. Ladra and U. A. Rozikov, *Evolution algebra of a bisexual population*, J. Algebra **378** (2013), 153–172.

[68] M. Ladra, P. Páez-Guillán, and A. Pérez-Rodríguez, *On the subalgebra lattice of solvable evolution algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **119** (2025), no. 3, Paper No. 83, 18 pp.

[69] M. Ladra and A. Pérez-Rodríguez, *A Frattini theory for evolution algebras*, Ric. Mat. (2025), DOI:10.1007/s11587-025-01027-y.

[70] M. Ladra and A. Pérez-Rodríguez, *Regular evolution algebras are closed under subalgebras*, C. R. Math. Acad. Sci. Paris **363** (2025), 1461–1465.

[71] M. Lincoln and D. Towers, *Frattini theory for restricted Lie algebras*, Arch. Math. (Basel) **45** (1985), no. 5, 451–457.

[72] Y. I. Lyubich, *Mathematical structures in population genetics*, Biomathematics, vol. 22, Springer-Verlag, Berlin, 1992.

[73] A. Makhlouf, *Algèbres associatives et calcul formel*, Theoret. Comput. Sci. **187** (1997), no. 1-2, 123–145.

[74] A. Makhlouf, *A comparison of deformations and geometric study of varieties of associative algebras*, Int. J. Math. Math. Sci. (2007), Art. ID 18915, 24 pp.

[75] A. Makhlouf and A. Pérez-Rodríguez, *On formal deformations and degenerations of evolution algebras*, arXiv:2512.07002 (2025).

[76] N. Maletesta, P. Páez-Guillán, and S. Siciliano, *Restricted Lie algebras having a distributive lattice of restricted subalgebras*, Linear Multilinear Algebra **69** (2021), no. 16, 3112–3120.

[77] J. Marchal, M. Acosta, M. Bullejos, R. Díaz de la Guardia, and A. Sánchez, *Sex chromosomes, sex determination, and sex-linked sequences in microtidae*, Cytogenet Genome Res **101** (2003), no. 3–4, 266–273.

[78] E. I. Marshall, *The Frattini subalgebra of a Lie algebra*, J. London Math. Soc. **42** (1967), 416–422.

[79] C. Martínez and J. Setó, *Bernstein algebras whose lattice of ideals is distributive*, Nonassociative algebra and its applications (São Paulo, 1998), Lecture Notes in Pure and Appl. Math., vol. 211, Dekker, New York, 2000, pp. 357–364.

[80] G. Mazzola, *The algebraic and geometric classification of associative algebras of dimension five*, Manuscripta Math. **27** (1979), no. 1, 81–101.

[81] E. C. Moore and R. B. Roberts, *Polygenic sex determination*, Current Biology **23** (2013), no. 12, R510–R512.

[82] F. Mukhamedov and I. Qaralleh, *On extendibility of evolution subalgebras generated by idempotents*, Mathematics **11** (2023), no. 12, Paper No. 2764, 18 pp.

[83] A. Nijenhuis and R. W. Richardson, Jr., *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc. **72** (1966), 1–29.

[84] A. Nijenhuis and R. W. Richardson, Jr., *Deformations of Lie algebra structures*, J. Math. Mech. **17** (1967), 89–105.

[85] P. Páez-Guillán, S. Siciliano, and D. A. Towers, *On the subalgebra lattice of a restricted Lie algebra*, Linear Algebra Appl. **660** (2023), 47–65.

[86] I. Paniello, *Markov evolution algebras*, Linear Multilinear Algebra **70** (2022), no. 19, 4633–4653.

[87] N. F. Parnell and J. T. Streelman, *Genetic interactions controlling sex and color establish the potential for sexual conflict in lake malawi cichlid fishes*, Heredity **110** (2012), no. 3, 239–246.

[88] M. L. Reed, *Algebraic structure of genetic inheritance*, Bull. Amer. Math. Soc. (N.S.) **34** (1997), no. 2, 107–130.

[89] T. Reis and P. Cadavid, *Derivations of evolution algebras associated to graphs over a field of any characteristic*, Linear Multilinear Algebra **70** (2022), no. 15, 2884–2897.

[90] U. A. Rozikov, *Evolution operators and algebras of sex linked inheritance*, Asia Pac. Math. Newsl. **3** (2013), no. 1, 6–11.

[91] U. A. Rozikov, *Population dynamics—algebraic and probabilistic approach*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

[92] U. A. Rozikov, S. K. Shoyimardonov, and R. Varro, *Gonosomal algebras and associated discrete-time dynamical systems*, J. Algebra **638** (2024), 153–188.

[93] U. A. Rozikov and J. P. Tian, *Evolution algebras generated by gibbs measures*, Lobachevskii J. Math. **32** (2011), no. 4, 270–277.

[94] U. A. Rozikov and R. Varro, *Dynamical systems generated by a gonosomal evolution operator*, Discontinuity, Nonlinearity, and Complexity **5** (2016), no. 2, 173–185.

[95] C. Scheiderer, *Intersections of maximal subalgebras in Lie algebras*, J. Algebra **105** (1987), no. 1, 268–270.

[96] R. Schmidt, *Subgroup lattices of groups*, De Gruyter Expositions in Mathematics, vol. 14, Walter de Gruyter & Co., Berlin, 1994.

[97] C. Seeley, *Degenerations of* 6*-dimensional nilpotent Lie algebras over* **C**, Comm. Algebra **18** (1990), no. 10, 3493–3505.

[98] S. Siciliano and D. A. Towers, *On the subalgebra lattice of a Leibniz algebra*, Comm. Algebra **50** (2022), no. 1, 255–267.

[99] J. P. Tian, *Evolution algebras and their applications*, Lecture Notes in Mathematics, vol. 1921, Springer, Berlin, 2008.

[100] J. P. Tian and P. Vojtěchovský, *Mathematical concepts of evolution algebras in non-Mendelian genetics*, Quasigroups Related Systems **14** (2006), no. 1, 111–122.

[101] D. A. Towers, *On the generators of a nilpotent non-associative algebra*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 545–550.

[102] D. A. Towers, *A Frattini theory for algebras*, Proc. London Math. Soc. (3) **27** (1973), 440–462.

[103] D. A. Towers, *Semi-modular subalgebras of a Lie algebra*, J. Algebra **103** (1986), no. 1, 202–207.

[104] N. Valenzuela and V. A. Lance, *Temperature-dependent sex determination in vertebrates*, Smithsonian Books, 2004.

[105] R. Varro, *Gonosomal algebra*, J. Algebra **447** (2016), 1–30.

[106] R. Varro, *Gonosomic algebras: an extension of gonosomal algebras*, arXiv:2510.04578 (2025).

[107] F. Veyrunes, P. Chevret, J. Catalan, R. Castiglia, J. Watson, G. Dobigny, T. J. Robinson, and J. Britton-Davidian, *A novel sex determination system in a close relative of the house mouse*, Proceedings of the Royal Society B: Biological Sciences **277** (2009), no. 1684, 1049–1056.

[108] A. Wörz-Busekros, *Algebras in genetics*, Lecture Notes in Biomathematics, vol. 36, Springer-Verlag, Berlin-New York, 1980.

[109] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Pure and Applied Mathematics, vol. 104, Academic Press, Inc., New York-London, 1982, Translated from the Russian by Harry F. Smith.

Journal permissions for reproducing publications

On the subalgebra lattice of solvable evolution algebras

[68] M. Ladra, P. Páez-Guillán, and A. Pérez-Rodríguez, *On the subalgebra lattice of solvable evolution algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **119** (2025), no. 3, Paper No. 83, 18 pp.

RightsLink

SPRINGER NATURE

On the subalgebra lattice of solvable evolution algebras

Author: Manuel Ladra et al

Publication:
Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas

Publisher: Springer Nature

Date: Jun 26, 2025

Open access policy: https://link.springer.com/article/10.1007/s13398-025-01752-x

A Frattini theory for evolution algebras

[69] M. Ladra and A. Pérez-Rodríguez, *A Frattini theory for evolution algebras*, Ric. Mat. (2025), DOI:10.1007/s11587-025-01027-y.

Regular evolution algebras are closed under subalgebras

[70] M. Ladra and A. Pérez-Rodríguez, *Regular evolution algebras are closed under subalgebras*, C. R. Math. Acad. Sci. Paris **363** (2025), 1461–1465.

A Diamond Open Access for *Comptes Rendus Mathématique*

A free publication for both readers and authors

Articles published in *Comptes Rendus Mathématique* are available on a free, immediate and permanent basis, also known as Diamond Open Access:

- Submission and publication: no submission or publication fees (Article Processing Charge or APC) are charged to authors or their institutions for the submission, review, editing and publication of their article;
- Consultation and download: no subscription fees are charged to consult the full text of articles online, nor is any financial contribution required to download articles.

Support and partners

Comptes Rendus Mathématique is published since 2020 with the centre Mersenne, a diamond open access scientific publishing infrastructure developed by Mathdoc, a support and research unit of CNRS and Université Grenoble Alpes. The centre Mersenne provides all the publishing tools and services that enable the editorial teams of *Comptes Rendus Mathématique* to manage, produce and distribute their publications.

Even if it is free for the readers as well as for the authors, the publication in open access diamond represents a significant cost for the editors. This virtuous economic model is made possible thanks to the recurrent technical, human and financial support of several institutions and research funders:

The Académie des sciences

The KOALA program of the TIB consortium (Leibniz Information Centre for Science and Technology University Library)

The Simone and Cino Del Duca Foundation

Open access policy: https://comptes-rendus.academie-sciences.fr/mathematique/page/diamond-open-access/

Gonosomal algebras and operators associated to genetic systems with a single male genotype

[22] Y. Cabrera Casado, M. Ladra, and A. Pérez-Rodríguez, *Gonosomal algebras and operators associated to genetic systems with a single male genotype*, arXiv: 2502.06299 (2025). To appear in Publicacions Matemàtiques.

50 ANYS PUBLICACIONS MATEMÀTIQUES UAB

Home | Editors | All issues | **Authors** | Ethics | Subscriptions | Alert | Search | Administration section

Publicacions Matemàtiques publishes original research papers and expository surveys in any area of Mathematics which represent a substantial contribution. Only articles with excellent reports will be accepted.

Format and Language. Authors are encouraged to prepare their papers with LaTeX and submit the PDF output by e-mail to: pubmat@uab.cat. The paper should also include the 2020 Mathematics Subject Classification which best represents the subject of the paper and a short list of keywords. The use of footnotes should be avoided.

We encourage submission of papers written in English (American or British English are accepted, but not a mixture of those). In any case, the paper should include a short informative abstract in English.

After acceptance of the article, authors will be required to provide the TeX files and, if applicable, all files for figures (preferably in EPS format).

Authors will receive page-proofs of the paper. A copy of the volume and a PDF file of the final version of the paper will be supplied free of charge.

Peer Review. Publicacions Matemàtiques follows a blind peer review process. Each submission is assessed by the Editorial Board and, if suitable, is sent to at least one external reviewer (typically two) for independent scientific evaluation. If the evaluations are inconsistent, or if for any other reason it is deemed necessary, the Editorial Board may send the article to a third evaluator. The time allocated to evaluate each manuscript depends on its length and the availability of the reviewers. The journal strives to keep the period elapsed between submission and decision to a minimum. In 2025, the average time between the reception of a paper and its publication was twenty two months, and the average time between the acceptance of a paper and its publication was eleven months.

The decision of acceptance or rejection is made by the Managing Editor taking into account the opinions of the reviewers. This may be:

- Publishable as it is (or with minor adjustments).
- Publishable after revision.
- Not accepted in the present form: there is the possibility of rewriting the paper in part or as a whole.
- Not publishable.

Authors should be aware that the time allocated for changes after receiving the reviewers' reports is of three months. After six months without any communication from the authors, the submission will be consider as withdrawn.

Periodicity. Two issues, constituting a single volume, are published each year. The first issue is published in January and the second in July.

Open Access. Publicacions Matemàtiques provides immediate open access to its content in PDF. This includes all issues from 1976.

It is understood that the authors will agree to sign a formal Consent to Publish in due course. All articles will be published with an appropriate Creative Commons License. The authors retain copyright.

Authors are not required to pay any article-processing charges (APCs).

Privacy Statement. Publicacions Matemàtiques follows the data protection policy of the UAB, in compliance with data protection laws.

Open access policy: https://mat.uab.cat/pubmat/publicacions/authors

USC UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

CAMPUS VIDA
CAMPUS DE EXCELENCIA INTERNACIONAL

The study of populations and the mechanisms regulating them is fundamental to understanding ecosystems, and analysing their evolution over time remains a central challenge in mathematics. Among the many frameworks for modelling population dynamics, this dissertation focuses on genetic algebras, a class of nonassociative algebras used to describe inheritance in genetics.

In particular, the thesis addresses two specific subclasses of genetic algebras: evolution algebras and gonosomal algebras. The former are studied from a purely algebraic perspective, with emphasis on their subalgebra lattices and deformation properties. The latter are analysed through their associated operators, whose trajectories and limit points are computed to derive the corresponding biological interpretations.