Existence of positive solutions of nonlinear second order Dirichlet problems perturbed by integral boundary conditions

Alberto Cabada, Lucía López-Somoza and Mouhcine Yousfi

Abstract In this paper we study a second-order nonlinear perturbed Dirichlet problem with integral boundary conditions. We obtain the exact expression of the Green's function related to the perturbed problem in terms of the Green's function of the homogeneous Dirichlet problem. Moreover, we characterize the set of parameters where the Green's function has constant sign (which, contrary to the homogeneous case, can be either positive or negative) on its square of definition. Finally, as an application, the existence of positive solutions is derived from fixed point theory applied to related operators defined on suitable cones in Banach spaces.

1 Introduction

In the present paper, we consider the following nonlinear equations

$$u''(t) + \gamma\, u(t) + f(t, u(t)) = 0, \quad t \in I \equiv [0, 1], \tag{1}$$

and

$$u''(t) + \gamma\, u(t) - f(t, u(t)) = 0, \quad t \in I, \tag{2}$$

subject to the integral boundary conditions

$$u(0) = \delta_1 \int_0^1 u(s)\, ds, \quad u(1) = \delta_2 \int_0^1 u(s) ds. \tag{3}$$

We will study these nonlinear problems for $\gamma \le \pi^2$, $\delta_1 \ge 0$ and $\delta_2 \ge 0$.

The existence of positive solutions will depend on the regularity of the function f and the different values of the real parameters γ, δ_1 and δ_2. More concisely, if the Green's function associated to the related linear part of the considered problem is positive, then we will deduce existence of positive solutions of problem (1), (3). We will ensure the existence of solutions of problem (2), (3) when such function is negative. So, first of all, we will do an exhaustive study of the sign of the Green's function depending on the values of the parameters γ, δ_1 and δ_2.

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We mention that problem (1), (3) with either $\delta_1 = 0$ or $\delta_2 = 0$, has been studied in [4]. So our results generalize the ones obtained in that paper. The main difference with the properties of the Green's function in this general case, is that in [4] it is proved that, depending on the values of the real parameters, either the Green's function is positive on $(0, 1) \times (0, 1)$ or it changes its sign on $I \times I$. This property is, in some sense, natural and expected, because it is the situation for the homogeneous case $\delta_1 = \delta_2 = 0$.

However, as we will see on this paper, we will prove that, in addition to being positive for small enough values of $\delta_1 + \delta_2$, the Green's function related to our problem is negative on $(0, 1) \times (0, 1)$ for some large enough values of $\delta_1 + \delta_2$ with $\delta_1 > 0$ and $\delta_2 > 0$. This is a, a priori, non expected situation, but it may be interpreted by the fact that in such a case we are considering a strong perturbation of the homogeneous case.

We mention that the interest in considering this boundary conditions relies on the fact that they appear in different real phenomena such as, among others, problems of blood flow, chemical engineering, thermoelastic or population dynamics, see for instance, [9, 10, 2, 6, 7].

The paper is divided in three sections. After the introduction we will study in Section 2 the linear problem and prove the main properties of the related Green's function. Section 3 is devoted to ensure the existence of solutions of nonlinear problems (1) and (2) coupled to (3). Moreover, some examples of the applicability of the obtained results are shown at the end of the paper.

2 Linear part: Green's function

In this section, we will study the sign of the Green's function related to the linear problem

$$u''(t) + \gamma\, u(t) + \sigma(t) = 0, \quad t \in I, \tag{4}$$

coupled to (3). To this end, in what follows we do the study distinguishing three cases depending on the sign of the value γ.

First, it is very simple to verify that problem $u''(t) + \gamma\, u(t) = 0$, coupled to boundary conditions (3) has a trivial solution if and only if $\gamma = (2k\pi)^2$, $k = 1, 2, \ldots$ or

$$\delta_1 + \delta_2 = \Delta(\gamma) := \begin{cases} \frac{\sqrt{-\gamma}}{\tanh(\frac{\sqrt{-\gamma}}{2})}, & \gamma < 0, \\ 2, & \gamma = 0, \\ \frac{\sqrt{\gamma}}{\tan(\frac{\sqrt{\gamma}}{2})}, & \gamma > 0. \end{cases} \tag{5}$$

We point out that in the homogeneous case ($\delta_1 = \delta_2 = 0$) the second condition ($\delta_1 + \delta_2 = \Delta(\gamma)$) is rewritten as $\Delta(\gamma) = 0$, which is equivalent to $\gamma = (k\pi)^2$ with k odd. However, the eigenvalues of the homogeneous case are $\gamma = (k\pi)^2$, $k = 1, 2, \ldots$. Therefore, we have two cases regarding the eigenvalues of the homogeneous problem: those with k even remain eigenvalues of the perturbed problem for any value of δ_1 and δ_2, meanwhile those with k odd are eigenvalues only when $\delta_1 + \delta_2 = 0$.

As we will see, we must take this in account when obtaining the expression of the Green's function for $\gamma > 0$.

In the sequel we derive the following property of symmetry with respect to the parameters δ_1 and δ_2.

Theorem 1 *Assume that the linear problem* (4) *coupled to the homogeneous boundary conditions* $u(0) = u(1) = 0$*, has a unique solution for any* $\sigma \in C(I)$*, that is,* $\gamma \neq (k\,\pi)^2$*,* $k = 1, 2, \ldots$ *and that* $\delta_1 + \delta_2 \neq \Delta(\gamma)$*. Let* $G_{\gamma,\delta_1,\delta_2}$ *be the Green's function related to problem* (4), (3)*. Then the following symmetry property holds*

$$G_{\gamma,\delta_1,\delta_2}(t, s) = G_{\gamma,\delta_2,\delta_1}(1 - t, 1 - s), \quad \forall\, (t, s) \in I \times I. \tag{6}$$

Proof From [5, Theorem 2.6], taking $C(u) = \int_0^1 u(s)\, ds$, it follows that the Green's function of problem (4), (3) is given by the expression

$$G_{\gamma,\delta_1,\delta_2}(t,s) = G_{\gamma,0,0}(t,s) + \frac{\delta_1\, w_1(t) + \delta_2\, w_2(t)}{1-\delta_1\, C(w_1) - \delta_2\, C(w_2)} \int_0^1 G_{\gamma,0,0}(r,s)\, dr, \tag{7}$$

where $G_{\gamma,0,0}$ is the Green's function of the homogeneous Dirichlet problem

$$u''(t) + \gamma\, u(t) + \sigma(t) = 0, \quad t \in I, \quad u(0) = u(1) = 0,$$

w_1 is the unique solution of

$$u''(t) + \gamma\, u(t) = 0, \quad t \in I, \quad u(0) = 1,\ u(1) = 0,$$

and w_2 is the unique solution of

$$u''(t) + \gamma\, u(t) = 0, \quad t \in I, \quad u(0) = 0,\ u(1) = 1.$$

It is immediate to verify that $w_2(t) = w_1(1-t)$ for all $t \in I$, therefore $C(w_1) = C(w_2)$.

First, note that if u is a solution of the homogeneous problem

$$u''(t) + \gamma\, u(t) + \sigma(t) = 0,\ t \in I, \quad u(0) = u(1) = 0,$$

then the function $v(t) = u(1-t)$ satisfies

$$v''(t) + \gamma\, v(t) + \sigma(1-t) = 0,\ t \in I, \quad v(0) = v(1) = 0.$$

Thus,

$$u(1-t) = \int_0^1 G_{\gamma,0,0}(t,s)\, \sigma(1-s)\, ds = \int_0^1 G_{\gamma,0,0}(t,1-s)\, \sigma(s)\, ds.$$

As a consequence

$$u(t) = \int_0^1 G_{\gamma,0,0}(1-t,1-s)\, \sigma(s)\, ds$$

or, which is the same,

$$G_{\gamma,0,0}(t,s) = G_{\gamma,0,0}(1-t,1-s),\ \forall (t,s) \in I \times I.$$

Then, using formula (7), we infer that

$$\begin{aligned}
G_{\gamma,\delta_2,\delta_1}(1-t,1-s) &= G_{\gamma,0,0}(1-t,1-s) + \frac{\delta_2\, w_1(1-t) + \delta_1\, w_2(1-t)}{1-\delta_2\, C(w_1) - \delta_1\, C(w_2)} \int_0^1 G_{\gamma,0,0}(r,1-s)\, dr \\
&= G_{\gamma,0,0}(t,s) + \frac{\delta_1\, w_1(t) + \delta_2\, w_2(t)}{1-\delta_1\, C(w_1) - \delta_2\, C(w_2)} \int_0^1 G_{\gamma,0,0}(1-r,1-s)\, dr \\
&= G_{\gamma,0,0}(t,s) + \frac{\delta_1\, w_1(t) + \delta_2\, w_2(t)}{1-\delta_1\, C(w_1) - \delta_2\, C(w_2)} \int_0^1 G_{\gamma,0,0}(r,s)\, dr \\
&= G_{\gamma,\delta_1,\delta_2}(t,s), \quad \text{for all } (t,s) \in I \times I.
\end{aligned}$$

Remark 1 In the case of $\gamma = (k\,\pi)^2$, for some $k \in \mathbb{N}$ odd, and $\delta_1 + \delta_2 \neq 0$, we have that

$$G_{(k\pi)^2,\delta_1,\delta_2}(t,s)=\lim_{\gamma\to(k\pi)^2} G_{\gamma,\delta_1,\delta_2}(t,s) \quad \forall\ (t,s)\in I\times I.$$

So, as a direct consequence, we have that equality (6) is also true in this situation.

2.1 Case $\gamma = 0$

In this subsection, we obtain the expression of the Green's function related to the linear problem

$$u''(t)+\sigma(t)=0, \quad t\in I, \tag{8}$$

subject to the integral boundary conditions (3). Moreover, we deduce some suitable properties of such function as, among others, the range of values of δ_1 and δ_2 for which it has constant sign on $I\times I$.

Theorem 2 *Let $\delta_1+\delta_2\neq 2\ (=\Delta(0))$ and $\sigma\in C(I)$, then problem (8), (3) has a unique solution $u\in C^2(I)$, which is given by the following expression*

$$u(t)=\int_0^1 G_{0,\delta_1,\delta_2}(t,s)\,\sigma(s)ds,$$

where

$$G_{0,\delta_1,\delta_2}(t,s)=\begin{cases} s\,\frac{(1-t)(2-\delta_2-\delta_1 s)+\delta_2(1-s)t}{2-\delta_1-\delta_2}, & 0\le s\le t\le 1,\\ (1-s)\,\frac{t(2-\delta_1-\delta_2+\delta_2 s)+\delta_1 s(1-t)}{2-\delta_1-\delta_2}, & 0\le t< s\le 1.\end{cases} \tag{9}$$

Proof According to equation (7), the Green's function of problem (8), (3) is given by

$$G_{0,\delta_1,\delta_2}(t,s)=G_{0,0,0}(t,s)+\frac{\delta_1\,w_1(t)+\delta_2\,w_2(t)}{1-\delta_1\,C(w_1)-\delta_2\,C(w_2)}\int_0^1 G_{0,0,0}(t,s)\,dt, \tag{10}$$

where

$$G_{0,0,0}(t,s)=\begin{cases} s\,(1-t), & 0\le s\le t\le 1,\\ t\,(1-s), & 0\le t<s\le 1,\end{cases}$$

is the Green's function related to the homogeneous Dirichlet problem

$$u''(t)+\sigma(t)=0,\ \ t\in I,\ \ u(0)=u(1)=0,$$

$w_1(t)=1-t$ and $w_2(t)=t$, $t\in I$.

The result is obtained by taking into account that $C(w_1)=C(w_2)=\displaystyle\int_0^1 t\,dt=\frac{1}{2}$, $C(G_{0,0,0}(\cdot,s))=\frac{1}{2}s\,(1-s)$ and expression (10). □

Next, we will state two lemmas related to the properties of the Green's function that will be useful to prove the existence of a positive solution of the nonlinear problems (1), (3) and (2), (3) with $\gamma=0$.

Lemma 1 *Let G_{0,δ_1,δ_2} be the Green's function related to problem (8), (3), given by expression (9). Then, for all $\delta_1+\delta_2\neq 2\ (=\Delta(0))$ the following properties are fulfilled:*

1. *$G_{0,\delta_1,\delta_2}(t,0)=G_{0,\delta_1,\delta_2}(t,1)=0$, for all $t\in I$.*
2. *$G_{0,\delta_1,\delta_2}(t,s)$ is continuous on $I\times I$.*
3. *$G_{0,\delta_1,\delta_2}(0,s)=0$ for all $s\in(0,1)$ if and only if $\delta_1=0$.*

4. *$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(0,s) > 0$ for all $s \in (0,1)$ if and only if $\delta_1 > 0$.*
5. *$G_{0,\delta_1,\delta_2}(1,s) = 0$ for all $s \in (0,1)$ if and only if $\delta_2 = 0$.*
6. *$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(1,s) > 0$ for all $s \in (0,1)$ if and only if $\delta_2 > 0$.*
7. *$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(s,s) > 0$ for all $s \in (0,1)$ if and only if $\delta_1 \le 2$ and $\delta_2 \le 2$.*
8. *$G_{0,\delta_1,\delta_2}(t,s) > 0$ for all $t,s \in (0,1)$ if and only if $0 \le \delta_1+\delta_2 < 2$, $0 \le \delta_1 \le 2$ and $0 \le \delta_2 \le 2$.*
9. *$G_{0,\delta_1,\delta_2}(t,s) < 0$ for all $(t,s) \in (0,1)\times(0,1)$ if and only if $\delta_1+\delta_2 > 2$, $0 < \delta_1 \le 2$ and $0 < \delta_2 \le 2$.*
10. *G_{0,δ_1,δ_2} changes its sign on $(0,1)\times(0,1)$ if and only if one of the following situations holds:*
    a) $\delta_1 < 0$.
    b) $\delta_2 < 0$.
    c) $\delta_1 > 2$.
    d) $\delta_2 > 2$.
11. *If $0 \le \delta_1+\delta_2 < 2$, $0 \le \delta_1 \le 2$ and $0 \le \delta_2 \le 2$, then $G_{0,\delta_1,\delta_2}(t,s) \le \frac{1}{2(2-\delta_1-\delta_2)}$, $\forall t,s \in I$.*
12. *If $\delta_1+\delta_2 > 2$, $0 < \delta_1 \le 2$ and $0 < \delta_2 \le 2$, then $G_{0,\delta_1,\delta_2}(t,s) \ge \frac{1}{2(2-\delta_1-\delta_2)}$, $\forall t,s \in I$.*

Proof Properties 1. and 2. are immediate from the expression of G_{0,δ_1,δ_2}. Let's now prove the remaining properties:

3. It follows from the equality $G_{0,\delta_1,\delta_2}(0,s) = \delta_1 \frac{s\,(1-s)}{2-\delta_1-\delta_2}$.
4. This statement follows from the following fact

$$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(0,s) = \delta_1\, s\,(1-s), \quad \text{for all } s \in (0,1). \tag{11}$$

5. It is immediate from the fact that $G_{0,\delta_1,\delta_2}(1,s) = \delta_2 \frac{s\,(1-s)}{2-\delta_1-\delta_2}$.
6. This statement follows from previous equality.
7. The following equality is fulfilled for all $s \in (0,1)$:

$$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(s,s) = s(1-s)\,(2-\delta_2+(\delta_2-\delta_1)s). \tag{12}$$

   Therefore, $(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(s,s) > 0$ if and only if $h(s) = 2-\delta_2+(\delta_2-\delta_1)s > 0$ for all $s \in (0,1)$, which is true if and only if $\delta_1 \le 2$ and $\delta_2 \le 2$.
8. Since $G_{0,\delta_1,\delta_2}(t,s)$ is linear on t, for all $s \in I$ fixed, $G_{0,\delta_1,\delta_2}(\cdot,s)$ attains its maximum and minimum either at $t=0, t=s$ or $t=1$.
   From Property 7., we have that $G_{0,\delta_1,\delta_2}(s,s) > 0$ for all $s \in (0,1)$, if and only if $\delta_1 \le 2$, $\delta_2 \le 2$ and $\delta_1+\delta_2 < 2$.
   From Property 4., we have that $G_{0,\delta_1,\delta_2}(0,s) > 0$ for all $s \in (0,1)$, if and only if $\delta_1 > 0$ and $\delta_1+\delta_2 < 2$.
   From Property 6., we have that $G_{0,\delta_1,\delta_2}(1,s) > 0$ for all $s \in (0,1)$, if and only if $\delta_2 > 0$ and $\delta_1+\delta_2 < 2$.
   As a consequence of the three previous assertions this property holds.
9. The proof is analogous to the proof of Property 8.
10. The proof is a direct consequence of Properties 8. and 9.
11. From Property 8., we know that $G_{0,\delta_1,\delta_2}(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$. As in Property 8., we know that the maximum values will be attained at $G_{0,\delta_1,\delta_2}(s,s)$, $G_{0,\delta_1,\delta_2}(0,s)$ and/or $G_{0,\delta_1,\delta_2}(1,s)$. Now, since

$$\begin{aligned}(2-\delta_1-\delta_2)G_{0,\delta_1,\delta_2}(s,s) &= s(1-s)\,(2-\delta_2-\delta_1 s+\delta_2 s)\\ &= s(1-s)(2-\delta_1-\delta_2)+\delta_1\, s(1-s)^2+\delta_2\, s^2(1-s)\\ &\le s(1-s)(2-\delta_1-\delta_2)+\delta_1\, s(1-s)+\delta_2\, s(1-s)\\ &= 2s(1-s) \le \frac{1}{2},\end{aligned}$$

$$(2-\delta_1-\delta_2)\, G_{0,\delta_1,\delta_2}(0,s) = \delta_1 s(1-s) \leq \frac{\delta_1}{4} \leq \frac{1}{2},$$

and

$$(2-\delta_1-\delta_2)\, G_{0,\delta_1,\delta_2}(1,s) = \delta_2 s(1-s) \leq \frac{\delta_2}{4} \leq \frac{1}{2},$$

the proof is concluded.

12. Making the same argument for the negative case ($\delta_1+\delta_2 > 2$, $0 < \delta_1 \leq 2$ and $0 < \delta_2 \leq 2$) we obtain that $G_{0,\delta_1,\delta_2}(t,s) \geq \frac{1}{2(2-\delta_1-\delta_2)}$. □

Remark 2 Properties 1, 2, 3, 4, 5, 6, 7, 8, 10 and 11 in previous lemma generalize the ones given in [4, Lemma 4] for the case $\delta_1 = 0$.

We point out that properties 9 and 12 have no sense for either $\delta_1 = 0$ or $\delta_2 = 0$ and cover the new situation in which the considered Green's function is negative on $(0,1)\times(0,1)$.

Now two inequalities for the positiveness of the Green's function are derived.

Lemma 2 *Assume that* $0 < \delta_1+\delta_2 < 2\,(=\Delta(0))$, $0 < \delta_1 < 2$ *and* $0 < \delta_2 < 2$. *Let* G_{0,δ_1,δ_2} *be the Green's function related to problem (8), (3), given by expression (9). Then, there are two real constants* $c_1 \leq 1 \leq c_2$ *such that:*

$$c_1\, G_{0,\delta_1,\delta_2}(1,s) \leq G_{0,\delta_1,\delta_2}(t,s) \leq c_2\, G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all } t,s\in I. \tag{13}$$

Proof Since there exist

$$\lim_{s\to 0^+} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = t+(1-t)\frac{2-\delta_1-\delta_2}{\delta_2} > 0, \quad \text{for all } t\in I$$

and

$$\lim_{s\to 1^-} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = \frac{t(2-\delta_1)+\delta_1(1-t)}{\delta_2} > 0, \quad \text{for all } t\in I,$$

we can continuously extend the strictly positive function $\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)}$ defined on $I\times(0,1)$ to the compact set $I\times I$. Thus, we have that the continuous positive expansion $\tilde{H}_{0,\delta_1,\delta_2}(t,s)$ attains a positive maximum c_2 and a positive minimum c_1 on $I\times I$, that is

$$c_1 \leq \tilde{H}_{0,\delta_1,\delta_2}(t,s) \leq c_2, \quad \text{for all } t,\ s\in I.$$

Now, since $G_{0,\delta_1,\delta_2}(1,s) > 0$ for all $s\in(0,1)$, we deduce inequalities (13). □

Remark 3 In the above proof we can take $c_1 = \min\left\{1, \frac{\delta_1}{\delta_2}\right\}$ and $c_2 = \frac{2}{\delta_2}$. Indeed, on the one hand we have that

$$\lim_{t\to s^+} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = \frac{2+(s-1)\delta_2-\delta_1 s}{\delta_2} \leq \frac{2}{\delta_2},$$

and, on the other hand,

$$\lim_{t\to 0^+} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = \frac{\delta_1}{\delta_2}, \quad \lim_{t\to 1^-} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = 1.$$

Moreover, since $\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)}$ is a non-negative, continuous and linear function with respect to t, we deduce

$$\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} \leq \max\left\{1, \frac{\delta_1}{\delta_2}, \frac{2}{\delta_2}\right\} = \frac{2}{\delta_2},$$

and
$$\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} \geq \min\left\{1, \frac{\delta_1}{\delta_2}, \frac{2}{\delta_2}\right\} = \min\left\{1, \frac{\delta_1}{\delta_2}\right\}.$$
As a corollary of the previous result we arrive at the following one:

Corollary 1 *Assume that* $0 < \delta_1 + \delta_2 < 2\,(= \Delta(0))$, $0 < \delta_1 < 2$ *and* $0 < \delta_2 < 2$. *Let* G_{0,δ_1,δ_2} *be the Green's function related to problem (8), (3), given by expression (9). Then, it holds that:*
$$\min_{t\in I} G_{0,\delta_1,\delta_2}(t,s) \geq \frac{c_1}{c_2} \max_{t\in I} G_{0,\delta_1,\delta_2}(t,s), \quad \textit{for all } s \in I.$$

Proof From Lemma 2 we have that
$$c_1\, G_{0,\delta_1,\delta_2}(1,s) \leq G_{0,\delta_1,\delta_2}(t,s), \quad \text{for all}\, t, s \in I,$$
and
$$\frac{1}{c_2}\, G_{0,\delta_1,\delta_2}(t,s) \leq G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all}\, t, s \in I.$$
Then,
$$\min_{t\in I} G_{0,\delta_1,\delta_2}(t,s) \geq c_1\, G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all } s \in I,$$
and
$$\frac{1}{c_2} \max_{t\in I} G_{0,\delta_1,\delta_2}(t,s) \leq G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all } s \in I.$$
Therefore,
$$\min_{t\in I} G_{0,\delta_1,\delta_2}(t,s) \geq c_1\, G_{0,\delta_1,\delta_2}(1,s) \geq \frac{c_1}{c_2} \max_{t\in I} G_{0,\delta_1,\delta_2}(t,s), \quad \text{for all}\, s \in I,$$
and the result holds. □

2.2 Case $\gamma > 0$

The aim of this subsection is to study the constant sign of the Green's function related to the following linear problem
$$u''(t) + m^2\, u(t) + \sigma(t) = 0, \quad t \in I, \tag{14}$$
subject to the integral boundary conditions (3).

On the following result the expression of the corresponding Green's function is obtained.

Theorem 3 *Let* $\delta_1 + \delta_2 \neq \frac{m\cos\left(\frac{m}{2}\right)}{\sin\left(\frac{m}{2}\right)}\,(= \Delta(m^2))$, $m > 0$, $m \neq 2\,k\,\pi$, $k = 1, 2, \ldots$, *and* $\sigma \in C(I)$. *Then problem (14), (3) has a unique solution* $u \in C^2(I)$, *which is given by the expression*
$$u(t) = \int_0^1 G_{m,\delta_1,\delta_2}(t,s)\,\sigma(s)\,ds,$$
where
$$G_{m,\delta_1,\delta_2}(t,s) = \begin{cases} G^1_{m,\delta_1,\delta_2}(t,s), & 0 \leq s \leq t \leq 1, \\ G^2_{m,\delta_1,\delta_2}(t,s), & 0 \leq t < s \leq 1. \end{cases} \tag{15}$$
Here, if $m \neq k\,\pi$, $k \in \mathbb{N}$ *odd,*

$$G^1_{m,\delta_1,\delta_2}(t,s) = \frac{\csc(m)\ \sin(ms)\ \sin(m-mt)}{m} + \frac{(-1+w_1(s)+w_2(s))(\delta_1\, w_1(t)+\delta_2\, w_2(t))}{m(m-(\delta_1+\delta_2)\tan(\frac{m}{2}))},$$

and

$$G^2_{m,\delta_1,\delta_2}(t,s) = \frac{\csc(m)\ \sin(mt)\ \sin(m-ms)}{m} + \frac{(-1+w_1(s)+w_2(s))(\delta_1\, w_1(t)+\delta_2\, w_2(t))}{m(m-(\delta_1+\delta_2)\tan(\frac{m}{2}))},$$

being $w_1(t) = \csc(m)\ \sin(m(1-t))$*,* $t \in I$ *and* $w_2(t) = \csc(m)\ \sin(mt)$*,* $t \in I$*.*

In the case of $m = k\,\pi$*, for some* $k \in \mathbb{N}$ *odd, we have that*

$$G^1_{k\,\pi,\delta_1,\delta_2}(t,s) = \lim_{m \to k\,\pi} G^1_{m,\delta_1,\delta_2}(t,s)$$

and

$$G^2_{k\,\pi,\delta_1,\delta_2}(t,s) = \lim_{m \to k\,\pi} G^2_{m,\delta_1,\delta_2}(t,s).$$

Proof Consider the case $m \neq k\,\pi$, $k \in \mathbb{N}$, and let w_1 be the unique solution of

$$w_1''(t) + m^2\, w_1(t) = 0, \quad t \in I, \qquad w_1(0) = 1, \quad w_1(1) = 0,$$

and w_2 be defined as the unique solution of

$$w_2''(t) + m^2\, w_2(t) = 0, \quad t \in I, \qquad w_2(0) = 0, \quad w_2(1) = 1.$$

It is immediate to verify that

$$w_2(t) = \csc(m)\ \sin(mt),$$

and

$$w_1(t) = w_2(1-t) = \csc(m)\ \sin(m-mt).$$

It is very well-known, see [3], that the Green's function related to the homogeneous case ($\delta_1 = \delta_2 = 0$) is given by the expression

$$G_{m,0,0}(t,s) = \begin{cases} \dfrac{\csc(m)\sin(ms)\sin(m-mt)}{m}, & 0 \le s \le t \le 1, \\[2mm] \dfrac{\csc(m)\sin(m-ms)\sin(mt)}{m}, & 0 \le t < s \le 1. \end{cases}$$

On the other hand, it is not difficult to verify that $C(w_1) = C(w_2) = \frac{\tan(\frac{m}{2})}{m}$ and

$$C(G_{m,0,0}(\cdot,s)) = \frac{-1+w_1(s)+w_2(s)}{m^2}.$$

Substituting those values in (7), we obtain the result.

Finally, for $m = k\pi$ with k odd, it is easy to see using the axiomatic definition of Green's function (see [3, Definition 1.4.1]) that the expression obtained by taking limits on G^1_m and G^2_m when m tends to $k\pi$ is a Green's function of problem (14), (3).

Remark 4 We point out that in previous theorem the expression of the Green's function when $\gamma = (k\pi)^2$, with k odd and $\delta_1 + \delta_2 \neq 0\,(=\ \Delta(\gamma))$, can not been obtained from expression (10) (as for the case $\gamma = (k\pi)^2$, k even) because in this situation, since $(k\pi)^2$ is an eigenvalue of problem $u''(t) = 0$, $t \in I$, $u(0) = u(1) = 0$, we have that $G_{\gamma,0,0}$ does not exist.

In the following result we describe the points in $I \times I$ where a constant sign Green's function of problem (14), (3) can vanish.

Lemma 3 *Let $\gamma < \pi^2$ and assume that problem (14), (3) has a unique solution for any $\sigma \in C(I)$. If G_{m,δ_1,δ_2} has constant sign on $I \times I$ and vanishes at some point (t_0, s_0), then either $s_0 = 0$, $s_0 = 1$, $t_0 = 0$, $t_0 = 1$ or $t_0 = s_0$.*

Proof Let us suppose that $(t_0, s_0) \in (0, 1) \times (0, 1)$, with $t_0 > s_0$. In such a case, from the definition of Green's function, we have that $u(t) := G_{\gamma,\delta_1,\delta_2}(t, s_0)$, $t \in I$, is the unique solution of the problem

$$u''(t) + \gamma\, u(t) = 0, \quad t \in (s_0, 1], \qquad u(t_0) = u'(t_0) = 0,$$

and so $G_{\gamma,\delta_1,\delta_2}(t, s_0) = 0$ for all $t \in (s_0, 1]$.

Now, from the properties of the Green's function, we have that

$$u(1) = \delta_2 \int_0^1 u(s)\, ds = 0$$

which implies that either $u(t) = 0$ for all $t \in I$, which contradicts that $u'(s_0^-) = 1 < 0 = u'(s_0^+)$, or u changes its sign on I, which contradicts the constant sign of the Green's function.

In the case $(t_0, s_0) \in (0, 1) \times (0, 1)$, with $t_0 < s_0$, from Theorem 1 and Remark 1 we deduce that if G_{m,δ_1,δ_2} has constant sign on $I \times I$ and vanishes at (t_0, s_0), then G_{m,δ_2,δ_1} will also have constant sign on $I \times I$ and vanish at $(1 - t_0, 1 - s_0)$ with $1 - t_0 > 1 - s_0$. Therefore, we are in the previous case and the proof is concluded. □

In the sequel we deduce some properties of the Green's function G_{m,δ_1,δ_2}.

Lemma 4 *Let G_{m,δ_1,δ_2} be the Green's function related to problem (14), (3), given by the expression (15). Then for all $\delta_1 + \delta_2 \neq \frac{m}{\tan\left(\frac{m}{2}\right)}$ $(= \Delta(m^2))$, $m > 0$, $m \neq 2\,k\,\pi$, $k = 1, 2, \ldots$ (we understand that $\Delta(\pi^2)(\equiv \frac{\pi}{\tan(\pi/2)} = 0)$, the following properties hold:*

1. *$G_{m,\delta_1,\delta_2}(t, 1) = G_{m,\delta_1,\delta_2}(t, 0) = 0$, for all $t \in I$.*
2. *$G_{m,\delta_1,\delta_2}(t, s)$ is continuous at $(t, s) \in I \times I$.*
3. *$G_{m,\delta_1,\delta_2}(0, s) = 0$ for all $s \in [0, 1]$ if and only if $\delta_1 = 0$.*
4. *$\left(m - (\delta_1 + \delta_2)\tan\left(\frac{m}{2}\right)\right) G_{m,\delta_1,\delta_2}(0, s) > 0$ for all $s \in (0, 1)$ if and only if $m \in (0, \pi)$ and $\delta_1 > 0$.*
5. *$G_{m,\delta_1,\delta_2}(1, s) = 0$ for all $s \in (0, 1)$ if and only if $\delta_2 = 0$.*
6. *$\left(m - (\delta_1 + \delta_2)\tan\left(\frac{m}{2}\right)\right) G_{m,\delta_1,\delta_2}(1, s) > 0$ for all $s \in (0, 1)$ if and only if $m \in (0, \pi)$ and $\delta_2 > 0$.*
7. *If $m \in (0, \pi)$, then $\left(m - (\delta_1 + \delta_2)\tan\left(\frac{m}{2}\right)\right) G_{m,\delta_1,\delta_2}(s, s) > 0$ for all $s \in (0, 1)$, $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$.*
8. *$G_{m,\delta_1,\delta_2}(t, s) > 0$ for all $t, s \in (0, 1)$ if and only if $0 \leq \delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$, $m \in (0, \pi]$, $\delta_1 \geq 0$ and $\delta_2 \geq 0$.*
9. *$G_{m,\delta_1,\delta_2}(t, s) < 0$ for all $t, s \in (0, 1)$ if and only if $\delta_1 + \delta_2 > \frac{m}{\tan\left(\frac{m}{2}\right)}$, $0 < \delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$, $0 < \delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $m \in (0, \pi)$.*
10. *G_{m,δ_1,δ_2} changes its sign on $(0, 1) \times (0, 1)$ if and only if one the following properties is fulfilled for $m > 0$, $m \neq 2\,k\,\pi$ and $k = 1, 2, \ldots$:*

    a. *$m > \pi$,*
    b. *$\delta_1 > \frac{m}{\tan\left(\frac{m}{2}\right)}$,*
    c. *$\delta_2 > \frac{m}{\tan\left(\frac{m}{2}\right)}$,*
    d. *$\delta_1 < 0$,*
    e. *$\delta_2 < 0$.*

Proof Properties 1. and 2. are immediate. Let's now see the others:

3. Let $s \in (0,1)$, then $G_{m,\delta_1,\delta_2}(0,s)=0$ if and only if
$$\delta_1(-1+w_1(s)+w_2(s))=0.$$
It is easy to see that the function
$$r_m(s):=-1+w_1(s)+w_2(s),$$
is positive on $(0,1)$ if and only if $m \in (0,\pi]$. Moreover, it is not identically zero on $(0,1)$ for any $m \neq 2\,k\,\pi$, $k=1,2,\ldots$.
Therefore $G_{m,\delta_1,\delta_2}(0,s)=0$ if and only if $\delta_1=0$.
4. Using expression (15), we have that
$$\left(m-(\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)G_{m,\delta_1,\delta_2}(0,s)=\frac{\delta_1}{m}r_m(s)>0,$$
for all $s \in (0,1)$, if and only if $m \in (0,\pi)$ and $\delta_1>0$.

5\. 6. These properties are immediately deduced from Properties 3. and 4. and the following equality deduced from (6) and Remark 1:
$$G_{m,\delta_1,\delta_2}(1,s)=G_{m,\delta_2,\delta_1}(0,1-s),\quad \forall s \in I.$$

7. Let us define the function
$$h(s,m,\delta_1,\delta_2):=m\ \sin(m)\ \left(-m+(\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)\,G_{m,\delta_1,\delta_2}(s,s).$$
So,
$$\frac{\partial}{\partial\delta_1}h(s,m,\delta_1,\delta_2)=(1-\cos(m))\sin(m-ms)>0,$$
and
$$\frac{\partial}{\partial\delta_2}h(s,m,\delta_1,\delta_2)=2\sin(ms)\sin^2\left(\frac{1}{2}(m-ms)\right)>0,$$
for all $s \in (0,1)$ and $m \in (0,\pi]$.
Moreover,
$$h\left(s,m,\frac{m}{\tan\left(\frac{m}{2}\right)},\frac{m}{\tan\left(\frac{m}{2}\right)}\right)=-4\sin^2\left(\frac{ms}{2}\right)\ \sin^2\left(\frac{m(1-s)}{2}\right)<0,$$
for all $s \in (0,1)$ and $m \in (0,\pi]$.
Therefore, $h(s,m,\delta_1,\delta_2)<0$ for all $s \in (0,1)$, $m \in (0,\pi]$, $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$, and the property is fulfilled.
8. From Lemma 3, for any $s \in (0,1)$ fixed, if function $G_{m,\delta_1,\delta_2}(\cdot,s)$ has constant sign, then it attains its maximum at minimum values at $t=0$, $t=s$ or $t=1$.
From Property 4., we have that $G_{m,\delta_1,\delta_2}(0,s)>0$ for all $s \in (0,1)$, if and only if $\delta_1+\delta_2<\frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_1>0$.
From Property 6., we have that $G_{0,\delta_1,\delta_2}(1,s)>0$ for all $s \in (0,1)$, if and only if $\delta_1+\delta_2<\frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2>0$.
From Property 7. we have that if $m \in (0,\pi]$ and $\delta_1+\delta_2<\frac{m}{\tan\left(\frac{m}{2}\right)}$ then $G_{m,\delta_1,\delta_2}(s,s)>0$ for all $s \in (0,1)$, provided that $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$.
Notice that if $\delta_1>0$ and $\delta_2>0$, the fact that $\delta_1+\delta_2<\frac{m}{\tan\left(\frac{m}{2}\right)}$ implies that $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$. So, we deduce that $G_{m,\delta_1,\delta_2}(t,s)>0$ for all $(t,s) \in (0,1)\times(0,1)$ for such values of the parameters.

If $G_{m,\delta_1,\delta_2}(s,s)$ is positive for other range of values we have that either $G_{m,\delta_1,\delta_2}(0,s)$ or $G_{m,\delta_1,\delta_2}(1,s)$ will be negative, and Property 8. holds.

9. The proof in this case is analogous to the previous one.
10. Arguing as in proof of [3, Theorems 1.8.5 and 1.8.6], one can verify that for any fixed values of t, s, δ_1 and δ_2, we have that $G_{m,\delta_1,\delta_2}(t,s)$ is monotone increasing with respect to m on the intervals of m where such function has constant sign on $I \times I$.
Thus, let $\delta_1 > 0$ and $\delta_2 > 0$ be such that $\delta_1 + \delta_2 < \frac{m_1}{\tan\left(\frac{m_1}{2}\right)}$ for some $m_1 \in (0,\pi)$, as a consequence, since function $\Delta(m^2) = \frac{m}{\tan\left(\frac{m}{2}\right)}$ is strictly positive and strictly monotone decreasing on $m \in (0,\pi)$, we have that there is a unique value $m_0 \in (m_1,\pi)$ such that
$$\delta_1 + \delta_2 = \frac{m_0}{\tan\left(\frac{m_0}{2}\right)} = \Delta(m_0^2)$$
and
$$0 < \delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)} \quad \text{for all } m \in (0,m_0).$$
Thus $G_{m,\delta_1,\delta_2}(t,s) > 0$ on $(0,1)\times(0,1)$ for all $m \in (0,m_0)$.
Therefore, if $\delta_1 > 0$ and $\delta_2 > 0$, we have that there is $m_2 \in (m_0,\pi)$ such that $\delta_1 \le \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \le \frac{m}{\tan\left(\frac{m}{2}\right)}$ for all $m \in (m_0,m_2]$. In consequence, $G_{m,\delta_1,\delta_2}(t,s) < 0$ on $(0,1)\times(0,1)$ if and only if $m \in (m_0,m_2]$ (See Figure 1).
For $m > m_2$ we have that G_{m,δ_1,δ_2} must change its sign on $I \times I$, on the contrary it contradicts the monotone increasing property of this function with respect to m.
In the case of $\delta_1 = 0$, we will be have $\delta_2 = \frac{m_0}{\tan\left(\frac{m_0}{2}\right)}$ and so, the Green's function G_{m,δ_1,δ_2} can never be negative on $(0,1)\times(0,1)$ and must change its sign for all $m > m_0$. The same holds for $\delta_2 = 0$ (See Figure 2). Both cases have been considered in [4]. □

Remark 5 We mention that Properties 1, 2, 5, 6, 8 and 10 are a generalization of the ones given in [4, Lemma 8]. Despite this, it is important to mention that the proof of Property 10 here is completely different to the simpler one given in that reference.

The rest of the cases have no sense for that situation.

Now we deduce some inequalities on the Green's function analogous to the ones given in Lemma 2.

Lemma 5 *Let* $0 < m < \pi$, $0 < \delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$ $(= \Delta(m^2))$ *and* G_{m,δ_1,δ_2} *be the Green's function of problem (14), (3) given by expression (15). Then, there are real constants* $c_3 > 0$ *and* $c_4 > 0$ *such that:*
$$c_3\, G_{m,\delta_1,\delta_2}(1,s) \le G_{m,\delta_1,\delta_2}(t,s) \le c_4\, G_{m,\delta_1,\delta_2}(1,s), \qquad \textit{for all } t,s \in I. \tag{16}$$

Proof If $s = 0$ or $s = 1$ the result follows from Lemma 4. Let then $t \in I$ be arbitrarily set.

On the one hand, we have that
$$\lim_{s\to 0^+} \frac{G_{m,\delta_1,\delta_2}(t,s)}{G_{m,\delta_1,\delta_2}(1,s)} = \frac{w_1(t)\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)}{\delta_2 \csc(m)\,(1-\cos(m))} + \frac{\delta_1\, w_1(t) + \delta_2\, w_2(t)}{\delta_2} > 0,$$
and on the other hand, since
$$w_1'(1) + w_2'(1) < 0$$
we have that

$$\lim_{s\to 1^-} \frac{G_{m,\delta_1,\delta_2}(t,s)}{G_{m,\delta_1,\delta_2}(1,s)} = \frac{-w_1(t)\left(m-(\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)}{\delta_2(w_1'(1)+w_2'(1))} + \frac{\delta_1\, w_1(t)+\delta_2\, w_2(t)}{\delta_2} > 0.$$

Taking into account Properties 6. and 7. of Lemma 4, we have that the strictly positive function $\frac{G_{m,\delta_1,\delta_2}(\cdot,s)}{G_{m,\delta_1,\delta_2}(1,s)}$ defined on $I\times(0,1)$ extends continuously to the compact set $I\times I$. Therefore, there exists $c_3>0$ and $c_4>0$, the minimum and the maximum, respectively, of such extension on $I\times I$. Therefore the inequalities (16) are fulfilled and we conclude the proof. □

Corollary 2 *Let $0<m<\pi$, $0<\delta_1+\delta_2<\frac{m}{\tan(\frac{m}{2})}$ $(=\Delta(m^2))$ and G_{m,δ_1,δ_2} be the Green's function of problem (14), (3) given by expression (15). Then, it holds that:*

$$\min_{t\in I} G_{m,\delta_1,\delta_2}(t,s) \ge \frac{c_3}{c_4} \max_{t\in I} G_{m,\delta_1,\delta_2}(t,s), \quad \textit{for all } t,s\in I.$$

Proof The proof is analogous to the one made in Corollary 1. □

2.3 Case $\gamma < 0$

Finally, we deal in this subsection with the study of the linear problem

$$u''(t) - m^2\, u(t) + \sigma(t) = 0, \quad t\in I, \tag{17}$$

subject to the integral boundary conditions (3).

For the construction of the Green's function we obtain the following result:

Theorem 4 *Let $\delta_1+\delta_2 \neq \frac{m}{\tanh(\frac{m}{2})}$ $(=\Delta(-m^2))$ and $\sigma\in C(I)$, then problem (17), (3) has a unique solution $u\in C^2(I)$, which is given by the expression*

$$u(t) = \int_0^1 G_{m,\delta_1,\delta_2}(t,s)\,\sigma(s)ds,$$

where

$$G_{m,\delta_1,\delta}(t,s) = \begin{cases} G^1_{m,\delta_1,\delta_2}(t,s), & 0\le s\le t\le 1,\\ G^2_{m,\delta_1,\delta_2}(t,s), & 0\le t< s\le 1,\end{cases} \tag{18}$$

with

$$G^1_{m,\delta_1,\delta_2}(t,s) = \frac{\sinh(ms)\,\sinh(m-mt)}{m\sinh(m)} - \frac{(-1+w_1(s)+w_2(s))(\delta_1\, w_1(t)+\delta_2\, w_2(t))}{m(m-(\delta_1+\delta_2)\tanh(m/2))},$$

and

$$G^2_{m,\delta_1,\delta_2}(t,s) = \frac{\sinh(mt)\,\sinh(m-ms)}{m\sinh(m)} - \frac{(-1+w_1(s)+w_2(s))(\delta_1\, w_1(t)+\delta_2\, w_2(t))}{m(m-(\delta_1+\delta_2)\tanh(\frac{m}{2}))},$$

being $w_1(t)=\frac{\sinh(m-mt)}{\sinh(m)}$, $t\in I$ and $w_2(t)=\frac{\sinh(mt)}{\sinh(m)}$, $t\in I$.

Proof It is proved analogously to Theorem 3 taking into account that in this case $w_1(t)=\frac{\sinh(m-mt)}{\sinh(m)}$, $w_2(t)=\frac{\sinh(mt)}{\sinh(m)}$, $C(w_1)=C(w_2)=\frac{\tanh(\frac{m}{2})}{m}$ and

$$G_{m,0,0}(t,s)=\begin{cases}\frac{\sinh(ms)\,\sinh(m-mt)}{m\sinh(m)}, & 0\le s\le t\le 1,\\ \frac{\sinh(mt)\,\sinh(m-ms)}{m\sinh(m)}, & 0\le t<s\le 1,\end{cases}$$

with

$$C(G_{m,0,0}(\cdot,s))=-\frac{-1+w_1(s)+w_2(s)}{m^2}.$$

We will now state the following results that characterize the constant sign of the Green's function on $I\times I$. We omit the proofs because they are analogous to the previous case.

Lemma 6 *Let G_{m,δ_1,δ_2} be the Green's function associated with problem (17), (3), given by expression (18). Then for all $\delta_1+\delta_2\neq\frac{m}{\tanh\left(\frac{m}{2}\right)}$ $(=\Delta(-m^2))$, $m>0$, the following properties hold:*

1. $G_{m,\delta_1,\delta_2}(t,0)=G_{m,\delta_1,\delta_2}(t,1)=0$, *for all $t\in I$.*
2. $G_{m,\delta_1,\delta_2}(t,s)$ *is continuous on $I\times I$.*
3. $G_{m,\delta_1,\delta_2}(0,s)=0$, *for all $s\in(0,1)$ if and only if $\delta_1=0$.*
4. $\left(m-(\delta_1+\delta_2)\tanh\left(\frac{m}{2}\right)\right)G_{m,\delta_1,\delta_2}(0,s)>0$ *for all $s\in(0,1)$ and $m>0$ if and only if $\delta_1>0$.*
5. $G_{m,\delta_1,\delta_2}(1,s)=0$, *for all $s\in(0,1)$ if and only if $\delta_2=0$.*
6. $\left(m-(\delta_1+\delta_2)\tanh\left(\frac{m}{2}\right)\right)G_{m,\delta_1,\delta_2}(1,s)>0$ *for all $s\in(0,1)$ and $m>0$ if and only if $\delta_2>0$.*
7. $\left(m-(\delta_1+\delta_2)\tanh\left(\frac{m}{2}\right)\right)G_{m,\delta_1,\delta_2}(s,s)>0$ *for all $s\in(0,1)$, $m>0$, $\delta_1\le\frac{m}{\tanh\left(\frac{m}{2}\right)}$ and $\delta_2\le\frac{m}{\tanh\left(\frac{m}{2}\right)}$.*
8. $G_{m,\delta_1,\delta_2}(t,s)>0$ *for all $t,s\in(0,1)$ if and only if $0\le\delta_1+\delta_2<\frac{m}{\tanh\left(\frac{m}{2}\right)}$, $0\le\delta_1$, $0\le\delta_2$ and $m>0$.*
9. $G_{m,\delta_1,\delta_2}(t,s)<0$ *for all $t,s\in(0,1)$ if and only if $\delta_1+\delta_2>\frac{m}{\tanh\left(\frac{m}{2}\right)}$, $0<\delta_1\le\frac{m}{\tanh\left(\frac{m}{2}\right)}$, $0<\delta_2\le\frac{m}{\tanh\left(\frac{m}{2}\right)}$ and $m>0$.*
10. G_{m,δ_1,δ_2} *changes its sign on $(0,1)\times(0,1)$ if and only if one the following properties is fulfilled:*
    a. $\delta_1>\frac{m}{\tanh\left(\frac{m}{2}\right)}$,
    b. $\delta_2>\frac{m}{\tanh\left(\frac{m}{2}\right)}$,
    c. $\delta_1<0$,
    d. $\delta_2<0$.

Remark 6 We point out that in order to prove Property 10 in previous Lemma, we must argue in a similar manner to Property 10 in Lemma 4. In this case, we must take into account that function Δ is strictly positive and strictly decreasing in $(-\infty,0)$ and (See Figures 1 and 2) that

$$\lim_{\gamma\to-\infty}\Delta(\gamma)=+\infty.$$

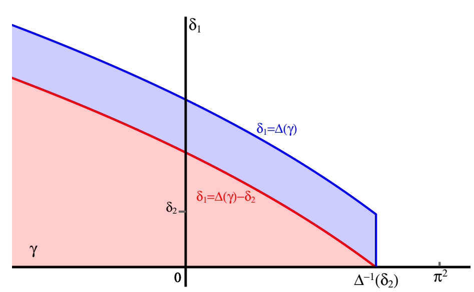

Fig. 1 Given $\delta_2>0$, the Green's function is positive on the red region, negative on the blue one, and changes its sign otherwise.

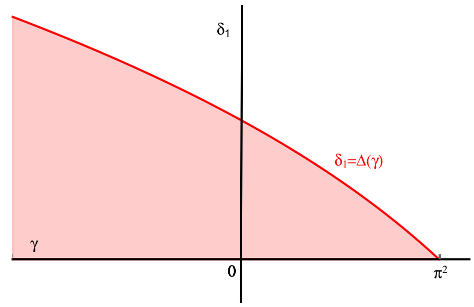

Fig. 2 If $\delta_2 = 0$, the Green's function is positive on the red region and changes its sign otherwise.

Lemma 7 *Let $m > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $0 < \delta_1 + \delta_2 < \frac{m}{\tanh(\frac{m}{2})}$ $(= \Delta(-m^2))$ and G_{m,δ_1,δ_2} be the Green's function of problem (17), (3) given by expression (18). Then there are two positive real constants c_5 and c_6 such that:*

$$c_5\, G_{m,\delta_1,\delta_2}(1,s) \leq G_{m,\delta_1,\delta_2}(t,s) \leq c_6\, G_{m,\delta_1,\delta_2}(1,s), \qquad \text{for all } t, s \in I. \tag{19}$$

Corollary 3 *Let $m > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $0 < \delta_1 + \delta_2 < \frac{m}{\tanh(\frac{m}{2})}$ $(= \Delta(-m^2))$ and G_{m,δ_1,δ_2} be the Green's function of problem (17), (3) given by expression (18). Then, it holds that:*

$$\min_{t\in[0,1]} G_{m,\delta_1,\delta_2}(t,s) \geq \frac{c_5}{c_6} \max_{t\in[0,1]} G_{m,\delta_1,\delta_2}(t,s), \qquad \text{for all } t, s \in I.$$

3 Nonlinear Problem

This section studies the existence of positive solutions of the nonlinear problems (1), (3) and (2), (3). We will ensure the existence of positive solutions of Problem (1), (3) when the related Green's function is positive and of Problem (2), (3) if it is negative. The existence results will be deduced from fixed point theory of integral operators defined in suitable cones. More concisely, we will use the classical Krasnoselskii's fixed point Theorem. The arguments are similar to the ones developed in [1].

We assume that the nonlinear part of equation satisfies the following regularity and constant sign condition:

(f) $\quad f : I \times [0,\infty) \to [0,\infty)$ is a continuous function.

Let $X \equiv (C(I), \|\cdot\|_\infty)$ be the real Banach space equipped with the supremum norm

$$\|u\|_\infty = \sup_{t\in I} |u(t)|, \qquad \text{for all } u \in X.$$

The following Krasnoselskii's fixed point Theorem [8] will be applied to the operator $T_{\gamma,\delta_1,\delta_2} : X \to X$, defined as

$$T_{\gamma,\delta_1,\delta_2} u(t) := \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\, f(s,u(s))\, ds, \quad t \in I, \tag{20}$$

to guarantee the existence of a fixed point of that operator.

Theorem 5 (Krasnoselskii)

Let X be a Banach space and $K \subset X$ a cone in X. Let $\Omega_1, \Omega_2 \subset X$ be open bounded sets such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ a compact operator that satisfies one of the following properties:

1. $\|T(u)\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ *and* $\|T(u)\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_2$.
2. $\|T(u)\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ *and* $\|T(u)\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_2$.

Then T has a fixed point at $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Note that the fixed points of operator $T_{\gamma,\delta_1,\delta_2}$ coincide with the solutions of problem (1), (3).

First, we consider the situation in which the related Green's function $G_{\gamma,\delta_1,\delta_2}$ is positive, that is:

$$\gamma < \pi^2 \text{ and } \delta_1,\ \delta_2 > 0 \text{ are such that } 0 < \delta_1 + \delta_2 < \Delta(\gamma), \tag{21}$$

where $\Delta : (-\infty, \pi^2) \to \mathbb{R}$ is the function defined by expression (5).

Next we define the cone K where we will apply Krasnoselskii's Theorem:

$$K = \{u \in X \,/\, \min\{u(t) : t \in I\} \geq \bar{c}\, \|u\|_\infty\},$$

where

$$\bar{c} = \begin{cases} \frac{c_1}{c_2}, & \gamma = 0, \\ \frac{c_3}{c_4}, & \gamma > 0, \\ \frac{c_5}{c_6}, & \gamma < 0, \end{cases}$$

with c_1, c_2, c_3, c_4, c_5 and c_6 given in inequalities (13), (16) and (19).

In this case, the following inequality is satisfied:

$$\min_{t \in I} G_{\gamma,\delta_1,\delta_2}(t,s) \geq \bar{c} \max_{t \in I} G_{\gamma,\delta_1,\delta_2}(t,s), \qquad \text{for all } s, t \in I. \tag{22}$$

Using inequalities (13), (16) and (19) again, we can guarantee that there are two real positive constants

$$h = \begin{cases} c_1, & \gamma = 0, \\ c_3, & \gamma > 0, \\ c_5, & \gamma < 0, \end{cases} \quad \text{and} \quad k = \begin{cases} c_2, & \gamma = 0, \\ c_4, & \gamma > 0, \\ c_6, & \gamma < 0, \end{cases}$$

such that

$$h\, G_{\gamma,\delta_1,\delta_2}(1,s) \leq G_{\gamma,\delta_1,\delta_2}(t,s) \leq k\, G_{\gamma,\delta_1,\delta_2}(1,s), \qquad \text{for all } t, s \in I. \tag{23}$$

Moreover, we will denote by

$$R = \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, ds = \begin{cases} \frac{\delta_2}{6(2-\delta_1-\delta_2)}, & \gamma = 0, \\ \frac{\delta_2\left(2\tan\left(\frac{\sqrt{\gamma}}{2}\right)-\sqrt{\gamma}\right)}{\gamma\left(\sqrt{\gamma}-(\delta_1+\delta_2)\tan\left(\frac{\sqrt{\gamma}}{2}\right)\right)}, & \gamma > 0, \\ \frac{\delta_2\left(2\tanh\left(\frac{\sqrt{-\gamma}}{2}\right)-\sqrt{-\gamma}\right)}{\gamma\left(\sqrt{-\gamma}-(\delta_1+\delta_2)\tanh\left(\frac{\sqrt{-\gamma}}{2}\right)\right)}, & \gamma < 0. \end{cases} \tag{24}$$

Remark 7 It is obvious that the value of $\bar{c}$, h, k and R will depend on those of γ, δ_1 and δ_2, but we will omit such dependence in the notation for the sake of simplicity. Moreover, it is clear that, when γ, δ_1 and δ_2 are fixed then $\bar{c}$, h, k and R are constants too.

Next, we will prove the following theorem to ensure the existence of positive solutions.

Theorem 6 *Assume that the positiveness condition of the Green's function* (21) *is fulfilled. Moreover, suppose that the following conditions hold:*

1. There exists $p > 0$ such that

$$f(t,u) \leq \frac{p}{k\,R}, \quad \text{for all } t \in I \text{ and } u \in [0,p].$$

2. There exists $q > 0$, with $q \neq p$ such that

$$f(t,u) \geq \frac{q}{h\,R}, \quad \text{for all } t \in I \text{ and } u \in [\bar{c}\,q, q].$$

Then problem (1), (3) has at least one positive solution $u \in K$, such that $\|u\|_\infty$ lies between p and q.

Proof First, since $G_{\gamma,\delta_1,\delta_2}(t,s) > 0$ for all $t, s \in (0,1)$, $f \geq 0$ and the fixed points of the operator $T_{\gamma,\delta_1,\delta_2}$ coincide with the solutions of problem (1), (3), we deduce that these solutions are nonnegative.

Now, we show that the operator $T_{\gamma,\delta_1,\delta_2}$ defined in (20) is compact and maps K into K.

Let us see that $T_{\gamma,\delta_1,\delta_2}$ maps K into K. For all $u \in K$, using inequality (22) we infer that, for all $t \in I$,

$$\begin{aligned} T_{\gamma,\delta_1,\delta_2}\,u(t) &= \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds \geq \bar{c}\int_0^1 \max_{t\in I}\,G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds \\ &\geq \bar{c}\,\max_{t\in I}\int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds. \end{aligned}$$

So, $T_{\gamma,\delta_1,\delta_2}\,u(t) \geq \bar{c}\,\|T_{\gamma,\delta_1,\delta_2}\,u\|_\infty$ for all $t \in I$, that is, $T_{\gamma,\delta_1,\delta_2}\,u \in K$.

On the other hand, since $G_{\gamma,\delta_1,\delta_2}$ and f are continuous, we have that operator $T_{\gamma,\delta_1,\delta_2}$ is continuous too.

Finally, we will prove that $T_{\gamma,\delta_1,\delta_2}$ maps bounded sets into relatively compact sets. Let $H \subset K$ be a bounded set. Then, using (23), it is easy to see that $T_{\gamma,\delta_1,\delta_2}(H)$ is bounded.

Let us show then the equicontinuity of $T_{\gamma,\delta_1,\delta_2}(H)$. Since H is bounded, there exists $r \in \mathbb{R}$, $r > 0$ such that $\|u\|_\infty \leq r$ for all $u \in H$. Let us take

$$M = \max_{t\in I,\,0\leq u\leq r} |f(t,u)|.$$

So, for all $t \in I$ and $u \in H$, we have that

$$\begin{aligned} |(T_{\gamma,\delta_1,\delta_2}\,u)'(t)| &= \left|\int_0^1 \frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s)\,f(s,u(s))\,ds\right| \leq \int_0^1 \left|\frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s)\right| |f(s,u(s))|\,ds \\ &\leq M\int_0^1 \left|\frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s)\right| ds. \end{aligned}$$

Using the regularity of the Green's function $G_{\gamma,\delta_1,\delta_2}$ we deduce that there exists $N \in \mathbb{R}$, $N > 0$ such that

$$M\int_0^1 \left|\frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s)\right| ds \leq N.$$

So, for all $t_1, t_2 \in I$, $t_1 < t_2$, the following inequality holds

$$|(T_{\gamma,\delta_1,\delta_2}u)(t_2) - (T_{\gamma,\delta_1,\delta_2}u)(t_1)| = \left|\int_{t_1}^{t_2} (T_{\gamma,\delta_1,\delta_2}u)'(s)ds\right| \leq \int_{t_1}^{t_2} |(T_{\gamma,\delta_1,\delta_2}u)'(s)|ds \leq N(t_2 - t_1).$$

Therefore, $T_{\gamma,\delta_1,\delta_2}(H)$ is an equicontinuous set in X. By Arzelà-Ascoli's Theorem, we deduce that $T_{\gamma,\delta_1,\delta_2}(H)$ is relatively compact, that is, $T_{\gamma,\delta_1,\delta_2} : K \to K$ is a compact operator.

Next, let us define the following sets of K

$$K_p = \{u \in X \,/\, \|u\|_\infty < p\} \quad \text{and} \quad K_q = \{u \in X \,/\, \|u\|_\infty < q\}.$$

From inequality (23) we infer that

$$\|T_{\gamma,\delta_1,\delta_2}\, u(t)\|_\infty = \max_{t\in I} \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\, f(s,u(s))\, ds \le k \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, f(s,u(s))\, ds.$$

Using Condition 1. we have, for all $u \in K \cap \partial K_p$, that

$$\|T_{\gamma,\delta_1,\delta_2}\, u(t)\|_\infty \le k \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, f(s,u(s))\, ds \le p = \|u\|_\infty.$$

On the other hand, for $u \in K \cap \partial K_q$, using Condition 2. we have that $q \ge u(t) \ge \bar{c}\|u\|_\infty = \bar{c}\, q$ for all $t \in I$ and from (23) we deduce that

$$\|T_{\gamma,\delta_1,\delta_2}\, u(t)\|_\infty \ge h \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, f(s,u(s))\, ds \ge q = \|u\|_\infty.$$

Thus, applying Krasnoselskii's Theorem 5, we obtain that $T_{\gamma,\delta_1,\delta_2}$ has a fixed point $u \in K$ such that $\|u\|_\infty$ lies between p and q. □

In the sequel, we present an example to illustrate the previous result.

Example

Consider the following problem

$$\begin{cases} u''(t) + u(t)\, e^{u^2(t)} = 0, \; t \in I, \\ u(0) = \dfrac{1}{3} \displaystyle\int_0^1 u(s)\, ds, \\ u(1) = \dfrac{1}{6} \displaystyle\int_0^1 u(s)\, ds. \end{cases} \tag{25}$$

In this case, $\gamma = 0$, $\delta_1 = \frac{1}{3}$, $\delta_2 = \frac{1}{6}$ and $\delta_1 + \delta_2 = \frac{1}{2} < 2$. As we have said in Remark 3 in this case we may consider $h = c_1 = 1$ and $k = c_2 = 12$ and, consequently, $\bar{c} = \frac{1}{12}$. Moreover, from (24) we know that

$$\int_0^1 G_{0,\frac{1}{3},\frac{1}{6}}(1,s)\, ds = \frac{1}{54}.$$

Moreover, $f(t,u) = u\, e^{u^2}$ is continuous on $I \times [0,\infty)$. Now, for $p \le \sqrt{\log\left(\frac{9}{2}\right)} \approx 1.226$ it holds that

$$f(t,u) \le p\, e^{p^2} \le \frac{9}{2}\, p \left(= \frac{p}{k \int_0^1 G_{0,\frac{1}{3},\frac{1}{6}}(1,s)\, ds} \right) \quad \forall\, t \in [0,1], \; u \in [0,p].$$

Similarly, for $q \ge 12\sqrt{\log 648} \approx 30.533$ it holds that

$$f(t,u) \geq \frac{1}{12}\, q\, e^{\frac{q^2}{144}} \geq 54\, q \left(= \frac{q}{h \int_0^1 G_{0,\frac{1}{3},\frac{1}{6}}(1,s)\, ds} \right) \quad \forall\, t \in [0,1],\ u \in \left[\frac{q}{12}, q\right].$$

Therefore, Theorem 6 guarantees the existence of a positive solution of problem (25).

3.1 Problem (2), (3)

To study the nonlinear problem (2), (3), we must use the Green's function related to

$$u''(t) + \gamma\, u(t) - \sigma(t) = 0, \quad t \in I,$$

coupled to the integral boundary conditions (3).

It is immediate to verify that such Green's function is indeed $-G_{\gamma,\delta_1,\delta_2}$.

So, by assuming the following condition:

$$\gamma < \pi^2,\ \delta_1 + \delta_2 > \Delta(\gamma),\ \delta_1 \leq \Delta(\gamma) \text{ and } \delta_2 \leq \Delta(\gamma), \tag{26}$$

we can ensure the positiveness of function $-G_{\gamma,\delta_1,\delta_2}$.

Therefore, the inequalities that we have used in the previous case, in Lemmas 2, 5 and 7, remain valid since it is clear that

$$c_1 \leq \frac{-G_{0,\delta_1,\delta_2}(t,s)}{-G_{0,\delta_1,\delta_2}(1,s)} \leq c_2, \qquad \text{for all } t \in I \text{ and } s \in (0,1),$$

$$c_3 \leq \frac{-G_{\gamma,\delta_1,\delta_2}(t,s)}{-G_{\gamma,\delta_1,\delta_2}(1,s)} \leq c_4, \quad \text{for } \gamma > 0 \qquad \text{and all } t \in I \text{ and } s \in (0,1),$$

and

$$c_5 \leq \frac{-G_{\gamma,\delta_1,\delta_2}(t,s)}{-G_{\gamma,\delta_1,\delta_2}(1,s)} \leq c_6, \quad \text{for } \gamma < 0 \qquad \text{and all } t \in I \text{ and } s \in (0,1).$$

Thus, we may define the constants $\bar{c}$, h, k and R (< 0) in the same way than before and inequalities (22) and (23) hold by simply changing $G_{\gamma,\delta_1,\delta_2}$ by $-G_{\gamma,\delta_1,\delta_2}$.

In this case, an existence result can also be proved by considering now values of γ, δ_1 and δ_2 for which $G_{\gamma,\delta_1,\delta_2}$ is negative (that is, $-G_{\gamma,\delta_1,\delta_2}$ is positive). The result is the following one.

Theorem 7 *Assume that condition* (26) *is fulfilled. Moreover, suppose that the following conditions hold:*

1. *There exists $p' > 0$ such that*

$$f(t,u) \leq \frac{p'}{-k\,R}, \quad \text{for all } t \in I \text{ and } u \in [0,p'].$$

2. *There exists $q' > 0$, with $q' \neq p'$ such that*

$$f(t,u) \geq \frac{q'}{-h\,R}, \quad \text{for all } t \in I \text{ and } u \in [\bar{c}\, q', q'].$$

Then problem (2), (3) has at least one positive solution $u \in K$, such that $||u||_\infty$ lies between p' and q'.

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