TRANSVERSAL HARD LEFSCHETZ THEOREM ON TRANSVERSELY SYMPLECTIC FOLIATIONS

JESÚS A. ÁLVAREZ LÓPEZ AND SEOUNG DAL JUNG

Abstract. We study the transversal hard Lefschetz theorem on a transversely symplectic foliation. This article extends the results of transversally symplectic flows (H.K. Pak, "Transversal harmonic theory for transversally symplectic flows", J. Aust. Math. Soc. 84 (2008), 233–245) to general transversely symplectic foliations.

1. Introduction

On a compact Riemannian manifold, the classical Hodge theory states that any cohomology class contains just one harmonic form because of $\mathcal{H}^r(M) \cong H^r(M)$, where $\mathcal{H}^r(M)$ and $H^r(M)$ are the harmonic space and de Rham cohomology group, respectively. But on sympleictic manifold, we can not define the harmonic form as in the Riemannian case.

In 1988, J.-L. Brylinski [4] introduced the notion of symplectic harmonic form on a symplectic manifold. That is, a differential form ϕ is *symplectic harmonic* if $d\phi = \delta\phi = 0$, where δ is a symplectic adjoint operator of d. He proved that on a compact Kähler manifold, any de Rham cohomology class has a symplectic harmonic form. However, this is not true on an arbitrary symplectic manifold. In fact, O. Mathieu [15] proved the following hard Lefschetz theorem on a symplectic manifold.

Theorem 1.1. *On a symplectic manifold (M, ω) of dimension $2n$, the following properties are equivalent:*

1. *Any cohomology class contains at least one harmonic form.*
2. *(Hard Lefschtez property.) For all $r \leq n$, the map $L^r : H^{n-r}(M) \to H^{n+r}(M)$ is surjective, where $L[\phi] = [\omega \wedge \phi]$ for any closed form ϕ.*

The classical Hodge theory on a Riemannian foliation was studied by F.W. Kamber and Ph. Tondeur [11] in 1988. It states that there is a canonical isomorphism

$$\mathcal{H}^r_B(\mathcal{F}) \cong H^r_B(\mathcal{F}) ,$$

where $\mathcal{H}^r_B(\mathcal{F})$ and $H^r_B(\mathcal{F})$ are the basic harmonic space and basic de Rham cohomology group, respectively. In a similar way, a version of Theorem 1.1 can be considered for transversely symplectic foliations.

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In 2008, H. Pak [17] introduced the transversely symplectic harmonic form and studied the transverse hard Lefschetz theorem on a transversally symplectic flow, obtaining the following version of Theorem 1.1 for flows.

Theorem 1.2. *Let $(\mathcal{F}, \omega)$ be a tense, transversally symplectic flow on a manifold M of dimension $2n+1$. Then the following properties are equivalent:*

(1) *Any basic cohomology class for $\mathcal{F}$ has a transversely symplectic harmonic form.*
(2) *(Transverse hard Lefschetz property.) For all $r \leq n$, the map $L^r : H_B^{n-r}(\mathcal{F}) \to H_B^{n+r}(\mathcal{F})$ is surjective, where $L[\phi] = [\omega \wedge \phi]$ for any basic closed form ϕ.*

Given a Riemannian metric on M, recall that the foliation $\mathcal{F}$ is said to be *isoparametric* (respectively, *minimal*) if its mean curvature form is basic (respectively, zero) (see Section 2). If this condition is satisfied for some Riemannian metric, then the foliation is said to be *tense* (respectively, *taut*). In 2011, L. Bak and A. Czarnecki [2, Corollary 12] extended Theorem 1.2 to taut foliations of arbitrary dimension. Recently, Y. Lin [13] has studied symplectic harmonic theory on transversely symplectic foliations by using the differential operator d_B and δ_T, which is different from δ_B, the symplectic formal adjoint of d_B (if $\mathcal{F}$ is minimal, then $\delta_T = \delta_B$). L. Bak and A. Czarnecki [2, Theorem 10] also gave another version of Theorem 1.2 using d_B and δ_T, without requiring tautness. See also the additional publications [8, 12, 14] about the transversal hard Lefschetz theorem.

In this paper, we study the transversal hard Lefschetz theorem on an arbitrary transversely symplectic foliation. That is, we extend Theorem 1.2 to an arbitrary foliation by using the modified symplectic Hodge theory. Let $d_\kappa = d_B - \frac{1}{2}\epsilon(\kappa)$ be the modified differential operator, where $\epsilon(\kappa)$ is the exterior multiplication by the mean curvature form κ, which satisfies $d_B\kappa = 0$ on a closed manifold. The symplectic adjoint of d_κ is $\delta_\kappa = \delta_B + \frac{1}{2}i(\kappa^\sharp)$, where $i(\kappa^\sharp)$ is the inner product by the symplectic dual vector field $\kappa^\sharp$ of κ. Since $d_\kappa^2 = 0$ on a closed manifold, we can define the modified basic cohomology group $H_\kappa^*(\mathcal{F}) = \ker d_\kappa / \operatorname{im} d_\kappa$, which was introduced by G. Habib and K. Richardson [7]. Let us point out that $H_\kappa^*(\mathcal{F})$ only depends on the basic cohomology class defined by κ. Also that, if $\mathcal{F}$ is isoparametric with any pair of Riemannian metrics on M, then the corresponding mean curvature forms induce the same basic cohomology class. Then our main result is the following generalization of Theorem 1.2.

Main Theorem (Cf. Theorem 4.14)**.** *Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation of codimension $2n$ on a closed manifold M^{p+2n}. If $\mathcal{F}$ is tense, then the following properties are equivalent:*

(1) *Any modified basic cohomology class contains at least one modified symplectic harmonic form.*
(2) *For any $r \leq n$, the homomorphism $L^r : H_\kappa^{n-r}(\mathcal{F}) \to H_\kappa^{n+r}(\mathcal{F})$ is surjective.*

Remark 1.3. (1) The main theorem yields Theorem 1.2 when $p = 1$ [17] and [2, Corollary 12] when $\mathcal{F}$ is minimal, respectively. Also, if $\kappa = 0$, then $\delta_k = \delta_T$, and therefore our main theorem agrees with [2, Theorem 10] and [13, Theorem 4.1] in this case.
(2) The transversal hard Lefschetz theorem on a transverse Kähler foliation was studied in [9] and [10]. Namely, it was proved that, on a transverse Kähler foliation on a compact Riemannian manifold, if the mean curvature form satisfies some condition, then the transversal hard Lefschetz property holds. From this fact, we can apply the

main theorem to any transverse Kähler foliation on a closed manifold; that is, any basic cohomology class for $\mathcal{F}$ has a transversely symplectic harmonic representative (cf. Corollary 5.3).

(3) In contrast to the case of closed symplectic manifolds, $H^*_\kappa(\mathcal{F})$ may be of infinite dimension in the Main Theorem, despite of the compactness of M. The essential ingredient of the proof is a natural structure of sl(2)-module on the space of modified transversely symplectic harmonic forms, $\ker(d_\kappa)\cap\ker(\delta_\kappa)$. This method works as well in the infinite dimensional setting.

2. Transversely symplectic foliations

Let M^{p+2n} be a smooth manifold of dimension $m = 2n+p$ with a foliation $\mathcal{F}$ of dimension p. Let Q be the distribution of rank $2n$ with a nondegenerate closed 2-form ω on Q such that $\ker\omega_x = T_x\mathcal{F}$ for any $x \in M$, that is, $T\mathcal{F} = \{ X \in TM \mid i(X)\omega = 0 \}$. Here $i(X)$ is the interior product by X. The form ω is said to be *transversely symplectic* with respect to $\mathcal{F}$. A foliation $\mathcal{F}$ is said to be *transversely symplectic foliation* if M admits a transversely symplectic form ω with respect to $\mathcal{F}$ [13]. The notation $(\mathcal{F},\omega)$ is used in this case. Trivially, the form ω is basic, that is, $\omega \in \Omega^2_B(\mathcal{F})$, where

$$\Omega^*_B(\mathcal{F}) = \{ \phi \in \Omega^*(M) \mid i(X)\phi = \theta(X)\phi = 0,\ \forall X \in \Gamma T\mathcal{F} \} .$$

Here $\theta(X)$ is the Lie derivative with respect to X. The space of basic vector fields is

$$\mathfrak{X}_B(\mathcal{F}) = \{ X \in \Gamma Q \mid [X, V] \in \Gamma T\mathcal{F},\ \forall V \in \Gamma T\mathcal{F} \} .$$

In particular, if $\mathcal{F}$ is the flow generated by a nonsingular vector field ξ such that $i(\xi)\omega = 0$, then $\mathcal{F}$ is called the *transversally symplectic flow* and denoted by $\mathcal{F}_\xi$. A transverse Kähler foliation is a transversely symplectic foliation with a basic Kähler form as a transverse symplectic form. For more examples, see [5, 13].

Let $\{v_1, \cdots, v_n, w_1, \cdots, w_n\}$ be a local basic symplectic frame of Q, i.e.,

$$\omega(v_a, v_b) = \omega(w_a, w_b) = 0 , \quad \omega(v_a, w_b) = \delta_{ab} .$$

Then ω is locally expressed as

$$\omega = \sum_{a=1}^{n} v^*_a \wedge w^*_a ,$$

where $\{v^*_a, w^*_a\}$ is the local dual basic frame, that is,

$$v^*_a(X) = \omega(X, w_a) , \quad w^*_a(X) = -\omega(X, v_a) ,$$

for any normal vector field $X \in \Gamma Q$. Any vector field $X \in \Gamma Q$ is expressed as

$$X = \sum_{a=1}^{n} \{\omega(X, w_a)v_a - \omega(X, v_a)w_a\} .$$

Let ∇ be a connection on Q. Then the *torsion vector field* τ_∇ is given by

$$\tau_\nabla = \sum_{a=1}^{n} T_\nabla(v_a, w_a) ,$$

where the torsion tensor T_∇ is defined by

$$T_\nabla(X, Y) = \nabla_X\pi(Y) - \nabla_Y\pi(X) - \pi[X, Y]$$

for any vector fields $X, Y \in \Gamma TM$. Here $\pi : TM \to Q$ is the projection. It is easy to prove that the vector field τ_∇ is well-defined; that is, it is independent to the choice of local symplectic frames of Q. A *transversely symplectic connection* ∇ on Q is one which satisfies $\nabla\omega = 0$; that is, for all $X \in \Gamma TM$ and $Y, Z \in \Gamma Q$,

$$X\omega(Y,Z) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)\ .$$

For the study of an ordinary symplectic manifold, see [3, 6].

Without loss of generality, we assume that $\mathcal{F}$ is oriented. So, given an auxiliary Riemannian metric on M with $Q = T\mathcal{F}^\perp$, there is a unique form $\chi_\mathcal{F} \in \Omega^p(M)$ whose restriction to the leaves is the volume form of the leaves, and such that with $i(Y)\chi_\mathcal{F} = 0$ for any $Y \in \Gamma Q$. The p-form $\chi_\mathcal{F}$ is called the *characteristic form* of $\mathcal{F}$. Locally, we can describe $\chi_\mathcal{F}$ as follows. Let $\{E_j\}$ $(j = 1, \cdots, p)$ be a local oriented orthonormal frame of $T\mathcal{F}$, and let $\{\alpha_j\}$ be the local 1-forms satisfying $\alpha_j(E_k) = \delta_{jk}$ and $\alpha_j(X) = 0$ for any $X \in Q$. Then $\chi_\mathcal{F} = \alpha_1 \wedge \cdots \wedge \alpha_p$; that is, $\chi_\mathcal{F}(E_1, \cdots, E_p) = 1$ and $i(Y)\chi_\mathcal{F} = 0$ for all $Y \in \Gamma Q$. Now the corresponding mean curvature form κ of $\mathcal{F}$ with respect to the auxiliary Riemannian metric is

$$\kappa = (-1)^{p+1} i(E_p) \cdots i(E_1) d\chi_\mathcal{F}\ ,$$

locally. Since $\varphi_0 := d\chi_\mathcal{F} + \kappa \wedge \chi_\mathcal{F}$ satisfies $i(E_p)\cdots i(E_1)\varphi_0 = 0$, we get $i(X_1)\cdots i(X_p)\varphi_0 = 0$ for all $X_j \in \Gamma T\mathcal{F}$. Now let

$$F^r\Omega^k := \{\phi \in \Omega^k(M) \mid i(X_1)\cdots i(X_{k-r+1})\phi = 0,\ \forall X_j \in \Gamma T\mathcal{F}\}\ .$$

Then $\varphi_0 \in F^2\Omega^{p+1}$, and we have the Rummler's formula [19]

$$d\chi_\mathcal{F} = -\kappa \wedge \chi_\mathcal{F} + \varphi_0\ . \tag{2.1}$$

Example 2.1. Let (M^{2n+1}, α) be a contact manifold of dimension $2n+1$, where α is a contact 1-form such that $\alpha \wedge (d\alpha)^n \neq 0$. Then we have the Reeb vector field ξ such that $i(\xi)\alpha = 1$ and $i(\xi)d\alpha = 0$, which determines a 1-dimensional foliation $\mathcal{F}_\alpha$, called the *contact flow*. It is trivial that the contact flow $\mathcal{F}_\alpha$ on a contact manifold (M, α) is a transversely symplectic foliation with the transverse symplectic form $\omega := d\alpha$. Moreover, $\mathcal{F}_\alpha$ is geodesible, i.e., $\kappa = i(\xi)d\chi_\mathcal{F} = 0$, where $\chi_\mathcal{F} = \alpha$. Note that $\varphi_0 = d\alpha (\neq 0)$ and $TM = T\mathcal{F}_\alpha \oplus Q$, where $Q = \ker\alpha$ (see [17]).

Example 2.2. Let (M^{2n+1}, η, Φ) be an almost cosymplectic manifold of dimension $2n+1$, that is, M admits a closed 1-form η and a closed 2-form Φ such that $\eta \wedge \Phi^n$ is a volume form on M [8]. Then we have a Reeb vector field ξ satisfying $i(\xi)\eta = 1$ and $i(\xi)\Phi = 0$. Let $\mathcal{F}_\eta$ be the flow generated by ξ, which is called the *cosymplectic flow*. Then the cosymplectic flow $\mathcal{F}_\eta$ is a transversely symplectic foliation with a transverse symplectic form Φ. The characteristic form $\chi_\mathcal{F}$ is given by $\chi_\mathcal{F} = \eta$. Since $d\eta = 0$, $\mathcal{F}_\eta$ is geodesible, i.e., $\kappa = i(\xi)d\chi_\mathcal{F} = 0$ and $\varphi_0 = 0$ on M. Moreover, since η is a closed 1-form, by the Frobenius theorem $Q = \ker\eta$ also defines a codimension 1-foliation $\mathcal{F}_\eta^\perp$ transversal to $\mathcal{F}_\eta$ (see [17]).

Proposition 2.3 (This is also proved in [18, Theorem 4.33])**.** *Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation on a closed oriented manifold. If $\mathcal{F}$ is taut, then $H_B^{2r}(\mathcal{F}) \neq 0$ for all r.*

Proof. Assume that $H_B^{2r}(\mathcal{F}) = 0$ for some r. Then $\omega^r = d\beta$ for some basic $(2r-1)$-form β. Since $\mathcal{F}$ is taut, we choose the metric such that $\kappa = 0$. So we get $d\chi_\mathcal{F} = \varphi_0$ and the normal degree of $\beta\wedge\omega^{n-r}\wedge\varphi_0$ is $2n+1$, so it is zero. Hence

$$\begin{aligned}0\neq\omega^n\wedge\chi_{\mathcal F}&=d\beta\wedge\omega^{n-r}\wedge\chi_{\mathcal F}=d(\beta\wedge\omega^{n-r}\wedge\chi_{\mathcal F})+\beta\wedge\omega^{n-r}\wedge d\chi_{\mathcal F}\\&=d(\beta\wedge\omega^{n-r}\wedge\chi_{\mathcal F})+\beta\wedge\omega^{n-r}\wedge\varphi_0=d(\beta\wedge\omega^{n-r}\wedge\chi_{\mathcal F})\ .\end{aligned}$$

Thus, by the Stokes' theorem, we get the contradiction

$$0\neq\int_M\omega^n\wedge\chi_{\mathcal F}=\int_M d(\beta\wedge\omega^{n-r}\wedge\chi_{\mathcal F})=0\ .$$ □

From Examples 2.1 and 2.2, we get the following.

Corollary 2.4. *Let $(\mathcal F,\omega)$ be a contact or cosymplectic flow on a closed manifold M^{2n+1}. Then $H^{2r}_B(\mathcal F)\neq 0$ for all $r=1,\dots,n$.*

Remark 2.5. In contrast to an ordinary symplectic manifold, the condition $H^{2r}_B(\mathcal F)\neq 0$ is not necessary for the existence of a transversely symplectic structure on a foliation. In fact, when $\mathcal F$ is Riemannian and M closed, since $\mathcal F$ is transversely oriented, it is nontaut if and only if $H^{2n}_B(\mathcal F)=0$ [1]. For example, consider the hyperbolic torus $T^3_A=T^2\times\mathbb R/\sim$, where $A\in\mathrm{SL}(2,\mathbb Z)$ and $(m,t)\sim(A(m),t+1)$ for any $m\in T^2$ and $t\in\mathbb R$. Then T^3_A has a transversely symplectic foliation $\mathcal F$ of codimension 2 such that $\mathcal F$ is nontaut and $H^2_B(\mathcal F)=0$ [10, Example 9.1].

3. Transversely symplectic harmonic forms

Let $(\mathcal F,\omega)$ be a transversely symplectic foliation. Let $\flat:Q\to Q^*$ be defined by $\flat(X)=i(X)\omega$. Since ω plays a role of a symplectic structure on the distribution Q, the map $\flat$ is an isomorphism. It is trivial that $\flat(v_a)=w_a^*$ and $\flat(w_a)=-v_a^*$. Let $\sharp=\flat^{-1}$. For any $\phi\in\Gamma Q^*$,

$$\phi=i(\phi^\sharp)\omega\ , \tag{3.1}$$

where $\phi^\sharp:=\sharp(\phi)$. From (3.1), we get

$$\omega(\phi^\sharp,\psi^\sharp)=i(\psi^\sharp)\phi \tag{3.2}$$

for all $\phi,\psi\in\Gamma Q^*$. Moreover ϕ and $\phi^\sharp$ can be locally expressed as

$$\phi=\sum_{a=1}^n\{\phi(w_a)\flat(v_a)-\phi(v_a)\flat(w_a)\}\ ,\quad \phi^\sharp=\sum_{a=1}^n\{\phi(w_a)v_a-\phi(v_a)w_a\}\ .$$

The map $\flat$ can be extended to an isomorphism $\flat:\Gamma\Lambda^rQ\to\Gamma\Lambda^rQ^*$, defined by

$$\flat(X_1\wedge\cdots\wedge X_r):=\flat(X_1)\wedge\cdots\wedge\flat(X_r)\ , \tag{3.3}$$

where $X_i\in\Gamma Q$ $(i=1,\dots,r)$. Similarly, $\sharp$ is extended to an isomorphism $\Gamma\Lambda^rQ^*\to\Gamma\Lambda^rQ$. Now let

$$\omega(\phi,\psi)=\det\big(\omega(\phi_i,\psi_j)\big)_{i,j=1,\dots,r}$$

for all r-forms $\phi=\phi_1\wedge\cdots\wedge\phi_r$ and $\psi=\psi_1\wedge\cdots\wedge\psi_r$ with $\phi_j,\psi_j\in\Gamma Q^*$ $(j=1,\dots,r)$, where $\omega(\phi_i,\psi_j)=\omega(\phi_i^\sharp,\psi_j^\sharp)$. Then

$$\omega(\phi,\psi)=(-1)^r\omega(\psi,\phi)\ ,\quad i(\phi^\sharp)\psi=\omega(\psi,\phi)\ , \tag{3.4}$$

where $i(\phi^\sharp)=i(\phi_r^\sharp)\cdots i(\phi_1^\sharp)$.

Lemma 3.1. *Let $(\mathcal{F},\omega)$ be transversely symplectic foliation on a smooth manifold M. Then $\flat$ defines an isomorphism $\mathfrak{X}_B(\mathcal{F})\cong\Omega^1_B(\mathcal{F})$.*

Proof. Let $X\in\mathfrak{X}_B(\mathcal{F})$. For any vector field $E\in\Gamma T\mathcal{F}$,

$$i(E)\flat(X)=i(E)i(X)\omega=-i(X)i(E)\omega=0\,. \tag{3.5}$$

Note that $\theta(Y)=di(Y)+i(Y)d$ and $[\theta(Y),i(Z)]=i([Y,Z])$ for any Y and Z. Hence

$$i(E)d(\flat(X))=i(E)di(X)\omega=i(E)\theta(X)\omega=\theta(X)i(E)\omega-i([X,E])\omega=0\,. \tag{3.6}$$

The last equality in (3.6) holds because $X\in\mathfrak{X}_B(\mathcal{F})$. By (3.5) and (3.6), $\flat(X)\in\Omega^1_B(\mathcal{F})$.

Conversely, let $\phi\in\Omega^1_B(\mathcal{F})$. Then there exists $X\in Q$ such that $\flat(X)=\phi$, i.e., $\phi=i(X)\omega$. For $E\in\Gamma T\mathcal{F}$, since $i(E)\omega=0$ and $d\phi$ is basic, we have

$$\begin{aligned}i([X,E])\omega&=\theta(X)i(E)\omega-i(E)\theta(X)\omega=-i(E)\theta(X)\omega\\&=-i(E)di(X)\omega=-i(E)d\phi=0\,;\end{aligned}$$

that is, $[X,E]\in\Gamma T\mathcal{F}$. Hence $X\in\mathfrak{X}_B(\mathcal{F})$. □

From Lemma 3.1 and (3.3), $\flat$ can be naturally extended to an isomorphism $\flat:\mathfrak{X}^r_B(\mathcal{F})\to\Omega^r_B(\mathcal{F})$, where $\mathfrak{X}^r_B(\mathcal{F})$ is the space of all foliated skew symmetric r-vector fields.

Now, let $T^*:\Gamma\Lambda Q^*\to\Gamma\Lambda Q^*$ denote the *symplectic adjoint operator* of any operator $T:\Gamma\Lambda Q^*\to\Gamma\Lambda Q^*$, which is given by

$$\omega(T\phi,\psi)=\omega(\phi,T^*\psi)$$

for any $\phi,\psi\in\Lambda Q^*$. The *transversal symplectic Hodge star operator* $\bar\star:\Lambda^rQ^*\to\Lambda^{2n-r}Q^*$ is defined by the formula

$$\phi\wedge\bar\star\,\psi=\omega(\phi,\psi)\nu$$

for any $\phi,\psi\in\Gamma\Lambda^rQ^*$, where $\nu=\omega^n/n!$ is the transversal volume form of $\mathcal{F}$.

Lemma 3.2. *For any $X\in\Gamma Q$, $\alpha\in\Gamma Q^*$ and $\phi\in\Gamma\Lambda^rQ^*$,*

$$(1)\ \bar\star\,\phi=i(\phi^\sharp)\nu\,,\quad(2)\ \bar\star\,\epsilon(X^\flat)\phi=(-1)^ri(X)\,\bar\star\,\phi\,,\quad(3)\ \epsilon(\alpha)^*=-i(\alpha^\sharp)\,,\quad(4)\ (\bar\star)^2=\mathrm{id}\,,$$

where $\epsilon(\alpha)=\alpha\wedge$ is the exterior product by α.

Proof. (1) This equality is proved by induction on r. For $\phi,\psi\in\Gamma Q^*$, we have

$$\phi\wedge i(\psi^\sharp)\nu=-i(\psi^\sharp)(\phi\wedge\nu)+(i(\psi^\sharp)\phi)\nu=(i(\psi^\sharp)\phi)\nu=\omega(\phi,\psi)\nu=\phi\wedge\bar\star\,\psi\,.$$

Assume that it holds for $r-1$. Let $\psi=\eta\wedge\beta$ for $\eta\in\Gamma\Lambda^{r-1}Q^*$ and $\beta\in\Gamma Q^*$. Then

$$\begin{aligned}\phi\wedge i(\psi^\sharp)\nu&=(-1)^ri(\beta^\sharp)\big(\phi\wedge i(\eta^\sharp)\nu\big)+(-1)^{r+1}i(\beta^\sharp)\phi\wedge i(\eta^\sharp)\nu\\&=(-1)^{r+1}i(\beta^\sharp)\phi\wedge i(\eta^\sharp)\nu=(-1)^{r+1}\big(i(\eta^\sharp)i(\beta^\sharp)\phi\big)\nu\\&=(-1)^{r+1}\big(i(\beta^\sharp\wedge\eta^\sharp)\phi\big)\nu=\big(i(\eta^\sharp\wedge\beta^\sharp)\phi\big)\nu\\&=\big(i(\psi^\sharp)\phi\big)\nu=\omega(\phi,\psi)\nu=\phi\wedge\bar\star\,\psi\,.\end{aligned}$$

Here, $\phi\wedge i(\eta^\sharp)\nu$ is zero because it is of degree $q+1$. The third equality holds because $i(\beta^\sharp)\phi$ is of degree $r-1$.

(2) This equality follows from (1).

(3) For any $\phi \in \Gamma\Lambda^{r-1}Q^*$ and $\psi \in \Gamma\Lambda^r Q^*$, we have

$$\begin{aligned}\omega(\epsilon(\alpha)\phi,\psi)\nu &= \epsilon(\alpha)\phi\wedge\bar\star\psi = (-1)^{r-1}\phi\wedge\epsilon(\alpha)\bar\star\psi = (-1)^{r-1}\phi\wedge\bar\star(\bar\star\epsilon(\alpha)\bar\star\psi)\\ &= \omega(\phi,(-1)^{r-1}\bar\star\epsilon(\alpha)\bar\star\psi)\nu = -\omega(\phi,i(\alpha^\sharp)\psi)\nu\ ,\end{aligned}$$

which proves (3).

(4) Note that, for any $\phi \in \Gamma\Lambda^r Q^*$ and $\psi \in \Gamma\Lambda^{q-r}Q^*$,

$$\phi\wedge\psi = \omega(\phi\wedge\psi,\nu)\nu\ .$$

Since $q = 2n$, we obtain

$$\phi\wedge(\bar\star)^2\psi = \omega(\phi,\bar\star\psi)\nu = \omega(\phi,i(\psi^\sharp)\nu)\nu = (-1)^r\omega(\psi\wedge\phi,\nu)\nu = (-1)^r\psi\wedge\phi = \phi\wedge\psi\ .$$

Hence $(\bar\star)^2 = \mathrm{id}$. □

Proposition 3.3. *Let $(\mathcal{F},\omega)$ be transversely symplectic foliation on a smooth manifold M. Then $\bar\star$ induces a homomorphism $\bar\star : \Omega^r_B(\mathcal{F}) \to \Omega^{2n-r}_B(\mathcal{F})$.*

Proof. Let $\phi \in \Omega^r_B(\mathcal{F})$. Since ν is basic, by Lemma 3.2 (1), it is trivial that $i(X)\bar\star\phi = 0$ for any $X \in \Gamma T\mathcal{F}$. On the other hand, for any $X \in \Gamma T\mathcal{F}$ and $K \in \mathfrak{X}^r_B(\mathcal{F})$, $[\theta(X), i(K)] = i(\theta(X)K)$ and $\theta(X)K \in \mathfrak{X}^r_B(\mathcal{F})$. Since ν is basic, for any $X \in \Gamma T\mathcal{F}$,

$$\theta(X)\bar\star\phi = \theta(X)i(\phi^\sharp)\nu = i(\phi^\sharp)\theta(X)\nu + i(\theta(X)\phi^\sharp)\nu = 0\ .$$

Hence $\bar\star\phi$ is basic. □

Let $d_B := d|_{\Omega^r_B(\mathcal{F})}$, and define δ_B on $\Omega^r_B(\mathcal{F})$ by

$$\delta_B\phi = (-1)^r\bar\star(d_B - \kappa\wedge)\bar\star\phi\ .$$

If $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$, then δ_B preserves the basic forms.

Proposition 3.4. *Let $(\mathcal{F},\omega)$ be a transversely symplectic foliation on a closed manifold M. If $\mathcal{F}$ is isoparametric, then*

$$\int_M \omega(d_B\phi,\psi)\mu_M = \int_M \omega(\phi,\delta_B\psi)\mu_M$$

for all basic forms $\phi \in \Omega^{r-1}_B(\mathcal{F})$ and $\psi \in \Omega^r_B(\mathcal{F})$, using the volume form $\mu_M = \nu\wedge\chi_{\mathcal{F}}$ on M.

Proof. Let $\phi \in \Omega_B^{r-1}(\mathcal{F})$ and $\psi \in \Omega_B^r(\mathcal{F})$. Since the normal degree of $\phi \wedge \bar\star\psi \wedge \varphi_0$ is $q+1$, it is zero. Hence, by the Stokes' theorem and the Rummler's formula (2.1), we have

$$\begin{aligned}\int_M \omega(d_B\phi,\psi)\mu_M &= \int_M d\phi \wedge \bar\star\psi \wedge \chi_{\mathcal{F}} \\ &= \int_M d(\phi\wedge\bar\star\psi\wedge\chi_{\mathcal{F}}) - (-1)^{r-1}\int_M \phi\wedge d\bar\star\psi\wedge\chi_{\mathcal{F}} + \int_M \phi\wedge\bar\star\psi\wedge d\chi_{\mathcal{F}} \\ &= (-1)^r\int_M \phi\wedge d\bar\star\psi\wedge\chi_{\mathcal{F}} - \int_M \phi\wedge\bar\star\psi\wedge\kappa\wedge\chi_{\mathcal{F}} + \int_M \phi\wedge\bar\star\psi\wedge\varphi_0 \\ &= (-1)^r\int_M \phi\wedge(\bar\star)^2 d\bar\star\psi\wedge\chi_{\mathcal{F}} + (-1)^{r+1}\int_M \phi\wedge(\bar\star)^2(\kappa\wedge\bar\star\psi)\wedge\chi_{\mathcal{F}} \\ &= (-1)^r\int_M \phi\wedge\bar\star(\bar\star d\bar\star - \bar\star\kappa\wedge\bar\star)\psi\wedge\chi_{\mathcal{F}} \\ &= (-1)^r\int_M \omega(\phi,\bar\star(d-\kappa\wedge)\bar\star\psi)\nu\wedge\chi_{\mathcal{F}} = \int_M \omega(\phi,\delta_B\psi)\mu_M\ .\end{aligned}$$ □

Remark 3.5. By (3.4), it is trivial that

$$\int_M \omega(\delta_B\phi,\psi)\mu_M = -\int_M \omega(\phi,d_B\psi)\mu_M\ .$$

Now, let $\delta_T = (-1)^r\bar\star d_B\bar\star$ on $\Omega_B^r(\mathcal{F})$. Since $(\bar\star)^2 = \mathrm{id}$, we have the following.

Lemma 3.6. *On $\Omega_B^r(\mathcal{F})$, we have*

$$d_B\bar\star = (-1)^r\bar\star\delta_T\ ,\quad \bar\star d_B = (-1)^r\delta_T\bar\star\ .$$

Lemma 3.7. *If $\mathcal{F}$ is isoparametric, i.e., $\kappa\in\Omega_B^1(\mathcal{F})$, then, on $\Omega_B^*(\mathcal{F})$,*

$$(1)\ \delta_B = \delta_T - i(\kappa^\sharp)\ ,\quad (2)\ \delta_T\epsilon(\kappa) = -\bar\star d_B i(\kappa^\sharp)\bar\star\ ,\quad (3)\ \epsilon(\kappa)\delta_T = -\bar\star i(\kappa^\sharp)d_B\bar\star\ .$$

In addition, if M is closed, then

$$\delta_T i(\kappa^\sharp) + i(\kappa^\sharp)\delta_T = 0\ . \tag{3.7}$$

Proof. By Lemma 3.2 (2), the equality (1) is trivial.

Let us prove (2) and (3). By Lemma 3.2 (2),

$$\begin{aligned}\delta_T\epsilon(\kappa)\phi &= (-1)^{r+1}\bar\star d_B\bar\star(\epsilon(\kappa)\phi) = -\bar\star d_B i(\kappa^\sharp)\bar\star\phi\ , \\ \epsilon(\kappa)\delta_T\phi &= (-1)^r\epsilon(\kappa)\bar\star d_B\bar\star\phi = (-1)^r\bar\star(\bar\star\epsilon(\kappa)\bar\star(d_B\bar\star\phi)) = -\bar\star i(\kappa^\sharp)d_B\bar\star\phi\ .\end{aligned}$$

On the other hand, if M is closed, then $\delta_B^2 = 0$ by Proposition 3.4. Hence

$$\delta_T i(\kappa^\sharp) + i(\kappa^\sharp)\delta_T = -\delta_B^2 = 0\ ,$$

which proves (3.7). □

By Lemma 3.7, we have the following.

Corollary 3.8. *If $\mathcal{F}$ is isoparametric, then*

$$\delta_T\epsilon(\kappa) + \epsilon(\kappa)\delta_T = -\bar\star\theta(\kappa^\sharp)\bar\star\ .$$

For later use, we prove the following about the mean curvature form.

Proposition 3.9. *If $\mathcal{F}$ is isoparametric on a closed manifold M, then $d\kappa = 0$.*

Proof. Let $\tilde{d} = d - \kappa\wedge$. Then $0 = \delta_B^2 = -\bar{\star}(\tilde{d})^2\bar{\star}$, and therefore $(\tilde{d})^2 = 0$ and $\tilde{d}(1) = -\kappa$. Hence

$$d\kappa = (\tilde{d} + \kappa\wedge)\kappa = \tilde{d}\kappa = -\tilde{d}(\tilde{d}1) = 0 \ .$$ □

4. Transversal hard Lefschetz theorem

Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation on a smooth manifold M. Now, we define the operator $L : \Gamma\Lambda^r Q^* \to \Gamma\Lambda^{r+2}Q^*$ by $L\phi = \omega \wedge \phi$ for any form $\phi \in \Gamma\Lambda^r Q^*$. Let $\Lambda = L^* : \Gamma\Lambda^r Q^* \to \Gamma\Lambda^{r-2}Q^*$ be the symplectic adjoint operator of L; that is,

$$\omega(L\phi, \psi) = \omega(\phi, \Lambda\psi) \ .$$

Lemma 4.1 (Cf. [17])**.** *Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation. Then, on $\Gamma\Lambda^r Q^*$,*

$$\Lambda = \bar{\star}\, L\, \bar{\star} = i(\omega^\sharp) \ , \quad L\,\bar{\star} = \bar{\star}\,\Lambda \ .$$

Proof. Let $\phi \in \Lambda^r Q^*$ and $\psi \in \Lambda^{r+2}Q^*$. Then

$$\omega(L\phi, \psi)\nu = L\phi \wedge \bar{\star}\psi = \phi \wedge L\,\bar{\star}\psi = \phi \wedge \bar{\star}(\bar{\star}\, L\,\bar{\star}\psi) = \omega(\phi, \bar{\star}\, L\,\bar{\star}\psi)\nu \ ,$$

which proves $\Lambda = \bar{\star}\, L\,\bar{\star}$. By Lemma 3.2, we have

$$\bar{\star}\, L\,\bar{\star}\phi = \bar{\star}(\omega \wedge \bar{\star}\phi) = i((\bar{\star}\phi)^\sharp)i(\omega^\sharp)\nu = i(\omega^\sharp)i((\bar{\star}\phi)^\sharp)\nu = i(\omega^\sharp)(\bar{\star})^2\phi = i(\omega^\sharp)\phi \ .$$

Finally, $\bar{\star}\,\Lambda = (\bar{\star})^2 L\,\bar{\star} = L\,\bar{\star}$. □

Remark 4.2. Since $\bar{\star}$ preserves the basic forms, L and Λ preserve the basic forms.

Lemma 4.3. *Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation on M. Then $[d_B, \Lambda] = \delta_T$ on $\Omega_B^*(\mathcal{F})$*

Proof. The proof is similar to the proof of [4, Theorem 2.2.1]. □

We introduce the operator $A : \Omega_B^*(\mathcal{F}) \to \Omega_B^*(\mathcal{F})$ defined by

$$A = \sum_{r=0}^{2n}(n - r)\pi_r \ , \tag{4.1}$$

where $\pi_r : \Omega_B^*(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ is the natural projection. Then we have the following [4, 20].

Lemma 4.4. *Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation on M. Then, on $\Omega_B(\mathcal{F})$:*

1. $[\Lambda, L] = A,\ [A, L] = -2L,\ [A, \Lambda] = 2\Lambda$;
2. $[L, d_B] = [\Lambda, \delta_T] = 0$, $[L, \epsilon(\kappa)] = [\Lambda, i(\kappa^\sharp)] = 0$;
3. $[A, \delta_B] = \delta_B,\ [L, \delta_T] = [A, d_B] = -d_B$; *and*
4. $[L, i(X)] = -\epsilon(X^\flat)$ *and* $[\Lambda, \epsilon(X^\flat)] = -i(X)$ *for any* $X \in \mathfrak{X}_B(\mathcal{F})$.

Proof. (1) Using Lemma 4.1, the proof is easy.

(2) Trivially, $[L, d_B] = [L, \epsilon(\kappa)] = 0$. By the definition of δ_T and Lemmas 3.2 and 4.1,

$$\Lambda\delta_T = (-1)^r\,\bar{\star}\, L d_B\,\bar{\star} = (-1)^r\,\bar{\star}\, d_B L\,\bar{\star} = (-1)^r\,\bar{\star}\, d_B\,\bar{\star}\,\Lambda = \delta_T\Lambda \ ,$$

$$\Lambda i(\kappa^\sharp) = (-1)^r\,\bar{\star}\, L\epsilon(\kappa)\,\bar{\star} = (-1)^r\,\bar{\star}\,\epsilon(\kappa) L\,\bar{\star} = (-1)^r\,\bar{\star}\,\epsilon(\kappa)\,\bar{\star}\,\Lambda = i(\kappa^\sharp)\Lambda \ ,$$

on $\Omega_B^r(\mathcal{F})$. So $[\Lambda, \delta_T] = [\Lambda, i(\kappa^\sharp)] = 0$.

(3) The proofs of $[A, \delta_B] = \delta_B$ and $[A, d_B] = -d_B$ are trivial by (4.1). By Lemma 4.3,

$$[L, \delta_T] = [\Lambda, L]d_B - d_B[\Lambda, L] = [A, d_B] .$$

(4) For any $X \in \Gamma Q$ and $\phi \in \Omega^r_B(\mathcal{F})$,

$$i(X)L\phi = i(X)(\omega \wedge \phi) = i(X)\omega \wedge \phi + Li(X)\phi ,$$

which proves the first equality. The second one follows because, by Lemmas 3.2 and 4.1, for any $\phi \in \Omega^r_B(\mathcal{F})$,

$$\begin{aligned}[\Lambda, \epsilon(X^\flat)]\phi &= \Lambda\epsilon(X^\flat)\phi - \epsilon(X^\flat)\Lambda\phi \\ &= (-1)^r \bar{\star} L(\bar{\star})^2 i(X) \bar{\star} \phi - (-1)^r \bar{\star} i(X)(\bar{\star})^2 L \bar{\star} \phi \\ &= (-1)^r \bar{\star} [L, i(X)] \bar{\star} \phi = (-1)^{r+1} \bar{\star} \epsilon(X^\flat) \bar{\star} \phi = -i(X)\phi .\end{aligned}$$

□

From Lemma 4.3, it is well known that it follows that $d_B\delta_T + \delta_T d_B = 0$. Hence

$$\Delta_B := d_B\delta_B + \delta_B d_B = -\theta(\kappa^\sharp) .$$

If $\mathcal{F}$ is minimal, then $d_B\delta_B + \delta_B d_B = 0$; that is, $\ker \Delta_B = \Omega_B(\mathcal{F})$. Thus we define the following.

Definition 4.5. A basic form ϕ is said to be a *transversely* (resp., *normally*) *symplectic harmonic form* if $d_B\phi = \delta_B\phi = 0$ (resp., $d_B\phi = \delta_T\phi = 0$).

Remark 4.6. Lemma 4.4 implies that $\{A, L, \Lambda\}$ spans the Lie algebra sl(2). Hence the space $\Omega^*_B(\mathcal{F})$ is a sl(2)-module on which A acts diagonally with only finitely many different eigenvalues. Hence we have the duality on transversely symplectic harmonic forms [17].

Let $\mathcal{H}^r_{SB}(\mathcal{F})$ (resp., $\mathcal{H}^r_{ST}(\mathcal{F})$) be the space of all transversely (resp., normally) symplectic harmonic forms on M; that is,

$$\begin{aligned}\mathcal{H}^r_{SB}(\mathcal{F}) &= \{\, \phi \in \Omega^r_B(\mathcal{F}) \mid d_B\phi = \delta_B\phi = 0 \,\} ,\\ \mathcal{H}^r_{ST}(\mathcal{F}) &= \{\, \phi \in \Omega^r_B(\mathcal{F}) \mid d_B\phi = \delta_T\phi = 0 \,\} .\end{aligned}$$

If the foliation is minimal, then $\mathcal{H}^*_{SB}(\mathcal{F}) = \mathcal{H}^*_{ST}(\mathcal{F})$.

Proposition 4.7. *Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation. Then $L^r : \mathcal{H}^{n-r}_{ST}(\mathcal{F}) \to \mathcal{H}^{n+r}_{ST}(\mathcal{F})$ is an isomorphism.*

Proof. Since $[L, d_B] = 0$ and $[L, \delta_T] = -d_B$, the proof is easy. □

Remark 4.8. By Proposition 4.7, Y. Lin [13] studied the existence of normally symplectic harmonic representatives in a basic cohomology class on a transversely symplectic foliation. But the operator δ_T is not a symplectic adjoint operator of d_B when the foliation is not minimal. So it is natural to consider the symplectic formal adjoint δ_B of d_B instead of δ_T. But we may have $[L, \delta_B] \neq d_B$ (in fact, $[L, \delta_B] = -d_B + \epsilon(\kappa)$). Hence, on an isoparametric folitaion, L^r may not preserve the transversely symplectic harmonic forms generally. But we can overcome this problem by modifying the operators d_B and δ_B as follows.

Now, we consider the modified operators

$$d_\kappa = d_B - \frac{1}{2}\epsilon(\kappa) , \quad \delta_\kappa = \delta_B + \frac{1}{2}i(\kappa^\sharp) .$$

It is trivial that, if $\mathcal{F}$ is isoparametric, then d_κ and δ_κ preserve the basic forms. By Lemmas 3.2 and 3.7, on $\Omega_B^r(\mathcal{F})$, we have

$$\delta_\kappa = \delta_T - \frac{1}{2}i(\kappa^\sharp) = (-1)^r \,\bar\star\, d_B \,\bar\star - \frac{1}{2}(-1)^r \,\bar\star\, \epsilon(\kappa)\,\bar\star = (-1)^r \,\bar\star\, d_\kappa \,\bar\star\ .$$

Proposition 4.9. *Let $(\mathcal{F},\omega)$ be a transversely symplectic foliation on a closed manifold M. If $\mathcal{F}$ is isoparametric, then δ_κ is the symplectic adjoint operator of d_κ, i.e., for all $\phi\in\Omega_B^r(\mathcal{F})$ and $\psi\in\Omega_B^{r+1}(\mathcal{F})$,*

$$\int_M \omega(d_\kappa\phi,\psi)\mu_M = \int_M \omega(\phi,\delta_\kappa\psi)\mu_M\ .$$

Proof. From Lemma 3.2 and Proposition 3.4, the proof follows easily. □

Lemma 4.10. *Let $(\mathcal{F},\omega)$ be a transversely symplectic foliation. If $\mathcal{F}$ is isoparametric, then*

$$[A,d_\kappa]=[L,\delta_\kappa] = -d_\kappa\ ,\quad [A,\delta_\kappa]=[d_\kappa,\Lambda]=\delta_\kappa\ ,\quad [L,d_\kappa]=[\Lambda,\delta_\kappa]=0\ . \tag{4.2}$$

Moreover, if M is closed, then $d_\kappa^2=\delta_\kappa^2=0$.

Proof. The proof of (4.2) follows from Lemma 4.4. From Proposition 3.9, we get $d_\kappa^2=0$, and therefore $\delta_k^2=0$ by Proposition 4.9. □

Note that $\Delta_k := d_\kappa\delta_\kappa+\delta_\kappa d_\kappa = 0$ because of $\delta_k=[d_k,\Lambda]$ by Lemma 4.10. Then $\ker\Delta_\kappa = \Omega_B(\mathcal{F})$. Hence we define the following.

Definition 4.11. A basic form ϕ is said to be a *modified symplectic harmonic form* if $d_\kappa\phi=0$ and $\delta_\kappa\phi=0$. And the *modified symplectic harmonic space* is defined by

$$\mathcal{H}_{SK}^r(\mathcal{F}) = \{\,\phi\in\Omega_B^r(\mathcal{F})\mid d_\kappa\phi=\delta_\kappa\phi=0\,\}\ .$$

Proposition 4.12. *Let $(\mathcal{F},\omega)$ be a transversely symplectic foliation on M^{p+2n}. If $\mathcal{F}$ is isoparametric, then $L^r:\mathcal{H}_{SK}^{n-r}(\mathcal{F})\to\mathcal{H}_{SK}^{n+r}(\mathcal{F})$ is an isomorphism.*

Proof. By Lemma 4.10, the operators L,Λ and A map modified symplectic harmonic forms to modified symplectic harmonic forms, and so, turn $\mathcal{H}_{SK}^*(\mathcal{F})$ into an sl(2)-module of finite A-type (for definition, see [20]). From [20, Corollary 2.6 and Corollary 2.9], the proof follows. This is similar to the proof of [13, Lemma 3.8]. □

On a closed manifold, $d_\kappa^2=0$ by Lemma 4.10. So the cohomology group $H_\kappa^r(\mathcal{F}) = \ker d_\kappa/\operatorname{im} d_\kappa$ (called as *modified basic cohomology group*) is defined on a closed manifold. See [7] for many properties of this modified basic cohomology. Let

$$\widetilde{H}_\kappa^r(\mathcal{F}) = \{\,[\phi]\in H_\kappa^r(\mathcal{F})\mid d_k\phi=\delta_k\phi=0\,\}\ .$$

A basic form ϕ is said to be *primitive* if $\Lambda\phi=0$. Then a basic form $\phi\in\Omega_B^r(\mathcal{F})$ is primitive if and only if $L^{n-r+1}\phi=0$. Let

$$P\Omega_B^r(\mathcal{F}) = \{\,\phi\in\Omega_B^r(\mathcal{F})\mid L^{n-r+1}\phi=0\,\}\ ,$$
$$PH_\kappa^r(\mathcal{F}) = \{[\phi]\in H_\kappa^r(\mathcal{F})\mid L^{n-r+1}[\phi]=0\}\ .$$

Theorem 4.13. *Let $(\mathcal{F},\omega)$ be a transversely symplectic foliation of codimension $2n$ on a closed manifold M^{p+2n}. If $\mathcal{F}$ is tense, then the following properties are equivalent:*

(1) *Any basic cohomology class contains at least one modified symplectic harmonic form, that is,* $\widetilde{H}^*_\kappa(\mathcal{F}) = H^*_\kappa(\mathcal{F})$.
(2) *For any* $r \le n$*, the homomorphism* $L^r : H^{n-r}_\kappa(\mathcal{F}) \to H^{n+r}_\kappa(\mathcal{F})$ *is surjective.*

Proof. The proof is similar to ones in [17] and [20]. In fact, assume $\widetilde{H}^*_\kappa(\mathcal{F}) = H^*_\kappa(\mathcal{F})$. So the canonical map $\mathcal{H}^r_{SK}(\mathcal{F}) \to H^r_\kappa(\mathcal{F})$ is surjective. Hence, by Proposition 4.12, it is trivial that $L^r : H^{n-r}_\kappa(\mathcal{F}) \to H^{n+r}_\kappa(\mathcal{F})$ is surjective.

Conversely, assume that for any $r \le n$, the map $L^r : H^{n-r}_\kappa(\mathcal{F}) \to H^{n+r}_\kappa(\mathcal{F})$ is surjective. By induction on r, we prove that $\widetilde{H}^r_\kappa(\mathcal{F}) = H^r_\kappa(\mathcal{F})$. First, it is trivial that $\widetilde{H}^0_\kappa(\mathcal{F}) = H^0_\kappa(\mathcal{F})$.

For $r = 1$, let $[\phi] \in H^1_\kappa(\mathcal{F})$. We have $d_k\phi = 0$ and $\delta_k\phi = [d_k, \Lambda]\phi = 0$ because $\Lambda\phi = 0$. Hence $\widetilde{H}^1_\kappa(\mathcal{F}) = H^1_\kappa(\mathcal{F})$.

Now, assume that this property holds for $s < n - r$, and let us show that

$$\widetilde{H}^{n-r}_\kappa(\mathcal{F}) = H^{n-r}_\kappa(\mathcal{F}) \; . \tag{4.3}$$

Trivially, $\widetilde{H}^{n-r}_\kappa(\mathcal{F}) \subset H^{n-r}_\kappa(\mathcal{F})$. Let $[\phi] \in H^{n-r}_\kappa(\mathcal{F})$. Since $L^{r+2} : H^{n-r-2}_\kappa(\mathcal{F}) \to H^{n+r+2}_\kappa(\mathcal{F})$ is surjective by assumption, there exists $[\psi] \in H^{n-r-2}_\kappa(\mathcal{F})$ such that $L^{r+1}[\phi] = L^{r+2}[\psi]$; that is, $L^{r+1}([\phi] - L[\psi]) = 0$. Hence $[\phi] - L[\psi] \in PH^{n-r}_\kappa(\mathcal{F})$. Since

$$[\phi] = ([\phi] - L[\psi]) + L[\psi] \; ,$$

we get

$$H^{n-r}_\kappa(\mathcal{F}) = PH^{n-r}_\kappa(\mathcal{F}) + \operatorname{im} L \; .$$

By induction and Lemma 4.10, it is well known that any class of $\operatorname{im} L$ contains a modified symplectic harmonic representative; that is, $\operatorname{im} L \subset \widetilde{H}^{n-r}_\kappa(\mathcal{F})$. Therefore it suffices to show

$$PH^{n-r}_\kappa(\mathcal{F}) \subset \widetilde{H}^{n-r}_\kappa(\mathcal{F}) \; . \tag{4.4}$$

Let $[\phi] \in PH^{n-r}_\kappa(\mathcal{F})$; that is, $L^{r+1}[\phi] = 0$. Then there exists some $\psi \in \Omega^{n+r+1}_B(\mathcal{F})$ such that

$$L^{r+1}\phi = d_\kappa\psi \; . \tag{4.5}$$

Moreover, by the representation theory of sl(2), the map $L^{r+1} : \Omega^{n-r-1}_B(\mathcal{F}) \to \Omega^{n+r+1}_B(\mathcal{F})$ is an isomorphism. Hence there exists $\eta \in \Omega^{n-r-1}_B(\mathcal{F})$ such that

$$\psi = L^{r+1}(\eta) \; . \tag{4.6}$$

From (4.5), (4.6) and Lemma 4.10, we get

$$L^{r+1}(\phi - d_\kappa\eta) = 0 \; ;$$

that is, $\Lambda(\phi - d_\kappa\eta) = 0$ [13, Lemma 3.4]. Hence

$$\delta_\kappa(\phi - d_\kappa\eta) = [d_\kappa, \Lambda](\phi - d_\kappa\eta) = 0 \; .$$

Therefore $\phi - d_\kappa\eta$ is a modified symplectic harmonic representative of $[\phi]$, showing (4.4). $\square$

5. Case of transverse Kähler foliations

In this section, we consider a transverse Kähler foliation $(\mathcal{F}, J, \omega)$ of codimension $q = 2n$ on a closed Riemannian manifol M. Here ω is a basic Kähler 2-form and J is a holonomy invariant almost complex structure on Q such that $\nabla J = 0$, where ∇ is the transversal Levi-Civita connection on Q, extended in the usual way to tensors [16].

Let $\Lambda_{\mathbb{C}} Q^*$ be the complexification of ΛQ^*. Then $\Lambda^1_{\mathbb{C}} Q^* = Q_{1,0} + Q_{0,1}$, where $Q_{1,0}$ and $Q_{0,1}$ are the eigenspaces of J with eigenvalues $-i$ and i, respectively. Write $\kappa_B = \kappa_B^{1,0} + \kappa_B^{0,1}$, with

$$\kappa_B^{1,0} = \frac{1}{2}(\kappa_B + iJ\kappa_B)\ , \quad \kappa_B^{0,1} = \overline{\kappa_B^{1,0}}\ ,$$

where κ_B is the basic part of the mean curvature form κ [1]. Note that there exists a bundle-like metric such that $\partial_B^* \kappa^{1,0} = 0$, where $d_B = \partial_B + \bar\partial_B$ on $\Omega_B^*(\mathcal{F}) \otimes \mathbb{C}$. But we do not expect that $\partial_B \kappa_B^{0,1}$ would be in general zero for any metric [10]. Hence we get the following theorem.

Theorem 5.1 (Hard Lefschetz Theorem [10])**.** *Let $(\mathcal{F}, J, \omega)$ be a transverse Kähler foliation of codimension $2n$ on a closed Riemannian manifold with compatible bundle-like metric. Suppose that the class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar\partial_B}(\mathcal{F})$ is trivial. Then the hard Lefschetz theorem holds for basic Dolbeault cohomology; that is, the map $L^r : H_B^{n-r}(\mathcal{F}) \to H_B^{n+r}(\mathcal{F})$ is an isomorphism.*

On the other hand, we have the following.

Theorem 5.2 (See [10])**.** *Let $(\mathcal{F}, J, \omega)$ be a transverse Kähler foliation of codimension $2n$ on a compact Riemannian manifold with compatible bundle-like metric. Then the following properties are equivalent:*

1. *The class $[\kappa_B] \in H^1_B(\mathcal{F})$ is trivial; that is, $(M, \mathcal{F})$ is taut.*
2. *The class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar\partial_B}(\mathcal{F})$ is trivial.*
3. *The hard Lefschetz theorem holds for basic Dolbeault cohomology.*

From Theorems 4.13, 5.1 and 5.2, we get the following corollary.

Corollary 5.3. *Let $(\mathcal{F}, J, \omega)$ be a transverse Kähler foliation of codimension $2n$ on a closed Riemannian manifold with compatible bundle-like metric. Suppose that the class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar\partial_B}(\mathcal{F})$ is trivial. Then any basic cohomology class for $\mathcal{F}$ has a transversely symplectic harmonic representative.*

Proof. Since $\left[\partial_B \kappa_B^{0,1}\right] = 0$ in $H^{1,1}_{\partial_B \bar\partial_B}(\mathcal{F})$, by Theorem 5.2, we know that $[\kappa_B] = 0$. So

$$H^*_\kappa(\mathcal{F}) = H^*_B(\mathcal{F}) = H^*_T(\mathcal{F})\ .$$

Hence by Theorems 4.13 and 5.1, any basic cohomology class has a transversely symplectic harmonic representative. □

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Departamento de Matemáticas, Facultade de Matemáticas, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain

E-mail address: `jesus.alvarez@usc.es`

Department of Mathematics, Jeju National University, Jeju 63243, Republic of Korea

E-mail address: `sdjung@jejunu.ac.kr`