UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
Departamento de Estatística e Investigación Operativa

ESSAYS ON COMPETITION AND COOPERATION IN GAME THEORETICAL MODELS

Julio González Díaz

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A mi familia, por hacerlo todo más fácil...

Un peldaño más

Preface

This thesis is the result of my first four years as a researcher in game theory. Nonetheless, my devotion for games, specially the zero-sum ones, is much older than that. I would say that it really began when I first saw my eldest brother playing chess with my father; by that time I was six years old. Both of them passed me the love for this game, which I still practice. Apart from chess, I have also *wasted* part of my leisure time over the last few years playing computer games, cards, and many other board and table games with my family and friends. It was not before the fifth year of my undergraduate studies in Mathematics that I realized that the scope of the theory of games goes far beyond simple (and not so simple) diversions.

My first formal approach to game theory was during a course taught by Ignacio García Jurado. After Ignacio's course, games were not just a hobby anymore. Hence, after finishing the degree, I joined the PhD program of the Department of Statistics and Operations Research with the idea of writing my thesis in game theory. Soon after that, Ignacio became my advisor. He is the one who has helped me most during these four years, not only because of his academic guidance, but also for being the main responsible for the fruitful years I have spent as a game theorist so far. Many thanks, Ignacio, for the time you have spent on me.

Many thanks, too, to my other advisor, Estela, for all the time she has devoted to this thesis; mainly through her co-authorship in Chapters 5, 6, and 7. Thanks for all the discussions, so central to the core of this thesis.

Joint research with different people has helped me to deepen into game theory and to understand many other aspects of a researcher's life. Hence, I am grateful to all my co-authors: Ignacio, Estela, Peter, Henk, Ruud, Marieke, and Antonio. Besides, special thanks to my *advanced mathematics consultants*: Roi and Carlitos for their helpful discussions that contributed to most of the Chapters of this thesis, mainly through Chapters 5 and 6.

I have also had the possibility of visiting some prestigious universities during these years. These stays have substantially influenced my formation not only as a researcher, but also in many other aspects of life. Because of this, I am indebted to Peter, Henk, Ruud, Arantza,... and all the people at CentER for the pleasant atmosphere I had during my three-month visit to Tilburg University. I am also indebted to Inés and Jordi for having invited me to visit the Unit of Economic Analysis of the Universitat Autònoma de Barcelona, and to the other members of the Department for their reception; I am specially grateful to the PhD students at IDEA for their

warm welcome, where Sergio and Joan deserve a special mention. Finally, I am deeply indebted to William for inviting me to visit the Department of Economics of Rochester University. My gratitude to all the members of the Department, to the PhD students, to Diego, Paula, Ricardo, Cagatay, and many others.

Moreover, William's influence on this thesis goes further than just the invitation to visit Rochester University. He has taught to me some of the *secrets* of correct (scientific) writing, and I have tried to follow his *credo* throughout this thesis. Unfortunately, it was already too late to implement his principles in some of the chapters of this thesis (in the others just blame me for my inaptitude).

I deeply appreciate the kind support from the group of Galician game theorists and from the people in the Department of Statistics and Operations Research.

I am also grateful to my two officemates, Rosa and Manuel. Because of them I have developed my research in a very comfortable environment. Also, thanks Manuel for your countless LaTeX recommendations.

Finally, I want to mention all the other PhD students at the Faculty of Mathematics for the enjoyable conversations and discussions during the daily coffee breaks. Thanks to Marco, Carlitos, Tere, Bea,....

Last, but not least, I have to render many thanks to my family and to my friends. They have provided me with a very pleasant and relaxed atmosphere during all these years.

Julio González Díaz
April 2005, Santiago de Compostela

Contents

Notations

Since many of the chapters within this thesis do not bear any relation to each other, all of them are self-contained. In order to do this, it can be the case that the same piece of notation is introduced in more than one chapter. Even though, the following symbols and notations are common for all the chapters.

$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^N$	The set of vectors whose coordinates are indexed by the elements of N
$\mathbb{R}_+$	The set of non-negative real numbers
$\mathbb{R}_{++}$	The set of positive real numbers
$T \subseteq S$	T is a subset of S
$T \subsetneq S$	T is a subset of S and T is not equal to S
2^N	The set of all subsets of N
$\lvert S \rvert$	The number of elements of S
$\text{co}(A)$	The convex hull of A

Part I

Noncooperative Game Theory

Introduction to Noncooperative Game Theory

This first Part of the dissertation deals with noncooperative game theoretical models. It consists of four Chapters, and in each of them we present and discuss a different issue.

Chapter 1 presents a noncooperative situation in which timing plays an important role. That is, not only the chosen strategies are relevant to obtain the payoffs of a given profile, but also the moment in which they are played influences the outcome. The games we define model the division of a cake by n-players as a special class of timing games. Our main result establishes the existence and uniqueness of a Nash equilibrium profile for each of these games. This Chapter is based on the paper González-Díaz et al. (2004) and generalizes to the n-player situation the 2-player results described in Hamers (1993).

Both Chapters 2 and 3 describe models within the scope of the repeated games literature. In Chapter 2, which is based on the paper González-Díaz (2003), we elaborate a little bit more on the extensively studied topic of the folk theorems. More specifically, we present a generalized Nash folk theorem for finitely repeated games with complete information. The main result in this Chapter refines the sufficient condition presented in Benoît and Krishna (1987), replacing it by a new one which turns out to be also necessary. Besides, this result also corrects a small flaw in Smith (1995). Moreover, our folk theorem is more general than the standard ones. The latter look for conditions under which the set of feasible and individually rational payoffs can be supported by Nash or subgame perfect equilibria. Our folk theorem also looks for such conditions, but we also characterize the set of achievable payoffs when those conditions are not met. On the other hand, in Chapter 3 we deepen in the topic of unilateral commitments; the research on this issue follows the lines in García-Jurado et al. (2000). We study the impact of unilateral commitments in the assumptions needed for the various folk theorems, showing that, within our framework, they can always be relaxed. These results imply that, when unilateral commitments are possible, we can support "cooperative" payoffs of the original game as the result of either a Nash or a subgame perfect equilibrium profile in situations in which they could not be supported within the classic framework. Chapter 3 is based on García-Jurado and González-Díaz (2005).

We conclude this first Part with Chapter 4. This Chapter deals with bankruptcy problems. We associate noncooperative bankruptcy games to bankruptcy situations and then we study the properties of the equilibria of such games. We show that each of these games has a unique Nash equilibrium payoff, which, moreover, is always supported by strong Nash equilibria. Besides, we show that for each bankruptcy rule and each bankruptcy situation, we can define a bankruptcy game whose Nash equilibrium payoff corresponds with the proposal of the given bankruptcy rule. This Chapter is based on García-Jurado et al. (2004).

Short Bibliography

Benoît, J.-P. and V. Krishna (1987): "Nash Equilibria of Finitely Repeated Games," *International Journal of Game Theory*, 16, 197–204. (Quoted in pp. 3)

García-Jurado, I. and J. González-Díaz (2005): "Unilateral Commitments in Repeated Games," Preprint. (Quoted in pp. 3)

García-Jurado, I., J. González-Díaz, and A. Villar (2004): "A Noncooperative Approach to Bankruptcy Problems," Preprint. (Quoted in pp. 3)

García-Jurado, I., L. Méndez-Naya, and F. Patrone (2000): "Unilateral Commitments in Finitely Repeated Games," *International Game Theory Review*, 2, 129–139. (Quoted in pp. 3)

González-Díaz, J. (2003): "Finitely Repeated Games: A Generalized Nash Folk Theorem," Tech. Rep. 03-08, Department of Statistics and OR, University of Santiago de Compostela, to appear in Games and Economic Behavior. (Quoted in pp. 3)

González-Díaz, J., P. Borm, and H. Norde (2004): "A Silent Battle over a Cake," Tech. rep., CentER discussion paper series. (Quoted in pp. 3)

Hamers, H. (1993): "A Silent Duel over a Cake," *Methods and Models of Operations Research*, 37, 119–127. (Quoted in pp. 3)

Smith, L. (1995): "Necessary and Sufficient Conditions for the Perfect Finite Horizon Folk Theorem," *Econometrica*, 63, 425–430. (Quoted in pp. 3)

Chapter 1
A Silent Battle over a Cake

Contents

1.1 Introduction

There are many strategic situations in which some agents face a decision problem in which timing is important. The literature on timing games has been devoted to analyze these situations and provide theoretical models to study the underlying strategic problem. A first approach to timing games appears in Karlin (1959) in the zero sum context. More recent contributions are Baston and Garnaev (2000) and Laraki et al. (2003). A classic example of timing game is the war of attrition, introduced in Smith (1974) and widely studied, for instance, in Hendricks et al. (1988). More specifically, consider the following war of attrition game. Two rival firms are engaged in a race to make a patentable discovery, and hence, as soon as one firm makes the discovery, all the previous effort made by the other firm turns out to be useless. This patent race model has been widely studied in literature (see, for instance, Fudenberg et al. (1983)). In this model it is assumed that, as soon as one of the firms leaves the race, the game ends. The motivation for this assumption is that, once there is only one firm in the race, the game reduces to a decision problem in which the remaining firm has to optimize its resources. Hence, the strategy of each firm consists of deciding, for each time t, whether to leave the race or not. Most of the literature in timing games models what we call *non-silent* timing games, that is, as soon as one player acts, the others are informed and the game ends.[1] In this Chapter, on the contrary, we provide a formal model for the *silent* situation. We use again the patent race to motivate our approach. Consider a situation in which two firms are engaged in a patent race and also in an advertising campaign. Suppose that one of the two rival firms, say firm 1, decides to leave the patent race. Then, it will probably be the case that firm 1 does not want firm 2 to realize that 1 is not in the race anymore; and therefore, firm 1 can get a more advantageous position for the advertising campaign. Moreover, if firm 2 does not realize about the fact that firm 1 has already left the race, it can also be the case that, having already firm 1 left the race, firm 2 leaves the race before making the discovery, benefiting again firm 1.

Next, we introduce our silent timing game. We consider the situation that n players have to divide a cake of size S. At time 0 player i has the initial right to receive the amount α_i, where it is assumed that $\sum_{i\in N}\alpha_i < S$. If player i claims his part at time $t > 0$ then he receives the discounted part $\delta^t\alpha_i$ of the cake, unless he is the last claimant in which case he receives the discounted remaining part of the cake $\delta^t(S - \sum_{j\neq i}\alpha_j)$. We refer to this game as a *cake sharing game.*

Hamers (1993) showed that 2-player cake sharing games always admit a unique Nash equilibrium. In this Chapter we consider cake sharing games that are slightly different from the games introduced in Hamers (1993). We first provide an alternative, but more direct, existence and uniqueness result for 2-player cake sharing games and we generalize this result to cake sharing games with more players.

It is worth to mention the similarities between our results and some well known results in all-

[1] An exception is Reinganum (1981), although her model is very different from ours.

pay auctions (Weber, 1985). At first glance, our model seems quite different from that of all-pay auctions, but it turns out to be the case that they have many similarities. Indeed, in this Chapter we show that the same kind of results obtained for the all-pay auction (Hilman and Riley, 1989; Baye et al., 1996) can be obtained for our timing game. Anyhow, even when both the results and also the arguments underlying some of the proofs are very similar, the two models are different enough so that our results can not be derived from those in the all-pay auctions literature.

This Chapter is organized as follows. In Section 1.2 we introduce the cake sharing games. In Sections 1.3 and 1.4 we deal with 2-player cake sharing games and more player cake sharing games, respectively.

1.2 The Model

In this Section we formally introduce the cake sharing games.

Let $N = \{1, \ldots, n\}$ be a set of players with $n \geq 2$, let $S > 0$, let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^N_+$ be such that $\alpha_1 + \cdots + \alpha_n < S$, and let $\delta \in (0, 1)$. Throughout this Chapter we assume that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. The number S is called the *size of the cake*, the vector α the *initial right vector* and δ the *discount factor*.

The *cake sharing game with pure strategies* associated with S, α, and δ, is the triple $\Gamma^{\text{pure}}_{S,\alpha,\delta} := (N, \{A_i\}_{i\in N}, \{\pi_i\}_{i\in N})$, where

- $A_i := [0, \infty)$ is the set of pure strategies of player $i \in N$,
- π_i is the payoff function of player $i \in N$, defined by:

$$\pi_i(t_1, \ldots, t_n) := \begin{cases} (S - \sum_{j\neq i} \alpha_j)\delta^{t_i} & t_i > \max_{j\neq i} t_j \\ \alpha_i \delta^{t_i} & \text{otherwise.} \end{cases}$$

Hence, if there is a unique last claimant, then he receives the discounted value of the cake that remains after that other players have taken their initial rights. If there is not a unique last claimant, then all players receive the discounted value of their initial rights. Note that the payoff functions defined above differ slightly from the payoff functions introduced in Hamers (1993), where, in case there is not a unique last claimant, the discounted value of the remaining cake is shared equally between the last claimants. This change in the model does not affect the results, but it helps to have cleaner proofs. [2]

One easily verifies that $\Gamma^{\text{pure}}_{S,\alpha,\delta}$ has no Nash equilibria. If there is a unique last claimant, then this player can improve his payoff by claiming a little bit earlier (and remaining the last

[2] Let us make some comments concerning the relation between the cake sharing game (CS) and the all-pay auctions model (AP). For simplicity, we think of the two player case. Setting aside the issue of timing, note the following differences: (i) Initial rights: in CS they depend on the player (α_i), in AP they are 0; (ii) in CS each player wants to get $1 - (\alpha_1 + \alpha_2)$, in AP the valuation of the object depends on the player; and (iii) In CS waiting till time t, each player is "paying" $\alpha_i - (\alpha_i)\delta^t$, *i.e.*, it depends on the player, in AP bidding v, each player is "paying" v. All the other strategic elements are analogous in the two models.

claimant). If there is no unique last claimant, then one of the last claimants can improve his payoff by claiming a little bit later (becoming the unique last claimant in this way). Hence, for an appropriate analysis of cake sharing games we need to consider mixed strategies.

Formally, a *mixed strategy* is a function $G : [0, \infty) \to [0, 1]$ satisfying:

- $G(0) = 0$,
- G is a nondecreasing function,
- G is left-continuous,
- $\lim_{x\to\infty} G(x) = 1$.

For a mixed strategy G we can always find a probability measure P on $[0, \infty)$ such that:[3]

$$\text{for each } x \in [0, \infty), \quad G(x) = P\big([0, x)\big). \tag{1.1}$$

On the other hand, every probability measure P on $[0, \infty)$ defines by formula (1.1) a mixed strategy G. Hence, the set of mixed strategies coincides with the set of probability measures on $[0, \infty)$.[4] Let $\mathcal{G}$ denote the set of all mixed strategies. We introduce now some other notations related to mixed strategy G:

- for each $x \in [0, \infty)$, we denote $\lim_{y\downarrow x} G(y)$, the probability of choosing an element in the closed interval $[0, x]$, by $G(x^+)$.
- if there is $x > 0$ such that for each pair $a, b \in [0, \infty)$, with $a < x < b$, we have $G(b) > G(a^+)$ (*i.e.*, the probability of choosing an element in (a, b) is positive), then x is an element of the support of G. If for each $b > 0$, $G(b) > 0$ (*i.e.*, the probability of choosing an element in $[0, b)$ is positive), then 0 is an element of the support of G. Let $S(G)$ be the support of the distribution function G. One easily verifies that $S(G)$ is a closed set.
- the set of jumps (discontinuities) of G is $J(G) := \{x \in [0, \infty) \,:\, G(x^+) > G(x)\}$, *i.e.*, the set of pure strategies which are chosen with positive probability.

If player i chooses pure strategy t and all other players choose mixed strategies $\{G_j\}_{j\neq i}$ then the expected payoff for player i is

$$\begin{aligned}\pi_i(G_1, \ldots, G_{i-1}, t, G_{i+1}, \ldots, G_n) &= \prod_{j\neq i} G_j(t)\, \delta^t (S - \sum_{j\neq i} \alpha_j) + (1 - \prod_{j\neq i} G_j(t))\, \delta^t \alpha_i \\ &= \delta^t(\alpha_i + (S - \sum_{j\in N} \alpha_j) \prod_{j\neq i} G_j(t)).\end{aligned}$$

[3] See Rohatgi (1976) for more details.

[4] An alternative way of defining mixed strategies G is as a nondecreasing, right-continuous function from $[0, \infty)$ to $[0, 1]$ with $\lim_{x\to\infty} G(x) = 1$. For such a function we can always find a probability measure P on $[0, \infty)$ such that for each $x \in [0, \infty)$, $G(x) = P\big([0, x]\big)$, *i.e.*, G is the (cumulative) distribution function corresponding to P. Although this equivalent approach seems more natural, it would lead to technical problems when computing Lebesgue-Stieltjes integrals later on.

If player i also chooses a mixed strategy G_i, whereas all other players stick to mixed strategies $\{G_j\}_{j\neq i}$, then the expected payoff for player i can be computed by use of the Lebesgue-Stieltjes integral:

$$\pi_i(G_1,\ldots,G_n) = \int \pi_i(G_1,\ldots,G_{i-1},t,G_{i+1},\ldots,G_n)dG_i(t). \tag{1.2}$$

Note that, with a slight abuse of notation, the functions π_i do not only denote payoffs to players when pure strategies are played, but also when mixed strategies are used.

The *cake sharing game* associated with S, α, and δ, is defined by the triple $\Gamma_{S,\alpha,\delta} := (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$, where

- $X_i := \mathcal{G}$ is the set of mixed strategies of player $i \in N$,
- π_i, defined by (1.2), is the (expected) payoff function of player $i \in N$.

Given a strategy profile $G = (G_1, G_2, \ldots, G_n) \in \mathcal{G}^n$, let $\pi_i^G(t)$ be the corresponding payoff $\pi_i(G_1,\ldots,G_{i-1},t,G_{i+1},\ldots,G_n)$. Hence, $\pi_i^G(t)$ is the expected payoff for player i when he plays the pure strategy t and all the other players act in accordance with G.

1.3 Two Players

In this Section we provide an alternative proof of the result of Hamers (1993) for 2-player cake sharing games. Our incentives for doing this job are threefold. First of all we want to recall that our model is slightly different from the model of Hamers (1993), and hence, a new proof is required. Secondly, our proof is more direct than Hamers' proof. Finally, our proof forms the basis for the results in Section 1.4 for cake sharing games with three or more players.

First, we derive a number of properties for Nash equilibria of n-player cake sharing games. The following Lemma shows that in a Nash equilibrium players do not put positive probability on a pure strategy $t > 0$.

Lemma 1.1. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game and let the profile $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Then, for each $i \in N$, $J(G_i) \cap (0,\infty) = \emptyset$.*

Proof. Let $i \in N$. We show that $J(G_i) \cap (0,\infty) = \emptyset$. Assume, without loss of generality, that $i = 1$. Suppose that $u \in J(G_1) \cap (0,\infty)$. If there is $i \neq 1$ such that $G_i(u^+) = 0$, then, for each $t \in [0,u]$, $\pi_1^G(t) = \delta^t\alpha_1$. Since the function $\pi_1^G(\cdot)$ is strictly decreasing on $[0,u]$, player 1 would be better off moving the probability in u to 0. Hence, for each $i \in N$, $G_i(u^+) > 0$. Now, for each $i \in N\backslash\{1\}$, consider the functions

$$\pi_i^G(t) = \delta^t(\alpha_i + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq i}G_j(t)).$$

Since G_1 is discontinuous at u, *i.e.*, $G_1(u^+) > G_1(u)$, there are $u_1 < u$, $u_2 > u$, and $\varepsilon > 0$ such that for each $i \neq 1$ and each $t \in [u_1, u]$,

$$\pi_i^G(u_2) - \pi_i^G(t) \geq \varepsilon.$$

If player $i \in N\backslash\{1\}$ puts positive probability on $[u_1, u]$, *i.e.*, if $G_i(u^+) > G_i(u_1)$, then he can increase his payoff by at least $\varepsilon(G_i(u^+) - G_i(u_1))$ by moving all this probability to u_2. Hence, for each $i \in N\backslash\{1\}$, we have $G_i(u^+) = G_i(u_1)$ and, for each $t \in [u_1, u]$, $G_i(t) = G_i(u)$. Hence, the function

$$\pi_1^G(t) = \delta^t(\alpha_1 + (S - \sum_{j\in N} \alpha_j) \prod_{j\neq 1} G_j(t))$$

is strictly decreasing on $[u_1, u]$. Now, player 1 can improve his payoff by moving some probability from u to u_1. □

Lemma 1.1 implies that, in a Nash equilibrium G, the players use mixed strategies which are continuous on $(0, \infty)$. Hence, for each $i \in N$ and each $t > 0$, we can write $G_i(t^+) = G_i(t)$. Moreover, the functions $\pi_i^G(\cdot)$ are continuous on $(0, \infty)$.

Lemma 1.2. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game and let the profile $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Let $i \in N$ and $t \in S(G_i)$. Then, there is $j \in N\backslash\{i\}$ such that $t \in S(G_j)$.*

Proof. Suppose that $t \notin \cup_{j\neq i} S(G_j)$. We distinguish between two cases:

Case 1: $t > 0$.

There are $t_1, t_2 > 0$, with $t_1 < t < t_2$, such that for each $j \neq i$, $G_j(t_2) = G_j(t_1)$.[5] Hence, for each $u \in [t_1, t_2]$ and each $j \neq i$, $G_j(u) = G_j(t_2)$. Hence, the function

$$\pi_i^G(u) = \delta^u(\alpha_i + (S - \sum_{j\in N} \alpha_j) \prod_{j\neq i} G_j(u))$$

is strictly decreasing on $[t_1, t_2]$. Since $t \in S(G_i)$, we have $G_i(t_2) > G_i(t_1^+)$, *i.e.*, player i puts positive probability on (t_1, t_2). Now, player i can strictly improve his payoff by moving all this probability to t_1.

Case 2: $t = 0$.

Let $b > 0$ be the smallest element in $\cup_{j\neq i} S(G_j)$ (recall that all the $S(G_j)$ are closed). Clearly, for each $j \neq i$, $G_j(b) = 0$. Again, if $G_i(b) > G_i(0^+)$, *i.e.*, if player i puts positive probability on $(0, b)$, then similar arguments as in Case 1 can be used to show that player i can strictly improve his payoff by moving this probability to 0. Hence, $G_i(b) = G_i(0^+)$ and hence, since $0 \in S(G_i)$, we have $G_i(0^+) > 0$. Moreover, for each $t \in (0, b]$, $G_i(t) = G_i(b)$ (this is relevant only for the

[5] For each $j \in N\backslash\{i\}$ there are $t_1^j, t_2^j > 0$, with $t_1^j < t < t_2^j$, such that $G_j(t_2^j) = G_j(t_1^j)$. Hence, we take $t_1 = \max_{j\in N\backslash\{i\}} t_1^j$ and $t_2 = \min_{j\in N\backslash\{i\}} t_2^j$.

case $n = 2$). Hence, for each $j \in N \backslash \{i\}$, the function

$$\pi_j^G(t) = \delta^t(\alpha_j + (S - \sum_{k\in N} \alpha_k) \prod_{k\neq j} G_k(t))$$

is strictly decreasing on $(0, b]$.

Let $a \in (0, b)$ and let $j \in N \backslash \{i\}$ be a player such that $b \in S(G_j)$. Let $\varepsilon := \pi_j^G(a) - \pi_j^G(b) > 0$. Since the function $\pi_j^G(\cdot)$ is continuous on $(0, \infty)$, we have that, for $\delta > 0$ sufficiently small,

$$\text{for each } t \in [b, b + \delta], \quad \pi_j^G(a) - \pi_j^G(t) > \frac{1}{2}\varepsilon.$$

Since $b \in S(G_j)$, $G_j(b + \delta) > 0 = G_j(b)$. Hence, player j can improve his payoff by moving the probability he assigns to $[b, b + \delta)$ to a. Contradiction. □

The following Lemma shows that if some pure strategy t does not belong to the support of any of the equilibrium strategies, then no pure strategy $t' > t$ belongs to the support of any of the equilibrium strategies either.

Lemma 1.3. *Let $G = (G_i)_{i\in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{S,\alpha,\delta}$. Let $t \in [0, \infty)$ be such that for each $j \in N$, $t \notin S(G_j)$. Then, for each $j \in N$, $(t, \infty) \cap S(G_j) = \emptyset$.*

Proof. Let $K := \cup_{j\in N} S(G_j)$. Clearly, K is closed and $t \notin K$. We have to show that $K \cap (t, \infty) = \emptyset$. Suppose that $K \cap (t, \infty) \neq \emptyset$. Let $t^* := \min\{u \in K : u > t\}$. Let $j^* \in N$ be such that $t^* \in S(G_{j^*})$. Since for each $j \in N$, $[t, t^*) \cap S(G_j) = \emptyset$, then we have that, for each $j \in N$, $G_j(t) = G_j(t^*)$. Hence, the functions G_j are constant on $[t, t^*]$. Now, since for each $u \in [0, \infty)$,

$$\pi_{j^*}^G(u) = \delta^u(\alpha_{j^*} + (S - \sum_{j\in N} \alpha_j) \prod_{j\neq j^*} G_j(u)),$$

then, the function $\pi_{j^*}^G(\cdot)$ is strictly decreasing on $[t, t^*]$. By the continuity of $\pi_{j^*}^G(\cdot)$ at t^*, for each $u \in [t^*, t^* + \varepsilon]$, with $\varepsilon > 0$ sufficiently small, we have $\pi_{j^*}^G(t) > \pi_{j^*}^G(u)$. Hence, G_{j^*} is constant on $[t^*, t^* + \varepsilon]$ as well, contradicting the fact that $t^* \in S(G_{j^*})$. □

Now, we provide specific results for 2-player cake sharing games. The following Lemma shows that, in a Nash equilibrium, the players use mixed strategies of which the supports coincide.

Lemma 1.4. *Let $\Gamma_{S,\alpha,\delta}$ be a 2-player cake sharing game and let $(G_1, G_2) \in \mathcal{G} \times \mathcal{G}$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Then, $S(G_1) = S(G_2)$.*

Proof. This result is just a consequence of Lemma 1.2. □

In the following Lemma we show that the supports of the strategies in a Nash equilibrium are compact intervals.

Lemma 1.5. *Let $\Gamma_{S,\alpha,\delta}$ be a 2-player cake sharing game and let $G = (G_1, G_2) \in \mathcal{G} \times \mathcal{G}$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Let $k := \log_\delta \frac{\alpha_2}{S-\alpha_1}$. Then, $S(G_1) = S(G_2) = [0, k]$.*

Proof. First, we show that $S(G_1) = S(G_2) \subseteq [0, k]$. For each $t \in (k, \infty)$, we have

$$\begin{aligned}
\pi_2^G(t) &= \delta^t(\alpha_2 + (S - \alpha_1 - \alpha_2)G_1(t)) \\
&\leq \delta^t(\alpha_2 + (S - \alpha_1 - \alpha_2)) \\
&= \delta^t(S - \alpha_1) \\
&< \delta^k(S - \alpha_1) \\
&= \alpha_2 \\
&= \pi_2^G(0).
\end{aligned}$$

If $G_2(k) = G_2(k^+) < 1$, *i.e.*, if player 2 puts positive probability on (k, ∞), then he can improve his payoff strictly by moving all this probability to 0. Hence $G_2(k) = 1$ and $S(G_1) = S(G_2) \subseteq [0, k]$.

Let k^* be the largest element in the closed set $S(G_1)$. Clearly, $k^* \leq k$. If $k^* = 0$, then (G_1, G_2) would be an equilibrium in pure strategies, a contradiction. Hence, $k^* > 0$. Now, by Lemma 1.3, $S(G_1) = S(G_2) = [0, k^*]$.

The only thing which remains to be shown is that $k^* = k$. Suppose that $k^* < k$. Now, for each $\tau \in (0, k - k^*)$,

$$\begin{aligned}
\pi_1^G(k^* + \tau) &= \delta^{k^*+\tau}(\alpha_1 + (S - \alpha_1 - \alpha_2)G_2(k^* + \tau)) \\
&= \delta^{k^*+\tau}(\alpha_1 + (S - \alpha_1 - \alpha_2)) \\
&= \delta^{k^*+\tau}(S - \alpha_2) \\
&> \delta^k(S - \alpha_2) \\
&= \frac{\alpha_2(S - \alpha_2)}{S - \alpha_1} \\
&\geq \alpha_1 \\
&= \pi_1^G(0),
\end{aligned}$$

where at the weak inequality we used that $\alpha_2(S - \alpha_2) \geq \alpha_1(S - \alpha_1)$. Hence, if $G_1(0^+) > 0$, *i.e.*, if player 1 plays pure strategy 0 with positive probability, then he can improve his payoff by moving some probability from 0 to pure strategy $k^* + \tau$. Hence, $G_1(0^+) = 0$. Now, there is $t \in (k^*, k)$ such that

$$\begin{aligned}
\pi_2^G(t) &= \delta^t(\alpha_2 + (S - \alpha_1 - \alpha_2)G_1(t)) \\
&= \delta^t(\alpha_2 + (S - \alpha_1 - \alpha_2)) \\
&= \delta^t(S - \alpha_1) \\
&> \delta^k(S - \alpha_1) \\
&= \alpha_2 \\
&= \pi_2^G(0).
\end{aligned}$$

Since $0 \in S(G_1)$ and $\pi_2^G(\cdot)$ is continuous at 0 (because $G_1(0^+) = 0$), player 2 can strictly improve his payoff by moving some probability from the neighborhood of 0 to t. Contradiction. Hence, $k^* = k$. □

Now, we are ready to prove the main theorem of this Section.

Theorem 1.1. *Let $\Gamma_{S,\alpha,\delta}$ be a 2-player cake sharing game and $k := \log_\delta \frac{\alpha_2}{S-\alpha_1}$. Define $G^* = (G_1^*, G_2^*) \in \mathcal{G} \times \mathcal{G}$ by*

$$\begin{aligned} G_1^*(t) &:= \begin{cases} \dfrac{\alpha_2 - \alpha_2\delta^t}{\delta^t(S-\alpha_1-\alpha_2)} & 0 \le t \le k \\ 1 & t > k, \end{cases} \\ G_2^*(t) &:= \begin{cases} 0 & t = 0 \\ \dfrac{\alpha_2(S-\alpha_2) - \alpha_1(S-\alpha_1)\delta^t}{\delta^t(S-\alpha_1)(S-\alpha_1-\alpha_2)} & 0 < t \le k \\ 1 & t > k. \end{cases} \end{aligned}$$

Then, G^* is the unique Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Moreover, the equilibrium payoffs are

$$\begin{aligned} \pi_1(G_1^*, G_2^*) &= \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1}, \\ \pi_2(G_1^*, G_2^*) &= \alpha_2. \end{aligned}$$

Proof. One easily verifies that

$$\pi_1^{G^*}(t) = \begin{cases} \alpha_1 & t = 0 \\ \dfrac{\alpha_2(S-\alpha_2)}{S-\alpha_1} & 0 < t \le k \\ \delta^t(S-\alpha_2) & t > k, \end{cases}$$

and

$$\pi_2^{G^*}(t) = \begin{cases} \alpha_2 & 0 \le t \le k \\ \delta^t(S-\alpha_1) & t > k. \end{cases}$$

Hence,

$$\pi_1(G_1^*, G_2^*) = \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1} \quad \text{and} \quad \pi_2(G_1^*, G_2^*) = \alpha_2.$$

Since for each $t \in [0, \infty)$,

$$\pi_1(t, G_2^*) \le \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1} \quad \text{and} \quad \pi_2(G_1^*, t) \le \alpha_2,$$

we have that G^* is a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$.

In order to show that there are no other Nash equilibria, let (G_1, G_2) be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. By Lemma 1.1, the strategies G_1 and G_2 are continuous on $(0, \infty)$. In the same way as in the proof of Lemma 1.5, we can show that $G_1(0^+) = 0$. Hence, the function $\pi_1^G(\cdot)$ is continuous on $(0, \infty)$ and the function $\pi_2^G(\cdot)$ is continuous on $[0, \infty)$. By Lemma 1.5, $S(G_1) = S(G_2) = [0, k]$. Hence, there are constants c and d such that

$$\begin{aligned} &\text{for each } t \in (0, k], \quad c = \pi_1^G(t) = \delta^t(\alpha_1 + (S-\alpha_1-\alpha_2)G_2(t)), \\ &\text{for each } t \in [0, k], \quad d = \pi_2^G(t) = \delta^t(\alpha_2 + (S-\alpha_1-\alpha_2)G_1(t)). \end{aligned}$$

Since $G_1(0) = 0$, $d = \pi_2^G(0) = \alpha_2$. Hence, for each $t \in [0, k]$,

$$G_1(t) = \frac{\alpha_2 - \alpha_2\delta^t}{\delta^t(S - \alpha_1 - \alpha_2)} = G_1^*(t).$$

Now, for each $t > k$, $G_1(t) = 1 = G_1^*(t)$. Hence, $G_1 = G_1^*$. Moreover, since $G_2(k) = 1$, $c = \pi_1^G(k) = \delta^k(S - \alpha_2) = \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1}$. Hence, for each $t \in (0, k]$

$$G_2(t) = \frac{\alpha_2(S - \alpha_2) - \alpha_1(S - \alpha_1)\delta^t}{\delta^t(S - \alpha_1)(S - \alpha_1 - \alpha_2)} = G_2^*(t).$$

Now, $G_2(0) = 0 = G_2^*(0)$ and, for each $t > k$, $G_2(t) = 1 = G_2^*(t)$. Hence, $G_2 = G_2^*$. This finishes the proof. □

1.4 More Players

In this Section we consider cake sharing games with more than two players. Again, we show that such games admit a unique Nash equilibrium.

First, we show that mixed strategies in a Nash equilibrium have a bounded support.

Lemma 1.6. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game, with $n \geq 3$. Let $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. For each $i \in N$, let*

$$k_i := \log_\delta \frac{\alpha_i}{1 - \sum_{j\neq i}\alpha_j}.$$

Then, $k_1 > k_2 > \cdots > k_n$ and, for each $i \in N$, $S(G_i) \subset [0, k_i]$. Moreover, $S(G_1) \subset [0, k_2]$.

Proof. Let $i, j \in N$ be such that $i > j$. Let $\gamma := \sum\limits_{l\neq i,j} \alpha_l$. Then,

$$\begin{aligned} \alpha_i(S - \gamma - \alpha_i) - \alpha_j(S - \gamma - \alpha_j) &= \alpha_i(S - \gamma - \alpha_i) - \alpha_i\alpha_j + \alpha_i\alpha_j - \alpha_j(S - \gamma - \alpha_j) \\ &= (\alpha_i - \alpha_j)(S - \sum_{l\in N}\alpha_l) \\ &> 0. \end{aligned}$$

Now,

$$\frac{\delta^{k_i}}{\delta^{k_j}} = \frac{\dfrac{\alpha_i}{S - \gamma - \alpha_j}}{\dfrac{\alpha_j}{S - \gamma - \alpha_i}} = \frac{\alpha_i(S - \gamma - \alpha_i)}{\alpha_j(S - \gamma - \alpha_j)} > 1, \tag{1.3}$$

and hence, $k_i < k_j$.

Now, for each $i \in N$ and each $t \in (k_i, \infty)$,

$$\begin{array}{rcl} \pi_i^G(t) & = & \delta^t(\alpha_i + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq i} G_j(t)) \\ & \leq & \delta^t(\alpha_i + (S - \sum_{j\in N}\alpha_j)) \\ & = & \delta^t(S - \sum_{j\neq i}\alpha_j) \\ & < & \delta^{k_i}(S - \sum_{j\neq i}\alpha_j) \\ & = & \alpha_i \\ & = & \pi_i^G(0). \end{array}$$

Repeating the reasoning in Lemma 1.5, if $G_i(k_i) = G_i(k_i^+) < 1$, then player i can strictly improve his payoff by moving all the probability in (k_i, ∞) to 0. Hence, for each $i \in N$, $G_i(k_i) = 1$. Now, for each $j \in N \backslash \{1\}$, $k_2 \geq k_j$. Hence, $G_j(k_2) \geq G_j(k_j) = 1$. Hence, $G_j(k_2) = 1$ and $S(G_j) \subset [0, k_2]$. Now, by Lemma 1.2, we have $S(G_1) \subset [0, k_2]$ as well. □

In the following Lemma we show that pure strategy 0 belongs to the support of every equilibrium strategy. Moreover, players $2, \ldots, n$ play this strategy with positive probability.

Lemma 1.7. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game. Let $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Then, for each $j \in N$, $0 \in S(G_j)$. Moreover, for each $j \in N\backslash\{1\}$, $G_j(0^+) > 0$.*

Proof. Suppose that there is $i \in N$ such that $0 \notin S(G_i)$. Let $s > 0$ be the smallest element in the closed set $S(G_i)$. Then, $[0, s) \cap S(G_i) = \emptyset$. Hence, for each $t \in [0, s]$, $G_i(t) = 0$. Hence, for each $j \in N\backslash\{i\}$ the function

$$\pi_j^G(t) = \delta^t(\alpha_j + (S - \sum_{k\in N}\alpha_k)\prod_{k\neq j} G_k(t)) = \alpha_j\delta^t$$

is strictly decreasing on $[0, s]$. Hence, for each $j \in N\backslash\{i\}$, $(0, s) \cap S(G_j) = \emptyset$. Let $s^* \in (0, s)$. Then, for each $j \in N$, $s^* \notin S(G_j)$. By Lemma 1.3, we have $(s^*, \infty) \cap S(G_i) = \emptyset$. Contradiction with $s \in S(G_i)$. Hence, for each $j \in N$, $0 \in S(G_j)$.

Now suppose $i \in N\backslash\{1\}$ is such that $G_i(0^+) = 0$. This implies that the function

$$\pi_1^G(t) = \delta^t(\alpha_1 + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq 1} G_j(t))$$

is continuous at 0. Let $k_2 := \log_\delta \dfrac{\alpha_2}{S - \sum_{j\neq 2}\alpha_j}$. By Lemma 1.6, for each $j \in N$, $G_j(k_2) = 1$. Hence,

$$\pi_1^G(k_2) = \delta^{k_2}(S - \sum_{j\neq 1}\alpha_j) = \frac{\alpha_2}{S - \sum_{j\neq 2}\alpha_j}(S - \sum_{j\neq 1}\alpha_j) > \alpha_1 = \pi_1^G(0).$$

By the continuity of $\pi_1^G(\cdot)$ at 0, for each $t \in [0, \varepsilon]$ with $\varepsilon > 0$ sufficiently small, we have $\pi_1^G(k_2) > \pi_1^G(t)$. Hence, $[0, \varepsilon) \cap S(G_1) = \emptyset$. Contradiction with $0 \in S(G_1)$. □

The following Lemma provides the equilibrium payoffs in a Nash equilibrium.

Lemma 1.8. *Let $G = (G_i)_{i\in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{S,\alpha,\delta}$ and let $\eta = (\eta_i)_{i\in N}$ be the corresponding vector of equilibrium payoffs. Then,*

$$\eta_1 = \frac{\alpha_2(S - \sum_{j\neq 1}\alpha_j)}{S - \sum_{j\neq 2}\alpha_j}$$

and, for each $j \in N\backslash\{1\}$, $\eta_j = \alpha_j$.

Proof. By Lemma 1.7, we have that, for each $j \in N\backslash\{1\}$, $G_j(0^+) > 0$. Hence, for each $j \in N\backslash\{1\}$, $\eta_j = \pi_j^G(0) = \alpha_j$. Again, let $k_2 := \log_\delta \frac{\alpha_2}{S-\sum_{j\neq 2}\alpha_j}$. By Lemma 1.6, we have that, for each $j \in N$, $G_j(k_2) = 1$. Hence,

$$\eta_1 \geq \pi_1^G(k_2) = \frac{\alpha_2(S - \sum_{j\neq 1}\alpha_j)}{S - \sum_{j\neq 2}\alpha_j}.$$

If $\eta_1 > \pi_1^G(k_2)$, then, by the continuity of $\pi_1^G(\cdot)$ at k_2, for $\gamma > 0$ sufficiently small, we have that, for each $t \in [k_2 - \gamma, k_2]$, $\eta_1 > \pi_1^G(t)$. Hence, $S(G_1) \subseteq [0, k_2 - \gamma]$. Now, player 2 can get more than α_2 by putting all his probability at $k_2 - \gamma + \varepsilon$ for $\varepsilon > 0$ small enough. □

In the following Lemma we show that in a Nash equilibrium players $3, \ldots, n$ claim their initial right immediately, *i.e.*, they play pure strategy 0.

Lemma 1.9. *Let $G = (G_i)_{i\in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{S,\alpha,\delta}$, with $n \geq 3$. Then, for each $i \in N\backslash\{1,2\}$ we have*

$$G_i(t) = \begin{cases} 0 & t = 0 \\ 1 & t > 0, \end{cases}$$

i.e., *G_i corresponds with pure strategy 0.*

Proof. Let $i \in N\backslash\{1,2\}$ and suppose that it is not true that

$$G_i(t) = \begin{cases} 0 & t = 0 \\ 1 & t > 0. \end{cases}$$

Let $t^* := \inf\{t : G_i(t) = 1\}$. Note that $t^* > 0$ and $t^* \in S(G_i)$. Moreover, by the continuity of G_i at t^*, $G_i(t^*) = 1$. Now, we have

$$\pi_2^G(t^*) = \delta^{t^*}(\alpha_2 + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq 2}G_j(t^*)) \leq \alpha_2 \tag{1.4}$$

since, otherwise, player 2 could deviate to pure strategy t^* obtaining strictly more than his equilibrium payoff α_2. Moreover, since $t^* \in S(G_i)$,

$$\pi_i^G(t^*) = \delta^{t^*}(\alpha_i + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq i}G_j(t^*)) = \alpha_i. \tag{1.5}$$

From (1.4) and (1.5) we have

$$\delta^{t^*}(S - \sum_{j\in N} \alpha_j) \prod_{j\neq 2} G_j(t^*) \leq \alpha_2(1 - \delta^{t^*}),$$

$$\delta^{t^*}(S - \sum_{j\in N} \alpha_j) \prod_{j\neq i} G_j(t^*) = \alpha_i(1 - \delta^{t^*}).$$

By Lemma 1.7, we have that, for each $j \in N$, $G_j(t^*) > 0$. Hence, dividing these two expressions we have

$$\frac{G_i(t^*)}{G_2(t^*)} \leq \frac{\alpha_2}{\alpha_i} < 1,$$

which leads to the conclusion that $G_2(t^*) > G_i(t^*) = 1$. Contradiction. □

As a consequence of the last result, the only possible Nash equilibrium in a cake sharing game is one in which players $3, \ldots, n$ play pure strategy 0 and players 1 and 2 play the game with total cake size $S - \sum_{i=3}^{n} \alpha_i$.

Theorem 1.2. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game, $n \geq 3$. Let $k_2 := \log_\delta \frac{\alpha_2}{S-\sum_{j\neq 2}\alpha_j}$. Define $G^* = (G_i^*)_{i\in N} \in \mathcal{G}^N$ by*

$$G_1^*(t) := \begin{cases} \dfrac{\alpha_2 - \alpha_2\delta^t}{\delta^t(S - \sum_{j\in N}\alpha_j)} & 0 \leq t \leq k_2 \\ 1 & t > k_2, \end{cases}$$

$$G_2^*(t) := \begin{cases} 0 & t = 0 \\ \dfrac{\alpha_2(S - \sum_{j\neq 1}\alpha_j) - \alpha_1\delta^t(S - \sum_{j\neq 2}\alpha_j)}{\delta^t(S - \sum_{j\in N}\alpha_j)(S - \sum_{j\neq 2}\alpha_j)} & 0 < t \leq k_2 \\ 1 & t > k_2, \end{cases}$$

$$\text{for each } i \in \{3, \ldots, n\}, \quad G_i^*(t) := \begin{cases} 0 & t = 0 \\ 1 & t > 0. \end{cases}$$

Then, G^* is the unique Nash equilibrium of $\Gamma_{S,\alpha,\delta}$.

Proof. Suppose $G = (G_i)_{i\in N} \in \mathcal{G}^N$ is a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. By Lemma 1.9, we have that, for each $i \in \{3, \ldots, n\}$, $G_i = G_i^*$. Hence, players $3, \ldots, n$ claim their initial rights immediately. Now, (G_1, G_2) is a Nash equilibrium of the 2-player cake sharing game with cake size $S - \sum_{i=3}^{n}\alpha_i$ and initial right vector (α_1, α_2). By Theorem 1.1, we have $G_1 = G_1^*$ and $G_2 = G_2^*$. Hence, $G = G^*$.

Now, we show that G^* is indeed a Nash equilibrium. Since, by Theorem 1.1, (G_1^*, G_2^*) is a Nash equilibrium of the 2-player cake sharing game with cake size $S - \sum_{i=3}^{n}\alpha_i$ and initial right vector (α_1, α_2), then players 1 and 2 can not gain by deviating unilaterally. Now, we show that players

$3, \ldots, n$ are not interested in deviating either. It suffices to show that for each $i \in \{3, \ldots, n\}$ and each $t \in [0, \infty)$, $\pi_i^{G^*}(t) \leq \alpha_i$. Let $i \in \{3, \ldots, n\}$ and $k_i := \log_\delta \frac{\alpha_i}{S - \sum_{j \neq i} \alpha_j}$. By Lemma 1.6, $k_i < k_2$. Hence, for each $t \in [k_2, \infty)$, we have

$$\begin{array}{rcl} \pi_i^{G^*}(t) & = & \delta^t(\alpha_i + (S - \sum_{j \in N} \alpha_j) \prod_{j \neq i} G_j^*(t)) \\ & = & \delta^t(\alpha_i + (S - \sum_{j \in N} \alpha_j)) \\ & = & \delta^t(S - \sum_{j \neq i} \alpha_j) \\ & \leq & \delta^{k_2}(S - \sum_{j \neq i} \alpha_j) \\ & \leq & \delta^{k_i}(S - \sum_{j \neq i} \alpha_j) \\ & = & \alpha_i. \end{array}$$

Hence, it suffices to show that for each $t \in [0, k_2]$, $\pi_i^{G^*}(t) \leq \alpha_i$. Note that for each $t \in [0, k_2]$ ($t = 0$ included), we have

$$\begin{array}{rcl} \pi_i^{G^*}(t) & = & \delta^t(\alpha_i + (S - \displaystyle\sum_{j \in N} \alpha_j) \prod_{j \neq i} G_j^*(t)) \\ & = & \delta^t(\alpha_i + (S - \displaystyle\sum_{j \in N} \alpha_j) G_1^*(t) G_2^*(t)) \\ & = & \delta^t \alpha_i + (\alpha_2 - \alpha_2 \delta^t) \dfrac{\alpha_2(S - \sum_{j \neq 1} \alpha_j) - \alpha_1 \delta^t (S - \sum_{j \neq 2} \alpha_j)}{\delta^t (S - \sum_{j \in N} \alpha_j)(S - \sum_{j \neq 2} \alpha_j)} \\ & = & \delta^t(\alpha_i + \dfrac{\alpha_1 \alpha_2}{S - \sum_{j \in N} \alpha_j}) + \delta^{-t} \dfrac{\alpha_2^2 (S - \sum_{j \neq 1} \alpha_j)}{(S - \sum_{j \in N} \alpha_j)(S - \sum_{j \neq 2} \alpha_j)} \\ & & -(\dfrac{\alpha_1 \alpha_2}{S - \sum_{j \in N} \alpha_j} + \dfrac{\alpha_2^2 (S - \sum_{j \neq 1} \alpha_j)}{(S - \sum_{j \in N} \alpha_j)(S - \sum_{j \neq 2} \alpha_j)}) \\ & = & a\delta^t + b\delta^{-t} + c, \end{array}$$

where

$$a = \alpha_i + \frac{\alpha_1 \alpha_2}{1 - \sum_{j \in N} \alpha_j}$$

$$b = \frac{\alpha_2^2 (1 - \sum_{j \neq 1} \alpha_j)}{(1 - \sum_{j \in N} \alpha_j)(1 - \sum_{j \neq 2} \alpha_j)}$$

$$c = -\frac{\alpha_1 \alpha_2}{1 - \sum_{j \in N} \alpha_j} - \frac{\alpha_2^2 (1 - \sum_{j \neq 1} \alpha_j)}{(1 - \sum_{j \in N} \alpha_j)(1 - \sum_{j \neq 2} \alpha_j)}.$$

Now, make the change of variables $x = \delta^t$. Then, it suffices to show that for the function $f : (0, \infty) \to \mathbb{R}$, defined by

$$\text{for each } x \in (0, \infty),\ f(x) := ax + \frac{b}{x} + c,$$

we have that, for each $x \in [\delta^{k_2}, 1]$, $f(x) \leq \alpha_i$. Since for each $x \in [\delta^{k_2}, 1]$, $f''(x) = \frac{2b}{x^3} > 0$, then the function f is convex on $[\delta^{k_2}, 1]$. Hence, $f(x) \leq \max\{f(\delta^{k_2}), f(1)\}$. Finally, since $f(1) = a + b + c = \alpha_i$ and $f(\delta^{k_2}) = \pi_i^{G^*}(k_2) \leq \alpha_i$, we are done. □

1.5 Concluding Remarks

Throughout this Chapter we assumed that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Scrutinizing the proofs of Lemmas 1.1-1.5 and Theorem 1.1 we may conclude that for 2-player cake sharing games the same result (existence and uniqueness of a Nash equilibrium) also holds in case $\alpha_1 = \alpha_2$. For cake sharing games with at least three players the existence result is still valid in the more general case $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ (and a Nash equilibrium is still provided by the profile described in Theorem 1.2). With few additional efforts we can show that this Nash equilibrium is unique if and only if $\alpha_2 < \alpha_3$.

Moreover, it would be interesting to study whether similar results to those in the all-pay auctions model hold for the different configurations of the initial right vector. If so, we would have a strong parallelism between the results of the two models that, in principle, are very far from each other.

Bibliography

Baston, V. J. and A. Y. Garnaev (2000): "On a Game in Manufacturing," *Mathematical Methods of Operations Research*, 52, 237–249. (Quoted in pp. 6)

Baye, M. R., D. Kovenock, and C. G. de Vries (1996): "The All-Pay Auction with Complete Information," *Economic Theory*, 8, 291–395. (Quoted in pp. 7)

Fudenberg, D., R. Gilbert, J. Stiglitz, and J. Tirole (1983): "Preemption, Leapfrogging and Competition in Patent Races," *European Economic Review*, 22, 3–31. (Quoted in pp. 6)

Hamers, H. (1993): "A Silent Duel over a Cake," *Methods and Models of Operations Research*, 37, 119–127. (Quoted in pp. 6, 7, 9)

Hendricks, K., A. Weiss, and C. Wilson (1988): "The War of Attrition in Continuous Time with Complete Information," *International Economic Review*, 29, 663–680. (Quoted in pp. 6)

Hilman, A. L. and J. G. Riley (1989): "Politically Contestable Rents and Transfers," *Economics and Politics*, 1, 17–39. (Quoted in pp. 7)

Karlin, S. (1959): *Mathematical Methods and Theory in games. Programming and Economics*, Addison-Wesley. (Quoted in pp. 6)

Laraki, R., E. Solan, and N. Vieille (2003): "Continuous-time Games of Timing," Discussion Paper 1363, Kellogg School of Management, Northwestern University. (Quoted in pp. 6)

Reinganum, J. (1981): "On the Diffusion of a New Technology: A Game Theoretic Approach," *The Review of Economic Studies*, 48, 395–405. (Quoted in pp. 6)

Rohatgi, V. K. (1976): *An Introduction to Probability Theory and Mathematical Statistics*, Wiley series in probability and mathematical statistics. (Quoted in pp. 8)

Smith, J. M. (1974): "The Theory of Games and Evolution in Animal Conflicts," *Journal of Theoretical Biology*, 47, 209–221. (Quoted in pp. 6)

Weber, R. J. (1985): "Auctions and Competitive Bidding," in *Fair Allocation*, ed. by H. P. Young, American Mathematical Society, Proceedings of symposia in applied mathematics, 143–170. (Quoted in pp. 7)

Chapter 2

Finitely Repeated Games: A Generalized Nash Folk Theorem

Contents

2.1 Introduction

Over the past thirty years, necessary and sufficient conditions have been published for numerous "folk theorems", asserting that the individually rational feasible payoffs of finitely or infinitely repeated games with complete information can be achieved by Nash or subgame perfect equilibria.[1] The original folk theorem was concerned about the Nash Equilibria of infinitely repeated games. This folk theorem stated that every individually rational feasible payoff of the original game can be obtained as a Nash Equilibrium of the repeated game; no assumption was needed for this result (a statement and proof of this result can be found in Fudenberg and Maskin (1986)). Then, the theorists turned to study subgame perfection in infinite horizon models and they found a counterpart of the previous result for undiscounted repeated games; again, no assumptions were needed (Aumann and Shapley, 1976; Rubinstein, 1979). A few years later, discount parameters were incorporated again into the model; in this case, some conditions were needed to get the perfect folk theorem (Fudenberg and Maskin, 1986). These conditions were refined in the mid-nineties (Abreu et al., 1994; Wen, 1994).

Together with the previous results, also the literature on finitely repeated games grew. The main results for finite horizon models obtained conditions for the Nash folk theorem (Benoît and Krishna, 1987), and also for the perfect one (Benoît and Krishna, 1985). This perfect folk theorem relied on the fact that mixed strategies were observable; the same result but without that assumption was obtained in the mid-nineties (Gossner, 1995). Assuming again observable mixed strategies, Smith (1995) obtained a *necessary and sufficient* condition for the arbitrarily close approximation of strictly rational feasible payoffs by subgame perfect equilibria with finite horizon: that the game have "recursively distinct Nash payoffs", a premise that relaxes the assumption in Benoît and Krishna (1985) that each player have multiple Nash payoffs in the stage game.

Smith claimed that this condition was also necessary for approximation of the individually rational feasible payoffs of finitely repeated games by Nash equilibria. In this Chapter we show that this is not so by establishing a similar but distinct sufficient condition that is weaker than both Smith's condition and the assumptions made by Benoît and Krishna (1987). Moreover, our condition is also necessary. Essentially, the difference between the subgame perfect and Nash cases hinges on the weakness of the Nash solution concept: in the Nash case it is not necessary for threats of punitive action against players who deviate from the equilibrium not to involve loss to the punishing players themselves, *i.e.*, threats need not be credible. The kind of equilibrium we define in this Chapter requires for its corresponding path ρ, to finish, for each player i, with a series Q_i of rounds in which i cannot unilaterally improve his stage payoff by deviation from ρ_i, and for this terminal phase to start with a series Q_i^0 of rounds in which the other players, regardless of the cost to themselves, can punish him effectively for any prior deviation by imposing a loss that wipes out any gains he may have made in deviating.

Many of the results mentioned above concern the approximability of the entire set of individ-

[1] The survey by Benoît and Krishna (1996) includes many of these results.

ually rational feasible payoffs. The main theorem in this Chapter is more general in that, for any game, it characterizes the set of feasible payoffs that are approximable.

Although subgame perfect equilibrium is a desirable refinement of Nash equilibrium, results for the latter are still needed for games in which the perfect folk theorem does not apply. Game G in Figure 2.1 shows that, indeed, this is the case for a generic class of games. The assumptions for the perfect folk theorem do not hold for game G. Moreover, Theorem 2 in Smith (1995) implies that $(3,3)$ is the unique payoff achievable via subgame perfect equilibrium in any repeated game such that G is its stage game. However, every feasible and individually rational payoff, (*e.g.*, (4,4)) can be approximated in Nash equilibrium in many of those repeated games (for small enough discount and big enough number of repetitions).

	L	R
T	3,3	6,2
B	2,6	0,0

Figure 2.1: A game for which the Nash folk theorem is needed.

We have structured this Chapter as follows. We introduce notation and concepts in Section 2.2. In Section 2.3 we state and prove the main result. Next, in Section 2.4 we are concerned about unobservable mixed strategies. Finally, we conclude in Section 2.5.

2.2 Basic Notation, Definitions and an Example

2.2.1 The Stage Game

A *strategic game* G is a triplet (N, A, φ), where:

- $N := \{1, \dots, n\}$ is the set of players,
- $A := \prod_{i \in N} A_i$ and A_i is the set of player i's strategies,
- $\varphi := (\varphi_1, \dots, \varphi_n)$ and $\varphi_i : A \to \mathbb{R}$ is the payoff function of player i.

Let $\mathcal{G}^N$ be the set of games with set of players N.

We assume that, for each $i \in N$, the sets A_i are compact and the functions φ_i are continuous. Let a_{-i} be a strategy profile for players in $N \backslash \{i\}$ and A_{-i} the set of such profiles. For each $i \in N$ and each $a_{-i} \in A_{-i}$, let $\mu_i(a_{-i}) := \max_{a_i \in A_i}\{\varphi_i(a_{-i}, a_i)\}$. Also, for each $i \in N$, let $v_i := \min_{a_{-i} \in A_{-i}}\{\mu_i(a_{-i})\}$. The vector $v := \{v_1, \dots, v_n\}$ is the *minimax payoff vector*. Let F be the set of *feasible* payoffs: $F := \text{co}\{\varphi(a) : a \in A\}$. Let $\bar{F}$ be the set of all *feasible and individually rational* payoffs:

$$\bar{F} := F \cap \{u \in \mathbb{R}^n : u \geq v\}.$$

To avoid confusion with the strategies of the repeated game, in what follows we refer to the strategies $a_i \in A_i$ and the strategy profiles $a \in A$ of the stage game as actions and action profiles, respectively.

2.2.2 The Repeated Game

Let $G(\delta, T)$ be the game consisting in the T-fold repetition of G with payoff discount parameter $\delta \in (0, 1]$. In this game we assume *perfect monitoring*, *i.e.*, each player can choose his action in the current stage in the light of all actions taken by all players in all previous stages. Let σ be a strategy profile of $G(\delta, T)$, and the action profile sequence $\rho = \{\rho^1, \ldots, \rho^T\}$ its corresponding *path*. Let $\varphi_i^t(\rho)$ be the stage payoff of player i at stage t when all players play in accordance with ρ. Then, player i's payoff in $G(\delta, T)$ when σ is played is his average discounted stage payoff: $\psi_i(\sigma) \equiv \psi_i(\rho) := ((1-\delta)/(1-\delta^T)) \sum_{t=1}^{T} \delta^{t-1}\varphi_i^t(\rho)$.[2]

2.2.3 Minimax-Bettering Ladders

Let M be an m-player subset of N. Let $A_M := \prod_{i\in M} A_i$ and let $G(a_M)$ be the game induced for the $n-m$ players in $N\backslash M$ when the actions of the players in M are fixed at $a_M \in A_M$. By abuse of language, if $i \in N\backslash M$, $a_M \in A_M$, and $\sigma \in A_{N\backslash M}$ we write $\varphi_i(\sigma)$ for i's payoff at σ in $G(a_M)$. A *minimax-bettering ladder* of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$, where $\mathcal{N}$ is a strictly increasing chain $\{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ of $h+1$ subsets of N $(h \geq 1)$, $\mathcal{A}$ is a chain of action profiles $\{a_{N_1} \in A_{N_1}, \ldots, a_{N_{h-1}} \in A_{N_{h-1}}\}$ and Σ is a chain $\{\sigma^1, \ldots, \sigma^h\}$ of Nash equilibria of $G = G(a_{N_0}), G(a_{N_1}), \ldots, G(a_{N_{h-1}})$, respectively, such that at σ^l the players of $G(a_{N_{l-1}})$ receiving payoffs strictly greater than their minimax payoff are exactly those in $N_l\backslash N_{l-1}$: for each $i \in N_l\backslash N_{l-1}$, $\varphi_i(\sigma^l) > v_i$, and for each $i \in N\backslash N_l$, $\varphi_i(\sigma^l) \leq v_i$.

Let the sets in $\mathcal{N}$ be the rungs of the ladder. In algorithmic terms, if the first $l-1$ rungs of the ladder have been constructed, then, for the l-th rung to exist, there must be $a_{N_{l-1}} \in A_{N_{l-1}}$ such that the game $G(a_{N_{l-1}})$ has an equilibrium σ^l. Moreover, σ^l has to be such that there are players $i \in N\backslash N_{l-1}$ for whom $\varphi_i(\sigma^l) > v_i$. Let $N_l\backslash N_{l-1}$ be this subset of players of $G(a_{N_{l-1}})$. The game played in the next step is defined by some action profile a_{N_l}. The set N_h is the *top rung of the ladder*. A ladder with top rung N_h is *maximal* if there is no ladder with top rung $N_{h'}$ such that $N_h \subsetneq N_{h'}$. A game G is *decomposable as a complete minimax-bettering ladder* if it has a minimax-bettering ladder with N as its top rung. We show below that being decomposable as a complete minimax-bettering ladder is a necessary and sufficient condition for it to be possible to approximate all payoff vectors in $\bar{F}$ by Nash equilibria of $G(\delta, T)$ for some δ and T. Clearly, being decomposable as a complete minimax-bettering ladder is a weaker property than the requirement in Smith (1995), that at each step $l-1$ of a similar kind of ladder there be action profiles $a_{N_{l-1}}$, $b_{N_{l-1}}$ such that the games $G(a_{N_{l-1}})$ and $G(b_{N_{l-1}})$ have Nash equilibria σ_a^l and σ_b^l with $\varphi_i(\sigma_a^l) \neq \varphi_i(\sigma_b^l)$ for a nonempty set of players (those in $N_l\backslash N_{l-1}$).

2.2.4 An Example

Let $G \in \mathcal{G}^N$, let L be a maximal ladder of G, and $N_{\max}$ its top rung. For each $i \in N$, let l_i be the unique integer such that $i \in N_{l_i}\backslash N_{l_i-1}$. In the equilibrium strategy profile constructed

[2] Or, $\psi_i(\sigma) \equiv \psi_i(\rho) := (1/T) \sum_{t=1}^{T} \varphi_i^t(\rho)$ if there are no discounts ($\delta = 1$).

in Theorem 2.1 below, the action profile sequence in the terminal phase Q_i referred to in the Introduction, consists of repetitions of $(a_{N_{l_i-1}}, \sigma^{l_i}), (a_{N_{l_i-2}}, \sigma^{l_i-1}), \ldots, (a_{N_2}, \sigma^2)$ and σ; and the σ^j are Nash equilibria of the corresponding games $G(a_{N_{j-1}})$. Since player i is a player in all these games, he can indeed gain nothing by unilateral deviation during this phase. In the potentially punishing series of rounds Q_i^0, the action profile sequence consists of repetitions of $(a_{N_{l_i-1}}, \sigma^{l_i})$, in which i obtains more than his minimax payoff, with the accompanying threat of punishing a prior unilateral deviation by i by minimaxing him instead.

L	l	m	r
T	0, 0, 3	0,-1, 0	0,-1, 0
M	-1, 0, 0	0,-1, 0	0,-1, 0
B	-1, 0, 0	0,-1, 0	0,-1, 0

R	l	m	r
T	0, 3,-1	0,-1,-1	1,-1,-1
M	-1, 0,-1	-1,-1,-1	0,-1,-1
B	-1, 0,-1	-1,-1,-1	0,-1,-1

Figure 2.2: A game that is decomposable as a complete minimax-bettering ladder

As an illustration of the above ideas, consider the three-player game G shown in Figure 2.2. Its minimax payoff vector is $(0,0,0)$, and its unique Nash equilibrium is the action profile $\sigma^1 = (T, l, L)$, with associated payoff vector $(0,0,3)$. Hence, $N_1 = \{3\}$; player 3 can be punished by 1 and 2 by playing one of his minimax profiles instead of playing $(T, l, \cdot)$. If player 3 now plays R ($a_{N_1} = R$), the resulting game $G(a_{N_1}) = G(R)$ has an equilibrium $\sigma^2 = (T, l)$ with payoff vector $(0,3)$. Hence, $N_2 = \{2,3\}$ and player 2 can be punished by 1 and 3 by playing one of his minimax profiles instead of playing $(T, \cdot, R)$. Finally if players 2 and 3 now play r and R ($a_{N_2} = (r, R)$), the resulting game $G(a_{N_2}) = G(r, R)$ has the trivial equilibrium $\sigma^3 = (T)$ with payoff 1 for player 1. Hence, player 1 can be punished by 2 and 3 if they play one of his minimax profiles instead of playing $(\cdot, r, R)$.

2.2.5 Further Preliminaries

As a consequence of the next Lemma we can unambiguously refer to the *top rung of a game G*.

Lemma 2.1. *Let $G \in \mathcal{G}^N$. Then, all its maximal ladders have the same top rung.*

Proof. Suppose there are maximal ladders $L = \{\mathcal{N}, \mathcal{A}, \Sigma\}$, $L' = \{\mathcal{N}', \mathcal{A}', \Sigma'\}$ with $\mathcal{N} = \{N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ and $\mathcal{N}' = \{N_0' \subsetneq N_1' \subsetneq \cdots \subsetneq N_k'\}$ such that $N_h \neq N_k'$. Assume, without loss of generality, that $N_k' \backslash N_h \neq \emptyset$. For each $j \in N_k'$, let l_j be the unique integer such that $j \in N_{l_j}' \backslash N_{l_j-1}'$. Let $i \in \operatorname{argmin}_{j \in N_k' \backslash N_h} l_j$. Then, $N_{l_i-1}' \subseteq N_h$. Let a_{N_h} be the action profile defined as follows:

$$\text{for each } j \in N, \quad (a_{N_h})_j = \begin{cases} (a'_{N'_{l_i-1}})_j & j \in N'_{l_i-1} \\ (\sigma'^{l_i})_j & j \in N_h \backslash N'_{l_i-1}, \end{cases}$$

where $\sigma'^{l_i} \in \Sigma'$ is an equilibrium of the game $G(a'_{N'_{l_i-1}})$ induced by the action profile $a'_{N'_{l_i-1}} \in \mathcal{A}'$.

Now, let σ^{h+1} be the restriction of σ'^{l_i} to $N \backslash N_h$. Since σ'^{l_i} is an equilibrium of $G(a'_{N'_{l_i-1}})$, and $N \backslash N_h \subseteq N \backslash N'_{l_i-1}$, σ^{h+1} is an equilibrium of $G(a_{N_h})$. Moreover, the set of players $j \in N \backslash N_h$ for whom $\varphi_j(\sigma^{h+1}) > v_j$ is $N'_{l_i} \backslash N_h$. Let $N_{h+1} := N'_{l_i} \backslash N_h$. Since N_{h+1} contains i, it is nonempty. Let $L'' = \{\mathcal{N}'', \mathcal{A}'', \Sigma''\}$ be the ladder defined by

- $\mathcal{N}'' = \{\mathcal{N}_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h \subsetneq N_{h+1}\}$,
- $\mathcal{A}'' = \{a_{N_1}, \ldots, a_{N_{h-1}}, a_{N_h}\}$,
- $\Sigma'' = \{\sigma^1, \ldots, \sigma^h, \sigma^{h+1}\}$.

The top rung of L'' strictly contains that of L. Hence, L is not maximal, which proves the Lemma. □

Let G be a game with set of players N and let $N' \subseteq N$. We say that $G \in \mathrm{TR}_{N'}(\mathcal{G}^N)$ if the top rung of any maximal ladder of G is N'. Hence, a game G is decomposable as a complete minimax-bettering ladder if and only if $G \in \mathrm{TR}_N(\mathcal{G}^N)$.

Let $G \in \mathrm{TR}_{N_{\max}}(\mathcal{G}^N)$ and $\hat{a} \in A_{N_{\max}}$. Let $\Lambda(\hat{a}) := \{\lambda = (\hat{a}, \sigma) \in A \ : \ \sigma$ Nash equilibrium of $G(\hat{a})\}$ and $\Lambda := \bigcup_{\hat{a} \in A_{N_{\max}}} \Lambda(\hat{a})$. Let $\varphi(\Lambda) := \{\varphi(\lambda) \ : \ \lambda \in \Lambda\}$. Let $\bar{F}_{N_{\max}}$ be the set of N_{max}*-attainable* payoffs of G: $\bar{F}_{N_{\max}} := \bar{F} \cap \operatorname{co} \varphi(\Lambda)$. Note that, by the definition of $N_{\max}$, for each $u \in \bar{F}_{N_{\max}}$ and each $i \in N \backslash N_{\max}$, $u_i = v_i$. Moreover, when $N_{\max} = N$ we have $\Lambda = A$ and $\bar{F}_{N_{\max}} = \bar{F}$.

Lemma 2.2. *Let $G \in \mathrm{TR}_{N_{max}}(\mathcal{G}^N)$. Then, the set $\bar{F}_{N_{max}}$ is closed.*

Proof. First, we show that Λ is closed. Let $\{(a_n, \sigma_n)\}$ be a sequence of action profiles in Λ with limit (a, σ). Since $A_{N_{\max}}$ is compact, $a \in A_{N_{\max}}$. Since φ is continuous, σ is a Nash equilibrium of $G(a)$. Hence, $(a, \sigma) \in \Lambda$.

The set $\varphi(\Lambda)$ is the image of a closed set under a continuous function. Since φ has a compact domain, $\varphi(\Lambda)$ is closed. Hence, $\bar{F} \cap \operatorname{co} \varphi(\Lambda)$ is closed. □

The promised result concerning the approximability of all payoffs in $\bar{F}$ by Nash equilibrium payoffs is obtained below as an immediate corollary of a more general theorem concerning the approximability of all payoffs in $\bar{F}_{N_{\max}}$. In this more general case, the collaboration of the players in $N_{\max}$ is secured by a strategy analogous to that sketched in the Example of Section 2.2.4, while the collaboration of the players in $N \backslash N_{\max}$ is also ensured because none of them is able to obtain any advantage by unilateral deviation from any action profile in Λ.

2.3 The Theorem

In the theorem that follows, the set of action profiles A may consist either of pure or mixed action profiles; in the latter case, we assume that all players are cognizant not only of the pure actions actually put into effect at each stage, but also of the mixed actions of which they are realizations.

We discuss unobservable mixed actions in Section 2.4. Also, we assume *public randomization*: at each stage of the repeated game, players can let their actions depend on the realization of an exogenous continuous random variable. The assumption of public randomization is without loss of generality. Given a correlated mixed action, its payoff can be approximated by alternating pure actions with the appropriate frequencies. More precisely, for each $u \in \bar{F}$ and each $\varepsilon > 0$, there are pure actions $a_1, \ldots, a_l$ such that $||u - (a_1 + \ldots + a_l)/l|| < \varepsilon$. Hence, if the discount parameter δ is close enough to 1, the same inequality is still true if we consider discounted payoffs. Then, since we state Theorem 2.1 in terms of approximated payoffs, public randomization assumption can be dispensed with.[3]

Theorem 2.1. *Let $G \in \mathrm{TR}_{N_{max}}(\mathcal{G}^N)$. Let $u \in F$. Then, a necessary and sufficient condition for there to be for each $\varepsilon > 0$, an integer T_0 and a positive real number $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, $G(\delta, T)$ has a Nash equilibrium payoff w such that $\|w - u\| < \varepsilon$ is that u be N_{max}-attainable (*i.e., *$u \in \bar{F}_{N_{max}}$).*

Proof. $\overset{\text{suffic}}{\Longleftarrow}$ Let $a \in \Lambda$ be an action profile of G such that $\varphi(a) = u$, and let $L = \{\mathcal{N}, \mathcal{A}, \Sigma\}$ be a maximal minimax-bettering ladder of G. By the definition of Λ, players in $N \backslash N_{\max}$ have no incentive for unilateral deviation from a. Let ρ be the following action profile sequence:

$$\rho := \{\underbrace{a, \ldots, a}_{T-T_0+q_0}, \underbrace{\lambda^h, \ldots, \lambda^h}_{q_h}, \underbrace{\lambda^{h-1}, \ldots, \lambda^{h-1}}_{q_{h-1}}, \ldots, \underbrace{\lambda^1, \ldots, \lambda^1}_{q_1}\},$$

where for each $l \in \{1, \ldots h\}$, $\lambda^l = (a_{N_{l-1}}, \sigma^l)$ with $a_{N_{l-1}} \in \mathcal{A}$ and $\sigma^l \in \Sigma$. Let $\varepsilon > 0$. Next, we obtain (in this order) values for $q_h, \ldots, q_1$, the discount δ_0, q_0, and T_0 to ensure that for each $T \geq T_0$ and each $\delta \in (\delta_0, 1]$, there is a Nash equilibrium of $G(\delta, T)$ whose path is ρ and such that $||\varphi(\rho) - u|| < \varepsilon$.

First, we calculate how many repetitions of $G(a_{N_{l_i-1}})$ are necessary for the players in $N \backslash \{i\}$ to be able to punish a player $i \in N_{\max}$ for prior deviation. For each action profile $\hat{a} \in A$, let $\bar{\mu}_i(\hat{a}) := \mu_i(\hat{a}_{-i}) - \varphi_i(\hat{a})$, *i.e.*, the maximum "illicit" profit that player i can obtain by unilateral deviation from $\hat{a}$. Let $\bar{\mu}_i = \max\{\bar{\mu}_i(a), \bar{\mu}_i((a_{N_{h-1}}, \sigma^h)), \ldots, \bar{\mu}_i(\sigma^1)\}$ and $m_i = \min\{\varphi_i(a) \,:\, a \in A\}$. Let $l_i \in \mathbb{N}$ be such that $i \in N_{l_i} \backslash N_{l_i - 1}$. Let $\delta_0 \in (0, 1)$ and let $q_h, \ldots, q_1$ be the natural numbers defined through the following iterative procedure:

Step 0:

For each $i \in N_h \backslash N_{h-1}$, let $r_i \in \mathbb{N}$ and $\delta_i \in (0, 1)$ be

$r_i := \min\{r \in \mathbb{N} \,:\, r(\varphi_i(\sigma^{l_i}) - v_i) > \bar{\mu}_i\}$,[4]

$\delta_i := \min\{\delta_i \in (0, 1) \,:\, \bar{\mu}_i - \sum_{t=1}^{r_i} \delta_i^t(\varphi_i(\sigma^{l_i}) - v_i) < 0\}$.

Let $q_h \in \mathbb{N}$ be

[3]For further discussion on public randomization refer to Fudenberg and Maskin (1991) and Olszewski (1997). Also, refer to Gossner (1995) for a paper in which public randomization is not assumed and the approximation procedure we described above is explicitly made (though discounts are not considered).

[4]The natural number r_i is such that, at each step, punishing player i during r_i stages suffices to wipe out any stage gain he could get by deviating from ρ when the discount is $\delta = 1$.

$q_h := \max\{r_i \,:\, i \in N_h \backslash N_{h-1}\}$.

Step k **(**$k < h$**):**

Let $T_k := \sum_{l=0}^{k-1} q_{h-l}$.

For each $i \in N_{h-k} \backslash N_{h-k-1}$, let $r_i \in \mathbb{N}$ and $\delta_i \in (0,1)$ be

$r_i := \min\{r \in \mathbb{N} \,:\, r(\varphi_i(\sigma^{l_i}) - v_i) > \bar{\mu}_i + T_k(v_i - m_i)\}$,

$\delta_i := \min\{\delta_i \in (0,1) \,:\, \bar{\mu}_i + \sum_{t=1}^{T_k} \delta_i^t (v_i - m_i) - \sum_{t=T_k+1}^{T_k+r_i} \delta_i^t (\varphi_i(\sigma^{l_i}) - v_i) < 0\}$.

Let $q_{h-k} \in \mathbb{N}$ be

$q_{h-k} := \max\{r_i \,:\, i \in N_{h-k} \backslash N_{h-k-1}\}$.

Step h**:**

$\delta_0 := \max_{i \in N} \delta_i$.

The natural numbers $q_h, \ldots, q_1$ and the discount δ_0 are such that for each $l \in \{1, \ldots, h\}$, q_l repetitions of $G(a_{N_{l-1}})$ suffice to allow any player in $N_l \backslash N_{l-1}$ to be punished. Next, we obtain the values for q_0 and T_0. Let q_0 be the smallest integer such that:

$$\left\| \frac{q_0\, \varphi(a) + q_h\, \varphi(\lambda^h) + \cdots + q_1\, \varphi(\lambda^1)}{q_0 + q_h + \cdots + q_1} - \varphi(a) \right\| < \varepsilon. \tag{2.1}$$

Let $T_0 := q_0 + q_1 + \cdots + q_h$. Let $T \geq T_0$ and $\delta \in [\delta_0, 1]$. We prescribe for $G(\delta, T)$ the strategy profile in which all players play according to ρ unless and until there is a unilateral deviation. In such a deviation occurs, the deviating player is minimaxed by all the others in the remaining stages of the game. It is straightforward to check that this profile is a Nash equilibrium of $G(\delta, T)$. Moreover, by inequality (2.1), its associated payoff vector w differs from u by less than $\frac{T_0}{T}\varepsilon$ if $\delta = 1$. Hence, the same observation is certainly true if $\delta < 1$, in which case payoff vectors of the early stages, $\varphi(a)$, receive greater weight than the payoff vectors of the endgame.

$\overset{\text{necess}}{\Longrightarrow}$ Let $u \notin \bar{F}_{N_{\max}}$. Suppose that $N_{\max} = N$. Then, $\bar{F}_{N_{\max}} = \bar{F}$. Hence, u is not individually rational. Hence, it can not be the payoff associated to any Nash equilibrium. Then, we can assume $N_{\max} \subsetneq N$. Since $\bar{F}_{N_{\max}}$ is a closed set, there is $\varepsilon > 0$ such that $\|w - u\| < \varepsilon$ implies $w \notin \bar{F}_{N_{\max}}$. Hence, if for some T and δ there is a strategy profile σ of $G(\delta, T)$ such that $\|\varphi(\sigma) - u\| < \varepsilon$, then $\varphi(\sigma) \notin \bar{F}_{N_{\max}}$. Hence, by the definition of $\bar{F}_{N_{\max}}$, there is at least one stage of $G(\delta, T)$ in which, with positive probability, σ prescribes an action profile not belonging to Λ . Let q be the last such stage and $\bar{a} = (\bar{a}_{N_{\max}}, \bar{a}_{N \backslash N_{\max}})$ the corresponding action profile. By the definition of $\bar{F}_{N_{\max}}$, $\bar{a}_{N \backslash N_{\max}}$ cannot be a Nash equilibrium of $G(\bar{a}_{N_{\max}})$. Hence, there is a player $j \in N \backslash N_{\max}$ who can increase his payoff in round q by deviating unilaterally from $\bar{a}$. Since, by the definition of q, σ assigns j a stage payoff of v_j in all subsequent rounds, this deviation cannot subsequently be punished. Hence, σ is not an equilibrium of $G(\delta, T)$. $\square$

Corollary 2.1. *Let $G \in \mathcal{G}^N$ be decomposable as a complete minimax-bettering ladder, (*i.e.*, $G \in \mathrm{TR}_N(\mathcal{G}^N)$). Then, for each $u \in \bar{F}$ and each $\varepsilon > 0$, there is $T_0 \in \mathbb{N}$ and $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash equilibrium payoff w of $G(\delta, T)$ with $\|w - u\| < \varepsilon$.*

Proof. $N = N_{\max} \Rightarrow \bar{F} = \bar{F}_{N_{\max}}$. Hence, this result is a consequence of Theorem 2.1. □

Corollary 2.2. *Let $G \in \mathcal{G}^N$ be not decomposable as a complete minimax-bettering ladder (*i.e., *$G \notin \mathrm{TR}_N(\mathcal{G}^N)$). Then, for each $T \in \mathbb{N}$, each $\delta \in (0,1]$, each $i \in N\backslash N_{max}$, and each Nash equilibrium σ of $G(\delta, T)$ we have $\varphi_i(\sigma) = v_i$.*

Proof. For each $u \in \bar{F}_{N_{\max}}$ and for each $i \in N\backslash N_{\max}$, $u_i = v_i$. Hence, this result follows by an argument paralleling the proof of necessity in Theorem 2.1. □

2.4 Unobservable Mixed Actions

In what follows, we drop the assumption that mixed actions are observable. Hence, if a mixed action is chosen by one player, the others can only observe its realization. To avoid confusion, for each game G, let G^u be the corresponding game with unobservable mixed actions. We need to introduce one additional piece of notation to distinguish between pure and mixed actions. Let A_i and S_i be the sets of player i's pure and mixed actions respectively (with generic elements a_i and s_i). Similarly, let A and S be the sets of pure and mixed action profiles. Hence, a game is now a triplet (N, S, φ).

The game G (or G^u) in Figure 2.3 illustrates some of the differences between the two frameworks. Although it is not entirely straightforward, it is not difficult to check that the minimax payoff of G is $v = (0, 0, 0)$. Let $s_3 = (0, 0.5, 0.5)$ be the mixed action of player 3 in which he plays L with probability 0, and M and R with probability 0.5. Let $\sigma^2 \in A_{\{1,2\}}$. Let $\mathcal{N} = \{\emptyset, \{3\}, N\}$, $\mathcal{S} = \{s_3\}$ and $\Sigma = \{(T, l, L), \sigma^2\}$. Then, $L = \{\mathcal{N}, \mathcal{S}, \Sigma\}$ is a complete minimax-bettering ladder of G regardless of σ^2 (note that in the game $G(s_3)$, for each $\sigma^2 \in A_{\{1,2\}}$, both players 1 and 2 receive the constant payoff 0.5). Hence, G satisfies the assumptions of Corollary 2.1, so every payoff in $\bar{F}$ can be approximated in Nash equilibrium.

L	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

Figure 2.3: A game where unobservable mixed actions make a difference

Consider now the game G^u. Let $u \in \bar{F}$, and let a be such that $\varphi(a) = u$ (recall that we assumed public randomization). If we follow the path ρ constructed in the proof of Theorem 2.1, there are natural numbers q_0, q_1, and q_2 such that ρ leads to play (i) a during the first q_0 stages, (ii) (σ^2, s_3) during the following q_2 stages, and (iii) (T,l,L) during the last q_1 stages. Let Q be the phase described in (ii). Since player 3 is not indifferent between the two actions in the support of s_3, we need a device to detect possible deviations from that support. But, once such a device has been chosen, it is not clear whether we can ensure that there are not realizations for the first

$q_2 - 1$ stages of Q that would allow player three to play L in the last stage of Q without being detected.[5]

Next, we revisit the results of Section 2.3 to understand the extent to which their counterparts hold. Unfortunately, we have not found a necessary and sufficient condition for the folk theorem under unobservable mixed actions, *i.e.*, we have not found an exact counterpart for Theorem 2.1. More precisely, as the previous example shows, unobservable mixed actions invalidate the proofs related to sufficiency conditions. On the other hand, proofs related to necessary conditions still carry over.

For the next result, we need to introduce a restriction on the ladders. The objective is to rule out situations as the one illustrated with Figure 2.3. Let $L = \{\mathcal{N}, \mathcal{S}, \Sigma\}$ be a ladder with $\mathcal{S} = \{s_{N_1}, \ldots, s_{N_{h-1}}\}$. L is a *p-ladder* if, for each $l \in \{1, \ldots, h-1\}$, $s_{N_l} \in A_{N_l}$. That is, at each rung of the ladder we only look at subgames obtained by fixing pure action profiles.[6]

Lemma 2.3. *Let $G \in \mathcal{G}^N$. Then, all its maximal p-ladders have the same top rung.*

Proof. Analogous to the proof of Lemma 2.1. □

Let G (or G^u) be a game with set of players N and let $N' \subseteq N$. We say that $G \in \mathrm{TR}^P_{N'}(\mathcal{G}^N)$ if the top rung of any maximal p-ladder of G is N'. Clearly, if $G \in \mathrm{TR}^P_{N'}(\mathcal{G}^N)$, then $G \in \mathrm{TR}_{N''}(\mathcal{G}^N)$ with $N' \subseteq N''$. The game G in Figure 2.3 provides an example in which the converse fails: $G \in \mathrm{TR}^P_{\{3\}}(\mathcal{G}^N)$ and $G \in \mathrm{TR}_{\{N\}}(\mathcal{G}^N)$. Let $G \in \mathrm{TR}^P_{N_{\max}}(\mathcal{G}^N)$ and a pure strategy $\hat{a} \in A_{N_{\max}}$. We can define $\bar{F}^P_{N_{\max}}$ paralleling the definition of $\bar{F}_{N_{\max}}$ in Section 2.2.

Next, we state the results. Note that the sets TR and $\bar{F}_{N_{\max}}$ are used for the necessity results and the sets TR^P and $\bar{F}^P_{N_{\max}}$ for the sufficiency ones.

Proposition 2.1 (Sufficient condition)**.** *Let $G^u \in \mathrm{TR}^P_{N_{max}}(\mathcal{G}^N)$. Then, for each $u \in \bar{F}^P_{N_{max}}$ and each $\varepsilon > 0$, there are $T_0 \in \mathbb{N}$ and $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash equilibrium payoff w of $G(\delta, T)$, with $\|w - u\| < \varepsilon$.*

Proof. Analogous to the proof of the sufficiency condition in Theorem 2.1. This is because, as far as a p-ladder is used to define the path ρ, whenever a player plays a mixed action, all the pure actions in its support are best replies to the actions of the others. □

Corollary 2.3. *Let $G^u \in \mathrm{TR}^P_N(\mathcal{G}^N)$. Then, for each $u \in \bar{F}$ and each $\varepsilon > 0$, there are $T_0 \in \mathbb{N}$ and $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash equilibrium payoff w of $G(\delta, T)$, with $\|w - u\| < \varepsilon$.*

Proof. $N = N_{\max} \Rightarrow \bar{F} = \bar{F}_{N_{\max}} = \bar{F}^P_{N_{\max}}$. Hence, this result is an immediate consequence of Proposition 2.1. □

[5] Game G^u partially illustrates why the arguments in Gossner (1995) cannot be easily adapted to our case. First, *mutatis mutandi*, he applies an existence of equilibrium theorem to the subgame in Q. If we want to do so, we need to ensure that players 1 and 2 get more than 0 in Q. Second, Gossner also uses the assumption of full-dimensionality of F to punish all the players who deviate during Q. We do not have that assumption and hence, it could be the case that we could not punish more than one player at the end of the game.

[6] Note that the games G and G^u have the same ladders and the same p-ladders.

Note that the folk theorem in Benoît and Krishna (1987) is a particular case of this corollary. Next two results show that the exact counterparts of the necessity results in Section 2.3 carry over.

Proposition 2.2 (Necessary condition). *Let* $G^u \in \mathrm{TR}_{N_{max}}(\mathcal{G}^N)$. *If* $N_{max} \subsetneq N$ *then, for each* $u \notin \bar{F}_{N_{max}}$ *there is* $\varepsilon > 0$ *such that for each* $T \in \mathbb{N}$ *and each* $\delta \in (0,1]$, $G(\delta, T)$ *does not have a Nash equilibrium payoff* w *such that* $\|w - u\| < \varepsilon$.

Proof. Analogous to the proof of the necessity condition in Theorem 2.1. □

Corollary 2.4. *Let* $G^u \notin \mathrm{TR}_N(\mathcal{G}^N)$. *Then, for each* $T \in \mathbb{N}$, *each* $\delta \in (0,1]$, *each* $i \in N \backslash N_{max}$, *and each Nash equilibrium* σ *of* $G(\delta, T)$ *we have* $\varphi_i(\sigma) = v_i$.

Proof. Analogous to the proof of Corollary 2.2. □

2.5 Concluding Remarks

Recall that Corollaries 2.1 and 2.3 hold for a wider class of games than the result obtained by Benoît and Krishna (1987). Moreover, Theorem 2.1 requires no use of the concept of effective minimax payoff, because non-equivalent utilities are irrelevant to the approximation of $N_{\max}$-attainable payoffs by Nash equilibria, in which there is no need for threats to be credible.[7]

Theorem 2.1 raises the question whether a similarly general result on the approximability of payoffs by equilibria also holds for subgame perfect equilibria. The main problem is to determine the subgame perfect equilibrium payoffs of players with "recursively distinct Nash payoffs" (Smith, 1995) when the game is not completely decomposable. Similarly, the results in Section 2.4 raise the question whether a necessary and sufficient condition exists for the Nash folk theorem under unobservable mixed strategies.

Finally, note that the results of this Chapter can be easily extended to the case in which each player has a different discount rate.

[7] See Wen (1994) and Abreu et al. (1994) for details on the effective minimax payoff and non-equivalent utilities respectively.

Bibliography

Abreu, D., P. K. Dutta, and L. Smith (1994): "The Folk Theorem for Repeated Games: A NEU Condition," *Econometrica*, 62, 939–948. (Quoted in pp. 22, 31)

Aumann, R. J. and L. S. Shapley (1976): "Long Term Competition, A Game Theoretic Analysis," Tech. rep., Hebrew University. (Quoted in pp. 22)

Benoît, J.-P. and V. Krishna (1985): "Finitely Repeated Games," *Econometrica*, 53, 905–922. (Quoted in pp. 22)

——— (1987): "Nash Equilibria of Finitely Repeated Games," *International Journal of Game Theory*, 16, 197–204. (Quoted in pp. 22, 31)

——— (1996): "The Folk Theorems for Repeated Games: A Synthesis," Tech. rep., EconWPA-9601001. (Quoted in pp. 22)

Fudenberg, D. and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54, 533–554. (Quoted in pp. 22)

——— (1991): "On the Dispensability of Public Randomization in Discounted Repeated Games," *Journal of Economic Theory*, 53, 428–438. (Quoted in pp. 27)

Gossner, O. (1995): "The Folk Theorem for Finitely Repeated Games with Mixed Strategies," *International Journal of Game Theory*, 24, 95–107. (Quoted in pp. 22, 27, 30)

Olszewski, W. (1997): "Perfect Folk Theorems. Does Public Randomization Matter?" *International Journal of Game Theory*, 27, 147–156. (Quoted in pp. 27)

Rubinstein, A. (1979): "Strong Perfect Equilibrium in SuperGames," *International Journal of Game Theory*, 9, 1–12. (Quoted in pp. 22)

Smith, L. (1995): "Necessary and Sufficient Conditions for the Perfect Finite Horizon Folk Theorem," *Econometrica*, 63, 425–430. (Quoted in pp. 22, 23, 24, 31)

Wen, Q. (1994): "The "Folk Theorem" for Repeated Games with Complete Information," *Econometrica*, 62, 949–954. (Quoted in pp. 22, 31)

Chapter 3

Unilateral Commitments in Repeated Games

Contents

3.1 Introduction

The impact of different kinds of commitments in noncooperative models has been widely studied in game theoretical literature. Most of the contributions wihtin this issue have been devoted to study delegation models; situations in which the players are represented by agents who play on their behalf. The concept of delegation, as well as other approaches to the idea of commitment, was already discussed in Schelling (1960). There has been an extensive research in the topic of delegation; see, for instance, Vickers (1985), Fershtman (1985), Sklivas (1987), and, more recently, Fershtman et al. (1991) and Caillaud et al. (1995). The model we present here is specially close to that in Fershtman et al. (1991). What they do is the following. For each 2-player strategic game, they associate a delegation game in which the agents play the original game on behalf of their principals. There is a first stage in which each principal, simultaneously and independently, provides his agent with a compensation scheme. In the second stage, the compensation schemes become common knowledge and the agents play the original game and have the payoffs given by the compensation schemes. Their main result is a folk theorem which, roughly speaking, states that if the original game has a Nash equilibrium, then every Pareto optimal allocation can be achieved as a subgame perfect equilibrium of the delegation game.

In this Chapter we study another model with commitments, not far from that of delegation, but with several differences. We begin with a motivation and then we informally introduce the model. There are many strategic situations in real life in which one of the players gets rid of some of his strategies: a department that makes the commitment of not hiring its own graduate students; a firm announcing a limited edition of a certain product; a party announcing, during the election campaign, that they are not going to make certain alliances regardless of the final result of the election;... We model these situations by what we call unilateral commitments (UC). This is not the first time that unilateral commitments are studied in literature (see, for instance, Faína-Medín et al. (1998) and García-Jurado et al. (2000)). To each n-player strategic game we associate its UC-extension as follows. There is a first stage in which each player, simultaneously and independently, chooses a subset of his set of strategies, *i.e.*, he makes a commitment. In the second stage, the commitments become common knowledge and the players play the original game with the restrictions given by the respective commitments.

After the latter (informal) presentation of our model, we can stress some similarities and differences with the model in Fershtman et al. (1991). We do not have principals and agents, *i.e.*, we have the same players in the two stages of our game. Nonetheless, our model is very close to that of delegation; consider the UC-extension of a game in which the strategies of player 1 belong to A_1 and he commits to play only strategies in A_1^c. Suppose, for simplicity, that all the payoffs of the original game are positive. This situation can be seen as a delegation game in which the principal chooses the following compensation scheme for his agent: (i) if the agent plays a strategy within A_1^c, then he receives some *fixed* proportion of his principal's payoff and (ii) if he plays some strategy not in A_1^c then he receives some fixed negative amount of money. Hence, because of (i), the agent has the same incentives of the principal in the second stage of the game, so we can think of him as the principal himself. It is worth to mention that, as far as the model in Fershtman et al. (1991) is concerned, the compensation schemes are restricted to functions that are weakly monotonic on the payoff received by the principal. On the contrary, the compensation scheme we have defined to "imitate" our commitment can be non-monotonic and, moreover, it can depend not only on the payoffs but also on the specific strategies leading to them. Anyhow, the monotonicity assumption in Fershtman et al. (1991) responded to technical reasons and it seems natural that the principal can sign "non-monotonic contracts" with his agent if both of them agree upon. One more similarity between the two models is the following: in the delegation game, the contracts are public and have to be regarded and, in our model, we take the same assumptions for the commitments.

From the discussion above, we can conclude that our model with unilateral commitments can be seen as a particular family of delegation games; a family in which only some specific compensation schemes are possible. Hence, in an economic situation in which there is room for the contracts needed for the delegation games, there is also room for the kind of commitments we define in this Chapter. Moreover, recall that in this Chapter we model n-player games and not only 2-player situations as it is common in the delegation games literature.

We devote this Chapter to study the implications of unilateral commitments within the framework of repeated games with complete information. We show that, when unilateral commitments are possible, it is easier to find both Nash and subgame perfect equilibria supporting the "cooperative" payoffs of the original game; indeed, most of the folk theorems do not need any assumption at all when unilateral commitments are considered.

The structure of this Chapter is as follows. In Section 3.2 we introduce the background concepts, the definition of a new equilibrium concept for extensive games, and the definition of the model with unilateral commitments. In Section 3.3 we present some folk theorems for repeated games when unilateral commitments are possible. In addition, we compare our results with those without unilateral commitments. Finally, we conclude in Section 3.4.

3.2 Notation

A *strategic game* G is a triplet (N, A, φ), where:

- $N := \{1, \dots, n\}$ is the set of players,
- $A := \prod_{i \in N} A_i$ and A_i is the set of player i's strategies,
- $\varphi := (\varphi_1, \dots, \varphi_n)$ and $\varphi_i : A \to \mathbb{R}$ is the payoff function of player i.

We assume that, for each $i \in N$, the sets A_i are compact and the functions φ_i are continuous. Let a_{-i} be a strategy profile for the players in $N \backslash \{i\}$ and A_{-i} the set of such profiles. For each $i \in N$ and each $a_{-i} \in A_{-i}$, let $\mu_i(a_{-i}) := \max_{a_i \in A_i} \{\varphi_i(a_{-i}, a_i)\}$. Also, for each $i \in N$, let $v_i := \min_{a_{-i} \in A_{-i}} \{\mu_i(a_{-i})\}$. The vector $v := \{v_1, \dots, v_n\}$ is the *minimax payoff vector*. Let

F be the set of *feasible* payoffs, $F := \text{co}\{\varphi(a) : a \in A\}$. Now, for each $i \in N$, let $p_{-i} \in \text{argmin}_{a_{-i}\in A_{-i}}\{\mu_i(a_{-i})\}$.

To avoid confusion with the strategies of the repeated game, in what follows we refer to the strategies $a_i \in A_i$ and the strategy profiles $a \in A$ of the stage game as actions and action profiles, respectively.

Next, given a game $G = (N, A, \varphi)$, we define the *repeated game* $G(\delta, T)$; the T-fold repetition of G with discount parameter $\delta \in (0, 1]$. A history at stage $t \in \{1, \ldots, T\}$ is defined as follows:

(i) for $t = 1$, an element of $A^0 = \{*\}$, where $*$ is any element not belonging to $\bigcup_{k\in\mathbb{N}} A^k$.

(ii) for $t \in \{2, \ldots, T\}$, an element of A^{t-1}.

The set of all histories is $H := \bigcup_{t=1}^{T} A^{t-1}$. In the repeated game we assume *perfect monitoring*, *i.e.*, each player can choose his action in the current stage in the light of all actions taken by all players in all previous stages. Hence, let $G(\delta, T)$ be the triplet (N, S, φ^δ), where:

- The set of players N remains the same.
- $S := \prod_{i\in N} S_i$ is the set of strategy profiles, where $S_i := A_i^H$, *i.e.*, the set of mappings from H to A_i. Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in S$ and $h \in H$; then, we denote the action profile $(\sigma_1(h), \ldots, \sigma_n(h))$ by $\sigma(h)$. A strategy profile $\sigma \in S$ recursively determines the sequence of action profiles $\pi(\sigma) \in A^T$ as follows: $\pi^1(\sigma) := \sigma(*)$ and, for each $t \in \{2, \ldots, T\}$, $\pi^t(\sigma) = \sigma(\pi^1(\sigma), \ldots, \pi^{t-1}(\sigma))$. We refer to $\pi(\sigma)$ as the *path* determined by σ.
- The payoff function φ^δ is defined as follows. Let $\sigma \in S$. Then, player i's payoff in $G(\delta, T)$ is his average discounted stage payoff:

$$\varphi_i^\delta(\sigma) := \frac{1-\delta}{1-\delta^T}\sum_{t=1}^{T}\delta^{t-1}\varphi_i(\pi^t(\sigma)).$$[1]

Finally, recall that, from our definitions, we only use pure actions. If mixed actions are to be taken into account for a given game, then we just define a new game having them as pure actions. Hence, we are implicitly assuming that, when working with mixed actions, they are observable, *i.e.*, the players do not only observe the realization of a mixed action, but also the randomization process that leads to such a realization.

3.2.1 Virtually Subgame Perfect Equilibria

A repeated game with perfect monitoring can be represented as an extensive game and, more specifically, as a multi-stage game with observed actions.[2] Subgame perfect equilibrium (Selten, 1965), shortly SPE, is probably the most important equilibrium concept within this class of games.

[1] If there are no discounts (*i.e.*, if $\delta = 1$), we have $\varphi_i^\delta(\sigma) := (1/T)\sum_{t=1}^{T}\varphi_i(\pi^t(\sigma))$.

[2] We model extensive games following the framework used in Kreps and Wilson (1982), except for the fact that we consider that the sets of nodes may be infinite.

Its main target is to disregard those Nash equilibria which are only possible if some players give credit to irrational plans of others. More formally, a SPE is a Nash equilibrium which, moreover, induces a Nash equilibrium in every subgame.

In this Section we introduce a new equilibrium concept for extensive games which is essential for this Chapter: the *virtually subgame perfect equilibrium*, shortly VSPE. This equilibrium concept has the same effect as subgame perfection, but it only concentrates on those subgames which are *relevant* for a given strategy profile; relevant in the sense that they are reachable if exactly one player deviates from the strategy profile in any subgame which has already been classified as relevant. Despite of being based on the same idea, SPE and VSPE are different concepts, the latter existing in many games which do not have SPE. Hence, VSPE is especially useful when dealing with extensive games having large trees. There are many extensive games without SPE, but still, they can have sensible equilibria. This is the case when the non-existence of SPE is because some subgames which are irrelevant for a certain strategy profile do not have Nash equilibria.

Let Γ be an extensive game and let x and σ be a *single-node* information set and a strategy profile, respectively. Then, Γ_x denotes the *subgame* of Γ that begins at node x and σ_x the restriction of σ to Γ_x. Now, let Γ be an extensive game, σ a strategy profile of Γ, and x a single-node information set. Then, the subgame Γ_x is *σ-relevant* if either (i) $\Gamma_x = \Gamma$, or (ii) there are a player i, a strategy σ_i', and a single-node information set y such that Γ_y is σ-relevant and node x is reached by $(\sigma_{-i}, \sigma_i')_y$.

Definition 3.1. *Let Γ be an extensive game. The strategy profile σ is a* virtually subgame perfect equilibrium *of Γ if for each σ-relevant subgame Γ_x, then σ_x is a Nash equilibrium of Γ_x.*

Let SPE(Γ) and VSPE(Γ) denote the sets of SPE and VSPE of game Γ, respectively. By definition, for each extensive game Γ, we have SPE(Γ) $\subseteq$ VSPE(Γ). However, the reciprocal is not true as the following example illustrates.

Example 3.1. *Consider the extensive game depicted in Figure 3.1.*

Let $\sigma = \big((D_1, a_1^i), (D_2, a_2^i)\big)$, with $i \in \{1,2\}$. Clearly, since the subgame that begins after playing (U_1, U_2) is σ-irrelevant, σ is a VSPE. However, this game does not have any SPE (in pure strategies). Moreover, the equilibrium σ is a sensible one.

Next, we point out one more relation between SPE and VSPE. Let Γ be an extensive game. Let σ and $\hat{\sigma}$ be two strategy profiles of Γ. Now, let $\bar{\sigma}$ be the strategy profile which consists of playing in accordance with σ in the σ-relevant subgames and in accordance with $\hat{\sigma}$ elsewhere. Then, the following statements hold:

(i) The payoffs associated with σ and $\bar{\sigma}$ coincide (they define the same path).

(ii) If $\sigma \in$ VSPE(Γ), then $\bar{\sigma} \in$ VSPE(Γ).

(iii) If $\sigma \in$ VSPE(Γ) and $\hat{\sigma} \in$ SPE(Γ), then $\bar{\sigma} \in$ SPE(Γ).

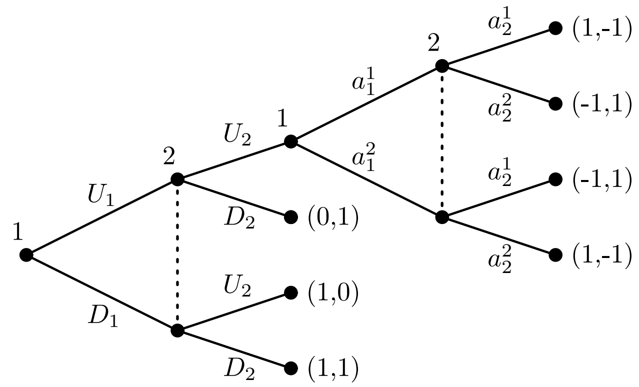

Figure 3.1: A game without SPE, but with VSPE.

Remark. In this Chapter we study a special family of multistage games with observed actions. The main reason why we need the concept of VSPE is that we work with pure strategies. Hence, although we mainly deal with finite extensive games with perfect recall, we cannot apply the general results for the existence of subgame perfect equilibria.

3.2.2 Unilateral Commitments

The main objective of this Chapter is to study the effect of unilateral commitments on the appearing of constructive behavior in repeated games. Given a game G, the corresponding game with unilateral commitments consists of adding an initial stage to G; in this new stage each player can commit not to play certain strategies of game G. Moreover, these commitments are made simultaneously and unilaterally. The fact that the commitments have to be unilateral is quite important; if players could condition their commitments on the commitments of the others, then we would be in a completely cooperative model, and hence, the players could easily achieve in equilibrium the cooperative payoffs of the game.

The problem of unilateral commitments, henceforth UC, has already been tackled in García-Jurado et al. (2000). They obtained a Nash folk theorem for finitely repeated games with UC. In this Chapter we deepen a little bit more in the impact of UC in the assumptions needed for the folk theorems. Next, following García-Jurado et al. (2000), we formally define the UC-extension of a game.

Given a game $G = (N, A, \varphi)$, we define the *UC-extension of* G, $U(G)$, as follows. There is a preliminary stage in which players choose, simultaneously and independently, a nonempty subset of their sets of strategies. Formally, each player $i \in N$ chooses $A_i^c \subseteq A_i$, where A_i^c has to be a compact set. This election is interpreted as a commitment to play strategies only in A_i^c. Then, this preliminary stage ends and the commitments of the players, A^c, are made public, *i.e.*, they become common knowledge. Finally, a reduced version of game G in which players have to respect their commitments is played. Note that, as we have already pointed out, this kind of commitments are unilateral because we do not allow them to be conditional on the other players' commitments. The compactness assumption for the sets A_i^c responds, as usually, to technical reasons; it ensures that the subgames starting after the stage of commitments belong to the class of games defined at the beginning of this Section. Note that, in the particular case in which the sets of strategies of the game under consideration are finite, the compactness requirement imposes no restriction at all. Throughout the rest of this Section, with a slight abuse of notation, given a set A, we use 2^A to denote the *set of compact subsets* of A. Now, $U(G) := (N, A^U, \varphi^U)$, where:

- The set of players N remains the same.
- $A^U := \prod_{i\in N} A_i^U$, where A_i^U is the set of all couples (A_i^c, α_i) such that
  - (i) $\emptyset \subsetneq A_i^c \subseteq A_i$,
  - (ii) $\alpha_i : \prod_{j\in N} 2^{A_j} \longrightarrow A_i$ and, for each $A^c \in \prod_{j\in N} 2^{A_j}$, $\alpha_i(A^c) \in A_i^c$.
- The payoff associated with a strategy profile (A^c, α) is $\varphi^U(A^c, \alpha) := \varphi(\alpha(A^c))$.

3.3 The Folk Theorems

The appearing of constructive behavior in repeated games has been widely treated in the game theoretical literature.[3] Given a game G, the classic Nash folk theorem for finitely repeated games (Benoît and Krishna, 1987) states that if the game G is such that, for each player i, there are two Nash equilibria that give i different payoffs, then every feasible and individually rational payoff of G can be approximated by a Nash equilibrium of $G(\delta, T)$ for big enough T and δ close enough to 1. Recently, González-Díaz (2003) introduced a new condition, namely that the game G is decomposable as a complete minimax-bettering ladder; this new condition, besides being weaker than the former, turned out to be both necessary and sufficient for the finite horizon Nash folk theorem.

Next, we state and prove a Nash folk theorem for finitely repeated games with unilateral commitments. This result, Theorem 3.1, is a variation of the main result in García-Jurado et al. (2000) to place it within our framework. More precisely, here we deal with utilities instead of with preferences, we allow for discounts, and we consider the set F instead of the set $\{\varphi(a) : a \in A\}$. We assume *public randomization*: at each stage of the repeated game, the players can let their actions depend on the realization of an exogenous continuous random variable. The assumption of public randomization is without loss of generality. Given a correlated action, its payoff can be approximated by alternating actions with the appropriate frequencies. More precisely, for each $u \in F$ and each $\varepsilon > 0$, there are actions $a_1, \ldots, a_l$ such that $||u - (a_1 + \ldots + a_l)/l|| < \varepsilon$. Hence, if the discount parameter δ is close enough to 1, the same inequality is still true if we consider discounted payoffs. Then, since we state Theorem 3.1 in terms of approximated payoffs, public randomization assumption can be dispensed with.[4]

[3]Refer to Benoît and Krishna (1996) for a complete survey on the topic.

[4]For further discussion on public randomization refer to Fudenberg and Maskin (1991) and Olszewski (1997).

Theorem 3.1. *Let* $G = (N, A, \varphi)$ *and let* v *be its minimax payoff vector. Let* $u \in F$, $u > v$. *Then, for each* $\varepsilon > 0$, *there are* $\delta_0 \in (0,1)$ *and* $T_0 \in \mathbb{N}$ *such that for each* $\delta \in [\delta_0, 1]$ *and each* $T \geq T_0$, *the game* $U(G(\delta, T))$ *has a Nash equilibrium payoff* w *such that* $\|w - u\| < \varepsilon$.

Proof. Let $G = (N, A, \varphi)$. Let $u \in F$ and let $\bar{a} \in A$ be a (possibly correlated) action profile such that $\varphi(\bar{a}) = u$. Now, for each $\delta \in (0,1]$ and each $T \in \mathbb{N}$, let $G(\delta, T) = (N, S, \varphi^\delta)$. We define the following strategy profile $(\bar{S}^c, \bar{\alpha})$ of $U(G(\delta, T))$:

(i) For each $i \in N$, $\bar{S}^c_i :=$ "If $\bar{a}$ is played in the first stage, then I play $\bar{a}_i$ forever".

(ii) For each $i \in N$ and each $S^c \in \prod_{j \in N} 2^{S_j}$, we define $\bar{\alpha}_i(S^c)$ as follows:

- If $S^c = \bar{S}^c$:
  - i plays $\bar{a}_i$ in the first stage.
  - If $\bar{a}$ is played in the first stage, then i plays $\bar{a}_i$ forever.
  - If in the first stage only player $j \neq i$ has deviated from $\bar{a}$, then, i plays $(p_{-j})_i$ forever.
  - Otherwise, i plays *ad libitum*.
- If $S^c = (S^c_j, \bar{S}^c_{-j})$, where $j \neq i$ and $S^c_j \neq \bar{S}^c_j$: i plays $(p_{-j})_i$ forever.
- Otherwise: i plays *ad libitum*.

Note that $\varphi^c(\bar{S}^c, \bar{\alpha}) = \varphi^\delta(\bar{\alpha}(\bar{S}^c)) = \varphi(\bar{a}) = u$. For each $i \in N$, let T_i be such that $T_i u_i > \mu_i(\bar{a}_{-i}) + (T-1)v_i$ and let $\delta_i \in (0,1)$ be such that $\sum_{t=1}^{T_i} \delta^{t-1} u_i > \mu_i(\bar{a}_{-i}) + \sum_{t=2}^{T_i} \delta^{t-1} v_i$. Finally, let $T_0 := \max_{i \in N} T_i$ and $\delta_0 := \max_{i \in N} \delta_i$.

Now, it is straightforward to check that for each $\delta \in [\delta_0, 1]$ and each $T \geq T_0$, the strategy profile $(\bar{S}^c, \bar{\alpha})$ is a Nash equilibrium of $U(G(\delta, T))$ whose payoff w is such that $\|w - u\| = 0 < \varepsilon$ (Note that we have obtained an exact result, *i.e.*, $w = u$ because of the public randomization assumption).[5] □

The main purpose for the rest of this Section is to state and prove a subgame perfect folk theorem with UC. The trick of the proof of Theorem 3.1, in which the strategies corresponding with many subgames were defined *ad libitum*, does not work for subgame perfection. Moreover, when dealing with unilateral commitments, we face extremely large game trees. They have many subgames, some of which may correspond to senseless commitments. Thus, we need to use the VSPE concept instead of the classical SPE. Theorem 3.1 says that, when unilateral commitments are possible, no condition is needed for the Nash folk theorem to hold. Note that the Nash equilibrium profile $(\bar{S}^c, \bar{\alpha})$ defined in the proof of Theorem 3.1 is neither a SPE nor a VSPE; this is because, in general, the punishments to a player who deviates from the commitment are not credible. Now, Proposition 3.1 shows that not only the proof of Theorem 3.1 fails when we write VSPE instead of Nash equilibrium, but also the result itself is false.

[5] The reader willing to deepen into the arguments of this proof is referred to García-Jurado et al. (2000).

Proposition 3.1. *The counterpart of Theorem 3.1 for VSPE does not hold.*

Proof. We do the proof by means of an example. Let $G = (N, A, \varphi)$ be the game defined in Figure 3.2. The game G does not have a Nash equilibrium. Moreover, $v = (1, 1)$ and $\varphi(U, L) =$

	L	R
U	10,11	1,10
D	11,0	0,1

Figure 3.2: A counterexample for Proposition 3.1

$(10, 11) > v$. However, for each $T \in \mathbb{N}$ and each $\delta \in (0, 1]$, $U(G(\delta, T))$ does not have a VSPE. Suppose, on the contrary, that there are $\delta \in (0, 1]$ and $T \in \mathbb{N}$ such that (S^c, α) is a VSPE of $U(G(\delta, T))$. If S^c contains a unique element, then one of the players can change his commitment to no commitment at all (*i.e.*, $S^c_i = S_i$ if i is such a player) and deviate from the strategy in the final stage. Hence, there is a last stage in which, according to the path defined by (S^c, α), one of the players is free to play any action. Let t be that stage and assume, without loss of generality, that, following the path of (S^c, α), player 1 can play both U and D at stage t. Moreover, let x^* be the corresponding single-node of $G(\delta, T)$.

Now, let player 2 deviate to the strategy $(\bar{S}^c_2, \bar{\alpha}_2)$ defined as follows: (i) $\bar{S}^c_2$:= "from stage $t+1$ on, I play according to the path defined by (S^c, α)" and (ii) for each $\hat{S}^c_1 \in 2^{S_1}$, $\bar{\alpha}_2(\hat{S}^c_1, \bar{S}^c_2) := \alpha_2(S^c)$. Let y be the single-node reached after $(S^c_1, \bar{S}^c_2)$ is played. By definition, $U(G(\delta, T))_y$ is a relevant subgame. Let now player 1 deviate, in $U(G(\delta, T))_y$, to the strategy $\bar{\alpha}_1$ defined as follows: for each $\hat{S}^c_2 \in 2^{S_2}$, $\bar{\alpha}_1(S^c_1, \hat{S}^c_2) := \alpha_1(S^c)$. The subgame $U(G(\delta, T))_y$ is such that, when playing according to (α_1, α_2), the single-node x^* of $G(\delta, T)$ is reached again at stage t. Hence, the subgame beginning at the corresponding single-node of $U(G(\delta, T))$, namely x, is relevant for (S^c, α). According to the commitments, both players can choose their two actions at x and, from the stage $t + 1$ on, player 2's actions are determined by the commitment. Now, it is immediate to check that the relevant subgame $U(G(\delta, T))_x$ does not have any Nash equilibrium. Hence, (S^c, α) cannot be a VSPE. □

In the counterexample we used in the proof above, we defined a game G with no Nash equilibrium. Moreover, for each $T \in \mathbb{N}$ and each $\delta \in (0, 1]$, the game $G(\delta, T)$ did not have any Nash equilibrium. On the other hand, we have the following positive result concerning the existence of VSPE for games with UC.

Proposition 3.2. *Let $G = (N, A, \varphi)$ and let $\bar{a} \in A$ be a Nash equilibrium of G. Then, the game $U(G)$ has a VSPE $(\bar{A}^c, \bar{\alpha})$ with payoff $\varphi(\bar{a})$.*

Proof. Let $(\bar{A}^c, \bar{\alpha})$ be such that for each $i \in N$, we have

(i) $\bar{A}^c_i = \{\bar{a}_i\}$,

(ii) for each $j \neq i$ and each $A^c_j \in 2^{A_j}$, $\bar{\alpha}_i(\bar{A}^c_{-j}, A^c_j) = \bar{a}_i$. Finally, for each $A^c_i \in 2^{A_i}$, $\bar{\alpha}_i(\bar{A}^c_{-i}, A^c_i) = \hat{a}_i$, where $\hat{a}_i \in \operatorname{argmax}_{a_i \in \bar{A}^c_i}\{\varphi_i(\bar{a}_{-i}, a_i)\}$.

It is immediate to check that each such strategy profile $(\bar{A}^c, \bar{\alpha})$ is a VSPE of $U(G)$. □

In view of Proposition 3.2, it is clear that every Nash folk theorem for finitely repeated games can be easily adapted to provide a subgame perfect folk theorem for finitely repeated games with unilateral commitments. More precisely, the necessary and sufficient condition for the Nash folk theorem in González-Díaz (2003), "that the game is decomposable as a complete minimax-bettering ladder", is a sufficient condition for the VSPE folk theorem with UC. The former condition implies among other things, the existence of a Nash equilibrium in the stage game G; this implication is all we use in this Chapter. Example 3.2 shows that such condition is not necessary.

Example 3.2. *Let $G = (N, A, \varphi)$ be the game defined in Figure 3.3.*

	L	M	R
U	10,10	0,1	1,0
D	11,0	1,1	0,2

Figure 3.3: A game without Nash equilibria

The game G does not have a Nash equilibrium. Hence, G it is not decomposable as a complete minimax-bettering ladder. Moreover, $v = (1, 2)$. Now, for each $T \in \mathbb{N}$ and each $\delta \in (0, 1]$, the payoff $(10, 10)$ can be supported by a VSPE of $U(G(\delta, T)) = (N, S^U, \varphi^U)$. To check this assertion, consider the strategy profile $(\bar{S}^c, \bar{\alpha})$ of $U(G(\delta, T))$ defined as follows: (i) $\bar{S}^c_1 :=$ "I play U in every stage", (ii) $\bar{S}^c_2 :=$ "I never play R", and (iii) for each $i \in \{1, 2\}$ and each $S^c_i \in 2^{S_i}$, $\bar{\alpha}(\bar{S}^c_{-i}, S^c_i)$ consists of playing, at each stage, the unique Nash equilibrium of the corresponding one stage game. Then, $(\bar{S}^c, \bar{\alpha})$ is a VSPE and $\varphi^U(\bar{S}^c, \bar{\alpha}) = (10, 10)$.

Proposition 3.2 says that we can use the UC to make actions credible, even actions that in the original game could be dominated. On the other hand, Example 3.2 shows that we can use the UC to go further than that. Hence, some more research is needed to find new sufficient conditions for the VSPE folk theorem; conditions weaker than the existence of a complete minimax-bettering ladder. Although we have made some research in this specific issue, we have not found any satisfactory condition. Nevertheless, we have found the following result, which, with the aid of Proposition 3.2, is straightforward.

Theorem 3.2. *Let $G = (N, A, \varphi)$ and let v be its minimax payoff vector. Let $u \in F$, $u > v$. Then, for each $\varepsilon > 0$, there are $\delta_0 \in (0, 1)$ and $T_0 \in \mathbb{N}$ such that for each $\delta \in [\delta_0, 1]$ and each $T \geq T_0$, the game $U(U(G(\delta, T)))$ has a VSPE with payoff w such that $\|w - u\| < \varepsilon$.*

Proof. Immediate from the combination of Theorem 3.1 and Proposition 3.2. □

Theorem 3.2 implies that, when two stages of commitments are possible, any feasible and individually rational payoff of the original game can be achieved as a VSPE of the repeated game with unilateral commitments. No assumption is needed for the original game, not even the existence of a Nash equilibrium.

Next, we briefly discuss the impact of Theorem 3.2 within the delegation framework discussed in the Introduction. First, from the point of view of our model with unilateral commitments, the game $U(U(G(\delta,T)))$ can be difficult to motivate. It is true that we get a very strong result for the set of equilibrium payoffs of this game, but the fact that we allow for commitments on commitments might have unnatural features in some models. The point is that, when we introduced unilateral commitments, we emphasized the fact that they were unilateral, *i.e.*, the commitments of one player could not be conditional on the other players' commitments; if we allow for two stages of commitments, then we are indirectly allowing for commitments on commitments, and hence, we achieve the same payoffs we could get with a cooperative model. On the other hand, if we reassess the delegation situation corresponding with our unilateral commitments model, and we do it in a similar way to that in the Introduction, then we have the following interpretation for the two stages of commitments. Consider a situation in which two firms are engaged in a competitive situation. Initially, the players are the presidents, and hence, in the first stage each president signs a contract with his principal in which the latter is committed not to play certain strategies and he will be paid proportionally to the payoff he finally gets. Then, in a second stage, a similar contract is signed between each principal and his agent. Finally, the agents play the original game but honoring the commitments. This situation has some important differences with the one stage situation: (i) the commitments that the president includes in the contract in the first stage can take into account the commitments that the principal will make with the agent at stage two, *i.e.*, the contract between each president and his principal also commits the latter on the commitments he can sign with his agent, (ii) in the second stage the principals, being consistent with the commitments of their contract with the principals and in view of the commitments made by the rivals, choose a new commitment for the agents, *i.e.*, a commitment on the commitment, and (iii) finally, the agents have to play being consistent with all the previous commitments. The hierarchical delegation model we have just described is quite natural and it is not difficult to think of real life situations with these sub-delegation structures. Hence, if such situations also correspond to some repeated game, then Theorem 3.2 says that, regardless of the properties of the underlying stage game, the "cooperative" (collusive) payoffs can be supported as a VSPE in the game with two stages of commitments.

3.3.1 Infinitely Repeated Games

Although we have not formally introduced the model with infinitely repeated games, the definitions can be immediately extended to encompass also this family of games; basically, replacing T by ∞ in the definition of history and in the subsequent ones. Now, within this new framework, Proposition 3.2 still carries over. Now, recall that the classic Nash folk theorem for infinitely repeated games (see, for instance, Fudenberg and Maskin (1986)) states that, if the discount is close enough to 1, every feasible and individually rational payoff can be achieved as a Nash equilibrium of the infinitely repeated game. Hence, if we combine this classic result with Proposition 3.2 we get the following Corollary:

Corollary 3.1. *Let $G = (N, A, \varphi)$ and let v be its minimax payoff vector. Let $u \in F$, $u > v$. Then, there is $\delta_0 \in (0, 1)$ such that for each $\delta \in [\delta_0, 1]$ the game $U(G(\delta, \infty))$ has a VSPE with payoff u.*

Proof. It is immediate from the combination of Proposition 3.2 with the classic Nash folk theorem for infinitely repeated games. □

3.3.2 The State of Art

Table 3.1 summarizes the results we have proved in this Chapter along with the classic folk theorems for repeated games with complete information. In particular, it shows the strength of Proposition 3.2, that allows to obtain many folk theorems for repeated games with unilateral commitments as immediate corollaries of the classic ones. Hence, by looking at Table 3.1, one easily understands the strength of unilateral commitments within this framework. Note that all the cells in the Table contain necessary and sufficient conditions, all of them but the one corresponding with the virtual subgame perfect folk theorem for finitely repeated games with unilateral commitments; some more research is still needed concerning this case.

	Without UC	**1 stage of UC**	**2 stages of UC**
Nash Theorem Infinite Horizon	None (Fudenberg and Maskin, 1986)	None (Prop. 3.2)	None (Prop. 3.2)
(Virtual) Perfect Th. Infinite Horizon	Non-Equivalent Utilities (Abreu et al., 1994)	None (Prop. 3.2)	None (Prop. 3.2)
Nash Theorem Finite Horizon	Minimax-Bettering Ladder (González-Díaz, 2003)	None (García-Jurado et al., 2000)	None (Prop. 3.2)
(Virtual) Perfect Th. Finite Horizon	Recursively-distinct Nash payoffs (Smith, 1995)	Minimax-Bettering Ladder (Prop. 3.2, only sufficient)	None (Th. 3.2)

Table 3.1: Necessary and Sufficient conditions for the folk theorems

3.4 Concluding Remarks

In this Chapter we have deepened in the literature of commitments. More specifically, we have studied the impact of unilateral commitments in the folk theorems for repeated games.

We want to emphasize again the following fact. Because of the way we have modeled unilateral commitments, it could seem that they are very far from the more standard models of commitment via delegation. But, as we pointed out in the Introduction and in the discussion of Theorem 3.2, unilateral commitments can be used to model situations in which there is a principal who signs a contract with his agent with two natural features: (i) The agent has committed not play certain strategies and (ii) among the remaining ones his payoff is proportional to that of the principal, *i.e.*, the agent can be thought of as a shareholder of the firm.

Moreover, we have shown that unilateral commitments have very strong implications within the literature of repeated games with complete information. They lead to new folk theorems in which the assumptions needed for the classic results have been notably relaxed.

Finally, there are several open questions that should be tackled in the future. One of them is to refine the conditions for the finite horizon perfect folk theorem with unilateral commitments. There is another important issue where some research is needed: the impact of unilateral commitments in repeated games with incomplete information.

Bibliography

Abreu, D., P. K. Dutta, and L. Smith (1994): "The Folk Theorem for Repeated Games: A NEU Condition," *Econometrica*, 62, 939–948. (Quoted in pp. 44)

Benoît, J.-P. and V. Krishna (1987): "Nash Equilibria of Finitely Repeated Games," *International Journal of Game Theory*, 16, 197–204. (Quoted in pp. 39)

——— (1996): "The Folk Theorems for Repeated Games: A Synthesis," Tech. rep., EconWPA-9601001. (Quoted in pp. 39)

Caillaud, B., B. Jullien, and P. Picard (1995): "Competing Vertical Structures: Precommitment and Renegotiation," *Econometrica*, 63, 621–466. (Quoted in pp. 34)

Faína-Medín, A., I. García-Jurado, L. Méndez-Naya, and J. Méndez-Naya (1998): "Unilateral Commitments in the Finitely Repeated Prisoners' Dilemma," *Annals of Operations Research*, 84, 187–194. (Quoted in pp. 34)

Fershtman, C. (1985): "Managerial Incentives as a Strategic Variable in a Duopolistic Environment," *International Journal of Industrial Organization*, 3, 245–253. (Quoted in pp. 34)

Fershtman, C., K. L. Judd, and E. Kalai (1991): "Observable Contracts: Strategic Delegation and Cooperation," *International Economic Review*, 32, 551–559. (Quoted in pp. 34, 35)

Fudenberg, D. and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54, 533–554. (Quoted in pp. 43, 44)

——— (1991): "On the Dispensability of Public Randomization in Discounted Repeated Games," *Journal of Economic Theory*, 53, 428–438. (Quoted in pp. 39)

García-Jurado, I., L. Méndez-Naya, and F. Patrone (2000): "Unilateral Commitments in Finitely Repeated Games," *International Game Theory Review*, 2, 129–139. (Quoted in pp. 34, 38, 39, 40, 44)

González-Díaz, J. (2003): "Finitely Repeated Games: A Generalized Nash Folk Theorem," Tech. Rep. 03-08, Department of Statistics and OR, University of Santiago de Compostela, to appear in Games and Economic Behavior. (Quoted in pp. 39, 42, 44)

Kreps, D. M. and R. Wilson (1982): "Sequential Equilibria," *Econometrica*, 50, 863–894. (Quoted in pp. 36)

Olszewski, W. (1997): "Perfect Folk Theorems. Does Public Randomization Matter?" *International Journal of Game Theory*, 27, 147–156. (Quoted in pp. 39)

Schelling, T. C. (1960): *The Strategy of Conflict*, Oxford University Press. (Quoted in pp. 34)

Selten, R. (1965): "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit," *Zeitschrift für die gesamte Staatswissenschaft*, 12, 301–324. (Quoted in pp. 36)

Sklivas, S. D. (1987): "The Strategic Choice of Managerial Incentives," *RAND Journal of Economics*, 18, 452–458. (Quoted in pp. 34)

Smith, L. (1995): "Necessary and Sufficient Conditions for the Perfect Finite Horizon Folk Theorem," *Econometrica*, 63, 425–430. (Quoted in pp. 44)

Vickers, J. (1985): "Delegation and the Theory of the Firm," *Economic Journal*, 95, 138–47. (Quoted in pp. 34)

Chapter 4

A Noncooperative Approach to Bankruptcy Problems

Contents

4.1 Introduction

To motivate the discussion think of the President of a University who has to allocate a given budget among a set N of Departments. Each Department, depending on its size, teaching duties, research commitments, etc. demands a total amount d_i. Most likely the aggregate amount claimed by the Departments, $\sum_{i \in N} d_i$, exceeds the budget available, E. This is a standard bankruptcy problem which requires devising some procedure to allocate what is available as a function of what is demanded.

More generally, a bankruptcy problem describes a case in which a planner has to allocate a given amount of a divisible good E among a set N of players, when their claims $(d_i)_{i \in N}$ exceed the available amount (*i.e.*, $\sum_{i \in N} d_i > E$). Most rationing situations can be given this form. Relevant examples are the execution of a will with insufficient assets, the allocation of a commodity with excess demand in a fixed price setting, the collection of a given amount of taxes, and, of course, the liquidation of a bankrupt firm among its creditors. Rationing problems encompass a wide range of distributive situations and are analytically very simple (indeed, a bankruptcy problem can be summarized by a triple (N, E, d)).

A solution to a bankruptcy problem, also called a rule, is a function that specifies, for each admissible problem (N, E, d), a vector $R(N, E, d) \in \mathbb{R}^n$ satisfying the following two restrictions: (i) for each $i \in N$, $0 \leq R_i(N, E, d) \leq d_i$ and (ii) $\sum_{i \in N} R_i(N, E, d) = E$. The first restriction says that no claimant gets more than he claims or less than zero. The second is an efficiency requirement: the available amount of the good is fully distributed. Alternative solutions correspond to the application of different ethical and procedural criteria.

The literature on bankruptcy problems is large and keeps growing. The main contributions refer to the analysis of different solutions following an axiomatic approach or translating to this context some of the standard solutions to coalitional games. The reader is referred to the works of Moulin (2002) and Thomson (2003) for comprehensive surveys of this literature.

There is also a number of contributions that study different strategic aspects of the problem (see Thomson (2003)). The first one already appears in O'Neill's seminal paper (see O'Neill (1982)), where the minimal overlap rule is analyzed as the Nash equilibrium of a noncooperative game. Chun (1989) introduces a different strategic consideration in the bankruptcy problem by allowing the players involved to propose solution concepts, rather than allocations. He devises a procedure that converges to the outcome associated with the constrained equal awards rule. A dual formulation is presented in Herrero (2003); in this case the procedure converges to the constrained equal losses rule. Sonn (1992) obtains the constrained equal awards rule as the limit of a process in which each player makes a proposal for someone else, who either accepts and leaves or rejects and takes the place of the proposer. Dagan et al. (1997) define a sequential game whose unique subgame perfect equilibrium outcome corresponds to the allocation of a given resource monotonic and consistent rule (an extension of the result in Serrano (1995)). The sequential game, which depends on the consistency assumption, can be summarized as follows. The player with the highest claim proposes a distribution of the amount available. The other players can either accept and leave, or else reject and leave, obtaining what the two-person rule recommends for the problem made of him and the proposer, with a budget made of the amounts allotted initially for them by the proposer. See also Dagan et al. (1999) for a further extension.

Along these lines, Corchón and Herrero (2004) discuss the implementation of bankruptcy rules when the proposals made by the players are bounded by their claims. Herrero et al. (2003) provide a experimental analysis of the strategic behavior in bankruptcy problems.

Somehow between the axiomatic and the strategic approaches there are those contributions on the manipulation of the rules via merging or splitting claims (see de Frutos (1999), Ju (2003) and Moreno-Ternero (2004)).

In this Chapter we provide a noncooperative support to the resolution of bankruptcy problems. The basic idea is the following. Each player is asked to declare what would be the minimal admissible reward he is ready to admit, given that there is not enough to satisfy all the demands. This is a familiar requirement in many discussions of this sort. It is not difficult to imagine the President of the University putting this question to the Heads of the different Departments: "Tell me what is the minimum you need to keep going; I'll try to ensure that you receive that amount and then we shall see how to allocate the rest". Of course this is a strategic situation and each player will declare the amount that maximizes his expected payoff. We propose here an elementary game form in which those who "demand less" are given priority in the distribution, such that in a Nash equilibrium all players "demand the same". Interestingly enough the equilibrium payoff vector is unique and so is, in many cases, the strategy profile. Moreover, the Nash equilibria of this game are all strong. By specifying properly what we mean by "demanding less" and "getting the same" through the rules of each particular game of this type, we obtain all the different solutions to bankruptcy problems as Nash equilibria.

We implicitly assume, as it is usual in this literature, that both the amount to divide and the claims are known by all the players. There are, however, two features that make this game form much simpler than those in other contributions: (i) no sequential procedure is involved (which, incidentally, makes the result independent on consistency) and (ii) each player's message only refers to his decision variable, in contrast with most of the results in which each player proposes a whole allocation. Moreover, our results apply to virtually all acceptable bankruptcy rules (where "unacceptable" rules are those that allow some player to get his full claim and, at the same time, give zero to some else).

The Chapter is organized as follows. Section 4.2 presents the formal model and the main results. Section 4.3 applies those results to the bankruptcy problem. It is shown that the allocation proposed by any acceptable bankruptcy rule can be obtained as the Nash equilibrium of a specific game within the family presented in Section 4.2. We conclude with a few final comments in Section 4.4.

4.2 The Model and the Main Results

A *bankruptcy problem* is a triple (N, E, d), where $N = \{1, 2, \ldots, n\}$ is a collection of players, $E > 0$ is the amount to divide, and $d \in \mathbb{R}^n_{++}$ is the vector of claims. The very nature of the problem under consideration implies that $\sum_{i\in N} d_i > E > 0$. A *bankruptcy rule* R is defined as a function mapping each bankruptcy problem (N, E, d) onto $\mathbb{R}^n_+$, such that for each $i \in N$, $R_i(N, E, d) \in [0, d_i]$ and $\sum_{i\in N} R_i(N, E, d) = E$. The rule R represents a sensible way of distributing the available amount E, with two natural restrictions. One is that no player gets more than he claims or less than zero, the other that the total amount E is divided among the players.

Our *noncooperative bankruptcy game* for the bankruptcy problem (N, E, d) is a strategic game with N as the set of players, who are endowed with strategy spaces D_i, and payoff functions $(\pi_i)_{i\in N}$ that describe what each player gets as a function of the joint strategy vector.

In our game, player i's strategy space is a closed interval $D_i = [0, m_i]$, for some scalar $m_i > 0$. Let α_i be a *strategy* for player i, $\alpha \in D = \prod_{i\in N} D_i$ a *strategy profile*, and α_{-i} an $(n-1)$-vector consisting of the strategies of all players other than player i. We interpret α_i as a message monotonically related to the amount of the total payoff that he declares admissible, given the rationing situation. For instance, α_i may describe the share of d_i that player i would be ready to accept. Or it may correspond to the loss he is ready to admit. We describe this message through a function $f_i : D_i \to [0, d_i]$ that specifies the relationship between the player's message and his intended reward. For instance $f_i(\alpha_i) = \alpha_i d_i$, when α_i corresponds to player i's admissible share. A strategy profile α is *feasible* if $\sum_{i\in N} f_i(\alpha_i) \le E$.

Concerning the functions f_i we assume:

Axiom 4.1. *For each $i \in N$, f_i is a monotone function that defines a bijection from $[0, m_i]$ to $[0, d_i]$.*

Axiom 4.2. *All f_i's are simultaneously increasing or decreasing.*

Note that Axiom 4.1 implies that the functions f_i are continuous and strictly monotone functions. Hence, we refer to the functions f_i as increasing or decreasing meaning strictly increasing or strictly decreasing, respectively. For simplicity we assume in Axiom 4.2 that the orientation of the messages of the players is uniform, which means that functions f_i are either all increasing or all decreasing. It follows from Axioms 4.1 and 4.2 that (i) if the functions f_i are increasing, then $f_i(0) = 0$ and $f_i(m_i) = d_i$ and (ii) if they are decreasing, then $f_i(0) = d_i$ and $f_i(m_i) = 0$. Let f be $(f_1, \ldots, f_n)$.

If the f_i's are increasing (decreasing) functions, we denote by $[i]$ the player whose message occupies the i-th position in the increasing (decreasing) ordering of messages. Although it is not important for the results of this Chapter, a tie-breaking rule must be considered in order to have a well defined reordering of the players. Here we consider the following: if there is a tie between two or more players, the ranking is made in increasing ordering of their indices within N. Hence:

- If the functions f_i are increasing: $\alpha_{[1]} \le \alpha_{[2]} \le \ldots \le \alpha_{[n]}$.

- If the functions f_i are decreasing: $\alpha_{[1]} \geq \alpha_{[2]} \geq \ldots \geq \alpha_{[n]}$.

Consider now the following procedure. Player i chooses his message α_i. If the profile $\alpha = (\alpha_i, \alpha_{-i})$ is feasible, then player i gets $f_i(\alpha_i)$ and, since each player obtains the payoff associated with the demand corresponding with his message, the game ends. If α is not feasible, then E is allocated among the players with lowest ranking. That is, let $[h]$ denote the smallest index for which $\sum_{[i]\leq[h]} f_{[i]}(\alpha_{[i]}) > E$. Then:

$$\pi_{[i]}(\alpha) = \begin{cases} f_{[i]}(\alpha_{[i]}) & [i] < [h] \\ E - \sum_{[i]\leq[h-1]} f_{[i]}(\alpha_{[i]}) & [i] = [h] \\ 0 & [i] > [h]. \end{cases}$$

We denote this noncooperative bankruptcy game by $\langle N, D, \pi\rangle$. Note that, under Axioms 4.1 and 4.2, the strict monotonicity of the functions f_i is translated, for each α_{-i} and provided that $\sum_{i\in N} f_i(\alpha_i) < E$, to the functions $\pi_i(\alpha_{-i}, \cdot)$. By construction, for each $\alpha \in D$, $\sum_{i\in N} \pi_i(\alpha) \leq E$.

We can define a *Nash equilibrium* of the game $\langle N, D, \pi\rangle$, associated with a bankrupcty problem (N, E, d), as a strategy profile $\alpha^* \in D$ such that for each $i \in N$ and each $\alpha_i \in D_i$, $\pi_i(\alpha^*) \geq \pi_i(\alpha^*_{-i}, \alpha_i)$.

A strategy profile $\alpha^* \in D$ is a *strong equilibrium* (Aumann, 1959) if there do not exist $T \subseteq N$ and $\alpha_T \in \prod_{i\in T} D_i$ such that for each $i \in T$, $\pi_i(\alpha^*) < \pi_i(\alpha^*_{N\setminus T}, \alpha_T)$.

Lemma 4.1. *Under Axioms 4.1 and 4.2, if $\alpha^* \in D$ is a Nash equilibrium of the bankruptcy game $\langle N, D, \pi\rangle$, then $\sum_{i\in N} \pi_i(\alpha^*) = E$,* i.e.*, all the Nash equilibria are efficient.*

Proof. Assume that the f_i's are increasing (the case of decreasing functions is fully symmetric). By the definition of $\langle N, D, \pi\rangle$, $\sum_{i\in N} \pi_i(\alpha^*) \leq E$. Moreover, if $\sum_{i\in N} \pi_i(\alpha^*) < E$, then there is $i \in N$ that can increase his claim and get a larger payoff. Hence, $\sum_{i\in N} \pi_i(\alpha^*) = E$. □

The following result is now obtained.

Proposition 4.1. *Let (N, E, d) be a bankruptcy problem and $\langle N, D, \pi\rangle$ an associated noncooperative bankruptcy game. The following statements are true under Axioms 4.1 and 4.2:*

(i) *There is a constant ρ such that the game has a Nash equilibrium in which $\alpha^*_i = \min\{\rho, m_i\}$,* i.e.*, each player selects the strategy in $[0, m_i]$ which is closer to ρ.*

(ii) *The equilibrium payoff for this game is unique.*

Proof. **Case 1:** Functions f_i are increasing.

Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be such that for each $x \in \mathbb{R}_+$, $F(x) = \sum_{i\in N} f_i(x_i)$, where $x_i = \min\{x, m_i\}$. The continuity of all f_i's implies the continuity of F. Hence, since (i) $F(0) = 0$, (ii) $F(\max_{i\in N}\{m_i\}) = \sum_{i\in N} d_i > E$, and iii) F is strictly increasing in the interval $[0, \max_{i\in N}\{m_i\}]$, there is a unique $\rho \in (0, \max_{i\in N}\{m_i\})$ such that $F(\rho) = E$.

Suppose that the profile α^* is a Nash equilibrium and that, for some fixed $i, j \in N$, we have $\alpha_j^* > \alpha_i^*$ and $m_i > \alpha_i^*$. If $\pi_j(\alpha^*) = 0$, then player j can ensure for himself a positive payoff with the strategy ε, for ε small enough. Hence, $\pi_j(\alpha^*)$ must be positive. Now, player i can obtain a greater payoff by switching to strategy $\alpha_i' \in (\alpha_i^*, \min\{\alpha_j^*, m_i\})$; player i has still a lower index than player j and, since f_i is increasing, his payoff increases. Hence, if α^* is a Nash equilibrium and $\alpha_j^* > \alpha_i^*$ for some $i, j \in N$, we have $\alpha_i^* = m_i$. Combining this with the fact that ρ is the unique positive real number for which $F(\rho) = E$, we get that the strategies $\alpha_i^* = \min\{\rho, m_i\}$ define a Nash equilibrium.

Note that there can exist $j \in N$, such that for each $i \neq j$, (i) $\alpha_i^* = m_i$, and (ii) $\alpha_j^* > \alpha_i^*$; in this case player j can change his strategy to a new $\alpha_j > \alpha_j^*$, obtaining a new Nash equilibrium of the game. Nonetheless, the payoff remains unchanged.

Case 2: Functions f_i are decreasing.

Take again the function F, now we have (i) $F(0) = \sum_{i \in N} d_i > E$ and (ii) $F(\max_{i \in N}\{m_i\}) = 0$. Define again ρ as the unique real number in $(0, \max_{i \in N}\{m_i\})$ such that $F(\rho) = E$. The situation is similar to the case with increasing functions: the profile α^* with $\alpha_i^* = \min\{\rho, m_i\}$ is again a Nash equilibrium.

Again, suppose that $\pi_i(\alpha^*) = 0$ for some $i \in N$. In this situation, if player i changes his strategy, no matter how, the new profile is still a Nash equilibrium. Nonetheless, the payoff remains unchanged. □

Moreover, all the Nash equilibria in the Proposition above are in fact strong equilibria, as the following Proposition shows.

Proposition 4.2. *Let (N, E, d) be a bankruptcy problem, $\langle N, D, \pi \rangle$ an associated noncooperative bankruptcy game, and α^* a strategy profile. Then, under Axioms 4.1 and 4.2, α^* is a Nash equilibrium if and only if α^* is a strong equilibrium.*

Proof. Since a strong equilibrium is a Nash equilibrium only one implication has to be proved. Assume that the functions f_i are increasing. Suppose that α^* is a Nash equilibrium which is not strong. Then, there are $T \subseteq N$ and $\alpha_T \in \prod_{j \in T} D_j$ such that for each $j \in T$, $\pi_j(\alpha^*) < \pi_j(\alpha^*_{N \setminus T}, \alpha_T)$. By Proposition 4.1, there is ρ such that for each $i \in N$, $\alpha_i^* = \min\{\rho, m_i\}$. Moreover, it is easy to check that, for each $i \in N$, $\pi_i(\alpha^*) = f_i(\alpha_i^*)$. Now, for each $j \in T$,

$$f_j(\alpha_j) \geq \pi_j(\alpha^*_{N \setminus T}, \alpha_T) > \pi_j(\alpha^*) = f_j(\alpha_j^*).$$

Hence, $\alpha_j > \alpha_j^*$. Hence, $\alpha_j^* < m_j$. Hence, $\alpha_j^* = \rho$ and $\alpha_j > \rho$. Now, since for each $i \in N \setminus T$, $\alpha_i^* \leq \rho$, then for each $j \in T$ and each $i \in N \setminus T$, we have $\alpha_j > \alpha_i^*$. Hence, for each $i \in N \setminus T$, $\pi_i(\alpha^*_{N \setminus T}, \alpha_T) = f_i(\alpha_i^*) = \pi_i(\alpha^*)$. Since $\sum_{i \in N} \pi_i(\alpha^*_{N \setminus T}, \alpha_T) \leq E$, then cannot be the case that, for each $j \in T$, $\pi_j(\alpha^*) < \pi_j(\alpha^*_{N \setminus T}, \alpha_T)$. In the decreasing case, a similar argument can be formulated. □

4.3 Bankrupcty Games and Bankruptcy Rules

Now, we illustrate how the results in Section 4.2 apply to the standard bankruptcy rules.

The *proportional rule*, P, which is probably the best known and most widely used solution concept, distributes awards proportionally to claims. It is defined as follows: for each (N, E, d), $P(N, E, d) = \lambda d$, with $\lambda = \frac{E}{\sum_{i\in N} d_i}$. It is easy to see that, if we take, for each $i \in N$, $D_i = [0, 1]$ and $f_i(\alpha_i) = \alpha_i d_i$, then the (unique) Nash equilibrium of the game produces the proportional solution to the bankruptcy problem.

The *constrained equal-awards rule, A*, applies an egalitarian principle on the awards received, provided no player gets more than he claims. It is defined as follows: for each (N, E, d) and each $i \in N$, $A_i(N, E, d) = \min\{d_i, \lambda\}$, where λ solves $\sum_{i\in N} \min\{d_i, \lambda\} = E$. By letting $D_i = [0, d_i]$ and $f_i(\alpha_i) = \alpha_i$, we get the constrained equal awards solution as the unique Nash equilibrium of the associated bankruptcy game.

The *constrained equal-loss rule*, L, is the dual of the latter. It distributes equally the difference between the amount available and the aggregate claims, with one proviso: no player ends up with a negative transfer. Namely, $L_i(N, E, d) = \max\{0, d_i - \lambda\}$, where λ solves $\sum_{i\in N} \max\{0, d_i - \lambda\} = E$. Taking $D_i = [0, d_i]$ and defining $f_i(\alpha_i) = d_i - \alpha_i$, we obtain the constrained equal-losses solution as the Nash equilibrium payoff of the game.

Aumann and Maschler (1985) introduced the Talmud rule as the consistent extension of the contested garment rule. It is defined as follows: for each (N, E, d) and each $i \in N$, $T_i(N, E, d) = \min\{\frac{1}{2}d_i, \lambda\}$ if $E \le \frac{1}{2}\sum_{i\in N} d_i$, and $T_i(N, E, d) = \max\{\frac{1}{2}d_i, d_i - \mu\}$ if $E \ge \frac{1}{2}\sum_{i\in N} d_i$, where λ and μ are chosen such that $\sum_{i\in N} T_i(N, E, d) = E$. If we let $D_i = [0, d_i]$ and

$$f_i(\alpha_i) = \begin{cases} \frac{1}{2}\alpha_i & E \le \frac{1}{2}\sum_{i\in N} d_i \\ \frac{1}{2}d_i - \alpha_i & E \ge \frac{1}{2}\sum_{i\in N} d_i, \end{cases}$$

the Nash equilibrium payoff of the game yields the allocation corresponding to the Talmud rule.

More generally, if we let $D_i = [0, d_i]$ and

$$f_i(\alpha_i) = \begin{cases} \theta\alpha_i & E \le \theta\sum_{i\in N} d_i \\ \theta d_i - \alpha_i & E \ge \theta\sum_{i\in N} d_i, \end{cases}$$

we generate the solutions corresponding to the TAL-family (Moreno-Ternero and Villar, 2003) which encompasses the constrained equal awards, the constrained equal losses, and the Talmud rule.[1]

These results illustrate on the applicability of this procedure to provide a noncooperative sup-

[1] The TAL-family consists of all rules with the following form: there is $\theta \in [0, 1]$ such that for each bankruptcy problem (N, E, c) and each $i \in N$,

$$R_i^\theta(N, E, d) = \begin{cases} \min\{\theta d_i, \lambda\} & E \le \theta\sum_{i\in N} d_i \\ \max\{\theta d_i, d_i - \mu\} & E \ge \theta\sum_{i\in N} d_i, \end{cases}$$

where and λ and μ are chosen such that $\sum_{i\in N} R_i^\theta(N, E, d) = E$.

port to the best known bankruptcy rules. But these results can actually be extended to virtually any meaningful rule. Consider now the following definition which introduces an extremely mild requirement on bankruptcy rules:

Definition 4.1. *A bankruptcy rule R is called* acceptable *if there are no bankruptcy problem (N, E, d) and players $i, j \in N$ such that $R_i(N, E, d) = 0$ and $R_j(N, E, d) = d_j$.*

Acceptable rules are those which never concede a player his claim in full whereas some other player gets nothing. Most of the rules which have been studied in the literature are acceptable.

The following proposition shows that for all acceptable bankruptcy rules there is a bankruptcy game whose equilibrium payoff coincides with the allocation proposed by the selected rule. Formally:

Proposition 4.3. *Let R be an acceptable bankruptcy rule and (N, E, d) a bankruptcy problem. Then, there is a noncooperative bankruptcy game $\langle N, D, \pi \rangle$, satisfying Axioms 4.1 and 4.2, whose unique equilibrium payoff coincides with $R(N, E, d)$.*

Proof. The proof consists of showing that we can define sets of strategies D_i and functions f_i in such a way that the result is a consequence of Proposition 4.1.

Let (N, E, d). To simplify notation we write R_i instead of $R_i(N, E, d)$. Since the rule is acceptable, either for each $i \in N$, $R_i > 0$, or for each $i \in N$, $R_i < d_i$. Next, we define the sets of strategies and the functions f_i.

Case 1: For each $i \in N$, $R_i > 0$. Let

$$m_i := \frac{R_1}{R_i} d_i \quad \text{and} \quad f_i(\alpha_i) := \frac{R_i}{R_1} \alpha_i.$$

It is clear that the functions f_i are monotone (in fact, increasing) and that, for each $i \in N$, f_i is a bijection mapping $[0, m_i]$ onto $[0, d_i]$.

By Proposition 4.1, the noncooperative bankruptcy game has a unique equilibrium payoff. Clearly, in this case, $\rho = R_1$. Hence, $\alpha^* = (R_1, \ldots, R_1)$ is a Nash equilibrium (which is, moreover, strong by Proposition 4.2); its associated payoff is $R(N, E, d)$.

Case 2: For each $i \in N$, $R_i < d_i$.

The reasoning is the same as before, except in that, now, we define:

$$m_i := \frac{d_1 - R_1}{d_i - R_i} d_i \quad \text{and} \quad f_i(\alpha_i) := d_i - \frac{d_i - R_i}{d_1 - R_1} \alpha_i.$$

Now, since $(d_1 - R_1, \ldots, d_1 - R_1)$ is a Nash equilibrium of this game and its associated payoff vector is $R(N, E, d)$, then Proposition 4.1 gives again the desired result. □

4.4 Concluding Remarks

We have presented in this Chapter a simple and intuitive game form which supports virtually all bankruptcy rules. The allocation proposed by each rule is obtained as the unique payoff vector corresponding to the Nash equilibrium of a specific game. In this respect, choosing the rules of the game (and most particularly the strategy space of the players) determines the bankruptcy rule that will emerge.

Interestingly enough, the game form that allows to implement those bankruptcy rules is a one-shot game in which every player sends a message concerning his own awards exclusively. Those messages refer to the cuts in their claims they might be ready to accept, given their claims and the existing shortage. The game form induces an equilibrium in which all players choose "the same" message. Selecting the nature of those messages (e.g. awards, shares, losses) amounts to deciding on the bankruptcy rule whose allocation will result (the equal awards-rule, the proportional rule, the equal-losses rule).

The game form proposed here implicitly assumes that all the data of the problem are public knowledge. In particular that the planner may know both the players' claims and the amount to divide. This is a natural assumption in most of the bankruptcy situations, where claims have to be eventually credited. The case of taxation problems may be an exception in this respect. Dagan et al. (1999) show that those problems are implementable when all players other than the planner know all the data of the problem. Even though this is an arguable assumption in this context, they also show an impossibility result when this is not the case (see also Corchón and Herrero (2004) on this point).

Bibliography

Aumann, R. J. (1959): "Acceptable Points in General Cooperative n-Person Games," in *Contributions to the theory of games IV*, ed. by A. Tucker and R. Luce, Princeton University Press, 287–324. (Quoted in pp. 51)

Aumann, R. J. and M. Maschler (1985): "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud," *Journal of Economic Theory*, 36, 195–213. (Quoted in pp. 53)

Chun, Y. (1989): "A Noncooperative Justification for the Egalitaria Surplus Sharing," *Mathematical Social Sciences*, 17, 245–261. (Quoted in pp. 48)

Corchón, L. and C. Herrero (2004): "A Decent Proposal," *Spanish Economic Review*, 6, 107–125. (Quoted in pp. 49, 55)

Dagan, N., R. Serrano, and O. Volij (1997): "A Noncooperative View of Consistent Bankrupcty Rules," *Games and Economic Behavior*, 18, 55–72. (Quoted in pp. 48)

——— (1999): "Feasible Implementation of Taxation Methods," *Review of Economic Design*, 4, 52–72. (Quoted in pp. 49, 55)

de Frutos, M. A. (1999): "Coalitional Manipulation in a Banfruptcy Problem," *Journal of Economic Theory*, 4, 255–272. (Quoted in pp. 49)

Herrero, C. (2003): "Equal Awards versus Equal Losses: Duality in Bankruptcy," in *Advances in Economic Design*, ed. by M. Sertel and S. Koray, Springer-Verlag. (Quoted in pp. 48)

Herrero, C., J. D. Moreno-Ternero, and G. Ponti (2003): "An Experiment of Bankruptcy," Tech. Rep. Working paper AD 2003-03, Ivie. (Quoted in pp. 49)

Ju, B.-G. (2003): "Manipulation via Merging and Splitting in Claims Problems," *Review of Economic Design*, 8, 205–215. (Quoted in pp. 49)

Moreno-Ternero, J. D. (2004): "Bankruptcy Rules and Coalitional Manipulation," Tech. rep., Yale University. (Quoted in pp. 49)

Moreno-Ternero, J. D. and A. Villar (2003): "The TAL-Family of Rules for Banruptcy Prolems," Tech. rep., University of Alicante. (Quoted in pp. 53)

Moulin, H. (2002): "Axiomatic Cost and Surplus Sharing," in *Handbood of Social Choice and Welfare*, ed. by K. Arrow, A. Sen, and K. Suzumura, Elsevier Science B.V., vol. 1. (Quoted in pp. 48)

O'Neill, B. (1982): "A Problem of Rights Arbitration from the Talmud," *Mathematical Social Sciences*, 2, 345–371. (Quoted in pp. 48)

Serrano, R. (1995): "Strategic Bargaining, Surplus Sharing Problems and the Nucleolus," *Journal of Mathematical Economics*, 24, 319–329. (Quoted in pp. 48)

Sonn, S. (1992): "Sequential Bargaining for Bankruptcy Problems," Preprint. (Quoted in pp. 48)

Thomson, W. (2003): "Axiomatic and Game Theoretic Analysis of Bankruptcy and Taxation Problems," *Mathematical Social Sciences*, 45, 249–297. (Quoted in pp. 48)

Part II

Cooperative Game Theory

Introduction to Cooperative Game Theory

This second Part is devoted cooperative game theory. We set the focus on the geometry underlying some of the best known solution concepts in the TU games literature. We describe the structure of this Part below.

The first three Chapters deal with the geometry of the core of a TU game. More specifically, we define a new solution concept for balanced games, the core-center, which is deeply studied in this Part of the dissertation. These three Chapters are based on the papers González-Díaz and Sánchez-Rodríguez (2003a,b). In Chapter 5 we formally introduce the core-center as the barycenter of the core and we carry out an analysis of the properties satisfied by this new allocation rule. The main focus is on the continuity property, which turns out to be a serious concern. We have also made an important effort studying the monotonicity properties of the core-center; the necessity of this effort comes from the existing negative results concerning the possibility of defining monotonic selections from the core of a TU game (Young, 1985; Housman and Clark, 1998). In Chapter 6 we combine some of the properties studied in Chapter 5 with an additivity property to obtain an axiomatic characterization of the core-center. Next, in Chapter 7, we develop some tools to establish a connection between the core-center and the Shapley value (Shapley, 1953) within the class of convex games. In this Chapter, we describe the formation of the core as the result of a dynamic process among coalitions. Based on this interpretation, we define the utopia games, a family of games associated with each TU game which naturally arise from the mentioned description. The utopia games are the corner stone for the connection between the core-center and the Shapley value.

Finally, in Chapter 8 we switch to the geometry underlying the τ value (Tijs, 1981). In this Chapter, which is based on the paper González-Díaz et al. (2005), we characterize the τ value as the barycenter of the edges of the core-cover of a quasi-balanced game (multiplicities have to be taken into account).

Summarizing, in this second Part we deepen in the geometry of the TU games. We do it by establishing some connections between set valued solutions and allocation rules. It is a well known property of the Shapley value the fact that it is the center of gravity of the vectors of marginal contributions. On the other hand, the nucleolus is many times referred to as the lexicographic center of the core. These two "central" properties of the Shapley value and the nucleolus have been used many times to motivate the use of these two allocation rules. Here, we add two more "central" relations, namely, (i) we introduce the core-center, defined as the center of gravity of the core, and hence, an allocation rule occupying a central position within the core and (ii) we show that the τ value lies, in general, in a central position inside the core-cover of a quasi-balanced game.

Short Bibliography

González-Díaz, J., P. Borm, R. Hendrickx, and M. Quant (2005): "A Geometric Characterisation of the Compromise Value," *Mathematical Methods of Operations Research*, 61. (Quoted in pp. 59)

González-Díaz, J. and E. Sánchez-Rodríguez (2003a): "The Core-Center and the Shapley Value: A Comparative Study," Reports in Statistics and Operations Research 03-10, University of Santiago de Compostela. (Quoted in pp. 59)

——— (2003b): "From Set-Valued Solutions to Single-Valued Solutions: The Centroid and the Core-Center," Reports in Statistics and Operations Research 03-09, University of Santiago de Compostela. (Quoted in pp. 59)

Housman and Clark (1998): "Core and Monotonic Allocation Methods," *International Journal of Game Theory*, 27, 611–616. (Quoted in pp. 59)

Shapley, L. S. (1953): "A Value for n-Person Games," in *Contributions to the theory of games II*, ed. by H. Kuhn and A. Tucker, Princeton: Princeton University Press, vol. 28 of *Annals of Mathematics Studies*. (Quoted in pp. 59)

Tijs, S. (1981): "Bounds for the Core and the τ-Value," in *Game theory and mathematical economics*, ed. by O. Moeschlin and D. Pallaschke, Amsterdam: North Holland Publishing Company, 123–132. (Quoted in pp. 59)

Young, H. (1985): "Monotonic Solutions of Cooperatives Games," *International Journal of Game Theory*, 14, 65–72. (Quoted in pp. 59)

Chapter 5

A Natural Selection from the Core of a TU game: The Core-Center

Contents

Introduction

In the framework of cooperative games with transferable utility there are several solution concepts that give rise to different ways of dividing the worth of the grand coalition, $v(N)$, among the players. Although solution concepts admit different classifications, we divide them into two groups: set-valued solutions and allocation rules (single-valued solutions). Roughly speaking, set-valued solutions provide a set of outcomes that can be infinite, finite, or even empty. The way to determine a set-valued solution can be seen as a procedure in which the set of all possible assignments is gradually reduced, until the final solution (not necessarily a singleton) is reached. This reduction is done by imposing some desirable properties that a solution should possess. Examples of this approach are the stable sets (von Neumann and Morgenstern, 1944), the core (Gillies, 1953), the kernel (Davis and Maschler, 1965), the bargaining sets (Aumann and Maschler, 1964) etc. On the other hand, one can establish some properties or axioms that determine a unique outcome for each game, this is known as an allocation rule. The Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969), and the τ-value (Tijs, 1981) are solutions of this type.

Each solution concept has its interpretation and attends to specific principles (fairness, equity, stability...) and all of them enrich the field of cooperative game theory. Besides, there have been many papers discussing on relations between allocation rules and set-valued solutions. Just to mention a couple of these relations: when the core of a game is nonempty the nucleolus selects an element inside it and, for the class of convex games, the Shapley value is in the core.

Our main purpose is to introduce a new allocation rule for balanced games summarizing all the information contained in the core, *i.e.*, obtain a single-valued solution from a set-valued one. This allocation should be a fair compromise among all the stable allocations selected by the core. In Maschler et al. (1979) it is shown that the nucleolus can be characterized as a “lexicographic center” for the core. With that in mind, we study the real center of the core, which we call the *core-center*, and discuss its game theoretical properties and interpretations. Now, we provide a natural motivation for this concept: assume that we have chosen all the efficient and stable allocations of a given game, *i.e.*, its core. If we want to select only one of all these outcomes as a proposal to divide $v(N)$, how to do it in a fair way? What we suggest is to select the expectation of a uniform distribution defined over the core of the game, in other words, its center of gravity. From the point of view of physics, the core-center is the point of the equilibrium of the core of a game. We have rewritten this notion in terms of a fairness (impartiality) property.

This Chapter deals with the axiomatic properties of the core-center. The main focus of this axiomatic study is in the continuity property, since it turns out to be the case that it is not easy to prove that the core-center is continuous. The problem of continuous selection from multi-functions has been widely studied in mathematics and Michael (1956) is a central paper in this literature. More specifically, the issue of selection from convex-compact-valued multi-functions (as the core) is discussed in Gautier and Morchadi (1992); they study, as an alternative to the barycentric selection, the Steiner selection, for which continuity is not a problem. Moreover, they briefly discuss the regularity problems one can face when working with the barycentric selection.

In this Chapter we show that, because of the special structure of the core of a TU game, the barycentric selection from the core (the core-center) is continuous as a function of the underlying game. Section 5.3 is entirely devoted to the discussion of the continuity of the core-center.

We also discuss in detail the monotonicity properties of the core-center. In fact, we show that, as far as monotonicity is concerned, there is a parallelism between the behavior of the core-center and that of the nucleolus.

As we have already said, this is not the first time that a central approach is used to obtain an allocation rule. It is widely known that the Shapley value is the center of gravity of the vectors of marginal contributions and, for convex games, it coincides with the center of gravity of the extreme points of the core (taking multiplicities into account). Besides, in González-Díaz et al. (2005) it is proved that the τ-value corresponds with the center of gravity of the edges of the core-cover (again multiplicities must be considered).

The structure of this Chapter is as follows. In Section 5.1 we introduce the preliminary game theoretical concepts. In Section 5.2, we define the core-center, provide some interpretations, and study its main properties. In Section 5.3 we discuss in depth the issue of the continuity of the core-center. Finally, we conclude in Section 5.4.

5.1 Game Theory Background

A transferable utility or TU game is a pair (N, v), where $N = \{1, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to every coalition $S \subseteq N$ a payoff $v(S)$. By convention, $v(\emptyset) = 0$. Since each game assigns a real value to each nonempty subset of N, it corresponds with a vector in $\mathbb{R}^{2^n-1}$. Let $|S|$ denote the number of elements of coalition S. Saving notation, when no ambiguity arises, we use i to denote $\{i\}$. Let G^n be the set of all n-player games.

Let $x \in \mathbb{R}^n$ be an allocation. Then, x is *efficient* if $\sum_{i=1}^n x_i = v(N)$ and x is individually rational if, for each $i \in N$, $x_i \geq v(i)$. Moreover, x is *stable* if for each $S \subsetneq N$, $\sum_{i \in S} x_i \geq v(S)$, *i.e.*, no coalition can improve by leaving the grand coalition. An *allocation rule* is a function which, given a game (N, v), selects an allocation in $\mathbb{R}^n$, *i.e.*,

$$\begin{array}{rccc} \varphi: & \Omega \subseteq G^n & \longrightarrow & \mathbb{R}^n \\ & (N,v) & \longmapsto & \varphi(N,v). \end{array}$$

Next, we define some properties for allocation rules. Let $(N, v) \in G^n$ and let φ be an allocation rule: φ is *continuous* if the function $\varphi : \mathbb{R}^{2^n-1} \to \mathbb{R}^n$ is continuous; φ is *efficient* if it always selects efficient allocations; φ is *individually rational* if it always selects individually rational allocations; φ is *scale invariant* if for each two games (N, v) and (N, w), and each $r \in \mathbb{R}$ such that, for each $S \subseteq N$, $w(S) = rv(S)$, then $\varphi(N, w) = r\varphi(N, v)$; φ is *translation invariant* if for each two games (N, v) and (N, w), and each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that for each $S \subseteq N$, $w(S) = v(S) + \sum_{i \in S} \alpha_i$, then $\varphi(N, w) = \varphi(N, v) + \alpha$; φ is *symmetric* if for each pair $i, j \in N$ such that for each $S \subset N \backslash \{i, j\}$, $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$, we have

$\varphi_i(N,v) = \varphi_j(N,v)$; φ satisfies *dummy player property* if for each $i \in N$ such that for each $S \subset N\backslash\{i\}$, $v(S \cup \{i\}) - v(S) = v(\{i\})$, we have $\varphi_i(N,v) = v(\{i\})$.

The core of a game (N,v) (Gillies, 1953), $C(N,v)$, is defined by $C(N,v) := \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)$ and, for each $S \subsetneq N$, $\sum_{i \in S} x_i \geq v(S)\}$. The class of games with a nonempty core is the class of balanced games. The class of games with nonempty core is the class of balanced games.The class of games with nonempty core is the class of balanced games.

5.2 The Core-Center: Definition, Interpretations, and Properties

The core of a game is the set of all the stable and efficient allocations. Now, if we consider that all these points are equally valuable, then it makes sense to think of the core as if it was endowed with a uniform distribution. The *core-center* summarizes the information of such a distribution of probability. Let $U(A)$ denote the uniform distribution defined over the set A and $E(\mathbb{P})$ the expectation of the probability distribution $\mathbb{P}$.

Definition 5.1. *Let (N,v) be a balanced game with core $C(N,v)$, the* core-center *of (N,v), $\mu(N,v)$, is defined as follows:*

$$\mu(N,v) := E\left[U(C(N,v))\right].$$

The idea underlying our motivation for the core-center can be summarized as follows: if in accordance with some properties and/or criteria we have selected a (convex) set of allocations (the core) and we want to choose one, and only one, of these allocations, why not to choose the center of the set?

Also, from the point of view of physics, if we think of the core as a homogeneous body, then the core-center selects its center of gravity. In physics, the center of gravity is a fundamental concept because it allows to simplify the study of a complex system just by reducing it to a point; for instance, the movement of a body can be analyzed by describing the movement of its center of gravity. Roughly speaking, the core-center is the unique point in the core such that all the core allocations are "balanced" with respect to it.

The fact that the core-center belongs to the core implies that it inherits many of the properties of the allocations in the latter. Hence, the core-center also satisfies, among others, the following properties:

- Efficiency
- Individual rationality
- Stability
- Symmetry
- Scale Invariance
- Translation Invariance
- Dummy player property.

All of them are straightforward, either because they are inherited from core properties or because they are a consequence of the properties of the center of gravity. In the rest of this Section, we first discuss a new property that the core-center satisfies, second, we discuss the monotonicity properties of the core-center, and, finally, we introduce a first approach to the issue of the continuity of the core-center.

5.2.1 A Fairness Property

In Dutta and Ray (1989) the problem of selecting an allocation in the core is studied assuming that all members of the society have subscribed to equality as a desirable end. In this context they propose an egalitarian allocation which is characterized in terms of Lorenz domination. The motivation for the core-center is from the angle of impartiality as opposed to that of egalitarianism. Assume that the situation the game models is the consequence of some previous efforts or investments made by the different agents, *i.e.*, the core allocations can be seen as the possible rewards arising from their contributions (possibly unequal) to a common purpose. In this situation the equity principle would not be a fair one. We present now a fairness property for the core-center to show the justice foundations it obeys to.

Let $D \subseteq \mathbb{R}^n$, $i \in \mathbb{N}$, and an allocation $x \in \mathbb{R}^n$. Let $W_i(x)$ denote the set of all allocations in D which are worse than x for i, and $B_i(x)$ is the set of all allocations which are better than x for i (Figure 5.1). Formally, $W_i(x) := \{y \in D \,:\, y_i < x_i\}$ and $B_i(x) := \{y \in D \,:\, y_i > x_i\}$. Also, let $E_i(x) := \{y \in D \,:\, y_i = x_i\}$.

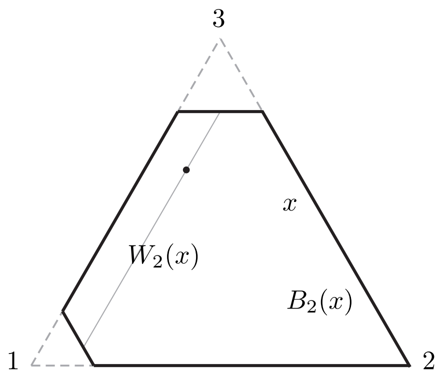

Figure 5.1: The sets $B_2(x)$ and $W_2(x)$ in the core of a game

Consider now the situation in which there is a probability distribution $\mathbb{P}$ defined over D (for the core-center the uniform distribution is defined over the core). Let $x \in \mathbb{R}^n$ and $i \in N$, the *relevance* (weight) for player i of a point $y \in D$, with respect to x, is the weight of the point according to $\mathbb{P}$, but re-scaled proportionally to $|x_i - y_i|$. The relevance for player i of y with respect to x depends on the weight of y according to $\mathbb{P}$ but also on the difference between x_i and y_i, *i.e.*, how good or bad y_i is compared to x_i.

Let $\mathbb{P}$ be a distribution of probability defined over $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. The *degree of satisfaction* of player i with respect to x is defined as the quotient

$$DS_i^{\mathbb{P}}(x) = \frac{\int_{W_i(x)} |x_i - y_i| d\mathbb{P}(y)}{\int_{B_i(x)} |x_i - y_i| d\mathbb{P}(y)} = \frac{\int_{W_i(x)} (x_i - y_i) d\mathbb{P}(y)}{\int_{B_i(x)} (y_i - x_i) d\mathbb{P}(y)}.$$

According to this definition, a degree of satisfaction of 1 for a player i with respect to an allocation x means that, in some way, he perceives the sets $B_i(x)$ and $W_i(x)$ as equal (with regard to $\mathbb{P}$ and x). A small observation concerning the indetermination 0/0 is needed, if both numerator and denominator in $DS_i^{\mathbb{P}}(x)$ are 0, *i.e.*, player i receives x_i in all the points in the support of $\mathbb{P}$, then, in line with the former comment, $DS_i^{\mathbb{P}}(x) = 1$. Besides, when the denominator takes value 0 but the numerator does not, there is no problem in letting the degree of satisfaction take the value $+\infty$.

Definition 5.2. *Let $\mathbb{P}$ be a distribution of probability defined over $\mathbb{R}^n$. An allocation $x \in \mathbb{R}^n$ is* impartial with respect to $\mathbb{P}$ *if for each pair $i, j \in N$, $DS_i^{\mathbb{P}}(x) = DS_j^{\mathbb{P}}(x)$.*

Definition 5.3. *Let φ be an allocation rule. For each $(N, v) \in G^n$, let $\mathbb{P}(N, v)$ be a distribution of probability defined over $\mathbb{R}^n$. Then, φ is* impartial with respect to $\mathbb{P}$ *if for each $(N, v) \in G^n$, $\varphi(N, v)$ is impartial with respect to $\mathbb{P}(N, v)$.*

Lemma 5.1. *Let (N, v) be a balanced game and let $U(N, v)$ be the uniform distribution defined over $C(N, v)$. Then, the core-center is the unique efficient allocation which is impartial with respect to $U(N, v)$.*

Proof. This Lemma is almost an immediate consequence of the properties of the center of gravity. Let $\bar{y}$ be the expectation of the uniform distribution defined over $C(N, v)$ and let x be an efficient allocation in $\mathbb{R}^n$. Let f be the density function associated with U. First, we show that for each $i \in N$, $DS_i^U(x) = 1$ if and only if $x_i = \bar{y}_i$, *i.e.*, for each $i \in N$,

$$\int_{W_i(x)} (x_i - y_i) f(y) dy = \int_{B_i(x)} (y_i - x_i) f(y) dy \Leftrightarrow x_i = \bar{y}_i.$$

Note that

$$\int_{\mathbb{R}^n} (x_i - y_i) f(y) dy = x_i \int_{\mathbb{R}^n} f(y) dy - \int_{\mathbb{R}^n} y_i f(y) dy = x_i - \bar{y}_i.$$

Moreover, since either the probability of $E_i(x)$ is 0 or for each $y \in C(N, v)$, $y_i = x_i$, we have $\int_{E_i(x)} (x_i - y_i) f(y) dy = 0$. Hence,

$$\int_{\mathbb{R}^n} (x_i - y_i) f(y) dy = \int_{W_i(x)} (x_i - y_i) f(y) dy - \int_{B_i(x)} (y_i - x_i) f(y) dy.$$

Now,

$$\int_{W_i(x)} (x_i - y_i) f(y) dy = \int_{B_i(x)} (y_i - x_i) f(y) dy \Leftrightarrow x_i = \bar{y}_i.$$

Now, suppose that there is a player $i \in N$ such that $DS_i^U(x) < 1$ (the case $DS_i^U(x) > 1$ is analogous). Then, $x_i < \bar{y}_i$. Now, because of the efficiency property, there is $j \neq i$ such that $x_j > \bar{y}_j$. Hence, by the first part of the proof, $DS_j^U(x) > DS_i^U(x)$. □

Note that in the previous proof we also showed that the unique outcome (not necessarily efficient) such that for each $i \in N$, $DS_i^U(x) = 1$ is the core-center. Hence, the unique case in which all players perceive the sets $B_i(x)$ and $W_i(x)$ as equal (with regard to U and x), is in that in which the core-center is chosen.

5.2.2 An Example

Consider the following 4-player game taken from Maschler et al. (1979),

$$v(S) = \begin{cases} 2 & S = N \\ 1 & 2 \leq |S| \leq 3 \text{ and } S \neq \{1,3\}, \{2,4\} \\ 0.5 & S = \{1,3\} \\ 0 & \text{otherwise.} \end{cases}$$

This is a quite symmetric game, actually, its main asymmetry hinges on the fact that coalition $\{1,3\}$ can obtain 0.5 whereas the coalition $\{2,4\}$ cannot obtain anything alone. In this game we have that the Shapley value is $(13/24, 11/24, 13/24, 11/24)$ and the nucleolus is $(0.5, 0.5, 0.5, 0.5)$. The core of this game is the segment joining the points $(1,0,1,0)$ and $(0.25, 0.75, 0.25, 0.75)$; and the core-center is its midpoint: the allocation $(5/8, 3/8, 5/8, 3/8)$.

The asymmetry of the game is reflected in the Shapley value: players 1 and 3 obtain a higher payoff, but not in the nucleolus which gives the same payoff to the four players. Besides, these two allocations lie within the core of the game. As one should expect, the core-center is very sensitive with the asymmetries which directly affect the structure of the core, and this is the case in this example.

5.2.3 Monotonicity

Next, we study the behavior of the core-center with respect to monotonicity. To do so, we define four different monotonicity properties. Let φ be an allocation rule. We say φ is *strongly monotonic* if for each pair $(N,v), (N,w) \in G^n$, and each $i \in N$ such that for each $S \subseteq N \backslash \{i\}$, $w(S \cup \{i\}) - w(S) \geq v(S \cup \{i\}) - v(S)$, we have $\varphi_i(N,w) \geq \varphi_i(N,v)$. Let $(N,v), (N,w) \in G^n$, let $T \subseteq N$ be such that $w(T) > v(T)$ and for each $S \neq T$, $w(S) = v(S)$: φ satisfies *coalitional monotonicity* if for each $i \in T$, $\varphi_i(N,w) \geq \varphi_i(N,v)$; φ satisfies *aggregate monotonicity* if $T = N$ implies that for each $i \in N$, $\varphi_i(N,w) \geq \varphi_i(N,v)$; φ satisfies *weak coalitional monotonicity* if $\sum_{i \in T} \varphi_i(N,w) \geq \sum_{i \in T} \varphi_i(N,v)$.

Young (1985) characterized the Shapley value as the unique strongly monotonic and symmetric allocation rule. Since the core-center is symmetric, it is not strongly monotonic. Also Young (1985) and Housman and Clark (1998) showed that if an allocation rule always selects an allocation in the core, it cannot not satisfy coalitional monotonicity when the number of players is greater than three. Hence, the core-center does not satisfy coalitional monotonocity. Things do not get better if we weaken the monotonicity property in the direction of aggregate monotonicity.

Proposition 5.1. *Let $n \geq 4$. Then, the core-center does dot satisfy aggregate monotonicity within the class of balanced games with n-players.*

Proof. The proof is made by means of an example when $n = 4$. If $n > 4$ the example can be adapted by adding dummy players. Let $(N, v) \in G^n$ be such that $N = \{1, 2, 3, 4\}$ and v is defined as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	0	0	0	0	0	1	1	1	1	0	1	1	1	2	2

Now, $C(N, v) = \{(0, 0, 1, 1)\}$ and hence, $\mu(N, v) = (0, 0, 1, 1)$. Let co(A) stand for the convex hull of the set A. Let (N, w) be such that $w(N) = 3$ and for each $S \neq N$, $w(S) = v(S)$. Then,

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$w(S)$	0	0	0	0	0	1	1	1	1	0	1	1	1	2	3

and

$$C(N, w) = \text{co}\{(1, 0, 1, 1), (0, 0, 2, 1), (0, 0, 1, 2), (0, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1), (1, 2, 0, 0)\}.$$

Next, we prove that the core-center does not satisfy aggregate monotonicity by showing that $\mu_3(N, v) > \mu_3(N, w)$. Let $(N, \hat{w})$ be the game defined as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$\hat{w}(S)$	0	0	0	0	0	1	1	1	1	0	1	1	2	2	3

with core

$$C(N, \hat{w}) = \text{co}\{(1, 0, 1, 1), (0, 0, 2, 1), (0, 0, 1, 2), (0, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1)\}.$$

The game $(N, \hat{w})$ only differs from (N, w) in the value for the coalition $\{1, 3, 4\}$. Figures 5.2 and 5.3 show the cores of (N, w) and $(N, \hat{w})$, respectively. Note that, because of the stronger restriction for coalition $\{1, 3, 4\}$, $C(N, \hat{w}) \subsetneq C(N, w)$. Now, $C(N, \hat{w})$ is symmetric with respect to the point $(0.5, 0.5, 1, 1)$, *i.e.*, $x \in C(N, \hat{w}) \Leftrightarrow -\big(x - (0.5, 0.5, 1, 1)\big) + (0.5, 0.5, 1, 1) \in C(N, \hat{w})$. Hence, $\mu(N, \hat{w}) = (0.5, 0.5, 1, 1)$.

Now, $C(N, w) \backslash C(N, \hat{w}) \subsetneq \text{co}\{(1, 1, 1, 0), (0, 1, 1, 1), (1, 1, 0, 1), (1, 2, 0, 0)\}$. Hence, for each $x \in C(N, w) \backslash C(N, \hat{w})$, $x_3 \leq 1$. Moreover, the volume of the points in $C(N, w) \backslash C(N, \hat{w})$ with the third coordinate smaller than 1 is positive. Hence, by the definition of the core-center, since $\mu_3(N, \hat{w}) = 1$, we have $\mu_3(N, w) < 1 = \mu_3(N, v)$. Hence, the core-center does not satisfy aggregate monotonicity. $\square$

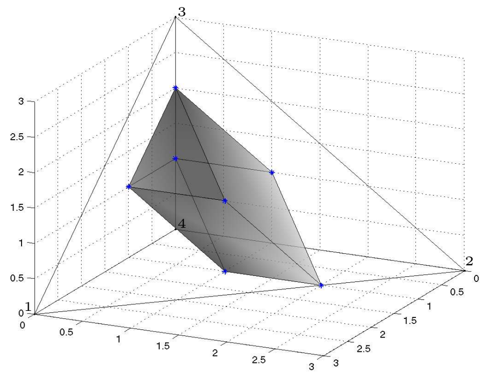

Figure 5.2: The core of the game (N, w)

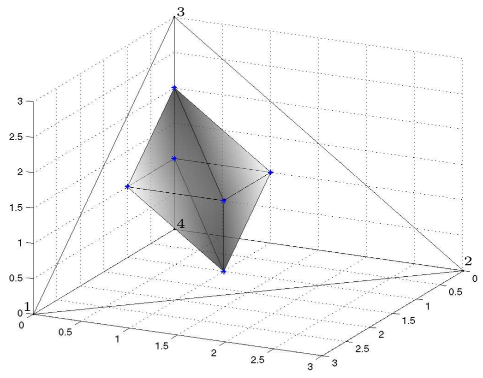

Figure 5.3: The core of the game $(N, \hat{w})$

The nucleolus (Schmeidler, 1969) also violates the three monotonicity properties we have studied so far. Zhou (1991) introduces the weak coalitional monotonicity and shows that the nucleolus satisfies it. This weakening of the coalitional monotonicity only requires that, when one coalition improves moving from (N, w) to (N, v) and there is no difference for all the other coalitions, then, the coalition as a whole (instead each player separately) has to be better off in the allocation selected for (N, v).

Proposition 5.2. *The core-center satisfies weak coalitional monotonicity.*

Proof. Let (N, v) and (N, w) be two balanced games as in the definition of weak coalitional monotonicity, *i.e.*, they only differ in the fact that $w(T) > v(T)$ for a given coalition T. If $T = N$ the result is immediately derived from the efficiency property. Hence, we can assume $T \subsetneq N$. If $C(N, w) = C(N, v)$ then $\mu(N, w) = \mu(N, v)$ and $\sum_{i \in T} \mu_i(N, w) \geq \sum_{i \in T} \mu_i(N, v)$. Hence, we can assume that $C(N, w) \subsetneq C(N, v)$. Let $x \in C(N, v) \backslash C(N, w)$ and $y \in C(N, w)$, then $\sum_{i \in T} y_i \geq w(T) > \sum_{i \in T} x_i$. Since the core-center is the expectation of the uniform distribution over the core, and passing from $C(N, v)$ to $C(N, w)$ we have removed the "bad" allocations for coalition T (as a whole), this coalition is better off in the core-center of (N, w). $\square$

Hence, the core-center and the nucleolus have an analogous behavior with respect to all monotonicity properties discussed in this Chapter.

5.2.4 Continuity

When introducing a new allocation rule, one of the first things to study is whether it is continuous or not. Intuitively, one could think that the center of gravity of the core of a game (N, v) varies continuously as a function of (N, v). Although the result is true, that intuition could lead to wrong arguments. The core is a set-valued mapping from $\mathbb{R}^{2^n - 1}$ to $\mathbb{R}^n$, and there is a huge literature studying the problem of continuous selection from set-valued mappings (see, for instance, Michael (1956)).

If two balanced games are close enough (as vectors of $\mathbb{R}^{2^n-1}$), then the corresponding cores are also close to each other (as sets). We are computing the center of gravity of these sets when they are endowed with the uniform distribution. Hence, the question is: are also the corresponding measures (associated with the uniform distribution) close to each other? This problem is not trivial at all. The following example shows what the problem is:

Example 5.1. *Consider the triangle with vertices* $(a, 0)$*,* $(-a, 0)$ *and* $(0, 1)$*. The center of gravity of this triangle is* $(0, 1/3)$*, no matter the value a takes. If we let a tend to* 0 *then, "in the limit", we get the segment joining the points* $(0, 0)$ *and* $(0, 1)$*, whose center of gravity is* $(0, 1/2)$*, which is not the limit of the centers of gravity.*

The problem with the continuity arises when the number of dimensions of the space under consideration is not fixed, *i.e.*, an $(n-2)$-polytope can be expressed (as a set) as the limit of $(n-1)$-polytopes. As we have shown in the previous example, the continuity property is quite sensitive to this kind of degenerations. Hence, this problem must be handled carefully, taking into account that the center of gravity of a convex polytope does not necessarily vary with continuity if degenerations are permitted. Even so, the following statement is true:

Theorem 5.1. *The core-center is continuous.*

The proof of this statement is quite technical. In Section 5.3 we formally introduce the problem along with the concepts needed for the proof.

5.2.5 Computation

The complexity of the computation of any allocation rule is a concept which also needs to be studied. Here, we provide some insights to this problem when working with the core-center. The computation of the center of gravity of a convex polytope is a problem which has been widely studied in computational geometry. There are many negative results concerning the complexity of this problem. In the case of the core-center, even if we are given a polynomial description of the game (*i.e.*, of the function v), the computation time can grow exponentially with the number of players. Basically, there are two ways for obtaining the center of gravity of a convex polytope. The classical one consists of the exact computation; many algorithms have already been developed for this issue, but all of them are exponential in the number of players. The second approach consists of using randomizing procedures to estimate the center of gravity. Roughly speaking, these procedures lead to algorithms which allow to obtain the estimations in polynomial time whenever we are able to find out whether a point belongs or not to the core in polynomial time; this is not a mild assumption, but it cannot be dispensed with.

5.3 Continuity of the Core-Center

5.3.1 The Problem

First, we introduce the exact formulation of the problem to be solved. Henceforth, we denote a game (N, v) by v. Note that in order to prove Theorem 5.1 it is enough to show that for each balanced game v, and each sequence of balanced games converging to v (under the usual convergence of vectors in $\mathbb{R}^{2^n-1}$), the associated sequence of the core-centers of the games converges to the core-center of v. Formally,

Theorem 5.2. *Let $\bar{v}$ be a balanced game and $\{v^t\}$ a sequence of balanced games such that $\lim_{t\to\infty} v^t = \bar{v}$. Then, $\lim_{t\to\infty} \mu(v^t) = \mu(\bar{v})$.*

Clearly, Theorems 5.1 and 5.2 are equivalent. The next Proposition, which is a weaker version of the previous Theorem contains the difficult part of the proof. Theorem 5.2, and hence Theorem 5.1, are an easy consequence.

Proposition 5.3. *Let $\bar{v}$ be a balanced game and $\{v^t\}$ a sequence of balanced games such that*

(i) *for each $t \in \mathbb{N}$, we have $\bar{v}(N) = v^t(N)$,*

(ii) $\lim_{t\to\infty} v^t = \bar{v}$.

Then, $\lim_{t\to\infty} \mu(v^t) = \mu(\bar{v})$.

In contrast with Theorem 5.2, where every possible sequence of games is considered, Proposition 5.3 only concerns specific sequences. Next, we prepare the ground for Proposition 5.3. We do it by stating and proving a general result. Then, Proposition 5.3 is easily derived. We make use of some measure theory and functional analysis results, which help us to place our result on a firm basis.

5.3.2 A New Framework

Next, we introduce a new framework in which we state and prove a general convergence result for uniform measures. Then, the main part of the proof of Proposition 5.3 is a particular case. The idea of the whole procedure can be summarized as follows: whenever we think about a balanced game and its core-center, we can just think of a polytope (its core) and its center of gravity. Similarly, whenever we have a polytope and its center of gravity, we can just think of the uniform measure defined over the polytope and the integral of the identity function with respect to it. Following this idea, if we want to prove that the core-center of a sequence of games converges to the core-center of the limit game (Theorem 5.2), it is enough to prove that the integrals over the corresponding uniform measures also converge.

Notation

A (convex) *polyhedron* is defined as the intersection of a finite number of closed halfspaces. A polyhedron P is an *m-polyhedron* if its dimension is m, *i.e.*, the smallest integer such that P is contained in an m-dimensional space. A (convex) *polytope* is a bounded polyhedron. Let M_λ^m stand for the Lebesgue measure on $\mathbb{R}^m$. Let $A \subseteq \mathbb{R}^m$ be a Lebesgue measurable set and let $m' \geq m$; we denote $M_\lambda^{m'}(A)$ by $\text{Vol}_{m'}(A)$, *i.e.*, the m'-dimensional volume of A; hence, if $A \subseteq \mathbb{R}^m$ and $m' > m$, then, $\text{Vol}_{m'}(A) = 0$. Let P be an m-polytope and $\mathcal{X}_P$ its characteristic function; let M_P be the Borel measure such that $M_P := \frac{1}{\text{Vol}_m(P)} \mathcal{X}_P M_\lambda^m$, *i.e.*, the uniform measure defined over polytope P.

Let u be a vector in $\mathbb{R}^m$. Let H_α^u be the following hyperplane normal to u, $H_\alpha^u := \{x \in \mathbb{R}^m : \sum_{j=1}^m u_j x_j = \alpha\}$. Let BH be the halfspace below hyperplane H. Let P be a polytope, then we say that hyperplane H is a *supporting hyperplane for* P if $H \cap P \neq \emptyset$ and BH contains P. Usually, a face of a polytope P is defined as (i) P itself, (ii) the empty set, or (iii) the intersection of P with some supporting hyperplane. With a slight abuse of language, we use the term *face* to designate only $(m-1)$-dimensional faces of an m-polytope. Let $\mathcal{F}(P)$ be the set of all faces of P and F be an arbitrary face.

Let P be an m-polytope. Then, the finite set of polytopes $\{P_1, \ldots, P_k\}$ is a *dissection* of P if (i) $P = \bigcup_{j=1}^k P_j$ and (ii) for each pair $\{j, j'\} \subseteq \{1, \ldots, k\}$, with $j \neq j'$, $\text{Vol}_m(P_j \cap P_{j'}) = 0$.

Next, we state, without proof, two elemental results.

Lemma 5.2. *Let P and P' be two m-polytopes such that $P' \subseteq P$. Then, P' belongs to some dissection of P.*

Lemma 5.3. *Let P be an m-polyhedron, let $u \in \mathbb{R}^m$, and let $\alpha, \beta \in \mathbb{R}$. Let $P \cap H_\alpha^u \neq \emptyset$ and $P \cap H_\beta^u \neq \emptyset$. Then, $P \cap H_\alpha^u$ is bounded if and only if $P \cap H_\beta^u$ is bounded.*

Let P be an m-polytope, let $r > 0$ be such that $P \subsetneq (-r, r)^m \subsetneq \mathbb{R}^m$. Let $R := [-r, r]^m$. The pair $(R, \mathcal{B})$, where $\mathcal{B}$ stands for the collection of Borel sets of R, is a measure space. Let $\mathcal{M}(R)$ be the set of all complex-valued regular Borel measures defined on $(R, \mathcal{B})$ and $\mathcal{M}^+(R)$ the subset of real-valued and positive Borel measures. Also, let $C(R)$ and $C^{\mathbb{R}}(R)$ be the sets of all continuous functions $f : R \to \mathbb{C}$ and $f : R \to \mathbb{R}$ respectively.

As a consequence of the Riesz Representation Theorem, $C(R)^* = \mathcal{M}(R)$, *i.e.*, $\mathcal{M}(R)$ is the dual of $C(R)$. This allows us to use the weak* topology (henceforth w^*) in $\mathcal{M}(R)$. According to this topology, a sequence of measures $\{M_t\}$ converges to a measure M if and only if for each $f \in C(R)$, $\lim_{t \to \infty} \int f dM_t = \int f dM$. For each $f \in C(R)$, and each measure $M \in \mathcal{M}(R)$, $\langle f, M \rangle$ denotes $\int f dM$.

Remark. We apologize for the readers that are not familiar with these concepts. They lead to a more consistent notation, cleaner statements, and less tedious proofs. Henceforth, convergence of a sequence of measures $\{M_t\}$ to a measure M under w^* just means that, for each continuous function f, the sequence of real numbers obtained by integration of f under the M_t's converges to the integral under M. Moreover, for notational convenience, we denote those integrals by $\langle f, M_t\rangle$ and $\langle f, M\rangle$, respectively.

The results

Next, we prove two technical lemmas.

Lemma 5.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $K \subsetneq \mathbb{R}$ a compact set. Then, the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := \max_{y\in K} f(x, y)$ is continuous.*

Proof. Suppose, on the contrary, that h is not continuous. Then, there is a sequence of real numbers $\{x_t\}$ such that (i) $\lim_{t\to\infty} x_t = x$, and (ii) the sequence $\{h(x_t)\}$ does not converge to $h(x)$. Let $y^* \in K$ be such that $f(x, y^*) = \max_{y\in K} f(x, y) = h(x)$.

For each $t \in \mathbb{N}$, let $y_t \in K$ be such that $h(x_t) = f(x_t, y_t)$. Since each $y_t \in K$, the sequence $\{y_t\}$ has a convergent subsequence. Assume, without loss of generality, that $\{y_t\}$ itself converges and let y' be its limit. Then,

$$f(x, y^*) = h(x) \overset{assumpt}{\neq} \lim_{t\to\infty} h(x_t) = \lim_{t\to\infty} f(x_t, y_t) \overset{f\,cont}{=} f(x, y').$$

Hence, $f(x, y^*) > f(x, y')$. Then, there is $\delta > 0$ such that

$$\left.\begin{array}{l} |x_t - x| < \delta \\ |y_t - y'| < \delta \end{array}\right\} \overset{f\,cont}{\Longrightarrow} f(x_t, y^*) - f(x_t, y_t) > 0, \text{ contradicting } h(x_t) = f(x_t, y_t).$$

□

Corollary 5.1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function and $K \subsetneq \mathbb{R}^l$, $1 < l < m$, a compact set. Then, the function $h : \mathbb{R}^{m-l} \to \mathbb{R}$ defined by $h(x) := \max_{y\in K} f(x, y)$ is continuous.*

Proof. The proof of Lemma 5.4 can be immediately adapted to this general case. □

Note that analogous results to Lemma 5.4 and Corollary 5.1 can be stated using min instead of max.

Lemma 5.5. *Let $M \in \mathcal{M}(R)$ and let $\{M_t\}$ be a sequence of measures in $\mathcal{M}(R)$ such that for each $f \in C^{\mathbb{R}}(R)$, $\lim_{t\to\infty}\langle f, M_t\rangle = \langle f, M\rangle$. Then, for each $f \in C(R)$, $\lim_{t\to\infty}\langle f, M_t\rangle = \langle f, M\rangle$.*

Proof. For each $f \in C(R)$, there exist functions f_1 and f_2 in $C^{\mathbb{R}}(R)$ such that for each $x \in R$, $f(x) = f_1(x) + f_2(x)i$. Then,

$$\langle f, M_t\rangle = \int f\, dM_t = \int f_1\, dM_t + i \int f_2\, dM_t \overset{t\to\infty}{\longrightarrow} \int f_1\, dM + i \int f_2\, dM = \langle f, M\rangle.$$

□

As a consequence of Lemma 5.5, to prove a convergence under w^*, it suffices to study functions in $C^{\mathbb{R}}(R)$.

Now we are ready to state the main result.

Theorem 5.3. *Let P be an m-polytope and R an m-dimensional cube $[-r,r]^m$ containing P in its interior. Let $u \in \mathbb{R}^m$. Let $\bar{\alpha} \in \mathbb{R}$ and let $\{\alpha_t\}$ be a sequence in $[\bar{\alpha},\infty)$ with limit $\bar{\alpha}$. Let $P_t := P \cap BH^u_{\alpha_t}$ and $\bar{P} := P \cap BH^u_{\bar{\alpha}}$. Then, $M_{P_t} \xrightarrow{w^*} M_{\bar{P}}$.*

Proof. Without loss of generality, we assume that $u = e^1 = (1,0,\dots,0)$ (otherwise a change of coordinates can be carried out) and that $\{\alpha_t\}$ is a decreasing sequence of positive numbers. If $\bar{P}$ is an m-polytope, there are no degeneracies and the result is straightforward. Hence, we assume that $\bar{P}$ is not an m-polytope. Hence, $\bar{\alpha} = \min_{x\in P} x_1$. Now, we distinguish two cases: $\bar{P}$ is an $(m-1)$-polytope, and $\bar{P}$ is an $(m-l)$-polytope, with $l > 1$ (multiple degeneracy).

Case 1: $\bar{P}$ is an $(m-1)$-polytope.
Let Q be the polyhedron defined as follows,

$$Q := \{y \in \mathbb{R}^m \ : \ y = x + \gamma e^1, \text{ where } x \in \bar{P} \text{ and } \gamma > 0\}.$$

Now, for each $t \in \mathbb{N}$, we define the auxiliary polytopes $Q_t := Q \cap BH^{e^1}_{\alpha_t}$. Also, let $\bar{Q} := Q \cap BH^{e^1}_{\bar{\alpha}}$ (see Figure 5.4). Note that, by definition, $\bar{Q} = \bar{P}$.

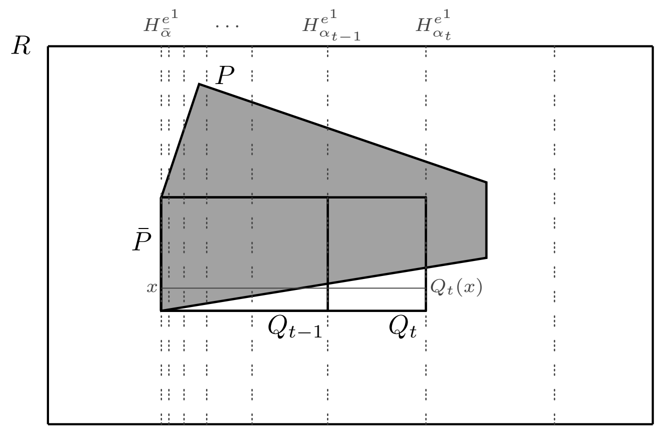

Figure 5.4: The Q_t polytopes

The proof is in three steps. In Step 1 we prove that the sequence of measures induced by the auxiliary polytopes, $\{M_{Q_t}\}$, converges to $M_{\bar{Q}}$. In Step 2, we study the relations between the volumes of $P_t \backslash Q_t$, $Q_t \backslash P_t$, and Q_t. Finally, in Step 3 we obtain the desired convergence result, *i.e.*, that of the sequence $\{M_{P_t}\}$ to $M_{\bar{P}}$. Recall that, by Lemma 5.5, we can restrict our attention to functions in $C^{\mathbb{R}}(R)$ whenever we have to prove some convergence under w^*.

Step 1: $M_{Q_t} \xrightarrow{w^*} M_{\bar{Q}}$.
We want to prove that for each $f \in C^{\mathbb{R}}(R)$, $\lim_{t\to\infty}\langle f, M_{Q_t}\rangle = \langle f, M_{\bar{Q}}\rangle$.
Step 1.a: Let $f \in C^{\mathbb{R}}(R)$ be such that there exists $c : [-r,r]^{m-1} \to \mathbb{R}$ with the following property: for each $x \in [-r,r]^m$, $f(x) = c(x_{-1})$. Let dx_{-1} stand for $dx_2 \dots dx_m$. Also, for each

$x \in \bar{Q}$ and each $t \in \mathbb{N}$, we define the 1-polytopes $Q_t(x) := \{y \in Q_t \ : \ y_{-1} = x_{-1}\}$. Note that, if $x \neq x'$, then $Q_t(x) \cap Q_t(x') = \emptyset$ and $\mathrm{Vol}_1(Q_t(x)) = \mathrm{Vol}_1(Q_t(x')) = \alpha_t - \bar{\alpha}$. Moreover, for each $x \in \bar{Q}$, f is constant in $Q_t(x)$. Then,

$$\begin{aligned}\langle f, M_{Q_t}\rangle &= \frac{1}{\mathrm{Vol}_m\,(Q_t)}\int_{\bar{Q}}\int_{Q_t(x)} c(x_{-1})dx_{-1}dx_1\\ &= \frac{\alpha_t-\bar{\alpha}}{\mathrm{Vol}_m\,(Q_t)}\int_{\bar{Q}} c(x_{-1})dx_{-1}\\ &= \frac{1}{\mathrm{Vol}_{m-1}\,(\bar{Q})}\int_{\bar{Q}} c(x_{-1})dx_{-1}\\ &= \langle f, M_{\bar{Q}}\rangle.\end{aligned}$$

Step 1.b: Let $f \in C^{\mathbb{R}}(R)$. Define the three auxiliary functions

$$\begin{aligned} f^*(x_1,x_{-1}) &:= f(\bar{\alpha},x_{-1}),\\ \overline{c}_t(x_1,x_{-1}) &:= \max_{z\in[\bar{\alpha},\alpha_t]} f(z,x_{-1}), \text{ and}\\ \underline{c}_t(x_1,x_{-1}) &:= \min_{z\in[\bar{\alpha},\alpha_t]} f(z,x_{-1}).\end{aligned}$$

By Corollary 5.1, functions $\overline{c}_t$ and $\underline{c}_t$ are continuous. Hence, by Step 1.a, we have $\langle \overline{c}_t, M_{Q_t}\rangle = \langle \overline{c}_t, M_{\bar{Q}}\rangle$ and $\langle \underline{c}_t, M_{Q_t}\rangle = \langle \underline{c}_t, M_{\bar{Q}}\rangle$. By the continuity of f, for each $x \in R$, $\lim_{t\to\infty} \underline{c}_t(x) = f^*(x) = \lim_{t\to\infty}\overline{c}_t(x)$. Let g be the constant function such that for each $x \in R$, $g(x) := \max_{x\in R}|f(x)|$. Since $\int g\, dM_{\bar{Q}} = \max_{x\in R}|f(x)|$, g is Lebesgue integrable with respect to $M_{\bar{Q}}$. Moreover, for each $x \in R$, $|\overline{c}_t(x)| \leq g(x)$ and $|\underline{c}_t(x)| \leq g(x)$. Since $M_{Q_t} \in \mathcal{M}^+(R)$, then $\langle \underline{c}_t, M_{Q_t}\rangle \leq \langle f, M_{Q_t}\rangle \leq \langle \overline{c}_t, M_{Q_t}\rangle$. Now, the Lebesgue's Dominated Convergence Theorem completes Step 1,

$$\begin{array}{ccccc} \langle \underline{c}_t, M_{Q_t}\rangle & \leq & \langle f, M_{Q_t}\rangle & \leq & \langle \overline{c}_t, M_{Q_t}\rangle\\ \| \text{ Step 1.a} & & & & \| \text{ Step 1.a}\\ \langle \underline{c}_t, M_{\bar{Q}}\rangle & & & & \langle \overline{c}_t, M_{\bar{Q}}\rangle\\ t\to\infty \ \downarrow \text{ Dom Conv} & & & & t\to\infty \ \downarrow \text{ Dom Conv}\\ \langle f^*, M_{\bar{Q}}\rangle & & & & \langle f^*, M_{\bar{Q}}\rangle\\ \| \ f^*(x)=f(x),\ x\in\bar{Q} & & & & \| \ f^*(x)=f(x),\ x\in\bar{Q}\\ \langle f, M_{\bar{Q}}\rangle & & & & \langle f, M_{\bar{Q}}\rangle. \end{array}$$

Hence, for each $f \in C^{\mathbb{R}}(R)$, $\lim_{t\to\infty}\langle f, M_{Q_t}\rangle = \langle f, M_{\bar{Q}}\rangle$.

Step 2: $\displaystyle\lim_{t\to\infty}\frac{\mathrm{Vol}_m(P_t\backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = \lim_{t\to\infty}\frac{\mathrm{Vol}_m(Q_t\backslash P_t)}{\mathrm{Vol}_m(Q_t)} = 0$ and $\displaystyle\lim_{t\to\infty}\frac{\mathrm{Vol}_m(P_t)}{\mathrm{Vol}_m(Q_t)} = 1$.

We show that $\lim_{t\to\infty}\frac{\mathrm{Vol}_m(P_t\backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = 0$, being the proof for $Q_t\backslash P_t$ analogous. By Lemma 5.2, there are polytopes $P_1^1,\ldots,P_1^k$, $k \geq 1$, such that $\{P_1^1,\ldots,P_1^k, Q_1\cap P_1\}$ is a dissection of P_1. Hence, $P_1\backslash Q_1 \subsetneq \cup_{j=1}^k P_1^j = \mathrm{co}(P_1\backslash Q_1)$. Note that $\mathrm{co}(P_1\backslash Q_1)$ coincides with the closure of $P_1\backslash Q_1$. Now, for each $t \in \mathbb{N}$ and each $j \in \{1,\ldots,k\}$, let $P_t^j := P_1^j \cap BH^{e^1}_{\alpha_t}$. Then, for each

$t \in \mathbb{N}$, $\{P_t^1, \ldots, P_t^k, Q_t \cap P_t\}$ is a dissection of P_t and $P_t \backslash Q_t \subsetneq \cup_{j=1}^k P_t^j = \text{co}(P_t \backslash Q_t)$. Hence, $\text{Vol}_m(P_t \backslash Q_t) \leq \sum_{j=1}^k \text{Vol}_m(P_t^j)$ (actually, they are equal).

Now, since $\text{Vol}_m(Q_t) = \text{Vol}_{m-1}(\bar{P})(\alpha_t - \bar{\alpha})$, $\text{Vol}_m(Q_t) = O(\alpha_t - \bar{\alpha})$, *i.e.*, $\text{Vol}_m(Q_t)$ is a linear function of $(\alpha_t - \bar{\alpha})$.[1] Let $j \in \{1, \ldots, k\}$, since $\bar{Q} = \bar{P}$, $P_1^j \cap BH_{\bar{\alpha}}^{e^1}$ is contained in some face of $\bar{P}$, *i.e.*, it is in the boundary of $\bar{P}$. Hence, $P_1^j \cap BH_{\bar{\alpha}}^{e^1}$ is, at most, an $(m-2)$-polytope. Hence, if for each $t \in N$, P_t^j is an m-polytope, we have that, in the limit, there is, at least, a 2-dimensional degeneracy. Hence, $\text{Vol}_m(P_t^j) = o((\alpha_t - \bar{\alpha})^2)$.[2] Now, since the number of polytopes in the dissection is finite, we have $\text{Vol}_m(P_t \backslash Q_t) = o((\alpha_t - \bar{\alpha})^2)$.

Finally, $\lim\limits_{t\to\infty} \dfrac{\text{Vol}_m(P_t \backslash Q_t)}{\text{Vol}_m(Q_t)} = \lim\limits_{t\to\infty} \dfrac{o((\alpha_t - \bar{\alpha})^2)}{O(\alpha_t - \bar{\alpha})} = 0$.

We turn now to $\frac{\text{Vol}_m(P_t)}{\text{Vol}_m(Q_t)}$. Since $P_t = Q_t \backslash (Q_t \backslash P_t) \cup (P_t \backslash Q_t)$, and $Q_t \backslash (Q_t \backslash P_t)$ and $P_t \backslash Q_t$ are disjoint sets, then $\text{Vol}_m(P_t) = \text{Vol}_m(Q_t) - \text{Vol}_m(Q_t \backslash P_t) + \text{Vol}_m(P_t \backslash Q_t)$. Hence,

$$\lim_{t\to\infty} \frac{\text{Vol}_m(P_t)}{\text{Vol}_m(Q_t)} = \lim_{t\to\infty} \Big(1 - \frac{\text{Vol}_m(Q_t \backslash P_t)}{\text{Vol}_m(Q_t)} + \frac{\text{Vol}_m(P_t \backslash Q_t)}{\text{Vol}_m(Q_t)}\Big) = 1.$$

Step 3: $M_{P_t} \xrightarrow{w^*} M_{\bar{P}}$.

$$\begin{aligned} \int f\, dM_{P_t} &= \int \frac{f \mathcal{X}_{P_t}}{\text{Vol}_m(P_t)} dM_\lambda^m \\ &= \frac{1}{\text{Vol}_m(P_t)} \int f(\mathcal{X}_{Q_t} - \mathcal{X}_{Q_t \backslash P_t} + \mathcal{X}_{P_t \backslash Q_t})\, dM_\lambda^m \\ &= \int \frac{f \mathcal{X}_{Q_t}}{\text{Vol}_m(P_t)}\, dM_\lambda^m - \int \frac{f \mathcal{X}_{Q_t \backslash P_t}}{\text{Vol}_m(P_t)}\, dM_\lambda^m + \int \frac{f \mathcal{X}_{P_t \backslash Q_t}}{\text{Vol}_m(P_t)}\, dM_\lambda^m. \end{aligned}$$

We want to show that both the second and the third addend tend to 0. We can assume that $\text{Vol}_m(Q_t \backslash P_t) \neq 0$, otherwise, $\int f \mathcal{X}_{Q_t \backslash P_t}\, dM_\lambda^m = 0$ and we are done with the corresponding addend. Similarly, we assume that $\text{Vol}_m(P_t \backslash Q_t) \neq 0$. Now,

$$\int f\, dM_{P_t} = A_1 - A_2 + A_3,$$

where,

$$\begin{aligned} A_1 &= \frac{\text{Vol}_m(Q_t)}{\text{Vol}_m(P_t)} \int \frac{f \mathcal{X}_{Q_t}}{\text{Vol}_m(Q_t)} dM_\lambda^m = \frac{\text{Vol}_m(Q_t)}{\text{Vol}_m(P_t)} \int f\, dM_{Q_t}, \\ A_2 &= \frac{\text{Vol}_m(Q_t \backslash P_t)}{\text{Vol}_m(P_t)} \int \frac{f \mathcal{X}_{Q_t \backslash P_t}}{\text{Vol}_m(Q_t \backslash P_t)} dM_\lambda^m = \frac{\text{Vol}_m(Q_t \backslash P_t)}{\text{Vol}_m(P_t)} \int f dM_{Q_t \backslash P_t}, \text{ and} \\ A_3 &= \frac{\text{Vol}_m(P_t \backslash Q_t)}{\text{Vol}_m(P_t)} \int \frac{f \mathcal{X}_{P_t \backslash Q_t}}{\text{Vol}_m(P_t \backslash Q_t)} dM_\lambda^m = \frac{\text{Vol}_m(P_t \backslash Q_t)}{\text{Vol}_m(P_t)} \int f\, dM_{P_t \backslash Q_t}. \end{aligned}$$

[1] We say that $f(t) = O(g(t))$ if there are $c_1, c_2 > 0$ and $t' \in N$ such that, for each $t > t'$, $c_1|g(t)| \leq |f(t)| \leq c_2|g(t)|$. The notation $f(t) = o(g(t))$ means that there is $c > 0$ and $t' \in N$ such that, for each $t > t'$, $|f(t)| \leq c|g(t)|$.

[2] Just because, roughly speaking, the volume of a polytope is a linear function of its "length" in each dimension.

Since $\int f dM_{Q_t \backslash P_t} \leq \max_{x \in R} f(x)$ and $\int f dM_{P_t \backslash Q_t} \leq \max_{x \in R} f(x)$, then, by Step 2, both A_2 and A_3 tend to 0. We move now to A_1. By Step 2, $\lim_{t \to \infty} \frac{\mathrm{Vol}_m(Q_t)}{\mathrm{Vol}_m(P_t)} = 1$. Since, by Step 1, $\lim_{t\to\infty} \int f\, dM_{Q_t} = \int f\, dM_{\bar{P}}$, we have $\lim_{t\to\infty} \int f\, dM_{P_t} = \int f\, dM_{\bar{P}}$.

Case 2: $\bar{P}$ is an $(m-l)$-polytope, $l > 1$.

We have multiple degeneracy. To study this case, new auxiliary polytopes Q_t and $\bar{Q}$ have to be defined, but the idea of the proof is the same. Assume that the degeneracies are in the first l components. Then, there exist $a_1, \ldots, a_l \in \mathbb{R}$ such that for each $x \in \bar{P}$, $x_1 = a_1, \ldots, x_l = a_l$. Let $\{F_1, \ldots, F_k\} \subseteq \mathcal{F}(P)$ be the set of the faces of P containing $\bar{P}$; since $\bar{P}$ is an $(m-l)$-polytope, $k \geq 2$. For each $j \in \{1, \ldots, k\}$, let H^j be the hyperplane containing F_j and assume, without loss of generality, that $P \subsetneq BH^j$. For each $i \in \{1, \ldots, m\}$, let $e^i \in \mathbb{R}^m$ be the i-th canonical vector. Let Q be the polyhedron defined as follows,

$$Q := \left\{ y \in \mathbb{R}^m \ : \begin{array}{l} \text{for each } j \in \{1, \ldots, k\},\ y \in BH^j \text{ and} \\ y = x + \sum_{i=1}^{l} \gamma_i e^i,\ \text{ where } x \in \bar{P} \text{ and, for each } i \in \{1, \ldots, l\}, \gamma_i > 0 \end{array} \right\}.$$

Now, for each $t \in \mathbb{N}$, we define the auxiliary polytopes $Q_t := Q \cap BH^{e^1}_{\alpha_t}$. Also, let $\bar{Q} := Q \cap BH^{e^1}_{\bar{\alpha}}$ (see Figure 5.5). Note that, by definition, $\bar{Q} = \bar{P}$. Since $Q_t \cap H^{e^1}_{\bar{\alpha}} = \bar{Q}$ is bounded, applying Lemma 5.3, we have that Q_t is bounded. Hence, each Q_t is indeed a polytope.

Now, all the steps in Case 1 can be adapted for the Q_t's. Only some minor (and natural) changes have to be made. Next, we go through these steps, stressing where modifications are needed.

Step 1: $M_{Q_t} \xrightarrow{w^*} M_{\bar{Q}}$.

Step 1.a: Let $x_L := (x_1, \ldots, x_l)$ and $x_{\bar{L}} := (x_{l+1}, \ldots, x_m)$. Let $f \in C^{\mathbb{R}}(R)$ be such that there exists $c : [-r, r]^{m-l} \to \mathbb{R}$ with the following property: $f(x_L, x_{\bar{L}}) = c(x_L)$. Also, for each $x \in \bar{Q}$ and each $t \in \mathbb{N}$, we define the l-polytope $Q_t(x) := \{y \in Q_t \ : \ y_{-L} = x_{-L}\}$ (Figure 5.6). Again, if $x \neq x'$, then $Q_t(x) \cap Q_t(x') = \emptyset$ and $\mathrm{Vol}_l(Q_t(x)) = \mathrm{Vol}_l(Q_t(x')) = \frac{\mathrm{Vol}_m(Q_t)}{\mathrm{Vol}_{m-l}(\bar{Q})}$. Moreover, for each $x \in \bar{Q}$, f is constant in $Q_t(x)$. The rest is analogous to Case 1.

Step 1.b: Let $f \in C^{\mathbb{R}}(R)$. Let $\hat{x} \in \bar{Q}$. For each $t \in \mathbb{N}$, we define the compact set $K_t := \{z \in \mathbb{R}^l \ : \ z = y_L, \text{ where } (y_L, y_{\bar{L}}) = y \in Q_t(\hat{x})\}$, *i.e.*, K_t is the projection of $Q_t(x)$ into $\mathbb{R}^l$. Note that the definition of K_t is independent of the selected $\hat{x} \in \bar{Q}$. Define the three auxiliary functions

$$\begin{array}{rcl} f^*(x_L, x_{\bar{L}}) & := & f(a_1, \ldots, a_l, x_{\bar{L}}), \\ \overline{c}_t(x_L, x_{\bar{L}}) & := & \max_{z \in K_t} f(z, x_{\bar{L}}), \text{ and} \\ \underline{c}_t(x_L, x_{\bar{L}}) & := & \min_{z \in K_t} f(z, x_{\bar{L}}). \end{array}$$

With these definitions Corollary 5.1 still applies. The rest is analogous to Case 1.

Step 2: $\lim\limits_{t \to \infty} \dfrac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = \lim\limits_{t \to \infty} \dfrac{\mathrm{Vol}_m(Q_t \backslash P_t)}{\mathrm{Vol}_m(Q_t)} = 0$ and $\lim\limits_{t \to \infty} \dfrac{\mathrm{Vol}_m(P_t)}{\mathrm{Vol}_m(Q_t)} = 1.$

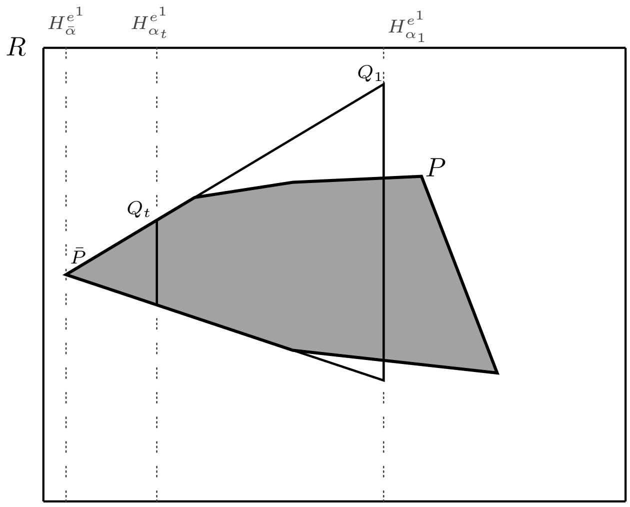

Figure 5.5: Defining the polytopes Q_t (P is a 2-polytope and $\bar{P}$ a 0-polytope)

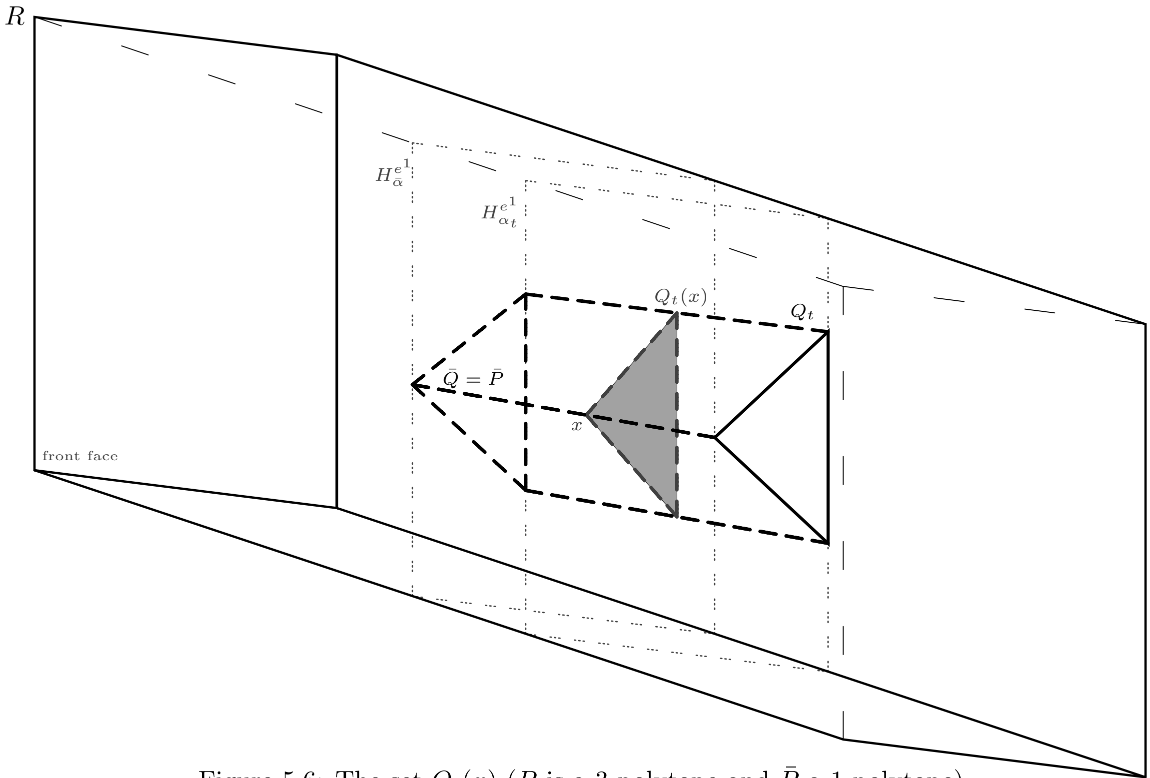

Figure 5.6: The set $Q_t(x)$ (P is a 3-polytope and $\bar{P}$ a 1-polytope)

Now, $P_1^j \cap BH_{\bar{\alpha}}^{e^1}$ is, at most, an $(m-(l+1))$-polytope; $\mathrm{Vol}_m(P_t \backslash Q_t) = o((\alpha_t - \bar{\alpha})^{l+1})$; and $\mathrm{Vol}_m(Q_t) = O(\alpha_t - \bar{\alpha})^l$. The rest is analogous to Case 1.

Step 3: $M_{P_t} \xrightarrow{w^*} M_{\bar{P}}$. Analogous to Case 1. □

Remark. Now, if we go back to Example 5.1, the one we used to illustrate the problem with continuity, we can wonder why the scheme of the proof above does not apply. The reason is that the vector u we used to define the sequence of polytopes was fixed for the whole proof and, in Example 5.1, we would need an infinite number of different vectors to construct the corresponding sequence of polytopes.

So far, measures M_P have belonged to $\mathcal{M}(R)$. These measures can be extended to (Borel) measures in $\mathbb{R}^m$ by letting the measure of each $A \subseteq \mathbb{R}^m$ be $M_P(A \cap R)$. With a slight abuse of notation, we also denote these extensions by M_P.

Corollary 5.2. *Let $P \subsetneq \mathbb{R}^m$ be an $(m-l)$-polytope, $0 \leq l \leq m$. Let $u \in \mathbb{R}^m$. Let $\bar{\alpha} \in \mathbb{R}$ and let $\{\alpha_t\}$ be a sequence in $[\bar{\alpha}, \infty)$ with limit $\bar{\alpha}$. Let $P_t := P \cap BH_{\alpha_t}^u$ and $\bar{P} := P \cap BH_{\bar{\alpha}}^u$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Then, $\lim_{t\to\infty} \int f\, dM_{P_t} = \int f\, dM_{\bar{P}}$.*

Proof. We distinguish two cases:

Case 1: $l = 0$. Let $r > 0$ be such that P is contained in the interior of $R = [-r, r]^m$. Let $f^R : R \to \mathbb{R}$ be the restriction of f to R. Then,

$$\int f\, dM_{P_t} = \int f^R\, dM_{P_t} \overset{\text{Th. 5.3}}{\longrightarrow} \int f^R\, dM_{\bar{P}} = \int f\, dM_{\bar{P}}.$$

Case 2: $l > 0$. There exist $a_1, \ldots, a_l \in \mathbb{R}$ such that $x \in P$ if and only if $x_1 = a_1, \ldots, x_l = a_l$. Let $R = a_1 \times \ldots \times a_l \times [-r, r]^{m-l}$ be such that P belongs to its interior. Now, everything in Theorem 5.3 can be adapted for the M_{P_t}'s, $M_{\bar{P}}$ and this new R. Hence, the same argument of Case 1 leads to the result. □

5.3.3 Back to Game Theory

Now, we turn back to the game theoretical framework and prove the results stated in Section 5.3.1.

Proof of Proposition 5.3. We distinguish two cases in this proof. In Case 1, only the value of a fixed coalition S varies throughout the sequence $\{v^t\}$. Next, in Case 2, all the coalitions but coalition N can change.

Case 1: There is $S \subsetneq N$ such that for each $T \neq S$ and each $t \in \mathbb{N}$, $\bar{v}(T) = v^t(T)$.

First, we define a new game whose core contains the cores of all v^t's and of $\bar{v}$. Let v be defined, for each $S \subseteq N$, by $v(S) := \min\{\bar{v}(S), \{v^t(S) : t \in \mathbb{N}\}\}$. Game v is well-defined because the set $\bar{v}(S) \cup \{v^t(S) : t \in \mathbb{N}\}$ is compact for each S (otherwise the sequence $\{v^t\}$ would not be convergent). Let $P := C(v)$, $P_t := C(v^t)$ and $\bar{P} := C(\bar{v})$. Clearly, by definition of v, P contains polytopes P_t and $\bar{P}$. Let $H_{v^t(S)}^S$ be the hyperplane of equation $\sum_{i\in S} x_i = v^t(S)$. Let

$e^S \in \mathbb{R}^n$ be such that $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ if $i \notin S$. Then, e^S is the normal vector of $H^S_{v^t(S)}$. The sequence $\{v^t(S)\}$ has limit $\bar{v}(S)$. Now, using the notation of Section 5.3.2, let $P_t = P \cap BH^{e^S}_{v^t(S)}$, and $\bar{P} = P \cap BH^{e^S}_{\bar{v}(S)}$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $h(x) := x$. Then, $\mu(v^t) = \int h \, dM_{P_t}$ and $\mu(\bar{v}) = \int h \, dM_{\bar{P}}$. For each $i \in \{1, \ldots, n\}$, the function $h_i : \mathbb{R}^n \to \mathbb{R}$ defined by $h_i(x) = (h(x))_i = x_i$ is continuous. Hence, by applying Corollary 5.2 to each h_i, we have $\lim_{t\to\infty} \mu(v^t) = \mu(\bar{v})$.

Case 2: For each $t \in \mathbb{N}$, $\bar{v}(N) = v^t(N)$. Only the value for the grand coalition is fixed now.

Assume that there are coalitions $S, \bar{S} \subsetneq N$ such that, for each $T \neq S, \bar{S}$ and each $t \in \mathbb{N}$, $\bar{v}(T) = v^t(T)$. Let $BH^{e^S}_{v^t(S)}$ and $BH^{e^{\bar{S}}}_{v^t(\bar{S})}$ be the corresponding halfspaces. The key now, is that we can change the order in which we make the intersections with the halfspaces, *i.e.*, the same polytope arises if we intersect first with a (BH^{e^S})-like halfspace and then with a $(BH^{e^{\bar{S}}})$-like one, or we intersect the other way around. Then, we can see the sequence of polytopes as a sequence with two indices. Polytope $P_{i,j}$ is obtained by intersecting P with the i^{th} (BH^{e^S})-like halfspace and the j^{th} $(BH^{e^{\bar{S}}})$-like halfspace. Polytope $\bar{P}$ is the "limit" of the polytopes $P_{i,j}$ when both i and j go to infinity. Hence, using Case 1 first for index i and second for index j, we prove the convergence of the centers of gravity. Intuitively, we carry out all the intersections with one halfspace, and then we do so with the other one (limits are inter-changeable). If there are more than two different types of halfspaces (more coalitions with non-fixed values throughout the sequence), the same argument works because there is always a finite number of such types. □

Proof of Theorem 5.2. Now we consider the general case, when the worths of all coalitions can vary along the sequence $\{v^t\}$.

Let $\varepsilon_t := \bar{v}(N) - v^t(N)$ (note that ε_t can be either positive or negative). For each $t \in \mathbb{N}$, let $\hat{v}^t$ be the auxiliary game such that for each $S \subseteq N$, $\hat{v}^t(S) = v^t(S) + \frac{|S|}{n}\varepsilon_t$. Now, for each $t \in \mathbb{N}$, we have (i) $\hat{v}^t(N) = \bar{v}(N)$, and (ii) $\mathrm{C}(\hat{v}^t)$ is obtained by translation of $C(v^t)$ by the vector $\frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t)$. Since $\{\varepsilon_t\}$ tends to 0, $\lim_{t\to\infty} v^t = \bar{v}$ implies that $\lim_{t\to\infty} \hat{v}^t = \bar{v}$. Hence, by Proposition 5.3, $\lim_{t\to\infty} \mu(\hat{v}^t) = \mu(\bar{v})$. Since the core-center is translation invariant, $\mu(v^t) = \mu(\hat{v}^t) - \frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t)$. Now, using again that $\{\varepsilon_t\} \to 0$ we have

$$\lim_{t\to\infty} \mu(v^t) = \lim_{t\to\infty} \left(\mu(\hat{v}^t) - \frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t) \right) = \lim_{t\to\infty} \mu(\hat{v}^t) - \lim_{t\to\infty} \frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t) = \mu(\bar{v}),$$

and the Theorem is proved. □

5.4 Concluding Remarks

In this Chapter we introduced a new allocation rule for the class of balanced games: the core-center. Then, we provided a detailed discussion of the axiomatic properties of the core-center. Special emphasis was made in two of them. First, we showed that the core-center does not satisfy some of the standard monotonicity properties; even though, we showed that it satisfies weak coalitional monotonicity and we established a certain parallelism with the nucleolus. Second, we deeply discussed the continuity property of the core-center. This property is finally derived from a more general result. Indeed, using Corollary 5.2, it can be shown that any allocation rule defined as the integral, with regard to the Lebesgue measure, of a continuous function (not necessarily the identity) is also continuous.

5.A Appendix (Classical Results in Measure Theory and Functional Analysis)

This appendix contains the statements of the main classic results used along the chapter. For a deeper discussion refer to Rudin (1966) and/or Conway (1990) and/or Billingsley (1968).

Let X be any set and let Ω be a σ-algebra of subsets of X. Hence, (X, Ω) is a *measurable space*. If X is a locally compact set and Ω denotes the smallest σ-algebra of subsets of X that contains the open sets, then sets in Ω are called Borel sets. Let $(X, \mathcal{B})$ denote this particular measurable space and let $M(X)$ be the set of complex-valued regular Borel measures on X.

5.A.1 The Riesz Representation Theorem

Let X be a locally compact set. Let $C_0(X)$ be the set of all continuous functions $f : X \to \mathbb{C}$ such that for each $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. Let $C_0(X)^*$ be the dual of $C_0(X)$. Next theorem shows that the dual of $C_0(X)$ is $M(X)$.

Theorem 5.4 (Riesz Representation Theorem[3])**.** *Let X be a locally compact space and $\mathcal{M} \in M(X)$. Let $F_{\mathcal{M}} : C_0(X) \to \mathbb{C}$ be such that $F_{\mathcal{M}}(f) := \int f \, d\mathcal{M}$. Then, $F_{\mathcal{M}} \in C_0(X)^*$ and the map $\eta \mapsto F_\eta$ is an isometric isomorphism of $M(X)$ onto $C_0(X)^*$.*

In this chapter $X = R$, a compact space. Hence, $C_0(R) = C(R)$, and the Riesz Representation Theorem can be used to conclude that $C(R)^* = M(R)$.

5.A.2 The Weak* Topology

Seminorms and generated topologies

Let X be a normed space. A natural metric can be defined considering the distance $d(x, y) := ||x - y||$. Let τ be the topology induced by this metric. This topology is usually referred to as the topology induced by the norm.

If X is a vector space over $\mathbb{K}$, a *seminorm* is a function $q : X \to [0, \infty)$ having the following properties,

(i) for each pair $x, y \in X$, $q(x + y) \leq q(x) + q(y)$,

[3] There are many similar theorems in literature which have been named "Riesz Representation Theorem", the one we present here has been extracted from Conway (1990).

(ii) for each α in $\mathbb{K}$, and each x in X, $q(\alpha x) = |\alpha| q(x)$.

A norm is a seminorm such that $q(x) = 0$ implies $x = 0$. Seminorms can be used to generate topologies, indeed, the weak* topology is defined in this way.

Let X be a vector space and $\{q_\alpha, \alpha \in I\}$ be a family of seminorms. The smallest translation invariant topology that makes all the q_α continuous, is the *topology generated by* $\{q_\alpha, \alpha \in I\}$. It can be proved that the collection $\{\bigcap_{k=1}^{\infty} q_{\alpha_k}^{-1}([0, \varepsilon_k)) \, : \, \varepsilon_1, \ldots, \varepsilon_n \in (0, \infty)\, , \, \alpha_1, \ldots, \alpha_n \in I\, , \, n \in \mathbb{N}\}$, is a local base in the origin for this topology (now, by the translation invariance property a base can be obtained). Under this topology, $V \subseteq X$ is open if and only if for each $x \in V$, there exist $\alpha_1, \ldots, \alpha_n \in I$, and $\varepsilon_1, \ldots, \varepsilon_n \in (0, \infty)$, such that $x + \bigcap_{k=1}^{\infty} q_{\alpha_k}^{-1}([0, \varepsilon_k)) \subseteq V$. Moreover, $\lim_{n\to\infty}\{x_n\} = 0$ if and only if for each $\varepsilon > 0$, and each $\alpha \in I$, there exists N_{α_ε} such that for each $n > N_{\alpha_\varepsilon}$, $x_n \in q_\alpha^{-1}([0, \varepsilon))$. Then, $\lim_{n\to\infty}\{x_n\} = 0$ if and only if for each $\alpha \in I$, $\lim_{n\to\infty} q_\alpha(x_n) = 0$. Since this topology is translation invariant, then $\lim_{n\to\infty}\{x_n\} = x$ if and only if $\lim_{n\to\infty}\{x_n - x\} = 0$.

Weak* topology

Next, the weak* topology, henceforth w^*, in introduced. Let X be a normed space, X^* its dual and τ and τ^* the topologies induced by the corresponding norms. For each $x \in X$, let q_x be the function in X^* such that for each $\varphi \in X^*$, $q_x(\varphi) = |\varphi(x)| = |\langle \varphi, x \rangle|$. Function q_x is a seminorm in X^*. The family of seminorms $\{q_x \, : \, x \in X\}$, generates a topology in X^*, w^*. Recall that w^* is, therefore, the smallest topology under which all the seminorms are continuous. It is also true that $w^* \leq \tau^*$; the equality holds if and only if $\dim(X^*) < \infty$.

Borel measures and the weak* topology

In this chapter $X = C(R)$ and $X^* = M(R)$. Consider the topology w^* in $M(R)$. The sequence of measures $\{\mathcal{M}_n\}$ converges to the measure $\mathcal{M}$ if and only if $\{\mathcal{M}_n - \mathcal{M}\}$ converges to the measure 0, *i.e.*, for each $f \in C(R)$, $\lim_{n\to\infty} q_f(\mathcal{M}_n - \mathcal{M}) = 0$. Then, the sequence converges if and only if for each $f \in C(R)$, $\lim_{n\to\infty} \int f d\mathcal{M}_n = \int f d\mathcal{M}$. The latter formulation is the appropriate for this chapter.

5.A.3 Lebesgue's Dominated Convergence Theorem

Let X be a measurable space, Y a topological space and f a mapping of X into Y. Then, f is a *measurable function* if for each open set V in Y, $f^{-1}(V)$ is a measurable set in X. Let $\mathcal{M} \in M(X)$, and let $L^1(\mathcal{M})$ be the collection of all complex measurable functions f on X for which $\int |f| d\mathcal{M} < \infty$. Functions in $L^1(\mathcal{M})$ are called *Lebesgue integrable functions with respect to* $\mathcal{M}$.

Theorem 5.5 (Lebesgue's Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of complex measurable functions on X such that for each $x \in X$, $f(x) := \lim_{n\to\infty} f_n(x)$ exists. If there is a* *function* $g \in L^1(\mathcal{M})$ *such that for each* $n \in \mathbb{N}$, *and all* $x \in X$, $|f_n(x)| \leq g(x)$, *then* $f \in L^1(\mathcal{M})$, $\lim_{n\to\infty} \int_X |f_n - f| \, d\mathcal{M} = 0$, *and* $\lim_{n\to\infty} \int_X f_n \, d\mathcal{M} = \int_X f \, d\mathcal{M}$.

In this chapter we apply this theorem to the sequences of functions $\{\underline{c}_t\}$ and $\{\overline{c}_t\}$ defined in Step 1.a of the proof of Theorem 5.3. It is easy to verify that they are, indeed, under the assumptions needed for the theorem. They are continuous and, since we work with Borel measures, they are also measurable. It is also easy to define a function dominating them because we work in a compact set. Finally, the punctual convergence in R is also fulfilled.

Bibliography

Aumann, R. J. and M. Maschler (1964): "The Bargaining Set for Cooperative Games," in *Advances in Game Theory*, ed. by M. Dresher, L. S. Shapley, and A. Tucker, Princeton University Press, vol. 52 of *Annals of Mathematical Studies*, 443–476. (Quoted in pp. 62)

Billingsley, P. (1968): *Convergence of Probability Measures*, New York: Wiley. (Quoted in pp. 81)

Conway, J. B. (1990): *A Course in Functional Analysis*, Springer-Verlag. (Quoted in pp. 81)

Davis, M. and M. Maschler (1965): "The Kernel of a Cooperative Game," *Naval Research Logistics Quarterly*, 12, 223–259. (Quoted in pp. 62)

Dutta, B. and D. Ray (1989): "A Concept of Egalitarianism under Participation Constraints," *Econometrica*, 57, 615–635. (Quoted in pp. 65)

Gautier, S. and R. Morchadi (1992): "A Selection of Convex-Compact-Valued Multi-Functions with Remarkable Properties: the Steiner Selection," *Numerical Functional Analysis and Optimization*, 13, 513–522. (Quoted in pp. 62)

Gillies, D. B. (1953): "Some Theorems on n-Person Games," Ph.D. thesis, Princeton. (Quoted in pp. 62, 64)

González-Díaz, J., P. Borm, R. Hendrickx, and M. Quant (2005): "A Geometric Characterisation of the Compromise Value," *Mathematical Methods of Operations Research*, 61. (Quoted in pp. 63)

Housman and Clark (1998): "Core and Monotonic Allocation Methods," *International Journal of Game Theory*, 27, 611–616. (Quoted in pp. 67)

Maschler, M., B. Peleg, and L. S. Shapley (1979): "Geometric Properties of the Kernel, Nucleolus, and Related Solution Concepts," *Mathematics of Operations Research*, 4, 303–338. (Quoted in pp. 62, 67)

Michael, E. (1956): "Continuous Selections. I," *The Annals of Mathematics*, 63, 361–382. (Quoted in pp. 62, 70)

Rudin, W. (1966): *Real and Complex Analysis*, McGraw-Hill. (Quoted in pp. 81)

Schmeidler, D. (1969): "The Nucleolus of a Characteristic Function Game," *SIAM Journal on Applied Mathematics*, 17, 1163–1170. (Quoted in pp. 62, 69)

Shapley, L. S. (1953): "A Value for n-Person Games," in *Contributions to the theory of games II*, ed. by H. Kuhn and A. Tucker, Princeton: Princeton University Press, vol. 28 of *Annals of Mathematics Studies*. (Quoted in pp. 62)

Tijs, S. (1981): "Bounds for the Core and the τ-Value," in *Game theory and mathematical economics*, ed. by O. Moeschlin and D. Pallaschke, Amsterdam: North Holland Publishing Company, 123–132. (Quoted in pp. 62)

von Neumann, J. and O. Morgenstern (1944): *Theory of Games and Economic Behavior*, Princeton: Princeton University Press. (Quoted in pp. 62)

Young, H. (1985): "Monotonic Solutions of Cooperatives Games," *International Journal of Game Theory*, 14, 65–72. (Quoted in pp. 67)

Zhou, L. (1991): "A Weak Monotonicity Property for the Nucleolus," *International Journal of Game Theory*, 19, 407–411. (Quoted in pp. 69)

Chapter 6

A Characterization of the Core-Center

Contents

6.1 Introduction

In González-Díaz and Sánchez-Rodríguez (2003), the core-center, a new allocation rule for the class of balanced games is introduced. That paper contains a first approach both to the study of the axiomatic properties of the core-center and to the search for an axiomatic characterization. This Chapter focuses in the latter. We formally develope and refine the characterization provided there.

The key property for the characterization is a weighted additivity, based on a principle of fairness with regard to the core, that we call fair additivity. This property, along with other standard properties in game theory leads to the axiomatic characterization of the core-center. It has a certain parallelism with the characterization of the Shapley value based on the additivity property. First, we prove the result for games with a simplicial core, which play the role of the unanimity games in Shapley's characterization. Second, we prove the result for arbitrary games by means of simplicial dissections of their cores.

In the fair additivity property the weights depend on the volumes of the cores. There are antecedents in game theory that look for this kind of fairness. One of the solutions for two person bargaining problems which depends on the whole feasible set is the Equal Area Solution. Anbarci and Bigelow (1994) interpreted equal area as equal concessions. Later Calvo and Peters (2000) looked at the underlying dynamic process.

The structure of this Chapter is as follows. In Section 6.2 we introduce the preliminary game theoretical concepts along with the definition of the core-center. In Section 6.3 we introduce and discuss the fair additivity property. In Section 6.4 we state and prove the characterization of the core-center. Finally, in the Appendix we provide rigorous proofs of some technical statements which have been skipped in the text; moreover, it also includes formal definitions and properties of some geometric concepts used along the Chapter.

6.2 Game Theory Background

A transferable utility or TU game is a pair (N, v), where $N := \{1, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to each coalition $S \subseteq N$ a payoff $v(S)$. By convention, $v(\emptyset) = 0$. Since each game assigns a real value to each nonempty subset of N, it corresponds with a vector in $\mathbb{R}^{2^n-1}$. Let $|S|$ be the number of elements of coalition S. Saving notation, when no ambiguity arises, we use i to denote $\{i\}$. Given a game (N, v), the *imputation set* is defined by $I(N, v) := \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)$ and, for each $i \in N$, $x_i \geq v(i)\}$.

Let $x \in \mathbb{R}^n$ be an allocation. Then, x is *efficient* if $\sum_{i=1}^n x_i = v(N)$. A game (N, v) is *superadditive* if for each $S, T \subseteq N$ such that $T \cap S = \emptyset$, we have $v(S \cup T) \geq v(S) + v(T)$. We restrict our attention to efficient allocations. Within this framework, it is widely accepted that superadditivity is quite a reasonable requirement for the game. This is because we expect the grand coalition to form, and then, share the amount $v(N)$ among the players; if the game was not superadditive this expectation might be unfounded. Hence, in the present Chapter we restrict to the class of superadditive TU games, denoted by G (G^n denotes the superadditive games with n players).

An *allocation rule* is a function which, given a game (N, v), selects an allocation in $\mathbb{R}^n$, *i.e.*,

$$\begin{array}{rccl} \varphi: & \Omega \subseteq G^n & \longrightarrow & \mathbb{R}^n \\ & (N,v) & \longmapsto & \varphi(N,v). \end{array}$$

Next, we define some properties for allocation rules. Let $(N, v) \in G^n$ and let φ be an allocation rule: φ is *continuous* if the function $\varphi : \mathbb{R}^{2^n-1} \to \mathbb{R}^n$ is continuous; φ is *efficient* if it always select efficient allocations; φ is *translation invariant* if for each two games (N, v) and (N, w), and each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that for each $S \subseteq N$, $w(S) = v(S) + \sum_{i\in S} \alpha_i$, then $\varphi(N, w) = \varphi(N, v) + \alpha$.

Next, we define some properties regarding symmetry. Let $(N, v) \in G^n$ and let $i, j \in N$. Then, i and j are *symmetric* if for each $S \subseteq N\backslash\{i, j\}$, $v(S \cup i) - v(S) = v(S \cup j) - v(S)$; i and j are *quasi-symmetric* if for each $S \subseteq N\backslash\{i, j\}$, $v(S\cup i)-(v(S)+v(i)) = v(S\cup j)-(v(S)+v(j))$. Now, (N, v) is *symmetric* if for each pair $i, j \in N$, i and j are symmetric; (N, v) is *quasi-symmetric* if for each pair $i, j \in N$, i and j are quasi-symmetric or, equivalently, a game is quasi-symmetric if the corresponding 0-normalized game is *symmetric*. Note that, for a symmetric game, $v(S)$ depends only on the cardinality of S (this gives an idea of the strength of this property). Quasi-symmetric games are important in this Chapter because their cores are symmetric sets from the geometric point of view.

Finally, we define two symmetry properties for an allocation rule. Let φ be an allocation rule. We say φ satisfies *weak symmetry* if for each symmetric game (N, v) and each pair $i, j \in N$, $\varphi_i(N, v) = \varphi_j(N, v)$; φ satisfies *extended weak symmetry* if for each quasi-symmetric game (N, v) and each pair $i, j \in N$, $\varphi_i(N, v) - v(i) = \varphi_j(N, v) - v(j)$. The extended weak symmetry property says that if for each pair $i, j \in N$, their contribution to any coalition differs only in $v(i) - v(j)$, then, the difference in the payoffs is also $v(i) - v(j)$. This property, besides being a symmetry property (it implies weak symmetry) has some flavor to translation invariance; roughly speaking, it says that the allocation rule satisfies weak symmetry and, moreover, translation invariance within the class of quasi-symmetric games. Next Lemma illustrates this point.

Lemma 6.1. *Translation invariance + weak symmetry ⇒ extended weak symmetry.*

Proof. Let φ be an allocation rule satisfying both translation invariance and weak symmetry. Let (N, v) be a quasi-symmetric game and $\alpha = (-v(1), \ldots, -v(n))$. Now, let $(N, w) \in G^n$ be such that, for each $S \subseteq N$, $w(S) = v(S) + \sum_{i\in S} \alpha_i$. Then (N, w) is symmetric. Hence, by weak symmetry, for each pair $i, j \in N$, $\varphi_i(N, w) = \varphi_j(N, w)$. Now, by translation invariance, we have $\varphi_i(N, v) + \alpha_i = \varphi_j(N, v) + \alpha_j$. Since $\alpha_i = -v(i)$ and $\alpha_j = -v(j)$, the result is proved. □

6.2.1 The Core and its Relatives

We introduce now the notions of core (Gillies, 1953) and strong ε-core (Maschler et al., 1979); both of them are based on efficiency and stability. An allocation $x \in \mathbb{R}^n$ is *stable* if there is no coalition $S \subseteq N$ such that $\sum_{i \in S} x_i < v(S)$, analogously, for each $\varepsilon \in \mathbb{R}$, x is ε*-stable* if there is no coalition $S \subseteq N$ such that $\sum_{i \in S} x_i < v(S) - \varepsilon$.

The core of a game (N, v), $C(N, v)$, is the set of all efficient and stable allocations

$$C(N,v) := \{x \in \mathbb{R}^n \; : \; \sum_{i \in N} x_i = v(N) \text{ and, for each } S \subsetneq N, \sum_{i \in S} x_i \geq v(S)\}.$$

The class of games with nonempty core is the class of balanced games. Let $BG \subsetneq G$ be the class of superadditive balanced games ($BG^n \subsetneq G^n$ denotes the set of superadditive balanced games with n players).

Let $\varepsilon \in \mathbb{R}$. The strong ε-core of a game (N, v), $C_\varepsilon(N, v)$ is the set of all efficient and ε-stable allocations:

$$C_\varepsilon(N,v) := \{x \in \mathbb{R}^n \; : \; \sum_{i \in N} x_i = v(N) \text{ and, for each } S \subsetneq N, \sum_{i \in S} x_i \geq v(S) - \varepsilon \; \}.$$

By definition, if $\varepsilon = 0$, $C_0(N, v) \equiv C(N, v)$. The *least core* of (N, v), $\mathcal{L}C(N, v)$, is the intersection of all nonempty strong ε-cores. Equivalently, let $\varepsilon_0(N, v)$ be the smallest ε such that $C_\varepsilon(N, v) \neq \emptyset$, then $\mathcal{L}C(N, v) = C_{\varepsilon_0(N,v)}(N, v)$.[1]

Let $(N, v) \in G^n$. The family of "shifted" games (N, v_ε) is defined by:[2]

$$v_\varepsilon(S) := \begin{cases} v(S) - \varepsilon & \emptyset \subsetneq S \subsetneq N \\ v(S) & S = \emptyset \text{ or } S = N. \end{cases}$$

Finally, we introduce a last concept related to balanced games: a balanced game (N, v) is *exact* (Schmeidler, 1972) if for each $S \subseteq N$, there is $x \in C(N, v)$ such that $\sum_{i \in S} x_i = v(S)$. Moreover, let $(N, v) \in BG^n$ with core $C(N, v)$, then, there is a unique exact game $(N, \hat{v})$ such that $C(N, \hat{v}) = C(N, v)$; this game is the *exact envelope* of (N, v). Note that if we have two exact games with the same core, then they are the same game. From the point of view of stability, we can say that $(N, \hat{v})$ throws away the redundant information of (N, v).

6.2.2 Some Geometric Considerations

For the sake of clarity, and if it does not entail confusion, henceforth we denote a game (N, v) by v. We need to introduce some notation and make some considerations regarding the underlying geometry of a TU game. We denote the efficient hyperplane by H_v^N; hence, $H_v^N := \{x \in \mathbb{R}^n \;:$

[1] In Maschler et al. (1979) it is shown that $\varepsilon_0(N, v)$ exists and is unique.

[2] This concept has also been taken from Maschler et al. (1979)

$\sum_{i \in N} x_i = v(N)\}$. All the sets we consider in this Chapter are contained in H_v^N and hence, we develop all our framework in an $(n-1)$-dimensional euclidean space.

A *(convex) polytope* P is the convex hull of a finite set of points $V = \{x^1, \ldots, x^k\}$ in $\mathbb{R}^n$, equivalently, it is a bounded subset of $\mathbb{R}^n$ which can be expressed as the intersection of a finite number of halfspaces. The core of a game, when nonempty, is a convex polytope (it is the intersection of halfspaces in H_v^N). An m-polytope is a polytope that lies in an m-dimensional space but there is no $(m-1)$-dimensional space containing it. Let P be an m-polytope and let $m' \geq m$, then, $\mathrm{Vol}_{m'}(P)$ denotes the m'-dimensional volume of P. Let P be an m-polytope. Then, a set of polytopes $\{P_1, \ldots, P_k\}$ define a *dissection* of P if (i) $P = \bigcup_{l=1}^k P_l$ and (ii) for each pair $l, l' \in \{1, \ldots, k\}$, with $l \neq l'$, $\mathrm{Vol}_m(P_l \cap P_{l'}) = 0$.

Lemma 6.2. *Except for the least core, all nonempty strong ε-cores are $(n-1)$-polytopes. The least core is always an m-polytope with $m < n-1$.*

Proof. The statement in this lemma has been taken from Maschler et al. (1979). Hence, we do not prove it. Anyway, not being a completely straightforward result, it is quite intuitive. □

Whenever the core of a game in G^n is an $(n-1)$-polytope, we say it is a *full dimensional* core. Otherwise, it is *degenerate*. By definition, all the restrictions in the core of a game are as follows: let $S \subsetneq N$, $R_v^S := \{x \in \mathbb{R}^n : \sum_{i \in S} x_i \geq v(S)\}$. Let R_v^S be a restriction, then we say that R_v^S is a $|S|$-restriction. The 1-restrictions play a special role in this Chapter; we call them *elemental restrictions*. We say a restriction is *redundant* in the core if removing it does not change the core. Conversely, the restrictions which are not redundant ones are *active* restrictions. Let H_v^S be the hyperplane associated with the restriction R_v^S, *i.e.*, $x \in H_v^S$ if and only if $x \in H_v^N$ and $\sum_{i \in S} x_i = v(S)$. Note that, because of the efficiency condition, the hyperplanes H_v^S have dimension $n-2$.

Lemma 6.3. *Let $v \in G^n$ and let $\emptyset \subsetneq S \subsetneq N$. Then, the hyperplanes H_v^S and $H_v^{N \backslash S}$ are parallel in H_v^N.*

Proof. Let $v \in G^n$ and $\emptyset \subsetneq S \subsetneq N$. Then, $H_v^S := \{x \in \mathbb{R}^n : \sum_{i \in S} x_i = v(S)\}$. We claim that there is $k \in \mathbb{R}$ such that H_v^S can be expressed as $\sum_{i \in N \backslash S} x_i = k$. Once this claim is proved, the statement of the Lemma is immediately derived. Since we work in H_v^N, $\sum_{i \in S} x_i + \sum_{i \in N \backslash S} x_i = v(N)$. If we impose the restriction $\sum_{i \in S} x_i = v(S)$, we have $v(S) + \sum_{i \in N \backslash S} x_i = v(N)$. Hence, $\sum_{i \in N \backslash S} x_i = v(N) - v(S) = k$. □

6.2.3 The Core-Center

The core of a game is the set of all the stable and efficient allocations. Now, if we consider that all these allocations are equally reasonable, then it makes sense to think of the core as if it was endowed with a uniform distribution. The *core-center* summarizes the information of such a distribution of probability. Let $U(A)$ be the uniform distribution defined over the set A and $E(\mathbb{P})$ the expectation of the probability distribution $\mathbb{P}$.

Definition 6.1. *Let (N,v) be a balanced game with core $C(N,v)$. The* core-center *of (N,v), $\mu(N,v)$, is defined as follows:*

$$\mu(N,v) := E\left[U(C(N,v))\right].$$

6.3 Fair Additivity

We devote this Section to the motivation and definition of the *fair additivity* property. This property is crucial in the characterization of the core-center we obtain in Section 6.4. First, before introducing the fair additivity property, we define a general family of allocation rules, that we call $\mathcal{T}$-solutions. Then, we say that an allocation rule satisfies the fair additivity property if it belongs to a special subfamily of $\mathcal{T}$-solutions.

6.3.1 $\mathcal{T}$-Solutions and $\mathcal{RT}$-Solutions

We need to introduce some concepts before formally defining what a $\mathcal{T}$-solution is. Let $v \in G^n$, $\emptyset \subsetneq T \subsetneq N$, and $k \geq v(T)$. We use constant k to define two games: $\overline{v}$, a good game for coalition T, and $\underline{v}$ a good game for coalition $N\backslash T$. Suppose that, because of some change in the situation underlying our TU game, coalition T alone can obtain k instead of $v(T)$. We define $\overline{v}$ as the game obtained when introducing this change in v:

$$\overline{v}(S) = \begin{cases} \max\{v(S), v(S\backslash T) + k\} & T \subseteq S \\ v(S) & \text{otherwise.} \end{cases}$$

We define $\bar{v}(S)$ as $\max\{v(S), v(S\backslash T) + k\}$ to ensure that $\overline{v}$ is a superadditive game. Superadditivity also implies that $v(T) \leq v(N) - v(N\backslash T)$. Hence, if we want game $\overline{v}$ to remain in the class of superadditive games, k must belong to the interval $[v(T), v(N) - v(N\backslash T)]$.

So, for value k, we have naturally defined a game $\overline{v}$ in which coalition T has improved with respect to v. Now, for this constant k, we define a game $\underline{v}$ in which coalition T is worst-off. We do it by letting coalition $N\backslash T$ improve, *i.e.*, defining $\underline{v}(N\backslash T) := v(N) - k$. The motivation for this definition comes from the idea of stability. Since $\overline{v}(T) = k$, coalition T should receive at least k in game $\overline{v}$. On the other hand, $\underline{v}(N\backslash T) = v(N) - k$ implies that coalition $N\backslash T$ should receive, at least, $v(N) - k$ and hence, coalition T should obtain at most k. This change leads to the superadditive game:

$$\underline{v}(S) = \begin{cases} \max\{v(S), v(S\backslash(N\backslash T)) + v(N) - k\} & N\backslash T \subseteq S \\ v(S) & \text{otherwise.} \end{cases}$$

At this point, we have defined two games: $\overline{v}$, in which coalition T has improved with respect to v, and $\underline{v}$, in which coalition $N\backslash T$ is the one that is better-off. A *cut* on the game v for coalition T at height $k \in [v(T), v(N) - v(N\backslash T)]$ is denoted by $\chi_{T,k}(v)$ and defined as the pair of games $\{\overline{v}, \underline{v}\}$. The reason for the name *cut* becomes clear when dealing with balanced games below.

An extra condition needs to be imposed on cuts, namely, if $v(T) = v(N) - v(N\backslash T)$, then no cut is permitted for coalition T. This last requirement is quite natural, if omitted, we could have a situation in which $\underline{v} = \overline{v} = v$, and the cut makes no sense. Note that, by definition, if $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$ and $\chi_{N\backslash T,v(N)-k}(v) = \{\overline{v}', \underline{v}'\}$, then $\overline{v} = \underline{v}'$ and $\underline{v} = \overline{v}'$. Lemma 6.4 shows that the games $\overline{v}$ and $\underline{v}$ are superadditive.

Lemma 6.4. *Let $v \in G^n$. Let $\emptyset \neq T \subsetneq N$ be such that $v(T) < v(N) - v(N\backslash T)$. Let $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$. Then, both $\overline{v}$ and $\underline{v}$ are superadditive games.*

Proof. Let $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$. We do the proof for the superadditivity of $\overline{v}$, being the one for $\underline{v}$ analogous (just think of the cut $\chi_{N\backslash T,v(N)-k}(v) = \{\overline{v}', \underline{v}'\}$ where $\overline{v}' = \underline{v}$). Let $S, S' \subseteq N$, $S \cap S' = \emptyset$. We want to show that $v(S) + v(S') \leq v(S \cup S')$. Now we have four possibilities:

(i) $T \nsubseteq S \cup S'$. Now, $\overline{v}(S) = v(S)$, $\overline{v}(S') = v(S')$, and $\overline{v}(S \cup S') = v(S \cup S')$. Hence, the result follows from the superadditivity of v.

(ii) $T \subseteq S \cup S'$, $T \nsubseteq S$, and $T \nsubseteq S'$. Now, $\overline{v}(S) = v(S)$, $\overline{v}(S') = v(S')$, and $\overline{v}(S \cup S') \geq v(S \cup S')$. Hence, the result follows from the superadditivity of v.

(iii) $T \nsubseteq S$ and $T \subseteq S'$. By definition of $\overline{v}$, $\overline{v}(S) = v(S)$ and $\overline{v}(S') = \max\{v(S'), v(S'\backslash T) + k\}$. If $\overline{v}(S') = v(S')$, then, since $\overline{v}(S \cup S') \geq v(S \cup S')$, we are done. Hence, we can assume that $\overline{v}(S') = v(S'\backslash T) + k$. Now, since $S \cap S' = \emptyset$, we have $T \cap S = \emptyset$ and hence, $(S \cup S')\backslash T = S \cup (S'\backslash T)$. Hence, by the definition of $\overline{v}$ and the superadditivity of v, $\overline{v}(S \cup S') \geq v((S \cup S')\backslash T) + k \geq v(S) + v(S'\backslash T) + k = \overline{v}(S) + \overline{v}(S')$.

(iv) $T \subseteq S$ and $T \nsubseteq S'$. Analogous to (iii). □

Next, we introduce the definition of $\mathcal{T}$-solution, where $\mathcal{T}$ stands for *trade-off.*

Definition 6.2. *An allocation rule φ is a $\mathcal{T}$-solution if for each game v and each cut $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$, there is $\alpha \in [0,1]$ such that*

$$\varphi(v) = \alpha\varphi(\overline{v}) + (1-\alpha)\varphi(\underline{v}).$$

The idea of a $\mathcal{T}$-solution is that $\varphi(v)$, the solution of the original game, must be a trade-off between $\varphi(\overline{v})$ and $\varphi(\underline{v})$. The result of a give and take between coalitions T and $N\backslash T$. The coefficient α measures how important $\overline{v}$ and $\underline{v}$ are for the original game v when φ is being considered. Once the allocation rule is fixed, the coefficient α is a function depending on the game v, the coalition T, and the constant k. Therefore, the concept of $\mathcal{T}$-solution is very general and dealing with the whole family of $\mathcal{T}$-solutions is not an easy task. Next, we impose a regularity condition on how the trade-off has to be made.

Let $v \in G^n$ and let $\{v_1, v_2\}$ be a cut on v. Now, a new cut $\{\overline{v}_2, \underline{v}_2\}$ can be defined on v_2. Hence, we have cut the original game v into the games $\{v_1, \overline{v}_2, \underline{v}_2\}$. The generalization of this idea leads to the definition of *dissection.* The collection of games $\mathcal{G} = \{v_1, v_2, \ldots, v_r\}$ is a dissection of v if it can be obtained by cutting successively game v. Now, if φ is a $\mathcal{T}$-solution, then there are constants $\alpha_1, \dots, \alpha_r$ such that $\sum_{i=1}^{r} \alpha_i = 1$, for each $i \in \{1, \dots, r\}$, $\alpha_i \geq 0$, and finally

$$\varphi(v) = \sum_{i=1}^{r} \alpha_i \varphi(v_i).$$

Again, each α_i measures the importance of v_i for the original game v within the dissection $\mathcal{G}$. Next, we impose a regularity condition on the $\mathcal{T}$-solutions.

Definition 6.3. *An allocation rule φ is an $\mathcal{RT}$-solution if it satisfies the following properties:*

(i) *φ is a $\mathcal{T}$-solution.*

(ii) (Translation Invariance) *Let $v \in G^n$. Let v_1 and v_2 be such that each of them belongs to some dissection of v. Let $\beta \in \mathbb{R}^n$ be such that for each $S \subseteq N$, $v_1(S) = v_2(S) + \sum_{i \in S} \beta_i$. Then, the coefficients associated with v_1 and v_2 in the corresponding dissections coincide.*[3]

The translation invariance requirement is quite natural and needs no motivation. It implies, in particular, that if a game v' belongs to two different dissections of a game v, then the corresponding coefficients associated with v' must coincide. Hence, let $v \in G^n$ and let $\mathcal{G} = \{v_1, \dots, v_r\}$, then we can interpret each coefficient α_i as dependent only on the original game v and on the game v_i; we do not need to know the cut that led to game v_i. The coefficients are now a function of v and v_i. Once φ and v are fixed, if v' belongs to some dissection of v then we denote the relevance of v' for game v by $\alpha_v(v')$.

6.3.2 $\mathcal{RT}$-Solutions and Balanced Games: Fair Additivity

Next, we show that being an $\mathcal{RT}$-solution has strong implications when working with balanced games. Besides, we introduce the fair additivity; a property that leads to $\mathcal{RT}$-solutions with an extra regularity condition within the class of balanced games. Next, Figure 6.1 and Lemma 6.5 illustrate why given a game v, $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$ is called a *cut.*

Lemma 6.5. *Let v be balanced game. Then, a cut $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$ has the following properties:*

(i) *$C(\overline{v}) \cup C(\underline{v}) = C(v)$.*

(ii) *If $C(v)$ is an $(n-1)$-polytope and $C(\overline{v}) \cap C(\underline{v}) \neq \emptyset$ then this intersection lies in an m-dimensional space with $m < n-1$,* i.e.*, its $(n-1)$-dimensional volume is 0.*

Proof. Let $\chi_{T,k}(v) = \{\overline{v}, \underline{v}\}$.

(i) "$\subseteq$" For each $S \subseteq N$, we have $\overline{v}(S) \geq v(S)$ and $\underline{v}(S) \geq v(S)$. Hence, $C(\overline{v}) \subseteq C(v)$ and $C(\underline{v}) \subseteq C(v)$. Hence, $C(\overline{v}) \cup C(\underline{v}) \subseteq C(v)$.

[3]Note that for each $v' \in \mathcal{G}$, $v'(N) = v(N)$. Hence, if for each $S \subseteq N$, $v_1(S) = v_2(S) + \sum_{i \in S} \beta_i$, then $\sum_{i \in N} \beta_i = 0$.

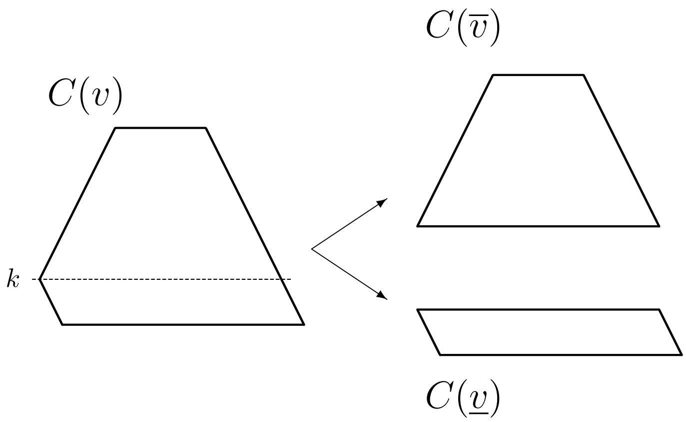

Figure 6.1: The cut $\chi_{3,k}(v)$ for the three-player balanced game v

"$\supseteq$" Let $x \in C(v)$. Suppose that $x \notin C(\overline{v}) \cup C(\underline{v})$. Then, since $x \notin C(\overline{v})$, there is $S \subsetneq N$ such that $T \subseteq S$ and $\sum_{i\in S} x_i < \overline{v}(S)$. Moreover, since $x \in C(v)$, we have $v(S) \leq \sum_{i\in S} x_i$ and hence, $\overline{v}(S) = v(S\backslash T) + k$. Similarly, since $x \notin C(\underline{v})$, there is $S' \subsetneq N$ such that $T \subseteq S'$ and $\sum_{i\in S'} x_i < \underline{v}(S') = v(S'\backslash(N\backslash T)) + v(N) - k$. Hence,

$$\sum_{i\in S} x_i + \sum_{i\in S'} x_i < v(S\backslash T) + k + v(S'\backslash(N\backslash T)) + v(N) - k = v(N) + v(S\backslash T) + v(S'\backslash(N\backslash T))$$

and, on the other hand,

$$\sum_{i\in S} x_i + \sum_{i\in S'} x_i = \sum_{i\in N} x_i + \sum_{i\in S\backslash T} x_i + \sum_{i\in S'\backslash(N\backslash T)} x_i \overset{x\in C(v)}{\geq} v(N) + v(S\backslash T) + v(S'\backslash(N\backslash T)).$$

Contradiction.

(ii) A cut $\chi_{T,k}(v)$ on game v leads to a cut in $C(v)$. This cut consists of taking the hyperplane $\sum_{i\in T} x_i = k$, that cuts it in two pieces (one of them can be empty if the hyperplane does not intersect the core). Once this consideration has been made the result follows immediately. □

Note that a cut on a game defines a unique cut on its core. Therefore, we use the expression cut to refer to both cuts on games and cuts on cores. Moreover, it is important to note that a dissection $\mathcal{G}$ of a game v induces, by Lemma 6.5, a dissection on its core. Let v be a balanced game and $\mathcal{G}$ a dissection of v. We say that $\mathcal{G}$ is a balanced dissection of v if for each $v' \in \mathcal{G}$, v' is balanced. Next, we introduce a strengthening of the $\mathcal{RT}$-property: the *fair additivity*.

Definition 6.4. *Let φ be an allocation rule. Let v be a balanced game. Let v' and v'' be two balanced games such that each of them belongs to some dissection of v. Then, φ satisfies* fair additivity with respect to the core *if:*

(i) φ *is a* $\mathcal{RT}$*-solution.*

(ii) $C(v') = C(v'')$ *implies that* $\alpha_v(v') = \alpha_v(v'')$.

The idea of this property is that, from the point of view of stability, games v' and v'' are equal. For instance, if we think of v' as the exact envelope of v'', then v' is obtained from v'' by removing redundant information. Hence, their relevance for game v should be the same. An immediate consequence of this property is that the coefficients $\alpha_v(v')$ can be denoted by $\alpha_v(C(v'))$, *i.e.*, the weight only depends on the core.

Example 6.1. *Let* v *be the game described in Table 6.1. Consider the cut* $\chi_{3,2}(v) = \{v_1, v_2\}$ *and then, the cut* $\chi_{3,2}(v_1) = \{v_3, v_4\}$*. The values associated to each of these games are also summarized in Table 6.1. After these two cuts, we have the following dissection of* v*:* $\mathcal{G} = \{v_2, v_3, v_4\}$*.*

S	v	$v_1\ (=\overline{v})$	$v_2\ (=\underline{v})$	$v_3\ (=\overline{v}_1)$	$v_4\ (=\underline{v}_1)$
{1}	0	0	0	0	0
{2}	0	0	0	0	0
{3}	0	2	0	2	2
{1,2}	6	6	8	6	8
{1,3}	6	6	6	6	6
{2,3}	6	6	6	6	6
N	10	10	10	10	10

Table 6.1: The game v and the cuts $\chi_{3,2}(v)$ and $\chi_{3,2}(v_1)$

Now, it is straightforward to check that $C(v), C(v_1)$*, and* $C(v_3)$ *are* 2*-polytopes. However,* $C(v_2)$ *and* $C(v_4)$ *are* 0*-polytopes; their core coincides with the point* $(4,4,2)$*. Now, if* φ *is a RT-solution, we have that* $\varphi(v) = \alpha_v(v_1)\varphi(v_1)+\alpha_v(v_2)\varphi(v_2)$ *and* $\varphi(v) = \alpha_v(v_2)\varphi(v_2)+\alpha_v(v_3)\varphi(v_3)+\alpha_v(v_4)\varphi(v_4)$ *with* $\alpha_v(v_1)+\alpha_v(v_2) = 1$ *and* $\alpha_v(v_2)+\alpha_v(v_3)+\alpha_v(v_4) = 1$*. Moreover, since* $v_1 = v_3$ *we also have* $\alpha_v(v_1) = \alpha_v(v_3)$*. Finally, if* φ *also satisfies fair additivity, although* $v_2 \neq v_4$*, we have that* $C(v_2) = C(v_4)$ *and hence* $\alpha_v(v_2) = \alpha_v(v_4)$*. Now, combining the two equations we easily conclude that* $\alpha_v(v_2) = \alpha_v(v_4) = 0$*,* i.e.*, the weight of the games with a degenerate core is* 0*.*

Lemma 6.6 shows that what happens in Example 6.1 is a general feature of the fair additivity property. More precisely, if we have a balanced dissection of a game with a full dimensional core, then fair additivity pins down to 0 the coefficients of the games whose core is degenerate.

Lemma 6.6. *Let* φ *be an allocation rule satisfying fair additivity. Let* v *be a balanced game and let* $\chi_{T,k}(v) = \{v_1, v_2\}$ *be a balanced dissection of* v*. If* $C(v)$ *is an* m*-polytope and there is* $i \in \{1,2\}$ *such that* $C(v_i)$ *is an* l*-polytope with* $l < m$*, then* $\alpha_v(v_i) = 0$*.*

Proof. Take a balanced game v, and a cut $\chi_{T,k}(v) = \{v_1, v_2\}$ such that $C(v)$ is an m-polytope and $C(v_2)$ is an l-polytope with $l < m$. Then, v_1 can be cut using again coalition T and height

k, *i.e.*, cut $\chi_{T,k}(v_1) = \{v_3, v_4\}$. Then,

$$\varphi(v) = \alpha_v(v_1)\varphi(v_1) + \alpha_v(v_2)\varphi(v_2)$$

and

$$\varphi(v) = \alpha_v(v_2)\varphi(v_2) + \alpha_v(v_3)\varphi(v_3) + \alpha_v(v_4)\varphi(v_4),$$

where $\alpha_v(v_1) + \alpha_v(v_2) = 1$, $\alpha_v(v_2) + \alpha_v(v_3) + \alpha_v(v_4) = 1$, and $v_3 = v_1$. Moreover, we claim that $C(v_4) = C(v_2)$. The restrictions in $C(v_4)$ are stronger than those defining $C(v_2)$ and hence, $C(v_4) \subseteq C(v_2)$. Now, suppose that there is $x \in C(v_2)\backslash C(v_4)$. Note that, since $C(v_2)$ is degenerate we have that $C(v_1) \cap C(v_2) = C(v_2)$. Hence, $C(v_2) \subsetneq C(v_1)$. Since $x \notin C(v_4)$, there is $S' \subsetneq N$ such that $\sum_{i \in S'} x_i < v_4(S')$. Now, for each $S \subseteq N$ such that $T \nsubseteq S$, we have $v_2(S) = v_4(S)$. Hence, since $x \in C(v_2)$, we have $T \subseteq S'$. Since we also have that $x \in C(v_1)$, then $\sum_{i \in S'} x_i \geq v_1(S')$. But, by definition of v_1 and v_4, $v_4(S') = v_1(S')$. Contradiction.

Now, by fair additivity, $\alpha_v(v_4) = \alpha_v(v_2)$ and $\alpha_v(v_3) = \alpha_v(v_1)$. Then, $\alpha_v(v_1) + \alpha_v(v_2) = 1$ and $\alpha_v(v_2) + \alpha_v(v_3) + \alpha_v(v_4) = \alpha_v(v_1) + 2\alpha_v(v_2) = 1$. Hence, $\alpha_v(v_2) = 0$. □

In the previous proof we used that, after the second cut, $C(v_4) = C(v_2)$. Note that, as Example 6.1 shows, it might be the case $v_4 \neq v_2$ and hence, fair additivity is needed. Next result describes $\mathcal{RT}$-solutions satisfying fair additivity property within the class of balanced games. The idea of Proposition 6.1 is similar to the idea of one classical result in measure theory, namely, "*If m is the Lebesgue measure, and η is a positive translation invariant Borel measure on $\mathbb{R}^k$ such that $\eta(K) < \infty$ for every compact set K, then there is a constant c such that $\eta(E) = cm(E)$ for all Borel sets $E \subset \mathbb{R}^k$*" (Rudin, 1966). Although Lemma 6.6 is a weaker result than Proposition 6.1, we obtain the two results independently, *i.e.*, we do not use Lemma 6.6 in the proof of Proposition 6.1.

Proposition 6.1. *Let φ be an allocation rule satisfying fair additivity. Let v and v' be two balanced games such that v' belongs to some dissection of v. If $C(v)$ is an m-polytope, then* $\alpha_v(v') = \frac{\mathrm{Vol}_m(C(v'))}{\mathrm{Vol}_m(C(v))}$.

Proof. Let v be a balanced game with an m-dimensional core. Let v' be a balanced game belonging to some dissection of v and let $w(v') := \alpha_v(v') \,\mathrm{Vol}_m(C(v))$. We claim that $w(v') = \mathrm{Vol}_m(C(v'))$. Next, we show that there is $\varepsilon > 0$ such that if $\mathrm{Vol}_m(C(v')) < \varepsilon$, then $w(v') = \mathrm{Vol}_m(C(v'))$. Suppose, on the contrary, that for each $\varepsilon > 0$, there is v_ε such that $w(v_\varepsilon) \neq \mathrm{Vol}_m(C(v_\varepsilon))$. Now, let $\{\varepsilon_l\}_{l \in \mathbb{N}}$ be a sequence of positive numbers with limit 0 and, for each $l \in \mathbb{N}$, let v_{ε_l} be such that $w(v_{\varepsilon_l}) \neq \mathrm{Vol}_m(C(v_{\varepsilon_l}))$. Now, either $|\{l \in \mathbb{N} : w(v_{\varepsilon_l}) > \mathrm{Vol}_m(C(v_{\varepsilon_l}))\}| = \infty$, or $|\{l \in \mathbb{N} : w(v_{\varepsilon_l}) < \mathrm{Vol}_m(C(v_{\varepsilon_l}))\}| = \infty$ (or both of them are true at the same time). Hence, we can consider the two following cases:

Case 1: For each $\varepsilon > 0$, there is v_ε such that $\mathrm{Vol}_m(C(v_\varepsilon)) < \varepsilon$ and $w(v_\varepsilon) > \mathrm{Vol}_m(C(v_\varepsilon))$.

Let $\varepsilon > 0$. Let v_ε be such that $w(v_\varepsilon) > \mathrm{Vol}_m(C(v_\varepsilon))$. Let $\delta = w(v_\varepsilon) - \mathrm{Vol}_m(C(v_\varepsilon))$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers with limit 0 and, for each $k \in \mathbb{N}$, let v_{ε_k} be such that

$w(v_{\varepsilon_k}) > \text{Vol}_m(C(v_{\varepsilon_k}))$. Now, we can define a balanced dissection $\mathcal{G}$ of v, $\mathcal{G} = v_\varepsilon \cup \mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_1$ is such that,

(i) For each $\hat{v} \in \mathcal{G}_1$, there is $k \in \mathbb{N}$ such that $\hat{v}$ is a translation of v_{ε_k} and hence, $C(\hat{v})$ is a translation of $C(v_{\varepsilon_k})$.

(ii) $\sum_{\hat{v}\in\mathcal{G}_1} \text{Vol}_m(C(\hat{v})) > \text{Vol}_m(C(v)) - \text{Vol}_m(C(v_\varepsilon)) - \delta$.

Now, by the translation invariance of the $\mathcal{RT}$-solution and the fair additivity, for each $\hat{v} \in \mathcal{G}_1$ there is $k \in \mathbb{N}$ such that, $w(\hat{v}) = w(v_{\varepsilon_k}) > \text{Vol}_m(C(v_{\varepsilon_k})) = \text{Vol}_m(C(\hat{v}))$. Hence,

$$\sum_{\hat{v}\in\mathcal{G}} w(\hat{v}) \geq w(v_\varepsilon) + \sum_{\hat{v}\in\mathcal{G}_1} w(\hat{v}) > \text{Vol}_m(C(v_\varepsilon)) + \delta + \text{Vol}_m(C(v)) - \text{Vol}_m(C(v_\varepsilon)) - \delta = \text{Vol}_m(C(v)).$$

Hence, $\sum_{\hat{v}\in\mathcal{G}} \alpha_v(\hat{v}) > 1$. Contradiction.

Case 2: For each $\varepsilon > 0$ there is v_ε such that $\text{Vol}_m(C(v_\varepsilon)) < \varepsilon$ and $w(v_\varepsilon) < \text{Vol}_m(C(v_\varepsilon))$.

We proof now that Case 2 implies Case 1. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers with limit 0 and, for each $k \in \mathbb{N}$, let v_{ε_k} be such that $w(v_{\varepsilon_k}) < \text{Vol}_m(C(v_{\varepsilon_k}))$. Let $\delta > 0$. Now, we can define a balanced dissection $\mathcal{G}_\delta$ of v, $\mathcal{G}_\delta = \mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_1$ is such that,

(i) For each $\hat{v} \in \mathcal{G}_1$, there is $k \in \mathbb{N}$ such that $C(\hat{v})$ is a translation of $C(v_{\varepsilon_k})$.

(ii) $\sum_{\hat{v}\in\mathcal{G}_1} \text{Vol}_m(C(\hat{v})) > \text{Vol}_m(C(v)) - \delta$.

Now, by the translation invariance of the $\mathcal{RT}$-solution and the fair additivity, for each $\hat{v} \in \mathcal{G}_1$, $w(\hat{v}) = w(v_{\varepsilon_k})$. Now, since $\sum_{\hat{v}\in\mathcal{G}} \alpha_v(\hat{v}) = 1$ implies that $\sum_{\hat{v}\in\mathcal{G}} w(\hat{v}) = \text{Vol}_m(C(v))$, there is $v_\delta \in \mathcal{G}_2$ such that $w(v_\delta) > \text{Vol}_m(C(v_\delta))$. Hence, we can construct a sequence $\{\delta_l\}_{l\in k}$ converging to 0 and such that for each $l \in \mathbb{N}$, there is v_{δ_l} such that $w(v_\delta) > \text{Vol}_m(C(v_\delta))$, Hence, Case 2 implies Case 1 and we are done.

Hence, we have shown that there is $\varepsilon > 0$ such that if $\text{Vol}_m(C(v')) < \varepsilon$, then $w(v') = \text{Vol}_m(C(v'))$. Now, for each balanced game v' belonging to a dissection of v, we can dissect v' in such a way that the cores of the games of the dissection cover $C(v')$ and their volumes do not exceed ε. Hence, $w(v') = \text{Vol}_m(C(v'))$. □

Corollary 6.1. *Let φ be an allocation rule satisfying fair additivity. Let v be a balanced game and $\mathcal{G} = \{v_1, \dots, v_r\}$ a balanced dissection of v. If $C(v)$ is an m-polytope, then*

$$\varphi(v) = \sum_{i=1}^{r} \alpha_v(v_i)\varphi(v_i), \quad \textit{where } \alpha_v(v_i) = \frac{\text{Vol}_m(C(v_i))}{\text{Vol}_m(C(v))}.$$

Proof. Immediate from Proposition 6.1 □

As a consequence of Proposition 6.1 and Corollary 6.1, if φ is a solution satisfying fair additivity, then we have completely characterized the coefficients associated with φ and a dissection of a balanced game. Indeed, we have shown that such weights are proportional to the volumes of the cores of the games in the dissection.

6.4 The Characterization

We state now the main result in this Chapter, a characterization of the core-center.

Theorem 6.1. *Let φ be an allocation rule satisfying*

T1) Efficiency

T2) Continuity

T3) Extended Weak Symmetry

T4) Fair Additivity with respect to the core.

Then, for each $v \in BG$, $\varphi(v) = \mu(v)$.

Let TG denote the subclass of games in which the four properties of Theorem 6.1 characterize the core-center. The proof of Theorem 6.1 will be focused in showing that $BG \subseteq TG$.

Next, we provide an outline of the proof with the main steps in which we have divided it:

Step 1 We show that extended weak symmetry and efficiency characterize the core-center when the core is simple enough (Section 6.4.1).

Step 2 We show that the four properties *T1-T4* characterize the core-center for the class of games with full dimensional core (Section 6.4.2).

Step 3 Finally, we show that the core of a balanced game can be approximated by full dimensional cores. Hence, the previous results along with the continuity property lead to the proof of Theorem 6.1 (Section 6.4.3).

In the first two steps we study full dimensional cores. We only deal with the degenerate case, when the core coincides with the least core, in the last step.

6.4.1 An Elemental Core

In this Subsection we show that, when the core is simple enough, the core-center can be characterized using extended weak symmetry and efficiency. Let $v \in BG^n$, and let A denote the set of active restrictions in $C(v)$. We say $C(v)$ is *elemental* if $A = \{x_i \geq v(i) \ : \ i \in N\} = \{\text{elemental restrictions}\}$. If $C(v)$ is an elemental core, then $C(v) = I(v)$.

Lemma 6.7. *Let $v \in BG^n$ be such that $C(v)$ is elemental. Then, $C(v)$ it is a regular simplex and its center of gravity is the allocation x such that, for each $i \in N$,*

$$x_i = \frac{v(N) - \sum_{j \in N} v(j)}{n} + v(i).$$

Proof. If $C(v)$ is elemental, $C(v) = I(v)$. For each $v \in G^n$, $I(v)$ is a regular simplex[4] with vertices $u^1, \ldots, u^n$ where, for each $i \in N$,

$$u^i = (v(1), v(2), \ldots, \overbrace{v(N) - \sum_{j \neq i} v(j)}^{i}, \ldots, v(n)).$$

Hence, to obtain the result, we only need to calculate the center of gravity of the simplex, *i.e.*, the average of the vertices. □

Lemma 6.8. *Let $v \in BG^n$. If $C(v) = I(v)$, then, v is quasi-symmetric.*

Proof. $I(v)$ is a regular $(n-1)$-simplex. Hence, if $C(v) = I(v)$, then, for each $S \subsetneq N$, with $|S| > 1$, the restrictions R_v^S are redundant. Let $S \subsetneq N$, $|S| > 1$. By superadditivity, $v(S) \geq \sum_{i \in S} v(i)$. Moreover, since R_v^S is redundant, $v(S) \leq \sum_{i \in S} v(i)$. Hence, $v(S) = \sum_{i \in S} v(i)$. Now, regardless of $v(N)$, the game is quasi-symmetric. □

Proposition 6.2. *Let $v \in BG^n$ be such that $C(v)$ is elemental. Let φ be an allocation rule satisfying efficiency and extended weak symmetry. Then, $\varphi(v) = \mu(v)$.*

Proof. Since v has an elemental core, $C(v) = I(v)$. By Lemma 6.8, v is quasi-symmetric. Now, by extended weak symmetry, we have that, for each pair $i, j \in N$, $\varphi_i(v) - v(i) = \varphi_j(v) - v(j)$. Hence, there is $k \in \mathbb{R}$ such that, for each $i \in N$, $\varphi_i(v) = k + v(i)$. The latter comment, along with the efficiency property, implies that, for each $i \in N$,

$$\varphi_i(v) = \frac{v(N) - \sum_{j \in N} v(j)}{n} + v(i).$$

Now, by Lemma 6.7, $\varphi(v)$ is the center of gravity of $C(v)$, *i.e.*, the core-center. Hence, $\varphi(v) = \mu(v)$. □

6.4.2 The Core is Full Dimensional

In this Section we combine Proposition 6.2 with the continuity and the fair additivity properties to show that a game with a non degenerate core belongs to TG.

At this point we know that games with an elemental core belong to TG. The class of games with an elemental core plays an important role in the forthcoming results. This role is similar to that of the unanimity games in the characterization of the Shapley value using additivity. The outline of this part of the proof is as follows. Let v be a balanced game and let $C(v)$ be full dimensional. First, successively cutting v, we obtain a dissection of $C(v)$; being this dissection primarily composed by small parallelepipeds. Second, we cut the games corresponding to these parallelepipeds, obtaining an elemental core inside each of them. Then, we successively repeat this procedure with the remaining non-elemental cores. Finally, we show that, using cuts, the

[4] Go over the Appendix to find a rigorous definition of a simplex and related concepts.

core of v can be covered with elemental cores (this is indeed a kind of triangulation). Finally, the fair additivity property leads to the conclusion of this part of the proof. Figure 6.2 illustrates this outline.

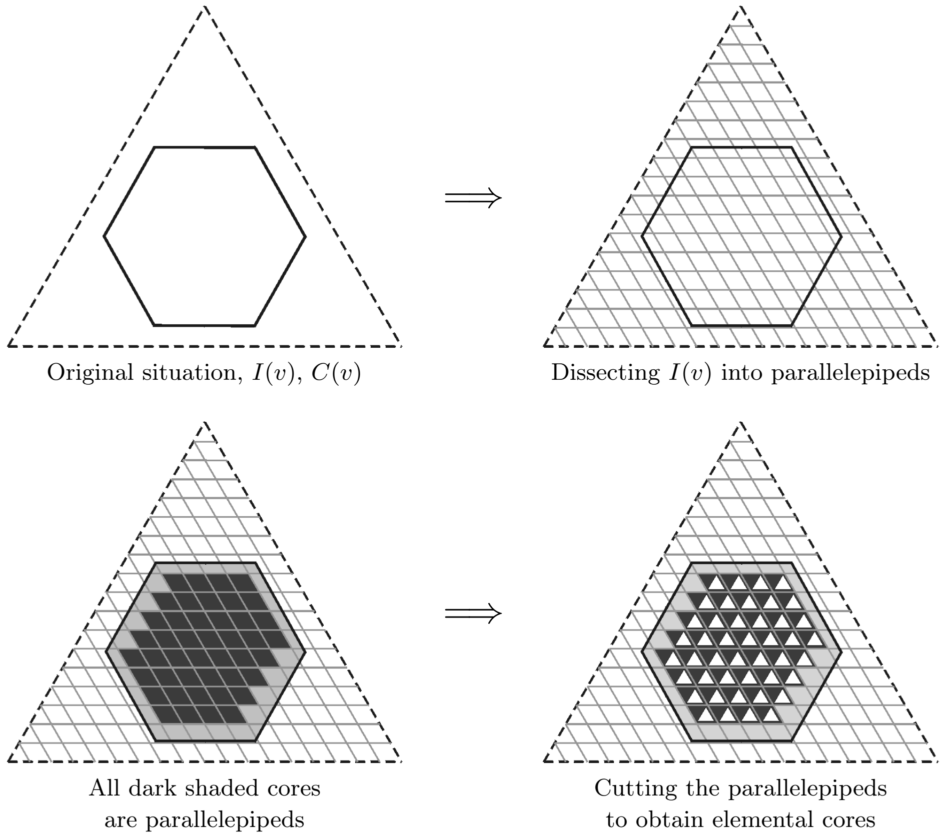

Figure 6.2: Scheme of the proof for full dimensional cores

Proposition 6.3. *Let $v \in BG^n$ be such that $C(v)$ is full dimensional. Let φ be an allocation rule satisfying the four properties T1-T4. Then, $\varphi(v) = \mu(v)$.*

Proof. Let φ be an allocation rule satisfying the properties *T1-T4*. Let $v \in BG^n$ be a game with a full dimensional core. The body of the proof consists of dissecting $C(v)$ into elemental cores; we do it in such a way that we can combine Proposition 6.2 with the continuity and the fair additivity properties to get $\varphi(v)$. Hence, we describe a procedure which "nearly" triangulates any full dimensional core. Henceforth, till the end of the proof, $\mathrm{Vol}(P)$ denotes the $(n-1)$-dimensional volume of polytope P.

First, we divide $I(v)$ into small $(n-1)$-parallelepipeds.[5] Let $i \in N$, i's payoff in his best allocation within $I(v)$ is $v(N) - \sum_{j \neq i} v(j)$, and in his worst one is $v(i)$; hence, regardless of i, the

[5] A rigorous definition of a k-parallelepiped can be found in the Appendix.

difference between these two payoffs is $v(N) - \sum_{j \in N} v(j)$. Let $L := v(N) - \sum_{j \in N} v(j)$. From now on, and for the sake of clarity, $v(i)$ is denoted by m_i. Let $q \in \mathbb{N}$ (for simplicity we assume $q > 2$), and let $\delta = L/q$. For each face in $I(v)$, different to that corresponding with the hyperplane $x_n = m_n$, we make $q + 1$ cuts on it, all of them parallel to the hyperplane in which that face lies. Hence, we partition $I(v)$ using the following hyperplanes: for each $i \in N$, $i \neq n$, and for each $k \in \{0, \dots, q\}$, $H^i_k := \{x \in \mathbb{R}^n \ : \ x_i = m_i + k\delta\}$. We use these hyperplanes to define a dissection of $C(v)$. Let $\chi_{i,k}$ be a cut, and let $\mathcal{G}$ be a collection of games. We denote by $\chi_{i,k}(\mathcal{G})$ the result of cutting successively all the cores of the games in $\mathcal{G}$ with the hyperplane $x_i = k$. Hence, $\chi_{i,k}(\{w, w', w'', \dots\}) = \{\overline{w}, \underline{w}, \overline{w}', \underline{w}', \overline{w}'', \underline{w}'', \dots\}$. It can be the case that some of these cuts is not permitted, *i.e.*, $k \notin [w(i), w(N) - w(N \backslash i)]$; besides, it is also possible that one of the games in $\chi_{i,k}(w)$ is not balanced. In the last two cases we take $\{w\}$ instead of $\chi_{i,k}(w)$, *i.e.*, we do not consider those cuts.

Let $v \in BG^n$ be such that $C(v)$ is full dimensonal. Let G^δ be the collection of games defined as follows:

Stage 0**:** We begin with the set of games $\mathcal{G}^0 \equiv \mathcal{G}^{0,q} = \{v\}$.

$\vdots$

Stage i, $i \in N$, $i \neq n$**:** We define the cuts for player i.

> **Step** $i.0$**:** We cut $I(v)$ with $x_i = m_i$; $\mathcal{G}^{i,0} = \chi_{i,m_i}(\mathcal{G}^{i-1,q})$.
>
> **Step** $i.1$**:** $\mathcal{G}^{i,1} = \chi_{i,m_i+\delta}(\mathcal{G}^{i,0})$.
>
> $\vdots$
>
> **Step** $i.k$**:** $\mathcal{G}^{i,k} = \chi_{i,m_i+k\delta}(\mathcal{G}^{i,k-1})$.
>
> $\vdots$
>
> **Step** $i.q$**:** $\mathcal{G}^{i,q} = \chi_{i,m_i+q\delta}(\mathcal{G}^{i,q-1})$.

Let $\mathcal{G}^\delta$ denote the set $\mathcal{G}^{n-1,q}$. In order to save notation, if no ambiguity arises, we denote $C(v')$ by C'. Now, $\bigcup_{v' \in \mathcal{G}^\delta} C' = C$ and, for each pair $v_1, v_2 \in \mathcal{G}^\delta$, $\text{Vol}(C(v_1) \cap C(v_2)) = 0$, *i.e.*, the cores of the games in G^δ define a dissection of C. It is quite intuitive that, for each $0 < \varepsilon < 1$, we can find $\delta > 0$ such that the sum of the volumes of the cores of games in $\mathcal{G}^\delta$ which are not parallelepipeds is, at most, $\varepsilon \,\text{Vol}(C)$ (note that ε is fixed now for all the proof). All these parallelepipeds are equal and they have positive $(n-1)$-dimensional volume. Let $\mathcal{G}^{NP} \subsetneq \mathcal{G}^\delta$ be the set of games such that their core is not a parallelepiped. The second part of the proof consists of cutting each one of the parallelepipeds to obtain an elemental core, a simplex, inside the parallelepiped. It is quite intuitive, and not difficult to check, that for $0 < \alpha < 1$ small enough, we can find a procedure which divides each parallelepiped P in such a way that the core of one of the resulting games is elemental and its volume is, at least, $\alpha \operatorname{Vol}(P)$.[6] Let $\mathcal{G}^{NE}$ be the set of games obtained in this second step such that their core is not elemental.

We can ensure now that at least a volume $\alpha(1-\varepsilon)\operatorname{Vol}(C)$ has been covered by elemental cores. Period 1 is finished. The procedure continues as follows. We begin period 2: for each game $v' \in \mathcal{G}^{NP} \cup \mathcal{G}^{NE}$, we repeat the procedure we have made for v (we have to find a new constant δ' which will probably be smaller than δ), covering at least a volume $\alpha(1-\varepsilon)\operatorname{Vol}(C')$ of its core with elemental cores. Note that the constant α keeps constant. This is because in this second period we obtain the same kind of parallelepipeds we had in the first one (but smaller). Hence, the procedure obtained to "put" a simplex inside each parallelepiped is the same, and the proportion of covered volume also remains unchanged.

Note that δ varies as the period changes but both α and ε keep constant. We claim that if we repeat successively this procedure, the volume of C which is not covered by elemental cores tends to 0. We begin with a volume $RV^0 = \operatorname{Vol}(C)$ which needs to be covered by elemental cores. After the first period, this volume has been reduced to $RV^1 = (1-\alpha)(1-\varepsilon)\operatorname{Vol}(C) + \varepsilon \operatorname{Vol}(C)$. Then, after t periods we have $RV^t = (a+b)^t \operatorname{Vol}(C)$, where $a = (1-\alpha)(1-\varepsilon)$ and $b = \varepsilon$. The proof of this statement is easily done by induction:

Case 1: $RV^1 = (1-\alpha)(1-\varepsilon)\operatorname{Vol}(C) + \varepsilon \operatorname{Vol}(C) = (a+b)\operatorname{Vol}(C)$.

Case t: Assume the result is true for this case (induction assumption).

Case t+1: Finally, we have $RV^{t+1} = (1-\alpha)(1-\varepsilon)RV^t + \varepsilon RV^t = aRV^t + bRV^t \overset{\text{induc}}{=} a(a+b)^t \operatorname{Vol}(C) + b(a+b)^t \operatorname{Vol}(C) = (a+b)^{t+1}\operatorname{Vol}(C)$.

Hence, since $a+b = (1-\alpha)(1-\varepsilon)+\varepsilon < 1$, we have $\lim_{t\to\infty} RV^t = \lim_{t\to\infty}(a+b)^t \operatorname{Vol}(C) = 0$. This means that, in the limit, this procedure defines an infinite dissection of $C(v)$. Let $\mathcal{G}^t$ be the collection of games after period t and $\mathcal{EG}^t$ those with an elemental core. Now, by the fair additivity of φ:

$$\varphi(v) = \sum_{v' \in \mathcal{G}^t} \frac{\operatorname{Vol}(C')}{\operatorname{Vol}(C)} \varphi(v') = \frac{1}{\operatorname{Vol}(C)} \Bigg(\sum_{v' \in \mathcal{EG}^t} \operatorname{Vol}(C')\varphi(v') + \sum_{v' \in \mathcal{G}^t \setminus \mathcal{EG}^t} \operatorname{Vol}(C')\varphi(v') \Bigg). \tag{6.1}$$

By Proposition 6.2, we have already characterized φ for the games in the first addend of the last term in Equation (6.1). Moreover, since φ is continuous, it is uniformly continuous in the set $B = \{w \in G^n : \text{for each } S \subseteq N, v(S) \le w(S) \le v(N)\}$. Hence, φ is bounded in B. Since all the games we have defined so far belong to B, we have $\lim_{t\to\infty} \sum_{v' \in \mathcal{G}^t \setminus \mathcal{EG}^t} \operatorname{Vol}(C')\varphi(v') = 0$. Now,

[6] We prove in the Appendix that the cuts divide the core of the original game in many parallelepipeds and that the proportion of the core covered by these sets is as close to one as needed. We also provide there an example of a procedure to "put" an elemental core inside each parallelepiped.

for each $t \in \mathbb{N}$, $\varphi(v) = \sum_{v' \in \mathcal{G}^t} \frac{\mathrm{Vol}(C')}{\mathrm{Vol}(C)} \varphi(v')$. Then,

$$\begin{aligned}
\varphi(v) &= \lim_{t\to\infty} \sum_{v' \in \mathcal{G}^t} \frac{\mathrm{Vol}(C')}{\mathrm{Vol}(C)} \varphi(v') \\
&= \frac{1}{\mathrm{Vol}(C)} \Bigg(\lim_{t\to\infty} \sum_{v' \in \mathcal{EG}^t} \mathrm{Vol}(C')\varphi(v') + \lim_{t\to\infty} \sum_{v' \in \mathcal{G}^t \setminus \mathcal{EG}^t} \mathrm{Vol}(C')\varphi(v') \Bigg) \\
&= \lim_{t\to\infty} \frac{1}{\mathrm{Vol}(C)} \sum_{v' \in \mathcal{EG}^t} \mathrm{Vol}(C')\varphi(v') \\
&\overset{\text{Prop 6.2}}{=} \lim_{t\to\infty} \frac{1}{\mathrm{Vol}(C)} \sum_{v' \in \mathcal{EG}^t} \mathrm{Vol}(C')\mu(v') \\
&= \mu(v).
\end{aligned}$$

□

6.4.3 The Core is Not Full Dimensional

Now, the core is an m-polytope with $1 \le m \le n-2$.

Proposition 6.4. *Let $v \in BG^n$ be such that $C(v)$ is not full dimensional. Let φ be an allocation rule satisfying the properties T1-T4. Then, $\varphi(v) = \mu(v)$.*

Proof. By Lemma 6.2, $C(v)$ is the least core of v. Let $\{v_{1/t}\}_{t\in\mathbb{N}}$ be a sequence of shifted games. Now, $\lim_{t\to\infty} v_{1/t} = v$. The core of $v_{1/t}$ coincides with the $\frac{1}{t}$-core of v. By Lemma 6.2, all these $\frac{1}{t}$-cores are full dimensional, and now, by Proposition 6.3 we know that these games have already been characterized. Hence,

$$\varphi(v) \overset{cont}{=} \lim_{t\to\infty} \varphi(v_{1/t}) \overset{\text{Prop 6.3}}{=} \lim_{t\to\infty} \mu(v_{1/t}) \overset{cont}{=} \mu(v).$$

□

Proof of Theorem 6.1. The assertion of the theorem follows from Propositions 6.2, 6.3, and 6.4. □

Next, we prove that the properties in Theorem 6.1 are tight. In order to do this we need a last Lemma.

Lemma 6.9. *Let v be a quasi-symmetric game. Then, $C(v)$ either is a point or is full dimensional.*

Proof. This is a geometric result. As we have already seen, the core of a quasi-symmetric game can be transformed in that of a symmetric game just using a translation. To prove this Lemma it suffices to show that the result is true for symmetric games. Hence, let v be a symmetric game, and assume that it has a degenerate core. Hence, there are $1 \le s \le n-1$ and an s-player coalition S, such that $v(S) + v(N\backslash S) = v(N)$. By efficiency and stability, we have that there is $k \in \mathbb{R}$ such that, for each $x \in C(v)$, $\sum_{i\in S} x_i = v(S) = k$ (this is the reason for the degeneration). Now, by symmetry, for each $x \in C(v)$ and each s-player coalition S', we have $\sum_{i \in S'} x_i = k$. If $s = 1$, we have that, for each $i \in N$ and each $x \in C(v)$, $x_i = k$ and we are done. Hence, we can assume that $s > 1$. Now, we claim that, for each $x \in C(v)$ and each $i \in N$, $x_i = k/s$. Suppose, on the contrary, that there are $x \in C(v)$ and $i \in N$ such that $x_i > k/s$. Hence, for each s-player coalition S containing i, there is $j \in S$ sucht that $x_j < k/s$. But this contradicts that, for each s-player coalition $\sum_{i \in S} x_i = k$ (just taking an s-player coalition with all these j's such that $x_j < k/s$ and the lower of the remaining to have s players). Hence, we have that, for each $i \in N$, $x_i = k/s$. Now, by efficiency, $k = sv(N)/n$. Hence $C(v) = \{x\}$, where, for each $i \in N$, $x_i = v(N)/n$. □

Proposition 6.5. *None of the properties used in Theorem* 6.1 *to characterize the core-center is redundant.*

Proof. Next, we show that if we remove one of these properties there are allocation rules different from the core-center satisfying the remaining ones.

Remove Fair Additivity: Both Shapley value and nucleolus satisfy efficiency, extended weak symmetry, and continuity.

Remove Efficiency: Take $k \neq 0$. The allocation rule $\varphi(v) = \mu(v) + (k, \ldots, k)$ where $\mu(v)$ denotes the core-center of the game v satisfies fair additivity, extended weak symmetry, and continuity.

Remove extended weak symmetry: The allocation rule $\varphi(v) = (v(N), 0, \ldots, 0)$ satisfies fair additivity, efficiency and continuity.

Remove continuity: This is the most complex situation, we need to distinguish different cases in the definition of our allocation rule φ:

The core is a single point: φ selects the point (the core-center).

The core is degenerate but not a single point: In this case the allocation φ selects the point $(v(N), 0, \ldots, 0)$.

The core is not degenerate: φ selects the core-center.

This allocation rule satisfies fair additivity, efficiency and extended weak symmetry. It satisfies fair additivity because of the following: if a cut divides a core in two new cores, and one of them is not full dimensional while the original was, then, the weight of this degenerate core is 0. Efficiency is straightforward. It also satisfies extended weak symmetry: in the non-degenerate case it coincides with the core-center so extended weak symmetry is met; in the degenerate case, as a consequence of Lemma 6.9 there are no quasi-symmetric games with degenerate core with more than one point and hence, extended weak symmetry can never be violated.

□

6.5 Concluding Remarks

In this Chapter we have presented a characterization of the core-center. The main result we stated uses three standard properties along with a new one, the fair additivity property. Recall that, according to the definitions of $\mathcal{T}$-solution and fair additivity, given a game we can "cut" it using any nonempty coalition different from the grand coalition. Nonetheless, the proofs we presented here involve cuts that only use 1-player coalitions and hence, we could state and proof a new characterization result, similar to Theorem 6.1, but with a weakened version of the fair additivity.

Although we have provided a first characterization of the core-center, more research is needed in order to find more convincing characterizations. One possibility is to deepen into the concepts of $\mathcal{T}$-solution and $\mathcal{RT}$-solution, and try to characterize the core-center without the additional restriction imposed by the fair additivity property. Similarly, the problem of finding an independent characterization that does not need the concept of $\mathcal{T}$-solution is still to be solved.

6.A Appendix

6.A.1 The Geometry of the Core-Center in Depth

-Definition of Simplex.

Let $\{a^0, a^1, \ldots, a^n\} \subsetneq \mathbb{R}^n$ be a geometrically independent set.[7] The *simplex* Δ^n spanned by $a^0, a^1, \ldots, a^n$ is the set of all $x \in \mathbb{R}^n$ such that $x = \sum_{i=0}^n t_i a^i$, where $\sum_{i=0}^n t_i = 1$ and, for each $i \in \{1, \ldots, n\}$, $t_i \geq 0$. Each a^i is a *vertex* of the n-simplex. The superscript n of Δ^n corresponds with the dimension of the simplex. An n-simplex is regular if the distance between any two vertices is constant.

The *barycenter of a simplex* Δ^n spanned by the points $a^0, a^1, \ldots, a^n$ is $\Theta(\Delta^n) := \sum_{i=0}^n \frac{a^i}{n+1}$. Let $m \leq n$, let Δ^m be an m-simplex contained in $\mathbb{R}^n$, and let $a^0, \ldots, a^m$ be its vertices. The m-dimensional volume of Δ^m, $\mathrm{Vol}_m(\Delta^m)$, can be computed in the following way: Let $B = (\beta_{ij})$ denote the $(m+1) \times (m+1)$ matrix given by $\beta_{ij} = \|a^i - a^j\|^2$. Then,

$$2^m (m!)^2 \, \mathrm{Vol}_m(\Delta^m)^2 = |\det(\hat{B})|,$$

where $\hat{B}$ is the $(m+2) \times (m+2)$ matrix obtained from B by bordering it with a top row $(0, 1, \ldots, 1)$ and left column $(0, 1, \ldots, 1)^T$. This is known as the Cayley-Menger determinant formula.[8]

[7] A set $\{a^0, a^1, \ldots, a^n\} \subsetneq \mathbb{R}^n$ is geometrically independent if for each vector $(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1}$, the equations $\sum_{i=0}^n t_i = 0$ and $\sum_{i=0}^n t_i a^i = (0, \ldots, 0)$, hold only if $t_0 = t_1 = \cdots = t_n = 0$. Note that $\{a^0, a^1, \ldots, a^n\}$ is geometrically independent if and only if the vectors $a^1 - a^0, \ldots, a^n - a^0$ are linearly independent.

[8] For references on this and other formulas for computing simplicial volumes look at Gritzman and Klee (1994).

-Definition of Parallelepiped.

Let $\{u^1, u^2, \ldots, u^m\}$ be m linearly independent vectors in $\mathbb{R}^n$, with $m \leq n$. The *m-parallelepiped* P_m spanned by $u^1, u^2, \ldots, u^m$ is the set of all $x \in \mathbb{R}^n$ such that $x = \sum_{i=1}^{m} t_i u^i$, where, for each $i \in \{1, \ldots, m\}$, $0 \leq t_i \leq 1$. Let A be the matrix whose rows are the vectors $u^1, u^2, \ldots, u^m$. The m-dimensional volume of P_m is $|\det A^T A|^{1/2}$.

Proof of the statements relative to Proposition 6.3 (*footnote 6*).

We divide this proof in three parts. First, we show that the procedure defined in the proof of Proposition 6.3 provides a dissection of $I(v)$; this dissection is mainly formed by $(n-1)$-parallelepipeds. Second, we show that for each $\varepsilon > 0$, there is $\delta > 0$ such that the sum of the the volumes of the parallelepipeds in the induced dissection of C is ε-close to $\mathrm{Vol}(C)$. Finally, we show a procedure to "put" an elemental core inside a parallelepiped. In order to make this proof more readable we assume, without loss of generality, that for each $i \in N$, $m_i = 0$.

- **The procedure described in the proof of Proposition 6.3 is a "quasi-dissection in parallelepipeds" of $I(v)$:**

Let $x \in I(v)$. There is $r = (r_1, \ldots, r_{n-1}) \in \mathbb{R}^{n-1}$ such that, (i) for each $i \in N$, $r_i \in \{0, \ldots, q\}$ and (ii) for each $i \in N$, $i \neq n$ we have $r_i\delta \leq x_i \leq (r_i + 1)\delta$; note that the second inequality is equivalent to $\sum_{j \neq i} x_j \geq v(N) - (r_i + 1)\delta$. Next, we find the parallelepiped corresponding with this vector r (note that the same point x can lie more than one parallelepiped at the same time). Let P be the parallelepiped spanned by the vectors $\{u_1, \ldots, u_{n-1}\}$, where $u_i = e_i\delta - e_n\delta$ (e_i denotes the ith vector of the canonical base in $\mathbb{R}^n$). Now, since the vectors $\{u_1, \ldots, u_{n-1}\}$ are independent, they generate a parallelepiped. Now, for each $x \in I(v)$, and an associated vector $r \in \mathbb{R}^{n-1}$, x lies in the parallelepiped $P^r := P + (r_1\delta, r_2\delta, \ldots, r_{n-1}\delta, v(N) - \sum_{i \neq n} r_i\delta)$. Note that we have also shown that all the parallelepipeds are equal (changing the translation we just move P onto a different position). But now, as it can be seen in Figure 6.2, a small amount of these parallelepipeds is not completely included in $I(v)$; those for whom the restriction $x_n \geq 0$ is not redundant. We show in the next step that this is not a problem.

- **The induced "quasi-dissection in parallelepipeds" in C can be arbitrarily tight:**

Next, we show that, for each $0 < \varepsilon < 1$, we can find $\delta > 0$ such that the sum of the volumes of the cores of the games in $\mathcal{G}^\delta$ which are not parallelepiped is, at most, $\varepsilon\,\mathrm{Vol}(C)$.

The situation we have is similar to that in Figure 6.2, most of the cores of games in $\mathcal{G}^\delta$ are strictly contained in C. Let $v' \in \mathcal{G}^\delta$ be such that C' is nonempty. There is a parallelepiped P' such that $C' = P' \cap C$. We want to show that, in most of the cases, we have $C' = P' \cap C = P'$ and C' is a parallelepiped (dark shaded zone in the third picture of Figure 6.2). Let d_δ be the maximum euclidean distance between any two points in P (since all the parallelepipeds are translations of each other, d_δ is common to all of them). By definition of P, $\lim_{\delta \to 0} d_\delta = 0$.[9]

[9]The maximum d_δ is achieved when $x = (0, \ldots, 0)$ and $y = \sum_{i=1}^{n-1} u_i = (\delta, \ldots, \delta, -(n-1)\delta)$. The distance between these two points is $(n(n-1))^{1/2}\delta$. Hence, it goes to 0 as δ does.

Each face of C is determined by a restriction R_v^S, where $\emptyset \subsetneq S \subsetneq N$. Hence, the maximum number of faces of the core of a game with n players is $f_n = 2^n - 2$. Let $\mathcal{F}(C)$ denote the set of all faces of C.

Let $y \in C$ be such that the distance from y to each face in $F(C)$ is more than d_δ. Then, y is inside a core C' such that $C' = P' \cap C = P'$ for some parallelepiped P', *i.e.*, C' is itself a parallelepiped. Hence, we can find an upper bound for the volume of the points $y \in C$ which are not in a parallelepiped. Let $F \in F(C)$. Let $B(F, \delta) := \{x \in \mathbb{R}^n : d(x, F) < d_\delta\}$. Now, $\lim_{\delta \to 0} B(F, \delta) = F$ and, since F lies in an $(n-2)$-dimensional space, $\mathrm{Vol}_{n-1}(F) = 0$. Now, since for each $F \in F(C)$, $B(F, \delta)$ is bounded, we have that $\mathrm{Vol}(B(F, \delta))$ goes to 0 as δ does. Hence, for each $0 < \varepsilon < 1$, we can find $\delta > 0$ such that for each $F \in F(C)$, $\mathrm{Vol}(B(F, \delta)) < \frac{\varepsilon}{f_n}$. Once one such δ has been chosen, if $y \in C$ but it is not in a parallelepiped, then it must lie in $B(F, \delta)$ for some face F of C. Hence, the total volume of these points is bounded from above by $\sum_{F \in \mathcal{F}(C)} \mathrm{Vol}(B(F, \delta) < \sum_{F \in \mathcal{F}(C)} \frac{\varepsilon}{f_n} = f_n \frac{\varepsilon}{f_n} = \varepsilon$.

- **Cutting a parallelepiped to obtain an elemental core:** Let v_r be the game such that its core is the parallelepiped P^r defined by $P + (r_1\delta, r_2\delta, \ldots, r_{n-1}\delta, v(N) - \sum_{i \neq n} r_i \delta)$. Let $\chi_{n,k}(v_r)$ be the cut where $k = v(N) - (1 + \sum_{i \neq n} r_i)\delta$. The game $\overline{v}_r$ has an elemental core Δ whose vertices are the following n extreme points:

$$\text{for each } i \in N, \quad p_i = (r_1\delta, r_2\delta, \ldots, r_{n-1}\delta, v(N) - \sum_{i \neq n} r_i\delta) + e_i\delta - e_n\delta.$$

The constant α used in the proof can be calculated as the quotient of the $(n-1)$-dimensional volumes of the simplex Δ and the parallelepiped P containing it. Making some computations with the formulas we introduced when we defined simplices and parallelepipeds we have

$$\mathrm{Vol}(\Delta) = \frac{\sqrt{n}}{(n-1)!}\delta^{n-1} \quad \text{and} \quad \mathrm{Vol}(P) = \sqrt{n}\delta^{n-1}.$$

Hence, $\alpha = \frac{\mathrm{Vol}(S)}{\mathrm{Vol}(P)} = \frac{1}{(n-1)!}$. Once n is fixed, α keeps constant. □

BIBLIOGRAPHY

ANBARCI, N. AND J. BIGELOW (1994): "The Area Monotonic Solution to the Cooperative Bargaining Problem," *Mathematical Social Sciences*, 28, 133–142. (Quoted in pp. 88)

CALVO, E. AND H. PETERS (2000): "Dynamics and Axiomatics of the Equal Area Bargaining Solution," *International Journal of Game Theory*, 29, 81–92. (Quoted in pp. 88)

GILLIES, D. B. (1953): "Some Theorems on n-Person Games," Ph.D. thesis, Princeton. (Quoted in pp. 90)

GONZÁLEZ-DÍAZ, J. AND E. SÁNCHEZ-RODRÍGUEZ (2003): "From Set-Valued Solutions to Single-Valued Solutions: The Centroid and the Core-Center," Reports in Statistics and Operations Research 03-09, University of Santiago de Compostela. (Quoted in pp. 88)

GRITZMAN, P. AND V. KLEE (1994): "On the Complexity of Some Basic Problems in Computational Convexity: Volume and Mixed volumes," in *Polytopes: Abstract, convex and computational*, ed. by T. Bisztriczky, P. McMullen, R. Schneider, and A. Weiss, Kluwer, Boston MA, 373–466. (Quoted in pp. 106)

MASCHLER, M., B. PELEG, AND L. S. SHAPLEY (1979): "Geometric Properties of the Kernel, Nucleolus, and Related Solution Concepts," *Mathematics of Operations Research*, 4, 303–338. (Quoted in pp. 90, 91)

RUDIN, W. (1966): *Real and Complex Analysis*, McGraw-Hill. (Quoted in pp. 97)

SCHMEIDLER, D. (1972): "Cores of Exact Games," *Journal of Mathematical Analysis and Applications*, 40, 214–225. (Quoted in pp. 90)

Chapter 7

The Core-Center and the Shapley Value: A Direct Connection for Convex Games

Contents

7.1 Introduction

In González-Díaz and Sánchez-Rodríguez (2003), a new allocation rule in the core of a TU game, the core-center, is defined for the class of balanced games. The core-center establishes a connection between a set-valued solution (the core) and a single-valued solution (the core-center) that summarizes all the information contained in the former. If we think of the core as a homogeneous body, then the core-center is its center of mass. In other words, it is the expectation of the uniform distribution defined over the core. Hence, the core-center summarizes the information of the core, inherits its properties, and, since it is a central point, it has also appealing motivations from the point of view of fairness. We refer to González-Díaz and Sánchez-Rodríguez (2003) for a detailed analysis and an axiomatic characterization of the core-center.

In the present Chapter we focus on the analysis of the core-center for convex games. A convex game has special properties: it is balanced; its core has a specially regular structure; the core coincides with the convex hull of the vectors of marginal contributions; the core is the unique stable set; the Weber set, the bargaining set, and the core coincide; the kernel coincides with the nucleolus... Moreover, the special geometric structure of the core of a convex game leads to the main finding of this Chapter: a direct relation between two different "centers" of the core, the Shapley value (Shapley, 1953) and the core-center. Recall that the Shapley value for convex games is the center of mass of the vectors of marginal contributions (the extreme points of the core).

One possible criticism for the core-center is that, since there can be different games with the same core, it does not necessarily use all the information of the characteristic function. But, for the class of convex games, two different games cannot have the same core. Hence, for convex games, since the core-center uses the information provided by all the allocations in the core, it also uses all the information of the characteristic function.

In this Chapter we analyze in detail the core of a game by means of a dynamic process among coalitions. Initially, we start with the imputation set; assuming that the players agree on the total amount to be shared and on their individual values. Then, each coalition S announces the value $v(S)$, that represents the utility that the coalition S can obtain independently of $N\backslash S$. Once the players accept that value, the set of stable allocations is reduced. The core is the final result of this process, once all the values of the characteristic function are considered. Once the process is finished, the imputation set can be dissected in several pieces, all of them being cores of games. The core of the original game is a piece of the dissection and the other pieces are cores of some particular games, that we call, utopia games. Each utopia game measures the loss experimented by a coalition once some other coalition announces its value. What we prove in this Chapter is that, for special classes of convex games, the core-center can be expressed as the Shapley value of a specific game, defined through the utopia games. We call this specific game the fair game.

Another important objective we have in this Chapter is to deepen in the study of the geometry of the core of a TU game and we hope that this contribution can help to a better understanding of the relation between the core and the Shapley value.

The outline of this Chapter is the following. In Section 7.2 we present the notation and background concepts. In the first part of Section 7.3, we introduce some geometrical concepts, and then we introduce the core-center. In Section 7.4 we define the utopia games and state the main results; we have moved some of the proofs of the results in this Section to the Appendix. Finally, we conclude in Section 7.5.

7.2 Preliminaries

A cooperative n-player game with transferable utility, shortly, a TU game, is a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a real number $v(S)$; by convention $v(\emptyset) = 0$. For each coalition S, $v(S)$ indicates what the players in S can get by cooperation among themselves. A player $i \in N$ is a *dummy player* if for each $S \subseteq N\backslash\{i\}$, $v(S \cup \{i\}) - v(S) = v(\{i\})$. Let G^n be the set of all n-player games. Given $S \subseteq N$, let $|S|$ be the number of players in S, and let $(S, v_S) \in G^{|S|}$ be the *subgame* such that, for each $T \subseteq S$, $v_S(T) := v(T)$.

A game (N, v) is *convex* if for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N\backslash\{i\}$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. The amount $v(S \cup \{i\}) - v(S)$ is called i's *marginal contribution* to coalition S. Convexity says that, for each $i \in N$, i's marginal contribution to the different coalitions does not decrease as the coalitions grow.

Let $(N, v) \in G^n$. The *efficiency* condition, $\sum_{i\in N} x_i = v(N)$, is used to define the *preimputation set*: $I^*(N, v) := \{x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N)\}$.

A *solution*, defined on some subdomain of G^n, is a correspondence ψ that associates, to each game (N, v) in the subdomain, a subset $\varphi(N, v)$ of its preimputation set $I^*(N, v)$.

The *imputation set*, $I(N, v)$, contains the *individually rational* preimputations, *i.e.*, $I(N, v) := \{x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N)$ and, for each $i \in N$, $x_i \geq v(\{i\})\}$. The imputation set is nonempty if and only if $v(N) \geq \sum_{i\in N} v(\{i\})$. If $v(N) = \sum_{i\in N} v(\{i\})$, then $I(N, v) = (v(\{1\}), \ldots, v(\{n\}))$. If, on the contrary, $v(N) > \sum_{i\in N} v(\{i\})$, then $I(N, v)$ is an $(n-1)$-dimensional simplex with extreme points $e^1, \ldots, e^n$, where $e^i_i(N, v) = v(N) - \sum_{j\neq i} v(\{j\})$ and, for each $j \neq i$, $e^i_j(N, v) = v(\{j\})$.

Let $(N, v) \in G^n$. The *core* (Gillies, 1953), is defined by $C(N, v) := \{x \in I(N, v) :$ for each $S \subseteq N$, $\sum_{i\in S} x_i \geq v(S)\}$. The allocations in the core satisfy the minimum requirements that any coalition might demand in the game. If for each $S \subseteq N$, $v(S) = \sum_{i\in S} v(\{i\})$, then (N, v) is *additive* and $C(N, v) = \{(v(\{1\}), \ldots, v(\{n\})\}$; if a game is additive, then every player is dummy. Moreover, if a game (N, v) is such that (i) for each $S \subsetneq N$, $v(S) = \sum_{j\in S} v(\{j\})$ and (ii) $v(N) > \sum_{i\in N} v(\{i\})$, then $C(N, v) = I(N, v)$. A game is *balanced* if it has a nonempty core. Let BG^n be the set of all n-player balanced games . A game has a *full dimensional* core if the latter has dimension $n - 1$.[1] Each convex game is balanced, but not every balanced game is convex. Let CG^n be the set of all n-player convex games.

[1]A polytope P has dimension m if P is contained in an m-dimensional euclidean space but, for each $m' < m$, no m'-dimensional space contains it.

Let $S \subseteq N$, and let $\Pi(S)$ be the set of all possible orderings of the elements in S, *i.e.*, bijective functions from $\{1, \dots, |S|\}$ to $\{1, \dots, |S|\}$. For each $i \in N$ and each $\sigma \in \Pi(N)$, let $P_\sigma(\{i\}) := \{j \in N : \sigma(j) < \sigma(i)\}$ be the set of *predecessors* of i with respect to σ.

Let $(N, v) \in G^n$ and $\sigma \in \Pi(N)$, the *marginal vector* associated with (N, v) and σ, $m^\sigma(N, v)$, is the vector such that, for each $i \in N$, $m_i^\sigma(N, v) := v(P_\sigma(\{i\}) \cup \{i\}) - v(P_\sigma(\{i\}))$.

An *allocation rule*, defined on some subdomain of G^n, is a function φ that associates, to each game (N, v) in the subdomain, a preimputation $\varphi(N, v)$. An allocation rule φ satisfies *additivity on the characteristic function* if for each pair of games, (N, v) and (N, w), then $\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w)$, where, for each $S \subseteq N$, $(v + w)(S) = v(S) + w(S)$.

Let $(N, v) \in G^n$. The *Shapley value*, Sh, is the average of the marginal vectors, *i.e.*,

$$\text{Sh}(N, v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, v).$$

The Shapley value satisfies, among many other properties, additivity on the characteristic function.

If (N, v) is a convex game, then the marginal vectors are the extreme points of its core, *i.e.*, $C(N, v) = \text{co}\{m^\sigma(N, v) : \sigma \in \Pi(N)\}$.[2] Then, for the class of convex games, the Shapley value is the barycenter of the extreme points of the core where the weight of each extreme point is the number of permutations that originate it. Usually, when working with convex games, the Shapley value is called, with an abuse of language, the barycenter of the core.

7.3 The Core-Center

Before defining the core-center, we introduce some notation regarding the geometrical concepts we deal with in this Chapter. A convex polytope P is the convex hull of a finite set of points in $\mathbb{R}^n$. Henceforth, we omit the word convex because we only deal with such polytopes. A polytope P is an *m-polytope* if its dimension is m, *i.e.*, the smallest integer such that P is contained in an m-dimensional space. The core of a game in BG^n is a polytope and its dimension is, at most, $n - 1$. Let P be an m-polytope and let $m' \geq m$, then, $\text{Vol}_{m'}(P)$ denotes the m'-dimensional volume of P.

Let $\{a^0, a^1, \dots, a^n\} \subsetneq \mathbb{R}^n$ be a geometrically independent set.[3] The *simplex* Δ^n spanned by $a^0, a^1, \dots, a^n$ is the set of all $x \in \mathbb{R}^n$ such that $x = \sum_{i=0}^n t_i a^i$, where $\sum_{i=0}^n t_i = 1$ and, for each $i \in \{1, \dots, n\}$, $t_i \geq 0$. Each a^i is a *vertex* of the n-simplex. The superscript n in Δ^n corresponds with the dimension of the simplex. An n-simplex is regular if the distance between any two vertices is constant.

[2]Given a set $A \subseteq \mathbb{R}^n$, we denote its convex hull by $\text{co}(A)$.

[3]A set $\{a^0, a^1, \dots, a^n\} \subsetneq \mathbb{R}^n$ is geometrically independent if for each vector $(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}$, the equations $\sum_{i=0}^n t_i = 0$ and $\sum_{i=0}^n t_i a^i = (0, \dots, 0)$, hold only if $t_0 = t_1 = \dots = t_n = 0$. Note that $\{a^0, a^1, \dots, a^n\}$ is geometrically independent if and only if the vectors $a^1 - a^0, \dots, a^n - a^0$ are linearly independent.

The *barycenter of a simplex* Δ^n spanned by the points $a^0, a^1, \ldots, a^n$ is $\Theta(\Delta^n) := \sum_{i=0}^{n} \frac{a^i}{n+1}$. Let $m \leq n$, let Δ^m be an m-simplex contained in $\mathbb{R}^n$, and let $a^0, \ldots, a^m$ be its vertices. The m-dimensional volume of Δ^m, $\mathrm{Vol}_m(\Delta^m)$, can be computed in the following way: Let $B = (\beta_{ij})$ denote the $(m+1) \times (m+1)$ matrix given by $\beta_{ij} = \|a^i - a^j\|^2$. Then,

$$2^m (m!)^2 \,\mathrm{Vol}_m(\Delta^m)^2 = |\det(\hat{B})|,$$

where $\hat{B}$ is the $(m+2)\times(m+2)$ matrix obtained from B by bordering it with a top row $(0, 1, \ldots, 1)$ and left column $(0, 1, \ldots, 1)^T$. This is known as the Cayley-Menger determinant formula.[4]

Let P be an m-polytope in $\mathbb{R}^n$. Then, the set of polytopes $\{P_1, \ldots, P_k\}$ defines a *dissection* of P if (i) $P = \bigcup_{l=1}^{k} P_l$ and (ii) for each pair $l, l' \in \{1, \ldots, k\}$, with $l \neq l'$, $\mathrm{Vol}_m(P_l \cap P_{l'}) = 0$.

Let $U(A)$ be the uniform distribution defined over the set A and $E(\mathbb{P})$ the expectation of the probability distribution $\mathbb{P}$. Let $(N, v) \in BG^n$. The *core-center* of (N, v), $\mu(N, v)$, is defined as follows:

$$\mu(N, v) := E\left[U(C(N, v))\right].$$

Next, we define an additivity property satisfied by the core-center that plays an essential role in this Chapter. Let $(N, v) \in BG^n$ such that $C(N, v)$ is an m-polytope, with $m \leq n - 1$. Let $\{v_1, \ldots, v_k\}$ be such that $\{C(N, v_1), \ldots, C(N, v_k)\}$ is a dissection of $C(N, v)$. Then, an allocation rule φ satisfies *w-additivity* (where "w" stands for weighted) if

$$\mathrm{Vol}_m(C(N, v))\varphi(N, v) = \sum_{l=1}^{k} \mathrm{Vol}_m(C(N, v_l)\varphi(N, v_l).$$

The w-additivity of the core-center is an immediate consequence of the classical properties of the barycenter of a set.

Next, we prove a series of simple results that establish a first connection between the core-center and the Shapley value.

Lemma 7.1. *Let $(N, v) \in G^n$, with $n \geq 2$, be such that $v(N) > \sum_{j \in N} v(\{j\})$. Then:*

(i) *For each $i \in N$, $\Theta_i(I(N, v)) = v(\{i\}) + \frac{v(N) - \sum_{k \in N} v(\{k\})}{n}$.*

(ii) *If for each $S \subsetneq N$, $v(S) = \sum_{i \in S} v(\{i\})$, then,*

$$\mu(N, v) = \mathrm{Sh}(N, v) = \Theta(I(N, v)).$$

(iii) $\mathrm{Vol}_{n-1}(I(N, v)) = \frac{\sqrt{n}}{(n-1)!}\left(v(N) - \sum_{j \in N} v(\{j\})\right)^{n-1}$.

Proof. (i) Follows from the formula for the barycenter of a simplex.

(ii) Follows from the following observations: v is convex and $C(N, v) = I(N, v)$.

(iii) It is a consequence of the formula for the volume of a simplex. □

[4]For references on this and other formulas for computing simplicial volumes look at Gritzman and Klee (1994).

Corollary 7.1. *Let $(N,v)\in G^2$ be such that $v(N)>v(\{1\})+v(\{2\})$. Then,*

$$\mathrm{Sh}(N,v)=\Theta(I(N,v))=\mu(N,v).$$

Moreover, they coincide with the barycenter (midpoint) of the segment joining the two points $\big(v(\{1\}),v(N)-v(\{1\})\big)$ and $\big(v(N)-v(\{2\}),v(\{2\})\big)$.

Proof. Immediate from Lemma 7.1. □

Let D_v be the set of dummy players of (N,v) and d_v its cardinality. Recall that, if $d_v=n$, then the game is additive.

Lemma 7.2. *Let $(N,v)\in G^n$. Then, the following statements are true:*

(i) $d_v\neq n-1$.

(ii) *Let $(N,v)\in BG^n$ and $d_v<n$. Then, $x_N\in C(N,v)$ if and only if (i) for each $i\in D_v, x_i=v(\{i\})$ and (ii) $x_{N\backslash D_v}\in C(N\backslash D_v,v_{N\backslash D_v})$.*

(iii) *Let $(N,v)\in CG^n$ and $d_v<n$. Then, $(N\backslash D_v,v_{N\backslash D_v})\in CG^{n-d_v}$,* i.e., *it is a convex game with full dimensional core.*

Proof. **a)** Suppose that $d_v\geq n-1$. Then, there is $i\in N$ such that $N\backslash\{i\}\subseteq D_v$. Suppose now that $i\notin D_v$. Let S be a minimal coalition among those such that $i\notin S$ and $v(S\cup\{i\})-v(S)\neq v(\{i\})$. Clearly, $S\neq\emptyset$. Let $j\in S$, then,

$$\begin{aligned} v((S\cup\{i\})\backslash\{j\}) &= v((S\cup\{i\})\backslash\{j\})-v(S\backslash\{j\})+v(S\backslash\{j\}) \\ &\underset{S\backslash\{j\}\subsetneq S}{=} v(\{i\})+\sum_{l\in S\backslash\{j\}} v(\{l\}). \end{aligned}$$

Now, $v(\{j\})=v(S\cup\{i\})-v((S\cup\{i\})\backslash\{j\})=v(S\cup\{i\})-v(\{i\})-\sum_{l\in S\backslash\{j\}}v(\{l\})$. Hence, since $\sum_{l\in S}v(\{l\})=v(S)$, we have $v(S\cup\{i\})-v(S)=v(\{i\})$, contradicting the definition of S. Hence, $i\in D_v$ and $d_v=n$.

b) Follows from the equality $v(N)=v(N\backslash D_v)+\sum_{j\in D_v}v(\{j\})$.

c) Each subgame of a convex game is a convex game. Hence, $(N\backslash D_v,v_{N\backslash D_v})$ is a convex game with no dummy players. Let (N,w) be a convex game. Since (i) for each $i\in N$, there is a marginal vector such that $m_i^\sigma=v(\{i\})$ and (ii) $C(N,w)$ is the convex hull of the marginal vectors, $C(N,w)$ has at least one point in each face of $I(N,w)$. Now, if (N,w) has no dummy players, then, for each $i\in N$, there is a marginal vector such that $m_i^{\sigma'}>v(\{i\})$. The full dimensionality of the core of each convex game with no dummy players follows from the combination of the two previous observations. □

Corollary 7.2. *Let (N,v) be a convex game with $n-2$ dummy players. Then,* $\mathrm{Sh}(N,v)=\mu(N,v)$.

Proof. It follows from Lemma 7.2 and Corollary 7.1. □

Remark. Lemma 7.2 shows that coalitions involving dummy players are not needed in order to compute the core-center. Then,

$$\mu_i(N,v) = \begin{cases} v(\{i\}) & i \in D_v \\ \mu_i(N\backslash D_v, v_{N\backslash D_v}) & i \notin D_v. \end{cases}$$

Note that the Shapley value also satisfies that for each $i \in D_v$, $\mathrm{Sh}_i(N,v) = v(\{i\})$.

Because of the previous Remark, we do not consider games with dummy players anymore. The main results in the next Section hold for convex games without dummy players or, equivalently, convex games with full dimensional core.

7.4 The Dynamic Process between Coalitions. The Utopia Games

The class of *exact* games (Schmeidler, 1972) is a subclass of BG^n. The main property of the games in this subclass is that, given two exact games, if they have the same core, then they are the same game, *i.e.*, no two distinct exact games have the same core. Hence, when working with exact games, the core uses all the information of the underlying game. Since the class of convex games is contained in the class of exact games, the last observation is also relevant to our framework: no two distinct convex games have the same core. This property reinforces the motivations for the core-center within the class of convex games. Since the core uses all the information of the game, why not to select the allocation rule that summarizes all the information of the core?

The results in this Section are for games with full dimensional core. Hence, when no confusion arises, $\mathrm{Vol}(P)$ denotes the $(n-1)$-dimensional volume of polytope P.

Let N be a set of players and suppose that the game (N,v) is gradually defined. First, the players agree on the amount $v(N)$ that is to be divided. Then, they agree on the individual values. Hence, only $v(N)$ and, for each $i \in N$, the value $v(\{i\})$ are determined. To formalize this step we define the game $(N, v_\emptyset)$, where the players do not gain anything by forming coalitions different from N,

$$v_\emptyset(S) = \begin{cases} \sum_{l \in S} v(\{l\}) & S \neq N \\ v(N) & S = N. \end{cases}$$

At this point, a fair allocation rule should provide some payoff in the imputation set, and without any more information, why not choose the center of the imputation set?

Suppose now that coalitions enter in the game and, for instance, players 1 and 2 announce that, together, they can get $v(\{1,2\})$. At this point, the set of "stable" points is $C_{1,2} = \{x \in I(N,v) : x_1 + x_2 \geq v(\{1,2\})\}$. Clearly, $C_{1,2}$ is a subset of the imputation set and, the larger is the difference $v(\{1,2\}) - (v(\{1\}) + v(\{2\}))$, the smaller is $C_{1,2}$. Next, we want measure how big is this set of "stable" points, which is contained in $I(N,v)$. One natural way is through the difference of the volumes, *i.e.*, $\mathrm{Vol}(I(N,v)) - \mathrm{Vol}(C_{1,2})$ (note that allocations in $I(N,v)\backslash C_{1,2}$ are the good ones for coalition $N\backslash\{1,2\}$). Now, we can repeat the argument with some other coalition different from $\{1,2\}$; we can compare the different coalitions and "measure" their differences.

Next, we introduce a new class of games: the *utopia games*. Roughly speaking, these games are such that the core of the utopia game for coalition S is precisely the set of allocations that are not "stable" after coalition $N\backslash S$ announces $v(N\backslash S)$. Formally, let (N,v) be a convex game, and let $T\in 2^N\backslash\emptyset$. Let $H\in 2^T\backslash\emptyset$. Then, we define the game $(N,v_T^H)\in G^n$ as follows:[5]

$$v_T^H(S)=\begin{cases} v((T\cap S)\cup(N\backslash T))-v(N\backslash T)+v(S\backslash(T\cap S)) & H\subseteq S\\ v((T\cap S)\cup(N\backslash T))-v(N\backslash T)+\sum\limits_{l\in S\backslash(T\cap S)} v(\{l\}) & \text{otherwise.}\end{cases}$$

Despite of the apparent complexity of this definition, the specific utopia games we look at in this Chapter allow for more transparent expressions.

It is easy to check that

- $v_T^H(\emptyset)=0$,
- $v_T^H(N)=v(N)$,
- if $T\cap S=\emptyset$, $v_T^H(S)=\sum_{l\in S} v(\{l\})$.

Next, we interpret the game (N,v_T^H). For each coalition $S\neq T$, the value $v_T^H(S)$ is the sum of two quantities. First, the marginal contribution of the players in S that are in T to $N\backslash T$. Second, the contribution of players in S that are not in T. Hence, what a coalition $S\subseteq T$ obtains in the game v_T^H is its marginal contribution to $N\backslash T$, *i.e.*, $v_T^H(S)=v(S\cup(N\backslash T))-v(N\backslash T)$; note that, if $S=T$, $v_T^H(T)=v(N)-v(N\backslash T)$.

Take now $S\subseteq N$ such that $\emptyset\subsetneq T\cap S\subsetneq S$. The contribution of the players in S that are not in T depends on the coalition H. Fixed H, if $H\subseteq S$, the contribution of players in $S\backslash(T\cap S)$ is the utility that they can guarantee themselves by joining together, *i.e.*, $v(S\backslash(T\cap S))$. On the other hand, if $H\nsubseteq S$, that contribution is computed by $\sum_{l\in S\backslash(T\cap S)} v(\{l\})$. Roughly speaking, players in H can be thought as the ones who have the key to allow for cooperation.

The main idea underlying the games v_T^H is that the players in T are the ones who have the power in the game, but always respecting the minimum rights of players in $N\backslash T$. In addition, the game also establishes, via coalition H, a hierarchical structure among players in T. As next proposition reads, these games are convex.

Proposition 7.1. *Let $(N,v)\in CG^n$, $T\in 2^N\backslash\emptyset$, and $H\in 2^T\backslash\emptyset$. Then, $(N,v_T^H)\in CG^n$.*

Proof. See the Appendix. □

Next, we study utopia games defined by coalitions with, at most, two players. For these special utopia games, the intuitions highlighted in the discussion above should become clearer.

[5] We do not use the subgames (S,v_S) anymore. Hence, no confusion can arise because of the notation for utopia games.

Let $i \in N$ and $T = \{i\}$; in this case $H = T$. Henceforth, we denote, for each $i \in N$, the game $(N, v_{\{i\}}^{\{i\}})$ by (N, v_i). The game (N, v_i) is the utopia game for player i, shortly, i-utopia game:

$$\text{For each } S \subseteq N, \quad v_i(S) = \begin{cases} v(N) - v(N\setminus \{i\}) + v(S\setminus\{i\}) & i \in S \\ \sum_{l \in S} v(\{l\}) & i \notin S. \end{cases}$$

In this game player i has the key for cooperation. The other players by themselves can only get the sum of their individual values.

We describe now the games v_T^H where T is a 2-player coalition, $T = \{i, j\} \subseteq N$ with $i \neq j$. To this extent, we define two games:

$$(N, v_{\{i,j\}}^{\{i\}}) \text{ and } (N, v_{\{i,j\}}^{\{j\}}),^{6}$$

that we denote by $(N, v_{(i,j)})$ and $(N, v_{(j,i)})$, respectively. The former, that we call (i, j)-utopia game is good for both players 1 and 2, but excellent for player 1:

$$\begin{aligned} v_{(i,j)}(S) &= \begin{cases} v((T\cap S) \cup (N\setminus T)) - v(N\setminus T) + v(S\setminus(T\cap S)) & i \in S \\ v((T\cap S) \cup (N\setminus T)) - v(N\setminus T) + \sum_{l \in S\setminus(T\cap S)} v(\{l\}) & i \notin S \end{cases} \\ &= \begin{cases} v(N) - v(N\setminus\{i,j\}) + v(S\setminus\{i,j\}) & i \in S, j \in S \\ v(N\setminus\{j\}) - v(N\setminus\{i,j\}) + v(S\setminus\{i\}) & i \in S, j \notin S \\ v(N\setminus\{i\}) - v(N\setminus\{i,j\}) + \sum_{l \in S\setminus\{j\}} v(\{l\}) & i \notin S, j \in S \\ \sum_{l \in S} v\{l\}) & i \notin S, j \notin S. \end{cases} \end{aligned}$$

Analogously, by interchanging the role of i and j, we can define the (j, i)-utopia game.

The next concept leads to a classification of the games in CG^n. This classification looks at the size of the smaller coalition, say S, such that $v(S) > \sum_{i \in S} v(\{i\})$, *i.e.*, joining together to form S is profitable for the players in S. Formally, let $(N, v) \in CG^n$, $n > 2$. Let $t \in \{1, \ldots, n-1\}$. Then, $(N, v) \in CG_t^n$ if (i) for each $S \subseteq N$ with $|S| \leq t$, $v(S) = \sum_{i \in S} v(\{i\})$ and (ii) there is a coalition S, $|S| = t + 1$, such that $v(S) > \sum_{i \in S} v(\{i\})$. On the other hand, if $v(N) = \sum_{i \in N} v(\{i\})$, then $(N, v) \in CG_n^n$. Let $(N, v) \in CG^n$, then there is $t \in \{1, \ldots, n\}$ such that $(N, v) \in CG_t^n$. Note that $I(N, v) = C(N, v)$ if and only if $(N, v) \in CG_t^n$, with $t \geq n - 1$, and then, by Lemma 7.1 both Shapley value and core-center coincide with the barycenter of the imputation set. Let $(N, v) \in CG_t^n$, with $t < n$, then, (i) the core restrictions originated by the m-player coalitions, $1 < m \leq t$, are redundant and (ii) there is at least one coalition with more than t players imposing a non-redundant restriction on $C(N, v)$.

Lemma 7.3. *Let* $(N, v) \in CG_{n-2}^n$, $n > 2$. *Then:*

[6] We could also define the game $(N, v_{\{i,j\}}^{\{i,j\}})$, where $H = T$. But, since we do not use it in our results, we skip its definition.

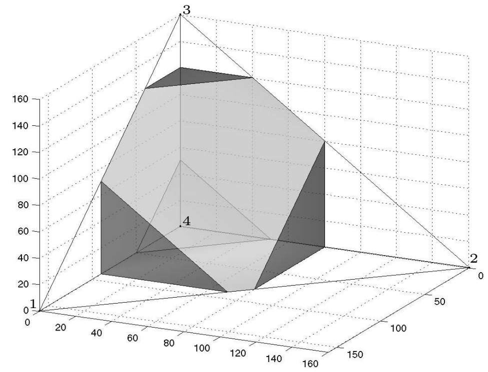

Figure 7.1: Core of a game in CG_2^4

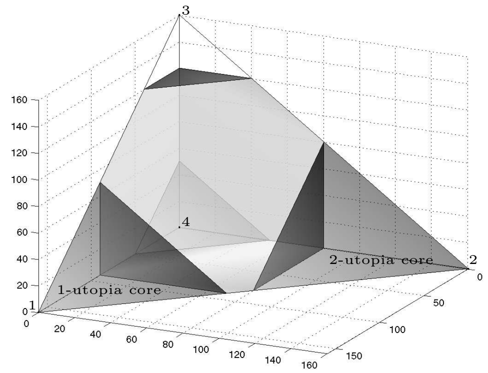

Figure 7.2: Cores of two utopia games

(i) *For each* $i \in N$, $C(N, v_i) = I(N, v_i)$ *and*

$$\frac{\text{Vol}(C(N, v_i))}{\text{Vol}(I(N, v))} = \left(\frac{v(N\backslash\{i\}) - \sum_{l \in N\backslash\{i\}} v(\{l\})}{v(N) - \sum_{l \in N} v(\{l\})}\right)^{n-1}.$$

(ii) *Let* $i, j \in N$, $i \neq j$. *Then,* $C(N, (v_i)_j) = m^\sigma(N, v)$, *where* $\sigma \in \Pi(N)$ *is such that* $\sigma(i) = n$ *and* $\sigma(j) = n - 1$.

(iii) $I(N, v) = \left(\bigcup_{i \in N} C(N, v_i)\right) \cup C(N, v)$.

(iv) $\text{Vol}(I(N, v)) = \sum_{i \in N} \text{Vol}(C(N, v_i)) + \text{Vol}(C(N, v))$.

Proof. See the Appendix. □

Let $(N, v) \in CG_{n-2}^n$. Let $p := \text{Vol}(C(N, v))$, $p_0 = \text{Vol}(C(N, v_\emptyset))$, and, for each $i \in N$, $p_i = \text{Vol}(C(N, v_i)$. Let the *fair game* associated with (N, v), (N, v^*), be defined as follows:

$$v^*(S) = \frac{1}{p}\big(p_0 v_\emptyset(S) - \sum_{i \in N} p_i v_i(S)\big).$$

Theorem 7.1. *Let* $(N, v) \in CG_{n-2}^n$, $n > 2$. *Then,* $\mu(N, v) = \text{Sh}(N, v^*)$.

Proof. The core-center satisfies w-additivity and, by Lemma 7.3, the imputation set can be dissected into $n + 1$ polytopes, the core of (N, v) and the cores of the utopia games. Hence, $\mu(N, v_\emptyset) = \frac{p}{p_0}\mu(N, v) + \sum_{i \in N} \frac{p_i}{p_0}\mu(N, v_i)$. By Lemma 7.1, $\mu(N, v_\emptyset) = \text{Sh}(N, v_\emptyset)$, and, for each

$i \in N$, $\mu(N, v_i) = \mathrm{Sh}(N, v_i)$. Hence,

$$\begin{aligned}\mu(N,v) &= \frac{p_0}{p}\big(\mu(N,v_\emptyset) - \sum_{i\in N}\frac{p_i}{p_0}\mu(N,v_i)\big) = \frac{p_0}{p}\,\mathrm{Sh}(N,v_\emptyset) - \sum_{i\in N}\frac{p_i}{p}\,\mathrm{Sh}(N,v_i)\\ &= \frac{1}{p}\big(p_0\,\mathrm{Sh}(N,v_\emptyset) - \sum_{i\in N} p_i\,\mathrm{Sh}(N,v_i)\big) = \mathrm{Sh}(N,v^*),\end{aligned}$$

where the last equality holds by the additivity of the Shapley value. □

Remark. This proof has the following feature: we start with the core-center of a game and, in two steps, using both the w-additivity of the core-center and the additivity of the Shapley value, we end up with the Shapley value of the fair game.

Corollary 7.3. *Let $(N, v) \in CG^3$. Then, $\mu(N, v) = \mathrm{Sh}(N, v^*)$.*

Proof. Immediate from Theorem 7.1. □

Remark. If $(N, v) \in CG^3$, the game (N, v^*) summarizes all the information of the core. The core of the fair game coincides with its imputation set and contains $C(N, v)$. Following the definition of (N, v^*) we have, for each $i \in N$, $v^*(\{i\}) = v(\{i\}) - \frac{p_i}{p}\big(v(N) - v(N\backslash\{i\}) + v(\{i\})\big)$. Hence,

$$\mu(N,v) = v^*(\{i\}) + \frac{1}{n}\big(v(N) - \sum_{k\in N} v^*(\{k\})\big).$$

Example 7.1. *Let $(N, v) \in G^3$ be such that, for each $i \in N$, $v(\{i\}) = 0$; $v(\{1, 2\}) = 2, v(\{1, 3\}) = v(\{2, 3\}) = 5$, and $v(N) = 10$. Then,*

S	$v_\emptyset$	v_1	v_2	v_3	$v_{\{1,2\}}$	$v_{\{1,3\}}$	$v_{\{2,3\}}$	v^*
$\{1\}$	0	5	0	0	5	2	0	−2.7174
$\{2\}$	0	0	5	0	5	0	2	−2.7174
$\{3\}$	0	0	0	8	0	5	5	−0.6957
$\{1,2\}$	0	5	5	0	10	2	2	−5.4348
$\{1,3\}$	0	5	0	8	5	10	5	−3.4130
$\{2,3\}$	0	0	5	8	5	5	10	−3.4130
N	10	10	10	10	10	10	10	10

and,

$$\begin{aligned}\mathrm{Sh}(N,v) &= (2.8333, 2.8333, 4.3333)\\ \mu(N,v) &= (2.6594, 2.6594, 4.6812).\end{aligned}$$

Let $r := p/p_0$ and, for each $i \in N$, $r_i := p_i/p_0$. In this example we have $r_1 = r_2 = 1/4$, $r_3 = 1/25$ and $r = 1 - (r_1 + r_2 + r_3) = 23/50$. Hence, players 1 and 2 are less powerful than player 3. Note *that* $C(N, v_{(1,2)}) = C(N, v_{(2,1)}) = (5, 5, 0)$. *Moreover,* $x \in C(N, v_{\{1,2\}})$ *if and only if*

$$\begin{array}{rcccl} v(\{1,3\}) - v(3) & \leq & x_1 & \leq & v(N) - v(\{2,3\}), \\ v(\{2,3\}) - v(3) & \leq & x_2 & \leq & v(N) - v(\{1,3\}), \text{ and} \\ & & x_3 & = & v(\{3\}). \end{array}$$

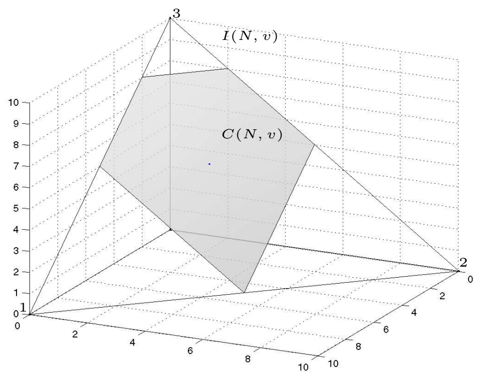

Figure 7.3: Core of a game in CG^3

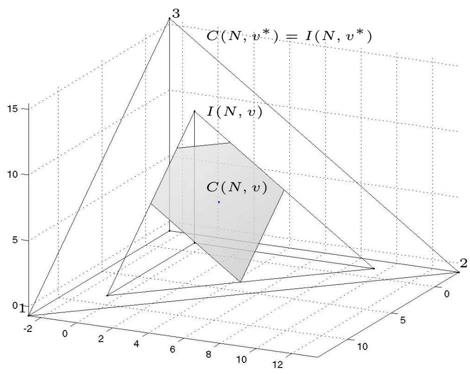

Figure 7.4: The core of the fair game

Since $\sum r_i = 1$ and, for each $i \in N$, $0 \leq r_i \leq 1$, the ratios r_i have the following interpretation. They determine a probability distribution over the cores of the utopia games and hence, over the imputation set; r is the probability that an allocation in $I(N, v)$ belongs to $C(N, v)$ and, for each $i \in N$, r_i is the probability that an allocation in $I(N, v)$ belongs to $C(N, v_i)$. Hence, the greater the core of the i-utopia game is, the worse is i's situation in the game. Roughly speaking, Figures 7.3 and 7.4 show that for (N, v), the "big" utopia cores are those of the utopia games of players 1 and 2. Hence, in the core of the fair game, the "bad" section that has been added for player 3 (with respect to the core of the original game) is smaller than those for players 1 and 2.

We turn now to study games in CG^n_{n-3}. We denote the game $(v_{(i,j)})_j$ by $v_{(i,j)_j}$.

Lemma 7.4. *Let* $(N, v) \in CG^n_{n-3}$, $n > 3$, *and let* $i \in N$. *Then,*

(i) $(N, v_i) \in CG^n_{n-t}$, $t < 3$.

(ii) *For each* $i, j \in N$, $i \neq j$, $C(N, (v_i)_j) = I(N, (v_i)_j)$.

(iii) $\mu(N, v_i) = \text{Sh}(N, (v_i)^*)$.

Proof. (i) It suffices to show that, for each $S \subseteq N$ such that $|S| \leq n - 2$, we have $v_i(S) =$

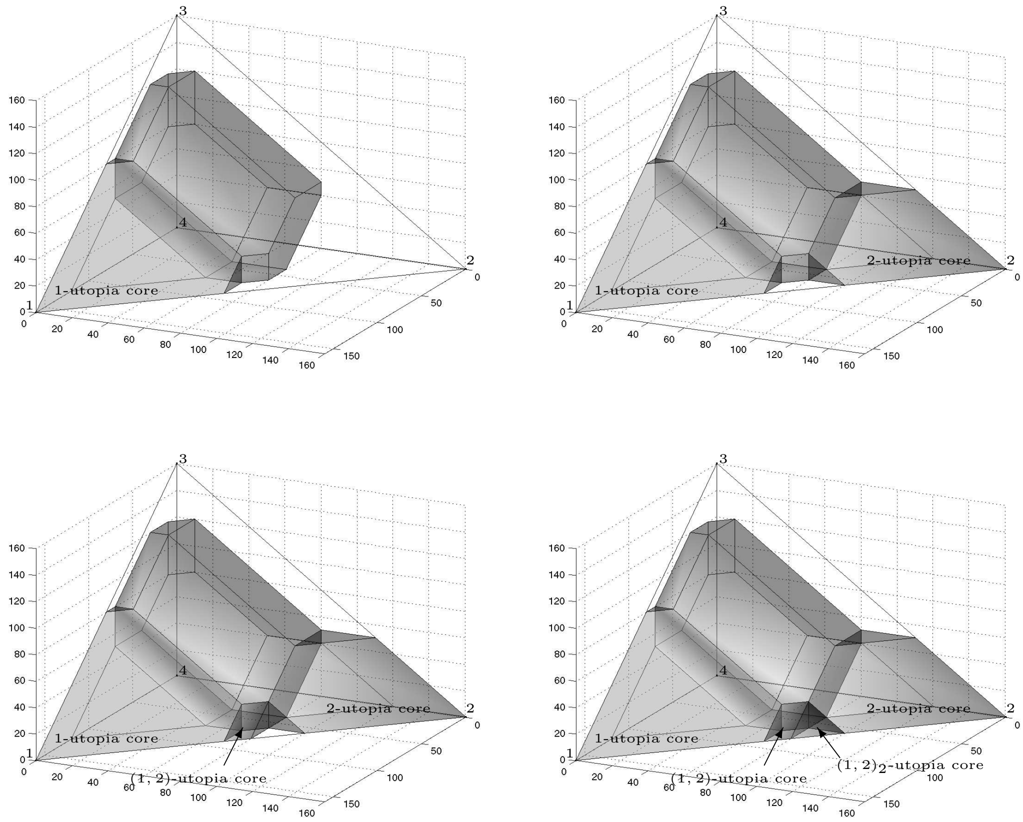

Figure 7.5: Example of a game in CG_3^4 with some of its utopia games

$\sum_{k\in S} v_i(\{k\})$. Now, for each $S \subseteq N$,

$$v_i(S) = \begin{cases} v(N) - v(N\setminus\{i\}) + v(S\setminus\{i\}) & i \in S, |S| \geq n-2 \\ v(N) - v(N\setminus\{i\}) + \sum_{l\in S\setminus\{i\}} v(\{l\}) & i \in S, |S| < n-2 \\ \sum_{l\in S} v(\{l\}) & i \notin S. \end{cases}$$

Hence, if $|S| < n-2$, the result is immediate; if $|S| = n-2$, since $|S\setminus\{i\}| = n-3$ we have $v(S\setminus\{i\}) = \sum_{l\in S\setminus\{i\}} v(\{l\})$. Hence, for each $S \subseteq N$ such that $|S| \leq n-2$, $v_i(S) = \sum_{k\in S} v_i(\{k\})$.

(ii) and (iii) follow from Lemma 7.3 and Theorem 7.1. □

Lemma 7.5. *Let $(N,v) \in CG^n_{n-3}$, $n > 3$. Let $i,j \in N$, $i \neq j$ and let $(N, v_{(i,j)})$ be the (i,j)-utopia game. Then, if $C(N, v_{(i,j)})$ is full dimensional we have*

(i) $\text{Sh}(N, v_{(i,j)}) = \mu(N, v_{(i,j)})$.

(ii) $\text{Vol}(C(N, v_{(i,j)})) =$

$$= \frac{\sqrt{n}}{(n-2)!}\Big(v(N\setminus\{i,j\}) - \sum_{l\in N\setminus\{i,j\}} v(\{l\})\Big)^{n-2}\Big(v(N) - v(N\setminus\{i\}) + v(N\setminus\{i,j\}) - v(N\setminus\{j\})\Big)$$

Proof. See the Appendix. □

Lemma 7.6. *Let $(N,v) \in CG^n_{n-3}$, $n > 3$. Let $i,j \in N$, $i \neq j$ and let $(N, v_{(i,j)})$ be the (i,j)-utopia game. Then,*

(i) *For each $S \subseteq N$, $v_{(i,j)_j}(S) = (v_j)_i(S)$ and $C(N, v_{(i,j)_j}) = C(N, (v_j)_i)$.*

(ii) $\mu(N, v_{(i,j)_j}) = \text{Sh}(N, v_{(i,j)_j})$.

(iii) *We have the following dissection of the set of imputations:*

$$I(N,v) = C(N, v_\emptyset) = \Big(\bigcup_{i\in N} C(N, v_i)\Big) \cup \Big(\bigcup_{i<j} \big(C(N, v_{(i,j)}) \cup C(N, v_{(i,j)_j})\big)\Big) \cup C(N,v).$$

Proof. See the Appendix. □

Let $(N,v) \in CG^n_{n-3}$. As before, let $p := \text{Vol}(C(N,v))$, $p_0 = \text{Vol}(C(N, v_\emptyset))$, and, for each $i \in N$, $p_i = \text{Vol}(C(N, v_i)$. Moreover, for each pair $i,j \in N$, let $p_{(i,j)} = \text{Vol}(C(N, v_{(i,j)}))$ and $p_{(i,j)_j} = \text{Vol}(C(N, v_{(i,j)_j}))$. Let the *fair game* associated with (N,v), (N, v^*), be defined as follows:

$$v^*(S) = \frac{1}{p}\Big(p_0 v_\emptyset(S) - \sum_{i\in N} p_i v_i(S) - \sum_{i\neq j} \frac{1}{2}\big(p_{(i,j)} v_{(i,j)}(S) + p_{(i,j)_j} v_{(i,j)_j}(S)\big)\Big).$$

Since for each $(N,v) \in CG^n_{n-2}$, the coefficients $p_{(i,j)}$ and $p_{(i,j)_j}$ are 0, this definition of fair game is consistent with the old one.

Let $r_{(i,j)} = p_{(i,j)}/p_0$ and $r_{(i,j)_j} = p_{(i,j)_j}/p_0$. Then,

$$r_{(i,j)} = \frac{\frac{\sqrt{n}}{(n-2)!}\Big(v(N\backslash\{i,j\})-\sum_{l\in N\backslash\{i,j\}} v(\{l\})\Big)^{n-2}\Big(v(N)-v(N\backslash\{i\})+v(N\backslash\{i,j\})-v(N\backslash\{j\})\Big)}{\frac{\sqrt{n}}{(n-1)!}\Big(v(N)-\sum_{l\in N} v(\{l\})\Big)^{n-1}}$$

$$= (n-1)\Big(\tfrac{v(N\backslash\{i,j\})-\sum_{l\in N\backslash\{i,j\}} v(\{l\})}{v(N)-\sum_{l\in N} v(\{l\})}\Big)^{n-1}\tfrac{v(N)-v(N\backslash\{i\})+v(N\backslash\{i,j\})-v(N\backslash\{j\})}{v(N)-\sum_{l\in N} v(\{l\})},$$

and,

$$r_{(i,j)_j} = \left(\frac{v(N\backslash\{i,j\})-\sum_{l\in N\backslash\{i,j\}} v(\{l\})}{v(N)-\sum_{l\in N} v(\{l\})}\right)^{n-1}.$$

Again, the numbers $r_{(i,j)}$ and $r_{(i,j)_j}$ can be interpreted as the probabilities that an allocation in $C(N, v_{(i,j)})$ or $C(N, v_{(i,j)_j})$, respectively, is chosen. According to our interpretation of the utopia games, if an allocation in $C(N, v_{(i,j)})$ is chosen, the coalition $\{i, j\}$ would receive an "utopic" payoff, *i.e.*, allocations in $C(N, v_{(i,j)})$ are the best for coalition $\{i, j\}$ within $I(N, v)$. Note that $r_{(i,j)} = r_{(j,i)}$ and $r_{(i,j)_j} = r_{(j,i)_i}$. This observation is crucial to understand the following result.

Lemma 7.7. *Let $(N, v) \in CG^n_{n-3}$, $n > 3$. Let $i, j \in N$, $i \neq j$ and let $(N, v_{(i,j)})$ be the (i, j)-utopia game. Then,*

$$r_{(i,j)}\mu(N, v_{(i,j)}) + r_{(i,j)_j}\mu(N,(v_{(i,j)})_{\{j\}}) = r_{(j,i)}\mu(N, v_{(j,i)}) + r_{(j,i)_i}\mu(N,(v_{(j,i)})_{\{i\}}).$$

Proof. This result is a consequence of the following equality:

$$C(N, v_{(i,j)}) \cup C(N, v_{(i,j)_j}) = C(N, v_{(j,i)}) \cup C(N, v_{(j,i)_i}). \qquad \square$$

Lemma 7.7 shows that, given any two players, there is an important symmetry between the two corresponding 2-player utopia games. Next, we state our main Theorem. It provides a direct relation between the core-center and the Shapley value of the fair game.

Theorem 7.2. *Let $(N, v) \in CG^n_{n-3}$, $n > 3$. Then, $\mu(N, v) = \mathrm{Sh}(N, v^*)$.*

Proof. Since

$$C(N, v_\emptyset) = \Big(\bigcup_{i\in N} C(N, v_i)\Big) \cup \Big(\bigcup_{i<j}\big(C(N, v_{(i,j)}) \cup C(N, v_{(i,j)_j})\big)\Big) \cup C(N, v),$$

we have

$$\begin{aligned}\mu(N, v_\emptyset) &= \sum_{i\in N}\tfrac{p_i}{p_0}\mu(N, v_i) + \sum_{i<j}\Big(\tfrac{p_{(i,j)}}{p_0}\mu(N, v_{(i,j)}) + \tfrac{p_{(i,j)_j}}{p_0}\mu(N, v_{(i,j)_j})\Big) + \tfrac{p}{p_0}\mu(N, v)\\ &= \sum_{i\in N}\tfrac{p_i}{p_0}\mu(N, v_i) + \sum_{i\neq j}\Big(\tfrac{p_{(i,j)}}{2p_0}\mu(N, v_{(i,j)}) + \tfrac{p_{(i,j)_j}}{2p_0}\mu(N, v_{(i,j)_j})\Big) + \tfrac{p}{p_0}\mu(N, v),\end{aligned}$$

where the last equality holds by Lemma 7.7. Now, by Lemmas 7.1 and 7.5,[7] and by the additivity of the Shapley value, we have,

$$\begin{aligned}\mu(N,v) &= \tfrac{p_0}{p}\mu(N,v_\emptyset) - \sum_{i\in N}\tfrac{p_i}{p}\mu(N,v_i) - \sum_{i\neq j}\Big(\tfrac{p_{(i,j)}}{2p}\mu(N,v_{(i,j)}) + \tfrac{p_{(i,j)_j}}{2p}\mu(N,v_{(i,j)_j})\Big)\\ &= \tfrac{p_0}{p}\,\mathrm{Sh}(N,v_\emptyset) - \sum_{i\in N}\tfrac{p_i}{p}\,\mathrm{Sh}(N,v_i) - \sum_{i\neq j}\Big(\tfrac{p_{(i,j)}}{2p}\,\mathrm{Sh}(N,v_{(i,j)}) + \tfrac{p_{(i,j)_j}}{2p}\,\mathrm{Sh}(N,v_{(i,j)_j})\Big)\\ &= \mathrm{Sh}(N,v^*). \qquad \square\end{aligned}$$

Corollary 7.4. *Let* $(N,v) \in CG^4$. *Then,* $\mu(N,v) = \mathrm{Sh}(N,v^*)$.

Proof. Immediate from Theorem 7.2. □

Remark. The fair game uses all the information of the original game. Suppose that we only have the value of the grand coalition and the values for the 1-player coalitions; with this information on the table, only the allocations within the set of imputations (the core of $(N,v_\emptyset)$) seem to be reasonable. Hence, the center of the imputation set would be a fair outcome, *i.e.*, every player gets what he would expect if an imputation were to be randomly picked. But, if we had all the characteristic function on the table, would it still be fair? Now, stability could turn to be a concern. Hence, players must take into account that each coalition S such that $v(S) > \sum_{i\in S} v(\{i\})$ is imposing a relevant constraint in the imputation set. The fair game uses all that information and, indirectly, measures its relevance. There is also a probabilistic interpretation for the 2-player coalitions: when two players, say i and j, form a coalition, there are two ways of ordering them within the coalition; we assign equal probability to each of them, *i.e.*, the orderings $\{i,j\}$ and $\{j,i\}$ are equally likely.

7.5 Concluding Remarks

The main question that arises from this Chapter is to know whether Theorem 7.2 can be extended to the entire class of convex games.

This Chapter shows that the utopia games use a lot of information of the underlying game. Since we have only defined them for convex games, a natural question is whether they can be defined for any balanced game. Then, the following step would be to wonder if the core-center of any balanced game can be expressed by means of these games. Even when we have provided many insights on the properties of these games, much more research on this topic is needed.

This Chapter also deepens in the motivations for the core-center: an allocation rule obtained in a natural way from a set-valued solution. Hence, as the final conclusion, just insist in the fact that the core-center provides a new focus on the search for connections between set-valued solutions and allocation rules. Of course, many things still remain to be explored.

[7] Lemma 7.5 is important for the games $v_{(i,j)}$ with a full dimensional core; otherwise, their volume would be 0, and hence, the corresponding addends would also be 0.

7.A Appendix

7.A.1 Proof of Proposition 7.1

Proof. We want to show that for each $R \subseteq S \subseteq N\backslash\{i\}$,

$$v_T^H(R \cup \{i\}) - v_T^H(R) \leq v_T^H(S \cup \{i\}) - v_T^H(S). \tag{7.1}$$

- **Case 1: $\boldsymbol{i \notin T}$.** We distinguish three possibilities:

a) $\boldsymbol{H \subseteq R \subseteq S}$. Since

$$\begin{array}{rrcl} \text{(i)} & v_T^H(S \cup \{i\}) - v_T^H(S) & = & v((S \cup \{i\})\backslash T) - v(S\backslash T), \\ \text{(ii)} & v_T^H(R \cup \{i\}) - v_T^H(R) & = & v((R \cup \{i\})\backslash T) - v(R\backslash T), \text{ and} \\ \text{(iii)} & (N, v) \text{ is convex,} & & \end{array}$$

inequality (7.1) holds.

b) $\boldsymbol{H \not\subseteq R}$ and $\boldsymbol{H \subseteq S}$. Since

$$\begin{array}{rrcl} \text{(i)} & v_T^H(S \cup \{i\}) - v_T^H(S) & = & v((S \cup \{i\})\backslash T) - v(S\backslash T), \\ \text{(ii)} & v_T^H(R \cup \{i\}) - v_T^H(R) & = & v(\{i\}), \text{ and} \\ \text{(iii)} & (N, v) \text{ is convex,} & & \end{array}$$

inequality (7.1) holds.

c) $\boldsymbol{H \not\subseteq S}$. Since $i \notin T$, $v_T^H(R \cup \{i\}) - v_T^H(R) = v_T^H(S \cup \{i\}) - v_T^H(S) = v(\{i\})$ and (7.1) holds again.

- **Case 2: $\boldsymbol{i \in T}$.** Again, we distinguish three possibilities:

a) $\boldsymbol{H \subseteq R \cup \{i\} \subseteq S \cup \{i\}}$. We distinguish two subcases: $i \notin H$ and $i \in H$.

a.1) $\boldsymbol{i \notin H}$. Since

$$\begin{array}{rrcl} \text{(i)} & v_T^H(S \cup \{i\}) - v_T^H(S) & = & v((T \cap S) \cup (N\backslash T) \cup \{i\}) - v((T \cap S) \cup (N\backslash T)), \\ \text{(ii)} & v_T^H(R \cup \{i\}) - v_T^H(R) & = & v((T \cap R) \cup (N\backslash T) \cup \{i\}) - v((T \cap R) \cup (N\backslash T)), \text{ and} \\ \text{(iii)} & (N, v) \text{ is convex,} & & \end{array}$$

inequality (7.1) holds.

a.2) $\boldsymbol{i \in H}$. Now,

$$\begin{array}{rrcl} \text{(i)} & v_T^H(S \cup \{i\}) - v_T^H(S) & = & v((T \cap S) \cup (N\backslash T) \cup \{i\}) - v((T \cap S) \cup (N\backslash T)) \\ & & + & v(S\backslash(T \cap S)) - \sum\limits_{l \in S\backslash(T \cap S)} v(\{l\}) \text{ and} \\ \text{(ii)} & v_T^H(R \cup \{i\}) - v_T^H(R) & = & v((T \cap R) \cup (N\backslash T) \cup \{i\}) - v((T \cap R) \cup (N\backslash T)) \\ & & + & v(R\backslash(T \cap R) - \sum\limits_{l \in R\backslash(T \cap R)} v(\{l\}). \end{array}$$

Moreover, by the convexity of (N,v) we have

(iii) $v((T\cap S)\cup(N\backslash T)\cup\{i\})-v((T\cap S)\cup(N\backslash T)) \geq v((T\cap R)\cup(N\backslash T)\cup\{i\})-v((T\cap R)\cup(N\backslash T))$.

Finally, since

$$v(S\backslash(T\cap S))-v(R\backslash(T\cap R)) \quad = \quad v(S\backslash(T\cap S))-v(R\backslash(T\cap S\cap R))$$

and

$$\begin{aligned}\sum_{l\in S\backslash(T\cap S)} v(\{l\})-\sum_{l\in R\backslash(T\cap R)} v(\{l\}) &= \sum_{l\in S\backslash(T\cap S)} v(\{l\})-\sum_{l\in R\backslash(T\cap S\cap R)} v(\{l\}) \\ &= \sum_{l\in S\backslash(R\cup(T\cap S\cap(N\backslash R))} v(\{l\}),\end{aligned}$$

we have

$$\text{iv)} \quad v(S\backslash(T\cap S))-\sum_{l\in S\backslash(T\cap S)} v(\{l\}) \quad \geq \quad v(R\backslash(T\cap R))-\sum_{l\in R\backslash(T\cap R)} v(\{l\}).$$

Now, the combination of equations i) to iv) yields (1).

b) $\boldsymbol{H\not\subseteq R\cup\{i\}}$ **and** $\boldsymbol{H\subseteq S\cup\{i\}}$. Again, we distinguish two subcases: $i\notin H$ and $i\in H$.

b.1) $\boldsymbol{i\notin H}$. Since

$$\begin{array}{rrcl}\text{(i)} & v_T^H(S\cup\{i\})-v_T^H(S) & = & v((T\cap S)\cup(N\backslash T)\cup\{i\})-v((T\cap S)\cup(N\backslash T)), \\ \text{(ii)} & v_T^H(R\cup\{i\})-v_T^H(R) & = & v((T\cap R)\cup(N\backslash T)\cup\{i\})-v((T\cap R)\cup(N\backslash T)), \text{ and} \\ \text{(iii)} & (N,v) \text{ is convex,} & & \end{array}$$

inequality (7.1) holds.

b.2) $\boldsymbol{i\in H}$. Now,

$$\begin{array}{rrcl}\text{(i)} & v_T^H(S\cup\{i\})-v_T^H(S) & = & v((T\cap S)\cup(N\backslash T)\cup\{i\})-v((T\cap S)\cup(N\backslash T)) \\ & & + & v(S\backslash(T\cap S))-\sum_{l\in S\backslash(T\cap S)} v(\{l\}) \text{ and} \\ \text{(ii)} & v_T^H(R\cup\{i\})-v_T^H(R) & = & v((T\cap R)\cup(N\backslash T)\cup\{i\})-v((T\cap R)\cup(N\backslash T)).\end{array}$$

Since $v(S\backslash(T\cap S))-\sum_{l\in S\backslash(T\cap S)} v(\{l\})\geq 0$, the arguments in case a.2) can be adapted.

c) $\boldsymbol{H\not\subseteq S\cup\{i\}}$. Since

$$\begin{array}{rrcl}\text{(i)} & v_T^H(S\cup\{i\})-v_T^H(S) & = & v((T\cap S)\cup(N\backslash T)\cup\{i\})-v((T\cap S)\cup(N\backslash T)), \\ \text{(ii)} & v_T^H(R\cup\{i\})-v_T^H(R) & = & v((T\cap R)\cup(N\backslash T)\cup\{i\})-v((T\cap R)\cup(N\backslash T)), \text{ and} \\ \text{(iii)} & (N,v) \text{ is convex,} & & \end{array}$$

inequality (7.1) holds.

We have discussed all possible cases. Hence, $(N, v_T^H) \in CG^n$. $\square$

7.A.2 Proof of Lemma 7.3

Proof. (i) Let $i \in N$. If $(N, v) \in CG^n_{n-t}$, with $0 \le t \le 2$. Then v_i can be defined as follows:

$$v_i(S) = \begin{cases} v(N) - v(N\backslash\{i\}) + v(S\backslash\{i\}) & i \in S, |S| \ge n-1 \\ v(N) - v(N\backslash\{i\}) + \sum_{l \in S\backslash\{i\}} v(\{l\}) & i \in S, |S| < n-1 \\ \sum_{l \in S} v(\{l\}) & i \notin S. \end{cases} \tag{7.2}$$

Let $\sigma \in \Pi(N)$ be such that $\sigma(i) = n$. Then, $m_i^\sigma(N, v_i) = v(N) - \sum_{l \in N\backslash\{i\}} v(\{l\})$ and, for each $k \ne i$, $m_k^\sigma(N, v_i) = v(\{k\})$. Let $j \in N$, $j \ne i$; then, for each $\sigma \in \Pi(N)$ such that $\sigma(j) = n$, we have $m_i^\sigma(N, v_i) = v(N) - v(N\backslash\{i\})$, $m_j^\sigma(N, v_i) = v(N\backslash\{i\}) - \sum_{l \ne i,j} v(\{l\})$, and, for each $k \ne i, j$, $m_k^\sigma(N, v_i) = v(\{k\})$. Hence, since (N, v_i) is convex, $C(N, v_i)$ coincides with the convex hull of the marginal vectors. Hence, for each $i \in N$, $C(N, v_i) = I(N, v_i)$. Finally, the ratio between the volumes follows from Lemma 7.1.

(ii) Let $i, j \in N$, $j \ne i$. Let $(N, v_i) \in CG^n_{n-t}$, with $0 \le t \le 2$. For each $S \subseteq N$ such that $|S| \le n-2$, we have $v_i(S) = \sum\limits_{l \in S} v_i(\{l\})$. Hence, following (7.2),

$$(v_i)_j(S) = \begin{cases} v_i(N) - v_i(N\backslash\{j\}) + v_i(S\backslash\{j\}) & j \in S, |S| \ge n-1 \\ v_i(N) - v_i(N\backslash\{j\}) + \sum_{l \in S\backslash\{j\}} v_i(\{l\}) & j \in S, |S| < n-1 \\ \sum_{l \in S} v_i(\{l\}) & j \notin S. \end{cases} \tag{7.3}$$

Now, since $|S\backslash\{i, j\}| = n-2$,

$$v_i(S\backslash\{j\}) = \begin{cases} v(N) - v(N\backslash\{i\}) + \sum_{l \in S\backslash\{i,j\}} v(\{l\}) & i \in S \\ \sum_{l \in S\backslash\{j\}} v(\{l\}) & i \notin S. \end{cases} \tag{7.4}$$

Moreover,

$$v_i(\{l\}) = \begin{cases} v(N) - v(N\backslash\{i\}) & l = i \\ v(\{l\}) & l \ne i. \end{cases} \tag{7.5}$$

Hence, using (7.4) and (7.5) in (7.3) we have

$$(v_i)_j(S) = \begin{cases} v(N) - \sum_{l \in N\backslash S} v(\{l\}) & i \in S, j \in S \\ v(N) - v(N\backslash\{i\}) + \sum_{l \in S\backslash\{i\}} v(\{l\}) & i \in S, j \notin S \\ v(N\backslash\{i\}) - \sum_{l \in N\backslash(S \cup \{i\})} v(\{l\}) & i \notin S, j \in S \\ \sum_{l \in S} v(\{l\}) & i \notin S, j \notin S. \end{cases}$$

Hence, $(N, (v_i)_j)$ is additive and $C\big(N, (v_i)_{\{j\}}\big) = m^\sigma(N, v)$, where $\sigma \in \Pi(N)$ is such that $\sigma(i) = n$ and $\sigma(j) = n-1$.

(iii) We study the two inclusions separately:

"$\supseteq$" It suffices to show that for each $i \in N$, $C(N, v_i) \subseteq I(N, v)$. Let $i \in N$ and $x \in C(N, v_i)$. Then, $x_i \geq v_i(\{i\}) = v(N) - v(N\backslash\{i\}) \geq v(\{i\})$ and, for each $k \in N\backslash\{i\}$, $x_k \geq v_i(\{k\}) = v(\{k\})$. Hence $x \in I(N, v)$.

"$\subseteq$" Let $x \in I(N, v)\backslash C(N, v)$. We claim that there is $i \in N$ such that $x \in C(N, v_i)$. Since $x \notin C(N, v)$ and $(N, v) \in CG^n_{n-2}$, there is $S \subsetneq N$, $|S| = n-1$, such that $\sum_{l \in S} x_l < v(S)$. Then, there is $i \in N$ such that $\sum_{l \in N\backslash\{i\}} x_l < v(N\backslash\{i\})$ and, by the efficiency condition, $x_i \geq v(N) - v(N\backslash\{i\})$. Moreover, since for each $j \neq i$, $x_j \geq v(\{j\})$, it is easy to check that $x \in C(N, v_i)$.

(iv) We prove that for each pair $i, j \in N$, $C(N, v_i) \cap C(N, v_j)$ and $C(N, v) \cap C(N, v_i)$ have volume 0. Then, the result follows from c). Let $i, j \in N$:

Claim 1: For each $i \in N$, $\text{Vol}(C(N, v) \cap C(N, v_i)) = 0$. Let $x \in C(N, v) \cap C(N, v_i)$, then

$$\begin{aligned} x \in C(N, v_i) \quad &\Rightarrow \quad x_i \geq v_i(\{i\}) = v(N) - v(N\backslash\{i\}) \text{ and} \\ x \in C(N, v) \quad &\Rightarrow \quad v(\{i\}) \leq x_i \leq v(N) - v(N\backslash\{i\}). \end{aligned}$$

Hence, for each $x \in C(N, v) \cap C(N, v_i)$, $x_i = v(N) - v(N\backslash\{i\})$. Hence, $C(N, v) \cap C(N, v_i)$ lies in an $(n-2)$-dimensional space.

Claim 2: For each pair $i, j \in N$, $\text{Vol}(C(N, v_i) \cap C(N, v_j)) = 0$. It suffices to show that $C(N, v_i) \cap C(N, v_j) \subseteq C(N, v) \cap C(N, v_i)$. Let $x \in C(N, v_i) \cap C(N, v_j)$. Suppose that $x \notin C(N, v)$. Since x is efficient and $x_i \geq v(N) - v(N\backslash\{i\})$, $x_j \geq v(N) - v(N\backslash\{j\})$, and for each $k \in N\backslash\{i, j\}$, $x_k \geq v(\{k\})$, then, there is $S \subsetneq N$, $|S| = n-1$, such that $\sum_{l \in S} x_l < v(S)$. Hence, either $i \in S$ or $j \in S$. Assume, without loss of generality that $S = N\backslash\{j\}$. Then

$$v(N\backslash\{j\}) > \sum_{l \in N\backslash\{j\}} x_l \geq v(N) - v(N\backslash\{i\}) + \sum_{l \in N\backslash\{i,j\}} v(\{l\}).$$

Since $(N, v) \in G^n_{n-2}$, $v(N\backslash\{i, j\}) = \sum_{l \in N\backslash\{i,j\}} v(\{l\})$. Hence, $v(N\backslash\{j\}) - \sum_{l \in N\backslash\{i,j\}} v(\{l\}) = v(N\backslash\{j\}) - v(N\backslash\{i, j\})$ and we have $v(N\backslash\{j\}) - v(N\backslash\{i, j\}) > v(N) - v(N\backslash\{i\})$, contradicting the convexity of (N, v). □

7.A.3 Proof of Lemma 7.5

Proof. (i) First, we obtain expressions for the marginal vectors associated with $(N, v_{(i,j)})$:

$$v_{(i,j)}(S) = \begin{cases} v(N) - v(N\backslash\{i, j\}) + v(S\backslash\{i, j\}) & i \in S, j \in S \\ v(N\backslash\{j\}) - v(N\backslash\{i, j\}) + v(S\backslash\{i\}) & i \in S, j \notin S \\ v(N\backslash\{i\}) - v(N\backslash\{i, j\}) + \sum_{l \in S\backslash\{j\}} v(\{l\}) & i \notin S, j \in S \\ \sum_{l \in S} v\{l\}) & i \notin S, j \notin S. \end{cases}$$

Let $\Pi_1(N) = \{\sigma \in \Pi(N) : \sigma(i) = n\}$. Then, for each $\sigma \in \Pi_1(N)$,

$$m_k^\sigma(N, v_{(i,j)}) = \begin{cases} v(N) - v(N\backslash\{i\}) + v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j\}} v(\{l\}) & k = i \\ v(N\backslash\{i\}) - v(N\backslash\{i,j\}) & k = j \\ v(\{k\}) & k \neq i, j. \end{cases}$$

Note that $|\Pi_1(N)| = (n-1)!$ and all these marginal vectors coincide.

Let $\Pi_2(N) = \{\sigma \in \Pi(N) : \sigma(j) < \sigma(i) \neq n\}$. Let $\sigma \in \Pi_2(N)$. Assume, without loss of generality that $\sigma(n) = n$. Then,

$$m_k^\sigma(N, v_{(i,j)}) = \begin{cases} v(N) - v(N\backslash\{i\}) + v(P_\sigma(i)\backslash\{j\}) - \sum_{l \in P_\sigma(i)\backslash\{j\}} v(\{l\}) & k = i \\ v(N\backslash\{i\}) - v(N\backslash\{i,j\}) & k = j \\ v((P_\sigma(k) \cup \{k\})\backslash\{i,j\}) - v((P_\sigma(k))\backslash\{i,j\}) & k \neq j, \sigma(k) > \sigma(i) \\ v(\{k\}) & k \neq j, \sigma(k) < \sigma(i). \end{cases}$$

Since $(N, v) \in CG^n_{n-3}$, then, for each coalition S such that $|S| \leq n-3$, $v(S) = \sum_{l \in S} v(\{l\})$. Hence,

$$m_k^\sigma(N, v_{(i,j)}) = \begin{cases} v(N) - v(N\backslash\{i\}) & k = i \\ v(N\backslash\{i\}) - v(N\backslash\{i,j\}) & k = j \\ v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j,n\}} v(\{l\}) & k = n \\ v(\{k\}) & k \neq i, j, n. \end{cases}$$

Let $\sigma \in \Pi(N)$ and $k \in N$ be such that $\sigma(j) < \sigma(i) < \sigma(k) = n$. Then, $\sigma \in \Pi_2(N)$ and there are $\frac{(n-1)!}{2}$ such permutations, all of them originating the same marginal vector. Moreover, varying k within $N\backslash\{i,j\}$, we obtain that $|\Pi_2(N)| = \frac{(n-1)!}{2}(n-2)$. Hence, the marginal vectors associated to permutations in $\Pi_2(N)$ define, at most, $n-2$ different points.

Let $\Pi_3(N) = \{\sigma \in \Pi(N) : \sigma(j) = n \text{ and } \sigma(i) = n-1\}$. Let $\sigma \in \Pi_3(N)$. Then,

$$m_k^\sigma(N, v_{(i,j)}) = \begin{cases} v(N\backslash\{j\}) - \sum_{l \in N\backslash\{i,j\}} v(\{l\}) & k = i \\ v(N) - v(N\backslash\{j\}) & k = j \\ v(\{k\}) & k \neq i, j. \end{cases}$$

Now, $|\Pi_3(N)| = (n-2)!$ and all these marginal vectors coincide.

Let $\Pi_{4_a}(N) = \{\sigma \in \Pi(N) : \sigma(i) < \sigma(j) < n\}$. Let $\sigma \in \Pi_{4_a}(N)$. Then,

$$m_k^\sigma(N, v_{(i,j)}) = \begin{cases} v(N\backslash\{j\}) - v(N\backslash\{i,j\}) & k = i \\ v(N) - v(N\backslash\{j\}) & k = j \\ v((P_\sigma(k) \cup \{k\})\backslash\{i\}) - \sum_{l \in P_\sigma(k)\backslash\{i\}} v(\{l\}) & \sigma(i) < \sigma(k) < \sigma(j) \\ v((P_\sigma(k)) \cup \{k\})\backslash\{i,j\}) - v((P_\sigma(k))\backslash\{i,j\}) & \sigma(k) > \sigma(j) \\ v(\{k\}) & \sigma(k) < \sigma(i). \end{cases}$$

Again, since $(N,v)\in CG^n_{n-3}$, the latter expression can be reduced to

$$m^\sigma_k(N,v_{(i,j)})=\begin{cases} v(N\backslash\{j\})-v(N\backslash\{i,j\}) & k=i\\ v(N)-v(N\backslash\{j\}) & k=j\\ v(N\backslash\{i,j\})-\sum_{l\in N\backslash\{i,j,k\}}v(\{l\}) & \sigma(k)=n\\ v(\{k\}) & \sigma(k)<n.\end{cases}$$

Now, $|\Pi_{4_a}(N)|=\frac{(n-1)!}{2}(n-2)$ with, at most, $n-2$ different marginal vectors.

Let $\Pi_{4_b}(N)=\{\sigma\in\Pi(N)\ :\ \sigma(j)=n \text{ and } \sigma(i)<n-1\}$. Let $\sigma\in\Pi_{4_b}(N)$. Again, the following expressions for the marginal vectors can be derived:

$$m^\sigma_k(N,v_{(i,j)})=\begin{cases} v(N\backslash\{j\})-v(N\backslash\{i,j\}) & k=i\\ v(N)-v(N\backslash\{j\}) & k=j\\ v(N\backslash\{i,j\})-\sum_{l\in N\backslash\{i,j,k\}}v(\{l\}) & \sigma(k)=n-1\\ v(\{k\}) & \sigma(k)<n-1.\end{cases}$$

Now, $|\Pi_{4_b}(N)|=(n-2)!(n-2)$ with, at most, $n-2$ different marginal vectors.

Let $\Pi_4(N)=\Pi_{4_a}(N)\cup\Pi_{4_b}(N)$. Then, $|\Pi_4(N)|=|\Pi_{4_a}(N)|+|\Pi_{4_b}(N)|=\frac{n^2-n-2}{2}(n-2)!$. Moreover, it is easy to check that the permutations in $\Pi_4(N)$ define, at most, $n-2$ different marginal points.

Finally, we have that the permutations in $\Pi(N)$ define, at most, $2n-2$ different points. For ease of exposition we assume that these points are different to each other.[8]

- **Computation of the Shapley value:**

$$Sh_i(N,v_{(i,j)})=\frac{1}{n!}\sum_{\sigma\in\Pi(N)}m^\sigma_i(N,v_{(i,j)})=\frac{1}{n!}\sum_{l=1}^{4}\sum_{\sigma\in\Pi_l(N)}m^\sigma_i(N,v_{(i,j)}).$$

Player i

$$\begin{aligned} Sh_i(N,v_{(i,j)}) &= \frac{(n-1)!}{n!}\Big(v(N)-v(N\backslash\{i\})+v(N\backslash\{i,j\})-\sum\nolimits_{l\in N\backslash\{i,j\}}v(\{l\})\Big)\\ &+ \frac{(n-1)!}{2n!}(n-2)\Big(v(N)-v(N\backslash\{i\})\Big)\\ &+ \frac{(n-2)!}{n!}\Big(v(N\backslash\{j\})-\sum\nolimits_{l\neq i,j}v(\{l\})\Big)\\ &+ \frac{n^2-n-2}{2n!}(n-2)!\Big(v(N\backslash\{j\})-v(N\backslash\{i,j\})\Big).\end{aligned}$$

[8] This assumption does not affect the algebra used in this proof, but it helps to get a better understanding of the geometric situation underlying the result.

Simplifying we have,

$$Sh_i(N, v_{(i,j)}) = \frac{v(N) + v(N\backslash\{j\}) - v(N\backslash\{i\})}{2} - \frac{1}{n-1}\Big(\frac{(n-3)}{2} v(N\backslash\{i,j\}) + \sum_{l \in N\backslash\{i,j\}} v(\{l\})\Big).$$

Player j:

$$Sh_j(N, v_{(i,j)}) = \frac{v(N\backslash\{i\}) - v(N\backslash\{i,j\}) + v(N) - v(N\backslash\{j\})}{2}.$$

Player $k \neq i, j$:

$$\begin{aligned} Sh_k(N, v_{(i,j)}) &= \frac{(n-1)!}{n!} v(\{k\} \\ &+ \frac{(n-1)!}{2n!}(n-3)v(\{k\}) + \frac{(n-1)!}{2n!}\Big(v(N\backslash\{i,j\}) - \textstyle\sum_{l \in N\backslash\{i,j,k\}} v(\{l\}\Big) \\ &+ \frac{(n-2)!}{n!} v(\{k\}) \\ &+ \frac{n^2 - n - 2}{2n!}(n-3)!\Big((n-3)v(\{k\}) + v(N\backslash\{i,j\}) - \textstyle\sum_{l \in N\backslash\{i,j,k\}} v(\{l\}\Big). \end{aligned}$$

Simplifying we have,

$$Sh_k(N, v_{(i,j)}) = v(\{k\} + \frac{1}{n-1}\Big(v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j\}} v(\{l\})\Big).$$

- **Computation of the core-center:**

First, we describe the geometry of $C(N, v_{(i,j)})$ (Figure 7.6 illustrates the geometry of the core in an example with four players). Note that, for player j,

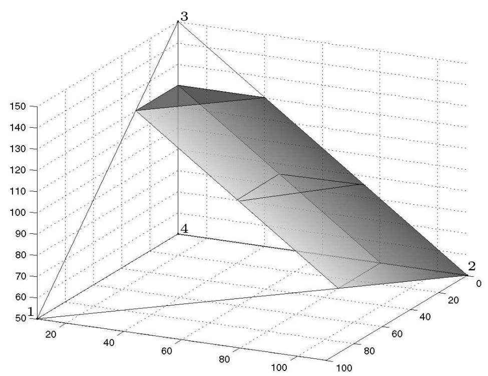

Figure 7.6: The core of the game $(N, v_{(i,j)})$

$$m_j^\sigma(N, v_{(i,j)}) = \begin{cases} v(N\backslash\{i\}) - v(N\backslash\{i,j\}) & \sigma \in \Pi_1(N) \cup \Pi_2(N) \\ v(N) - v(N\backslash\{j\}) & \sigma \in \Pi_3(N) \cup \Pi_4(N). \end{cases}$$

Hence, the marginal vectors lie either in the hyperplane $x_j = v(N) - v(N\backslash\{j\})$ or in the hyperplane $x_j = v(N\backslash\{i\}) - v(N\backslash\{i,j\})$ and there are, at most, $n-1$ different marginal vectors in each one of the two hyperplanes. Now, if there is a pair of coincident points in one of the two hyperplanes, then there is also a pair of coincident points in the other one; inducing a degeneracy in the core. Hence, since $C(N, v_{(i,j)})$ is full dimensional, these $n-1$ points in each hyperplane have to be different.

Let $k \in N\backslash\{i,j\}$ and $\sigma \in \Pi(N)$. We have already shown that either $m_k^\sigma(N, v_{(i,j)}) = v(\{k\})$ or $m_k^\sigma(N, v_{(i,j)}) = v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j,k\}} v(\{l\})$. Now, let $\sigma \in \Pi_3(N) \cup \Pi_4(N)$. Then, $m^\sigma(N, v_{(i,j)})$ lies in the hyperplane $v(N) - v(N\backslash\{j\})$. Moreover, there are $\sigma_1, \sigma_2 \in \Pi_3(N) \cup \Pi_4(N)$ such that (i) $m_k^{\sigma_1}(N, v_{(i,j)}) = v(\{k\})$ and (ii) $m_k^{\sigma_2}(N, v_{(i,j)}) = v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j,k\}} v(\{l\})$. An analogous observation is true for the hyperplane $x_j = v(N\backslash\{i\}) - v(N\backslash\{i,j\})$ and a pair of permutations $\sigma_1', \sigma_2' \in \Pi_1(N) \cup \Pi_2(N)$.

Next, we take the $n-1$ marginal vectors in the hyperplane $x_j = v(N) - v(N\backslash\{j\})$ and we show that they span an $(n-2)$-simplex. The same can be done for the $n-1$ points in $x_j = v(N\backslash\{i\}) - v(N\backslash\{i,j\})$.

Let $k \in N\backslash\{i,j\}$ and let u^k be the marginal vector lying on the hyperplane $x_j = v(N) - v(N\backslash\{j\})$ in which player k is better off. The coordinates of each u^k are the following:

$$(u^k)_l = \begin{cases} v(N\backslash\{j\}) - v(N\backslash\{i,j\}) & l = i \\ v(N) - v(N\backslash\{j\}) & l = j \\ v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j,k\}} v(\{l\}) & l = k. \\ v(\{l\}) & l \neq i,j,k. \end{cases}$$

Besides, let u^0 be the remaining marginal vector in $x_j = v(N) - v(N\backslash\{j\})$:

$$(u^0)_l = \begin{cases} v(N\backslash\{j\}) - \sum_{l \in N\backslash\{i,j\}} v(\{l\}) & l = i \\ v(N) - v(N\backslash\{j\}) & l = j \\ v(\{l\}) & l \neq i,j. \end{cases}$$

Now, since the vectors $u^k - u^0$ are linearly independent, the $n-1$ marginal vectors in $x_j = v(N) - v(N\backslash\{j\})$ define a geometrically independent set in $\mathbb{R}^n$. Hence, they span an $(n-2)$-simplex.

Now, for the hyperplane $x_j = v(N\backslash\{i\}) - v(N\backslash\{i,j\})$ we define the same $n-1$ points but with the following differences: (i) the j-th coordinate is $v(N\backslash\{i\}) - v(N\backslash\{i,j\})$ and (ii) the i-th coordinate is changed to recover efficiency (iii) the remaining coordinates remain unchanged. Now, since $(N, v_{(i,j)})$ is convex, $C(N, v_{(i,j)}) = \text{co}\{m^\sigma(N,v) : \sigma \in \Pi(N)\}$. Hence, there is an $(n-2)$-simplex Δ^{n-2} such that, for each $t \in \mathbb{R}$, either the intersection of $C(N, v_{(i,j)})$ with the hyperplane $x_j = t$ is empty or it is a translation of Δ^{n-2}. Hence,

$$\mu_j(N, v_{(i,j)}) = \frac{v(N\backslash\{i\}) - v(N\backslash\{i,j\}) + v(N) - v(N\backslash\{j\})}{2},$$

and it coincides with the corresponding coordinate of the Shapley value.

Now, because of the symmetries in $C(N, v_{(i,j)})$, to compute the core-center it suffices to compute the barycenter of the simplex generated by the $n-1$ points on the hyperplane $x_j = \mu_j(N, v_{(i,j)})$. Hence, for each $k \in N\backslash\{i,j\}$,

$$\mu_k(N, v_{(i,j)}) = v(\{k\}) + \frac{1}{n-1}\Big(v(N\backslash\{i,j\}) - \sum_{l \in N\backslash\{i,j\}} v(\{l\})\Big) = \mathrm{Sh}_k(N, v_{(i,j)}).$$

Finally, because of the efficiency property of both the Shapley value and the core-center, we have $Sh_i(N, v_{(i,j)}) = \mu_i(N, v_{(i,j)})$.

(ii) Immediate from the description of $C(N, v_{(i,j)})$ we have made above. □

7.A.4 Proof of Lemma 7.6

Proof. (i) Since

$$v_j(S) = \begin{cases} v(N) - v(N\backslash\{j\}) + v(S\backslash\{j\}) & \text{if } j \in S \\ \sum_{l \in S} v(\{l\}) & \text{if } j \notin S \end{cases}$$

and

$$v_{(i,j)}(S) = \begin{cases} v(N) - v(N\backslash\{i,j\}) + v(S\backslash\{i,j\}) & i \in S, j \in S \\ v(N\backslash\{j\}) - v(N\backslash\{i,j\}) + v(S\backslash\{i\}) & i \in S, j \notin S \\ v(N\backslash\{i\}) - v(N\backslash\{i,j\}) + \sum_{l \in S\backslash\{j\}} v(\{l\}) & i \notin S, j \in S \\ \sum_{l \in S} v(\{l\}) & i \notin S, j \notin S. \end{cases}$$

Then,

$$\begin{aligned} (v_{(i,j)})_j(S) &= \begin{cases} v_{\{i,j\}}(N) - v_{\{i,j\}}(N\backslash\{j\}) + v_{\{i,j\}}(S\backslash\{j\}) & j \in S \\ \sum_{l \in S} v_{\{i,j\}}(\{l\}) & j \notin S \end{cases} \\ &= \begin{cases} v(N) - v(N\backslash\{i,j\}) + v(S\backslash\{i,j\}) & i \in S, j \in S \\ v(N) - v(N\backslash\{j\}) + \sum_{l \in S\backslash\{j\}} v(\{l\}) & i \notin S, j \in S \\ v(N\backslash\{j\}) - v(N\backslash\{i,j\}) + \sum_{l \in S\backslash\{j,i\}} v(\{l\}) & i \in S, j \notin S \\ \sum_{l \in S} v(\{l\}) & i \notin S, j \notin S. \end{cases} \end{aligned}$$

Finally,

$$(v_j)_i)(S) = \begin{cases} v_j(N) - v_j(N\backslash\{i\}) + v_j(S\backslash\{i\}) & i \in S \\ \sum_{l\in S} v_j(\{l\}) & i \notin S. \end{cases}$$
$$= \begin{cases} v(N) - v(N\backslash\{i,j\}) + v(S\backslash\{i,j\}) & i \in S, j \in S \\ v(N\backslash\{j\}) - v(N\backslash\{i,j\}) + \sum_{l\in S\backslash\{i\}} v(\{l\}) & i \in S, j \notin S \\ v(N) - v(N\backslash\{j\}) + \sum_{l\in S\backslash\{j\}} v(\{l\}) & i \notin S, j \in S \\ \sum_{l\in S} v(\{l\}) & i \notin S, j \notin S. \end{cases}$$

(ii) It follows from the combination of (i) and Lemma 7.4.

The proofs of (iii) follow similar lines to the proofs of their counterparts in Lemma 7.3. □

Bibliography

Gillies, D. B. (1953): “Some Theorems on n-Person Games,” Ph.D. thesis, Princeton. (Quoted in pp. 113)

González-Díaz, J. and E. Sánchez-Rodríguez (2003): “From Set-Valued Solutions to Single-Valued Solutions: The Centroid and the Core-Center,” Reports in Statistics and Operations Research 03-09, University of Santiago de Compostela. (Quoted in pp. 112)

Gritzman, P. and V. Klee (1994): “On the Complexity of Some Basic Problems in Computational Convexity: Volume and Mixed volumes,” in *Polytopes: Abstract, convex and computational*, ed. by T. Bisztriczky, P. McMullen, R. Schneider, and A. Weiss, Kluwer, Boston MA, 373–466. (Quoted in pp. 115)

Schmeidler, D. (1972): “Cores of Exact Games I,” *Journal of Mathematical Analysis and Applications*, 40, 214–225. (Quoted in pp. 117)

Shapley, L. S. (1953): “A Value for n-Person Games,” in *Contributions to the theory of games II*, ed. by H. Kuhn and A. Tucker, Princeton: Princeton University Press, vol. 28 of *Annals of Mathematics Studies*. (Quoted in pp. 112)

Chapter 8

A Geometric Characterization of the Compromise Value

Contents

8.1 Introduction

Most game-theoretic solution concepts that have been proposed in the literature are defined on the basis of or characterized by properties. These properties are usually formulated in terms of individual payoffs and reflect notions like monotonicity and rationality. For some values, there exist additional characterizations in terms of geometry. The best-known example is the Shapley value (Shapley, 1953), which is the barycenter of the vectors of marginal contributions.

For some classes of games, there exist nice geometric expressions for the *compromise* or τ *value* (Tijs, 1981). In particular, the compromise value is the barycenter of the extreme points of the core cover in big boss games (Muto et al., 1988) and 1-convex games (Driessen, 1988).

In this Chapter, we extend the $APROP$ rule for bankruptcy problems (Curiel et al., 1987) to the whole class of compromise admissible (or quasi-balanced) games (Tijs, 1981). This extended rule, which we call τ^*, turns out to be the barycenter of the *edges* of the core cover, which is our main result. Moreover, τ^* and the compromise value coincide for most of quasi-balanced games.

This Chapter is organized as follows. In Section 8.2, we extend the $APROP$ rule and define the barycenter ζ of the edges of the core cover. In Section 8.3, we state our main result and give an overview of the proof, which consists of six steps. Finally, in Section 8.4, we prove our main result.

8.2 The τ^* Value

A *transferable utility* or *TU game* is a pair (N, v), where $N = \{1, \dots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to every coalition $S \subseteq N$ a payoff $v(S)$. By convention, $v(\emptyset) = 0$.

Following Tijs and Lipperts (1982), the *utopia vector* of a game (N, v), $M(v) \in \mathbb{R}^N$, is defined, for each $i \in N$,by

$$M_i(v) := v(N) - v(N \backslash \{i\}).$$

The *minimum right vector* $m_i(v) \in \mathbb{R}^N$ is defined, for each $i \in N$, by

$$m_i(v) := \max_{S \subseteq N, i \in S} \{v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)\}.$$

The *core cover* of a game (N, v) consists of those allocations of $v(N)$ according to which every player receives at most his utopia payoff and at least his minimal right:

$$CC(v) := \{x \in \mathbb{R}^N \; : \; \sum_{i \in N} x_i = v(N), \; m(v) \leq x \leq M(v)\}.$$

A game is *compromise admissible* if it has a nonempty core cover. We denote the class of compromise admissible games with player set N by CA^N. An *allocation rule* on a subclass

$A \subseteq CA^N$ is a function $\varphi : A \to \mathbb{R}^N$ assigning to each $v \in A$ a payoff vector $\varphi(v) \in \mathbb{R}^N$. Moreover, we say that φ is *efficient* if $\sum_{i \in N} \varphi_i(v) = v(N)$.

The *compromise value* or τ value (Tijs, 1981) is the rule on CA^N defined as the point on the line segment between $m(v)$ and $M(v)$ that is efficient:

$$\tau(v) := \lambda m(v) + (1 - \lambda) M(v),$$

where $\lambda \in [0, 1]$ is such that $\sum_{i \in N} \tau_i = v(N)$.

A *bankruptcy problem* is a triple (N, E, c), where $E \geq 0$ is the estate to be divided and $c \in \mathbb{R}^N_+$ with $\sum_{i \in N} c_i \geq E$ is the vector of claims. The corresponding *cooperative bankruptcy game* $(N, v_{E,c})$ is defined, for each $S \subseteq N$, by $v_{E,c}(S) = \max\{E - \sum_{i \in N \setminus S} c_i, 0\}$. We denote the class of bankruptcy problems with player set N by BR^N. The class of corresponding games is a proper subclass of CA^N. A *bankruptcy rule* is a function $f : BR^N \to \mathbb{R}^N$ assigning to every bankruptcy problem $(N, E, c) \in BR^N$ a payoff vector $f(E, c) \in \mathbb{R}^N_+$ such that $\sum_{i \in N} f_i(E, c) = E$.

In the literature, many bankruptcy rules have been proposed. One interesting question is how these can be extended in a natural way to the whole class of compromise admissible games. In this Chapter, we consider the *proportional rule* and the *adjusted proportional rule* (Curiel et al., 1987). The proportional rule $PROP$ simply divides the estate proportional to the claims, *i.e.*, for each $(N, E, c) \in BR^N$ and each $i \in N$,

$$PROP_i(E, c) = \frac{c_i}{\sum_{j \in N} c_j} E.$$

The adjusted proportional rule $APROP$ first gives each player $i \in N$ his minimal right $m_i(E, c) = \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\}$ and the remainder is divided using the proportional rule, where each player's claim is truncated to the estate left:

$$APROP(E, c) = m(E, c) + PROP(E', c'),$$

where $E' = E - \sum_{i \in N} m_i(E, c)$ and for each $i \in N$, $c'_i = \min\{c_i - m_i(E, c), E'\}$.

The compromise value can be seen as an extension of the $PROP$ rule:

$$\tau(v) = m(v) + PROP(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)).$$

Note that it follows from the definition of compromise admissibility that the argument of $PROP$ is indeed a bankruptcy problem.

Similarly, we can extend the $APROP$ rule:

$$\tau^*(v) = m(v) + APROP(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)).$$

To simplify the expression for τ^*, we show that the minimum rights in the associated bankruptcy problem equal 0. Let $v \in CA^N$, $E = v(N) - \sum_{i \in N} m_i(v)$, $c = M(v) - m(v)$, and $i \in N$. Then,

$$\begin{aligned} E - \sum_{j \in N \setminus \{i\}} c_j &= v(N) - \sum_{i \in N} m_i(v) - \sum_{j \in N \setminus \{i\}} (M_j(v) - m_j(v)) \\ &= v(N) - \sum_{j \in N \setminus \{i\}} M_j(v) - m_i(v) \\ &\leq 0, \end{aligned}$$

since $m_i(v) \geq v(N) - \sum_{j \in N \setminus \{i\}} M_j(v)$. Hence, $m_i(E, c) = \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\} = 0$. As a result, we have

$$\tau^*(v) = m(v) + PROP(E', c'), \tag{8.1}$$

where $E' = v(N) - \sum_{i \in N} m_i(v)$ and for each $i \in N$, $c'_i = \min\{M_i(v) - m_i(v), E'\}$.

It follows that for a game $v \in CA^N$ such that for each $i \in N$, $M_i(v) - m_i(v) \leq v(N) - \sum_{j \in N} m_j(v)$, τ^* coincides with the compromise value τ. The requirement involved in the former inequality is quite natural: it says that if we give all the players their minimum rights, then the adjusted utopia value of a player i, $M_i(v) - m_i(v)$, cannot exceed the state left, *i.e.*, $v(N) - \sum_{j \in N} m_j(v)$. Moreover, this natural requirement is usually met when dealing with quasi-balanced games, and hence, the τ and τ^* values coincide for a relevant subclass of the class quasi-balanced games.

The extended rule τ^* turns out to be a kind of barycenter of the core cover, which is the main result of this Chapter. To define this barycenter rule ζ, we need to introduce some more concepts. A *permutation* on N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(p)$ denotes the player at position p, and consequently, $\sigma^{-1}(i)$ denotes the position of player i. The set of all permutations on N is denoted by $\Pi(N)$. $\sigma^{i,j}$ denotes the permutation obtained from σ by switching players i and j. Two permutations σ and $\sigma^{\sigma(p),\sigma(p+1)}$ are called *permutation neighbors*. The set of permutation neighbors of σ is denoted by $\Pi^\sigma(N)$.

The core cover is a polytope whose extreme points are called *larginal vectors* or *larginals*. The larginal $\ell^\sigma \in \mathbb{R}^N$ associated with the ordering $\sigma \in \Pi(N)$ (Quant et al., 2003) is defined, for each $p \in \{1, \ldots, n\}$, by

$$\ell^\sigma_{\sigma(p)}(v) := \begin{cases} M_{\sigma(p)}(v) & \text{if } \sum_{k=1}^{p} M_{\sigma(k)}(v) + \sum_{k=p+1}^{n} m_{\sigma(k)}(v) \leq v(N), \\ m_{\sigma(p)}(v) & \text{if } \sum_{k=1}^{p-1} M_{\sigma(k)}(v) + \sum_{k=p}^{n} m_{\sigma(k)}(v) > v(N), \\ v(N) - \sum_{k=1}^{p-1} M_{\sigma(k)}(v) - \sum_{k=1}^{n} m_{\sigma(k)}(v) & \text{otherwise.} \end{cases}$$

Note that two permutations that are neighbors yield larginals which are adjacent extreme points of the core cover (possibly coinciding), which we also call permutation neighbors.

We define the ζ rule as the following weighted average of the larginal vectors:[1]

$$\zeta(v) = \frac{\sum_{\sigma\in\Pi(N)} w^{\sigma}(v)\ell^{\sigma}(v)}{\sum_{\sigma\in\Pi(N)} w^{\sigma}(v)}, \tag{8.2}$$

where

$$w^{\sigma}(v) = \frac{1}{\sqrt{2}} \sum_{\tau\in\Pi^{\sigma}(N)} d(\ell^{\sigma}(v), \ell^{\tau}(v))$$

equals the sum of the euclidean distances between $\ell^{\sigma}(v)$ and all its permutation neighbors, divided by the common factor $\sqrt{2}$. Now, we claim that the ζ value can be viewed as the barycenter of the edges of the core cover, taking the multiplicities into account. Next, we briefly discuss former statement. The constant $\frac{1}{\sqrt{2}}$ is a common factor in both the numerator and the denominator of Equation 8.2, hence, for the forthcoming arguments we can forget about it. Now, note that the weight associated with each larginal, namely l^{σ}, corresponds with the sum of the distances from l^{σ} to each of its neighbors (many of this distances can be 0). In particular, if l^{τ} is a neighbor of l^{σ}, we are computing the product $d(l^{\sigma}, l^{\tau})l^{\sigma}$ and, similarly, we are also computing $d(l^{\sigma}, l^{\tau})l^{\tau}$. Hence, we are counting twice the length of each edge. Now, recall that we can compute the barycenter of the edges of the core cover (taking multiplicities into account) just by the weighted average the midpoints of the edges, where the weight of each midpoint corresponds with the length of the original edge. Suppose now that we divide the numerator and the denominator of Equation 8.2 by 2. Now, for the edge joining l^{σ} and l^{τ} we have $d(l^{\sigma}, l^{\tau})\frac{l^{\sigma}+l^{\tau}}{2}$, *i.e.*, we have reduced the edge to its midpoint and the weight is the length of the edge. Since we can repeat the same argument for each pair of neighboring larginals, we have already provided a justification for the claim we made above for the ζ value.

To simplify the proofs later on, we first show that both τ^* and ζ satisfy the properties (SEQ) and (RTRUNC). Two games v and $\hat{v}$ are *strategically equivalent* if there exists a real number $k > 0$ and a vector $\alpha \in \mathbb{R}^N$ such that for each $S \subseteq N$,

$$\hat{v}(S) = kv(S) + \sum_{i\in S} \alpha_i. \tag{8.3}$$

A function $\varphi : CA^N \to \mathbb{R}^N$ is *relatively invariant with respect to strategic equivalence* (SEQ) if for each pair of games $v, \hat{v} \in CA^N$ such that (8.3) holds for some $k > 0$ and $\alpha \in \mathbb{R}^N$, we have

$$\varphi(\hat{v}) = k\varphi(v) + \alpha.$$

It is well-known that both the utopia vector M and the minimum right vector m satisfy (SEQ).

Proposition 8.1. *The τ^* rule and the ζ rule satisfy (SEQ).*

Proof. The proof for τ^* is straightforward and therefore omitted.

[1] In the degenerate case where $M = m$, the core cover consists of a single point, which we define to be ζ. Otherwise, there are at least two different larginals and the denominator is positive.

It readily follows from (SEQ) of m and M that for each $\sigma \in \Pi(N)$, ℓ^σ also satisfies (SEQ). Let $v, \hat{v} \in CA^N$, $k > 0$ and $\alpha \in \mathbb{R}^N$ such that (8.3) holds. Let $\sigma \in \Pi(N)$. Then,

$$\begin{aligned} w^\sigma(\hat{v}) &= \frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(\ell^\sigma(\hat{v}), \ell^\tau(\hat{v})) \\ &= \frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(k\ell^\sigma(v) + \alpha, k\ell^\tau(v) + \alpha) \\ &= k\frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(\ell^\sigma(v), \ell^\tau(v)) \\ &= kw^\sigma(v). \end{aligned}$$

Now,

$$\begin{aligned} \zeta(\hat{v}) &= \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(\hat{v})\ell^\sigma(\hat{v})}{\sum_{\sigma \in \Pi(N)} w^\sigma(\hat{v})} \\ &= \frac{k\sum_{\sigma \in \Pi(N)} w^\sigma(v)(k\ell^\sigma(v) + \alpha)}{k\sum_{\sigma \in \Pi(N)} w^\sigma(v)} \\ &= k\zeta(v) + \alpha. \end{aligned}$$

Hence, ζ satisfies (SEQ). □

A rule $\varphi : CA^N \to \mathbb{R}^N$ satisfies the *restricted truncation property* (RTRUNC) if, for each $v \in CA^N$ such that $m(v) = 0$, it holds that, for each $\hat{v} \in CA^N$ such that (i) $\hat{v}(N) = v(N)$, (ii) $m(\hat{v}) = 0$, and (iii) $M_i(\hat{v}) = \min\{M_i(v), v(N)\}$, we have $\varphi(\hat{v}) = \varphi(v)$. The idea behind (RTRUNC) is that if a player's utopia value (or, in bankruptcy terms, his claim) is higher than the value of the grand coalition (the estate), his payoff according to φ should not by influenced by truncating this claim.

Proposition 8.2. *The τ^* rule and the ζ rule satisfy (RTRUNC).*

Proof. Let $v \in CA^N$ with $m(v) = 0$. Then (8.1) reduces to

$$\tau^*(v) = PROP(v(N), (\min\{M_i(v), v(N)\})_{i \in N}).$$

From this it immediately follows that τ^* satisfies (RTRUNC).

For the ζ rule, it suffices to note that truncating the utopia vector has no influence on the larginal vectors. □

8.3 Main Result

In this Section, we present our main result: the equality between τ^* and ζ on CA^N. After dealing with some simple cases, we present a six step outline of the proof, which we give in Section 8.4.

Theorem 8.1. *Let $v \in CA^N$. Then*

$$\tau^*(v) = \zeta(v).$$

As a result of Proposition 8.1, it suffices to show the equality for each game $v \in CA^N$ such that $m(v) = 0$. Next, we can use Proposition 8.2 and conclude that we have to show that for each $v \in CA^N$ such that (i) for each $i \in N$, $M_i(v) \leq v(N)$ and (ii) $m(v) = 0$ we have [2,3]

$$PROP(v(N), M(v)) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v) \ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}.$$

In case there are only two players, the equality between τ^* and ζ follows from $M_1(v) = M_2(v) = v(N)$.

If $M_i(v) = 0$ for a player $i \in N$, then we have $\tau_i^*(v) = \zeta_i(v) = 0$. Now, let $\sigma_{N \backslash \{i\}} \in \Pi(N \backslash \{i\})$ be defined, for each $h, j \in N \backslash \{i\}$, by $\sigma^{-1}_{N \backslash \{i\}}(h) < \sigma^{-1}_{N \backslash \{i\}}(j) \Leftrightarrow \sigma^{-1}(h) < \sigma^{-1}(j)$. Then, for each $\sigma \in \Pi(N)$, the payoff to the players in $N \backslash \{i\}$ according to $\ell^\sigma(v)$ equals their payoff in the situation without player i (*i.e.*, the situation with player set $N \backslash \{i\}$, utopia vector $M_{N \backslash \{i\}}(v)$ and the same amount $v(N)$ to be distributed) according to the larginal corresponding to the restricted permutation $\sigma_{N \backslash \{i\}}$. It is readily verified that also the total weight of each larginal (taking multiplicities into account) is the same in the game with and without player i. Hence, we can omit player i from the game and establish equality between τ^* and ζ for the remaining players.[4]

We establish the equality between τ^* and ζ by combining the permutations in the numerator and denominator in (8.2) into so-called *chains*. In the denominator, these chains allow us to combine terms in such a way that the total weight can be expressed as a simple function of $M(v)$. In the numerator, we construct an iterative procedure to find an expression for the weighted larginals, in which the chains allow us to keep track of changes that occur from one iteration to the next.

The proof of Theorem 8.1 consists of six steps:

Step 1 We first find an expression for the weight of each permutation. We do this by introducing the concept of *pivot* and classifying each permutation in terms of its pivot and its neighbors' pivots.

Step 2 Using the concept of pivot, we introduce chains, which constitute a partition of the set of all permutations. Then, we use the results of the previous step to compute the total weight of each chain.

[2] Note that the condition $M_i(v) \leq v(N)$ is necessary and sufficient to have $M_i(v) = \max_{\sigma \in \Pi(N)} \ell_i^\sigma(v)$. Only in this case, the utopia vector can be reconstructed from the core cover.

[3] The denominator is zero if and only if $M(v) = 0$ $(= m(v))$. In this degenerate case equality between τ^* and ζ is trivial and we therefore assume $M(v) > 0$.

[4] Geometrically, the core cover, which lies in the hyperplane $M_i(v) = 0$, is projected into a space which is one dimension lower.

Step 3 We define a family of auxiliary functions f^{ij} and g^{ij}. We use them to show that each player "belongs" to the same number of chains. As a result, we use our expression of the previous step to compute the total of all the weights, *i.e.*, the denominator in (8.2).

Step 4 In the numerator, we partition the set of chains on the basis of the first player in each permutation. Within each part, we compute the total weighted payoff to all the players. For the first player, this total weighted payoff can easily be computed.

Step 5 We prove the expression for the payoffs to the other players using an iterative argument, varying the utopia vector while keeping $v(N)$ constant. We start with a utopia vector for which our expression is trivial and lower this vector step by step until we reach $M(v)$. In each step of the iteration, (generically) only two chains change and using this, we show that the total weighted payoff to each player who is not first does not change as function of the utopia vector.

Step 6 Combining the previous three steps, we derive an expression for ζ and show that this equals τ^*.

8.4 Proof of the Main Result

Throughout this Section, let $v \in CA^N$ be such that $|N| \geq 3$; $m(v) = 0$; $M(v) > 0$; and, for each $i \in N$, $v(N) \geq M_i(v)$. To prove Theorem 8.1 is suffices to show that for this game v we have

$$PROP(v(N), M(v)) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v) \ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}.$$

Since v is fixed for the rest of the Section, we suppress it as argument and write M rather than $M(v)$, etc. Moreover, we denote the weight $w^\sigma(v)$ by $w(\sigma)$.

Step 1: pivots

Let $\sigma \in \Pi(N)$. Player $\sigma(p)$ with $p \geq 2$ is called the *pivot* in ℓ^σ if $\ell^\sigma_{\sigma(p-1)} = M_{\sigma(p-1)}$, $\ell^\sigma_{\sigma(p)} > 0$ and $\ell^\sigma_{\sigma(p+1)} = 0$. The pivot of a larginal is the player who gets a lower amount according this larginal if the amount $v(N)$ is decreased slightly. In the boundary case where $M_{\sigma(1)} = v(N)$, $v(N)$ cannot be decreased without violating the condition $M_{\sigma(1)} \leq v(N)$; in this case, player $\sigma(2)$ is defined to be the pivot, being the player who gets a higher amount if $v(N)$ is increased slightly. Note that $m = 0$ implies that $\sum_{j \in N \setminus \{i\}} M_j \geq v(N)$ and hence, player $\sigma(n)$ can never be the pivot.

In the following example, we introduce a game which we use throughout this Chapter to illustrate the various concepts.

Example 8.1. *Consider the game* (N, v) *with* $N = \{1, \ldots, 5\}$, $v(N) = 10$, *and* $M = (5, 7, 1, 3, 4)$.

For this game, we have $\tau^* = \zeta = \frac{1}{2}M$*. Take* σ_1 *to be the identity permutation. Then,*

$$\ell^{\sigma_1} = (5, 5, 0, 0, 0)$$

and player 2 *is the pivot.*

For a permutation $\sigma \in \Pi(N)$, we define p_σ to be the position at which the pivot is located.[5] We define $\sigma^L = \sigma^{\sigma(p_\sigma - 1), \sigma(p_\sigma)}$ to be the *left neighbor* of σ and $\sigma^R = \sigma^{\sigma(p_\sigma), \sigma(p_\sigma + 1)}$ to be the right neighbor of σ. It follows from the definition of pivot that the left and right neighbors of ℓ^σ are the only two permutation neighbors that can give rise to a larginal different from ℓ^σ.

Recall that the weight of ℓ^σ, $w(\sigma)$, equals the sum of the (euclidean) distances between ℓ^σ and all its permutation neighbors. In line with the previous paragraph, we only have to take the left and right neighbors into account. Hence,

$$w(\sigma) = \frac{1}{\sqrt{2}} \left[d(\ell^\sigma, \ell^{\sigma^L}) + d(\ell^\sigma, \ell^{\sigma^R}) \right].$$

We classify the larginals into four categories, depending on the pivot in the left and right neighbors. Let $\sigma = (\ldots, h, i, j, \ldots)$ be a permutation with pivot i. Then the four types are given in the following table:

Type	Pivot in σ^L	Pivot in σ	Pivot in σ^R
PPP	i	i	i
$-PP$	h	i	i
$PP-$	i	i	j
$-P-$	h	i	j

We can now determine the weight of each larginal, depending on its type. Take $\sigma \in \Pi(N)$ to be the identity permutation and assume that ℓ^σ is of type $PP-$ and has pivot i. Then,

$$\begin{aligned}
\ell^\sigma &= (M_1, \ldots, M_{i-2}, M_{i-1}, v(N) - \sum_{j=1}^{i-1} M_j, 0, \ldots, 0), \\
\ell^{\sigma^L} &= (M_1, \ldots, M_{i-2}, 0, v(N) - \sum_{j=1}^{i-2} M_j, 0, \ldots, 0), \text{ and} \\
\ell^{\sigma^R} &= (M_1, \ldots, M_{i-2}, M_{i-1}, 0, v(N) - \sum_{j=1}^{i-1} M_j, 0, \ldots, 0).
\end{aligned}$$

[5] As with *neighbors*, we use the term *pivot* as a property of a permutation as well as a property of the corresponding larginal.

Hence,

$$\begin{aligned}
d(\ell^\sigma, \ell^{\sigma^L}) &= \sqrt{2M_{i-1}^2} = \sqrt{2}M_{i-1}, \\
d(\ell^\sigma, \ell^{\sigma^R}) &= \sqrt{2(v(N) - \sum_{j=1}^{i-1} M_j)^2} = \sqrt{2}(v(N) - \sum_{j=1}^{i-1} M_j), \text{ and} \\
w(\sigma) &= (v(N) - \sum_{j=1}^{i-2} M_j).
\end{aligned}$$

Doing these calculations for each type and arbitrary $\sigma \in \Pi(N)$, we obtain the following weights:

Type	$w(\sigma)$
PPP	$M_{\sigma(p_\sigma-1)} + M_{\sigma(p_\sigma+1)}$
$-PP$	$\sum_{k=1}^{p_\sigma+1} M_{\sigma(k)} - v(N)$
$PP-$	$v(N) - \sum_{k=1}^{p_\sigma-2} M_{\sigma(k)}$
$-P-$	$M_{\sigma(p_\sigma)}$

Example 8.2. *Let σ_1 be the identity permutation. Then, we have (the player with ˆ is the pivot):*

$$\begin{aligned}
\sigma_1 &= (1, \hat{2}, 3, 4, 5) & \ell^{\sigma_1} &= (5, 5, 0, 0, 0) \\
\sigma_1^L &= (2, \hat{1}, 3, 4, 5) & \ell^{\sigma_1^L} &= (3, 7, 0, 0, 0) \\
\sigma_1^R &= (1, 3, \hat{2}, 4, 5) & \ell^{\sigma_1^R} &= (5, 4, 1, 0, 0).
\end{aligned}$$

Hence, ℓ^{σ_1} is of type $-PP$. The weight of σ_1 equals

$$\begin{aligned}
w(\sigma_1) &= \frac{1}{\sqrt{2}}\left(d(\sigma_1, \sigma_1^L) + d(\sigma_1, \sigma_1^R)\right) \\
&= 2 + 1 \\
&= 3.
\end{aligned}$$

Indeed, we have that $w(\sigma_1) = \sum_{k=1}^{p_{\sigma_1}+1} M_{\sigma_1(k)} - v(N) = M_1 + M_2 + M_3 - v(N) = 5 + 7 + 1 - 10 = 3$, as the table shows.

Step 2: chains

A *chain* of length q and with pivot i is a set of q permutations $\Gamma = \{\sigma_1, \ldots, \sigma_q\}$ such that

(i) for each $m \in \{1, \ldots, q-1\}$, $(\sigma_m)^R = \sigma_{m+1}$;

(ii) for each $m \in \{1, \ldots, q\}$, i is pivot in σ_m; and

(iii) i is not pivot in σ_1^L and σ_q^R.

If $q = 1$, then it follows from the definitions of the four types that σ_1 is of type $-P-$. If $q > 1$, then σ_1 is of type $-PP$, σ_m is of type PPP for each $m \in \{2, \ldots, q-1\}$, and σ_q is of type $PP-$. Observe that the set of all chains, which we denote by $\mathcal{C}$, constitutes a partition of the set of permutations $\Pi(N)$.

Let σ^* denote the permutation on the $n-1$ players obtained from σ by removing the pivot. Now, we characterize the chains in the following Lemma.

Lemma 8.1. *$\sigma_1 \in \Pi(N)$ and $\sigma_2 \in \Pi(N)$ are in the same chain if and only if $\sigma_1^* = \sigma_2^*$.*

Given the weights of the larginal vectors, depending on the type, we can easily compute the weight of a chain Γ, which is simply defined as the total weight of its elements, *i.e.*, $w(\Gamma) = \sum_{\sigma \in \Gamma} w(\sigma)$.

Lemma 8.2. *Let $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}$. Then,*

$$w(\Gamma) = \sum_{k=p_{\sigma_1}}^{p_{\sigma_1}+q-1} M_{\sigma_1(k)}.$$

Proof. Let $p = p_{\sigma_1}$. We have (for $q \geq 5$; for smaller chains the proof is similar):

$$\begin{array}{rcrrrrcr}
w(\sigma_1) & = & \sum_{k=1}^{p-1} M_{\sigma_1(k)} & - & v(N) & + & M_{\sigma_1(p)} & + \; M_{\sigma_1(p+1)} \\
w(\sigma_2) & = & & & & + & M_{\sigma_1(p+1)} & + \; M_{\sigma_1(p+2)} \\
w(\sigma_3) & = & & & & + & M_{\sigma_1(p+2)} & + \; M_{\sigma_1(p+3)} \\
\vdots & = & & & & & \vdots & \vdots \\
w(\sigma_{q-1}) & = & & & & + & M_{\sigma_1(p+q-2)} & + \; M_{\sigma_1(p+q-1)} \\
w(\sigma_q) & = & -\sum_{k=1}^{p-1} M_{\sigma_1(k)} & + & v(N) & - & \sum_{k=p+1}^{p+q-2} M_{\sigma_1(k)} & + \\
\hline
w(\Gamma) & = & & & & & \sum_{k=p}^{p+q-1} M_{\sigma_1(k)} &
\end{array}$$

□

We say that player $i \in N$ *belongs* to chain $\Gamma = \{\sigma_1, \ldots, \sigma_q\}$ if $i \in \{\sigma_1(p_{\sigma_1}), \ldots, \sigma_1(p_{\sigma_1}+q-1)\}$, *i.e.*, if his position is not constant throughout the chain. Note that if a player does belong to a chain, his utopia payoff contributes to its weight. We define $C(i)$ to be the set of chains to which i belongs. By $P(i) \subseteq C(i)$ we denote the set of chains in which i is pivot and by $\bar{P}(i) = C(i) \backslash P(i)$ its complement. For each $\Lambda \in \bar{P}(i)$, we denote the permutation in Λ in which i is immediately before the pivot by λ_{bi} and the permutation in which i is immediately after the pivot by λ_{ai}.

Example 8.3. *Since player 2 is not the pivot in σ_1^L, σ_1 is the first permutation of a chain. This chain Γ consists of σ_1, $\sigma_2 = \sigma_1^R$ and $\sigma_3 = \sigma_2^R$, all having player 2 as pivot. In line of Lemma 8.1, we have $\sigma_1^* = \sigma_2^* = \sigma_3^* = (1, 3, 4, 5)$. Players 2, 3 and 4 belong to Γ and $w(\Gamma) = M_2 + M_3 + M_4 = 11$.*

Step 3: denominator

In this step, we derive an expression for the denominator in ζ. We do this by showing that each player belongs to the same number of chains, *i.e.*, for each $i, j \in N$,

$$|C(i)| = |C(j)|. \tag{8.4}$$

If $M_i = M_j$, then this is trivial, so throughout this step, let $i, j \in N$ be such that $M_i > M_j$. We prove only one part of (8.4):

$$|P(j)| + |\bar{P}(j)| \leq |P(i)| + |\bar{P}(i)|. \tag{8.5}$$

The proof of the reverse inequality goes along similar lines, as we indicate later on.

An immediate consequence of Lemma 8.4 is that it follows from $M_i > M_j$ that $|P(i)| \geq |P(j)|$ and $|\bar{P}(j)| \geq |\bar{P}(i)|$. We establish (8.5) in Proposition 8.3 by partnering all the chains in $P(j)$ to some of the chains in $P(i)$ and partnering all the chains $\bar{P}(i)$ to some of the chains in $\bar{P}(j)$. We then show that for every chain in $\bar{P}(j)$ which has no partner in $\bar{P}(i)$, we can find a chain in $P(i)$ which has no partner in $P(j)$.

To partner the various chains, we define two functions. First, we define f^{ij}:

$$\begin{array}{rcl} P(j) & \stackrel{f^{ij}}{\rightarrow} & P(i) \\ \Delta & \mapsto & f(\Delta) = \Lambda, \end{array}$$

where $\Delta = \{\delta_1, \ldots, \delta_q\}$ and Λ is the chain to which $\delta_1^{i,j}$ belongs. Note that the function f^{ij} is well-defined: since $M_i > M_j$, player i is indeed the pivot in $\delta_1^{i,j}$ and hence, in Λ.

Similarly, we define the function g^{ij}:

$$\begin{array}{rcl} \bar{P}(i) & \stackrel{g^{ij}}{\rightarrow} & \bar{P}(j) \\ \Lambda & \mapsto & g(\Lambda) = \Delta, \end{array}$$

where for each $\Lambda \in \bar{P}(i)$, Δ is the chain containing $\lambda_{bi}^{i,j}$.[6]

In the following Lemma, we show that g^{ij} is well-defined, *i.e.*, that the chain Δ thus constructed is indeed an element of the range of g^{ij}, $\bar{P}(j)$.

Lemma 8.3. *The function g^{ij} is well-defined.*

Proof. Denote the pivot player in λ_{bi} (and hence, λ_{ai}) by h. Observe that as a result of $M_i > M_j$, player h cannot coincide with j. Distinguish between the following two cases:

[6]By $\lambda_{bi}^{i,j}$ we mean $(\lambda_{bi})^{i,j}$, *i.e.*, the permutation which is obtained by switching i and j in the permutation in Λ where i is immediately before the pivot.

- i is before j in λ_{bi}:

$$\lambda_{ai} = (\dots, \hat{h}, i, \dots, j, \dots) \quad \lambda_{ai}^{i,j} = (\dots, \hat{h}, j, \dots, i, \dots)$$
$$\lambda_{bi} = (\dots, i, \hat{h}, \dots, j, \dots) \quad \lambda_{bi}^{i,j} = (\dots, j, \hat{h}, \dots, i, \dots).$$

Since h is pivot in λ_{ai}, it immediately follows that h is also pivot in $\lambda_{ai}^{i,j}$. Player j cannot be the pivot in $\lambda_{bi}^{i,j}$, because i is before the pivot in λ_{bi} and $M_i > M_j$. Combining this with the fact that h is pivot in $\lambda_{ai}^{i,j}$, h is also pivot in $\lambda_{bi}^{i,j}$. But then $\lambda_{ai}^{i,j}$ belongs to the same chain Δ as $\lambda_{bi}^{i,j}$. From this, $\Delta \in C(j)$, and because j is not the pivot in Δ, $\Delta \in \bar{P}(j)$.

- j is before i in λ_{bi}:

$$\lambda_{ai} = (\dots, j, \dots, \hat{h}, i, \dots) \quad \lambda_{ai}^{i,j} = (\dots, i, \dots, \hat{h}, j, \dots)$$
$$\lambda_{bi} = (\dots, j, \dots, i, \hat{h}, \dots) \quad \lambda_{bi}^{i,j} = (\dots, i, \dots, j, \hat{h}, \dots).$$

Since h is pivot in λ_{bi}, we immediately have that h is pivot in $\lambda_{bi}^{i,j}$. Because of this, the pivot in $\lambda_{ai}^{i,j}$ cannot be before h. It can also not be after h, because h is pivot in λ_{ai} and $M_i > M_j$. By the same argument as in the first case, $\Delta \in \bar{P}(j)$.

From these two cases, we conclude that g^{ij} is well-defined. □

For our partnering argument to hold, we need that the functions f^{ij} and g^{ij} are injective. This is shown in the following Lemma.

Lemma 8.4. *The functions f^{ij} and g^{ij} are injective.*

Proof. To see that f^{ij} is injective, let $\Delta, \tilde{\Delta} \in P(j)$ be such that $f^{ij}(\Delta) = f^{ij}(\tilde{\Delta})$. By construction, i is pivot in both $f^{ij}(\Delta)$ and $f^{ij}(\tilde{\Delta})$, so i is pivot in both $\delta_1^{i,j}$ and $\tilde{\delta}_1^{i,j}$. Since by assumption these permutations are in the same chain, by Lemma 8.1 we have $(\delta_1^{i,j})^* = (\tilde{\delta}_1^{i,j})^*$. But since j is pivot in both δ_1 and $\tilde{\delta}_1$, it follows that $\delta_1^* = \tilde{\delta}_1^*$. So, δ_1 and $\tilde{\delta}_1$ are in the same chain and $\Delta = \tilde{\Delta}$.

For injectivity of g^{ij}, let $\Lambda, \tilde{\Lambda} \in \bar{P}(i)$ be such that $g^{ij}(\Lambda) = g^{ij}(\tilde{\Lambda})$. Then $\lambda_{bi}^{i,j}$ and $\tilde{\lambda}_{bi}^{i,j}$ are in the same chain. By the same arguments as used before, j is just before the pivot in both permutations and hence, $\lambda_{bi}^{i,j} = \tilde{\lambda}_{bi}^{i,j}$. From this, we conclude $\lambda_{bi} = \tilde{\lambda}_{bi}$ and $\Lambda = \tilde{\Lambda}$. □

From Lemma 8.4, we conclude

$$|P(j)| \leq |P(i)|$$

and

$$|\bar{P}(i)| \leq |\bar{P}(j)|.$$

With these inequalities, we can now apply our partnering argument to prove that each player belongs to the same number of chains.

Proposition 8.3. *Let $i, j \in N$. Then $|C(i)| = |C(j)|$.*

Proof. If $M_i = M_j$, then the statement is trivial. Hence, assume without loss of generality that $M_i > M_j$.

We only show (8.5). Let $\Delta \in \bar{P}(j)$ be such that there exists no $\Lambda \in \bar{P}(i)$ with $g^{ij}(\Lambda) = \Delta$. Denote the pivot in Δ by h and distinguish between the following three cases:

- $h \neq i$ and i is after j in δ_{bj}:

$$\begin{aligned} \delta_{aj} &= (\dots, \hat{h}, j, \dots, i, \dots) \quad & \delta_{aj}^{i,j} &= (\dots, \hat{h}, i, \dots, j, \dots) \\ \delta_{bj} &= (\dots, j, \hat{h}, \dots, i, \dots) \quad & \delta_{bj}^{i,j} &= (\dots, \hat{\imath}, h, \dots, j, \dots). \end{aligned}$$

  Of course, h is also the pivot in $\delta_{aj}^{i,j}$. If h were the pivot in $\delta_{bj}^{i,j}$, then $\delta_{aj}^{i,j}$ and $\delta_{bj}^{i,j}$ would belong to the same chain $\Lambda \in \bar{P}(i)$. But then $g^{ij}(\Lambda) = \Delta$, which is impossible by assumption. Since $M_i > M_j$, player i must be the pivot in $\delta_{bj}^{i,j}$. The chain to which $\delta_{bj}^{i,j}$ belongs cannot be an image under f^{ij}, since it is obtained by switching i and j in a permutation in which j is not the pivot. Furthermore, two different starting chains $\Delta, \tilde{\Delta} \in \bar{P}(j)$ cannot give rise to one single chain containing $\delta_{bj}^{i,j}$ and $\tilde{\delta}_{bj}^{i,j}$, because both permutations are of type $PP-$ or $-P-$ and there can be only one such permutation in a chain.

- $h \neq i$ and i is before j in δ_{bj}:

$$\begin{aligned} \delta_{aj} &= (\dots, i, \dots, \hat{h}, j, \dots) \quad & \delta_{aj}^{i,j} &= (\dots, j, \dots, h, \hat{\imath}, \dots) \\ \delta_{bj} &= (\dots, i, \dots, j, \hat{h}, \dots) \quad & \delta_{bj}^{i,j} &= (\dots, j, \dots, i, \hat{h}, \dots). \end{aligned}$$

  Again, it easily follows that h is pivot in $\delta_{bj}^{i,j}$ and by the same argument as in the first case, i must be pivot in $\delta_{aj}^{i,j}$. Also, the chain to which $\delta_{aj}^{i,j}$ belongs cannot be an image under f^{ij} and two different starting chains $\Delta, \tilde{\Delta} \in \bar{P}(j)$ cannot give rise to one single chain containing $\delta_{aj}^{i,j}$ and $\tilde{\delta}_{aj}^{i,j}$, because both permutations are of type $-PP$ or $-P-$. Moreover, the chains constructed in this second case, containing $\delta_{aj}^{i,j}$, must differ from the chains constructed in the first case, containing $\delta_{bj}^{i,j}$, as a result of the relative positions of h and j.

- $h = i$:

$$\begin{aligned} \delta_{aj} &= (\dots, \hat{\imath}, j, \dots) \quad & \delta_{aj}^{i,j} &= (\dots, j, \hat{\imath}, \dots) \\ \delta_{bj} &= (\dots, j, \hat{\imath}, \dots) \quad & \delta_{bj}^{i,j} &= (\dots, \hat{\imath}, j, \dots). \end{aligned}$$

  Obviously, i is pivot in both $\delta_{aj}^{i,j}$ and $\delta_{bj}^{i,j}$. So, these two permutations belong to the same chain $\Lambda \in P(i)$. Again Λ cannot be an image under f^{ij}, and since $\Lambda = \Delta$, different starting chains give rise to different Λ's. Finally, since j belongs to the "new" chains constructed in this case, they must differ from the chains in the first two cases.

Combining the three cases, for every element of $\bar{P}(j)$ that is not an image under g^{ij} of any chain in $\bar{P}(i)$, we have found a different element of $P(i)$ that is not an image under f^{ij} of any chain in $P(j)$. Together with Lemma 8.4, $|P(j)| + |\bar{P}(j)| \leq |P(i)| + |\bar{P}(i)|$.

Similarly, by taking $\Lambda \in P(i)$ such that there exists no $\Delta \in P(j)$ with $\Lambda = f^{ij}(\Delta)$, one can prove the reverse inequality of (8.5). Combining the two inequalities, we obtain $|C(i)| = |C(j)|$. □

Using the previous proposition, we can compute the total weight of all larginals.

Proposition 8.4. $\sum_{\sigma \in \Pi(N)} w(\sigma) = (n-1)! \sum_{i \in N} M_i$.

Proof. Since each of the n players belongs to the same number of chains and there are $n!$ permutations making up the chains, each player belongs to $\frac{n!}{n} = (n-1)!$ chains. But then the statement immediately follows from Lemma 8.2. □

Step 4: numerator, first player

Now we turn our attention to the numerator of ζ. For this, we partition the set of chains into subsets with the same starting player:

$$\mathcal{C}_k = \{\{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C} \, : \, \sigma_1(1) = k\}.$$

Note that since player k is by definition never the pivot in σ_1, he is also the first player in $\sigma_2, \ldots, \sigma_q$. It is easily verified that $\{\mathcal{C}_k\}_{k \in N}$ is indeed a partition of $\mathcal{C}$.

For a chain $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}$, we define L^Γ to be the weighted sum of its corresponding larginals:

$$L^\Gamma = \sum_{k=1}^{q} w(\sigma_k) \ell^{\sigma_k}.$$

We compute the numerator in (8.2) by combining the permutations that belong to the same $\mathcal{C}_k$, $k \in N$. We derive, for each player $i \in N$, an expression for $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma$. In this step, we consider the special case where $i = k$, while in the next step we compute the payoff to the other players.

Lemma 8.5. *For each $i \in N$, $\sum_{\Gamma \in \mathcal{C}_i} L_i^\Gamma = (n-2)! M_i \sum_{j \in N \setminus \{i\}} M_j$.*

Proof. In a similar way as in Proposition 8.3, we can show that for each $j, k \in N \setminus \{i\}$, $|\mathcal{C}_i \cap C(j)| = |\mathcal{C}_i \cap C(k)|$. Analogous to Proposition 8.4, we then have $\sum_{\sigma \in \Pi(N): \sigma(1) = i} w(\sigma) = (n-2)! \sum_{j \in N \setminus \{i\}} M_j$. Since player i always gets M_i at the first position, the statement follows. □

Step 5: numerator, other players

In this step, we finish the expression for the numerator in ζ by computing, for each $i \in N$, $i \neq k$, $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma$. First, in a similar way as in Lemma 8.2, one can compute the total weighted larginal for each chain, as is done in the next Lemma.

Lemma 8.6. *Let $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in P(i)$. Then for $j = \sigma(s)$ we have*

$$L_j^\Gamma = \begin{cases} w(\Gamma)M_j & \text{if } s < p_{\sigma_1}, \\ (v(N) - \sum_{k=1}^{p_{\sigma_1}-1} M_{\sigma_1(k)})M_j & \text{if } j = i, \\ (v(N) - \sum_{k=1,k\neq p_{\sigma_1}}^{s-1} M_{\sigma_1(k)} + \sum_{k=s+1}^{p_{\sigma_1}+q-1} M_{\sigma_1(k)})M_j & \text{if } \Gamma \in \bar{P}(j), \\ 0 & \text{if } s > p_{\sigma_1} + q - 1. \end{cases}$$

Example 8.4. *Of course, $L_1^\Gamma = w(\Gamma)M_1 = 11 \cdot 5 = 55$ and $L_5^\Gamma = 0$. For player 2, the pivot, we have*

$$\begin{aligned} L_2^\Gamma &= w(\sigma_1)(v(N) - M_1) + w(\sigma_2)(v(N) - M_1 - M_3) \\ &\quad + w(\sigma_3)(v(N) - M_1 - M_3 - M_4) \\ &= 3 \cdot (10 - 5) + 4 \cdot (10 - 5 - 1) + 4 \cdot (10 - 5 - 1 - 3) \\ &= 35. \end{aligned}$$

Indeed, this equals $(v(N) - \sum_{k=1}^{p_{\sigma_1}-1} M_{\sigma_1(k)})M_2 = (10 - 5) \cdot 2$, as stated in Lemma 8.6.

For player 3, which belongs to Γ but is not the pivot, we have

$$\begin{aligned} L_3^\Gamma &= w(\sigma_1) \cdot 0 + w(\sigma_2)M_3 + w(\sigma_3)M_3 \\ &= 0 + 4 \cdot 1 + 4 \cdot 1 \\ &= 8, \end{aligned}$$

which equals the expression in Lemma 8.6. For player 4, the computation is similar.

Lemma 8.7. *For each $i, k \in N, i \neq k$, we have $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma = (n-2)!(v(N) - M_k)M_i$.*

Proof. We prove the assertion using an iterative procedure, varying the utopia payoffs while keeping $v(N)$ constant. We denote the utopia vector in iteration t by M^t and throughout the procedure, this vector satisfies all our assumptions. We first show that the statement holds for $M^1 = (v(N), \ldots, v(N)) \geq M$. Then we iteratively reduce the components of the utopia vector one by one until we, after finitely many steps, end up in M. For every M^t, we show that for the corresponding (induced) set of chains, the total weighted payoff to i is as stated, *as function of the utopia vector.*

Step 1

Take $M^1 = (v(N), \ldots, v(N))$. Then all chains consist of one permutation, in which the second player is the pivot. Player i gets 0 if he is after the pivot and $v(N) - M_k^1$ if he is the pivot. There are $(n-2)!$ chains in which the latter occurs, each having weight M_i^1. Hence, $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma = (n-2)!(v(N) - M_k^1)M_i^1$.

Step t

Suppose that the statement holds for utopia vector M^{t-1}. If $M^{t-1} = M$, then we are finished. Otherwise, there exists a $j \in N$ such that $M_j^{t-1} > M_j$. We now reduce j's utopia payoff until one of the chains changes, or until M_j is reached.

A chain changes if in one of its permutations, the pivot changes. Obviously, this can only happen if player j is before the pivot. Because in the first permutation of each chain the gap between what the pivot gets and his utopia payoff is smallest, this permutation is the first to change. Denoting this gap corresponding to $\sigma \in \Pi(N)$ by $\gamma(\sigma)$, *i.e.*,

$$\gamma(\sigma) = M^{t-1}_{\sigma(p_\sigma)} - (v(N) - \sum_{k=1}^{p_\sigma - 1} M^{t-1}_{\sigma(k)}),$$

the first chain changes when j's utopia payoff is decreased by

$$\gamma = \min\{\gamma(\sigma_1) \ : \ \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}_k, \sigma_1^{-1}(j) \leq p_{\sigma_1}\}. \tag{8.6}$$

Assume, for the moment, that the corresponding argmin is unique and denote its first permutation by $\hat{\sigma}$.

If $\gamma \geq M_j^{t-1} - M_j$, then decreasing j's utopia payoff from M_j^{t-1} to M_j does not result in any change in the chains. In this case, the statement holds for M_j^t defined, for each $h \in N \backslash \{j\}$, by $M_j^t = M_j, M_h^t = M_h^{t-1}$. Proceed to step $t+1$.

Otherwise, define the second-highest gap $\tilde{\gamma}$ by

$$\tilde{\gamma} = \min\{\gamma(\sigma_1) \ : \ \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}_k, \sigma_1^{-1}(j) \leq p_{\sigma_1}, \gamma(\sigma_1) > \gamma(\hat{\sigma})\}$$

and take $M_j^t = M_j^{t-1} - (\gamma + \varepsilon)$, where $\varepsilon \in (0, \tilde{\gamma} - \gamma)$ and for each $h \in N \backslash \{j\}$, $M_h^t = M_h^{t-1}$. We show that the statement holds for this new utopia vector.

As mentioned before, $\hat{\sigma}$ is the first in a chain, say $\Gamma \in \mathcal{C}_k$. So, $\hat{\sigma}$ must be either of type $-P-$ or $-PP$. Define $s = \hat{\sigma}^{-1}(i)$ and distinguish between the two cases:

- $\hat{\sigma}$ is of type $-P-$:

  $\hat{\sigma}^R$ belongs to another chain, say $\Delta \in \mathcal{C}_k$ with length q. Note that the players $\hat{\sigma}(p_{\hat{\sigma}} - q + 1), \ldots, \hat{\sigma}(p_{\hat{\sigma}} - 1)$ and $\hat{\sigma}(p_{\hat{\sigma}} + 1)$ belong to Δ. When the pivot changes in $\hat{\sigma}$, this permutation joins Δ, as type $PP-$, forming chain $\Delta \cup \{\hat{\sigma}\}$. Hence, chain Γ disappears and the length of Δ is increased by one, while the other chains are not affected. So, it suffices to show that $L_i^{\Gamma, t-1} + L_i^{\Delta, t-1}$ as function of M^{t-1} equals $L_i^{\Delta \cup \{\hat{\sigma}\}, t}$ as function of M^t. Using Lemma 8.6, we have:

– $1 < s < p_{\hat{\sigma}} - q + 1$:

$$\begin{aligned} L_i^{\Gamma,t-1} &= M^{t-1}_{\hat{\sigma}(p_{\hat{\sigma}})} M_i^{t-1} \text{ (i is before Γ)}, \\ L_i^{\Delta,t-1} &= (M^{t-1}_{\hat{\sigma}(p_{\hat{\sigma}}+1)} + \sum_{k=p_{\hat{\sigma}}-q+1}^{p_{\hat{\sigma}}-1} M^{t-1}_{\hat{\sigma}(k)}) M_i^{t-1} \text{ (i is before Δ)}, \\ L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (\sum_{k=p_{\hat{\sigma}}-q+1}^{p_{\hat{\sigma}}+1} M^{t}_{\hat{\sigma}(k)}) M_i^{t} \text{ (i is before $\Delta\cup\{\hat{\sigma}\}$)}. \end{aligned}$$

– $s = p_{\hat{\sigma}}$:

$$\begin{aligned} L_i^{\Gamma,t-1} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-1} M^{t-1}_{\hat{\sigma}(k)}) M_i^{t-1} \text{ ($\Gamma \in P(i)$)}, \\ L_i^{\Delta,t-1} &= 0 \text{ (i is after Δ)}, \\ L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-1} M^{t}_{\hat{\sigma}(k)}) M_i^{t} \text{ (i is last in $\Delta\cup\{\hat{\sigma}\}$)}. \end{aligned}$$

– $p_{\hat{\sigma}} - q + 1 \leq s < p_{\hat{\sigma}}$:

$$\begin{aligned} L_i^{\Gamma,t-1} &= M^{t-1}_{\hat{\sigma}(p_{\hat{\sigma}})} M_i^{t-1} \text{ (i is before Γ)}, \\ L_i^{\Delta,t-1} &= (v(N) - \sum_{k=1}^{s-1} M^{t-1}_{\hat{\sigma}(k)} + \sum_{k=s+1}^{p_{\hat{\sigma}}-1} M^{t-1}_{\hat{\sigma}(k)}) M_i^{t-1} \text{ ($\Delta \in \bar{P}(i)$)}, \\ L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{k=1}^{s-1} M^{t}_{\hat{\sigma}(k)} + \sum_{k=s+1}^{p_{\hat{\sigma}}} M^{t}_{\hat{\sigma}(k)}) M_i^{t} \text{ ($\Delta\cup\{\hat{\sigma}\} \in \bar{P}(i)$)}. \end{aligned}$$

– $s = p_{\hat{\sigma}} + 1$:

$$\begin{aligned} L_i^{\Gamma,t-1} &= 0 \text{ (i is after Γ)}, \\ L_i^{\Delta,t-1} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-q} M^{t-1}_{\hat{\sigma}(k)}) M_i^{t-1}, \text{ ($\Delta \in P(i)$)}, \\ L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-q} M^{t}_{\hat{\sigma}(k)}) M_i^{t} \text{ ($\Delta\cup\{\hat{\sigma}\} \in P(i)$)}. \end{aligned}$$

– $s > p_{\hat{\sigma}} + 1$:

$$L_i^{\Gamma,t-1} = L_i^{\Delta,t-1} = L_i^{\Delta\cup\{\hat{\sigma}\},t} = 0 \text{ (i is after all three chains)}.$$

It is readily checked that in all cases, $L_i^{\Gamma,t-1} + L_i^{\Delta,t-1}$ as function of M^{t-1} equals $L_i^{\Delta\cup\{\hat{\sigma}\},t}$ as function of M^t.

- $\hat{\sigma}$ is $-PP$:
  $\hat{\sigma}^R$ belongs to the same chain as $\hat{\sigma}$. When the pivot changes in $\hat{\sigma}$, this permutation forms a new chain of length one. In the same manner as in the previous case, we can show that the total weighted payoff to i as function of the utopia vector in these two chains remains the same.

So, from these two cases, we conclude that the statement holds for the new set of chains induced by the (lower) utopia vector M^t. Proceed to step $t+1$.

We assumed that the minimal gap in (8.6) is obtained for a unique permutation, $\hat{\sigma}$. Suppose now that there exists another permutation, $\tilde{\sigma}$, with this minimal gap. Since both $\hat{\sigma}$ and $\tilde{\sigma}$ are of type $-P-$ or $-PP$, they must belong to different chains Γ and $\tilde{\Gamma}$. Also the two corresponding "neighboring" chains Δ and $\tilde{\Delta}$ are different, and different from Γ and $\tilde{\Gamma}$. Hence, we can consider the analysis in step t for $\hat{\sigma}$ and $\tilde{\sigma}$ separately to prove the statement.

Finally, our procedure stops after finitely many steps, because in all the changes, the pivot concerned moves towards the back of a permutation. □

Step 6: final

In this final step, we combine our previous results to prove the main theorem.

Proof of Theorem 8.1: Let $i \in N$. Then applying Lemmas 8.5 and 8.7 yields

$$\begin{aligned}
\sum_{\sigma\in\Pi(N)} w(\sigma)\ell_i^\sigma &= \sum_{\Gamma\in\mathcal{C}} L_i^\Gamma \\
&= \sum_{j\in N\setminus\{i\}} \sum_{\Gamma\in\mathcal{C}_k} L_i^\Gamma + \sum_{\Gamma\in\mathcal{C}_i} L_i^\Gamma \\
&= \sum_{k\in N\setminus\{i\}} (n-2)!(v(N)-M_k)M_i + (n-2)!M_i \sum_{k\in N\setminus\{i\}} M_k \\
&= (n-1)!v(N)M_i.
\end{aligned}$$

Then, using Proposition 8.4, we have

$$\begin{aligned}
\zeta_i &= \frac{\sum_{\sigma\in\Pi(N)} w(\sigma)\ell_i^\sigma}{\sum_{\sigma\in\Pi(N)} w(\sigma)} \\
&= \frac{(n-1)!v(N)M_i}{(n-1)!\sum_{j\in N} M_j} \\
&= \frac{v(N)}{\sum_{j\in N} M_j} M_i.
\end{aligned}$$

Hence, $\tau^* = \zeta$. □

8.5 Concluding Remarks

As we already stated in Section 8.2, for the class of compromise admissible games in which for each $i \in N$, $M_i(v) - m_i(v) \leq v(N) - \sum_{j \in N} m_j(v)$, τ^* coincides with the compromise value τ. As a result, Theorem 8.1 gives a geometric characterization of the latter on this class of games. Moreover, in Section 8.2 we also provided a motivation for the previous requirement, showing that it is quite natural.

This geometric property of the compromise value with respect to the core-cover can be added to the already existing ones in TU games. Hence, we have the following results: the Shapley value is the center of gravity of the vectors of marginal contributions, besides, for convex games it is the center of gravity of the extreme points of the core (now multiplicities have to be taken into account); the nucleolus is the lexicographic center of the core; the core-center is, by definition, the center of gravity of the core; and now, for the class of games we have mentioned above, the compromise value is the center of gravity of the edges of the core-cover (again, multiplicities have to be taken into account).

BIBLIOGRAPHY

CURIEL, I., M. MASCHLER, AND S. TIJS (1987): "Bankruptcy Games," *Zeitschrift für Operations Research (Now: Mathematical Methods of Operations Research)*, 31, 143–159. (Quoted in pp. 140, 141)

DRIESSEN, T. (1988): *Cooperative Games, Solutions and Applications*, Dordrecht: Kluwer Academic Publishers. (Quoted in pp. 140)

MUTO, S., M. NAKAYAMA, J. POTTERS, AND S. TIJS (1988): "On Big Boss Games," *Economic Studies Quarterly*, 39, 303–321. (Quoted in pp. 140)

QUANT, M., P. BORM, J. REIJNIERSE, AND B. VAN VELZEN (2003): "Compromise Stable TU-Games," CentER Discussion Paper 2003-55, Tilburg University, Tilburg, The Netherlands. (Quoted in pp. 142)

SHAPLEY, L. S. (1953): "A Value for n-Person Games," in *Contributions to the theory of games II*, ed. by H. Kuhn and A. Tucker, Princeton: Princeton University Press, vol. 28 of *Annals of Mathematics Studies*. (Quoted in pp. 140)

TIJS, S. (1981): "Bounds for the Core and the τ-Value," in *Game theory and mathematical economics*, ed. by O. Moeschlin and D. Pallaschke, Amsterdam: North Holland Publishing Company, 123–132. (Quoted in pp. 140, 141)

TIJS, S. AND F. LIPPERTS (1982): "The Hypercube and the Core Cover of n-Person Cooperative Games," *Cahiers du Centre d'Études de Researche Opérationelle*, 24, 27–37. (Quoted in pp. 140)

Resumen en Castellano

Resumen en Castellano

La teoría de juegos se ha dedicado, desde sus orígenes en los años 20 del siglo pasado, a dotar de estructura matemática a aquellas situaciones de la vida real en las que varios agentes (jugadores), entre los cuales existe un conflicto de intereses, compiten (juegan) estratégicamente. Situaciones como las que acabamos de comentar se repiten continuamente en la vida real; esto explica el impacto de la teoría de juegos en campos tan diversos como las ciencias políticas, la psicología, la biología, la inteligencia militar,... y, sobre todo, la economía. Los primeros trabajos en teoría de juegos se deben al matemático francés Emile Borel, quien, en 1921, publicó una serie de trabajos acerca de este tema. Sin embargo, serían John von Neumann y Oskar Morgenstern quienes en su libro *Theory of Games and Economic Behavior* revolucionaron la forma de estudiar teoría económica. Años después, en 1950, John Nash introdujo un nuevo concepto de equilibrio en teoría de juegos. Este nuevo concepto permitió, junto con otras herramientas desarrolladas en teoría de juegos, dotar a la teoría económica de un rigor del que había carecido hasta entonces. Los teoremas de punto fijo usados en los resultados de Nash fueron aplicados para obtener nuevos resultados relativos al equilibrio general, piedra angular de los estudios microeconómicos.

El propio John Nash fue el primero en establecer formalmente las diferencias entre juegos no cooperativos y juegos cooperativos; siendo la principal diferencia entre ambos que, en los modelos cooperativos, los jugadores pueden realizar acuerdos vinculantes y firmar contratos que después han de ser respetados. Esta misma separación la hemos adoptado en esta tesis, dedicando la primera parte de la misma a estudiar modelos no cooperativos y la segunda a los cooperativos.

Juegos No Cooperativos

En esta primera parte de la tesis, que comprende los Capítulos del 1 al 4, describimos y estudiamos varios modelos no cooperativos.

En el Capítulo 1 estudiamos una famila especial de juegos no cooperativos. Los conocidos como "timing games" (Karlin, 1959) modelan situaciones en las que no sólo las estrategias elegidas son importantes, sino también el momento en el que se juegan ha de ser tenido en cuenta. Muchos modelos conocidos en teoría de juegos pertenecen a esta familia. De entre ellos destacan los modelos de "war of attrition" o guerras de desgaste (Smith, 1974). El ejemplo más conocido, y al que deben el nombre, es el de dos depredadores peleando por la misma presa: en cuanto uno se retire de la pelea, la presa será para el otro, de modo que la estrategia consiste en elegir el momento en el que abandonar la lucha. La mayoría de los modelos de esta clase estudian situaciones en las que, una vez que uno de los agentes "actúa" (en el ejemplo anterior abandona la lucha), el otro es inmediatamente informado. Nosotros discutimos un modelo en el que suponemos que éste no es el caso. Consideremos una situación en la que dos empresas rivales están investigando para hacer el mismo descubrimiento patentable (véase, por ejemplo, Fudenberg et al. (1983)): sólo una de las dos conseguirá la patente, y toda la inversión realizada por la otra habrá sido en balde. En este caso, la estrategia en cada instante del tiempo consiste en seguir invirtiendo o abortar la investigación. En este modelo parece natural que una vez que una de las empresas abandona la investigación la otra no sea informada. Esto es importante en situaciones en las que las empresas en cuestión compiten en varios frentes al mismo tiempo. En este caso, si una de las empresas deja de invertir en uno de los frentes, entonces no va a querer que la otra firma pueda redistribuir su presupuesto para ser más competitiva en los frentes aún abiertos.

En este Capítulo, basado en el trabajo González-Díaz et al. (2004), presentamos una nueva clase de timing games y, siguiendo la motivación anterior, los llamamos "silent timing games". Esta primera aproximación a esta nueva clase de juegos la realizamos a través de los "cake sharing games", en los cuales la división de un pastel entre varios jugadores es modelada como un timing game. La idea es la siguiente. Supongamos que tenemos dos jugadores, cada uno de los cuales con derecho a un cuarto del pastel. El juego consiste en decir en qué instante del tiempo se desea recibir el pastel; el jugador más paciente recibirá su parte de pastel y, a mayores, la parte sobrante después de haberle dado al otro la parte a la que tenía derecho en el momento en el que la pidió. Esto es hecho de un modo silencioso, es decir, un jugador no tiene nunca información de lo que ha hecho el otro. Una vez que se introducen descuentos en el modelo, el dilema de los jugadores será cuánto arriesgar de su derecho inicial para conseguir la parte sobrante del pastel. Los principales resultados de este Capítulo extienden los obtenidos para juegos bipersonales en Hamers (1993) al caso general en el que hay n-jugadores. Más concretamente, probamos la existencia y unicidad del equilibrio de Nash para estos juegos.

En el Capítulo 2 realizamos un giro dentro de los juegos no cooperativos. Este Capítulo se enmarca dentro del campo de los juegos repetidos con información completa. Durante los últimos treinta años, se han publicado multitud de condiciones necesarias y suficientes para los llamados "folk theorems". Estos resultados aseguran que, bajo ciertas condiciones, todos los pagos factibles e individualmente racionales del juego de partida pueden ser obtenidos en equilibrio, de Nash o perfecto en subjuegos, en el juego repetido (ya sea una cantidad finita o infinita de veces). En el primero de estos "folk theorems" se probó que para juegos infinitamente repetidos no se necesita ninguna hipótesis para poder sustentar cualquier pago factible e individualmente racional en equilibrio de Nash (este resultado puede encontrarse en Fudenberg y Maskin (1986)). Impulsados por este resultado inicial, se siguieron buscando nuevos "folk theorems", ya fuese cambiando el horizonte infinito por el finito o bien reemplazando el concepto de equilibrio de Nash por el de equilibrio perfecto en subjuegos. Una buena recopilación de todos estos trabajos puede encontrarse en Benoît y Krishna (1996).

En este Capítulo, nosotros nos centramos en juegos finitamente repetidos con información completa y en el equilibrio de Nash. El resultado clásico en este contexto fue obtenido en Benoît y Krishna (1987), donde prueban una condición suficiente para el "Nash folk theorem": que el juego de partida tenga, para cada jugador, al menos dos pagos de Nash distintos. En este Capítulo, basado en el trabajo González-Díaz (2003), presentamos una nueva condición: que el juego se pueda descomponer como una "complete minimax-bettering ladder". Esta condición es más débil que la introducida en Benoît y Krishna (1987). Probamos, además, que no sólo es suficiente para el “folk theorem” en equilibrio de Nash, sino que también es necesaria. Además, también caracterizamos el conjunto de pagos que se pueden obtener en equilibrio de Nash en el caso de que la condición antes mencionada no se cumpla. Este último enfoque es nuevo en esta literatura ya que, tradicionalmente, los resultados se han centrado en buscar condiciones bajo las cuales todos los pagos factibles e individualmente racionales pueden obtenerse en equilibrio en el juego repetido. Nosotros, además de hacer eso, estudiamos cuáles serán los pagos que se pueden obtener en equilibrio cuando esas condiciones no se cumplen.

En el Capítulo 3, aunque seguimos dentro del marco de los juegos repetidos, nos alejamos ligeramente del enfoque clásico. Introducimos, dentro de este marco, el concepto de compromiso. El impacto de diferentes tipos de compromisos en modelos no cooperativos ha sido ampliamente estudiado y, de entre estos estudios, destacan los juegos de delegación. En estos modelos se estudian aquellas situaciones en las que los jugadores pueden “contratar” agentes para que participen en el juego en su lugar. Estos juegos fueron discutidos inicialmente en Schelling (1960), donde se pueden encontrar diversas motivaciones para los mismos. El tipo de compromisos que nosotros introducimos es formalmente distinto de los modelos de delegación, pero la idea subyacente es la misma. El interés de estos modelos con compromisos radica en estudiar hasta qué punto se pueden conseguir en equilibrio pagos que, sin la ayuda de estos compromisos, serían inestables. Nosotros discutimos el concepto de compromisos unilaterales, concepto introducido inicialmente en García-Jurado et al. (2000) y cuya idea es la siguiente: en una primera etapa del juego los jugadores pueden deshacerse, simultánea e independientemente, de algunas de sus estrategias en el juego repetido. Después, estos compromisos se hacen públicos y el juego repetido comienza. En otras palabras, antes de empezar a jugar, los jugadores realizan una serie de compromisos que después han de respetar durante el desarrollo del juego.

En este Capítulo, basado en el trabajo García-Jurado y González-Díaz (2005), inicialmente discutimos la relación entre los compromisos unilaterales y los modelos de delegación. Después, obtenemos una serie de “folk theorems” para juegos repetidos con compromisos unilaterales. Finalmente, realizamos un análisis comparativo de nuestros resultados con los resultados clásicos en juegos repetidos para evaluar el impacto de los compromisos a la hora de obtener equilibrios del juego repetido que sustenten pagos “cooperativos” del juego original. Del mismo modo, también discutimos brevemente nuestros resultados en comparación con los obtenidos para juegos de delegación y damos una posible aplicación de los compromisos unilaterales dentro de la literatura de los modelos de delegación.

En el Capítulo 4 realizamos un nuevo giro y estudiamos problemas de bancarrota (O’Neill, 1982) con un enfoque no cooperativo. Los problemas de bancarrota surgieron para modelar situaciones en las que una empresa se declara en quiebra y hay que repartirse el dinero que queda entre los acreedores. Por supuesto, la propia naturaleza del problema implica que el dinero restante no es suficiente para satisfacer las demandas de los acreedores. A pesar de la sencillez del problema y de su aparente simplicidad matemática, la literatura en problemas de bancarrota ha crecido tremendamente en los últimos 20 años. También el enfoque de estos problemas desde el punto de vista no cooperativo ha sido ampliamente abordado en esta literatura (veánse, por ejemplo, Thomson (2003); Dagan et al. (1997); Sonn (1992)).

En este Capítulo, basado en García-Jurado et al. (2004), definimos una familia de juegos no cooperativos de tal manera que a cada problema de bancarrota podemos asignarle juegos dentro de esta familia. En una primera parte demostramos que todos los equilibrios de Nash de cada uno de estos juegos tienen el mismo pago y que, además, dicho equilibrio de Nash es también un equilibrio fuerte. Después, demostramos que, dados un problema y una regla de bancarrota, podemos encontrar un juego dentro de nuestra familia cuyo pago en equilibrio se corresponde con la propuesta de la regla para el citado problema.

Juegos Cooperativos

La segunda parte de la Tesis está dedicada a estudiar situaciones en las que los distintos jugadores pueden firmar entre ellos contratos y realizar acuerdos vinculantes. Esto hace que, a diferencia de lo que pasaba en los juegos no cooperativos, el concepto de equilibrio no sea relevante, la estabilidad de las soluciones vendrá garantizada por el carácter vinculante de los acuerdos y la obligatoriedad de cumplir los contratos. La subclase más estudiada dentro de los modelos de juegos cooperativos se corresponde con los juegos con utilidad transferible (TU). Los juegos TU modelan situaciones en las que los contratos y acuerdos contemplan la posibilidad de transferir dinero (u otra variable que actúe como numeraria) entre los distintos agentes.

Nuestro estudio en esta parte de la Tesis se centra, principalmente, en estudiar la geometría que subyace bajo algunos de los conceptos de solución más relevantes en la literatura de juegos TU. Los tres primeros Capítulos de esta segunda parte se basan en los trabajos González-Díaz y Sánchez-Rodríguez (2003a,b). Dichos trabajos se centran en el estudio del core de un juego TU (Gillies, 1953); del core-center, un nuevo concepto de solución introducido en esta tesis; y, finalmente, de su relación con el valor de Shapley (Shapley, 1953). En el Capítulo 5 se introduce formalmente el core-center, un nuevo concepto de solución para juegos equilibrados, definido como el centro de gravedad del core de un juego TU. El core es, con el permiso del valor de Shapley, el concepto de solución más importante en juegos cooperativos. La idea detrás de este concepto de solución está centrada en la estabilidad y la eficiencia: hay que proponer un reparto que, siendo eficiente, proponga una asignación que ninguna de las coaliciones pueda bloquear, es decir, que ninguna coalición pueda irse por su cuenta y salir ganando con respecto a la asignación propuesta. Una vez aceptado que el core es un concepto de solución muy natural, nosotros planteamos el siguiente razonamiento: si estamos de acuerdo en que hay que elegir una asignación dentro del core (es decir, estable y eficiente), y dado que el core de un juego TU es un conjunto convexo, ¿por qué no quedarnos con su centro de gravedad?. En este capítulo discutimos diversas motivaciones para esta solución y llevamos a cabo un estudio de las propiedades que verifica. De entre estas propiedades destacan las propiedades de monotonía, cuyo estudio está fuertemente marcado por los resultados negativos de los trabajos de Young (1985) y de Housman y Clark (1998); trabajos en los que se prueba una fuerte incompatibilidad entre las propiedades de monotonía y las reglas de asignación que siempre escogen elementos del core. Por otro lado, también juega un papel importante la continuidad, ya que una buena parte del Capítulo 5 está dedicada a probar que, efectivamente, el core-center es una regla de asignación continua. El Capítulo 6 está íntegramente dedicado a la caracterización axiomática del core-center. Para ello se introduce una nueva propiedad, llamada "fair additivity"; esta es una aditividad ponderada, en la que aparecen unos coeficientes que miden la importancia de los juegos teniendo en cuenta la estructura de sus cores. Posteriormente, combinando esta propiedad de aditividad con algunas de las propiedades discutidas en el capítulo anterior, obtenemos una caracterización axiomática del core-center. Estas propiedades son: fair additivity, simetría, continuidad, y eficiencia. Además, la demostración de este resultado guarda un cierto paralelismo con la caracterización del valor de Shapley utilizando la propiedad de aditividad. En una primera parte de la demostración se prueba el resultado para juegos cuyo core en un simplex; estos juegos tienen un papel similar al de los juegos de unanimidad en la caracterización del valor de Shapley, y son usados posteriormente para probar el resultado para juegos cuyos cores son polítopos arbitrarios. Finalmente, en el Capítulo 7 se estudia, ya dentro de la clase de juegos convexos, la relación entre el valor de Shapley y el core-center. En este Capítulo juegan un papel muy importante los llamados juegos de utopía. Primero describimos el proceso de formación del core de un juego TU como un proceso dinámico entre las distintas coaliciones de jugadores. Después, basados en esta interpretación, definimos una serie de juegos, los juegos de utopía, que surgen de un modo natural de la motivación anterior. Posteriormente, probamos que, para ciertas subclases de juegos, el core-center y el valor de Shapley de los juegos de utopía coinciden. Este primer resultado nos permite probar un resultado más general que establece una conexión bastante directa entre el valor de Shapley y el core-center.

Finalmente, en el Capítulo 8 nos alejamos un poco de lo estudiado en el resto de capítulos de esta parte. Aunque seguimos estudiando la geometría de los juegos TU, el concepto de solución con el que trabajamos es el τ valor (Tijs, 1981). Esta regla de asignación pretende ser un compromiso entre lo mínimo que se debe conceder a cada jugador de acuerdo a la situación dada y lo máximo a lo que éste puede aspirar. Una vez que estos valores se definen para cada jugador tenemos dos vectores, el de mínimos derechos y el de máximas aspiraciones. El τ valor se define como la única asignación eficiente en la recta que une estos dos puntos. También a partir del vector de mínimos derechos y el de máximas aspiraciones se define el core-cover: un conjunto que juega para el τ valor un papel similar al que juega el conjunto de Weber con respecto del valor de Shapley. En este Capítulo demostramos que hay una fuerte relación entre el τ valor y la geometría del core-cover. Más específicamente, probamos que, para la mayoría de los juegos para los que está definido, el τ valor es el centro de gravedad de las aristas del core-cover, teniendo en cuenta las multiplicidades de las mismas. Este resultado recuerda al que relaciona a los vectores de contribuciones marginales con el valor de Shapley, ya que este último es el centro de gravedad de los mismos (teniendo en cuenta las multiplicidades). Llamemos ahora τ^* valor a la solución que consiste en seleccionar el centro de gravedad de las aristas del core-cover. Entonces, como ya hemos dicho, esta solución coincidirá casi siempre con el τ valor. Pero además, si estudiamos el juego de bancarrota asociado con cada juego TU, nos encontramos con que el τ valor del juego original se corresponde con la asignación propuesta por la regla proporcional y, por otro lado, el τ^* valor se corresponde con la regla proporcional ajustada (Curiel et al., 1987).

Conclusiones

En la primera parte de esta tesis hemos discutido distintos modelos no cooperativos. Para ellos hemos encontrado resultados relativos, principalmente, a la existencia y unicidad de equilibrios de Nash en las distintas situaciones. En el Capítulo 1 hemos obtenido un teorema de existencia y unicidad de equilibrio de Nash en una clase de "timing games" que generaliza la discutida en Hamers (1993). En el Capítulo 2 hemos obtenido una condición necesaria y suficiente para el "Nash folk theorem" para juegos finitamente repetidos con información completa. Este resultado refina el obtenido en Benoît y Krishna (1987). Además, nuestro resultado es más general que los habituales en esta literatura. Esto es porque no sólo busca condiciones necesarias y suficientes para que todos los pagos factibles e individualmente racionales puedan ser obtenidos en equilibrio, sino que también caracterizamos el conjunto de pagos que se pueden obtener en equilibrio de Nash en el caso de que tales condiciones no se cumplan. El Capítulo 3 también ha girado en torno a los juegos repetidos. En él hemos estudiado los distintos "folk theorems" cuando al juego repetido se le añade una etapa inicial en la que los jugadores pueden adoptar compromisos unilaterales. En el Capítulo 4, último de la primera parte de la tesis, hemos presentado una aproximación no cooperativa a los modelos de bancarrota. En ella hemos presentado una familia de juegos que se pueden asociar a cada problema de bancarrota y que nos permiten obtener en equilibrio el reparto propuesto por cualquier regla de asignación.

La segunda parte de la tesis ha tratado sobre juegos cooperativos. La mayor parte de esta segunda parte la hemos centrado en el estudio de una nueva regla de asignación para juegos TU, el core-center. En el Capítulo 5 hemos realizado un estudio en profundidad de las propiedades que verifica el core-center; en el Capítulo 6 hemos presentado una caracterización axiomática; finalmente, en el Capítulo 7 hemos llevado a cabo un análisis que nos ha permitido establecer una conexión entre el core-center y el valor de Shapley para juegos convexos. Además, en el Capítulo 8 hemos obtenido una caracterización geométrica del τ valor. Finalemte, destacar también que hemos profundizado en la geometría de los juegos TU. Lo hemos hecho estableciendo conexiones entre varias reglas de asignación y varios conceptos de solución multivaluadas. Es muy conocido que, por un lado, el valor de Shapley ocupa una posición central dentro del conjunto de Weber y, por otro, que el nucleolo se conoce también como el centro lexicográfico del core. A estas relaciones nosotros hemos añadido la relación entre el core y el core-center, definido como el centro de gravedad del primero y, por otro lado, hemos demostrado que el τ valor ocupa, en general, una posición central en el core-cover de un juego quasi-equilibrado.

Líneas abiertas

Finalmente, destacamos las siguientes líneas de investigación que serían una continuación natural de los distintos capítulos presentados en esta tesis:

- Profundizar en la literatura de los "timing games" para estudiar las posibles implicaciones y aplicaciones de los resultados presentados en el Capítulo 1.
- Estudiar si es posible aplicar las ideas del Capítulo 2 para afinar/extender otros "folk theorems" en juegos repetidos.
- Seguir estudiando el impacto de los compromisos unilaterales en las hipótesis necesarias para los "folk theorems".
- Ampliar el horizonte de estudio dentro de los juegos repetidos a aquellos con información incompleta.
- Siguiendo con las ideas presentadas en el Capítulo 4, buscar algún modelo de implementación descentralizado.
- Seguir estudiando la geometría del core de un juego TU y profundizar en las propiedades del core-center.
- Buscar otras caracterizaciones del core-center.
- Analizar las conexiones entre soluciones tipo conjunto y soluciones puntuales mediante el uso de centroides.
- Seguir analizando las relaciones existentes entre core-center y las demás soluciones clásicas para juegos TU.
- Estudiar el problema de la computación del core-center.

Bibliografía del Resumen

Benoît, J.-P. y V. Krishna (1987): "Nash Equilibria of Finitely Repeated Games," *International Journal of Game Theory*, 16, 197–204. (Quoted in pp. 164, 168)

——— (1996): "The Folk Theorems for Repeated Games: A Synthesis," Tech. rep., EconWPA-9601001. (Quoted in pp. 164)

Curiel, I., M. Maschler, y S. Tijs (1987): "Bankruptcy Games," *Zeitschrift für Operations Research (Now: Mathematical Methods of Operations Research)*, 31, 143–159. (Quoted in pp. 168)

Dagan, N., R. Serrano, y O. Volij (1997): "A Noncooperative View of Consistent Bankrupcty Rules," *Games and Economic Behavior*, 18, 55–72. (Quoted in pp. 166)

Fudenberg, D., R. Gilbert, J. Stiglitz, y J. Tirole (1983): "Preemption, Leapfrogging and Competition in Patent Races," *European Economic Review*, 22, 3–31. (Quoted in pp. 163)

Fudenberg, D. y E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54, 533–554. (Quoted in pp. 164)

García-Jurado, I. y J. González-Díaz (2005): "Unilateral Commitments in Repeated Games," Preprint. (Quoted in pp. 165)

García-Jurado, I., J. González-Díaz, y A. Villar (2004): "A Noncooperative Approach to Bankruptcy Problems," Preprint. (Quoted in pp. 166)

García-Jurado, I., L. Méndez-Naya, y F. Patrone (2000): "Unilateral Commitments in Finitely Repeated Games," *International Game Theory Review*, 2, 129–139. (Quoted in pp. 165)

Gillies, D. B. (1953): "Some Theorems on n-Person Games," Ph.D. thesis, Princeton. (Quoted in pp. 166)

González-Díaz, J. (2003): "Finitely Repeated Games: A Generalized Nash Folk Theorem," Tech. Rep. 03-08, Department of Statistics and OR, University of Santiago de Compostela, to appear in Games and Economic Behavior. (Quoted in pp. 164)

González-Díaz, J., P. Borm, y H.Ñorde (2004): "A Silent Battle over a Cake," Tech. rep., CentER discussion paper series. (Quoted in pp. 164)

González-Díaz, J. y E. Sánchez-Rodríguez (2003a): "The Core-Center and the Shapley Value: A Comparative Study," Reports in Statistics and Operations Research 03-10, University of Santiago de Compostela. (Quoted in pp. 166)

——— (2003b): "From Set-Valued Solutions to Single-Valued Solutions: The Centroid and the Core-Center," Reports in Statistics and Operations Research 03-09, University of Santiago de Compostela. (Quoted in pp. 166)

Hamers, H. (1993): "A Silent Duel over a Cake," *Methods and Models of Operations Research*, 37, 119–127. (Quoted in pp. 164, 168)

Housman y Clark (1998): "Core and Monotonic Allocation Methods," *International Journal of Game Theory*, 27, 611–616. (Quoted in pp. 166)

KARLIN, S. (1959): *Mathematical Methods and Theory in games. Programming and Economics*, Addison-Wesley. (Quoted in pp. 163)

O'NEILL, B. (1982): "A Problem of Rights Arbitration from the Talmud," *Mathematical Social Sciences*, 2, 345–371. (Quoted in pp. 165)

SCHELLING, T. C. (1960): *The Strategy of Conflict*, Oxford University Press. (Quoted in pp. 165)

SHAPLEY, L. S. (1953): "A Value for n-Person Games," en *Contributions to the theory of games II*, editado por H. Kuhn y A. Tucker, Princeton: Princeton University Press, vol. 28 de *Annals of Mathematics Studies*. (Quoted in pp. 166)

SMITH, J. M. (1974): "The Theory of Games and Evolution in Animal Conflicts," *Journal of Theoretical Biology*, 47, 209–221. (Quoted in pp. 163)

SONN, S. (1992): "Sequential Bargaining for Bankruptcy Problems," Preprint. (Quoted in pp. 166)

THOMSON, W. (2003): "Axiomatic and Game Theoretic Analysis of Bankruptcy and Taxation Problems," *Mathematical Social Sciences*, 45, 249–297. (Quoted in pp. 166)

TIJS, S. (1981): "Bounds for the Core and the τ-Value," en *Game theory and mathematical economics*, editado por O. Moeschlin y D. Pallaschke, Amsterdam: North Holland Publishing Company, 123–132. (Quoted in pp. 167)

YOUNG, H. (1985): "Monotonic Solutions of Cooperatives Games," *International Journal of Game Theory*, 14, 65–72. (Quoted in pp. 166)

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