Tese de Doutoramento

FRACTIONAL DIFFERENTIAL EQUATIONS

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Asdo. ______________________________

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Don Juan José Nieto Roig, catedrático do Departamento de Análise Matemática, Estatística e Optimización da Universidade de Santiago de Compostela, informa que a memoria titulada:

FRACTIONAL DIFFERENTIAL EQUATIONS

foi realizada baixo a sú dirección por Don Johnatan Pimentel Sunio, estimando que o interesado cumpre as condicións para optar ao grao de Doutor en Ciencias Matemáticas, polo que solicitan que sexa admitida a trámite para a súa lectura e defensa pública.

En Santiago de Compostela, a 23 de Novembro de 2016.

O director,

Prof. Dr. Juan José Nieto Roig

O doutorando,

Johnatan Pimentel Sunio

Abstract

ECUACIONES DIFERENCIALES DE ORDEN FRACCIONARIO

El cálculo fraccionario, la rama de la matemática que trata derivadas e integrales de orden no entero, que comenzó como una curiosidad matemática durante la época de Leibniz y a través de los años se ha convertido en un campo de investigación muy dinámico. Matemáticos famosos, como Riemann, Liouville, Grunwald, Euler, Lagrange, Caputo y otros pusieron el fundamento de la teoría moderna preparando así el camino para el desarrollo del cálculo fraccionario y ser una de las corrientes actuales dentro de las matemáticas. En la actualidad, este campo de estudio aún se está desarrollando. Nuevos conceptos e ideas tales como, por ejemplo, la formulación de Caputo-Fabrizio, han surgido y aplicaciónes en tan variados campos tan viscoelasticidad , flujo de fluidos , reología , etc., han emergido. En esta Tesis abordaremos nuevos problemas y cuestiones dentro de esta temática.
Palabras Clave: Cálculo fraccionario, Derivada fraccionaria de Caputo, Ecuación diferencial, Inclusión, Teorema de punto fijo

ECUACIÓNS DIFERENCIAIS DE ORDE FRACCIONARIA

O cálculo fracionário, a rama da matemática que trata sobre derivadas e integrais de orde non enteiro, que comezou como unha curiosidade matemática durante o tempo de Leibniz e ao longo dos anos converteuse nun campo moi dinámico de busca. Matemáticos famosos como Riemann, Liouville, Grunwald, Euler, Lagrange, Caputo e outros lanzaron as bases da teoría moderna abrindo o camiño para o desenvolvemento de cálculo fraccionário e ser unha das tendencias actuais en matemáticas . Actualmente, este campo de estudo aínda está en desenvolvemento. Novos conceptos e ideas, como, por exemplo, a formulación de Caputo-Fabrizio, xurdiron e aplicacións en ámbitos tan variados como a viscoelasticidade , o fluxo de fluído , reolóxicas , etc. , ten emerxido. Nesta Tese abordaremos novos problemas e cuestións dentro desta temática.
Palabras Chave: Cálculo fracionário, Derivada fracionária de Caputo, Ecuación diferencial, Inclusión, Teorema do punto fixo

FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional calculus, the branch of mathematics dealing with derivatives and integrals of non-integer order, began as a mere mathematical curiosity during the time of Leibniz but through the years has developed into a very dynamic field of research. Famous mathematicians such as Riemann, Liouville, Grunwald, Euler, Lagrange, Caputo and others laid the foundation of the modern theory thus paving the way for fractional calculus to enter mainstream mathematics. At present, this field of study is still developing rapidly. New concepts and ideas such as the Caputo-Fabrizio formulation for example, have emerged and applications in such varied fields as viscoelasticity, fluid flow, rheology, etc., have arisen. In this thesis we will address new problems and issues within this area.
Keywords: Fractional calculus, Caputo fractional derivative, Differential equation, Inclusion, Fixed point theorem

Contents

Chapter 1

Introduction

Fractional calculus traces its origins to the beginnings of calculus itself. In 1675, Leibniz invented the symbol $\frac{d^n y}{dx^n}$ to denote the nth derivative of y with respect to x, the meaning of which holds for the non-negative integer n.

In a letter to Leibniz, L'hopital asked the question as to the meaning of $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$. Leibniz replied saying that "this will lead to an apparent paradox from which, one day, useful consequences will be drawn." [46]

Perhaps the very first definition of fractional derivatives was made by Leonard Euler when he wrote in 1730 that when n is a positive integer, the ratio $\frac{d^n p}{dx^n}$, p a function of x, can always be expressed algebraically. Indeed, for $n \leq m$

$$\begin{aligned} \frac{d^n x^m}{dx^n} &= (m)(m-1)(m-2)...(m-n+1)x^{m-n} \\ &= \frac{m!}{(m-n)!}x^{m-n} \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n} \end{aligned}$$

He proposed that it might be possible to interpolate if the order n of the derivative is a fraction.

Lacroix later showed in his 1819 book "Traité du Calcul Différenciel et du Calcul Intégral" that for $m = 1$, $n = \frac{1}{2}$,

$$\begin{aligned} \frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} &= \sqrt{\frac{4x}{\pi}} \\ &= \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} \end{aligned}$$

which is exactly the same result obtained using the present-day Riemann-Liouville definition of the fractional derivative.

For his part in the development of fractional calculus, Fourier in 1822 started with the integral representation of $f(x)$ given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz)dp$$

and made the following generalization:

$$\frac{d^\mu f(x)}{dx^\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} p^\mu \cos(px - pz + \frac{\mu\pi}{2})dp,$$

adding that μ could take any arbitrary value whether positive or negative. In his definition, one can see that the existence of the the fractional derivative or integral depends on the convergence of the improper integrals.
Ross [46, 47, 45] attributes the first application of fractional calculus to Abel. In 1823, Abel solved the integral equation

$$\frac{1}{\Gamma(\mu)}\int_0^x \frac{\phi(t)}{(x-t)^{1-\mu}}dt = f(x) \qquad\qquad 0 < \mu < 1,$$

which arises in connection with the tautochrone problem: A bead on a frictionless wire starts from rest at some point and slides down under the influence of gravity. What should the shape of the wire be so that the amount of time it makes the bead to descend to its lowest point is independent of its starting point?
Abel obtained the solution

$$\phi(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^\mu}dt.$$

According to Butzer and Westphal, [13] Abel did not actually obtain his solution using fractional calculus but merely showed that it could be written as a fractional derivative. However, Abel's ideas played an enormous role in the further development of fractional calculus.
It is Liouville who is generally credited to have laid the more substantial groundwork for the theory of fractional calculus as we know it today. Between 1832 and 1855, he published a series of papers on the subject. Liouville's first definition of the fractional derivative was based on the formula differentiating an exponential function:

$$\frac{d^m e^{ax}}{dx^n} = a^m e^{ax}.$$

He considered functions which can be written as the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$$

and defined the derivative as

$$\frac{d^\mu f(x)}{dx^\mu} = \sum_{n=0}^{\infty} c_n a_m^\mu e^{a_n x} \qquad \mu \in \mathbb{C}.$$

This definition is clearly too restrictive as it depends on the convergence of the series. He also derived the formula

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_0^\infty \phi(x+\alpha)\alpha^{\mu-1} d\mu \qquad -\infty < x < \infty, \Re(\mu) > 0.$$

If we let $\tau = x + \alpha$, then

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_x^\infty \phi(\tau)(\tau - x)^{\mu-1} d\tau.$$

This formula is what is now known as the Liouville form of fractional integration with the factor $(-1)^\mu$ being omitted. Liouville applied these formulas to solve various problems in electrodynamics, mechanics and geometry. It is also worthwhile to note that in both Fourier's and Liouville's definitions, the fractional derivatives take the form of an integral.
Grunwald in 1867 and Letnikov in 1868 introduced what is now known as the Grunwald-Letnikov fractional derivative. Their idea was to start with the ordinary derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

and apply this recursively to obtain higher-order derivatives. For example, the second-order derivative would be:

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \end{aligned}$$

In general, we have

$$f^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{0\le m\le n} (-1)^m \binom{n}{m} f[x + (m-n)h]$$

where

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}.$$

By allowing n to be any real number, the Grunwald-Letnikov derivative is obtained:

$$D^{\alpha}f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\frac{x-a}{h}} (-1)^m \binom{\alpha}{m} f(x-mh)$$

Nowadays, this formula is very useful in numerical calculations of fractional derivatives.

Riemann developed his theory of fractional calculus while he was a student but it was published posthumously in 1876. Riemann sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$\frac{d^{-\mu}f(x)}{dx^{-\mu}} = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} f(t)dt + \psi(x).$$

He felt the need to add the complementary function $\psi(x)$ to deal with the ambiguity of the lower limit of integration c, which only created even confusion as to what is meant by it.

It was Laurent's work in 1884 which lead to what is now known as the Riemann-Liouville integral :

$$_cD_x^{-\mu}f(x) = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} f(t)dt.$$

We see that if $c = 0$, we have Riemann's definition with the complementary function equal to 0. If $c = -\infty$, the integral is equivalent to Liouville's definition.

Through the centuries, many eminent mathematicians contributed to the development of fractional calculus. The list includes H. Holmgren, A. K. Grunwald, A. V. Letnikov, H. Laurent, P. A. Nekrassov, A. Krug, J. Hadamard, O. Heaviside, S. Pincherle, G. H. Hardy, J. E. Littlewood, H. Weyl, P. Lévy, A.Marchaud, H. T. Davis, E. L. Post, A. Zygmund, E. R. Love, A. Erdelyi, H. Kober, D. V. Widder, M. Riesz, W. Feller, M.A. AlBassam, L.S. Bosanquet, P.L. Butzer, M.M. Dzherbashyan, A. Erdelyi, T.M. Flett, Ch. Fox, S.G. Gindikin, S.L. Kalla, LA. Kipriyanov, H. Kober, P.I.Lizorkin, E.R. Love, A.C. McBride, M. Mikolas, S.M. Nikol'skii, K. Nishimoto, LI. Ogievetskii, R.O. O'Neil, T.J. Osier, S. Owa, B. Ross, M. Saigo, I.N. Sneddon, H.M. Srivastava, A.F. Timan, U. Westphal, A. Zygmund, M. Caputo and others. [13, 51, 52]

The modern formulation of the Riemann-Liouville integral is given by

$$I_a^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the classical Gamma function.

On the other hand, the Riemann-Liouville fractional derivative of order α of a function $f : [a, b] \to \mathbb{R}$ is defined by

$$^{rl}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds.$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.
Some mathematicians had some misgiving about the Riemann-Liouville definition. In particular, there was an apparent difficulty in providing physical interpretations of initial conditions involving the derivative. In 1969, Caputo introduced a new definition of fractional derivative that would permit physically interpretable initial conditions. The Caputo fractional derivative of order α of a function f is defined by

$$({}^{c}D_a^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.
At present, there are numerous formulations for fractional derivatives and integrals in the literature such as the Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, Marchaud, Miller-Ross, etc. The monographs of Kilbas et al [31], Kiryakova [32], Miller and Ross [39], Poblubny [43], Oldham and Spanier [41], Samko et al [48] and the refences therein detail some of the recent advances in the field.
The object of this study is to investigate differential equations and inclusions involving fractional derivatives of the Caputo type.
In particular, we investigate for $T > 0$ and $1 < q \leq 2$ the following class of fractional differential equations

$$^{c}D^q x(t) = f(t, x(t)), \quad t \in [0, T], \tag{1.0.0.1}$$

where ${}^{c}D^q$ denotes the Caputo fractional derivative of order q and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$. We study (1.0.0.1) subject to two families of boundary conditions:

(i) Mixed boundary conditions

$$Tx'(0) = -ax(0) - bx(T) \quad Tx'(T) = bx(0) + dx(T), \tag{1.0.0.2}$$

(ii) Closed boundary conditions

$$x(T) = \alpha x(0) + \beta T x'(0), \quad Tx'(T) = \gamma x(0) + \delta T x'(0), \tag{1.0.0.3}$$

where $a, b, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ are given constants.

We derive the corresponding Green's function to express the solution of the boundary value problem as an equivalent integral expression and prove the existence of solutions.

We extend the discussion to fractional differential inclusion problem

$$ {}^{c}D^{q}x(t) \in F(t, x(t)), \quad t \in [0, T], \; T > 0, \quad 1 < q \leq 2, \tag{1.0.0.4} $$

subject to two families of boundary conditions (1.0.0.2) and (1.0.0.3). where $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.
Using results for multivalued maps and some fixed point theorems, we establish the existence of solutions for (1.0.0.4) for the cases where F is convex, F is not necessarily convex and F is nonconvex.

Finally, we study a boundary value problem that models a thermostat insulated at one end and with the controller at the other end:

$$ -{}^{c}D^{\alpha}u(t) = f(t, u(t)), \quad t \in [0, 1], $$

where $1 < \alpha \leq 2$, ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α and $f \in C([0, 1] \times [0, \infty), [0, \infty))$ subject to the boundary conditions:

$$ u'(0) = 0, \quad \beta\, {}^{c}D^{\alpha-1}u(1) + u(\eta) = 0, $$

where $\beta > 0$, $0 \leq \eta \leq 1$ are given constants.

Mathematical models of physical processes are useful if positive solutions exist. In this regard, we establish conditions for the existence of positive solutions for the boundary value problem.

Chapter 2

Preliminaries

In this chapter we discuss the necessary mathematical tools we need in the succeeding chapters. We look at some essential properties of fractional differential operators, limiting our scope to the Riemann-Liouville and Caputo versions. We also review some of the basic properties of multivalued maps which are crucial in our results regarding fractional differential inclusions.

2.1 Function spaces

Let $C[a,b]$ denote the Banach space of continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm $\|f\| = \sup |f(t)| : t \in [a,b]$.

Define for $t \in [a,b]$, $f_r(t) = (t-a)^r f(t)$. Let $C_r[a,b], r \geq 0$ be the space of all functions f such that $f_r \in C[a,b]$.

$C_r[a,b]$, endowed with the norm $\|f\|_r = \sup\,(t-a)^r |f(t)| : t \in [a,b]$, is a Banach space.

Let $L^1([a,b],\mathbb{R})$ be the Banach space of measurable functions $f : [a,b] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|f\|_{L^1} = \int_a^b |f(t)|dt$.

Definition 2.1.1. *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any finite set of pairwise non-intersecting intervals $[a_k,b_k] \subset [a,b],\ k = 1,2,\dots,n,$ such that*

$$\sum_{k=1}^{n} (b_k - a_k) < \delta,$$

then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

The space of absolutely continuous functions is denoted by $AC[a,b]$.

Theorem 2.1.2. *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if and only if f' exists almost everywhere on $[a,b]$, $f' \in L^1[a,b]$, and*

$$f(t) = f(a) + \int_a^t f'(s)ds \quad \textit{for all } t \in [a,b]. \tag{2.1.0.1}$$

Definition 2.1.3. *A function $f : [a,b] \to \mathbb{R}$ is said to be in $AC^n[a,b]$, $n = 1, 2, \ldots$ if f has continuous derivatives up to order $n-1$ on $[a,b]$ and $f^{(n-1)} \in AC[a,b]$. We note that in particular, $AC^1[a,b] = AC[a,b]$.*

Theorem 2.1.4. *[48] The space $AC^n[a,b]$ consists of those and only those functions f which can be represented in the form*

$$f(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f^{(n-1)}(s)ds + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k, \tag{2.1.0.2}$$

where $f^{(n-1)} \in L^1[a,b]$.

2.2 Special Functions

Definition 2.2.1. The Gamma function $\Gamma(\cdot)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

where $\Re(z) > 0$.

Although the integral formulation for the Gamma function holds only for $\Re(z) > 0$, the definition can be extended by analytic continuation to all complex numbers except the non-positive integers (where the function has simple poles). For positive integer values n, the Gamma function becomes $\Gamma(n) = n!$ and thus can be seen as an extension of the factorial function to complex values.
An important property of the gamma function $\Gamma(z)$ is that it satisfies : $\Gamma(z+1) = z\Gamma(z)$.

Definition 2.2.2. The beta function $B(\cdot,\cdot)$ is given by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \; (\Re(z) > 0, \; \Re(w) > 0).$$

To extend the definition of the beta function to the entire complex plane, we use the following formula which expresses the beta function in terms of the gamma function:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

from which it also follows that $B(z,w) = B(w,z)$.

2.3 The Riemann-Liouville Integral

First, we recall the well-known Cauchy formula for n-fold integrals:

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_2} dx_3 \cdots \int_a^{x_{n-1}} f(x_n)dx_n = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

where n is a non-negative integer.
To define the Riemann-Liouville fractional integral, we generalize this formula by letting n take values other than the non-negative integers, and noting at the same time that the factorial function is a special case of the Gamma function $\Gamma(\cdot)$.

Definition 2.3.1 ([31, 43])**.** *The fractional integral of order* $\alpha > 0$ *of the function* $f : [a,b] \longrightarrow \mathbb{R}$ *is defined by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \qquad (2.3.0.1)$$

where $\Gamma(\cdot)$ *is the classical Gamma function.*

If f is continuous in $(0,T]$ and integrable on any subinterval of $[0,T]$, we see that the expression is integrable.
We let $a < \epsilon < t$ and write

$$\int_a^t (t-s)^{\alpha-1} f(s)ds = \int_a^\epsilon (t-s)^{\alpha-1} f(s)ds + \int_\epsilon^t (t-s)^{\alpha-1} f(s)ds.$$

The first integral on the right satisfies

$$\int_a^\epsilon |(t-s)^{\alpha-1} f(s)|ds \leq A_1 \int_a^\epsilon |f(s)|ds$$

where $A_1 = \sup_{s\in[a,\epsilon]}(t-s)^{\alpha-1}$. Hence it exists since $(t-s)^{\alpha-1}$ is bounded on $[a,\epsilon]$ and f is integrable on $[a,\epsilon]$.
On the other hand, the second integral on the right satisfies

$$\int_{\epsilon}^{t} |(t-s)^{\alpha-1} f(s)| ds \leq A_2 \int_{\epsilon}^{t} (t-s)^{\alpha-1} ds$$

where $A_2 = \sup_{s\in[\epsilon,t]} f(s)$ which exists since $f(s)$ is bounded on $[\epsilon, t]$. Also, $(t-s)^{\alpha-1}$ is integrable on $[\epsilon, t]$ although in the case where $0 < a < 1$, we end up with an improper integral.
Clearly, if $f \in C_r[a,b]$, $r < \alpha$ then $I_a^\alpha f \in C[a,b]$ and $I_a^\alpha f(a) = 0$.
If $f \in C_\alpha[a,b]$, then $I_a^\alpha f$ is bounded at a but if $f \in C_r[a,b]$, $\alpha < r < 1$ then we may expect $I_a^\alpha f$ to be unbounded at a.

Remark 2.3.2. Although the above definition does not permit that $\alpha = 0$, for consistency we define $I_a^0 = \lim_{\alpha\to 0^+} I_a^\alpha f(t)$ Under suitable conditions on the function $f(t)$ it is easy to see that $\lim_{\alpha\to 0^+} I_a^\alpha f(t) = f(t)$, and so we have $I_a^0 = I$ where I is the identity operator.

Indeed, if we have $f(t) \in AC^1(a,b)$, then integrating by parts we obtain

$$I_a^\alpha f(t) = \frac{(t-a)^\alpha f(a)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-s)^\alpha f'(s) ds$$

and thus,

$$\lim_{\alpha\to 0^+} I_a^\alpha f(t) = f(a) + \int_a^t f'(s) ds = f(t).$$

Remark 2.3.3. In [44], Podlubny provides an interesting interpretation of the Riemann-Liouville fractional integral.
To begin, we rewrite $I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$ as

$$\begin{cases} I_0^\alpha f(t) = \int_0^t f(s) dg_t(s), \\ g_t(s) = \frac{1}{\Gamma(\alpha+1)} \left\{ t^\alpha - (t-s)^\alpha \right\} \end{cases} \tag{2.3.0.2}$$

For a fixed t, (2.3.0.2) becomes a Riemann-Stieltjes integral. Now, we take the axes s, g, and f. In the plane $\langle s, g\rangle$, we plot the function $g_t(s)$ for $0 \leq s \leq t$. A "fence" of varying height $f(s)$ is then built with its base on the curve $g_t(s)$ so that the top edge of the "fence" is the three dimensional curve $(s, g_t(s), f(s))$, $0 \leq s \leq t$.
This "fence" is then projected onto the planes $\langle s, f\rangle$ and (g, f).
We have the following conclusions:

- The area of the projection onto the plane $\langle s, f \rangle$ corresponds to the value of the integral $I_0^1 f(t) = \int_0^t f(s)ds,\ t \geq 0$,or the area under the curve $f(s)$.
- The area of the projection onto the plane $\langle g, f \rangle$ corresponds to the value of (2.3.0.2).

If $\alpha = 1$, then $g_t(s) = s$ and so the two projections coincide, which shows that even geometrically, classical definite integration is a particular case of the Riemann-Liouville fractional integration.
As to a physical interpretation of the fractional integral (2.3.0.2), we have the following:
The fractional integral $I_0^\alpha f(t)$ of the function $f(s)$ may be interpreted as the real distance passed by a moving object, for which the local values of its speed $f(s)$ and the local values of its time s have been recorded. Here we consider two time scales: one is homogeneous, i.e., the geometrically equal intervals of the time axis are considered as corresponding to equal time intervals, while the other is inhomogeneous, i.e., the "ticks" on the time axis don't come at equal intervals. The function $T = g_t(s)$ describes, at each individual time instance t, the relationship between the locally recorded time s which flows equably on a homogeneous time scale and the cosmic time T which flows non-equably.

Example 2.3.4. The Riemann-Liouville fractional integral of the power function $(t-a)^r \quad r > -1$.
By definition,

$$I_a^\alpha (t-a)^r = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^r ds.$$

Using the change of variables $s = a + \tau(t-a)$ we get,

$$I_a^\alpha (t-a)^r = \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+r} \int_0^1 (1-\tau)^{\alpha-1}\tau^r d\tau$$

Observe that the integral is equivalent to $B(\alpha-1, r)$ where $B(\cdot,\cdot)$ is the classical Beta function. Thus,

$$\begin{aligned} I_a^\alpha (t-a)^r &= \frac{1}{\Gamma(\alpha)} B(\alpha-1, r)(t-a)^{\alpha+r} \\ &= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(r+1)}{\Gamma(\alpha+r+1} (t-a)^{\alpha+r} \\ &= \frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)} (t-a)^{\alpha+r} \end{aligned}$$

Example 2.3.5. The Riemann-Liouville fractional integral of $e^{\lambda t}$
Here let us take the lower terminal $a = 0$.

$$I_0^\alpha e^{\lambda t} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\lambda s} ds.$$

Using the change of variables $x = t - s$ we get,

$$\begin{aligned} I_0^\alpha e^{\lambda t} &= \frac{1}{\Gamma(\alpha)} \int_0^t (x)^{\alpha-1} e^{\lambda(t-x)} dx \\ &= \frac{e^{\lambda t}}{\Gamma(\alpha)} \int_0^t (x)^{\alpha-1} e^{-\lambda x} dx \\ &= t^\alpha e^{\lambda t} \gamma^*(\alpha, \lambda t). \end{aligned}$$

where $\gamma^*(\cdot,\cdot)$ is the incomplete Gamma function.

Theorem 2.3.6. *Let $f(t)$ and $g(t)$ be functions such that both $I_a^\alpha f(t)$ and $I_a^\alpha g(t)$ exist. The following basic properties of the Riemann-Liouville integrals hold:*

(ii) *Linearity*

$$I_a^\alpha[\lambda f(t) + \beta g(t)] = \lambda I_a^\alpha f(t) + \beta I_a^\alpha g(t), \ \forall \lambda, \beta \in \mathbb{C};$$

(i) *Backward Compatibility*

$$\lim_{\alpha \to n} I_a^\alpha f(t) = I_a^n f(t)$$

where $I^n (n \in \mathbb{N})$ is the classical operator for the n-fold integration;

(iii) *Index Law*

$$I_a^\alpha[I_a^\beta f(t)] = I_a^{\alpha+\beta} f(t);$$

holds at every point if $f(t) \in C([a,b])$, and holds almost everywhere if $f(t) \in L^1(a,b)$.

(iv) *Commutativity*

$$I_a^\alpha[I_a^\beta f(t)] = I_a^\beta[I_a^\alpha f(t)].$$

We prove the index law:

$$\begin{aligned} I_a^\alpha[I_a^\beta f(t)] &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\{ \frac{1}{\Gamma(\beta)} \int_a^s (s-x)^{\beta-1} f(x) dx \right\} ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t dx \int_x^t (t-s)^{\alpha-1} (s-x)^{\beta-1} f(x) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(x) dx \int_x^t (t-s)^{\alpha-1} (s-x)^{\beta-1} ds \end{aligned}$$

We see that if $g(t) = (t-x)^{\beta-1}$ then the integral with respect to s is just $\Gamma(\alpha) I_x^{\alpha} g(t) = \frac{1}{\Gamma(\alpha+\beta)}(t-x)^{\alpha+\beta-1}$. Hence

$$\begin{aligned} I_a^{\alpha}[I_a^{\beta} f(t)] &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-x)^{\alpha+\beta-1} f(x)dx \\ &= I_a^{\alpha+\beta} f(t). \end{aligned}$$

Observe that it follows from the index law that $I_a^{n+\alpha} f(t) = I_a^n [I_a^{\alpha} f(t)]$ for $n \in \mathbb{N}, \alpha > 0$ which implies that $\dfrac{d^n}{dt^n} I_a^{n+\alpha} f(t) = I_a^{\alpha} f(t)$

2.4 Fractional Derivatives

From the definition of the Riemann-Liouville fractional integral, the fractional derivative is obtained not by replacing α with $-\alpha$ because the integral $\int_a^t (t-s)^{-\alpha-1} f(s)\, ds$ is, in general, divergent. Instead, differentiation of arbitrary order is defined as the composition of ordinary differentiation D^n and fractional integration, i.e.,

$$D_a^{\alpha} f(x) = D^n I_a^{n-\alpha}\,, \ n = [\alpha] + 1$$

resulting in the Riemann-Liouville fractional derivative or

$$D_a^{\alpha} f(x) = I_a^{n-\alpha} D^n f(x)\,, \ n = [\alpha] + 1$$

which gives the Caputo fractional derivative.

2.4.1 The Riemann-Liouville Fractional Derivative

Definition 2.4.1 ([31, 43])**.** The Riemann-Liouville fractional derivative of order α of a function $f : [a, b] \to \mathbb{R}$ is defined by

$$^{rl}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds.$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Indeed, it follows from the definition that $^{rl}D_a^{\alpha} f(t) = D^n I_a^{n-\alpha} f(t)$.

Example 2.4.2. Riemann-Liouville fractional derivative of the power function $(t-a)^r \quad r > -1$.

$$\begin{aligned} {}^{rl}D_a^{\alpha}(t-a)^r &= \frac{d^n}{dt^n} I_a^{n-\alpha}(t-a)^r \\ &= \frac{\Gamma(r+1)}{\Gamma(r+n-\alpha+1)} \frac{d^n}{dt^n}(t-a)^{r+n-\alpha} \\ &= \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(t-a)^{r-\alpha} \end{aligned}$$

Remark 2.4.3. If we let $r = 0$ in the previous example, we see that the Riemann-Liouville fractional derivative of a constant is not 0. In fact,

$$ {}^{rl}D_a^{\alpha} 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} $$

Remark 2.4.4. On the other hand, for $j = 1, 2, \ldots, [\alpha] + 1$,

$$ {}^{rl}D_a^{\alpha}(t-a)^{\alpha-j} = 0. $$

We could say that $(t-a)^{\alpha-j}$ plays the same role in Riemann-Liouville fractional differentiation as a constant does in classical integer-ordered differentiation.

As a result, we have the following fact:

Lemma 2.4.5. *Let $\alpha > 0$ and $n = [\alpha] + 1$.*

$$ {}^{rl}D_a^{\alpha} f(t) = 0 \Longleftrightarrow f(t) = \sum_{j=1}^{n} c_j (t-a)^{\alpha-j}, \tag{2.4.1.1} $$

where $c_j \in \mathbb{R}$, $(j = 0, 1, 2, \ldots, n)$ are arbitrary constants.

The next result describes ${}^{rl}D_a^{\alpha}$ in the space $AC^n([a, b])$.

Lemma 2.4.6. *Let $\alpha \geq 0$, and $n = [\alpha]+1$. If $f \in AC^n([a, b])$ then the fractional derivative ${}^{rl}D_a^{\alpha} f$ exists almost everywhere on $[a, b]$ and can be represented in the form:*

$$ {}^{rl}D_a^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a) + \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \tag{2.4.1.2} $$

In particular, if $\alpha \in (0, 1)$ and $f \in AC([a, b])$, then

$$ {}^{rl}D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(a)}{(t-a)^{\alpha}} + \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds \right] \tag{2.4.1.3} $$

Proof.

$$ {}^{rl}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds $$

We integrate by parts to get

$$ {}^{rl}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{(t-a)^{n-\alpha} f(a)}{n-\alpha} + \int_a^t \frac{(t-s)^{n-\alpha}}{n-\alpha} f'(s)ds \right] $$

Repeating the process $n-1$ times, we obtain

$$ \begin{aligned} {}^{rl}D_a^{\alpha} f(t) &= \frac{d^n}{dt^n} \left[\sum_{k=0}^{n-1} \frac{(t-a)^{n+k-\alpha} f^{(k)}(a)}{\Gamma(n+k-\alpha+1)} + \frac{1}{\Gamma(2n-\alpha)} \int_a^t (t-s)^{2n-\alpha-1} f^{(n)}(s)ds \right] \\ &= \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds \end{aligned} $$

□

2.4.2 The Caputo Fractional Derivative

In the late 1960's an alternative definition was proposed by Caputo [16, 15] in order to avoid the apparent limitations of the Riemann-Liouville derivative in dealing with differential equations modelling real-life processes.

Definition 2.4.7 ([31, 43])**.** The Caputo fractional derivative of order α of a function f is defined by

$$ ({}^{c}D_a^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, $$

where $n = [\alpha] + 1$.

In relation to the Riemann-Liouville fractional integral, we can see that ${}^{c}D_a^{\alpha} f(t) = I_a^{n-\alpha} f^{(n)}(t)$.

Example 2.4.8. The Caputo derivative of the power function $(t-a)^r,\ r \geq 0$.

In the case where $r \leq n-1,\ r \in \mathbb{N}$ where $n = [\alpha]+1$. , we observe that $\frac{d^n}{dt^n}(t-a)^r = 0$.
It therefore follows directly ${}^{c}D_a^{\alpha}(t-a)^r = 0$

If $r > n-1,\ r \in \mathbb{R}$,

$$
\begin{aligned}
{}^{c}D_a^{\alpha}(t-a)^r &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\frac{d^n}{ds^n}(s-a)^r ds \\
&= \frac{\Gamma(r+1)}{\Gamma(r-n+1)\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}(s-a)^{r-n} ds
\end{aligned}
$$

Using the substitution $s = a + \tau(t-a)$ leads us to

$$
\begin{aligned}
{}^{c}D_a^{\alpha}(t-a)^r &= \frac{\Gamma(r+1)}{\Gamma(r-n+1)\Gamma(n-\alpha)}(t-a)^{r-\alpha}\int_0^1 (1-\tau)^{n-\alpha-1}(\tau)^{r-n} d\tau \\
&= \frac{\Gamma(r+1)}{\Gamma(r-n+1)\Gamma(n-\alpha)}(t-a)^{r-\alpha}B(r-n+1, n-\alpha) \\
&= \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(t-a)^{r-\alpha}
\end{aligned}
$$

Hence,

$$
{}^{c}D_a^{\alpha}(t-a)^r = \begin{cases} 0 & \textit{if } r \leq n-1,\ r \in \mathbb{N}, \\ \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(t-a)^{r-\alpha} & \textit{if } r > n-1,\ r \in \mathbb{R}. \end{cases} \tag{2.4.2.1}
$$

We see that consistent with classical integer-ordered derivatives, for any constant C, ${}^{c}D_a^{\alpha}C = 0, \quad \alpha > 0.$
We also recognize from (2.4.2.1) that:

Lemma 2.4.9. *Let $\alpha > 0$ and $n = [\alpha] + 1$.*

$$
{}^{c}D_a^{\alpha}f(t) = 0 \Longleftrightarrow f(t) = \sum_{j=1}^{n} c_j(t-a)^{n-j}, \tag{2.4.2.2}
$$

where $c_j \in \mathbb{R},\ j = 0, 1, 2, \ldots, n$ are arbitrary constants.

Remark 2.4.10. We note that if $f \in AC^n([a,b])$, then Lemma 2.4.6 is equivalent to saying that

$$
{}^{rl}D_a^{\alpha}f(t) = \sum_{k=0}^{n-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(a) + {}^{c}D_a^{\alpha}f(t)
$$

In particular, if $0 < \alpha < 1$, we have

$$
{}^{rl}D_a^{\alpha}f(t) = \frac{(t-a)^{\alpha}}{\Gamma(1-\alpha)}f(a) + {}^{c}D_a^{\alpha}f(t).
$$

Clearly, we see that if $f^{(k)}(a) = 0$ for $k = 0, 1, 2, \ldots, n-1$ then we have ${}^{rl}D_a^{\alpha}f(t) = {}^{c}D_a^{\alpha}f(t)$.

2.4.3 Some Properties of Fractional Derivatives

In [42], the authors formulated a criteria that distinguish fractional derivatives from other operators. According to the authors, the following properties must be satisfied in order to be considered a fractional derivative:

- The derivative is linear.
- The zero order derivative of a function returns the function to itself, that is,$D^0 f(t) = f(t)$.
- When the order of the fractional derivative is integer n, the derivative reduces to the classical derivative of order n.
- The corresponding fractional integral satisfies the index law, that is, $I^\alpha I^\beta f(t) = I^{\alpha+\beta} F(t)$.
- The derivative satisfies a rule that generalizes the Leibniz rule : $D^n(fg) = \sum_{k=0}^{n} (D^{n-k} f)(D^k g)$.

By these criteria, both the Riemann-Liouville and Caputo formulations are considered fractional derivatives.

Lemma 2.4.11. *Linearity*

$$D_a^\alpha[\lambda f(t) + \beta g(t)] = \lambda D_a^\alpha f(t) + D_a^\alpha g(t), \ \forall \lambda, \beta \in \mathbb{R}, \ \alpha > 0$$

where D_a^α is either the Riemann-Liouville or the Caputo fractional derivative.

Lemma 2.4.12. *Let $n - 1 < \alpha < n, \ n \in \mathbb{N}$. Then for the Riemann-Liouville derivative, we have*

$$\lim_{\alpha \to (n-1)^+} {}^{rl}D_a^\alpha f(t) = f^{(n-1)}(t)$$

$$\lim_{\alpha \to n^-} {}^{rl}D_a^\alpha f(t) = f^{(n)}(t)$$

On the other hand, for the Caputo derivative, the following hold

$$\lim_{\alpha \to (n-1)^+} {}^{c}D_a^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(a)$$

$$\lim_{\alpha \to n^-} {}^{c}D_a^\alpha f(t) = f^{(n)}(t)$$

Unlike with the Riemann-Liouville fractional integral operator, the index law and commutative properties do not hold for either the Riemann-Liouville fractional derivative operator and the Caputo fractional derivative operator. That is, in general, for $\alpha, \beta \in \mathbb{R}^+$,

$$ {}^{rl}D_a^{\alpha}\, {}^{rl}D_a^{\beta} f(t) \neq {}^{rl}D_a^{\beta}\, {}^{rl}D_a^{\alpha} f(t) \neq {}^{rl}D_a^{\alpha+\beta} f(t) $$

$$ {}^{c}D_a^{\alpha}\, {}^{c}D_a^{\beta} f(t) \neq {}^{c}D_a^{\beta}\, {}^{c}D_a^{\alpha} f(t) \neq {}^{c}D_a^{\alpha+\beta} f(t) $$

For example, let us consider $f(t) = t^{\frac{1}{2}}$. We have

$$ {}^{rl}D_0^{\frac{3}{2}}\, {}^{rl}D_0^{\frac{1}{2}} t^{\frac{1}{2}} = {}^{rl}D_0^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{2}\right) = -\frac{1}{4} t^{-\frac{3}{2}} $$

and

$$ {}^{rl}D_0^{\frac{3}{2}+\frac{1}{2}} t^{\frac{1}{2}} = D^{(2)} t^{\frac{1}{2}} = -\frac{1}{4} t^{-\frac{3}{2}}. $$

However

$$ {}^{rl}D_0^{\frac{1}{2}}\, {}^{rl}D_0^{\frac{3}{2}} t^{\frac{1}{2}} = {}^{rl}D_0^{\frac{1}{2}}(0) = 0. $$

Similarly, $f(t) = t^2$, we have

$$ {}^{c}D_0^{\frac{3}{2}}\, {}^{c}D_0^{\frac{1}{2}} t^{\frac{3}{2}} = {}^{c}D_0^{\frac{3}{2}} \left(\frac{3\sqrt{\pi}t}{4}\right) = 0 $$

and

$$ {}^{c}D_0^{\frac{1}{2}}\, {}^{c}D_0^{\frac{3}{2}} t^{\frac{3}{2}} = {}^{c}D_0^{\frac{1}{2}} \left(\frac{3\sqrt{\pi}}{4}\right) = 0 $$

But

$$ {}^{c}D_0^{\frac{1}{2}+\frac{3}{2}} t^{\frac{3}{2}} = D^{(2)} t^{\frac{3}{2}} = -\frac{1}{4} t^{-\frac{1}{2}}. $$

We recall that, in the case of integer-ordered derivatives and integrals, we have the following property: $D^n I^n f(t) = f(t)$ but $I^n D^n f(t) \neq f(t)$ where $D^n, I^n,\ n \in \mathbb{N}$ are the operators for the n-fold differentiation and n-fold integration, respectively.
In other words, the operator D^n is left-inverse to the corresponding integral operator I^n but is not right-inverse. For $f : [a,b] \to \mathbb{R} \in AC^n([a,b]),\ n \in \mathbb{N}$, what we do know is that

$$ I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(t-a)^k}{k!}. $$

In the case of fractional integrals and derivatives, we have an analogous property.

Lemma 2.4.13. *Let* $\alpha > 0$ *and let* $f(t) \in L^p([a,b]),\ (1 \leq p \leq \infty)$ *then*

$$ {}^{rl}D_a^{\alpha} I_a^{\alpha} f(t) = f(t) $$

holds almost everywhere on $[a,b]$.

Lemma 2.4.14. *Let $\alpha > 0$ and let $f(t) \in L^\infty([a,b])$ or $f(t) \in C([a,b])$ then*

$$^cD_a^\alpha I_a^\alpha f(t) = f(t)$$

Lemma 2.4.15. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f \in L^1([a,b])$ such that $I_a^{(n-\alpha)} f(t) \in AC^n[a,b]$ then*

$$I_a^\alpha \, {}^{rl}D_a^\alpha f(t) = f(t) - \sum_{k=1}^{n} \frac{^{rl}D_a^{\alpha-k} f(a)}{\Gamma(\alpha - k + 1)} (t-a)^{\alpha-k}$$

In particular, if $0 < \alpha \leq 1$, then

$$I_a^\alpha \, {}^{rl}D_a^\alpha f(t) = f(t) - \frac{I_a^{1-\alpha}(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}.$$

Lemma 2.4.16. *Let $\alpha > 0$ and $n = [\alpha]+1$. If $f(t) \in AC^n[a,b]$ or $f(t) \in C^n[a,b]$, then*

$$I_a^\alpha \, {}^cD_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(a)}{k!} (t-a)^k.$$

In particular, if $0 < \alpha \leq 1$ and $f \in AC[a,b]$ or $f \in C[a,b]$, then

$$I_a^{\alpha c} D_a^\alpha f(t) = f(t) - f(a).$$

We end this section with the Leibniz formula for the Riemann-Liouville and Caputo fractional derivatives.

Lemma 2.4.17. *Leibniz Formula for Riemann-Liouville Derivatives*
Let $\alpha > 0$ and $n-1 < \alpha < n$, $n \in \mathbb{N}$. Then

$$^{rl}D_a^\alpha (fg)(t) = \sum_{k=0}^{n-1} \binom{\alpha}{k} {}^{rl}D_a^k f(t) \, {}^{rl}D_a^{\alpha-k} g(t) + \sum_{k=0}^{\infty} \binom{\alpha}{k} {}^{rl}D_a^k f(t) I_a^{k-\alpha} g(t)$$

Lemma 2.4.18. *Leibniz Formula for Caputo Derivatives*
Let $0 < \alpha < 1$. Then

$$^cD_a^\alpha (fg)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} g(a)[f(t) - f(a)] + {}^cD_a^\alpha g(t) f(t) + \sum_{k=1}^{\infty} \binom{\alpha}{k} {}^cD_a^k f(t) I_a^{k-\alpha} g(t)$$

2.5 Fractional Differential Equations

In the past few decades, an increasing number of researchers have started using fractional differential equations to model real-life problems. The main advantage of fractional derivatives over the classical integer-ordered derivatives is that mathematical models involving fractional derivatives provide a good description of the memory and hereditary properties of various materials and processes. Indeed, in order to calculate the classical integer-ordered derivative of a function $f(t)$ at a particular point t_0, it is sufficient to know f in an arbitrarily small neighborhood of t_0. On the other hand, we observe from the given definitions of both the Riemann-Liouville and Caputo derivatives that the fractional derivative of a function $f(t)$ defined on an interval $[a, b]$ evaluated at a particular point t_0 depends not only on the local conditions at t_0 but also on all the history of the function throughout the entire interval $[a, t_0]$.
To illustrate this let us consider this simple example:
Let $h(t) : [0, T] \to \mathbb{R}$ be continuous, and $0 < \alpha < 1$. The solution $\phi(t)$ of the boundary value problem

$$ {}^cD_0^\alpha u(t) = h(t) \quad u(0) = u_0 $$

is given by the integral expression

$$ \phi(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. $$

On the other hand, the solution $\psi(t)$ of the boundary value problem

$$ {}^cD_a^\alpha u(t) = h(t) \quad u(a) = \phi(a) \quad 0 < a < T $$

is given by the integral expression

$$ \psi(t) = \phi(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds. $$

We would expect the solutions $\phi(t)$ and $\psi(t)$ to coincide for $t > a$ as is the case if α were equal to 1 but because of the memory property of the fractional derivative, this is not the case. To see this, suppose $\phi(t) \equiv \psi(t)$ for $t > a$. Then

$$ \begin{aligned} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds &= \phi(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds \\ &= u_0 + \frac{1}{\Gamma(\alpha)} \left(\int_0^a (a-s)^{\alpha-1} h(s) ds + \int_a^t (t-s)^{\alpha-1} h(s) ds \right) \end{aligned} $$

Simplifying, we end up with

$$\int_0^a (t-s)^{\alpha-1} h(s)ds = \int_0^a (a-s)^{\alpha-1} h(s)ds$$

for all $t > a$ which obviously leads to a contradiction.
Moreover, it is also important to note that the solution of a fractional differential equations may differ qualitatively from its integer-order analogue. For example, it has been shown in some recent papers [50, 30, 5] that periodic functions have non-periodic fractional derivatives and integrals. As a result, some integer-ordered differential equations may admit periodic solutions while their fractional counterparts do not. Nonetheless, in [6] the authors have shown that under certain conditions, the fractional integrals and derivatives of periodic functions satisfy quasi-periodic properties.
The last two results in the previous section, Lemma 2.4.15 and Lemma 2.4.16 play a major role in the solution of differential equations involving the Riemann-Liouville and the Caputo derivatives. To illustrate, let us assume that $g(t)$ is a given function such that $I_a^\alpha g(t)$ exists. The solution of the differential equation

$$^{rl}D_a^\alpha f(t) = g(t)$$

is given by

$$f(t) = I_a^\alpha g(t) + \sum_{j=1}^{n} c_j (t-a)^{\alpha-j}$$

with $n = [\alpha] + 1$ and for some constants c_j.
By Lemma 2.4.15 we see that the constants are of the form $c_j = \dfrac{^{rl}D_a^{\alpha-j} f(a)}{\Gamma(\alpha - j + 1)}$ and therefore in order to obtain a unique solution, it is natural to prescribe the value of the derivatives $^{rl}D_a^{\alpha-j}$, $j = 1, 2, \ldots, n$ at $t = a$.
Analogously, the unique solution of the differential equation

$$^{c}D_a^\alpha h(t) = g(t)$$

is given by

$$f(t) = I_a^\alpha g(t) + \sum_{j=1}^{n} c_j (t-a)^{j}$$

and with $n = [\alpha] + 1$ and for some constants c_j.
In this case, according to Lemma 2.4.16, the constants take the form $c_j = \dfrac{f^j(a)}{j!}$ and what we need to obtain the unique solution of the differential equation are the values of $f(a), f'(a), \ldots, f^{(n)}(a)$.

In mathematical models describing physical processes Caputo derivatives were usually preferred because the physical interpretation of the initial conditions is clear and easily measured. For example, one may interpret $f(t)$ as the displacement at time t, $f'(t)$ and $f''(t)$ would be the corresponding velocity and acceleration, respectively. However, Heymans and Podlubny [26] demonstrated using several models how it is possible to provide physical interpretations of initial conditions involving fractional derivatives. They argued that there is no need for experimental evaluation of initial conditions involving fractional derivatives of some function $f(t)$. Instead, we may consider another function $g(t)$ related to $f(t)$ via a basic physical law and measure its initial values.
Nowadays, both types of fractional derivatives and initial conditions are widely used in the literature as can be seen in the survey articles by Agarwal et al [2, 1]. For example, in the paper by Belmikki et al [11] the authors investigated a class of fractional differential equations of the Riemann-Liouville type and provided the correct formulation of the initial condition.
Research in the field of fractional differential equations have developed continuously through the years. Mathematical models involving fractional derivatives such as the Ebola epidemic model in [7] for example, provide new insights into real world problems. Moreover, classical results are being reconsidered in the context of fractional calculus, for example, the logistic equation in [4]. An inspection of the articles in specialized journals such as Fractional Calculus and Applied Analysis, Progress in Fractional Differentiation and Applications and in other scientific journals also attest to the rapid growth of the field.

2.6 Multivalued Maps

Let us recall some basic concepts of multivalued maps.
By $\mathcal{P}(X)$we denote the family of all non-empty subsets of a set X.

For a normed space $(X, \|.\|)$, let

- $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$,
- $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
- $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
- $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$,

- $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition 2.6.1. *Let $\mathcal{P}(X)$ be the family of all nonempty subsets of a set X and let Ω be a non-empty set. A mapping $F : \Omega \to \mathcal{P}(X)$ is called a multivalued map.*

Definition 2.6.2. *A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.*

The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{||y|| : y \in G(x)\}\} < \infty$).

Definition 2.6.3. *G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.*

G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$.

Definition 2.6.4. *The set $\Sigma_F \subset X \times Y$, defined by*

$$\Sigma_F = \{(x, y) : x \in X,\ y \in F(x)\}$$

is said to be the graph of F.

F is a closed graph if Σ_F is closed in $X \times Y$.

Lemma 2.6.5. *If the multivalued map G is completely continuous with non-empty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.*

Definition 2.6.6. *G has a fixed point if there is $x \in X$ such that $x \in G(x)$. We denote the fixed point set of the multivalued map G by FixG.*

Definition 2.6.7. *A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function*

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.6.8. *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if*

(i) $t \longmapsto F(t,x)$ *is measurable for each* $x \in \mathbb{R}$*;*

(ii) $x \longmapsto F(t,x)$ *is upper semicontinuous for almost all* $t \in [0,T]$*;*

(iii) *for each* $q > 0$*, there exists* $\varphi_q \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_q(t)$$

for all $\|x\|_\infty \le q$ *and for a. e.* $t \in [0,T]$.

Definition 2.6.9. *For each* $y \in C([0,T],\mathbb{R})$*, we define the set of selections of* F *by*

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Definition 2.6.10. *Let* E *be the Banach space,* X *a non-empty closed subset of* E *and* $G : X \to \mathcal{P}(E)$ *a multivalued map with closed values.*
G *is lower semi-continuous (l.s.c.) if the set* $\{y \in X : G(y) \cap B \neq \emptyset\}$ *is open for any open set* B *in* E.

Definition 2.6.11. *Let* A *be a subset of* $[0,T] \times \mathbb{R}$*.* A *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable if* A *belongs to the* $\sigma-$*algebra generated by all sets of the form* $\mathcal{J} \times \mathcal{D}$*, where* $\mathcal{J}$ *is Lebesgue measurable in* $[0,T]$ *and* $\mathcal{D}$ *is Borel measurable in* $\mathbb{R}$.

Definition 2.6.12. *A subset* $\mathcal{A}$ *of* $L^1([0,T],\mathbb{R})$ *is decomposable if for all* $u, v \in \mathcal{A}$ *and measurable* $\mathcal{J} \subset [0,T] = J$*, the function* $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$*, where* $\chi_{\mathcal{J}}$ *stands for the characteristic function of* $\mathcal{J}$.

Definition 2.6.13. *Let* $F : [0,T]. \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *be a multivalued map with non-empty compact values. Define a multivalued operator* $\mathcal{F} : C([0,T] \times \mathbb{R}) \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ *associated with* F *as*

$$\mathcal{F}(x) = \{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\},$$

which is called the Nymetzki operator associated with F.

Definition 2.6.14. *Let* $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *be a multivalued map with non-empty compact values. We say* F *is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator* $\mathcal{F}$ *is lower semi-continuous and has non-empty closed and decomposable values.*

Let (X, d) be a metric space. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

where $d(a, B) = \inf_{b \in B} d(a, b)$. H_d is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space ([33].

Definition 2.6.15. *A multivalued map $N : X \to P_{cl}(X)$ is called*

(**a**) $\overline{\gamma}-$*Lipschitz if and only if there exists $\overline{\gamma} > 0$ such that*

$$H_d(N(x), N(y)) \leq \overline{\gamma} d(x, y) \text{ for each } x, y \in X;$$

(**b**) *a contraction if and only if it is $\overline{\gamma}-$Lipschitz with $\overline{\gamma} < 1$.*

Lemma 2.6.16. *([36]) Let X be a Banach space. Let $F : [0, T] \times \mathbb{R} \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0, T], X)$ to $C([0, T], X)$. Then the operator*

$$\Theta \circ S_F : C([0, T], X) \to P_{cp,c}(C([0, T], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0, T], X) \times C([0, T], X)$.

Lemma 2.6.17. *([12]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, T], \mathbb{R}))$ be a multivalued operator which is lower semi-continuous (l.s.c.) and has non-empty closed and decomposable values.. Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, T], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.6.18. *([19]) If Γ_1 and Γ_2 are compact valued measurable multivalued maps then the multivalued map $t \longmapsto \Gamma_1(t) \bigcap \Gamma_2(t)$ is measurable. If Γ_n is a sequence of compact valued measurable multivalued maps then the multivalued map $t \longmapsto \bigcap \Gamma_n(t)$ is measurable, and, if $\overline{\bigcup \Gamma_n(t)}$ is compact, then $t \longmapsto \bigcup \Gamma_n(t)$ is measurable.*

Lemma 2.6.19. *([19]) Let X be a separable metric space, $(T, \mathcal{T})$ a measurable space, Γ a multivalued map from T to complete non empty subsets of X. If for each open set U in X, $\Gamma(U) := \{t | \Gamma(t) \cap U \neq \phi\}$ belongs to $\mathcal{T}$, then Γ admits a measurable selection.*

The books of Aubin and Cellina [8], Aubin and Frankowska [9], Deimling [21], and Hu and Papageorgiou [49] detail more properties of multivalued maps.

2.7 Some Fixed Point Theorems

Definition 2.7.1. *Let X be a Banach space and $M \subset X$ be a closed subset. An operator $T : M \to X$ is said to be contractive if there exists $\lambda \in (0, 1]$ such that*

$$\|Tu - Tv\| \leq \lambda \|u - v\|$$

for all $u, v \in M$.

Theorem 2.7.2. *(Banach Fixed Point Theorem) Let X be a Banach space and $M \subset X$ be a closed subset. If the operator $T : M \to M$ is contractive then T has a unique fixed point in M.*

Definition 2.7.3. *Let E be a real Banach space and P a subset of E. P is called a cone if:*

- **(i)** *P is closed, non-empty and $P \neq 0$,*
- **(ii)** *$ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,*
- **(iii)** *$P \cap (-P) = 0$.*

To visualize, we may think of E as the three-dimensional space $\mathbb{R}$ with the euclidean norm, and P the infinite circular cone with its vertex at the origin.

The next result is the known Guo-Krasnosel'skii fixed point theorem [25]:

Lemma 2.7.4. *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that*

- **(i)** *$\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$; or*
- **(ii)** *$\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$.*

Then the operator P has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7.5. *([20]) Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Lemma 2.7.6 (Nonlinear Alternative of Leray-Schauder Type [23])**.** *Let X be a Banach space and C a nonempty convex subset of X. Let U be a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ be a continuous and compact operator. Then either*

- *(a) T has fixed points or*

(b) There exist $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u = \lambda T(u)$.

We also have the nonlinear alternative for multivalued maps:

Lemma 2.7.7 (Nonlinear Alternative of Leray-Schauder Type [23])**.** *Let* X *be a Banach space and* C *a nonempty convex subset of* X*. Let* U *be a relatively open subset of* C *with* $0 \in U$ *and* $T : \overline{U} \to P_{cp,c}(X)$ *be an upper semicontinuous and compact map. Then either*

(a) T *has fixed points or*

(b) There exist $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u \in \lambda T(u)$.

Chapter 3

Fractional Differential Equations and Inclusions

The following results are from [3].
We consider for $T > 0$ and $1 < q \leq 2$ the following fractional differential equation

$$^{c}D^{q}x(t) = f(t, x(t)), \quad t \in [0, T], \tag{3.0.0.1}$$

where $^{c}D^{q}$ denotes the Caputo fractional derivative of order q and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$. We study (3.0.0.1) subject to two families of boundary conditions:
(i) Mixed boundary conditions

$$Tx'(0) = -ax(0) - bx(T) \quad Tx'(T) = bx(0) + dx(T), \tag{3.0.0.2}$$

(ii) Closed boundary conditions

$$x(T) = \alpha x(0) + \beta Tx'(0), \quad Tx'(T) = \gamma x(0) + \delta Tx'(0), \tag{3.0.0.3}$$

where $a, b, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ are given constants.

Here we remark that the boundary conditions (3.0.0.2) interpolate between Neumann ($a = b = d = 0$) and Dirichlet ($a = b = d = \infty$) boundary conditions while (3.0.0.3) include quasi-periodic boundary conditions ($\beta = \gamma = 0$) and interpolate between periodic ($\alpha = \delta = 1,\ \beta = \gamma = 0$) and antiperiodic $\alpha = \delta = -1,\ \beta = \gamma = 0$) boundary conditions. Notice that Zaremba boundary conditions $x(0) = 0, x'(T) = 0$ can be considered either as mixed boundary conditions with $a = \infty, b = d = 0$ or as quasi-periodic boundary conditions with $\alpha = \infty,\ \gamma = \delta = 0$. For more details on Zaremba boundary conditions, see [29, 53, 10].

Lemma 3.0.1. *For $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, a unique solution of the boundary value problem* (3.0.0.1) *and* (3.0.0.2) *is given by*

$$x(t) = \int_0^T G_1(t,s) f(s, x(s)) ds,$$

where $G_1(t,s)$ is the Green's function given by

$$G_1(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} - \\ \dfrac{[T(b+d)+(b^2-ad)t](T-s)^{q-1}}{\Delta_1 T \Gamma(q)} \\ -\dfrac{[(a+b)t-(1+b)T](T-s)^{q-2}}{\Delta_1 \Gamma(q-1)}, & 0 \le s \le t, \\ -\dfrac{[T(b+d)+(b^2-ad)t](T-s)^{q-1}}{\Delta_1 T \Gamma(q)} \\ -\dfrac{[(a+b)t-(1+b)T](T-s)^{q-2}}{\Delta_1 \Gamma(q-1)}, & t \le s \le T, \end{cases} \tag{3.0.0.4}$$

with

$$\Delta_1 = (1+b)(b+d) - (a+b)(d-1) \neq 0. \tag{3.0.0.5}$$

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds - c_0 - c_1 t. \tag{3.0.0.6}$$

In view of the relations ${}^cD^q\ I^q x(t) = x(t)$ and $I^q\ I^p x(t) = I^{q+p} x(t)$ for $q, p > 0, x \in L(0,T)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s) ds - c_1.$$

Using the boundary conditions (3.0.0.2) in(3.0.0.6), we find that

$$c_0 = \frac{1}{\Delta_1}\Big\{(b+d)\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - T(1+b)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big\},$$

$$c_1 = \frac{1}{\Delta_1}\Big\{\frac{(b^2-ad)}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + (a+b)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big\},$$

where Δ_1 is given by (3.0.0.5). Substituting the values of c_0 and c_1 in (3.0.0.6), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &- \frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &+ [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big\}. \\ &= \int_0^T G_1(t,s) f(s,x(s))ds, \end{aligned}$$

where $G_1(t,s)$ is given by (3.0.0.4). This completes the proof.

Lemma 3.0.2. *For $f \in C([0,T]\times\mathbb{R},\mathbb{R})$, the unique solution of the boundary value problem* (3.0.0.1) *and* (3.0.0.3) *is given by*

$$x(t) = \int_0^T G_2(t,s) f(s,x(s))ds,$$

where $G_2(t,s)$ is the Green's function given by

$$G_2(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} - \dfrac{[T(1-\delta)+\gamma t](T-s)^{q-1}}{\Delta_2 T\Gamma(q)} \\ -\dfrac{[(1-\alpha)t-(1-\beta)T](T-s)^{q-2}}{\Delta_2\Gamma(q-1)}, & 0 \le s \le t, \\ -\dfrac{[T(1-\delta)+\gamma t](T-s)^{q-1}}{\Delta_2 T\Gamma(q)} \\ -\dfrac{[(1-\alpha)t-(1-\beta)T](T-s)^{q-2}}{\Delta_2\Gamma(q-1)}, & t \le s \le T, \end{cases} \tag{3.0.0.7}$$

with

$$\Delta_2 = \gamma(1-\beta)+(1-\alpha)(1-\delta) \neq 0. \tag{3.0.0.8}$$

Proof. We do not provide the proof as it is similar to that of Lemma 3.0.1.

3.1 Existence of Solutions

In relation to the problems (3.0.0.1) and (3.0.0.2), and (3.0.0.1) and (3.0.0.3), we define

$$\mu_1 = \frac{1}{\Gamma(q+1)}\Big\{1 + \frac{|b+d+b^2-ad| + q|a-1|}{\Delta_1}\Big\}, \tag{3.1.0.1}$$

$$\mu_2 = \frac{1}{\Gamma(q+1)}\Big\{1 + \frac{|1-\delta+\gamma| + q|\alpha-\beta|}{\Delta_2}\Big\}, \tag{3.1.0.2}$$

where Δ_1 and Δ_2 are given by (3.0.0.5) and (3.0.0.8) respectively.

Theorem 3.1.1. *Assume that there exist constants* $0 \leq \kappa < \frac{1}{\mu_1}$ *and* $M > 0$ *such that* $|f(t,x)| \leq \frac{\kappa}{\Gamma^q}|x| + M$ *for all* $t \in [0,T], x \in C[0,T]$. *Then the boundary value problem* (3.0.0.1) *and* (3.0.0.2) *has at least one solution.*

Proof. Using Lemma 3.0.1, the problem (3.0.0.1) and (3.0.0.2) can be transformed into a fixed point problem as

$$x = \mathcal{F}x, \tag{3.1.0.3}$$

where $\mathcal{F} : C[0,T] \to C[0,T]$ is given by

$$\begin{aligned}(\mathcal{F}x)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\\ &- \frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\\ &+ [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big\}, t \in [0,T].\end{aligned}$$

Thus we just need to prove the existence of at least one solution $x \in C[0,T]$ satisfying (3.1.0.3). Define a suitable ball $B_R \subset C[0,T]$ with radius $R > 0$ as

$$B_R = \{x \in C[0,T] : \max_{t\in[0,T]} |x(t)| < R\},$$

where R will be fixed later. Then, it is sufficient to show that $\mathcal{F} : \overline{B}_R \to C[0,T]$ satisfies

$$x \neq \lambda\mathcal{F}x, \quad \forall\ x \in \partial B_R \ \text{ and } \ \forall\ \lambda \in [0,1].) \tag{3.1.0.4}$$

Let us set

$$H(\lambda, x) = \lambda\mathcal{F}x, \quad x \in C(\mathbb{R}) \quad \lambda \in [0,1].$$

Then, by the Arzela-Ascoli theorem, $h_\lambda(x) = x - H(\lambda,x) = x - \lambda\mathcal{F}x$ is completely continuous. If (3.1.0.4) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda\mathcal{F}, B_R, 0)\\ &= \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0)\\ &= \deg(I, B_R, 0)\\ &= 1 \neq 0, \quad 0 \in B_r,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = x - \lambda \mathcal{F} x = 0$ for at least one $x \in B_R$. In order to prove (3.1.0.4), we assume that $x = \lambda \mathcal{F} x$ for some $\lambda \in [0,1]$ and for all $t \in [0,T]$ so that

$$\begin{aligned}
|x(t)| &= |\lambda \mathcal{F} x(t)| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&+ \frac{|T(b+d)+(b^2-ad)t|}{T|\Delta_1|} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&+ \frac{|(a+b)t-(1+b)T|}{|\Delta_1|} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))| ds \\
&\leq \Big(\frac{\kappa}{T^q}|x| + M\Big) \Big[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds \\
&+ \frac{|T(b+d)+(b^2-ad)t|}{T|\Delta_1|} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \\
&+ \frac{|(a+b)t-(1+b)T|}{|\Delta_1|} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \Big] \\
&\leq \Big(\frac{\kappa}{T^q}|x| + M\Big) \frac{T^q}{\Gamma(q+1)} \Big[1 + \frac{|b+d+b^2-ad| + q|a-1|}{\Delta_1} \Big],
\end{aligned}$$

which, on taking norm ($\sup_{t\in[0,T]} |x(t)| = \|x\|$) and using (3.1.0.1) yields

$$\|x\| \leq \frac{MT^q\mu_1}{(1-\kappa\mu_1)}.$$

Letting $R = \frac{MT^q\mu_1}{(1-\kappa\mu_1)} + 1$, (3.1.0.4) holds. This completes the proof.

Example 2.1. Consider the following boundary value problem

$$\begin{aligned}
&{}^cD^q x(t) = \frac{1}{(4\pi)} \sin(\frac{2\pi x}{T^q}) + \frac{|x|}{1+|x|}, \quad t \in [0,T],\ 1 < q \leq 2, \\
&Tx'(0) = -2x(0) - x(T) \quad Tx'(T) = x(0) + x(T).
\end{aligned} \tag{3.1.0.5}$$

Clearly

$$\Big|f(t,x)\Big| = \Big|\frac{1}{(4\pi)} \sin(\frac{2\pi x}{T^q}) + \frac{|x|}{1+|x|}\Big| \leq \frac{1}{2T^q}\|x\| + 1,$$

with $\kappa = \frac{1}{2} < \frac{4\Gamma(q+1)}{5+q}$ for $1 < q \leq 2$ and $M = 1$. Thus, the conclusion of Theorem 3.1.1 applies to the problem (3.1.0.5) .

Theorem 3.1.2. *Assume that there exist constants* $0 \leq \overline{\kappa} < \frac{1}{\mu_2}$ *and* $\overline{M} > 0$ *such that* $|f(t,x)| \leq \frac{\overline{\kappa}}{T^q}|x| + \overline{M}$ *for all* $t \in [0,T], x \in C[0,T]$. *Then the boundary value problem* (3.0.0.1) *and* (3.0.0.3) *has at least one solution.*

Proof. Using Lemma 3.0.2 together with the arguments employed in the proof of Theorem 3.1.1, the proof can easily be constructed. We omit the details.

Remark 3.1.3. *For positive constants* N_1, N_2, *we can modify the assumption on the nonlinear function* $f(t,x)$ *in Theorem 3.1.1 and Theorem 3.1.2 respectively as*

$$|f(t,x)| \leq \frac{N_1}{T^q \mu_1}, \quad \forall t \in [0,T], \quad x \in [-N_1, N_1], \tag{3.1.0.6}$$

$$|f(t,x)| \leq \frac{N_2}{T^q \mu_2}, \quad \forall t \in [0,T], \quad x \in [-N_2, N_2], \tag{3.1.0.7}$$

where μ_1, μ_2 *are respectively given by* (3.1.0.1) *and* (3.1.0.2).

Remark 3.1.4. *We list down some interesting situations arising from the results.*

- **(i)** *The results for a nonlinear boundary value problem of fractional order* $q \in (1,2]$ *with quasi-periodic (quasi-antiperiodic) boundary conditions follow as a special case of Theorem 3.1.2 by taking* $\beta = \gamma = 0$.
- **(ii)** *The results for an anti-periodic boundary value problem of fractional differential equations of order* $q \in (1,2]$ *can be obtained by taking* $\alpha = -1 = \delta, \ \beta = \gamma = 0$.
- **(iii)** *For* $q = 2$, *we obtain new results for second order boundary value problems with mixed and closed boundary conditions. In this case, the Green's functions* $G_1(t,s)$ *and* $G_2(t,s)$ *take the form*

$$G_1(t,s) = \begin{cases} (t-s) - \dfrac{[T(b+d) + (b^2 - ad)t](T-s)}{T\ \Delta_1} \\ -\dfrac{(a+b)t - (1+b)T}{\Delta_1}, & 0 \leq s \leq t \leq T, \\ \\ -\dfrac{[T(b+d) + (b^2 - ad)t](T-s)}{T\ \Delta_1} \\ -\dfrac{(a+b)t - (1+b)T}{\Delta_1}, & 0 \leq t \leq s \leq T, \end{cases}$$

with

$$\Delta_1 = (1+b)(b+d) - (a+b)(d-1) \neq 0.$$

$$G_2(t,s) = \begin{cases} (t-s) - \dfrac{[T(1-\delta)+\gamma t](T-s)}{T\ \Delta_2} \\ -\dfrac{(1-\alpha)t-(1-\beta)T}{\Delta_2}, & 0 \le s \le t \le T, \\ \\ -\dfrac{[T(1-\delta)+\gamma t](T-s)}{T\ \Delta_2} \\ -\dfrac{(1-\alpha)t-(1-\beta)T}{\Delta_2}, & 0 \le t \le s \le T, \end{cases}$$

with

$$\Delta_2 = \gamma(1-\beta) + (1-\alpha)(1-\delta) \neq 0.$$

The Green's functions $G_2(t,s)$ for the second order anti-periodic boundary value problem ($\alpha = -1 = \delta$, $\beta = \gamma = 0$) is

$$G_2(t,s) = \begin{cases} \frac{1}{4}(-T-2t+2s), & 0 \le t < s \le T, \\ \\ \frac{1}{4}(-T+2t-2s), & 0 \le s \le t \le T. \end{cases}$$

3.2 Fractional Differential Inclusions

In this section, we consider the fractional differential inclusions

$$^cD^q x(t) \in F(t, x(t)), \quad t \in [0,T],\ T > 0, \quad 1 < q \le 2, \qquad (3.2.0.1)$$

where $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. We will study the existence of solutions for (3.2.0.1) subject to two families of boundary conditions (3.0.0.2) and (3.0.0.3).

3.2.1 The Convex Case

Theorem 3.2.1. *Assume that*

($\mathbf{H_1}$) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has compact and convex values;*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T], \mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty) \text{ for each } (t,x) \in [0,T] \times \mathbb{R};$$

$(\mathbf{H_3})$ *there exists a number* $M_1 > 0$ *such that*

$$\frac{\Gamma(q+1)M_1}{T^q\Big(1+\frac{|b+d+b^2-ad|+q|a-1|}{\Delta_1}\Big)\psi(M_1)\|p\|_{L^\infty}} > 1.$$

Then the boundary value problem (3.2.0.1) *and* (3.0.0.2) *has at least one solution on* $[0,T]$.

Proof. Define an operator $\Omega : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ as

$$\begin{aligned}\Omega(x) = \Big\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ -\frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ +[(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds\Big\}, \quad f \in S_{F,x}\Big\}.\end{aligned}$$

We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray- Schauder type .

The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0,T],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$\begin{aligned}h_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\ -\frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\ +[(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)ds\Big\}, \quad i=1,2.\end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,T]$, we have

$$\begin{aligned}&[\omega h_1 + (1-\omega)h_2](t) \\ &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega)f_2(s)]ds \\ &-\frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}[\omega f_1(s)+(1-\omega)f_2(s)]ds \\ &+[(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}[\omega f_1(s)+(1-\omega)f_2(s)]ds\Big\}.\end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$.

Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\|_\infty \leq r\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$
\begin{aligned}
h(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\
& - \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\
& + [(a+b)t-(1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
|h(t)| \leq & \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |f(s)|ds \\
& + \Big|\frac{1}{|\Delta_1|}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T}\Big| \int_0^T \frac{|T-s|^{q-1}}{\Gamma(q)} |f(s)|ds \\
& + |(a+b)t-(1+b)T| \int_0^T \frac{|T-s|^{q-2}}{\Gamma(q-1)} |f(s)|ds \Big\} \\
\leq & \frac{T^{q-1}}{\Gamma(q)}\Big(1 + \frac{|b+d+b^2-ad|}{|\Delta_1|}\Big) \int_0^T \varphi_r(s)ds + \frac{|a-1|T^q}{\Gamma(q)|\Delta_1|} p(t)\psi(\|x\|_\infty)
\end{aligned}
$$

Thus,

$$
\|h\|_\infty \leq \frac{T^{q-1}}{\Gamma(q)}\Big(1 + \frac{|b+d+b^2-ad|}{|\Delta_1|}\Big)\|\varphi_r\|_{L^1} + \frac{|a-1|T^q}{\Gamma(q)|\Delta_1|}\|p\|_\infty \psi(r)
$$

Now we show that Ω maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,T],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$
\begin{aligned}
&|h(t'')-h(t')|\\
&=\Big|\int_0^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds-\int_0^{t'}\frac{(t'-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&-\frac{1}{\Delta_1}\Big\{\frac{[(b^2-ad)(t''-t')]}{T}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&+(a+b)(t''-t')\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds\Big\}\Big|\\
&\le\Big|\int_0^{t'}\frac{[(t''-s)^{q-1}-(t'-s)^{q-1}]}{\Gamma(q)}f(s)ds\Big|+\Big|\int_{t'}^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|\\
&+\Big|\frac{(t''-t')}{\Delta_1}\Big\{\frac{(b^2-ad)}{T}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s)ds+(a+b)\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds\Big\}\Big|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_{r'}$ as $t''-t'\to 0$. As Ω satisfies the above three assumptions, therefore it follows by the Ascoli-Arzela theorem that $\Omega : C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.
In our next step, we show that Ω has a closed graph. Let $x_n\to x_*, h_n\in\Omega(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega(x_*)$. Associated with $h_n\in\Omega(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$
\begin{aligned}
h_n(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_n(s)ds\\
&-\frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f_n(s)ds\\
&+[(a+b)t-(1+b)T]\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f_n(s)ds\Big\}.
\end{aligned}
$$

Thus we have to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in[0,T]$,

$$
\begin{aligned}
h_*(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds\\
&-\frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f_*(s)ds\\
&+[(a+b)t-(1+b)T]\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f_*(s)ds\Big\}.
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &- \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ [(a+b)t-(1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \Big\}. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t) - h_*(t)\| \\ &= \Big\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \\ &- \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \\ &+ [(a+b)t-(1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (f_n(s) - f_*(s))ds \Big\} \Big\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, it follows by Lemma 2.6.16 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &- \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &+ [(a+b)t-(1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \Big\} \end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let x be a solution of (3.0.0.1). Then there exists $f \in L^1([0,T],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &- \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ [(a+b)t-(1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \Big\}. \end{aligned}$$

In view of (H_2), for each $t \in [0,T]$, we obtain

$$|x(t)| \leq \frac{T^q}{\Gamma(q+1)}\Big(1 + \frac{|b+d+b^2-ad| + q|a-1|}{|\Delta_1|}\Big)\psi(\|x\|_\infty)p(t).$$

Consequently, we have

$$\frac{\Gamma(q+1)\|x\|_\infty}{T^q\Big(1 + \frac{|b+d+b^2-ad|+q|a-1|}{|\Delta_1|}\Big)\psi(\|x\|_\infty)\|p\|_{L^\infty}} \leq 1,$$

In view of (H_3), there exists M_1 such that $\|x\|_\infty \neq M_1$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) : \|x\|_\infty < M_1 + 1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,T],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (3.2.0.1) and (3.0.0.2). This completes the proof.

Theorem 3.2.2. *Assume that* $(H_1), (H_2)$ *and the following condition hold:*

($\mathbf{H_1}$) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is* L^1-*Carathéodory and has compact and convex values;*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|_\infty) \text{ for each}(t,x) \in [0,T]\times\mathbb{R};$$

($\overline{\mathbf{H}}_\mathbf{3}$) *there exists a number* $\overline{M}_1 > 0$ *such that*

$$\frac{\Gamma(q+1)\overline{M}_1}{T^q\Big(1 + \frac{|1-\delta+\gamma|+q|\alpha-\beta|}{|\Delta_2|}\Big)\psi(\overline{M}_1)\|p\|_{L^\infty}} > 1.$$

Then the boundary value problem (3.2.0.1) *and* (3.0.0.3) *has at least one solution on* $[0,T]$.

Proof. Define an operator $\Omega : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ as

$$\Omega(x) = \Big\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds$$
$$- \frac{1}{\Delta_2}\Big\{ \frac{[T(1-\delta)+\gamma t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds$$
$$+ [(1-\alpha)t - (1-\beta)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \Big\}, \quad f \in S_{F,x} \Big\}.$$

The rest of the proof employs the same arguments used in the proof of Theorem 3.2.1.

3.2.2 The Nonconvex Case

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [12] for lower semicontinuous maps with decomposable values.

Theorem 3.2.3. *Assume that the following conditions hold:*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty)$ *for each* $(t,x) \in [0,T]\times\mathbb{R}$;

($\mathbf{H_3}$) *there exists a number* $M_1 > 0$ *such that*

$$\frac{\Gamma(q+1)M_1}{T^q\Big(1 + \frac{|b+d+b^2-ad|+q|a-1|}{\Delta_1}\Big)\psi(M_1)\|p\|_{L^\infty}} > 1.$$

($\mathbf{H_4}$) $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(**a**) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable,*

(**b**) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,T]$;

($\mathbf{H_5}$) *for each* $\sigma > 0$, *there exists* $\varphi_\sigma \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \varphi_\sigma(t)$$

for all x *such that* $\|x\|_\infty \le \sigma$ *and for a.e.* $t \in [0,T]$.

Then the boundary value problem (3.2.0.1) *and* (3.0.0.2) *has at least one solution on* $[0, T]$.

Proof. In [22], the authors proved that if (H_4) and (H_5) hold then F is of l.s.c. type.
Then from Lemma 2.6.17, there exists a continuous function $f : C([0, T], \mathbb{R}) \to L^1([0, T], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0, T], \mathbb{R})$. Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad t \in [0, T], \ T > 0, \quad 1 < q \leq 2, \\ Tx'(0) = -ax(0) - bx(T) \quad Tx'(T) = bx(0) + dx(T), \end{cases} \tag{3.2.2.1}$$

Observe that if $x \in C^2([0, T])$ is a solution of (3.2.2.1), then x is a solution to the problem (3.2.0.1) and (3.0.0.2). In order to transform the problem (3.2.2.1) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ &- \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d) + (b^2 - ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ &+ [(a+b)t - (1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(x(s))ds \Big\} \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof follows form that of Theorem 3.2.1 and the proof is complete.

Theorem 3.2.4. *Assume that the following conditions hold:*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *and a function* $p \in C([0, T], \mathbb{R}^+)$ *such that*

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \leq p(t)\psi(\|x\|_\infty) \ \textit{for each}(t, x) \in [0, T] \times \mathbb{R};$$

($\overline{\mathbf{H}}_\mathbf{3}$) *there exists a number* $\overline{M}_1 > 0$ *such that*

$$\frac{\Gamma(q+1)\overline{M}_1}{T^q\Big(1 + \frac{|1-\delta+\gamma|+q|\alpha-\beta|}{|\Delta_2|}\Big)\psi(\overline{M}_1)\|p\|_{L^\infty}} > 1.$$

($\mathbf{H_4}$) $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that* (**a**) *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable,*

(**b**) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t\in[0,T]$;*

(**H$_5$**) *for each $\sigma>0$, there exists $\varphi_\sigma \in L^1([0,T],\mathbb{R}_+)$ such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \varphi_\sigma(t)$$

for all x such that $\|x\|_\infty \le \sigma$ and for a.e. $t\in[0,T]$.

Then the boundary value problem (3.2.0.1) *and* (3.0.0.3) *has at least one solution on* $[0,T]$.

Proof. The proof is similar to that of Theorem 3.2.3.

Next we prove the existence of solutions for the problem (3.2.0.1) with a non-convex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [20].

Theorem 3.2.5. *Assume that the following conditions hold:*

(**H$_6$**) *$F:[0,T]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ is such that $F(.,x):[0,T]\to P_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$.*

(**H$_7$**) *$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x-\bar{x}|$ for almost all $t\in[0,T]$ and $x,\bar{x}\in\mathbb{R}$ with $m\in C([0,T],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,T]$.*

Then the problem (3.2.0.1) *and* (3.0.0.2) *has at least one solution on* $[0,T]$ *if*

$$\frac{T^q\|m\|_{L^\infty}}{\Gamma(q+1)}\Big(1+\frac{|b+d+b^2-ad|+q|a-1|}{|\Delta_1|}\Big)<1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x\in C([0,T],\mathbb{R})$ since by the assumption (H_6), it follows from Lemma 2.6.19 that F has a measurable selection.

Now we show that the operator Ω satisfies the assumptions of Lemma 2.6.17 To show that $\Omega(x)\in P_{cl}((C[0,T],\mathbb{R}))$ for each $x\in C([0,T],\mathbb{R})$, let $\{u_n\}_{n\ge 0}\in\Omega(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C([0,T],\mathbb{R})$. Then $u\in C([0,T],\mathbb{R})$ and there exists $v_n\in S_{F,x}$ such that, for each $t\in[0,T]$,

$$\begin{aligned} u_n(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds \\ & - \frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds \\ & + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)ds\Big\}. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) \to u(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds \\ &- \frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds \\ &+ [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)ds\Big\}. \end{aligned}$$

Hence $u \in \Omega(x)$.

Next we show that there exists $\gamma_1 < 1$ such that

$$H_d(\Omega(x),\Omega(\bar{x})) \leq \gamma_1 \|x-\bar{x}\|_\infty \quad \text{for each} \quad x,\bar{x} \in C([0,T],\mathbb{R}).$$

Let $x,\bar{x} \in C([0,T],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds \\ &- \frac{1}{\Delta_1}\Big\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)ds \\ &+ [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_1(s)ds\Big\}. \end{aligned}$$

By (H_7), we have

$$H_d(F(t,x),F(t,\bar{x})) \leq m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w| \leq m(t)|x(t)-\bar{x}(t)|, \quad t \in [0,T].$$

Define $U : [0,T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \leq m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t,\bar{x}(t))$ is measurable by Lemma 2.6.18, there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [0,T]$, we have $|v_1(t)-v_2(t)| \leq m(t)|x(t)-\bar{x}(t)|$.

For each $t \in [0, T]$, let us define

$$\begin{aligned} h_2(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s) ds \\ & - \frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_2(s) ds \\ & + [(a+b)t-(1+b)T] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_2(s) ds \Big\}. \end{aligned}$$

Thus

$$\begin{aligned} & |h_1(t) - h_2(t)| \\ & \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)| ds \\ & \quad + \Big|\frac{1}{\Delta_1}\Big\{ \frac{[T(b+d)+(b^2-ad)t]}{T\Delta_1}\Big| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)| ds \\ & \quad + \Big|\frac{[(a+b)t-(1+b)T]}{\Delta_1}\Big| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |v_1(s) - v_2(s)| ds \Big\}. \\ & \leq \frac{T^{q-1}}{\Gamma(q)} \Big(1 + \frac{|b+d+b^2-ad| + (q-1)|a-1|}{\Delta_1}\Big) \int_0^T m(s) ds \|x - \overline{x}\|. \end{aligned}$$

Hence

$$\|h_1(t) - h_2(t)\|_\infty \leq \frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)} \Big(1 + \frac{|b+d+b^2-ad| + (q-1)|a-1|}{\Delta_1}\Big) \|x - \overline{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x), \Omega(\bar{x})) & \leq \gamma_1 \|x - \bar{x}\|_\infty \\ & \leq \frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)} \Big(1 + \frac{|b+d+b^2-ad| + (q-1)|a-1|}{\Delta_1}\Big) \|x - \overline{x}\|_\infty. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 2.7.5 that Ω has a fixed point x which is a solution of (3.2.0.1)-(3.0.0.2). This completes the proof.

Theorem 3.2.6. *Assume that the following conditions hold:*

($\mathbf{H_6}$) $F : [0, T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(., x) : [0, T] \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

($\mathbf{H_7}$) $H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ *for almost all* $t \in [0, T]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in C([0, T], \mathbb{R}^+)$ *and* $d(0, F(t, 0)) \leq m(t)$ *for almost all* $t \in [0, T]$.

Then the problem (3.1)-(1.3) has at least one solution on $[0, T]$ *if*

$$\frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)}\left(1+\frac{|1-\delta+\gamma|+(q-1)|\alpha-\beta|}{\Delta_2}\right) < 1.$$

Proof. We do not provide the proof as it can easily be traced on the pattern of the proof of Theorem 3.2.5.

Example Consider the following inclusion boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t, x(t)), \quad t \in [0,1], \\ x'(0) = -x(0) - \frac{1}{3}x(1) \quad x'(1) = \frac{1}{3}x(0) + \frac{2}{3}x(1), \end{cases} \tag{3.2.2.2}$$

where $q = 3/2,\ T = 1,\ a = 1, b = 1/3,\ d = 2/3$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = [\frac{x^3}{x^3+3} + t^3 + 3, \frac{x}{x+1} + t + 1].$$

For $f \in F$, we have

$$|f| \leq \max(\frac{x^3}{x^3+3} + t^3 + 3, \frac{x}{x+1} + t + 1) \leq 5, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 5 = p(t)\psi(\|x\|_\infty), \quad x \in \mathbb{R},$$

with $p(t) = 1,\ \psi(\|x\|_\infty) = 5$. Further, using the condition

$$\frac{\Gamma(q)M_1}{T^{q-1}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{\Delta_1}\right)\psi(M_1)\|p\|_{L^1}} > 1,$$

we find that $M_1 > \frac{25}{2\sqrt{\pi}}$. Clearly, all the conditions of Theorem 3.2.1 are satisfied. So there exists at least one solution of the problem (3.2.2.2) on [0,1].

Chapter 4

Positive Solutions of a Fractional Thermostat Model

In this chapter we present the results from [40].
Infante and Webb [27] studied the nonlocal boundary value problem

$$-u'' = f(t,u), \quad t \in (0,1), \quad u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0.$$

which models a thermostat insulated at $t = 0$, with the controller at $t = 1$ adding or discharging heat depending on the temperature detected by the sensor at $t = \eta$. Using fixed point index theory and some results on their work on Hammerstein integral equations [28, 55], they obtained results on the existence of positive solutions of the boundary value problem. In particular, they have shown that if $\beta \geq 1 - \eta$, then positive solutions exist under suitable conditions on f. This type of boundary value problem was earlier investigated by Guidotti and Merino [24] for the linear case with $\eta = 0$ where they have shown a loss of positivity as β decreases. In this chapter, we consider the following fractional analogue of the thermostat model

$$-\,{}^cD^\alpha u(t) = f(t, u(t)), \quad t \in [0,1], \tag{4.0.0.1}$$

where $1 < \alpha \leq 2$, ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α and $f \in C([0,1] \times [0,\infty), [0,\infty))$ subject to the boundary conditions:

$$u'(0) = 0, \qquad \beta\ {}^cD^{\alpha-1}u(1) + u(\eta) = 0, \tag{4.0.0.2}$$

where $\beta > 0$, $0 \leq \eta \leq 1$ are given constants.

We point out that for $\alpha = 2$, we recover the second-order problem of [27] . We use the properties of the corresponding Green's function and the Guo-Krasnosel'skii fixed point theorem to show the existence of positive solutions of (5)-(6) under the condition that the nonlinearity f is either sublinear or superlinear.

We start by solving an auxiliary problem to get an expression for the Green's function of the boundary value problem (5)-(6).

Lemma 4.0.1. *Suppose $f \in C[0,1]$. A function $u \in C[0,1]$ is a solution of the boundary value problem*

$$-{}^{c}D^{\alpha}u(t) = f(t), \;\; u'(0) = 0, \;\; \beta\, {}^{c}D^{\alpha-1}u(1) + u(\eta) = 0, \quad t \in [0,1]$$

if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,s)f(s)ds,$$

where $G(t,s)$ is the Green's function (depending on α) given by

$$G(t,s) = \beta + H_{\eta}(s) - H_t(s) \tag{4.0.0.3}$$

and for $r \in [0,1]$, $H_r : [0,1] \to \mathbb{R}$ is defined as $H_r(s) = \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \leq r$ and $H_r(s) = 0$ for $s > r$.

Proof. Using Lemma 2.4.16 we have, for some constants $c_0, c_1 \in \mathbb{R}$,

$$u(t) = -I_0^{\alpha}f(t) + c_0 + c_1 t = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + c_0 + c_1 t. \tag{4.0.0.4}$$

In view of Lemma 2.1 we obtain

$$u'(t) = -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)ds + c_1$$

Since $u'(0) = 0$, we find that $c_1 = 0$.
It also follows that

$${}^{c}D^{\alpha-1}u(t) = -I^1 u(t).$$

Using the boundary condition $\beta\, {}^{c}D^{\alpha-1}u(1) + u(\eta) = 0$, we get

$$c_0 = \beta \int_0^1 f(s)ds + \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds.$$

Finally, substituting the values of c_0 and c_1 in (4.0.0.4), we have

$$\begin{aligned} u(t) &= \beta \int_0^1 f(s)ds + \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\ &= \int_0^1 G(t,s)f(s)ds, \end{aligned}$$

where $G(t,s)$ is given by (4.0.0.3). This completes the proof. □

Remark 4.0.2. *We observe that H_r is continuous on $[0,1]$ for any $r \in [0,1]$. Thus, $G(t,s)$ given by (4.0.0.3) is continuous on $[0,1] \times [0,1]$.*

Remark 4.0.3. *By taking $\alpha = 2$, we get*

$$u(t) = \beta \int_0^1 f(s)ds + \int_0^1 (\eta - s)f(s)ds - \int_0^t (t-s)f(s)ds = \int_0^1 G(t,s)f(s)ds$$

and $G(t,s)$ in this case coincides with the one obtained in [27] for the boundary value problem

$$-u''(t) = f(t), \quad u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0.$$

4.1 Existence of Solutions

Theorem 4.1.1. *Let $f \in C([0,1] \times [0,\infty), [0,\infty))$ such that for $u, v \in C[0,1]$ $|f(s,u(s)) - f(s,v(s))| \leq L \|u - v\|$ where $L > 0$.*
If $L \leq \dfrac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha+1) + \eta^\alpha + 1}$ then the boundary value problem (5)-(6) has a unique solution.

Proof. We define the operator $T : C[0,1] \to C[0,1]$ as

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds$$

where $G(t,s)$ is defined by (4.0.0.3).
It is clear from Lemma 4.0.1 that the fixed points of the operator T coincide with the solutions of problem (5)-(6).
Let $u, v \in C[0,1]$. Then for each $t \in [0,1]$ we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &\leq \beta \int_0^1 |f(s,u(s)) - f(s,v(s))|ds + \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s)) - f(s,v(s))|ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s)) - f(s,v(s))|ds \\
&\leq L\beta \|u-v\| + \frac{L\|u-v\|}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1}ds + \frac{L\|u-v\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds \\
&\leq L\beta \|u-v\| + \frac{L\|u-v\|\eta^\alpha}{\alpha\Gamma(\alpha)} + \frac{L\|u-v\|t^\alpha}{\alpha\Gamma(\alpha)} \\
&\leq L\left(\frac{\beta\Gamma(\alpha+1) + \eta^\alpha + 1}{\Gamma(\alpha+1)}\right)\|u-v\|
\end{aligned}$$

Thus,

$$\|Tu - Tv\| \leq L\left(\frac{\beta\Gamma(\alpha+1)+\eta^{\alpha}+1}{\Gamma(\alpha+1)}\right)\|u-v\|.$$

Therefore T is a contraction and by the Banach fixed point theorem, T has a unique fixed point.

□

4.2 Existence of Positive Solutions

Remark 4.2.1. *We observe that for each fixed point $s \in [0,1]$, $\frac{\partial G}{\partial t} = 0$ for $t \leq s$ and $\frac{\partial G}{\partial t} < 0$ for $t > s$ and deduce that $G(t,s)$ is a decreasing function of t. It then follows that*

$$\max_{t\in[0,1]} G(t,s) = G(0,s) = \begin{cases} \beta, & s > \eta, \\ \frac{\beta\Gamma(\alpha)+(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \leq \eta. \end{cases}$$

and

$$\min_{t\in[0,1]} G(t,s) = G(1,s) = \begin{cases} \frac{\beta\Gamma(\alpha)-(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s > \eta, \\ \frac{\beta\Gamma(\alpha)+(\eta-s)^{\alpha-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \leq \eta. \end{cases}$$

Consequently, by looking at the behavior of $G(t,s)$ with respect to s we get

$$\min_{t,s\in[0,1]} G(t,s) = \frac{\beta\Gamma(\alpha)-(1-\eta)^{\alpha-1}}{\Gamma(\alpha)}$$

and

$$\max_{t,s\in[0,1]} G(t,s) = \frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}.$$

To establish the existence of positive solutions of problem (5)-(6), we will show that $G(t,s)$ satisfies the following property introduced by Lan and Webb in [35]:

(**A**)There exists a measurable function $\phi : [0,1] \to [0,\infty)$, a subinterval $[a,b] \subseteq [0,1]$ and a constant $\lambda \in [0,1]$ such that

$$|G(t,s)| \leq \phi(s), \quad \forall t,s \in [0,1]$$

and

$$G(t,s) \geq \lambda\phi(s), \quad \forall t \in [a,b], \quad \forall s \in [0,1].$$

Lemma 4.2.2. *If $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$ then $G(t,s) > 0$ for all $t,s \in [0,1]$ and $G(t,s)$ satisfies property* (**A**).

Proof. If $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$ then $G(t,s) > 0$ for all $t,s \in [0,1]$. We choose $[a,b] = [0,1]$ and we have

$$|G(t,s)| = G(t,s) \leq \frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)} := \phi(s)$$

and

$$G(t,s) \geq \lambda\phi(s), \quad \forall s,t \in [0,1]$$

where

$$\lambda = \frac{\beta\Gamma(\alpha)-(1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha)+\eta^{\alpha-1}}. \tag{4.2.0.1}$$

□

Lemma 4.2.3. *If $\beta\Gamma(\alpha) = (1-\eta)^{\alpha-1}$, then $G(t,s) \geq 0$ for all $t,s \in [0,1]$ and $G(t,s)$ satisfies property* (**A**).

Proof. We choose $[a,b] = [0,b]$ with $\eta \leq b < 1$. Following the arguments in the previous lemma, we have

$$|G(t,s)| \leq \frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)} := \phi(s) \quad \forall t,s \in [0,1]$$

Also, by taking

$$\lambda = \frac{\beta\Gamma(\alpha)-(b-\eta)^{\alpha-1}}{\beta\Gamma(\alpha)+\eta^{\alpha-1}},$$

we obtain

$$G(t,s) \geq \lambda\phi(s), \quad \forall t \in [0,b], \ \forall s \in [0,1].$$

□

Lemma 4.2.4. *If $\beta\Gamma(\alpha) < (1-\eta)^{\alpha-1}$, then $G(t,s)$ changes sign on $[0,1]\times[0,1]$ and $G(t,s)$ satisfies property* (**A**).

Proof. We choose $[a,b] = [0,b]$ with $\eta \leq b < 1$ such that $\beta\Gamma(\alpha) > (b-\eta)^{\alpha-1}$. We have

$$|G(t,s)| \leq \max\left\{\frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}, \frac{(1-\eta)^{\alpha-1}-\beta\Gamma(\alpha)}{\Gamma(\alpha)}\right\} := \phi(s) \quad \forall t,s \in [0,1]$$

and

$$G(t,s) \geq \lambda\phi(s), \quad \forall t \in [0,b], \ \forall s \in [0,1]$$

where

$$\lambda = \min\left\{\frac{\beta\Gamma(\alpha)-(b-\eta)^{\alpha-1}}{\beta\Gamma(\alpha)+\eta^{\alpha-1}}, \frac{\beta\Gamma(\alpha)-(b-\eta)^{\alpha-1}}{(1-\eta)^{\alpha-1}-\beta\Gamma(\alpha)}\right\}.$$

□

We set,

$$f_0 = \lim_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \; f_0^* = \lim_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$f_\infty = \lim_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \; f_\infty^* = \lim_{u\to \infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}.$$

We now state the main results.

Theorem 4.2.5. Let $f(s,u(s)) \in C([0,1]\times[0,\infty),[0,\infty))$. Assume that one of the following conditions is satisfied:

(i) (Sublinear case)

$$\lim_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty \text{ and } \lim_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0.$$

(ii) (Superlinear case)

$$\lim_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0 \text{ and } \lim_{u\to \infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty.$$

If $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$ then the problem (5)-(6) admits at least one positive solution.

Proof. Let $C[0,1]$ be the Banach space of all continuous real-valued functions on $[0,1]$ endowed with the usual supremum norm $\|\cdot\|$.
We define the operator $T : C[0,1] \to C[0,1]$ as

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds$$

where $G(t,s)$ is defined by (4.0.0.3).
It is clear from Lemma 4.0.1 that the fixed points of the operator T coincide with the solutions of problem (5)-(6).
We now define the cone

$$P = \left\{ u | u \in C[0,1], \; u(t) \geq 0, \; \min_{t\in[0,1]} u(t) \geq \lambda \|u\| \right\}$$

where λ is given by (4.2.0.1).
This type of cone has been used by [14],[25],[34],[38].

First, we show that $T(P) \subset P$:

It follows from the continuity and the non-negativity of the functions G and f on their domains of definition that if $u \in P$ then $Tu \in C[0,1]$ and $Tu(t) \geq 0$ for all $t \in [0,1]$.
For a fixed $u \in P$ and for all $t \in [0,1]$, the fact that $G(t,s)$ satisfies property (**A**) leads to the following inequalities :

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \lambda \int_0^1 \phi(s)f(s,u(s))ds \\ &\geq \lambda \int_0^1 \max_{t\in[0,1]} G(t,s)f(s,u(s))ds \\ &\geq \lambda \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))ds \\ &= \lambda \|Tu\| . \end{aligned}$$

Hence $T(P) \subset P$.

We now show that $T : P \to P$ is completely continuous:

In view of the continuity of the functions G and f, the operator $T : P \to P$ is continuous.
Let $\Omega \subset P$ be bounded, that is, there exists a positive constant $M > 0$ such that $\|u\|_\infty \leq M$ for all $u \in \Omega$. Define

$$L = \max_{0\leq t\leq 1,\ 0\leq u\leq M} |f(t,u)| + 1.$$

Then for all $u \in \Omega$, we have

$$|Tu(t)| \leq \int_0^1 G(t,s)f(s,u(s))ds \leq L \int_0^1 G(t,s)ds$$

for all $t \in [0,1]$. That is, the set $T(\Omega)$ is bounded.

For each $u \in \Omega$ and $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| =& \left| -\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \right| \\
\leq& \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) |f(s,u(s))|\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |f(s,u(s))|\, ds \\
\leq& \frac{L}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right) \\
=& \frac{L}{\alpha\Gamma(\alpha)} \left(-(t_2-t_1)^{\alpha} + t_2^{\alpha} - t_1^{\alpha} + (t_2-t_1)^{\alpha}\right) \\
=& \frac{L}{\Gamma(\alpha+1)} \left(t_2^{\alpha} - t_1^{\alpha}\right)
\end{aligned}$$

Clearly, the right-hand side of the above inequalities tends to 0 as $t_1 \to t_2$ and therefore the set $T(\Omega)$ is equicontinuous. It follows from the Arzela-Ascoli theorem that the operator $T : P \to P$ is completely continuous.
We now consider the two cases:
(i.) Sublinear case ($f_0 = \infty$ and $f_\infty = 0$)
Since $f_0 = \infty$ there exists $\rho_1 > 0$ such that $f(t,u) \geq \delta_1 u$ for all $0 < u \leq \rho_1$ where δ_1 satisfies

$$\delta_1 \left(\frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)} \right) \geq 1 \tag{4.2.0.2}$$

We take $u \in P$ such that $\|u\| = \rho_1$, then we have the following inequalities

$$\begin{aligned}
Tu =& \int_0^1 G(t,s) f(s,u(s))ds \\
\geq& \delta_1 \int_0^1 G(t,s)u(s)ds \\
\geq& \delta_1 \|u\| \left(\frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)} \right) \\
\geq& \|u\|
\end{aligned}$$

Let $\Omega_1 = \{u \in C[0,1]|, \;\; \|u\| < \rho_1\}$. Hence we have $\|Tu\| \geq \|u\|, \;\; u \in P \cap \partial\Omega_1$. Since $f(t,\cdot)$ is a continuous function on $[0,\infty)$ we can define the function:

$$\tilde{f}(t,u) = \max_{z \in [0,u]} \{f(t,z)\}.$$

It is clear that $\tilde{f}(t,u)$ is non-decreasing on $(0,\infty)$ and since $f_\infty = 0$, we have (see [54])

$$\lim_{u\to\infty}\left\{\max_{t\in[0,1]}\frac{\tilde{f}(t,u)}{u}\right\} = 0.$$

Therefore there exists $\rho_2 > \rho_1 > 0$ such that $\tilde{f}(t,u) \leq \delta_2 u$ for all $u \geq \rho_2$ where δ_2 satisfies

$$\delta_2\left(\frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}\right) \leq 1. \tag{4.2.0.3}$$

Define $\Omega_2 = \{u \in C[0,1]|, \ \ \|u\| < \rho_2\}$ and let $u \in P$ such that $\|u\| = \rho_2$. Then,

$$\begin{aligned}
Tu &= \int_0^1 G(t,s)f(s,u(s))ds \\
&\leq \int_0^1 G(t,s)\tilde{f}(s,\|u\|)ds \\
&\leq \delta_2\|u\|\left(\frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}\right) \\
&\leq \|u\|.
\end{aligned}$$

Hence we have $\|Tu\| \leq \|u\|, \quad u \in P \cap \partial\Omega_2$.
Thus, by the first part of the Guo-Krasnosel'skii fixed point theorem, we conclude that (5)-(6) has at least one positive solution.

(ii.) Superlinear case ($f_0^* = 0$ and $f_\infty^* = \infty$)
Let $\delta_2 > 0$ be given as in (4.2.0.3).
Since $f_0^* = 0$ there exists a constant $r_1 > 0$ such that $f(t,u) \leq \delta_2 u$ for $0 \leq u \leq r_1$.
Take $u \in P$ such that $\|u\| = r_1$. Then we have,

$$\begin{aligned}
Tu &= \int_0^1 G(t,s)f(s,u(s))ds \\
&\leq \delta_2\int_0^1 G(t,s)u(s)ds \\
&\leq \delta_2\|u\|\left(\frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}\right) \\
&\leq \|u\|.
\end{aligned}$$

If we let $\Omega_1 = \{u \in C[0,1]|\,\|u\| < r_1\}$, we see that $\|Tu\| \leq \|u\|$, for $u \in P\cap\partial\Omega_1$.
Now, since $f_\infty^* = \infty$, there exists $r > 0$ such that $f(t,u) \geq \delta_1 u$ for all $u \geq r$ where δ_1 is as in (4.2.0.2).
Define $\Omega_2 = \{u \in C[0,1]|\,\|u\| < r_2\}$ where $r_2 = \max(2r_1, \frac{r}{\lambda})$. Then $u \in P$ and $\|u\| = r_2$ imply that

$$\min u(t) \geq \lambda\|u\| = \lambda r_2 \geq r,$$

and so we obtain

$$\begin{aligned} Tu &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \delta_1 \int_0^1 G(t,s)u(s)ds \\ &\geq \delta_1 \|u\| \left(\frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right) \\ &\geq \|u\| \end{aligned}$$

This shows that $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. We conclude by the second part of the Guo-Krasnosel'skii fixed point theorem that (5)-(6) has at least one positive solution $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Theorem 4.2.6. Let $f(s,u(s)) \in C([0,1] \times [-\infty,+\infty),[0,\infty))$. Assume that one of the following conditions is satisfied:

(i) (Sublinear case)

$$\lim_{u\to 0} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty \text{ and } \lim_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0.$$

(ii) (Superlinear case)

$$\lim_{u\to 0} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0 \text{ and } \lim_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty.$$

If $\beta\Gamma(\alpha) \leq (1-\eta)^{\alpha-1}$ then the problem (5)-(6) admits a solution which is positive on an interval $[0,b] \subset [0,1]$.

Remark 4.2.7. *To prove Theorem 4.2.6, we employ similar arguments as in the proof of Theorem 4.2.5. In this case, we use the cone*

$$P = \left\{u|u \in C[0,1],\ \min_{t\in[0,b]} u(t) \geq \lambda \|u\|\right\}$$

where b and λ are defined in Lemma 4.2.3 for the case where $\beta\Gamma(\alpha) = (1-\eta)^{\alpha-1}$, and in Lemma 4.2.4 for the case where $\beta\Gamma(\alpha) < (1-\eta)^{\alpha-1}$.

Example 4.2.8. *Consider the fractional boundary value problem*

$$\begin{cases} -{}^C D^{\frac{3}{2}} u(t) = t^2 e^{-u(t)} + \sqrt{u(t)}, & t \in [0,1], \\ u'(0) = 0, & \frac{4}{5} {}^C D^{\frac{1}{2}} u(1) + u(\frac{3}{4}) = 0, \end{cases} \tag{4.2.0.4}$$

which is problem (5)-(6) *with $\alpha = \frac{3}{2}$, $\beta = \frac{4}{5}$, $\eta = \frac{3}{4}$ and $f(t,u(t)) = t^2 e^{-u(t)} + \sqrt{u(t)}$.*

First, we note that $u = 0$ is not a solution of (4.2.0.4).
Clearly, $f_0 = \infty$ and $f_\infty = 0$, and we also have $\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1} = \frac{2\sqrt{\pi}}{5} - \frac{1}{2} \approx 0.20898 > 0$.
We take

$$\lambda = \frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha) + \eta^{\alpha-1}} = \frac{\frac{2\sqrt{\pi}}{5} - \frac{1}{2}}{\frac{2\sqrt{\pi}}{5} + \frac{\sqrt{3}}{2}} = \frac{4\sqrt{\pi} - 5}{4\sqrt{\pi} + 5\sqrt{3}} \approx 0.13269$$

and consider the cone $P = \left\{u | u \in C[0,1],\ u(t) \geq 0,\ \min_{t\in[0,1]} u(t) \geq \lambda \|u\|\right\}$.
By the first part of Theorem 4.2.5, we conclude that the boundary value problem (4.2.0.4) has a positive solution in the cone P.

Chapter 5

Conclusions and Future Work

We have shown the existence of solutions of the class of fractional differential equations

$$^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \; T > 0, \quad 1 < q \leq 2,$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and the class of fractional differential inclusions

$$^cD^q x(t) \in F(t, x(t)), ,$$

where $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$,
subject to two families of boundary conditions:
(i) Mixed boundary conditions

$$Tx'(0) = -ax(0) - bx(T) \quad Tx'(T) = bx(0) + dx(T),$$

(ii) Closed boundary conditions

$$x(T) = \alpha x(0) + \beta T x'(0), \qquad Tx'(T) = \gamma x(0) + \delta T x'(0),$$

where $a, b, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ are given constants.

We derived the corresponding Green's functions to express the solution as an equivalent integral expressions and proved the existence of solutions.
For the inclusions, we used results for multivalued maps and some fixed point theorems and established the existence of solutions for the cases where F is convex, F is not necessarily convex or F is nonconvex.
As an application of fractional calculus, we studied a boundary value problem that models a thermostat insulated at one end and with the controller at the other end:

$$-\,^cD^\alpha u(t) = f(t, u(t)), \quad t \in [0, 1],$$

where $1 < \alpha \leq 2$, ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α and $f \in C([0,1] \times [0,\infty), [0,\infty))$, subject to the boundary condition:

$$u'(0) = 0, \quad \beta\, {}^cD^{\alpha-1}u(1) + u(\eta) = 0,$$

where $\beta > 0$, $0 \leq \eta \leq 1$ are given constants.
Mathematical models of physical processes necessitate the existence of positive functions that satisfy the model. For this reason, we established conditions for the existence of positive solutions for the thermostat model.

New formulations of fractional derivatives and integrals have been presented recently. One notable example is the Caputo-Fabrizio fractional derivative [17, 18, 37] which uses an exponential kernel. An area for future work is to use this new derivative to solve boundary value problems and reformulate classical models such as the logistic model, competition models, etc.

In [42], the authors laid out a set of criteria that must be satisfied by an operator to be considered a fractional derivative. Another possible future work is to develop a new definition of fractional derivative that satisfies the given criteria. This could be done either by defining an entirely new derivative or by modifying one of the known definitions of fractional derivatives.

Resumen

El cálculo fraccionario, que empezó como una simple curiosidad matemática en el siglo XVII, ha evolucionado rápidamente a lo largo de los años. Realizando una simple búsqueda sobre el tema en la literatura, nos encontramos varias posibles formulaciones para las derivadas e integrales fraccionarias así como diversas aplicaciones a la física, ingeniería y otras áreas. El desarrollo de esta parte del Análisis Matemático continua a día de hoy a medida que nuevos conceptos y estrategias van surgiendo en esta área.

En el primer capítulo, se presenta una breve introducción histórica del cálculo fraccionario. Desde un punto de vista histórico, el cálculo fraccionario puede interpretarse como la extensión del concepto de diferenciación e integración de un orden entero n a un orden arbitrario α, donde α es un número real o complejo:

$$\frac{d^n f}{dx^n} \to \frac{d^\alpha f}{dx^\alpha},$$
$$I^n f \to I^\alpha f.$$

En este capítulo hacemos un seguimiento de la evolución del cálculo fraccionario desde Leibniz hasta el presente, así como comentamos las contribuciones de Euler, Lacroix, Liouville, Fourier, Riemann, Laurent y otros eminentes matemáticos en las formulaciones de lo que hoy en día se conocen como derivadas e integrales fraccionarias.

El segundo capítulo establece los preliminares necesarios para los siguientes capítulos. Las dos formulaciones más comunes en la literatura de derivadas e integrales fraccionarias son las definiciones de Riemann-Liouville y Caputo. En este capítulo, estudiamos las definiciones anteriormente mencionadas, así como sus principales propiedades. Estudiamos también las propiedades de las aplicaciones multivaluadas, las cuales juegan un papel fundamental en las inclusiones fraccionarias. Por último, se recogen algunos resultados previos sobre ecuaciones diferenciales con derivadas fraccionarias e inclusiones pertinentes para el estudio.

En el tercer capítulo, estudiamos una clase de ecuaciones diferenciales fraccionarias e inclusiones. En particular, investigamos para $T > 0$ y $1 < q \leq 2$ la siguiente clase de ecuaciones diferenciales fraccionarias:

$$^{c}D^{q}x(t) = f(t, x(t)), \quad t \in [0, T], \tag{1}$$

donde $^{c}D^{q}$ denota la derivada fraccionaria de Caputo de orden q y $f : [0, T] \times \mathbb{R} \to \mathbb{R}$. Estudiamos (1) sujeto a dos familias de condiciones de frontera:

(i) Condiciones de frontera mixtas

$$Tx'(0) = -ax(0) - bx(T) \quad Tx'(T) = bx(0) + dx(T), \tag{2}$$

(ii) Condiciones de frontera cerradas

$$x(T) = \alpha x(0) + \beta Tx'(0), \quad Tx'(T) = \gamma x(0) + \delta Tx'(0), \tag{3}$$

donde $a, b, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ son unas constantes fijadas.

Las condiciones de frontera (2) se interpolan entre las condiciones de frontera de Neumann ($a = b = d = 0$) y las de Dirichlet ($a = b = d = \infty$). Por otro lado, las condiciones de frontera (3) incluyen condiciones de frontera cuasi-periódicas ($\beta = \gamma = 0$) y se interpolan entre las condiciones de frontera periódicas ($\alpha = \delta = 1, \beta = \gamma = 0$) y antiperiódicas ($\alpha = \delta = -1, \beta = \gamma = 0$). Cabe resaltar que las condiciones de frontera de Zaremba ($x(0) = 0, x'(T) = 0$) pueden considerarse tanto como condiciones de frontera mixtas ($a = \infty, b = d = 0$) o como condiciones de frontera cuasi-periódicas ($\alpha = \infty, \gamma = \delta = 0$).

Para probar la existencia de solución para este tipo de ecuaciones,

- Obtenemos la función de Green para el correspondiente problema de frontera para cada una de las condiciones de frontera posibles. Esto permite tratar el problema de frontera como un problema de punto fijo.

- Probamos la existencia de solución en una bola adecuada $B_R \subset C[0, T]$ usando la teoría del grado de Leray-Schauder y algunas técnicas estándar.

Después, generalizamos los resultados obtenidos al problema de inclusión diferencial fraccionaria

$$^{c}D^{q}x(t) \in F(t, x(t)), \quad t \in [0, T], \ T > 0, \quad 1 < q \leq 2, \tag{4}$$

sujeto a las dos familias de condiciones de frontera, (2) y (3), donde $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ es una compact-value map, y $\mathcal{P}(\mathbb{R})$ es la familia formada por todos los subconjuntos no vacíos de $\mathbb{R}$.
Establecemos la existencia de solución para el problema (4) para los casos en los que F es convexa, F es no necesariamente convexa y F es no convexa.

Para el caso convexo, establecemos las siguientes hipótesis:

(H_1) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ es L^1-Carathéodory y toma valores convexos y compactos;

(H_2) existe una función continua y creciente $\psi : [0,\infty) \to (0,\infty)$ y una función $p \in C([0,T], \mathbb{R}^+)$ tal que

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|_\infty)$$

para cada $(t,x) \in [0,T] \times \mathbb{R}$;

Imponemos además la condición

(H_3) existe un número $M_1 > 0$ tal que

$$\frac{\Gamma(q+1)M_1}{T^q\Big(1 + \frac{|b+d+b^2-ad|+q|a-1|}{\Delta_1}\Big)\psi(M_1)\|p\|_{L^\infty}} > 1.$$

correspondiente a (2).

Análogamente, añadimos la condición

($\overline{H}_3$) existe un $\overline{M}_1 > 0$ tal que

$$\frac{\Gamma(q+1)\overline{M}_1}{T^q\Big(1 + \frac{|1-\delta+\gamma|+q|\alpha-\beta|}{|\Delta_2|}\Big)\psi(\overline{M}_1)\|p\|_{L^\infty}} > 1.$$

correspondiente a (3).

La demostración de la existencia de solución en este caso está basada en la alternativa no lineal de Leray y Schauder para aplicaciones multivaluadas.

En el caso en el que F es no necesariamente convexa, suponemos que las condiciones (H_2)-(H_3), (H_2)-($\overline{H}_3$) se satisfacen y en lugar de la condición (H_1), utilizamos las siguientes hipótesis

(H_4) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ es una multivaluada con valores compactos tal que (**a**) $(t,x) \longmapsto F(t,x)$ es $\mathcal{L}\otimes\mathcal{B}$ medible,

(**b**) $x \longmapsto F(t,x)$ semicontinua inferiormente para cada $t \in [0,T]$;

(H_5) para cada $\sigma > 0$, existe $\varphi_\sigma \in L^1([0,T],\mathbb{R}_+)$ tal que

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \varphi_\sigma(t)$$

para todo $\|x\|_\infty \leq \sigma$ y casi todo $t \in [0,T]$.

Las condiciones (H_4) y (H_5) permiten utilizar un Teorema de selección de Bressan y Colombo para aplicaciones semicontinuas inferiormente, que junto con la alternativa no lineal de Leray y Schauder, nos permiten probar la existencia de al menos una solución del problema.

Por último, en el caso en el que F sea no convexo aplicamos un Teorema de punto fijo para aplicaciones multivaluadas de Cavitz y Nadler. Así, obtenemos los siguientes resultados: si se verifica las siguientes hipótesis

(H_6) $F : [0,T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ es tal que $F(\cdot,x) : [0,T] \to P_{cp}(\mathbb{R})$ es medible para cada $x \in \mathbb{R}$.

(H_7) $H_d(F(t,x),F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ en casi todo punto $t \in [0,T]$ y $x,\bar{x} \in \mathbb{R}$ con $m \in C([0,T],\mathbb{R}^+)$ y $d(0,F(t,0)) \leq m(t)$ en casi todo punto $t \in [0,T]$.

entonces el problema (4)-(2) tiene al menos una solución en $[0,T]$ si

$$\frac{T^q\|m\|_{L^\infty}}{\Gamma(q+1)}\Big(1+\frac{|b+d+b^2-ad|+q|a-1|}{|\Delta_1|}\Big) < 1.$$

De forma similar, supongamos que las hipótesis (H_6) y (H_7) se verifican. Entonces el problema (4)-(3) tiene al menos una solución en $[0,T]$ si

$$\frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)}\Big(1+\frac{|1-\delta+\gamma|+(q-1)|\alpha-\beta|}{\Delta_2}\Big) < 1.$$

En el cuarto capítulo, estudiamos un modelo unidimensional fraccionario de un termostato. Infante y Webb estudiaron el problema de frontera no local

$$-u'' = f(t,u), \quad t \in (0,1), \quad u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0.$$

que modeliza un termostato aislado en $t = 0$, en el que el controlador en tiempo $t = 1$ aumenta o disminuye el calor dependiendo de la temperatura detectada por el sensor en tiempo $t = \eta$. Este tipo de problema de frontera fue inicialmente investigado por Guidotti y Merino para el caso lineal con $\eta = 0$.

Nosotros proponemos el siguiente modelo fraccional análogo:

$$-\,{}^{c}D^{\alpha}u(t) = f(t, u(t)), \quad t \in [0, 1], \tag{5}$$

donde $1 < \alpha \leq 2$, ${}^{c}D^{\alpha}$ denota la derivada fraccionaria de Caputo de orden α y $f \in C([0, 1] \times [0, \infty), [0, \infty))$ sujeta a las siguientes condiciones de frontera:

$$u'(0) = 0, \quad \beta\, {}^{c}D^{\alpha-1}u(1) + u(\eta) = 0, \tag{6}$$

siendo $\beta > 0$, $0 \leq \eta \leq 1$ constantes fijadas. Cabe resaltar que en el caso $\alpha = 2$, recuperamos el problema de segundo orden.

Los modelos matemáticos de procesos físicos, para tener una utilidad práctica, necesitan poder garantizar la existencia de funciones positivas satisfaciendo el modelo. Nuestro objetivo en este capítulo es establecer condiciones de existencia de soluciones positivas para los problemas de frontera (5)- (6).
En este capítulo, seguimos el siguiente esquema :

- Encontramos la función de Green $G(t, s)$ correspondiente al problema de frontera (5)- (6) para transformarlo en un problema de punto fijo.

- Probamos la existencia de al menos una solución del problema (5)- (6) usando el Teorema del punto fijo de Banach.

- Analizamos la función de Green para probar la existencia de una función medible $\phi : [0, 1] \to [0, \infty)$, un subintervalo $[a, b] \subseteq [0, 1]$ y una constante $\lambda \in [0, 1]$ tales que $|G(t, s)| \leq \phi(s), \quad \forall t, s \in [0, 1]$ y $G(t, s) \geq \lambda\phi(s), \quad \forall t \in [a, b], \quad \forall s \in [0, 1]$.

- Probamos la existencia de una solución positiva para el problema (5)- (6) en el cono $P = \left\{u | u \in C[0, 1],\ u(t) \geq 0,\ \min_{t\in[0,1]} u(t) \geq \lambda \|u\|\right\}$. Esto se consigue bajo la condición de que f sea o bien superlineal o bien sublineal. La demostración de este resultado está basada en el Teorema del punto fijo de Krasnosel'skii- Gao.

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