Spectral characterization of the constant sign derivatives of Green's function related to two point boundary value conditions

Alberto Cabada, Lucía López-Somoza and Mouhcine Yousfi

[1] CITMAga, 15782, Santiago de Compostela, Galicia, Spain

[2] Departamento de Estatística, Análise Matemática e Optimización

Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain.

alberto.cabada@usc.es; lucia.lopez.somoza@usc.es; yousfi.mouhcine@usc.es

Abstract

In this paper we will consider a n-order linear operator $T_n[M]$, depending on a real parameter M, coupled to different two-point boundary conditions, and we will study the set of parameters for which certain partial derivatives of the related Green's function are of constant sign. We will do it without using the explicit expression of the Green's function. In particular, the set of parameters for which the derivatives of the Green's function have constant sign will be an interval whose extremes are characterized as the first eigenvalues of the studied operator related to suitable boundary conditions. As a consequence of the main result, we will be able to give sufficient conditions to ensure that the derivatives of Green's function cannot be nonpositive (nonnegative). These characterizations and the obtained results can be used to deduce the existence of solutions of nonlinear problems under additional conditions on the nonlinear part. In order to illustrate the obtained results, some examples are given.

AMS Subject Classifications: 34B05, 34B08, 34B09, 34B15, 34B18, 34B27.
Keywords: Green's function, Two-point boundary conditions, Spectral theory, Comparison results, Constant sign.

1 Introduction

Many papers in the literature have studied the qualitative properties of solutions of differential equations coupled with different boundary conditions. In particular, several techniques have been used to ensure the existence, non-existence and multiplicity of solutions. Among them, we highlight the method of lower and upper solutions, coupled with monotone iterative techniques which, as it is very well-known, are equivalent to the constant sign of the Green's function of the corresponding linear problem. Moreover, the method of constructing suitable cones in

Banach spaces to apply the well-known Krasnosel'skiĭ's fixed-point theorem or the classical index theory, see for instance [15, 22, 23, 28], is a powerful tool to deduce the existence of positive solutions of the nonlinear problem. Such cones are constructed using constant sign properties of the Green's function and some of its derivatives. So, the more information we have about such constant signs, we may construct smaller cones, that allow us to ensure the positiveness of both the solutions we are looking for, and some of its derivatives. Thus, for instance, we may prove the existence of positive and convex solutions if both the Green's function and its second partial derivative with respect to t are nonnegative on their square of definition.

The study of the constant sign of the Green's function has been studied for many authors on the literature. Usually some sufficient conditions are obtained to deduce the constant sign of such function and some of its partial derivatives with respect to the first variable. Different kind of two-point boundary conditions have been considered and, usually, the differential linear operator is assumed to be disconjugate or disfocal on a suitable interval. We refer to the reader the classical works [6, 17–19, 24–26] and the recent ones of Almenar and Jódar [1–5] and references therein.

To be concise, in this paper, we consider the following n-order linear differential equation

$$T_n[M]\,u(t) \equiv u^{(n)}(t) + M\,u(t) = 0\,,\ \ t \in I := [a,b]\,, \tag{1.1}$$

where $M \in \mathbb{R}$ is a real parameter, coupled to the two-point boundary conditions

$$u^{(\sigma_1)}(a) = \quad \cdots \quad = u^{(\sigma_k)}(a) \quad = 0, \tag{1.2}$$
$$u^{(\varepsilon_1)}(b) = \quad \cdots \quad = u^{(\varepsilon_{n-k})}(b) = 0, \tag{1.3}$$

with $k \in \{1, \dots, n-1\}$ and the sets of indices $\{\sigma_1, \dots, \sigma_k\}$, $\{\varepsilon_1, \dots, \varepsilon_{n-k}\} \subset \{0, \dots, n-1\}$, satisfying that

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n-1\,, \quad 0 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leq n-1\,.$$

More concretely, this paper is devoted to study the set of parameters M for which certain partial derivatives of the Green's function g_M, related to problem (1.1)–(1.3), are of constant sign. In particular, we develop our study without using the explicit expression of the Green's function since, in general, it is not possible to obtain such expression, specially for non constant coefficients. Moreover, it is important to point out that the spectral characterization of the constant sign of the related Green's function to the general n-th order linear differential operator, with non constant coefficients, and coupled to the boundary conditions (1.2)–(1.3), has been obtained in [12]. Such existence results generalize the ones obtained by the same authors in [10] for the $(k, n-k)$ boundary conditions. The main assumption on that references consists on the disconjugacy property of the considered operator for a given value of the parameter $\bar{M}$ and the, so-called, strongly inverse positive (negative) property of the Green's function. Under these assumptions, it is obtained the exact interval of parameters M for which the Green's function has constant sign. It is important to point out that such interval does not coincide with the interval of parameters for which the considered equation is disconjugate (see [11] for details). It must be noted that the arguments developed in references [10, 12] do not hold for the partial derivatives of the Green's function, because such derivatives are not the solution of the homogeneous linear equation. This is the reason why we must restrict now our study to the operator $T_n[M]$.

For the particular case of equation (1.1), together with the boundary conditions (1.2)–(1.3), the interval of parameters for which the Green's function has constant sign has been characterized in [12, Theorem 8.1]. Also for boundary conditions of the type $(k, n-k)$, i.e., $\sigma_j = j-1$ for all $j = 1, \dots, k$, and $\varepsilon_j = j-1$ for all $j = 1, \dots, n-k$, which are a particular case of the previous ones, a characterization of the constant sign intervals of the Green's function is given in [10]. Moreover, the exact expression of the eigenvalues that characterizes these intervals up to $n = 8$ is obtained.

As we have mention before, the main assumption is the disconjugacy character of the considered operator. In our case, it is very well known that the operator $T_n[M]$, is disconjugate for the value $M = 0$ (see [10, 14, 16]) at any arbitrary interval $[a, b]$. In the literature, sufficient conditions have been given to ensure the disconjugacy character of linear operators, we refer to [14, 16, 27, 29] and the references therein. A characterization of this property is proved in [11], in which the optimal extremes of the intervals are given as the eigenvalues of suitable $(k, n-k)$ problems.

It is in this context where the corresponding spectral theory arises: we will prove that the constant sign interval is characterized by the first eigenvalues of the studied operator, related to suitable boundary conditions.

The study starts from conditions (1.2)–(1.3) and constructs the conditions satisfied by the derivatives of g_M, defining the necessary hypotheses that we need to obtain the main results, and deducing the relationships between the adjoint spaces of the respective related spaces. This way, we will obtain results that characterize the constant sign interval of the corresponding derivatives.

The novelty of the work is that the results generalize [12, Theorem 8.1] for the particular operator $T_n[M]$ defined in (1.1). Also, as a consequence of these results, we will give sufficient conditions under which the derivatives of the Green's function cannot be of constant sign. Furthermore, we will obtain a necessary condition that will allow us to ensure that the corresponding derivatives are strictly positive (strictly negative) on the interior of its square of definition. We point out that, in general, the interval of parameters M where the corresponding partial derivative has constant sign does not coincide with the interval of values for which the equation is disconjugate.

The paper is organized as follows: in a preliminary Section 2 we introduce the fundamental concepts, the results and the main hypotheses that are needed in the development of the article. Section 3 is devoted to prove the results that characterize the constant sign of the derivatives of the Green's function by distinguishing three cases. The main results are proved through spectral theory. This section also includes a corollary that provides sufficient conditions for derivatives to be nonpositive or nonnegative, a necessary condition to ensure the negative (positive) sign of certain derivatives with respect to t of g_M, and some examples of application of our main results. Finally, Section 4 includes an application to ensure the existence of positive solutions of nonlinear problems.

2 Preliminaries, hypotheses and main assumptions

In this section, we will make a survey of several results and properties that will be used along the paper. The main part of these results are proved in [12], and many of them are given in [10].

Let us consider the n^{th}-order linear differential equation (1.1) coupled to the two-point boundary conditions (1.2)–(1.3).

Definition 2.1. We say that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy property (N_a) if

$$\sum_{\sigma_j<h} 1+\sum_{\varepsilon_j<h} 1 \geq h\,,\quad \forall h\in\{1,\dots,n-1\}\,.$$

***Remark* 2.2.** We point out that condition (N_a) is a classical condition assumed when studying the sign of the Green's function coupled to a linear differential operator and with boundary conditions (1.2)–(1.3), and its corresponding partial derivatives. See for instance [6]. In reference [2] the authors denote such property as *admissible conditions*.

***Notation* 2.3.** Let us denote

$$\alpha=\min\{i\in\{0,\dots,n-1\}:\ \ i\notin\{\sigma_1,\dots,\sigma_k\}\}\ , \tag{2.1}$$

$$\beta=\min\{i\in\{0,\dots,n-1\}:\ \ i\notin\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\}\ . \tag{2.2}$$

We introduce the following set of functions related to the boundary conditions (1.2)–(1.3):

$$X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}=\Big\{u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_k)}(a)=u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0\Big\}\,.$$

Also, we define the following sets of functions:

$$X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}=\Big\{u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_{k-1})}(a)=0\,,\ u^{(\alpha)}(a)=0\,,u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0\Big\}\,,$$

$$X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}=\Big\{u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_k)}(a)=0\,,\ u^{(\alpha)}(a)=0\,,u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k-1})}(b)=0\Big\}\,,$$

$$X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}=\Big\{u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_{k-1})}(a)=0\,,u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0\,,\ u^{(\beta)}(b)=0\Big\}\,,$$

$$X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}=\Big\{u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_k)}(a)=0\,,u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k-1})}(b)=0\,,\ u^{(\beta)}(b)=0\Big\}\,.$$

Definition 2.4. Let us say that the operator $T_n[M]$ satisfies the property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, there exists the following decomposition:

$$T_0\,u(t)=u(t)\,,\quad T_l\,u(t)=\frac{d}{dt}\left(\frac{T_{l-1}\,u(t)}{v_l(t)}\right),\ l=1,\dots,n\,,\ t\in I, \tag{2.3}$$

where $v_l>0$ on I, $v_l\in C^n(I)$, are such that

$$T_n[M]\,u(t)=v_1(t)\,\dots\,v_n(t)\,T_n\,u(t)\,,\ t\in I\,,$$

and, moreover, such decomposition satisfies the following equalities for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$T_{\sigma_1}\, u(a) = \quad \cdots \quad = T_{\sigma_k}\, u(a) \quad = 0\,, \tag{2.4}$$
$$T_{\varepsilon_1}\, u(b) = \quad \cdots \quad = T_{\varepsilon_{n-k}}\, u(b) = 0\,. \tag{2.5}$$

***Remark* 2.5.** In [14, Chapter 3] it is proved that a linear differential equation admits a decomposition of the type (2.3) in some interval I if and only if such linear equation is disconjugate in I, i.e., every non trivial solution of the equation has less than n zeros on I, with multiple zeros being counted according to their multiplicity.

We point out that condition (T_d) introduced in Definition 2.4 can be rewritten as: "Operator $T_n[M]$ is disconjugate in I and satisties the additional boundary conditions (2.4) – (2.5)." We note that such boundary conditions may not hold for some disconjugate equation.

***Remark* 2.6.** Note that decomposition (2.3) is not unique, it depends on the choice of v_k for $k = 1, \dots, n$. Throughout this work, for the operator $T_n[0]\, u(t) = u^{(n)}(t)$ we choose the following decomposition:

$$T_0\, u(t) = u(t)\,, \quad T_k\, u(t) = \frac{d}{dt}\left(\frac{T_{k-1}\, u(t)}{v_k(t)}\right)\,,\ k = 1, \dots, n\,,\ \ t \in I,$$

where $v_k \equiv 1$ on I. That is, $T_k\, u(t) = u^{(k)}(t)$, $t \in I$. In particular:

$$T_n[0]\, u(t) = v_1(t)\, \dots\, v_n(t)\, T_n\, u(t)\,,\ t \in I\,,$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{aligned} T_{\sigma_1}\, u(a) = u^{(\sigma_1)}(a) = 0\,, \quad \dots\,, \quad T_{\sigma_k}\, u(a) &= u^{(\sigma_k)}(a) \quad = 0\,, \\ T_{\varepsilon_1}\, u(b) = u^{(\varepsilon_1)}(b) = 0\,, \quad \dots\,, \quad T_{\varepsilon_{n-k}}\, u(b) &= u^{(\varepsilon_{n-k})}(b) = 0\,, \end{aligned}$$

or, which is the same, $T_n[0]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Lemma 2.7. *[12, Lemma 3.8] Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies the property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Then $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfy the property (N_a) if, and only if, $M = 0$ is not an eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Let us denote by G_M the Green's matrix related to the equivalent n-dimensional first order system associated with (1.1)–(1.3), which is given by the expression

$$G_M(t,s) = \begin{pmatrix} g_1(t,s) & g_2(t,s) & \cdots & g_{n-1}(t,s) & g_M(t,s) \\ \\ \frac{\partial}{\partial t}\, g_1(t,s) & \frac{\partial}{\partial t}\, g_2(t,s) & \cdots & \frac{\partial}{\partial t}\, g_{n-1}(t,s) & \frac{\partial}{\partial t}\, g_M(t,s) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_1(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_2(t,s) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_{n-1}(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_M(t,s) \end{pmatrix}, \tag{2.6}$$

where g_M is the scalar Green's function related to operator $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ (see [7, Section 1.4] for details). Moreover, by the axiomatic definition of G_M, we have that for all $s \in (a,b)$, the following equality holds

$$\frac{\partial}{\partial t}\, G_M(t,s) = A\, G_M(t,s), \quad \text{for all } t \in I \backslash \{s\}\,, \tag{2.7}$$

where $A = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline -M & 0 \end{array}\right)$ denotes the matrix that defines the n-dimensional system equivalent to problem (1.1)–(1.3), being I_{n-1} the $(n-1)$ dimensional identity matrix. In addition, G_M satisfies that

$$B\, G_M(a,s) + C\, G_M(b,s) = 0, \;\; \text{for all } \; s \in (a,b), \tag{2.8}$$

with $B,\; C \in \mathcal{M}_{n\times n}$, defined by $b_{j,1+\sigma_j} = 1$ for $j = 1,\dots,k$ and $c_{j+k,1+\varepsilon_j} = 1$ for $j = 1,\dots,n-k$; otherwise, $b_{ij} = 0$ and $c_{ij} = 0$.

In particular, it is easy to see, using the expressions (2.6) and (2.7) (see [7, Section 1.4] for details), that $g_M \in C^{n-2}(I \times I)$. Moreover, it is a C^n function on the triangles $a \le s < t \le b$ and $a \le t < s \le b$, it satisfies, as a function of t for any $s \in (a,b)$ fixed, the two-point boundary value conditions (1.2)-(1.3), and it solves equation (1.1) for all $t \in I \setminus \{s\}$.

Moreover, for any $t \in (a,b)$ it satisfies that

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\, g_M(t^+,t) = \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_M(t^-,t) + 1. \tag{2.9}$$

Furthermore, studying the matrix G_M, it is proved in [10, page 13] that, for this particular case, $g_{n-j}(t,s)$ may be expressed as a function of $g_M(t,s)$ as follows:

$$g_{n-j}(t,s) = (-1)^j\, \frac{\partial^j}{\partial s^j}\, g_M(t,s), \quad j = 1,\dots,n-1. \tag{2.10}$$

On the other hand, the adjoint operator of $T_n[M]$ follows the expression

$$T_n^*[M]v(t) \equiv (-1)^n\, v^{(n)}(t) + M\, v(t), \tag{2.11}$$

and it is defined on the domain

$$D(T_n^*[M]) = \left\{ v \in C^n(I) \;\; | \sum_{i=0}^{n-1} (-1)^{n-1-i}\, v^{(n-1-i)}(b)\, u^{(i)}(b) = \sum_{i=0}^{n-1} (-1)^{n-1-i}\, v^{(n-1-i)}(a)\, u^{(i)}(a) \right\},$$

for all $u \in D(T_n[M])$.

Let us denote by $g_M^*(t,s)$ the Green's function related to operator $T_n^*[M]$. The following equality is satisfied (see [7, section 1.4])

$$g_M^*(t,s) = g_M(s,t), \quad (t,s) \in I \times I. \tag{2.12}$$

Thus, if we define the following operator

$$\widehat{T}_n[(-1)^n M] := (-1)^n T_n^*[M]\,, \tag{2.13}$$

we have, according to the above equality, that

$$\widehat{g}_{(-1)^n M}(t,s) = (-1)^n g_M^*(t,s) = (-1)^n g_M(s,t), \tag{2.14}$$

where $\widehat{g}_{(-1)^n M}(t,s)$ is the scalar Green's function related to operator $\widehat{T}_n[(-1)^n M]$ in $D\left(T_n^*[M]\right)$.

As a particular case of the results obtained in [12, section 4], we can deduce that

$$D(T_n^*[M]) \equiv X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}} = X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}},$$

being $\{\delta_1,\dots,\delta_k\}, \{\tau_1,\dots,\tau_{n-k}\} \subset \{0,\dots,n-1\}$, such that $\delta_i < \delta_{i+1}$ and $\tau_j < \tau_{j+1}$, for $i = 1,\dots,k-1$ and $j = 1,\dots,n-k-1$, satisfying:

$$\begin{aligned}
&\{\sigma_1,\dots,\sigma_k, n-1-\tau_1,\dots, n-1-\tau_{n-k}\} = \{0,\dots,n-1\},\\
&\{\varepsilon_1,\dots,\varepsilon_{n-k}, n-1-\delta_1,\dots, n-1-\delta_k\} = \{0,\dots,n-1\}.
\end{aligned}$$

***Notation* 2.8.** Let us define η, γ in the following way:

$$\eta = \min\{i \in \{0,\dots,n-1\} : \;\; i \notin \{\tau_1,\dots,\tau_{n-k}\}\}\,, \tag{2.15}$$

$$\gamma = \min\{i \in \{0,\dots,n-1\} : \;\; i \notin \{\delta_1,\dots,\delta_k\}\}\,. \tag{2.16}$$

Analogously, let us denote by $\widehat{G}(t,s)$ the Green's matrix associated with the equivalent n-dimensional problem of $\widehat{T}_n[(-1)^n M]\, v(t) = 0$, $v \in X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}}$, which is given by

$$\widehat{G}(t,s) = \begin{pmatrix}
\widehat{g}_1(t,s) & \cdots & \widehat{g}_{n-1}(t,s) & \widehat{g}_{(-1)^n M}(t,s) \\
\frac{\partial}{\partial t}\,\widehat{g}_1(t,s) & \cdots & \frac{\partial}{\partial t}\,\widehat{g}_{n-1}(t,s) & \frac{\partial}{\partial t}\,\widehat{g}_{(-1)^n M}(t,s) \\
\vdots & \cdots & \vdots & \vdots \\
\frac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_1(t,s) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{n-1}(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{(-1)^n M}(t,s)
\end{pmatrix}, \tag{2.17}$$

where $\widehat{g}_{(-1)^n M}(t,s)$ is the scalar Green's function related to operator $\widehat{T}_n[(-1)^n M]$ in $X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}}$. Similarly to (2.10), we can deduce that

$$\widehat{g}_{n-j}(t,s) = (-1)^j \frac{\partial^j}{\partial s^j}\,\widehat{g}_{(-1)^n M}(t,s)\,. \tag{2.18}$$

Moreover, by definition, $\widehat{G}(t,s)$ satisfies the equality

$$\widehat{B}\,\widehat{G}(a,s) + \widehat{C}\,\widehat{G}(b,s) = 0, \;\; \text{for all} \;\; s \in (a,b), \tag{2.19}$$

where $\widehat{B}$, $\widehat{C} \in \mathcal{M}_{n\times n}$, defined as $(\widehat{B})_{i,\,\tau_i+1} = 1$ for $i = 1,\dots,n-k$ and $(\widehat{C})_{j+n-k,\,1+\delta_j} = 1$ for $j = 1,\dots,k$; otherwise, $(\widehat{B})_{i\,j} = 0$ and $(\widehat{C})_{i\,j} = 0$.

***Remark* 2.9.** From the previous definitions, as it is pointed out in [12, Remarks 4.1 and 4.2] we have that $\alpha = n-1-\tau_{n-k}$, $\beta = n-1-\delta_k$, $\eta = n-1-\sigma_k$ and $\gamma = n-1-\varepsilon_{n-k}$.

Example 2.10. Let us consider operator $T_4[M]$ coupled with the boundary conditions

$$u(0) = u'(0) = u''(0) = u''(1) = 0.$$

In this case, $\{\sigma_1,\sigma_2,\sigma_3\} = \{0,1,2\}, \{\varepsilon_1\} = \{2\}, \{\tau_1\} = \{0\}$ and $\{\delta_1,\delta_2,\delta_3\} = \{0,2,3\}$. Thus, we deduce that $X^{*\{2\}}_{\{0,1,2\}} = X^{\{0,2,3\}}_{\{0\}}$. Moreover, $\alpha = 3\,, \beta = 0$, $\eta = 1$ and $\gamma = 1$.

Definition 2.11. Operator $T_n[M]$ is said to be inverse positive (negative) on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if every $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ such that $T_n[M]\,u \geq 0$ on I, satisfies that $u \geq 0$ ($u \leq 0$) on I.

The above definition is equivalent to the constant sign of Green's function g_M, as it is shown in the following result whose proof is analogous to the one made in [12, Theorem 2.9].

Theorem 2.12. *Operator $T_n[M]$ is inverse positive (negative) in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if, and only if, the Green's function related to problem* (1.1)–(1.3) *is non-negative (non-positive) on its square of definition.*

Definition 2.13. Operator $T_n[M]$ is said to be strongly inverse positive (negative) in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if every $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ such that $T_n[M]\,u \gneqq 0$ on I, satisfies that $u > 0$ (< 0) on (a,b) and, moreover, $u^{(\alpha)}(a) > 0$ (< 0) and $(-1)^\beta u^{(\beta)}(b) > 0$ (< 0), where α and β are defined in (2.1) and (2.2), respectively.

Analogously to Theorem 2.12, we have the following characterization:

Theorem 2.14. *Operator $T_n[M]$ is strongly inverse positive (negative) in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if, and only if, the Green's function related to problem* (1.1)–(1.3) *satisfies the following properties:*

- $g_M(t,s) > 0\,(< 0)$ *a.e. on* (a,b).
- $\frac{\partial^\alpha}{\partial t^\alpha}\, g_M(t,s) > 0\,(< 0)$ *for a.e.* $s \in (a,b)$.
- $(-1)^\beta \frac{\partial^\beta}{\partial t^\beta}\, g_M(t,s) > 0\,(< 0)$ *for a.e.* $s \in (a,b)$.

Let $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ be such that $T_n[M]\,u(t) = 0$, $t \in I$. Then, differentiating equation (1.1) with respect to t, as a direct consequence of equations (2.6) and (2.7) we deduce that, for any fixed $s \in (a,b)$, the q-th derivative $v_s^q[M](t) := \dfrac{\partial^q}{\partial t^q}\, g_M(t,s)$, with $1 \leq q \leq n-1$, satisfies the equation $T_n[M]\, v_s^q[M](t) = 0$, for all $t \in I \setminus \{s\}$, together with the new boundary conditions

$$(v_s^q)^{(\mu_1^q)}[M](a) = \quad \cdots \quad = (v_s^q)^{(\mu_k^q)}[M](a) \quad = 0\,, \tag{2.20}$$

$$(v_s^q)^{(\rho_1^q)}[M](b) = \quad \cdots \quad = (v_s^q)^{(\rho_{n-k}^q)}[M](b) = 0\,. \tag{2.21}$$

Here, the sets of indices $\{\mu_1^q, \ldots, \mu_k^q\},\ \{\rho_1^q, \ldots, \rho_{n-k}^q\} \subset \{0, \ldots, n-1\}$, satisfy that

$$0 \leq \mu_1^q < \mu_2^q < \cdots < \mu_k^q \leq n-1\,, \quad 0 \leq \rho_1^q < \rho_2^q < \cdots < \rho_{n-k}^q \leq n-1\,,$$

and are defined in the following way:

- If $A_q = \{i \in \{1, \ldots, k\} /\ \ \sigma_i \geq q\} \neq \varnothing$, then

  $$\mu_1^q = \sigma_j - q,\ \mu_2^q = \sigma_{j+1} - q,\ \ldots, \mu_{k-(j-1)}^q = \sigma_k - q,\ \mu_{k-(j-2)}^q = \sigma_1 - q + n, \ldots,\ \mu_k^q = \sigma_{j-1} - q + n\,,$$

  with

  $$j = \min A_q \geq 1. \tag{2.22}$$

- If $A_q = \varnothing$, then

  $$\mu_i^q = \sigma_i - q + n, \ \ i = 1, \ldots, k.$$

- If $B_q = \{i \in \{1, \ldots, n-k\} /\ \ \varepsilon_i \geq q\} \neq \varnothing$, then

  $$\rho_1^q = \varepsilon_r - q,\ \rho_2^q = \varepsilon_{r+1} - q,\ \ldots, \rho_{n-k-(r-1)}^q = \varepsilon_{n-k} - q,\ \rho_{n-k-(r-2)}^q = \varepsilon_1 - q + n, \ldots,$$
  $$\rho_{n-k}^q = \varepsilon_{r-1} - q + n,$$

  with

  $$r = \min B_q \geq 1. \tag{2.23}$$

- If $B_q = \varnothing$, then

  $$\rho_i^q = \varepsilon_i - q + n, \ \ i = 1, \ldots, n-k.$$

***Remark* 2.15.** The above process of calculating the boundary conditions of $v_s^q[M]$ is valid for both $M \neq 0$ and $M = 0$.

Taking into account previous computations, we may define ϕ as the function which maps the space $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$ into the space of the boundary conditions that satisfies the first derivative, that is, $X_{\{\mu_1^1,\ldots,\mu_k^1\}}^{\{\rho_1^1,\ldots,\rho_{n-k}^1\}}$. This way, we can extend this definition to the q-th derivative as

$$\phi^q = \phi \circ \overset{q)}{\cdots} \circ \phi : X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}} \longrightarrow X_{\{\mu_1^q,\ldots,\mu_k^q\}}^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}.$$

In particular, we observe that ϕ^n is the identity function.

Example 2.16. Consider the operator $T_5[M]$ defined in $X_{\{0,2,4\}}^{\{0,2\}}$. Applying the above definition of the spaces of the derivatives we obtain the following sequence of spaces:

$$X_{\{0,2,4\}}^{\{0,2\}} \xrightarrow{\ \phi\ } X_{\{1,3,4\}}^{\{1,4\}} \xrightarrow{\ \phi\ } X_{\{0,2,3\}}^{\{0,3\}} \xrightarrow{\ \phi\ } X_{\{1,2,4\}}^{\{2,4\}} \xrightarrow{\ \phi\ } X_{\{0,1,3\}}^{\{1,3\}} \xrightarrow{\ \phi\ } X_{\{0,2,4\}}^{\{0,2\}}.$$

Analogously to the Notation 2.3, let us define α^q, β^q as follows:

$$\alpha^q = \min\left\{i \in \{0,\dots,n-1\} : \ i \notin \{\mu_1^q,\dots,\mu_k^q\}\right\}, \tag{2.24}$$
$$\beta^q = \min\left\{i \in \{0,\dots,n-1\} : \ i \notin \{\rho_1^q,\dots,\rho_{n-k}^q\}\right\}. \tag{2.25}$$

***Remark* 2.17.** It is important to point out that for any $q \geq 1$, the function $v_s^q[M]$ is not the Green's function related to operator $T_n[M]$ in $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$. This is due to the fact that $v_s^q[M] \in C^{n-2-q}(I \times I)$.

***Remark* 2.18.** The operator $T_n[0]$ satisfies the property (T_d) in $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$. Indeed, from Remark 2.6, we know that operator $T_n[0]\,u(t) = u^{(n)}(t)$ satisfies the following decomposition:

$$T_0\,u(t) = u(t)\,, \quad T_k\,u(t) = \frac{d}{dt}\left(\frac{T_{k-1}\,u(t)}{v_k(t)}\right)\,,\ k = 1,\dots,n\,,$$

where $v_k \equiv 1$ on I and

$$T_n[0]\,u(t) = v_1(t)\,\dots\,v_n(t)\,T_n\,u(t)\,,\ t \in I\,,$$

that is, $T_k\,u(t) = u^{(k)}(t)$.

Moreover, this decomposition satisfies, for every $u \in X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$:

$$\begin{aligned} T_{\mu_1^q}\,u(a) &= u^{(\mu_1^q)}(a) = 0\,, \quad \dots, \quad T_{\mu_k^q}\,u(a) = u^{(\mu_k^q)}(a) = 0\,,\\ T_{\rho_1^q}\,u(b) &= u^{(\rho_1^q)}(b) = 0\,, \quad \dots, \quad T_{\rho_{n-k}^q}\,u(b) = u^{(\rho_{n-k}^q)}(b) = 0\,. \end{aligned}$$

Next we present a result concerning the spectrum of the spaces $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$.

Lemma 2.19. *$\bar{\lambda} \neq 0$ is an eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, $\bar{\lambda}$ is an eigenvalue of $T_n[0]$ in $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$ for all $q \in \{1,\dots,n-1\}$.*

Proof. Suppose that $\bar{\lambda} \neq 0$ is an eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. So, there is $u \not\equiv 0$ in I such that $u^{(n)}(t) + \bar{\lambda}\,u(t) = 0$, $t \in I$ and $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Since $u \not\equiv 0$ in I we have that $u^{(n)} \not\equiv 0$ in I. Therefore, $u^{(j)} \not\equiv 0$ in I for all $j \in \{1,\dots,n-1\}$.

Taking into account that $T_n[0]\,u^{(j)}(t) = u^{(j+n)}(t) = -\bar{\lambda}\,u^{(j)}(t)$ for all $t \in I$ and $j \in \mathbb{N}$, we have that $u^{(j)}$ is an eigenvector related to the eigenvalue $\bar{\lambda}$ such that $u^{(j)} \in X_{\{\mu_1^j,\dots,\mu_k^j\}}^{\{\rho_1^j,\dots,\rho_{n-k}^j\}}$ for all $j \in \{1,\dots,n-1\}$.

For $j = n$, we have that $\mu_i^n = \sigma_i$ for all $i \in \{1,\dots,k\}$ and $\rho_l^n = \varepsilon_l$ for all $l \in \{1,\dots,n-k\}$. Therefore, the result holds. □

Remark **2.20.** As a direct consequence of Remark 2.6, we have, from Lemma 2.7, that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a) if and only if $M=0$ is not an eigenvalue of $u^{(n)}$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Remark **2.21.** Note that if $\lambda=0$ is not an eigenvalue of $T_n[0]$ in $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$, then the spaces $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ have the same spectrum. However, it is very important to point out that $\lambda=0$ can appear as an eigenvalue in some space $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$, as it is shown in the following example.

Example 2.22. Let us consider again the Example 2.16. As we know, the following sequence of spaces is fulfilled:

$$X^{\{0,2\}}_{\{0,2,4\}} \xrightarrow{\phi} X^{\{1,4\}}_{\{1,3,4\}} \xrightarrow{\phi} X^{\{0,3\}}_{\{0,2,3\}} \xrightarrow{\phi} X^{\{2,4\}}_{\{1,2,4\}} \xrightarrow{\phi} X^{\{1,3\}}_{\{0,1,3\}} \xrightarrow{\phi} X^{\{0,2\}}_{\{0,2,4\}}.$$

In this case, we have that $\lambda=0$ is not an eigenvalue of $T_5[0]$ in $X^{\{0,2\}}_{\{0,2,4\}}$, $X^{\{0,3\}}_{\{0,2,3\}}$ and $X^{\{1,3\}}_{\{0,1,3\}}$, while $\lambda=0$ is an eigenvalue of $T_5[0]$ in $X^{\{1,4\}}_{\{1,3,4\}}$ and $X^{\{2,4\}}_{\{1,2,4\}}$.

Now, we will show how to determine the adjoint space of $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ from the adjoint space of $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. We arrive to the next result.

Theorem 2.23. *The adjoint space of* $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ *is given by* $X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$, *where the indices* $\{\tau_1^q,\dots,\tau_{n-k}^q\},\{\delta_1^q,\dots,\delta_k^q\}\subset\{0,\dots,n-1\}$ *satisfy*

$$0\le\tau_1^q<\tau_2^q<\cdots<\tau_{n-k}^q\le n-1\,,\quad 0\le\delta_1^q<\delta_2^q<\cdots<\delta_k^q\le n-1\,,$$

and are defined as follows:

- *If* $C_q=\{i\in\{1,\dots,n-k\}\,/\;\;\tau_i+q\le n-1\}\neq\varnothing$, *then*

  $\tau_1^q=\tau_{l+1}+q-n$, $\tau_2^q=\tau_{l+2}+q-n$, $\dots$, $\tau_{n-k-l}^q=\tau_{n-k}+q-n$, $\tau_{n-k-l+1}^q=\tau_1+q,\dots,$ $\tau_{n-k}^q=\tau_l+q$,

  with

  $$l=\max C_q. \tag{2.26}$$

- *If* $C_q=\varnothing$, *then*

  $$\tau_i^q=\tau_i+q-n,\;\; i=1,\dots,n-k.$$

- *If* $D_q=\{i\in\{1,\dots,k\}\,/\;\;\delta_i+q\le n-1\}\neq\varnothing$, *then*

  $\delta_1^q=\delta_{p+1}+q-n$, $\delta_2^q=\delta_{p+2}+q-n$, $\dots$, $\delta_{k-p}^q=\delta_k+q-n$, $\delta_{k-p+1}^q=\delta_1+q,\dots,\delta_k^q=\delta_p+q$,

  with

  $$p=\max D_q. \tag{2.27}$$

- *If $D_q = \varnothing$, then*

$$\delta_i^q = \delta_i + q - n, \ \ i = 1, \ldots, k.$$

As a consequence, the following diagram

$$\begin{array}{ccc} X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}} & \xrightarrow{\phi^q} & X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}} \\ \downarrow * & & \downarrow * \\ X^{\{\delta_1,\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}\}} & \xleftarrow[\phi^q]{} & X^{\{\delta_1^q,\ldots,\delta_k^q\}}_{\{\tau_1^q,\ldots,\tau_{n-k}^q\}} \end{array} \tag{2.28}$$

is commutative.

Proof. We will do the proof only for τ_i^q. For δ_i^q, the proof would be analogous.

Now, from the definition of τ_i, we have that $\tau_i < \tau_{i+1}$ for all $i \in \{1, \ldots, n-k-1\}$ and

$$\{\sigma_1, \ldots, \sigma_k, n-1-\tau_{n-k}, \ldots, \ n-1-\tau_1\} \equiv \{0, \ldots, n-1\}.$$

Taking into account the definition of $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$ we have that

$$\{\sigma_1, \ldots, \sigma_k, n-1-\tau_{n-k}, \ldots, \ n-1-\tau_1\} \xrightarrow{\phi^q} \{\mu_1^q, \ldots, \mu_k^q, n-1-\tau_{n-k}^q, \ldots, \ n-1-\tau_1^q\},$$

with $\{\mu_1^q, \ldots, \mu_k^q, n-1-\tau_{n-k}^q, \ldots, \ n-1-\tau_1^q\} \equiv \{0, \ldots, n-1\}$ and τ_i^q fulfilling that

$$0 \leq \tau_1^q < \tau_2^q < \cdots < \tau_{n-k}^q \leq n-1.$$

To calculate them, we distinguish two cases:

1) If $C_q \neq \varnothing$, then:

   - If $n-1-\tau_i \geq q$ (i.e, $i \leq l$), $n-1-\tau_i$ becomes $n-1-\tau_i-q$ by applying ϕ^q.
   - If $n-1-\tau_i < q$ (i.e, $i > l$), $n-1-\tau_i$ becomes $n-1-\tau_i-q+n$ by applying ϕ^q.

   Reordering the elements, we obtain that:

$$\begin{aligned} 0 \leq n-1-\tau_l-q < n-1-\tau_{l-1}-q < \cdots < n-1-\tau_1-q \\ < n-1-\tau_{n-k}-q+n < \cdots < n-1-\tau_{l+1}-q+n \leq n-1, \end{aligned}$$

   or, equivalently,

$$0 \leq \tau_{l+1}+q-n < \tau_{l+2}+q-n < \cdots < \tau_{n-k}+q-n < \tau_1+q < \cdots < \tau_l+q \leq n-1\,.$$

   Taking

   $\tau_1^q = \tau_{l+1}+q-n$, $\tau_2^q = \tau_{l+2}+q-n$, $\ldots, \tau_{n-k-l}^q = \tau_{n-k}+q-n$, $\tau_{n-k-l+1}^q = \tau_1+q, \ldots,$ $\tau_{n-k}^q = \tau_l + q\,,$

   the first statement yields.

2) If $C_q = \varnothing$, then $n-1-\tau_i$ becomes $n-1-\tau_i-q+n$ by applying ϕ^q and we obtain that

$$0 \le n-1-\tau_{n-k}-q+n < \cdots < n-1-\tau_2-q+n < n-1-\tau_1-q+n \le n-1,$$

and, reordering, we infer that

$$0 \le \tau_1+q-n < \tau_2+q-n < \cdots < \tau_{n-k}+q-n \le n-1.$$

Taking $\tau_i^q = \tau_i+q-n, \;\; i=1,\dots,n-k$, the second statement yields.

Therefore, $X^{*\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}} = X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$. □

***Remark* 2.24.** Note that, unlike the direct arrows between the leading spaces, the arrows between the adjoint spaces are in the inverse direction.

Example 2.25. Consider the operator $T_4[M]$ coupled with the boundary conditions

$$u(a) = u''(a) = u'(b) = u''(b).$$

In this case, $u \in X^{\{1,2\}}_{\{0,2\}}$ and $X^{*\{1,2\}}_{\{0,2\}} = X^{\{0,3\}}_{\{0,2\}}$. Using the diagram (2.28), we get that

$$\begin{array}{ccccccc}
X^{\{1,2\}}_{\{0,2\}} & \xrightarrow{\phi} & X^{\{0,1\}}_{\{1,3\}} & \xrightarrow{\phi} & X^{\{0,3\}}_{\{0,2\}} & \xrightarrow{\phi} & X^{\{2,3\}}_{\{1,3\}} \\
{\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} \\
X^{\{0,3\}}_{\{0,2\}} & \xleftarrow[\phi]{} & X^{\{0,1\}}_{\{1,3\}} & \xleftarrow[\phi]{} & X^{\{1,2\}}_{\{0,2\}} & \xleftarrow[\phi]{} & X^{\{2,3\}}_{\{1,3\}}
\end{array}$$

Using the fact that the adjoint operators have the same spectrum as the original one, from Lemma 2.19, we arrive at the following result.

Lemma 2.26. *$\lambda^* \neq 0$ is an eigenvalue of $T_n[0]$ in $X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}}$ if and only if λ^* is an eigenvalue of $T_n[0]$ in $X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$ for all $q \in \{0,\dots,n-1\}$. Moreover, for each fixed $q \in \{0,\dots,n-1\}$, $\bar\lambda \in \mathbb{R}$ is an eigenvalue of $T_n[0]$ in $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ if and only if $\bar\lambda$ is an eigenvalue of $T_n[0]$ in $X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$.*

Following the line of Definition 2.13 and the characterization of Theorem 2.14, we introduce the following concept for $v_s^q[M]$.

Definition 2.27. Let $q \in \{0,\dots,n-1\}$ be fixed. We say that the function $v_s^q[M]$ is strongly positive (strongly negative) on $I \times I$ if it satisfies the following properties:

- $v_s^q[M] > 0\,(<0)$ a.e. on (a,b).
- $(v_s^q)^{(\alpha^q)}[M](a) > 0\,(<0)$ for a.e. $s \in (a,b)$, with α^q defined in (2.24).

- $(-1)^{\beta^q}(v_s^q)^{(\beta^q)}[M](b) > 0\,(< 0)$ for a.e. $s \in (a,b)$, with β^q defined in (2.25).

Let us consider the following two conditions on g_M introduced in [7, pages 78 and 86] as follows:

(P_g) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (a,b)$ and $k_1,\ k_2 \in L^1(I)$, such that $0 < k_1(s) < k_2(s)$ for a.e. $s \in I$, satisfying

$$\phi(t)\, k_1(s) \le g_M(t,s) \le \phi(t)\, k_2(s)\,, \quad \text{for a.e. } (t,s) \in I \times I\,.$$

(N_g) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (a,b)$ and $k_1,\ k_2 \in L^1(I)$, such that $k_1(s) < k_2(s) < 0$ for a.e. $s \in I$, satisfying

$$\phi(t)\, k_1(s) \le g_M(t,s) \le \phi(t)\, k_2(s)\,, \quad \text{for a.e. } (t,s) \in I \times I\,.$$

As a particular case of [12, Theorem 5.1], the following result is attained.

Theorem 2.28. *Suppose that $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a). Then the following properties are fulfilled:*

- *If $n-k$ is even, then $T_n[0]$ is strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, the related Green's function, g_0, satisfies (P_g).*
- *If $n-k$ is odd, then $T_n[0]$ is strongly inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, the related Green's function, g_0, satisfies (N_g).*

Let us define $\{\sigma_1^q,\dots,\sigma_{c_q}^q\}, \{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\} \subset \{0,\dots,n-1\}$, in the following way:

$$\{\sigma_1^q,\dots,\sigma_{c_q}^q\} \;=\; \{\sigma_1 - q,\dots,\sigma_k - q\} \cap \{0,\dots,n-q-1\}, \tag{2.29}$$
$$\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\} \;=\; \{\varepsilon_1 - q,\dots,\varepsilon_{n-k} - q\} \cap \{0,\dots,n-q-1\}. \tag{2.30}$$

***Remark* 2.29.** Taking into account that function $v_s^q(t) = \frac{\partial^q}{\partial t^q}\, g_0(t,s)$ is the Green's function related to operator $T_{n-q}[0]$ in the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$, then operator $T_{n-q}[0]$ is strongly inverse positive (negative) in the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$ if and only if it satisfies the Definition 2.13 with the same α^q and β^q since

$$\mu_1^q = \sigma_1^q,\, \mu_2^q = \sigma_2^q,\, \dots, \mu_{k-(j-1)}^q = \sigma_{c_q}^q,$$

and

$$\rho_1^q = \varepsilon_1^q,\, \rho_2^q = \varepsilon_2^q,\, \dots, \rho_{n-k-(r-1)}^q = \varepsilon_{d_q}^q.$$

If $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies the condition (N_a) then $c_q + d_q \le n-q$. Moreover, if $c_q + d_q < n-q$ then $\{\sigma_1^q,\dots,\sigma_{c_q}^q\} - \{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}$ does not satisfy (N_a), and in this case $\lambda = 0$ is an eigenvalue of $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$. We will show now that $c_q + d_q = n-q$ is a necessary condition for the q-th partial derivative of the Green's function to have constant sign.

Lemma 2.30. *Let $q \in \{1,\dots,n-1\}$. Suppose that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a). If there exists $M \in \mathbb{R}$ such that $\frac{\partial^q}{\partial t^q} g_M$ has constant sign on $I \times I$, then $c_q + d_q = n-q$ and g_M has constant sign on $I \times I$.*

Proof. If $v_s^q[M](t) = \frac{\partial^q}{\partial t^q} g_M(t,s)$ has constant sign, then $v_s^{q-1}[M](t) = \frac{\partial^{q-1}}{\partial t^{q-1}} g_M(t,s)$ is a monotone and continuous function. Thus, $v_s^{q-1}[M]$ has at most one zero on I.

By recurrence, we have that $v_s^{q-l}[M](t) = \frac{\partial^{q-l}}{\partial t^{q-l}} g_M(t,s)$ has at most l zeros on I. In particular, $g_M(\cdot,s)$ has at most q zeros on I.

Now, since condition (N_a) implies that $g_M(\cdot,s)$ has at least q zeros of order smaller or equal to q on the boundary, we have, arguing as in the proof of [12, Theorem 8.1, Step 5], that the existence of any of such zeros implies that function $g_M(\cdot,s)$ lost one zero on the interior of I. As a consequence, we deduce that:

- $g_M(\cdot,s)$ has exactly q zeros of order smaller or equal to q on the boundary, that is, $c_q + d_q = n-q$.
- $g_M(\cdot,s)$ cannot have any zero on (a,b), that is, g_M has constant sign.

□

Next, for any $q \in \{1,\dots,n-1\}$ we determine the positive (negative) sign of $v_s^q(t) := v_s^q[0](t)$ on the interval I.

Theorem 2.31. *Let $q \in \{1,\dots,n-1\}$. Suppose that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a) and $c_q + d_q = n-q$. The following properties are fulfilled:*

(a) If $c_q \geq 1$ and $d_q \geq 1$, then v_s^q is strongly positive on $I \times I$ if $n-q-c_q$ is even and v_s^q is strongly negative on $I \times I$ if $n-q-c_q$ is odd.

(b) If $c_q = n-q$ and $d_q = 0$, then $v_s^q(t) > 0, \dots, v_s^{n-1}(t) > 0$ in $(s,b]$, $v_s^q(t) = \cdots = v_s^{n-2}(t) = 0$ in $[a,s]$ and $v_s^{n-1}(t) = 0$ in $[a,s)$.

(c) If $c_q = 0$ and $d_q = n-q$, then $v_s^q(t) > 0$ in $[a,s)$ if $q = n-l$ with l even and $v_s^q(t) < 0$ in $[a,s)$ if $q = n-l$ with l odd. Moreover, $v_s^q(t) = \cdots = v_s^{n-2}(t) = 0$ in $[s,b]$ and $v_s^{n-1}(t) = 0$ in $(s,b]$.

Proof. (a) In this case, it easy to verify that the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$ satisfies the condition (N_a) since

$$\sum_{\sigma_j - q < h} 1 + \sum_{\varepsilon_j - q < h} 1 \geq h\,, \quad \forall h \in \{1,\dots,n-q-1\}\,.$$

So, taking into account the fact that v_s^q is the Green's function related to operator $T_{n-q}[0]$ in the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$, we apply Theorem 2.28, and we have proved the first statement.

(b) Since $v_s^{(n)}(t) = 0$ for all $t \neq 0$, we have, by (2.9) and $v_s^{(n-1)}(a) = 0$, that $\frac{\partial^{n-1}}{\partial t^{n-1}}\, g_0(t,s) = 0$ if $t < s$ and $\frac{\partial^{n-1}}{\partial t^{n-1}}\, g_0(t,s) = 1$ if $t > s$.

Since $v_s^h(a) = 0$ for all $q \leq h \leq n-2$, and $g_0 \in C^{n-2}(I \times I)$, we deduce that $\frac{\partial^h}{\partial t^h}\, g_0(t,s) = 0$ if $t < s$ and $\frac{\partial^h}{\partial t^h}\, g_0(t,s) > 0$ if $t > s$. So, we conclude the proof of second claim.

(c) In this case arguing as in the previous case, since $v_s^{(n)}(b) = 0$, we have that $\frac{\partial^{n-1}}{\partial t^{n-1}}\, g_0(t,s) = 0$ if $t > s$ and $\frac{\partial^{n-1}}{\partial t^{n-1}}\, g_0(t,s) = -1$ if $t < s$.

As consequence, since $v_s^h(b) = 0$ for all $q \leq h \leq n-2$, and $g_0 \in C^{n-2}(I \times I)$, we deduce, by recurrence, that $\frac{\partial^h}{\partial t^h} g_0(t,s) = 0$ for all $t > s$ and $(-1)^{n-h}\frac{\partial^h}{\partial t^h} g_0(t,s) > 0$ for all $t < s$. Thus, the third assertion holds.

□

As for the set of parameters of M in which the function $v_s^q[M]$, $q \in \{1, \dots, n-1\}$ maintains constant sign and monotony with respect to M, we have the following results:

Lemma 2.32. *Let q_1, $q_2 \in \{1, \dots, n-1\}$, $q_1 < q_2$, be such that $c_{q_1} + d_{q_1} = n - q_1$. Suppose that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfies condition (N_a). If there exists $M \in \mathbb{R}$ such that $\frac{\partial^{q_2}}{\partial t^{q_2}} g_M$ has constant sign on $I \times I$, then $\frac{\partial^{q_1}}{\partial t^{q_1}} g_M$ has constant sign on $I \times I$.*

Proof. Reasoning as in Lemma 2.30, we deduce that $v_s^{q_1}[M](t) = \frac{\partial^{q_1}}{\partial t^{q_1}} g_M(t,s)$ has at most $q_2 - q_1$ zeros on I.

Now, condition $c_{q_1} + d_{q_1} = n - q_1$ implies that $v_s^{q_1}[M]$ has at least $q_2 - q_1$ zeros of order belonging to the set $\{q_1, \dots, q_2 - 1\}$ on the boundary. Thus, arguing as in the proof of Lemma 2.30 again, we deduce that $v_s^{q_1}[M]$ cannot have any zero on (a,b). Thus, $v_s^{q_1}[M]$ has constant sign. □

Moreover, we recall the following facts:

- If $n-k$ is even, then from Theorem 2.28 and [7, Lemma 1.8.33] we have that $g_M(t,s) > 0$ for all $(t,s) \in (a,b) \times (a,b)$ if and only if $M \in (\lambda_1, +\infty)$ or $M \in (\lambda_1, \lambda_2]$, and it is monotone decreasing with respect to M on such interval, where $\lambda_1 < 0$ is the first eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\lambda_2 > 0$ (if the interval is bounded) is not an eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. The value of λ_2 is characterized in [12, Sect.7 and Sect.8].
- If $n-k$ is odd, then from Theorem 2.28 and [7, Lemma 1.8.25] we have that $g_M(t,s) < 0$ for all $(t,s) \in (a,b) \times (a,b)$ if and only if $M \in (-\infty, \lambda_1)$ or $M \in [\overline{\lambda_2}, \lambda_1)$ and it is monotone decreasing with respect to M on such interval, where $\lambda_1 > 0$ is the first eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\overline{\lambda_2} < 0$ (if the interval is bounded) is not an eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. The value of $\overline{\lambda_2}$ is characterized in [12, Sect. 7 and Sect. 8].

Therefore, as a direct consequence of Theorem 2.28, Lemma 2.30 and [8, Theorem 3.6], the following result is deduced.

Lemma 2.33. *The following properties hold:*

- *If* $n-k$ *is even and* $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ *on* $(a,b)\times(a,b)$ *for some* $M \in \mathbb{R}$, *then* $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ *if and only if* $M \in (\lambda_1, M_q]$, *with* $M_q \leq \lambda_2$. *Moreover,* $\frac{\partial^q}{\partial t^q} g_M$ *is monotone decreasing with respect to* $M \in (\lambda_1, M_q]$.

- *If* $n-k$ *is even and* $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ *on* $(a,b)\times(a,b)$ *for some* $M \in \mathbb{R}$, *then* $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ *if and only if* $M \in (\lambda_1, M_q]$, *with* $M_q \leq \lambda_2$. *Moreover,* $\frac{\partial^q}{\partial t^q} g_M$ *is monotone increasing with respect to* $M \in (\lambda_1, M_q]$.

- *If* $n-k$ *is odd and* $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ *on* $(a,b)\times(a,b)$ *for some* $M \in \mathbb{R}$, *then* $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ *if and only if* $M \in [M_q, \lambda_1)$, *with* $M_q \geq \overline{\lambda_2}$. *Moreover,* $\frac{\partial^q}{\partial t^q} g_M$ *is monotone increasing with respect to* $M \in [M_q, \lambda_1)$.

- *If* $n-k$ *is odd and* $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ *on* $(a,b)\times(a,b)$ *for some* $M \in \mathbb{R}$, *then* $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ *if and only if* $M \in [M_q, \lambda_1)$, *with* $M_q \geq \overline{\lambda_2}$. *Moreover,* $\frac{\partial^q}{\partial t^q} g_M$ *is monotone decreasing with respect to* $M \in [M_q, \lambda_1)$.

Let $q \in \{0, \ldots, n-1\}$ be fixed. Suppose that $(-1)^{n-q-c_q}\, v_s^q$ is strongly positive on $I \times I$. Then, for each $s \in (a,b)$, we obtain the following limits:

$$\begin{aligned}
\ell_1^q(s) \; &:= \; \lim_{t\to a^+} \frac{(-1)^{n-q-c_q}\, v_s^q(t)}{(t-a)^{\alpha^q}\,(b-t)^{\beta^q}} = \frac{(-1)^{n-q-c_q}\,(v_s^q)^{(\alpha^q)}(a)}{\alpha^q!\,(b-a)^{\beta^q}}\,,\\
\ell_2^q(s) \; &:= \; \lim_{t\to b^-} \frac{(-1)^{n-q-c_q}\, v_s^q(t)}{(t-a)^{\alpha^q}\,(b-t)^{\beta^q}} = \frac{(-1)^{n-q-c_q-\beta^q}\,(v_s^q)^{(\beta^q)}(b)}{\beta^q!\,(b-a)^{\alpha^q}}.
\end{aligned}$$

For each $s \in (a,b)$, let us consider the following function defined on I by

$$\tilde{u}_s^q(t) = \begin{cases} \ell_1^q(s)\,, & t = a,\\ \dfrac{(-1)^{n-q-c_q}\, v_s^q(t)}{(t-a)^{\alpha^q}\,(b-t)^{\beta^q}}\,, & t \in (a,b),\\ \ell_2^q(s)\,, & t = b. \end{cases}$$

It is clear that $\tilde{u}_s^q > 0$ on $[a,b]$ for all $s \in (a,b)$.

Since $g_0 \in C^{n-2}(I \times I)$, $\frac{\partial^{n-1}}{\partial t^{n-1}} g_0 \in C^\infty((I\times I) \setminus \{(t,t) \;/\; t \in I\})$ and there exists $\lim\limits_{s\to t^\pm} \frac{\partial^{n-1}}{\partial t^{n-1}} g_0(t,s) \in \mathbb{R}$, we deduce that there exists $K^q > 0$ such that $\tilde{u}_s^q(t) \leq K^q$ for every $(t,s) \in I \times I$ and $q \in \{1, \ldots, n-1\}$. Therefore, the following functions

$$\begin{aligned}
\tilde{k}_1^q(s) \; &= \; \min_{t\in I} \tilde{u}_s^q(t)\,, \quad s \in I\,,\\
\tilde{k}_2^q(s) \; &= \; \max_{t\in I} \tilde{u}_s^q(t)\,, \quad s \in I\,,
\end{aligned}$$

are continuous on I and positive in (a,b).

Taking $\phi(t) = (t-a)^{\alpha^q}\,(b-t)^{\beta^q} > 0$ on (a,b), the function $v_s^q(t)$ satisfies the condition (P_g) if $n-q-c_q$ is even, with $k_1(s) = \tilde{k}_1^q(s)$ and $k_2(s) = \tilde{k}_2^q(s)$, and condition (N_g) if $n-q-c_q$ is odd, with $k_1(s) = -\tilde{k}_2^q(s)$ and $k_2(s) = -\tilde{k}_1^q(s)$.

3 Study of the constant sign of the derivatives of the Green's function

In this section, for any fixed $q \in \{1, \ldots, n-1\}$, we will give a proof of the main result that characterizes the constant sign of the partial derivative with respect to t of order q of the Green's function related to operator $T_n[M]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.

We distinguish three cases:

(a) $c_q \geq 1$ and $d_q \geq 1$,

(b) $c_q = n - q$ and $d_q = 0$,

(c) $c_q = 0$ and $d_q = n - q$,

where c_q and d_q are defined in (2.29) and (2.30) and satisfy $c_q + d_q = n - q$.

3.1 Case (a): $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$

Lemma 3.1. *Let $q \in \{1, \ldots, n-1\}$ be fixed. If $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$, then the indices $z = k - (j-1)$ and $h = n - k - (r-1)$ are such that*

$$\mu_z^q + q + \eta = n - 1, \quad \text{and} \quad \rho_h^q + q + \gamma = n - 1,$$

with j and r defined in (2.22) *and* (2.23)*, η and γ defined in* (2.15) *and* (2.16)*.*

Proof. By the definition of $\mu_z^q = \mu_{k-(j-1)}^q$, since $c_q \geq 1$, we have that $\mu_z^q = \sigma_k - q \geq 0$. Similarly, by the definition of $\rho_h^q = \rho_{n-k-(r-1)}^q$, since $d_q \geq 1$, we have that $\rho_h^q = \varepsilon_{n-k} - q \geq 0$. Using Remark 2.9, we deduce that

$$\mu_z^q + q + \eta = \sigma_k - q + q + n - 1 - \sigma_k = n - 1,$$

and

$$\rho_h^q + q + \gamma = \varepsilon_{n-k} - q + q + n - 1 - \varepsilon_{n-k} = n - 1.$$

□

Lemma 3.2. *Let $q \in \{1, \ldots, n-1\}$ be fixed. If $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$, then $z = p$ and $h = l$ with z and h defined in Lemma 3.1, p and l defined in* (2.27) *and* (2.26)*.*

Proof. First, note that $z = |A_q|$ is the cardinal of the set $A_q = \{i \in \{1, \ldots, k\} /\ \sigma_i \geq q\}$ and $p = |D_q|$ is the cardinal of the set $D_q = \{i \in \{1, \ldots, k\} /\ \delta_i + q \leq n-1\}$. Moreover, by (2.29), it is clear that c_q corresponds with the number of σ's that are equal or greater than q, that is, $z = c_q$.

On the other hand, we have that

$$\{\varepsilon_1, \ldots, \varepsilon_{n-k}, n-1-\delta_k, \ldots, n-1-\delta_1\} \equiv \{0, \ldots, n-1\}.$$

Among these n elements (all the numbers between 0 and $n-1$) there are $n-q$ elements that are equal or greater than q. By (2.30), we have that the number of ε's that are equal or greater than q is d_q. Therefore, the number of δ's such that $n-1-\delta_i \geq q$ is $n-q-d_q$, that is, $p = n-q-d_q$. Since, $c_q + d_q = n-q$, we have that $z = p$. In an analogous way, we can prove that $h = l$. □

***Remark* 3.3.** It should be noted that in the previous proof, since j and r depend on q, we have that z and h also depend on q, but we will omit such dependence in the notation for the sake of simplicity.

***Remark* 3.4.** Indices z and h depend on the space $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$ and are unique. Indeed, for the rest of the components of space $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$ we have that $(\mu_i^q + q + \eta) \bmod n < n-1$ for $i \neq z$ with $i \in \{1,\dots,k\}$ and $(\rho_l^q + q + \gamma) \bmod n < n-1$ for $l \neq h$ with $l \in \{1,\dots,n-k\}$.

Now, we present some results that provide sufficient conditions to ensure the constant sign of solutions of (1.1) in the spaces $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$. Using Lemma 2.18, these results can be proved analogously to [12, Proposition 6.7] and [12, Proposition 6.9], respectively, and so we omit the proofs.

Let us consider the following spaces

$$X_2 := \begin{cases} X_{\{\mu_1^q,\dots,\mu_{z-1}^q,\mu_{z+1}^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q|\beta^q\}}, & \text{if } z \neq 1, \\ \\ X_{\{\mu_2^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q|\beta^q\}}, & \text{if } z = 1, \end{cases} \tag{3.1}$$

$$X_3 := \begin{cases} X_{\{\mu_1^q,\dots,\mu_{z-1}^q,\mu_{z+1}^q,\dots,\mu_k^q|\alpha^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}, & \text{if } z \neq 1, \\ \\ X_{\{\mu_2^q,\dots,\mu_k^q|\alpha^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}, & \text{if } z = 1, \end{cases} \tag{3.2}$$

$$X_4 := \begin{cases} X_{\{\mu_1^q,\dots,\mu_k^q|\alpha^q\}}^{\{\rho_1^q,\dots,\rho_{h-1}^q,\rho_{h+1}^q,\dots,\rho_{n-k}^q\}}, & \text{if } h \neq 1, \\ \\ X_{\{\mu_1^q,\dots,\mu_k^q|\alpha^q\}}^{\{\rho_2^q,\dots,\rho_{n-k}^q\}}, & \text{if } h = 1, \end{cases} \tag{3.3}$$

and

$$X_5 := \begin{cases} X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{h-1}^q,\rho_{h+1}^q,\dots,\rho_{n-k}^q|\beta^q\}}, & \text{if } h \neq 1, \\ \\ X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_2^q,\dots,\rho_{n-k}^q|\beta^q\}}, & \text{if } h = 1, \end{cases} \tag{3.4}$$

where z and h are defined in Lemma 3.1.

Proposition 3.5. *Let $q \in \{1, \dots, n-1\}$. Suppose that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$. If $u \in C^n(I)$ is a solution of (1.1) on (a, b), satisfying the boundary conditions:*

$$u^{(\mu_1^q)}(a) = \cdots = u^{(\mu_{z-1}^q)}(a) = u^{(\mu_{z+1}^q)}(a) = \cdots = u^{(\mu_k^q)}(a) = 0\,, \tag{3.5}$$

$$u^{(\rho_1^q)}(b) = \cdots = u^{(\rho_{n-k}^q)}(b) = 0\,, \tag{3.6}$$

then it does not have any zero on (a, b) provided that one of the following assertions is satisfied:

- *Let $n - k$ be even:*
  - *If $k > 1$, $\mu_z^q \neq z - 1$ and $M \in [\lambda_3^q, \lambda_2^q]$, where:*
    - *$\lambda_3^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_3.*
    - *$\lambda_2^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_2.*
  - *If $k > 1$, $\mu_z^q = z - 1$ and $M \in [\lambda_1, \lambda_2^q]$, where:*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1, \dots, \sigma_k\}}^{\{\varepsilon_1, \dots, \varepsilon_{n-k}\}}$.*
    - *$\lambda_2^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_2.*
  - *If $k = 1$, $\mu_1^q \neq 0$ and $M \in [\lambda_3^q, +\infty)$, where:*
    - *$\lambda_3^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\alpha^q\}}^{\{\rho_1^q, \dots, \rho_{n-1}^q\}}$, where $\alpha^q = 0$.*
  - *If $k = 1$, $\mu_1^q = 0$ and $M \in [\lambda_1, +\infty)$, where:*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1, \dots, \varepsilon_{n-1}\}}$.*
- *Let $n - k$ be odd:*
  - *If $k > 1$, $\mu_z^q \neq z - 1$ and $M \in [\lambda_2^q, \lambda_3^q]$, where:*
    - *$\lambda_3^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_3.*
    - *$\lambda_2^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_2.*
  - *If $k > 1$, $\mu_z^q = z - 1$ and $M \in [\lambda_2^q, \lambda_1]$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1, \dots, \sigma_k\}}^{\{\varepsilon_1, \dots, \varepsilon_{n-k}\}}$..*
    - *$\lambda_2^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_2.*
  - *If $k = 1$, $\mu_1^q \neq 0$ and $M \in (-\infty, \lambda_3^q]$, where:*
    - *$\lambda_3^q > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\alpha^q\}}^{\{\rho_1^q, \dots, \rho_{n-1}^q\}}$, where $\alpha^q = 0$.*
  - *If $k = 1$, $\mu_1^q = 0$ and $M \in (-\infty, \lambda_1]$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1, \dots, \varepsilon_{n-1}\}}$.*

Proposition 3.6. *Let $q \in \{1, \dots, n-1\}$. Suppose that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$. If $u \in C^n(I)$ is a solution of (1.1) on (a, b) satisfying the boundary conditions:*

$$u^{(\mu_1^q)}(a) = \cdots = u^{(\mu_k^q)}(a) = 0\,, \tag{3.7}$$
$$u^{(\rho_1^q)}(b) = \cdots = u^{(\rho_{h-1}^q)}(b) = u^{(\rho_{h+1}^q)}(b) = \cdots = u^{(\rho_{n-k}^q)}(b) = 0\,, \tag{3.8}$$

then it does not have any zero on (a, b) provided that one of the following assertions is satisfied:

- *Let $n - k$ be even:*
  - *If $\rho_h^q \neq h - 1$ and $M \in [\lambda_5^q, \lambda_4^q]$, where:*
    - *$\lambda_5^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_5.*
    - *$\lambda_4^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_4.*
  - *If $\rho_h^q = h - 1$ and $M \in [\lambda_1, \lambda_4^q]$, where:*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_4^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_4.*
- *Let $n - k$ be odd:*
  - *If $k < n - 1$, $\rho_h^q \neq h - 1$ and $M \in [\lambda_4^q, \lambda_5^q]$, where:*
    - *$\lambda_5^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_5.*
    - *$\lambda_4^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_4.*
  - *If $k < n - 1$, $\rho_h^q = h - 1$ and $M \in [\lambda_4^q, \lambda_1]$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_4^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_4.*
  - *If $k = n - 1$, $\rho_1^q \neq 0$ and $M \in (-\infty, \lambda_5^q]$, where:*
    - *$\lambda_5^q > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\mu_1^q,\dots,\mu_{n-1}^q\}}^{\{\beta^q\}}$, where $\beta^q = 0$.*
  - *If $k = n - 1$, $\rho_1^q = 0$ and $M \in (-\infty, \lambda_1]$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.*

Example 3.7. Let us consider the operator $T_4[M]$ with the boundary conditions:

$$u'(0) = u'''(0) = u(1) = u'(1) = 0.$$

The hypotheses of Propositions 3.5 and 3.6 are satisfied only for the value $q = 1$. Moreover, $z = 2$, $h = 1$, and $v_s^1[M]$ satisfies the conditions of the space $X_{\{0,2\}}^{\{0,3\}}$.

From Proposition 3.5, we can affirm that any nontrivial solution of the problem

$$T_4[M] = 0,\ t \in [0,1],\quad u(0) = u(1) = u'''(1) = 0, \tag{3.9}$$

does not have any zero on $(0,1)$ for $M \in [\lambda_3^1, \lambda_2^1]$, where $\lambda_3^1 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$ and $\lambda_2^1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{1\}}^{\{0,1,3\}}$.

The eigenvalues of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$ are given by $-\lambda^4$, where λ is a positive solution of $\sin(\lambda) = 0$. The smallest positive solution of this equation is π, so $\lambda_3^1 = -\pi^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$.

Similarly, the eigenvalues of $T_4[0]$ in $X_{\{1\}}^{\{0,1,3\}}$ are given by λ^4, where λ is a positive solution of

$$\left(-1 + e^{\sqrt{2}\,m}\right) \cos\left(\frac{m}{\sqrt{2}}\right) + \left(1 + e^{\sqrt{2}\,m}\right) \sin\left(\frac{m}{\sqrt{2}}\right) = 0.$$

Denoting by m_2 the smallest positive solution of this equation, we deduce that $\lambda_2^1 = m_2^4 \approx 3.34^4$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0\}}^{\{0,1,3\}}$.

It is easy to verify that the solutions of problem (3.9) are given as multiples of the following expression:

- If $M = -m^4 < 0$,

$$\begin{aligned} u(t) = &\cosh(m - m\,t)\,\sin(m) - \sin(m\,t) - \cosh(m)\,\sin(m - m\,t) - \cos(m - m\,t)\,\sinh(m) \\ &+ \sinh(m\,t) + \cos(m)\,\sinh(m - m\,t). \end{aligned}$$

- If $M = 0$,

$$u(t) = t^2 - t.$$

- If $M = m^4 > 0$,

$$\begin{aligned} u(t) = &e^{-\frac{m\,t}{\sqrt{2}}} \left(e^{\sqrt{2}\,m} \left(-1 + e^{\sqrt{2}\,m\,t}\right) \cos\left(\frac{m(-2+t)}{\sqrt{2}}\right) - e^{\sqrt{2}\,m} \left(-1 + e^{\sqrt{2}\,m}\right) \cos\left(\frac{m\,t}{\sqrt{2}}\right)\right) \\ &+ e^{-\frac{m\,t}{\sqrt{2}}} \left(-1 + e^{\sqrt{2}\,m}\right) \left(-e^{\sqrt{2}\,m} + e^{\sqrt{2}\,m\,t}\right) \sin\left(\frac{m\,t}{\sqrt{2}}\right). \end{aligned}$$

Analogously, from Proposition 3.6, we conclude that any solution of the problem

$$T_4[M]\,u(t) = 0,\ t \in [0,1],\ \ u(0) = u''(0) = u'''(1) = 0, \tag{3.10}$$

does not have any zero on $(0,1)$ for $M \in [\lambda_1, \lambda_4^1]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{1,3\}}^{\{0,1\}}$ and $\lambda_4^1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{3\}}$.

The eigenvalues of $T_4[0]$ in $X_{\{1,3\}}^{\{0,1\}}$ are given by $-\lambda^4$, where λ is a positive solution of

$$\sin(\lambda) + \cos(\lambda)\,\tanh(\lambda) = 0\,.$$

Now, if m_1 is the smallest positive solution of this equation, then $\lambda_1 = -m_1^4 \approx -2.36^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$.

On the other hand, the eigenvalues of $T_4[0]$ in $X_{\{0,1,2\}}^{\{3\}}$ are given by λ^4, where λ is a positive solution of $\cos\left(\frac{m}{\sqrt{2}}\right) = 0$. The smallest positive solution of this equation is $\frac{\sqrt{2}}{2}\pi$, so $\lambda_4^1 = \left(\frac{\sqrt{2}}{2}\pi\right)^4$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{3\}}$.

It is immediate to show that the solutions of problem (3.10) are given as multiples of:

- If $M = -m^4 < 0$,

$$u(t) = \sin(m\,t) + \cos(m)\,\operatorname{sech}(m)\sinh(m\,t).$$

- If $M = 0$,

$$u(t) = t.$$

- If $M = m^4 > 0$,

$$\begin{aligned} u(t) = & e^{-\frac{m\,t}{\sqrt{2}}}\left(\left(-1+e^{\sqrt{2}\,m\,(1+t)}\right)\cos\left(\frac{m(-1+t)}{\sqrt{2}}\right)+\left(-e^{\sqrt{2}\,m}+e^{\sqrt{2}\,m\,t}\right)\cos\left(\frac{m(1+t)}{\sqrt{2}}\right)\right)\\ &+e^{-\frac{m\,t}{\sqrt{2}}}\left(\left(1+e^{\sqrt{2}\,m\,(1+t)}\right)\sin\left(\frac{m(-1+t)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}\,m}+e^{\sqrt{2}\,m\,t}\right)\sin\left(\frac{m(1+t)}{\sqrt{2}}\right)\right). \end{aligned}$$

***Remark* 3.8.** It should be noted that in the above propositions we have used the fact (proven in Lemma 2.19) that the first nonzero eigenvalues of the spaces $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$ coincide.

Let us define the following spaces

$$X_2^* := \begin{cases} X_{\{\tau_1,\,\dots,\,\tau_{n-k}|\eta\}}^{\{\delta_1,\,\dots,\,\delta_{z-1},\,\delta_{z+1},\,\dots,\,\delta_k\}}, & \text{if } z \neq 1,\\ \\ X_{\{\tau_1,\,\dots,\,\tau_{n-k}|\eta\}}^{\{\delta_2,\,\dots,\,\delta_k\}}, & \text{if } z = 1, \end{cases} \tag{3.11}$$

and

$$X_4^* := \begin{cases} X_{\{\tau_1,\,\dots,\,\tau_{h-1},\,\tau_{h+1},\,\dots,\,\tau_{n-k}\}}^{\{\delta_1,\,\dots,\,\delta_k|\gamma\}}, & \text{if } h \neq 1,\\ \\ X_{\{\tau_2,\,\dots,\,\tau_{n-k}\}}^{\{\delta_1,\,\dots,\,\delta_k|\gamma\}}, & \text{if } h = 1. \end{cases} \tag{3.12}$$

Lemma 3.9. *$\lambda \neq 0$ is an eigenvalue of $T_n[0]$ in X_2^* defined in (3.11) if and only if λ is an eigenvalue of $T_n[0]$ in X_2. In particular, $(\lambda_2^*)^q = \lambda_2^q$ where $(\lambda_2^*)^q \neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_2^* and $\lambda_2^q \neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_2.*

Proof. Suppose that $z \neq 1$ (the case $z = 1$ is proved similarly). By the definition of adjoint space of $X_{\{\mu_1^q,\,\dots,\,\mu_k^q\}}^{\{\rho_1^q,\,\dots,\,\rho_{n-k}^q\}}$ we have that

$$\{\mu_1^q,\,\dots,\,\mu_{z-1}^q,\,\mu_z^q,\,\mu_{z+1}^q,\,\dots,\,\mu_k^q,\,n-1-\tau_{n-k}^q,\,\dots,\,n-1-\tau_1^q\} = \{0,\,\dots,\,n-1\},$$

and
$$\{\rho_1^q,\,\ldots,\,\rho_{n-k}^q,\,n-1-\delta_k^q,\,\ldots,\,n-1-\delta_1^q\}=\{0,\,\ldots,\,n-1\}.$$
Since $\mu_z^q=\sigma_k-q$ and $\beta^q=n-1-\delta_k^q$, we deduce that

$$X^{\{\rho_1^q,\ldots,\rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q,\ldots,\mu_{z-1}^q,\mu_{z+1}^q,\ldots,\mu_k^q\}} \xrightarrow{\ *\ } X^{\{\delta_1^q,\ldots,\delta_{k-1}^q\}}_{\{\tau_1^q,\ldots,\tau_{n-k}^q|n-1-\sigma_k+q\}}. \tag{3.13}$$

Using that $\eta=n-1-\sigma_k$, the definition of $X^{\{\delta_1^q,\ldots,\delta_k^q\}}_{\{\tau_1^q,\ldots,\tau_{n-k}^q\}}$ and the diagram (2.28), we infer that
$$X^{\{\delta_1^q,\ldots,\delta_{k-1}^q\}}_{\{\tau_1^q,\ldots,\tau_{n-k}^q|n-1-\sigma_k+q\}} \xrightarrow{\ \phi^q\ } X^{\{\delta_1,\ldots,\delta_{p-1},\delta_{p+1},\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}|\eta\}}.$$
From Lemma 3.2 we know that $p=z$ and therefore

$$X^{\{\delta_1^q,\ldots,\delta_{k-1}^q\}}_{\{\tau_1^q,\ldots,\tau_{n-k}^q|n-1-\sigma_k+q\}} \xrightarrow{\ \phi^q\ } X^{\{\delta_1,\ldots,\delta_{z-1},\delta_{z+1},\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}|\eta\}}. \tag{3.14}$$

Considering Lemma 2.19 and the fact that the adjoint spaces have the same eigenvalues, from (3.13) and (3.14), we obtain that the spaces $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q,\ldots,\mu_{z-1}^q,\mu_{z+1}^q,\ldots,\mu_k^q\}}$ and $X^{\{\delta_1,\ldots,\delta_{z-1},\delta_{z+1},\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}|\eta\}}$ have the same eigenvalues. In particular, $(\lambda_2^*)^q=\lambda_2^q$. □

Using similar argument to the previous result, we arrive to the next one.

Lemma 3.10. *$\bar\lambda\neq 0$ is an eigenvalue of $T_n[0]$ in X_4^* defined in (3.12) if and only if $\bar\lambda$ is an eigenvalue of $T_n[0]$ in X_4. In particular, $(\lambda_4^*)^q=\lambda_4^q$ where $(\lambda_4^*)^q\neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_4^* and $\lambda_4^q\neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_4.*

Next, we state and prove our main result, which gives the characterization of the set of values of the parameter M where $v_s^q[M]$ is of constant sign.

Theorem 3.11. *Let $q\in\{1,\ldots,n-1\}$. Suppose that $\{\sigma_1,\ldots,\sigma_k\}-\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q+d_q=n-q$, $c_q\geq 1$ and $d_q\geq 1$. The following properties are fulfilled:*

- *Let $2\leq k\leq n-2$ and $n-k$ be even:*
    - *If $n-q-c_q$ is even, then $v_s^q[M]$ is strongly positive on $I\times I$ if, and only if, $M\in(\lambda_1,\lambda^q]$, and if $n-q-c_q$ is odd, then $v_s^q[M]$ is strongly negative on $I\times I$ if, and only if, $M\in(\lambda_1,\lambda^q]$ where:*
        - *$\lambda_1<0$ is the biggest negative eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$.*
        - *$\lambda^q>0$ is the minimum between:*
            - *· $\lambda_2^q>0$, the least positive eigenvalue of $T_n[0]$ in X_2 defined in (3.1).*
            - *· $\lambda_4^q>0$, the least positive eigenvalue of $T_n[0]$ in X_4 defined in (3.3).*

- *Let* $2 \le k \le n-2$ *and* $n-k$ *be odd:*
  - *If* $n-q-c_q$ *is even, then* $v_s^q[M]$ *is strongly positive on* $I \times I$ *if, and only if,* $M \in [\lambda^q, \lambda_1)$*, and if* $n-q-c_q$ *is odd, then* $v_s^q[M]$ *is strongly negative on* $I \times I$ *if, and only if,* $M \in [\lambda^q, \lambda_1)$ *where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda^q < 0$ *is the maximum between:*
      - $\lambda_2^q < 0$*, the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.
      - $\lambda_4^q < 0$*, the biggest negative eigenvalue of* $T_n[0]$ *in* X_4.
- *Let* $k = 1$ *and* $n > 2$ *be odd:*
  - *If* $n-q-1$ *is even, then* $v_s^q[M]$ *is strongly positive on* $I \times I$ *if, and only if,* $M \in (\lambda_1, \lambda_2^q]$*, and if* $n-q-1$ *is odd, then* $v_s^q[M]$ *is strongly negative on* $I \times I$ *if, and only if,* $M \in (\lambda_1, \lambda_2^q]$ *where:*
    - $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.
    - $\lambda_2^q > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* X_2.
- *Let* $k = 1$ *and* $n > 2$ *be even:*
  - *If* $n-q-1$ *is even, then* $v_s^q[M]$ *is strongly positive on* $I \times I$ *if, and only if,* $M \in [\lambda_2^q, \lambda_1)$*, and if* $n-q-1$ *is odd, then* $v_s^q[M]$ *is strongly negative on* $I \times I$ *if, and only if,* $M \in [\lambda_2^q, \lambda_1)$*, where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.
    - $\lambda_2^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.
- *If* $k = n-1$*,* $n > 2$ *and* $n-q-c_q = 1$*, then* $v_s^q[M]$ *is strongly negative on* $I \times I$ *if, and only if,* $M \in [\lambda_2^q, \lambda_1)$*, where*
  - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.
  - $\lambda_2^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.

Proof. First, it has been shown in Theorem 2.31 that v_s^q satisfies the property (P_g) if $n-q-c_q$ is even and the property (N_g) if $n-q-c_q$ is odd. In addition, from Lemma 2.33 we know that if $v_s^q[M]$ has constant sign then the first eigenvalue λ_1 of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is one of the extremes of the interval where $v_s^q[M]$ holds that sign.

Again, taking into account Lemma 2.33 on the monotony of $v_s^q[M]$ with respect to M, the constant sign starts/ends at the first eigenvalue λ_1 of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Thus, from Theorems 2.28 and 2.31 we conclude that:

- If $n-k$ and $n-q-c_q$ are even, and $M \le 0$, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$.

- If $n-k$ is even, $n-q-c_q$ is odd and $M \le 0$, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$.
- If $n-k$ is odd, $n-q-c_q$ is even and $M \ge 0$, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in [0, \lambda_1)$.
- If $n-k$ and $n-q-c_q$ are odd, and $M \ge 0$, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in [0, \lambda_1)$.

We now need to determine the other extreme of the constant sign interval of $v_s^q[M]$ using Definition 2.27, Propositions 3.5 and 3.6. We divide the proof into five steps assuming that $n-k$ and $n-q-c_q$ are even, and $2 \le k \le n-2$. For the rest of cases the proof is done in an analogous way.

Step 1. Study of $v_s^q[M]$ at $s=a$.

Step 2. Study of $v_s^q[M]$ at $s=b$.

Step 3. Study of $v_s^q[M]$ at $t=a$.

Step 4. Study of $v_s^q[M]$ at $t=b$.

Step 5. Study of $v_s^q[M]$ on $(a,b) \times (a,b)$.

Let us denote by

$$g_M(t,s) = \begin{cases} g_M^1(t,s)\,, & a \le s \le t \le b\,, \\ \\ g_M^2(t,s)\,, & a < t < s < b\,, \end{cases}$$

the Green's function related to $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Step 1. Study of the function $v_s^q[M]$ at $s=a$.

Let us consider the function

$$w_{M,q}(t) = \frac{\partial^\eta}{\partial s^\eta}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a},$$

where η has been defined in (2.15).

If $\eta > 0$, since $\{0,\dots,\eta-1\} \subset \{\tau_1,\dots,\tau_{n-k}\}$, using the equality (2.12) we obtain that

$$g_M^1(t,a) = \frac{\partial}{\partial s} g_M^1(t,a) = \dots = \frac{\partial^{\eta-1}}{\partial s^{\eta-1}} g_M^1(t,a) = 0\,, \quad \text{for all } t \in (a,b].$$

Therefore, for all $t \in (a,b]$ and $q \in \{1,\dots,n-1\}$ we infer that

$$\frac{\partial^q}{\partial t^q} g_M^1(t,a) = \frac{\partial^q}{\partial t^q}\Big(\frac{\partial}{\partial s} g_M^1(t,s)\Big)_{|s=a} = \dots = \frac{\partial^q}{\partial t^q}\Big(\frac{\partial^{\eta-1}}{\partial s^{\eta-1}} g_M^1(t,s)\Big)_{|s=a} = 0\,,$$

and, equivalently,

$$\frac{\partial^q}{\partial t^q} g_M^1(t,a) = \frac{\partial}{\partial s}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a} = \cdots = \frac{\partial^{\eta-1}}{\partial s^{\eta-1}}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a} = 0\,.$$

Note that it is necessary that $w_{M,q} > 0$ to guarantee the strongly positive sign of the function $v_s^q[M]$. Indeed, if there exists $t^* \in [a,b]$, such that $w_{M,q}(t^*) < 0$, then there exists $\rho(t^*) > a$ such that $v_s^q[M](t^*) < 0$ for all $s \in (a, \rho(t^*))$, which contradicts the strongly positive sign.

Since the function $v_s^q[M]$ is a solution of $T_n[M]\, v_s^q[M](t) = 0$ for $t \neq s$, we have that

$$\frac{\partial^\eta}{\partial s^\eta}\left(T_n[M]\, v_s^q[M](t)\right)_{|s=a} = T_n[M]\, w_{M,q}(t) = 0\,, \quad t \in (a,b]\,,$$

that is, the function $w_{M,q}$ solves the equation $T_n[M]\, u(t) = 0$, $t \in (a,b]$.

It remains to obtain the boundary conditions satisfied by the function $w_{M,q}$.

Using the matrix of the Green's function given in (2.6), the equality (2.8), the expression of g_{n-j}, $j = 1, \ldots, n-1$ given in (2.10), and the following identity

$$\frac{\partial}{\partial t}\Big(\frac{\partial^q}{\partial t^q}\, G_M(t,s)\Big) = A\, \frac{\partial^q}{\partial t^q} G_M(t,s), \quad \text{for all } t \in I\setminus\{s\}\,,\ q \in \{1,\ldots,n-1\},$$

it follows that

$$\frac{\partial^{\mu_1^q+q}}{\partial t^{\mu_1^q+q}} g_M^2(t,s)_{|t=a} = 0,\ -\frac{\partial^{\mu_1^q+q+1}}{\partial t^{\mu_1^q+q}\partial s} g_M^2(t,s)_{|t=a} = 0,\ \ldots,\ (-1)^\eta \frac{\partial^{\mu_1^q+q+\eta}}{\partial t^{\mu_1^q+q}\partial s^\eta} g_M^2(t,s)_{|t=a} = 0\,.$$

Since $(\eta + \mu_1^q + q) \mod n < n-1$, then we have that none of the previous partial derivatives is an element of the diagonal of $\frac{\partial^q}{\partial t^q} G_M(t,s)$, and taking into account that the above equalities are satisfied for $s = a$, by continuity, we deduce that

$$\begin{cases} \dfrac{\partial^{\mu_1^q+q}}{\partial t^{\mu_1^q+q}} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\[2ex] -\dfrac{\partial^{\mu_1^q+q+1}}{\partial t^{\mu_1^q+q}\partial s} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\ \vdots \\ (-1)^\eta \dfrac{\partial^{\mu_1^q+q+\eta}}{\partial t^{\mu_1^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,. \end{cases}$$

Therefore,

$$w_{M,q}^{(\mu_1^q)}(a) = \frac{\partial^{\mu_1^q+q+\eta}}{\partial t^{\mu_1^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,.$$

Making a similar reasoning we deduce that

$$w_{M,q}^{(\mu_2^q)}(a) = \cdots = w_{M,q}^{(\mu_{z-1}^q)}(a) = w_{M,q}^{(\mu_{z+1}^q)}(a) = \cdots = w_{M,q}^{(\mu_k^q)}(a) = 0\,.$$

For μ_z^q, we have the following equalities

$$\frac{\partial^{\mu_z^q+q}}{\partial t^{\mu_z^q+q}} g_M^2(t,s)_{|t=a} = 0,\ -\frac{\partial^{\mu_z^q+q+1}}{\partial t^{\mu_z^q+q}\partial s} g_M^2(t,s)_{|t=a} = 0,\ \ldots,\ (-1)^\eta \frac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^2(t,s)_{|t=a} = 0\,.$$

By assumption, we have that $\eta + \mu_z^q + q = n-1$, so $\dfrac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^2(t,s)_{|t=a}$ is an element of the diagonal of $\dfrac{\partial^q}{\partial t^q} G_M(t,s)$, and taking into account again that the previous equalities are satisfied for $s=a$ and that $\dfrac{\partial^q}{\partial t^q} G_M(t,s)$ has a jump equal to 1 on the diagonal we infer that

$$\begin{cases} \dfrac{\partial^{\mu_z^q+q}}{\partial t^{\mu_z^q+q}} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\[2ex] -\dfrac{\partial^{\mu_z^q+q+1}}{\partial t^{\mu_z^q+q}\partial s} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\[2ex] \vdots \\[2ex] (-1)^\eta \dfrac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 1\,. \end{cases}$$

Therefore,

$$w_{M,q}^{(\mu_z^q)}(a) = \frac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = (-1)^\eta\,.$$

Let us see what conditions $w_{M,q}$ satisfies at $t=b$. To do this, proceeding with the previous argument we obtain that

$$\frac{\partial^{\rho_1^q+q}}{\partial t^{\rho_1^q+q}} g_M^1(t,s)_{|t=b} = 0,\ -\frac{\partial^{\rho_1^q+q+1}}{\partial t^{\rho_1^q+q}\partial s} g_M^1(t,s)_{|t=b} = 0,\ \ldots,\ (-1)^\eta \frac{\partial^{\rho_1^q+q+\eta}}{\partial t^{\rho_1^q+q}\partial s^\eta} g_M^1(t,s)_{|t=b} = 0\,.$$

The above equalities are satisfied at $s=a$ and since $b \neq a$ we have that

$$w_{M,q}^{(\rho_1^q)}(b) = \frac{\partial^{\rho_1^q+q+\eta}}{\partial t^{\rho_1^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(b,a)} = 0\,.$$

Analogously we obtain that

$$w_{M,q}^{(\rho_2^q)}(b) = \cdots = w_{M,q}^{(\rho_{n-k}^q)}(b) = 0\,.$$

If $\eta = 0$ then, from the matrix argument we made earlier, it follows that the function $w_{M,q}(t) = \frac{\partial^q}{\partial t^q} g^1_M(t,a)$ satisfies the same boundary conditions (3.5)–(3.6).

Then, from Proposition 3.5 and Theorem 2.31 we conclude that $w_{M,q} > 0$ on (a,b) for all $M \in [0, \lambda_2^q]$.

Now, let us see that $v_s^q[M]$ cannot be positive for $M > \lambda_2^q$.

Suppose that there exists $\widehat{M} > \lambda_2^q$ such that $v_s^q[\widehat{M}]$ is positive. Then, by monotony of $v_s^q[M]$ we have that $w_{\widehat{M},q} \le w_{M,q} \le w_{\lambda_2^q,q}$ for every $M \in [\lambda_2^q, \widehat{M}]$. In particular, it occurs that $w^{(\beta^q)}_{\widehat{M},q}(b) \ge w^{(\beta^q)}_{M,q}(b) \ge w^{(\beta^q)}_{\lambda_2^q,q}(b) = 0$ if β^q is even and $w^{(\beta^q)}_{\widehat{M},q}(b) \le w^{(\beta^q)}_{M,q}(b) \le w^{(\beta^q)}_{\lambda_2^q,q}(b) = 0$ if β^q is odd.

If $w^{(\beta^q)}_{\widehat{M},q}(b) \neq 0$, then there exists $\rho > 0$ such that $w_{\widehat{M},q}(t) < 0$ for all $t \in (b-\rho, b)$, which contradicts our assumption. Hence,

$$0 = w^{(\beta^q)}_{\widehat{M},q}(b) = w^{(\beta^q)}_{M,q}(b) = w^{(\beta^q)}_{\lambda_2^q,q}(b)\,, \quad \forall M \in [\lambda_2^q, \widehat{M}],$$

and this fact contradicts the discrete character of the spectrum.

Thus, we conclude that, if $M \in [0, \lambda_2^q]$, then

$$\forall t \in (a,b)\,, \quad \exists \rho^q(t) > 0 \;\mid\; v_s^q[M](t) > 0 \;\forall s \in (a, a+\rho^q(t))\,.$$

Moreover, if $M > \lambda_2^q$, then $v_s^q[M]$ is not positive.

Step 2. Study of the function $v_s^q[M]$ at $s = b$.

We study the following function by applying a similar reasoning to Step 1. In this case we consider

$$y_{M,q}(t) = \frac{\partial^\gamma}{\partial s^\gamma}\Big(\frac{\partial^q}{\partial t^q} g^2_M(t,s)\Big)_{|s=b}\,,$$

where γ has been defined in (2.16).

Using analogous arguments to Step 1, we obtain that if $\gamma > 0$, then

$$\frac{\partial^q}{\partial t^q} g^2_M(t,b) = \frac{\partial}{\partial s} g^2_M(t,s)_{|s=b} = \cdots = \frac{\partial^{\gamma-1}}{\partial s^{\gamma-1}}\Big(\frac{\partial^q}{\partial t^q} g^2_M(t,s)\Big)_{|s=b} = 0\,.$$

Moreover, if γ is even, then $y_{M,q} \ge 0$, and if γ is odd, then $y_{M,q} \le 0$.

Again, we have that

$$T_n[M]\, y_{M,q}(t) = 0\,, \quad \forall\, t \in [a,b).$$

In the same way than above, studying $\dfrac{\partial^q}{\partial t^q} G_M(t,s)$ to determine the boundary conditions of $y_{M,q}$, we infer that

$$\begin{aligned} y^{(\mu_1^q)}_{M,q}(a) = \cdots = y^{(\mu_k^q)}_{M,q}(a) &= 0\,,\\ y^{(\rho_1^q)}_{M,q}(b) = \cdots = y^{(\rho_{h-1}^q)}_{M,q}(b) = y^{(\rho_{h+1}^q)}_{M,q}(b) = \cdots = y^{(\rho_{n-k}^q)}_{M,q}(b) &= 0\,,\\ y^{(\rho_h^q)}_{M,q}(b) &= (-1)^{\gamma+1}\,. \end{aligned}$$

Analogously, if $\gamma = 0$, then $y_{M,q}(t) = \frac{\partial^q}{\partial t^q} g_M^2(t,b)$ satisfies the conditions (3.7)–(3.8) by an argument similar to the case $\gamma > 0$.

Then, from Proposition 3.6 and Theorem 2.31, we conclude that $y_{M,q} > 0$ on (a,b) if γ is even, and $y_{M,q} < 0$ on (a,b) if γ is odd, for all $M \in [0, \lambda_4^q]$.

As in the previous step, it can be seen that for $M > \lambda_4^q$, the function $v_s^q[M]$ is never positive. Therefore, we conclude that, if $M \in [0, \lambda_4^q]$, then

$$\forall t \in (a,b)\,, \quad \exists \rho^q(t) > 0 \ \mid \ v_s^q[M](t) > 0 \ \forall s \in (b - \rho^q(t), b)\,.$$

Moreover, if $M > \lambda_4^q$, then $v_s^q[M]$ cannot be positive.

Step 3. Study of the function $v_s^q[M]$ at $t = a$.

Let us denote

$$\widehat{g}_{(-1)^n M}(t,s) = \begin{cases} \widehat{g}^1_{(-1)^n M}(t,s)\,, & a \le s \le t \le b\,, \\ \\ \widehat{g}^2_{(-1)^n M}(t,s)\,, & a < t < s < b\,, \end{cases}$$

the Green's function related to $\widehat{T}_n[(-1)^n M]$ in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}} = X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}}$.

In this case, reasoning similarly to Step 1, we study the following function:

$$\widehat{w}_{M,q}(t) = (-1)^n \frac{\partial^{\alpha^q+q}}{\partial\, s^{\alpha^q+q}} \widehat{g}^1_{(-1)^n M}(t,s)_{|s=a}\,,$$

which is equivalent, using equality (2.14), to study the function

$$\widehat{w}_{M,q}(s) = \frac{\partial^{\alpha^q+q}}{\partial\, t^{\alpha^q+q}} g_M^2(t,s)_{|t=a} = \frac{\partial^{\alpha^q}}{\partial\, t^{\alpha^q}} \Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s) \Big)_{|t=a}\,, \quad s \in I\,.$$

In this case, if $\alpha^q > 0$, using that $\{0, \dots, \alpha^q - 1\} \subset \{\mu_1^q, \dots, \mu_k^q\}$, we have

$$\frac{\partial^q}{\partial t^q} g_M^2(a,s) = \frac{\partial}{\partial t} \Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s) \Big)_{|t=a} = \cdots = \frac{\partial^{\alpha^q-1}}{\partial t^{\alpha^q-1}} \Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s) \Big)_{|t=a} = 0\,, \quad \forall s \in (a,b).$$

As in the Steps 1 and 2, we can deduce that

$$\widehat{T}_n[(-1)^n M]\, \widehat{w}_{M,q}(t) = 0, \ \ t \in (a,b].$$

Using the arguments of Step 1, we can affirm that if there exists $t^* \in (a,b)$ such that $\widehat{w}_{M,q}(t^*) < 0$, then $v_s^q[M]$ is not positive.

By the first row of (2.19) and (2.18) we deduce that:

$$\frac{\partial^{\tau_1}}{\partial t^{\tau_1}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} = 0,\ -\frac{\partial^{\tau_1+1}}{\partial t^{\tau_1} \partial s} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} = 0,\ \dots, (-1)^{\alpha^q+q} \frac{\partial^{\tau_1+\alpha^q+q}}{\partial t^{\tau_1} \partial s^{\alpha^q+q}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} = 0\,.$$

Since $(\tau_1+\alpha^q+q) \mod n < n-1$, none of the previous partial derivatives is an element of the diagonal of the matrix Green's function. Hence the previous equalities are satisfied for $s=a$, and we obtain:

$$\left\{\begin{array}{rl} \dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} & = 0\,, \\ \\ -\dfrac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} & = 0\,, \\ \\ \vdots & \\ \\ (-1)^{\alpha^q+q}\dfrac{\partial^{\tau_1+\alpha^q+q}}{\partial t^{\tau_1}\partial s^{\alpha^q+q}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} & = 0\,. \end{array}\right.$$

So, we conclude that $\widehat{w}^{(\tau_1)}_{M,q}(a) = 0\,.$

Analogously we obtain that

$$\widehat{w}^{(\tau_2)}_{M,q}(a) = \cdots = \widehat{w}^{(\tau_{h-1})}_{M,q}(a) = \widehat{w}^{(\tau_{h+1})}_{M,q}(a) = \cdots = \widehat{w}^{(\tau_{n-k})}_{M,q}(a) = 0.$$

For τ_h, we have the following equalities

$$\frac{\partial^{\tau_h}}{\partial t^{\tau_h}}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} = 0,\ -\frac{\partial^{\tau_h+1}}{\partial t^{\tau_h}\partial s}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} = 0,\ \ldots,(-1)^{\alpha^q+q}\frac{\partial^{\tau_h+\alpha^q+q}}{\partial t^{\tau_h}\partial s^{\alpha^q+q}}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} = 0\,.$$

In this case, since $\tau_h+\alpha^q+q = n-1$, we reach a diagonal element of $\widehat{G}(t,s)$ given in (2.17), and as consequence we obtain the following equalities for $s=a$:

$$\left\{\begin{array}{rl} \dfrac{\partial^{\tau_h}}{\partial t^{\tau_h}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} & = 0\,, \\ \\ -\dfrac{\partial^{\tau_h+1}}{\partial t^{\tau_h}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} & = 0\,, \\ \\ \vdots & \\ \\ (-1)^{\alpha^q+q}\dfrac{\partial^{\tau_h+\alpha^q+q}}{\partial t^{\tau_h}\partial s^{\alpha^q+q}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} & = 1\,. \end{array}\right.$$

Therefore, $\widehat{w}^{(\tau_h)}_{M,q}(a) = (-1)^{n-\alpha^q-q}\,.$

Now, let us study the behavior of $\widehat{w}_{M,q}$ at $t=b$. Studying the $(n-k+1)^{\text{th}}$ row of (2.19), we have for all $s \in (a,b)$:

$$\frac{\partial^{\delta_1}}{\partial t^{\delta_1}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} = 0, -\frac{\partial^{\delta_1+1}}{\partial t^{\delta_1} \partial s} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} = 0, \dots, (-1)^{\alpha^q+q} \frac{\partial^{\delta_1+\alpha^q+q}}{\partial t^{\delta_1} \partial s^{\alpha^q+q}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} = 0\,.$$

The above equalities are satisfied at $s=a$ and, since $b \neq a$, we have that $\widehat{w}^{(\delta_1)}_{M,q}(b) = 0$. Analogously we obtain that

$$\widehat{w}^{(\delta_2)}_{M,q}(b) = \cdots = \widehat{w}^{(\delta_k)}_{M,q}(b) = 0.$$

Note that if $\alpha^q = 0$ then, from the same argument than before, we deduce that the function $\widehat{w}_{M,q}(s) = \frac{\partial^q}{\partial t^q} g^2_M(a,s)$ satisfies the same boundary conditions:

$$\begin{aligned} \widehat{w}^{(\tau_2)}_{M,q}(a) = \cdots = \widehat{w}^{(\tau_{h-1})}_{M,q}(a) = \widehat{w}^{(\tau_{h+1})}_{M,q}(a) = \quad \cdots \quad &= \widehat{w}^{(\tau_{n-k})}_{M,q}(a) = 0\,, \\ \widehat{w}^{(\delta_2)}_{M,q}(b) = \quad \cdots \quad &= \widehat{w}^{(\delta_k)}_{M,q}(b) \quad = 0\,. \end{aligned}$$

From (2.11) and (2.13) we deduce that

$$\widehat{T}_n[0]\, v(t) := (-1)^n\, T_n^*[0]\, v(t) = T_n[0]\, v(t).$$

Hence, using similar arguments to Step 1, we conclude that $\widehat{w}_{M,q} > 0$ on (a,b) for all $M \in [0, (\lambda_4^*)^q]$, where $(\lambda_4^*)^q > 0$ is the least positive eigenvalue of $\widehat{T}_n[0] \equiv T_n[0]$ in X_4^* defined in (3.12).

From Lemma 3.10, we know that $(\lambda_4^*)^q = \lambda_4^q$. Thus, from this step we have that, if $M \in [0, \lambda_4^q]$, then

$$\forall s \in (a,b)\,, \quad \exists \rho^q(s) > 0 \;\mid\; v_s^q[M](t) > 0 \quad \forall t \in (a, a+\rho^q(s))\,.$$

Step 4. Study of the function $v_s^q[M]$ at $t=b$.

In this case, we study the following function by applying similar reasoning to Step 3

$$\widehat{y}_{M,q}(t) = (-1)^n \frac{\partial^{\beta^q+q}}{\partial\, s^{\beta^q+q}} \widehat{g}^2_{(-1)^n M}(t,s)_{|s=b}\,,$$

which is equivalent, using equality (2.14), to studying the function

$$\widehat{y}_{M,q}(s) = \frac{\partial^{\beta^q+q}}{\partial\, t^{\beta^q+q}} g^1_M(t,s)_{|t=b} = \frac{\partial^{\beta^q}}{\partial\, t^{\beta^q}} \Big(\frac{\partial^q}{\partial t^q} g^2_M(t,s) \Big)_{|t=b}\,.$$

Since $\{0,\dots,\beta^q\} \subset \{\rho_1^q,\dots,\rho_{n-k}\}$ if $\beta^q > 0$, we obtain that

$$\frac{\partial^q}{\partial t^q} g_M(b,t) = \frac{\partial}{\partial t} \Big(\frac{\partial^q}{\partial t^q} g_M(t,s) \Big)_{|t=b} = \cdots = \frac{\partial^{\beta^q-1}}{\partial t^{\beta^q-1}} \Big(\frac{\partial^q}{\partial t^q} g_M(t,s) \Big)_{|t=b} = 0\,.$$

Moreover, using analogous arguments to Step 1 we obtain that $\widehat{y}_{M,q} \geq 0$ if β^q is even and $\widehat{y}_{M,q} \leq 0$ if β^q is odd.

Again, we have that

$$\widehat{T}_n[(-1)^n M]\, \widehat{y}_{M,q}(t) = 0\,, \quad \forall\, t \in [a,b)\,.$$

An analogous study to Step 3 leads us to the fact that $\widehat{y}_{M,q}$ satisfies the boundary conditions

$$\begin{aligned} \widehat{y}_{M,q}^{(\tau_1)}(a) = \cdots = \widehat{y}_{M,q}^{(\tau_{n-k})}(a) &= 0\,, \\ \widehat{y}_{M,q}^{(\delta_1)}(b) = \cdots = \widehat{y}_{M,q}^{(\delta_{z-1})}(b) = \widehat{y}_{M,q}^{(\delta_{z+1})}(b) = \cdots = \widehat{y}_{M,q}^{(\delta_k)}(b) &= 0\,, \\ y_{M,q}^{(\delta_z)}(b) &= (-1)^{n-\beta^q-q+1}\,. \end{aligned} \tag{3.15}$$

Analogously, if $\beta^q = 0$, then $\widehat{y}_{M,q}(s) = \frac{\partial^q}{\partial t^q} g_M^2(b,s)$ satisfies the conditions (3.15).

Thus, we deduce that $\widehat{y}_{M,q} > 0$ on (a,b) if β^q is even and $\widehat{y}_{M,q} < 0$ on (a,b) if β^q is odd for all $M \in [0, (\lambda_2^*)^q]$, where $(\lambda_2^*)^q > 0$ is the least positive eigenvalue of $\widehat{T}_n[0] \equiv T_n[0]$ in X_2^* defined in (3.11).

Again, from Lemma 3.9 we have that $(\lambda_2^*)^q = \lambda_2^q$. Therefore, we conclude from this step that, if $M \in [0, \lambda_2^q]$, then

$$\forall s \in (a,b)\,, \quad \exists \rho^q(s) > 0 \ \mid \ v_s^q[M](t) > 0 \quad \forall t \in (b - \rho^q(t), b)\,.$$

Step 5. Study of the function $v_s^q[M]$ on $(a,b) \times (a,b)$.

In this step we verify that the function $v_s^q[M](t) > 0$ for all $(t,s) \in I \times I$ if M belongs to the given intervals. Let us denote $u_M^s(t) = g_M(t,s)$, for all $s \in (a,b)$.

The function $v_s^q[M](t)$ is in $C^{n-2-q}(I)$ and it satisfies the boundary conditions (2.20)-(2.21).

Using the definition of $v_s^q[M]$, the following equality holds for all $s \in (a,b)$:

$$T_{n-q}[0]\, v_s^q[M](t) = -M\, u_M^s(t)\,, \quad \forall\ t \in I \setminus \{s\}. \tag{3.16}$$

Let us see that for the values of the parameter M for which $v_s^q[M](t)$ has constant sign on I, it cannot have a double zero in (a,b). In particular, this implies that the change of the sign must occur at $t = a$ or $t = b$, and the result would be proven.

We do the proof for $n-k$ even (the case $n-k$ odd is similarly proved). In this case, we study the behavior of $v_s^q[M](t)$ for $M > 0$ and $u_M^s \geq 0$. From (3.16), we have that

$$\frac{d^{n-q}}{dt^{n-q}}\, v_s^q[M](t) = \frac{\partial^n}{\partial t^n}\, g_M(t,s) \leq 0.$$

Therefore, using the condition (T_d), since $v_1 = \cdots = v_n = 1$, we have that

$$\frac{d^{n-q-1}}{dt^{n-q-1}}\, v_s^q[M](t) = \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_M(t,s)$$

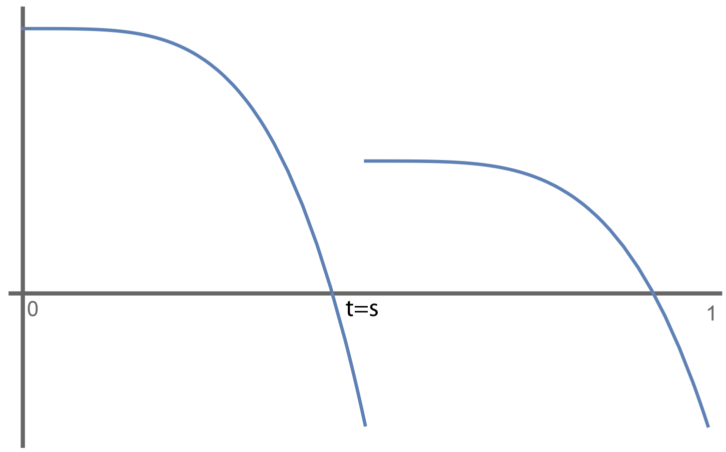

Figure 1: $\dfrac{d^{n-q-1}}{dt^{n-q-1}}\, v_s^q[M](t)$, maximal oscillation with $t \in I = [0,1]$.

is a decreasing function, with two continuous components and a jump $+1$ at $t = s$. Then, it has at most two zeros on I (see Figure 1).

Now, we have that

$$\frac{d^{n-q-2}}{dt^{n-q-2}}\, v_s^q[M](t) = \frac{\partial^{n-2}}{\partial t^{n-2}}\, g_M(t,s)$$

is a continuous function with at most four zeros on I (see Figure 2).

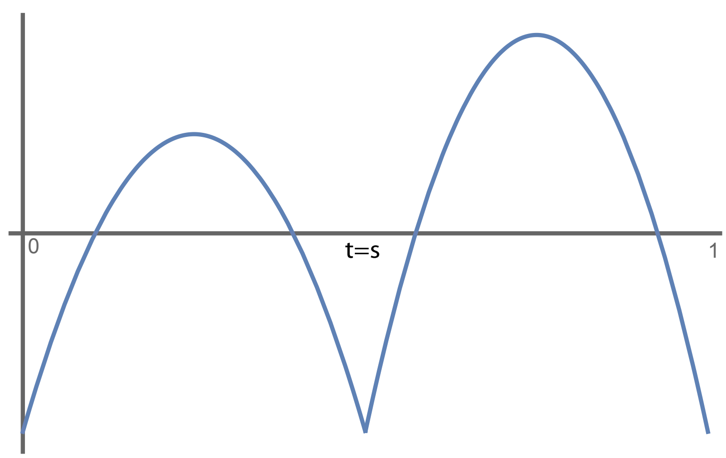

Figure 2: $\dfrac{d^{n-q-1}}{dt^{n-q-1}}\, v_s^q[M](t)$, maximal oscillation with $t \in I = [0,1]$.

By recurrence, we have that

$$\frac{d^{n-q-l}}{dt^{n-q-l}}\, v_s^q[M](t) = \frac{\partial^{n-l}}{\partial t^{n-l}}\, g_M(t,s)$$

has at most $l+2$ zeros on I. In particular, $v_s^q[M]$ has at most $n-q+2$ zeros on I. Moreover, since

$$\begin{aligned}(v_s^q)^{(\sigma_1^q)}[M](a) = \quad \cdots \quad &= (v_s^q)^{(\sigma_{c_q}^q)}[M](a) = 0,\\ (v_s^q)^{(\varepsilon_1^q)}[M](b) = \quad \cdots \quad &= (v_s^q)^{(\varepsilon_{d_q}^q)}[M](b) = 0,\end{aligned}$$

$v_s^q[M]$ has $c_q+d_q = n-q$ zeros on the boundary. Thus, $v_s^q[M]$ has at most two zeros on (a,b).

Now, making an argument similar to the one made in [12, Theorem 5.1], we deduce that the maximal oscillation (that is, the maximal number of zeros) is possible if and only if

$$\begin{cases} (v_s^q)^{(\alpha^q)}[M](a) \le 0\,, & \text{if } n-q-c_q \text{ is even,} \\ \\ (v_s^q)^{(\alpha^q)}[M](a) \ge 0\,, & \text{if } n-q-c_q \text{ is odd.} \end{cases} \tag{3.17}$$

Since we are in the case $n-q-c_q$ even, the maximal oscillation is only possible if $(v_s^q)^{(\alpha^q)}[M](a) \le 0$. However, by definition, $(v_s^q)^{(\alpha^q)}[M](a) \neq 0$ and, since $v_s^q[M]$ is non-negative and α^q denotes the order of the smallest derivative at $t=a$ which is different from zero, necessarily $(v_s^q)^{(\alpha^q)}[M](a) > 0$. Therefore the maximal oscillation can not happen (that is, $v_s^q[M]$ cannot have two zeros on (a,b)). Thus, $v_s^q[M]$ has at most one simple zero on (a,b), which is not possible due to the constant sign of $v_s^q[M]$, and we conclude that $v_s^q[M]$ does not have any zero on (a,b).

From this step we deduce that if $M>0$ and $u_M^s \ge 0$, then $v_s^q[M] > 0$ on (a,b).

In conclusion, from the previous steps the result is proved. □

In the sequel, we present some examples of application of the previous result.

Example 3.12. Consider again the fourth order operator $T_4[M]$ coupled with the boundary conditions $X_{\{0,1,2\}}^{\{2\}}$ defined in Example 2.10.

The functions $\frac{\partial}{\partial t} g_M$ and $\frac{\partial^2}{\partial t^2} g_M$ satisfy the conditions of the spaces $X_{\{0,1,3\}}^{\{1\}}$ and $X_{\{0,2,3\}}^{\{0\}}$, respectively.

In this case $k=3$ and $n-k=1$ is odd. For $q=1$, we have that $c_q=2$, $d_q=1$, $z=2\neq 1$ and $\beta^1=0$. From Theorem 3.11, it follows that $\frac{\partial}{\partial t} g_M$ is strongly negative on $[0,1]\times[0,1]$ if, and only if, $M \in [\lambda_2^1, \lambda_1)$, where

* $\lambda_1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$,
* $\lambda_2^1 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,3\}}^{\{0,1\}}$.

The eigenvalues of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$ are given by λ^4, where λ is a positive solution of

$$\tanh\left(\lambda\sqrt{2}\right) + \tan\left(\frac{\lambda}{\sqrt{2}}\right) = 0.$$

Let us denote by m_1 the smallest positive solution of this equation. Then, $\lambda_1 = m_1^4$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$.

In the same way, the eigenvalues of $T_4[0]$ in $X_{\{0,3\}}^{\{0,1\}}$ are given by $-\lambda^4$, where λ is a positive solution of $\sin(\lambda) = 0$.

In this case, $\lambda_{min}^1 = \pi$ is the smallest positive solution of this equation. Then, $\lambda_2^1 = -(\lambda_{min}^1)^4 = -\pi^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,3\}}^{\{0,1\}}$.

By numerical approach, it can be seen that $\lambda_1 = m_1^4 \approx 3.34^4$.

For $q = 2$, we have that $c_q = n-q-1 = 1$, $d_q = 1$, $z = 1$ and $\beta^2 = 1$. Again, from Theorem 3.11 it follows that $\frac{\partial^2}{\partial t^2} g_M$ is strongly negative on $[0,1] \times [0,1]$ if, and only if, $M \in [\lambda_2^2, \lambda_1)$, where

* $\lambda_1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$,

* $\lambda_2^2 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$.

The eigenvalues of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$ are given by $-\lambda^4$, where λ is the least positive solution of

$$2e^{2\lambda} + (e^{2\lambda} + 1)\cos(\lambda) = 0.$$

Denoting the smallest positive solution of this equation by λ_{min}^2, we have that $\lambda_2^2 = -(\lambda_{min}^2)^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$.

In this case, $\lambda_2^2 = -(\lambda_{min}^2)^4 \approx -1.87^4$.

Example 3.13. Consider the fourth order operator $T_4[M]$ defined in the space of the boundary conditions $X_{\{0,1\}}^{\{1,3\}}$.

In this case $k = 2$ and $n-k$ is even. For $q = 1$ we have that $c_q = 1$, $d_q = n-q-c_q = 2$, $z = 1$, $h = 2$ and $\alpha^1 = \beta^1 = 1$. Moreover, the function $\frac{\partial}{\partial t} g_M$ satisfies the conditions of the space $X_{\{0,3\}}^{\{0,2\}}$.

From Theorem 3.11 we obtain that $\frac{\partial}{\partial t} g_M$ is strongly positive on $[0,1] \times [0,1]$ if, and only if, $M \in (\lambda_1, \lambda^1]$, where

* $\lambda_1 < 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{1,3\}}$,

* $\lambda^1 > 0$ is the minimum between:

  · $\lambda_2^1 > 0$, the least positive eigenvalue of $T_4[0]$ in $X_{\{3\}}^{\{0,1,2\}}$.

  · $\lambda_4^1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,3\}}^{\{0\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -2.36^4$, $\lambda_2^1 \approx 2.22^4$ and $\lambda_4^1 \approx 4.44^4$. Therefore, $\frac{\partial}{\partial t} g_M$ is strongly positive on $[0,1] \times [0,1]$ if, and only if, $M \in (-2.36^4, 2.22^4]$.

As a consequence of Theorem 3.11 we arrive at the following result.

Corollary 3.14. *Let $q \in \{1, \dots, n-1\}$. Suppose that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$. If either $\sigma_k = q + c_q - 1$ or $\varepsilon_{n-k} = q + d_q - 1$, we have the following properties:*

- *If $n - q - c_q$ is even, then there is not any $M \in \mathbb{R}$ such that $v_s^q[M]$ is strongly negative on $I \times I$.*
- *If $n - q - c_q$ is odd, then there is not any $M \in \mathbb{R}$ such that $v_s^q[M]$ is strongly positive on $I \times I$.*

Proof. If $\sigma_k = q + c_q - 1$, then $\eta = n - 1 - \sigma_k = n - q - c_q$.

We consider

$$w_{M,q}(t) = \frac{\partial^\eta}{\partial s^\eta}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a},$$

defined in Step 1 of the proof of Theorem 3.11.

We know that this function satisfies the following boundary conditions:

$$\begin{aligned} w_{M,q}^{(\mu_1^q)}(a) = \cdots = w_{M,q}^{(\mu_{z-1}^q)}(a) = w_{M,q}^{(\mu_{z+1}^q)}(a) = \cdots = w_{M,q}^{(\mu_k^q)}(a) &= 0\,, \\ w_{M,q}^{(\rho_1^q)}(b) = \cdots = w_{M,q}^{(\rho_{n-k}^q)}(b) &= 0\,, \\ w_{M,q}^{(\mu_z^q)}(a) = (-1)^\eta &= (-1)^{n-q-c_q}\,. \end{aligned}$$

Hence, if $n - q - c_q$ is even, then there exists $\rho > 0$ such that $w_{M,q}(t) > 0$ for all $t \in (a, a+\rho)$. So, $v_s^q[M]$ cannot be negative for any real M.

Now, if $n - q - c_q$ is odd, then there exists $\rho > 0$ such that $w_{M,q}(t) < 0$ for all $t \in (a, a+\rho)$. Thus, $v_s^q[M]$ cannot be positive for any $M \in \mathbb{R}$.

Analogously, if $\varepsilon_{n-k} = q + d_q - 1$, then $\gamma = n - \varepsilon_{n-k} - 1 = n - q - d_q = c_q$. In this case, we consider the function

$$y_{M,q}(t) = \frac{\partial^\gamma}{\partial s^\gamma}\Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s)\Big)_{|s=b},$$

defined in Step 2 of the proof of Theorem 3.11.

We know that for all $M \in \mathbb{R}$, the function $y_{M,q}$ satisfies the following boundary conditions:

$$\begin{aligned} y_{M,q}^{(\mu_1^q)}(a) = \cdots = y_{M,q}^{(\mu_k^q)}(a) &= 0\,, \\ y_{M,q}^{(\rho_1^q)}(b) = \cdots = y_{M,q}^{(\rho_{h-1}^q)}(b) = y_{M,q}^{(\rho_{h+1}^q)}(b) = \cdots = y_{M,q}^{(\rho_{n-k}^q)}(b) &= 0\,, \\ y_{M,q}^{(\rho_h^q)}(b) &= (-1)^{c_q+1}\,. \end{aligned}$$

Hence, if $n - q - c_q$ and c_q are even, then there exists $\rho > 0$ such that $y_{M,q}(t) > 0$ for all $t \in (b-\rho, b)$. So, $v_s^q[M]$ cannot be negative for any real M.

Moreover, if $n - q - c_q$ is even and c_q odd, then there exists $\rho > 0$ such that $y_{M,q}(t) < 0$ for all $t \in (b-\rho, b)$. So, $v_s^q[M]$ cannot be positive for any real M.

Now, if $n-q-c_q$ is odd and c_q even, then there exists $\rho>0$ such that $y_{M,q}(t)<0$ for all $t\in(b-\rho,b)$. So, $v_s^q[M]$ cannot be positive for any real M.

Finally, if $n-q-c_q$ and c_q are odd, then there exists $\rho>0$ such that $y_{M,q}(t)>0$ for all $t\in(b-\rho,b)$.

As consequence, $v_s^q[M]$ cannot be negative for any real M. □

Now, we give a necessary condition for the nonpositive (nonnegative) sign of $v_s^q[M]$. In particular, we will give an interval in which we can ensure that $v_s^q[M]$ has nonpositive (nonnegative) sign and whose infimum or supremum is given by the first eigenvalue of $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Theorem 3.15. *Let $q\in\{1,\dots,n-1\}$. Suppose that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q+d_q=n-q$, $c_q\geq 1$ and $d_q\geq 1$. If $\mu_z^q\neq z-1$ and $\rho_h^q\neq h-1$, then the following properties are fulfilled:*

- *Let $n-k$ be even:*
  - *If $n-q-c_q$ is even and $v_s^q[M]$ is nonpositive on $I\times I$, then $M\in[\lambda_*^q,\lambda_1)$, and if $n-q-c_q$ is odd and $v_s^q[M]$ is nonnegative on $I\times I$, then $M\in[\lambda_*^q,\lambda_1)$, where*
    - *$\lambda_1<0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_*^q<0$ is the maximum between:*
      - *$\lambda_3^q<0$ is the biggest negative eigenvalue of $T_n[0]$ in X_3 defined in (3.2).*
      - *$\lambda_5^q<0$, the biggest negative eigenvalue of $T_n[0]$ in X_5 defined in (3.4).*
- *Let $n-k$ be odd:*
  - *If $n-q-c_q$ is even and $v_s^q[M]$ is nonpositive on $I\times I$, then $M\in(\lambda_1,\lambda_*^q]$, and if $n-q-c_q$ is odd and $v_s^q[M]$ is nonnegative on $I\times I$, then $M\in(\lambda_1,\lambda_*^q]$ where*
    - *$\lambda_1>0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_*^q>0$ is the minimum between:*
      - *$\lambda_3^q>0$, the biggest negative eigenvalue of $T_n[0]$ in X_3.*
      - *$\lambda_5^q>0$, the biggest negative eigenvalue of $T_n[0]$ in X_5.*

Proof. Since $\mu_z^q\neq z-1$ and $\rho_h^q\neq h-1$ we have, from Propositions 3.5 and 3.6, that there are eigenvalues λ_3^q and λ_5^q with corresponding related eigenfunctions of constant sign. We suppose that both $n-k$ and $n-q-c_q$ are even. For the other cases the proof is done in an analogous way.

Let us assume that there exists $M^*\notin[\lambda_*^q,\lambda_1)$, such that $v_s^q[M^*]$ is nonpositive on $I\times I$. From Theorem 3.11, we can affirm that $M^*<\lambda_1$. So, $M^*<\lambda_*^q$ and for all $M\in[M^*,\lambda_1)$ the function $v_s^q[M]$ is nonpositive on $I\times I$ because the set where it takes nonpositive values is an interval. By the monotone decreasing character of $v_s^q[M]$ we infer that

$$v_s^q[\lambda_*^q](t)\leq v_s^q[M](t)\leq v_s^q[M^*](t)\leq 0.$$

So, in particular

$$w_{\lambda_*^q,q}(t) \leq w_{M,q}(t) \leq w_{M^*,q}(t) \leq 0\,,$$

and

$$\begin{cases} y_{\lambda_*^q,q}(t) \leq y_{M,q}(t) \leq y_{M^*,q}(t) \leq 0\,, & \text{if } \gamma \text{ is even,} \\ \\ 0 \leq y_{M^*,q}(t) \leq y_{M,q}(t) \leq y_{\lambda_*^q,q}(t)\,, & \text{if } \gamma \text{ is odd.} \end{cases}$$

If $\lambda_*^q = \lambda_3^q$, then $w_{\lambda_*^q,q}^{(\alpha^q)}(a) = 0$. So, we conclude that, for all $M \in [M^*, \lambda_*^q)$, $w_{M,q}^{(\alpha^q)}(a) = 0$, which contradicts the discrete character of the spectrum of $T_n[0]$ in X_3.

If $\lambda_*^q = \lambda_5^q$, then $y_{\lambda_*^q,q}^{(\beta^q)}(b) = 0$. So, we conclude that, for all $M \in [M^*, \lambda_*^q)$, $y_{M,q}^{(\beta^q)}(b) = 0$, which contradicts the discrete character of the spectrum of $T_n[0]$ in X_5.

This way we arrive to a contradiction, and thus the result is proved. □

Next, we present an example to illustrate the previous result.

Example 3.16. Consider the operator $T_5[M]$ defined in the space of the boundary conditions $X_{\{0,2,3\}}^{\{1,3\}}$.

For $q = 1$ we have that $c_q = 2$, $n - q - c_q = d_q = 2$ is even, $z = h = 2$, $\alpha^1 = 0$ and $\beta^1 = 1$. Moreover, the function $\frac{\partial}{\partial t} g_M$ satisfies the conditions of the space $X_{\{1,2,4\}}^{\{0,2\}}$. From Theorem 3.15, if $\frac{\partial}{\partial t} g_M$ is nonpositive on $[0,1] \times [0,1]$, then $M \in [\lambda_*^1, \lambda^1)$, where

* $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_5[0]$ in $X_{\{0,2,3\}}^{\{1,3\}}$,
* $\lambda_*^1 < 0$ is the maximum between:
    - $\lambda_3^1 < 0$, the biggest negative eigenvalue of $T_5[0]$ in $X_{\{0,1,4\}}^{\{0,2\}}$.
    - $\lambda_5^1 < 0$ is the biggest negative eigenvalue of $T_5[0]$ in $X_{\{1,2,4\}}^{\{0,1\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -2.23^5$, $\lambda_3^1 \approx -3.67^5$ and $\lambda_5^1 \approx -2.88^5$. Thus, if $\frac{\partial}{\partial t} g_M$ is nonpositive on $[0,1] \times [0,1]$, then $M \in [-2.88^5, -2.23^5)$.

3.2 Case (b): $c_q = n - q$ and $d_q = 0$

In this subsection we deal with the constant sign of $v_s^q[M]$ for the case $c_q = n - q$ and $d_q = 0$. We arrive to the next theorem.

Theorem 3.17. *Let $q \in \{1, \ldots, n-1\}$. Suppose that $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n - q$, $c_q = n - q$ and $d_q = 0$. The following properties are fulfilled:*

- *If $n - k$ is even, then $v_s^q[M]$ is nonnegative on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*

- *If $n-k$ is odd, then $v_s^q[M]$ is nonnegative on $I \times I$ if, and only if, $M \in [0, \lambda_1)$, where $\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Proof. Let us assume that $n-k$ is even. For the case $n-k$ odd the proof is done in an analogous way. We know, from Theorem 2.31, that $v_s^q \in X_{\{0,\dots,c_q\}}$ satisfies an initial problem and it is nonnegative on $[a,b]$. Moreover, $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) > 0$ if $s < t$ and $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) = 0$ if $s > t$.

Taking into account Lemma 2.33, we obtain that $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) > 0$ on $(a,b) \times (a,b)$ if and only if $M \in (\lambda_1, M_q]$, with $M_q > \lambda_1$. Moreover, $\frac{\partial^q}{\partial t^q} g_M$ is monotone decreasing with respect to $M \in (\lambda_1, M_q]$.

Let us see that $M_q = 0$. Using the monotone decreasing character of $v_s^q[M]$ with respect to $M \in (\lambda_1, M_q]$, since $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) = 0$ if $s > t$, we have that $v_s^q[M_q](t) < v_s^q(t) = 0$ if $M_q > 0$. On the other hand, if $s < t$ then there exists $(t^*, s^*) \in (a,b) \times (a,b)$ with $s^* < t^*$ such that $0 < v_{s^*}^q[M_q](t^*) < v_{s^*}^q(t^*)$ if $M_q > 0$. Thus, $v_s^q[M]$ changes sign for $M_q > 0$ and this completes the proof. □

Example 3.18. Consider the operator $T_4[M]$ coupled with the boundary conditions in the space $X_{\{2,3\}}^{\{0,1\}}$.

In this case $n-k$ is even. For $q = 2$ we have that $c_q = n - q = 2$ and $d_q = 0$. Then, by Theorem 3.17 we have that $\frac{\partial^2}{\partial t^2} g_M$ is nonnegative on $[0,1] \times [0,1]$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -1.8751^4$. Thus, $\frac{\partial^2}{\partial t^2} g_M$ is nonnegative on $[0,1] \times [0,1]$ if, and only if, $M \in (-1.8751^4, 0]$.

For $q = 3$ we have that $c_q = n - q = 1$ and $d_q = 0$. Again, by Theorem 3.17 we have that $\frac{\partial^3}{\partial t^3} g_M$ is nonnegative on $[0,1] \times [0,1]$ if, and only if, $M \in (-1.8751^4, 0]$.

3.3 Case (c): $c_q = 0$ and $d_q = n - q$

In this section, arguing in a similar manner than in the previous one, we can characterize the constant sign of $v_s^q[M]$ for the case $c_q = 0$ and $d_q = n - q$.

Using Theorem 2.31, Lemma 2.33, and reasoning in an analogous way to the proof of Theorem 3.17, we get to the next result.

Theorem 3.19. *Let $q \in \{1, \dots, n-1\}$. Suppose that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n - q$, $c_q = 0$ and $d_q = n - q$. Then, the following properties are fulfilled:*

- *Let $n-k$ be even:*
  - *If $n-q$ is even, then $v_s^q[M]$ is nonnegative on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *If $n-q$ is odd, then $v_s^q[M]$ is nonpositive on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

- *Let $n-k$ be odd:*
  - *If $n-q$ is even, then $v_s^q[M]$ is nonnegative on $I\times I$ if, and only if, $M\in[0,\lambda_1)$, where $\lambda_1>0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *If $n-q$ is odd, then $v_s^q[M]$ is nonpositive on $I\times I$ if, and only if, $M\in[0,\lambda_1)$, where $\lambda_1>0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Example 3.20. Consider the operator $T_6[M]$ coupled with the boundary conditions in the space $X_{\{0,2\}}^{\{1,3,4,5\}}$.

In this case $n-k=4$ is even. For $q=3$ we have that $c_q=0$ and $d_q=n-q=3$ is odd. Then, by Theorem 3.19 we have that $\frac{\partial^3}{\partial t^3}g_M$ is nonpositive on $[0,1]\times[0,1]$ if, and only if, $M\in(\lambda_1,0]$, where $\lambda_1<0$ is the biggest negative eigenvalue of $T_6[0]$ in $X_{\{0,2\}}^{\{1,3,4,5\}}$.

By numerical approximation, we obtain that $\lambda_1\approx-1.953^6$. Thus, $\frac{\partial^3}{\partial t^3}g_M$ is nonpositive on $[0,1]\times[0,1]$ if, and only if, $M\in(-1.953^6,0]$.

For $q=4$ we have that $c_q=0$ and $d_q=n-q=2$ is even. Again, from Theorem 3.19 we have that $\frac{\partial^4}{\partial t^4}g_M$ is nonnegative on $[0,1]\times[0,1]$ if, and only if, $M\in(-1.953^6,0]$.

Finally, for $q=5$ we have that $c_q=0$ and $d_q=n-q=1$ is odd. Then, $\frac{\partial^5}{\partial t^5}g_M$ is nonpositive on $[0,1]\times[0,1]$ if, and only if, $M\in(-1.953^6,0]$.

4 Application to Nonlinear Problems

In this section we will study the existence of a nontrivial positive solution of the following nonlinear problem

$$\begin{cases} T_n[M]u(t)=f\left(t,u(t),\dots,u^{(n-1)}(t)\right),\ t\in I,\\ u^{(\sigma_1)}(a)=\dots=u^{(\sigma_k)}(a)=0,\\ u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0,\end{cases} \tag{4.1}$$

with $f:I\times[0,\infty)\times\mathbb{R}^n\to[0,\infty)$ a continuous function that satisfies some adequate conditions that we will detail later.

The existence of positive solutions of the nonlinear problem (4.1) will be deduced from the index theory applied to compact operators defined in suitable cones.

Suppose that there are r derivatives of constant sign of the Green's function g_M related to the corresponding linear problem of (4.1), with $r\in\{1,\dots,n-1\}$. Let us denote by $q_1,\dots,q_r\in\{1,\dots,n-1\}$ the indices of such derivatives, with

$$q_1<q_2<\dots<q_r.$$

Moreover, let us denote by

$$S:=S_1\cup S_2^I\cup S_2^F\cup\{0\}$$

where

$$S_1:=\{\{q_1,\dots,q_l\}\ /\ \ c_{q_l}+d_{q_l}=n-q_l, c_{q_l}\ge 1\ \text{ and }\ d_{q_l}\ge 1\},$$

$$S_2^I := \{\{q_{l+1},\dots,q_r\} \ / \ c_{q_{l+1}} = n - q_{l+1} \text{ and } d_{q_{l+1}} = 0\},$$
$$S_2^F := \{\{q_{l+1},\dots,q_r\} \ / \ c_{q_{l+1}} = 0 \text{ and } d_{q_{l+1}} = n - q_{l+1}\},$$

and $\{0\}$ denoting the constant sign of the Green's function g_M.

It is clear that either $S_2^I = \varnothing$ or $S_2^F = \varnothing$. We assume that $S_2^I = \varnothing$ and $S_2^F \neq \varnothing$ (for the other cases, i.e., $S_2^F = \varnothing$ and either $S_2^I = \varnothing$ or $S_2^I \neq \varnothing$, the arguments are analogous).

Along this section, we will assume that $n-k$ is even (for $n-k$ odd an analogous reasoning could be made). We know, from Lemma 2.32 and Theorem 3.11, that for any $q \in S_1$, $\frac{\partial^q}{\partial t^q} g_M$ has constant sign for all $M \in (\lambda_1, \lambda^{q_l}]$ (the constant sign interval of the largest derivative of S which belongs to S_1), with λ_1 and λ^{q_l} characterized in Theorem 3.11.

Moreover, for all $M \in (\lambda_1, \lambda^{q_l}]$ we have that

$$g_M(t,s) \geq 0 \text{ and } (-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s) \geq 0 \text{ for all } i \in \{1,\dots,l\},$$

where $d_{q_i} = n - q_i - c_{q_i}$ for $i = 1,\dots,l$.

Let us consider the following condition introduced in [13, page 182] as follows:

(P_{g_1}) Suppose that there are continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a,b)$ and $0 < k_1(t) < k_2(t)$ for all $t \in (a,b)$, satisfying:

$$\phi(s)\, k_1(t) \leq G(t,s) \leq \phi(s)\, k_2(t)\,, \quad \text{for all } (t,s) \in I \times I,$$

where G is a suitable integral kernel of certain integral operator.

Using the characterization of [12, Theorem 8.1] and Theorem 3.11, with a similar argument to the one made in [13, Theorem 5.2], the following result is proved.

Lemma 4.1. *Suppose that $n-k$ is even. Then, for any $M \in (\lambda_1, \lambda^{q_l})$, the functions g_M, $(-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M$, $i \in \{1,\dots,l\}$, satisfy the condition (P_{g_1}), that is, there exist continuous functions ϕ, k_1, k_2, $0 < k_1(t) < k_2(t)$ for all $t \in (a,b)$, satisfying*

$$\phi(s)\, k_1(t) \leq g_M(t,s) \leq \phi(s)\, k_2(t), \quad \text{for all } (t,s) \in I \times I,$$

and $k_1^{q_i}$, $k_2^{q_i}$, with $0 < k_1^{q_i}(t) < k_2^{q_i}(t)$ for all $t \in (a,b)$ and $i \in \{1,\dots,l\}$ such that

$$\phi(s)\, k_1^{q_i}(t) \leq (-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s)(t,s) \leq \phi(s)\, k_2^{q_i}(t), \quad \text{for all } (t,s) \in I \times I, \quad i = 1,\dots,l,$$

where $\phi(s) = (s-a)^\eta\,(b-s)^\gamma$, with η and γ defined in (2.15) *and* (2.16).

Proof. We make the proof only for the function g_M. The same argument holds for the functions $(-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M$, $i = 1,\dots,l$, by taking the same function ϕ and using Steps 1 and 2 of Theorem 3.11.

For all $M \in (\lambda_1, \lambda^{q_l})$, we know from Steps 1 and 2 of [12, Theorem 8.1] that

$$\frac{\partial^\eta}{\partial t^\eta} g_M(t,s)_{|s=a} > 0 \quad \text{and} \quad (-1)^\gamma \frac{\partial^\gamma}{\partial t^\gamma} g_M(t,s)_{|s=b} > 0, \text{ for all } t \in (a,b).$$

Let us define the following function

$$v_M^t(s) = \frac{g_M(t,s)}{(s-a)^\eta\,(b-s)^\gamma}.$$

It is clear that $v_M^t(s) > 0$ on $(a,b)\times(a,b)$ for all $M \in (\lambda_1, \lambda^{q_l})$. Moreover, for each $t \in (a,b)$ we have that

$$h_1(t) = \lim_{s\to a^+} \frac{g_M(t,s)}{(s-a)^\eta\,(b-s)^\gamma} = \frac{\frac{\partial^\eta}{\partial t^\eta} g_M(t,s)_{|s=a}}{\eta!\,(b-a)^\gamma} \in (0,\infty)\,,$$

$$h_2(t) = \lim_{s\to b^-} \frac{g_M(t,s)}{(s-a)^\eta\,(b-s)^\gamma} = \frac{(-1)^\gamma\,\frac{\partial^\gamma}{\partial t^\gamma} g_M(t,s)_{|s=b}}{\gamma!\,(b-a)^\eta} \in (0,\infty).$$

For each $t \in (a,b)$, let us define $\tilde{v}_M^t$ as the continuous extension of v_M^t to the interval I, that is

$$\tilde{v}_M^t(s) = \begin{cases} h_1(t)\,, & s = a, \\ v_M^t(s)\,, & s \in (a,b), \\ h_2(t)\,, & s = b. \end{cases}$$

Then, $\tilde{v}_M^t(s) > 0$ on $[a,b]$ for all $t \in (a,b)$. Therefore, the following functions

$$k_1(t) = \min_{s\in I} \tilde{v}_M^t(s)\,, \quad t \in I,$$

$$k_2(t) = \max_{s\in I} \tilde{v}_M^t(s)\,, \quad t \in I,$$

are continuous on I and positive on (a,b).

Taking $\phi(s) = (s-a)^\eta\,(b-s)^\gamma > 0$ on (a,b), the function g_M satisfies the condition (P_{g_1}). □

Let us consider $I_1 = [a_1, b_1]$ and $I_1^{q_i} = [a_1^{q_i}, b_1^{q_i}]$, $i = 1,\ldots,l$, subintervals of I such that $|k_1(t)| > 0$ for all $t \in I_1$ and $|k_1^{q_i}(t)| > 0$ for all $t \in I_1^{q_i}$. Notice that, due to their continuity on I, such functions have constant sign on their corresponding intervals. Furthermore, let us define

$$k_1 = \max_{t\in I} |k_1(t)|,\ \ m_1 = \min_{t\in I_1} |k_1(t)|,\ \ k_2 = \max_{t\in I} |k_2(t)|,$$

$$k_1^{q_i} = \max_{t\in I} |k_1^{q_i}(t)|,\ \ k_2^{q_i} = \max_{t\in I} |k_2^{q_i}(t)|,$$

for $i = 1,\ldots,l$.

We assume that the nonlinear part of equation satisfies the following conditions:

(H_1) $$\liminf_{\max\{|x_1|,\ldots,|x_n|\}\to 0}\ \min_{t\in I} \frac{f(t,x_1,\ldots,x_n)}{|x_1|+\cdots+|x_n|} = +\infty.$$

(H_2) $$\limsup_{\min\{|x_1|,\ldots,|x_n|\}\to \infty}\ \max_{t\in I} \frac{f(t,x_1,\ldots,x_n)}{|x_1|+\cdots+|x_n|} = 0.$$

Let $X \equiv (C^n(I), ||\cdot||)$ be the real Banach space endowed with the norm

$$||u|| = \max\{||u||_\infty, \dots, ||u^{(n)}||_\infty\}, \text{ for all } u \in X,$$

where $||u||_\infty = \max_{t\in I}\{|u(t)|\}$, and consider operator $\mathcal{L}_M : X \to X$ defined as

$$\mathcal{L}_M\, u(t) := \int_a^b g_M(t,s)\, f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds, \quad t \in I. \tag{4.2}$$

We will apply the fixed point index theory to the operator $\mathcal{L}_M$ to guarantee the existence of a fixed point of such operator.

Let us define the following cone:

$$K = \left\{ u \in X : \begin{array}{l} u(t) \geq 0, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq 0,\, i \in \{1,\dots,l\},\, (-1)^{n-q_j}\, u^{(q_j)}(t) \geq 0,\, j \in \{l+1,\dots,r\},\, t \in I, \\ u(t) \geq \dfrac{k_1(t)}{k_2}\|u\|_\infty,\, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq \dfrac{k_1^{q_i}(t)}{k_2^{q_i}}\|u^{(q_i)}\|_\infty,\, i \in \{1,\dots,l\},\, t \in I \end{array} \right\}.$$

***Remark* 4.2.** Note that if $S_2^F = \varnothing$ and $S_2^I \neq \varnothing$ we take the following cone defined as:

$$K^* = \left\{ u \in X : \begin{array}{l} u(t) \geq 0, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq 0,\, i \in \{1,\dots,l\},\, u^{(q_j)}(t) \geq 0,\, j \in \{l+1,\dots,r\},\, t \in I, \\ u(t) \geq \dfrac{k_1(t)}{k_2}\|u\|_\infty,\, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq \dfrac{k_1^{q_i}(t)}{k_2^{q_i}}\|u^{(q_i)}\|_\infty,\, i \in \{1,\dots,l\},\, t \in I \end{array} \right\}.$$

On the other hand, if $S_2^I = \varnothing$ and $S_2^F = \varnothing$ we have that $l = r$ and we work with the cone defined as:

$$\bar{K} = \left\{ u \in X : \begin{array}{l} u(t) \geq 0, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq 0,\, i \in \{1,\dots,l\},\, t \in I, \\ u(t) \geq \dfrac{k_1(t)}{k_2}\|u\|_\infty,\, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq \dfrac{k_1^{q_i}(t)}{k_2^{q_i}}\|u^{(q_i)}\|_\infty,\, i \in \{1,\dots,l\},\, t \in I \end{array} \right\}.$$

***Remark* 4.3.** Note that we can take the following family of cones $K_{q_m} \subset K$ defined as:

$$K_{q_m} = \left\{ u \in X : \begin{array}{l} u(t) \geq 0, \dots, (-1)^{d_{q_m}}\, u^{(q_m)}(t) \geq 0,\, t \in I, \\ u(t) \geq \dfrac{k_1(t)}{k_2}\|u\|_\infty, \dots, (-1)^{d_{q_m}}\, u^{(q_m)}(t) \geq \dfrac{k_1^{q_m}(t)}{k_2^{q_m}}\|u^{(q_m)}\|_\infty,\, t \in I \end{array} \right\},$$

if $0 \leq m \leq l$ and

$$K_{q_m} = \left\{ u \in X : \begin{array}{l} u(t) \geq 0, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq 0,\, i \in \{1,\dots,l\},\, (-1)^{n-q_j}\, u^{(q_j)}(t) \geq 0,\, j \in \{l+1,\dots,m\},\, t \in I, \\ u(t) \geq \dfrac{k_1(t)}{k_2}\|u\|_\infty,\, (-1)^{d_{q_i}}\, u^{(q_i)}(t) \geq \dfrac{k_1^{q_i}(t)}{k_2^{q_i}}\|u^{(q_i)}\|_\infty,\, i \in \{1,\dots,l\},\, t \in I \end{array} \right\}$$

if $l < m \leq r$.

***Remark* 4.4.** If $n-k$ is odd, using a similar argument to Lemma 4.1, we obtain that functions g_M and $(-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M$, $i \in \{1,\dots,l\}$, satisfy the following condition (with obvious notation):

(N_{g_1}) Suppose that there are continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a,b)$ and $k_1(t) < k_2(t) < 0$ for all $t \in (a,b)$, satisfying:

$$\phi(s)\, k_1(t) \le G(t,s) \le \phi(s)\, k_2(t)\,, \quad \text{for all } (t,s) \in I \times I\,.$$

In this case, we would look for a nontrivial positive solution of the following nonlinear problem

$$\begin{cases} T_n[M]u(t) + f\left(t, u(t), \dots, u^{(n-1)}(t)\right) = 0, \ t \in I, \\ u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \end{cases}$$

In order to prove an existence result of problem 4.1, we introduce the following definition and preliminary results.

Definition 4.5. Let X be a Banach space, $\Omega \subset X$ open and $T : \Omega \to X$ a continuous map. We say that T is compactly fixed if the set of fixed points of T is compact.

We will denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

Next lemma compiles some classical results regarding the fixed point index formulated in [20, Theorems 6.2, 7.3 and 7.11] in a more general framework.

In particular, given X a Banach space, $K \subset X$ a cone and $\Omega \subset K$ an arbitrary open subset, $\partial\,\Omega$ will denote the boundary of Ω in the relative topology in K, induced by the topology of X.

Lemma 4.6. *Let X be a Banach space, $K \subset X$ a cone and $\Omega \subset K$ an arbitrary open subset with $0 \in \Omega$. Assume that $T\colon \overline{\Omega} \to K$ is a compact and compactly fixed operator such that $x \ne Tx$ for all $x \in \partial\,\Omega$.*

Then the fixed point index $i_K(T,\Omega)$ has the following properties:

1. *If $x \ne \mu\, Tx$ for all $x \in \partial\,\Omega$ and for every $\mu \le 1$, then $i_K(T,\Omega) = 1$.*
2. *If Ω is bounded and there exists $e \in K \setminus \{0\}$ such that $x \ne Tx + \lambda\, e$ for all $x \in \partial\,\Omega$ and all $\lambda > 0$, then $i_K(T,\Omega) = 0$.*
3. *If $i_K(T,\Omega) \ne 0$, then T has a fixed point in Ω.*
4. *If Ω_1 and Ω_2 are two open and disjoint sets such that* $\mathrm{Fix}(T) \subset \Omega_1 \cup \Omega_2 \subset \Omega$*, then*

$$i_K(T,\Omega) = i_K(T,\Omega_1) + i_K(T,\Omega_2).$$

***Remark* 4.7.** Note that, in Item 2 in previous lemma, it is required that Ω is bounded. However, the other assertions hold for an arbitrary open set, which might be unbounded.

Using Items 1 and 2 in Lemma 4.6, it is possible to deduce the following corollary. The proof would be analogous to that of [21, Theorem 2.3.3].

Corollary 4.8. *Let X be a Banach space, $K \subset X$ a cone and $\Omega \subset K$ an open set such that $0 \in \Omega$. Assume that $T\colon \overline{\Omega} \to K$ is a compact and compactly fixed operator such that $x \neq Tx$ for all $x \in \partial\,\Omega$. Then*

1. *If $T\,x \not\geq x$ for all $x \in \partial\,\Omega$ then $i_K(T, \Omega) = 1$.*
2. *If Ω is bounded and, moreover, $T\,x \not\leq x$ for all $x \in \partial\,\Omega$, then $i_K(T, \Omega) = 0$.*

Next, we prove the following theorem to ensure the existence of positive solutions.

Theorem 4.9. *Suppose that conditions (H_1) and (H_2) hold. Then, the nonlinear problem* (4.1) *has at least a nontrivial solution $u \in K$ for all $M \in (\lambda_1, \bar{\lambda}]$, where*

$$\bar{\lambda} := \begin{cases} 0, & \text{if } l < r, \\ \lambda^{q_l} & \text{if } l = r. \end{cases}$$

Proof. Consider the operator $\mathcal{L}_M$ defined in (4.2). Since $g_M \geq 0$ for all $M \in (\lambda_1, \bar{\lambda}]$, $f \geq 0$ and the fixed points of the operator $\mathcal{L}_M$ coincide with the solutions of problem (4.1), we deduce that these solutions are nonnegative.

We show that $\mathcal{L}_M$ is well-defined in K, that is, $\mathcal{L}_M(K) \subset K$.

Let $u \in K$, it is immediate to verify that $\mathcal{L}_M\,u \geq 0$ on I. Moreover, we have that:

$$(-1)^{d_{q_i}}\,(\mathcal{L}_M\,u)^{(q_i)}(t) = \int_a^b (-1)^{d_{q_i}}\,\frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s)\;f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\,ds \geq 0,$$

$$\begin{aligned}
\mathcal{L}_M\,u(t) :&= \int_a^b g_M(t,s)\;f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\,ds \\
&\geq \int_a^b k_1(t)\,\phi(s)\;f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\,ds \\
&= \frac{k_1(t)}{k_2}\int_a^b k_2\,\phi(s)\;f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\,ds \\
&\geq \frac{k_1(t)}{k_2}\int_a^b \left\{\max_{t\in I} g_M(t,s)\right\}\;f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\,ds \\
&\geq \frac{k_1(t)}{k_2}\max_{t\in I}\left\{\int_a^b g_M(t,s)\,f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\,ds\right\} = \frac{k_1(t)}{k_2}\,\|\mathcal{L}_M\,u\|_\infty,
\end{aligned}$$

and

$$
\begin{aligned}
(-1)^{d_{q_i}} \left(\mathcal{L}_M\, u\right)^{(q_i)}(t) &= \int_a^b (-1)^{d_{q_i}} \, \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s) \; f\left(s, u(s), \dots, u^{(n-1)}(s)\right) \, ds \\
&\geq \int_a^b k_1^{q_i}(t)\, \phi(s) \; f\left(s, u(s), \dots, u^{(n-1)}(s)\right) \, ds \\
&= \frac{k_1^{q_i}(t)}{k_2^{q_i}} \int_a^b k_2^{q_i}\, \phi(s) \; f\left(s, u(s), \dots, u^{(n-1)}(s)\right) \, ds \\
&\geq \frac{k_1^{q_i}(t)}{k_2^{q_i}} \int_a^b \left\{ \sup_{t\in I} (-1)^{d_{q_i}} \, \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s) \right\} f\left(s, u(s), \dots, u^{(n-1)}(s)\right) \, ds \\
&\geq \frac{k_1^{q_i}(t)}{k_2^{q_i}} \sup_{t\in I} \left\{ \int_a^b (-1)^{d_{q_i}} \, \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s)\, f\left(s, u(s), \dots, u^{(n-1)}(s)\right) \, ds \right\} \\
&= \frac{k_1^{q_i}(t)}{k_2^{q_i}} \, \|(-1)^{d_{q_i}} \left(\mathcal{L}_M\, u\right)^{(q_i)}\|_\infty = \frac{k_1^{q_i}(t)}{k_2^{q_i}} \, \|\left(\mathcal{L}_M\, u\right)^{(q_i)}\|_\infty,
\end{aligned}
$$

for all $t \in I$ and $i \in \{1, \dots, l\}$.

On the other hand, for $j \in \{l+1, \dots, r\}$ and $t \in I$, we have that

$$
(-1)^{n-q_j} \left(\mathcal{L}_M\, u\right)^{(q_j)}(t) = \int_a^b (-1)^{n-q_j} \, \frac{\partial^{q_j}}{\partial t^{q_j}} g_M(t,s) \; f\left(s, u(s), \dots, u^{(n-1)}(s)\right) \, ds \geq 0.
$$

Thus, $\mathcal{L}_M(K) \subset K$.

By the continuity of f and using standard techniques, one can prove that operator $\mathcal{L}_M$ is compact.

Consider $u \in K \cap \partial\Omega_1$. Let us choose

$$
\epsilon_1 > \frac{k_2}{m_1^2 \displaystyle\int_{a_1}^{b_1} \phi(s)\, ds}.
$$

From assumption (H_1), there exists $p > 0$ such that when $\|u\| < p$ we have that

$$
f\left(t, u(t), \dots, u^{(n-1)}(t)\right) \geq \epsilon_1 \left(|u(t)| + \cdots + |u^{(n-1)}(t)|\right), \quad \text{for all } \; t \in I.
$$

Now, we show that $\mathcal{L}_M\, u \not\preceq u$ for all $u \in K \cap \partial\Omega_1$, with

$$
\Omega_1 = \{u \in K : \; \|u\| < p\},
$$

for some $p > 0$. Here $\preceq$ denotes the order induced by the cone K.

Then, we have that for $t \in I_1$,

$$
\begin{aligned}
\mathcal{L}_M\, u(t) &= \int_a^b g_M(t,s)\; f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \\
&\geq \int_a^b k_1(t)\,\phi(s)\; f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \\
&\geq \int_{a_1}^{b_1} k_1(t)\,\phi(s)\; f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \\
&\geq m_1 \int_{a_1}^{b_1} \phi(s)\; f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \\
&\geq m_1\, \epsilon_1 \int_{a_1}^{b_1} \phi(s)\; \left(|u(s)| + \cdots + |u^{(n-1)}(s)|\right)\, ds \\
&\geq \frac{m_1\,\epsilon_1}{k_2} \int_{a_1}^{b_1} \phi(s)\, k_1(s)\, \|u\|_\infty\, ds \\
&\geq \frac{m_1^2\,\epsilon_1}{k_2}\, \|u\|_\infty \int_{a_1}^{b_1} \phi(s)\, ds > \|u\|_\infty.
\end{aligned}
$$

Therefore, $\mathcal{L}_M\, u(t) > u(t)$ for all $t \in I_1$ and so it is proved that $\mathcal{L}_M\, u \not\leq u$ for all $u \in K \cap \partial\Omega_1$. As a consequence (see [21, Theorem 2.3.3]), we have that

$$i_K(\mathcal{L}_M,\, \Omega_1) = 0.$$

On the other hand, the regularity of the Green's function g_M allows us guarantee that there exist $N_0, N_{q_i} \in \mathbb{R}$, $i \in \{1, \dots, r\}$, with $N_0 > 0$ and $N_{q_i} > 0$, such that

$$\max_{t \in I} \int_a^b g_M(t,s)\, ds \leq N_0 \;\text{ and }\; \sup_{t \in I} \int_a^b (-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s)\, ds \leq N_{q_i}, \quad i \in \{1, \dots, r\}.$$

Let us choose

$$\epsilon_2 < \min\left\{ \frac{1}{n\, N_0}, \frac{1}{n\, N_{q_i}} : \quad i \in \{1, \dots, r\} \right\}.$$

By hypothesis (H_2), there exists $\tilde{M} > 0$ such that if $\min\left\{|u(t)|, \dots, |u^{(n-1)}(t)|\right\} \geq \tilde{M}$ we have that

$$f\left(t, u(t), \dots, u^{(n-1)}(t)\right) \leq \epsilon_2 \left(|u(t)| + \cdots + |u^{(n-1)}(t)|\right) \leq n\, \epsilon_2\, \|u\|, \;\text{ for all }\; t \in I.$$

Consider $q > \{p, \tilde{M}\}$ and let us define the following subset of K

$$\Omega_2 = \bigcup_{i=0}^{r} \left\{ u \in K : \min_{t \in I} |u^{(q_i)}(t)| < q \right\}.$$

Since the set Ω_2 is unbounded in the cone K, as we have pointed out earlier, the fixed point index of operator $\mathcal{L}_M$ with respect to Ω_2 is only defined in the case that the set of fixed points of operator $\mathcal{L}_M$ in Ω_2, that is, $(\mathrm{id} - \mathcal{L}_M)^{-1}(\{0\}) \cap \Omega_2$, is compact.

Arguing as in the proof of [9, Theorem 3.2], we can assume that $i_K(\mathcal{L}_M, \Omega_2)$ can be defined in this case. Indeed, since $\mathrm{id} - \mathcal{L}_M$ is a continuous operator, it is obvious that $(\mathrm{id} - \mathcal{L}_M)^{-1}(\{0\}) \cap \Omega_2$ is closed. Moreover, if such set is unbounded, we would have infinite fixed points of operator $\mathcal{L}_M$ on Ω_2 and, as a direct consequence, problem (4.1) would have an infinite number of positive solutions on Ω_2. Thus, we may assume that such set is bounded, and, from this hypothesis, it is not difficult to verify that $(\mathrm{id} - \mathcal{L}_M)^{-1}(\{0\}) \cap \Omega_2$ is equicontinuous.

Now, we prove that $\|\mathcal{L}_M\, u\| \le \|u\|$ for all $u \in K \cap \partial\Omega_2$.

If $u \in K \cap \partial\Omega_2$, then

$$\min\left\{\min_{t\in I} |u^{(q_i)}(t)| \,:\, i \in \{0,\dots,r\}\right\} = q > \tilde{M}.$$

Therefore,

$$\begin{aligned}\|\mathcal{L}_M\, u\|_\infty &= \max_{t\in I} \int_a^b g_M(t,s)\, f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \le n\,\epsilon_2\, \|u\| \max_{t\in I} \int_a^b g_M(t,s)\, ds \\ &\le n\,\epsilon_2\, \|u\|\, N_0 < \|u\|,\end{aligned}$$

and

$$\begin{aligned}\|(\mathcal{L}_M\, u)^{(q_i)}\|_\infty &= \sup_{t\in I} \left| \int_a^b \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s)\, f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \right| \\ &\le n\,\epsilon_2\, \|u\| \sup_{t\in I} \int_a^b (-1)^{d_{q_i}} \frac{\partial^{q_i}}{\partial t^{q_i}} g_M(t,s)\, f\left(s, u(s), \dots, u^{(n-1)}(s)\right)\, ds \\ &\le n\,\epsilon_2\, \|u\|\, N_{q_i} < \|u\|,\end{aligned}$$

for all $i \in 1, \dots, r$.

Thus, $\|\mathcal{L}_M\, u\| < \|u\|$ for all $u \in K \cap \partial\Omega_2$ and, as a consequence (see [20, Corollary 7.4]), we deduce that

$$i_K(\mathcal{L}_M,\, \Omega_2) = 1.$$

Therefore, we conclude that the operator $\mathcal{L}_M$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ which is a solution of problem (4.1).

Thus, in both situations ($(\mathrm{id} - \mathcal{L}_M)^{-1}(\{0\}) \cap \Omega_2$ bounded or unbounded), we may ensure that there is a nontrivial solution of problem (4.1) on the cone K. □

In the sequel, we present an example to illustrate our result.

Example 4.10. Let us consider again the space studied in Example 3.13: $X_{\{0,1\}}^{\{1,3\}}$. In this case, we have from [12, Theorem 8.1] that $g_M \ge 0$ on $[0,1]\times[0,1]$ if, and only if, $M \in (\lambda_1, \lambda_2]$, where

* $\lambda_1 < 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{1,3\}}$,

* $\lambda_2 > 0$ is the minimum between:

  · $\lambda_2' > 0$, the least positive eigenvalue of $T_4[0]$ in $X_{\{0\}}^{\{0,1,3\}}$.

  · $\lambda_2'' > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{1\}}$.

Since ${X^*}_{\{0,1,2\}}^{\{1\}} = X_{\{0\}}^{\{0,1,3\}}$, we have that $X_{\{0\}}^{\{0,1,3\}}$ and $X_{\{0,1,2\}}^{\{1\}}$ have the same eigenvalues. So, $\lambda_2' = \lambda_2''$. The eigenvalues of $T_4[0]$ in $X_{\{0\}}^{\{0,1,3\}}$ are given by λ^4, where λ is a positive solution of the equation $\sin\left(\frac{m}{\sqrt{2}}\right) = 0$.

Then, $m_1 = \pi\sqrt{2}$ is the smallest positive solution of this equation and so $\lambda_2' = \lambda_2'' = 4\pi^4 \approx 389.636$ is the least positive solution of $T_4[0]$ in $X_{\{0\}}^{\{0,1,3\}}$.

By numerical approach, we have that $\lambda_1 \approx -2.36^4 \approx -31.2852$. Thus, $g_M \geq 0$ on $[0,1] \times [0,1]$ if, and only if, $M \in (-2.36^4, 4\pi^4]$.

Therefore, the function g_M satisfies the property (P_{g_1}) for all $M \in (-2.36^4, 4\pi^4]$.

We know from Example 3.13 that $\frac{\partial}{\partial t} g_M(t,s) \geq 0$ on $[0,1]\times[0,1]$ if, and only if, $M \in (-2.36^4, 2.22^4]$. So, the function $\frac{\partial}{\partial t} g_M$ satisfies the property (P_{g_1}) for all $M \in (-2.36^4, 2.22^4]$.

In particular, both $g_0(t,s)$ and $\frac{\partial}{\partial t} g_0(t,s)$ satisfy the property (P_{g_1}).

The function $\frac{\partial^2}{\partial t^2} g_M(t,s)$ changes sign because $\frac{\partial}{\partial t} g_M(t,s)$ vanishes at points $t = 0$ and $t = 1$.

On the other hand, we have from Theorem 3.19 that $\frac{\partial^3}{\partial t^3} g_M(t,s)$ is nonpositive on $[0,1] \times [0,1]$ if, and only if, $M \in (\lambda_1, 0]$. Thus, $\frac{\partial^3}{\partial t^3} g_M(t,s) \leq 0$ on $[0,1]\times[0,1]$ if, and only if, $M \in (-2.36^4, 0]$.

Let us study the following nonlinear problem for the particular case of $M = 0$

$$\begin{cases} u^{(4)}(t) = (t^4+1)\left(e^{-\sqrt{(u(t))^2+(u'(t))^2+(u''(t))^2+(u'''(t))^2}} + \dfrac{1}{\ln(e+|u(t)|+|u'(t)|+|u''(t)|+|u'''(t)|)}\right), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ u'(1) = u'''(1) = 0. \end{cases} \tag{4.3}$$

In this case, the function $f(t,x_1,x_2,x_3,x_4) = (t^4+1)\left(e^{-\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} + \frac{1}{\ln(e+|x_1|+|x_2|+|x_3|+|x_4|)}\right)$ is continuous and satisfies conditions (H_1) and (H_2).

From the adjoint boundary conditions, we have that $\eta = 2$ and $\gamma = 0$ and so $\phi(s) = s^2$.

By direct integration, we calculate the expressions of the Green's functions and we have that $g_0(t,s)$ is given by

$$g_0(t,s) = \begin{cases} -\dfrac{1}{12} s^2 \left(2s + 3(t-2)t\right), & 0 \leq s \leq t \leq 1, \\ \dfrac{1}{12} t^2 \left(-3(s-2)s - 2t\right), & 0 \leq t < s \leq 1. \end{cases}$$

Moreover, we have that

$$\tilde{v}(t,s) = \begin{cases} -\dfrac{1}{12} \left(2s + 3(t-2)t\right), & 0 \leq s \leq t < 1, \\ \dfrac{1}{12} \dfrac{t^2}{s^2} \left(-3(s-2)s - 2t\right), & 0 < t < s \leq 1,\ t \neq 1, \end{cases}$$

where $\tilde{v}(t,s)$ is the continuous extension of $\frac{g_0(t,s)}{\phi(s)}$ to $(0,1)\times[0,1]$.

Since

$$\frac{\partial}{\partial s}\tilde{v}(t,s) = \begin{cases} -\dfrac{1}{6}, & 0 \le s \le t \le 1, \\ \dfrac{1}{6}\,\dfrac{t^2}{s^3}\,(2t-3s)\,, & 0 < t < s \le 1, \end{cases}$$

is negative for all $(t,s)\in(0,1)\times[0,1]$, we have that $\tilde{v}(t,s)$ is decreasing as a function of s for all $t\in(0,1)$. So,

$$k_1(t) = \min_{s\in I}\tilde{v}(t,s) = \tilde{v}(t,1) = \frac{t^2}{12}\,(3-2t)\,,$$

$$k_2(t) = \max_{s\in I}\tilde{v}(t,s) = \tilde{v}(t,0) = \frac{1}{4}\,t\,(2-t)\,.$$

Taking into account that these two functions are increasing on $[0,1]$, we have that

$$k_1 = k_1(1) = \frac{1}{6} \text{ and } k_2 = k_2(1) = \frac{1}{4}.$$

Let us choose $I_1 = [\frac{1}{6},1]$. Then, we have that

$$m_1 = \min_{t\in I_1} k_1(t) = k_1\left(\frac{1}{6}\right) = \frac{1}{162}.$$

The first derivative of $g_0(t,s)$ is given by the expression

$$\frac{\partial}{\partial t}\,g_0(t,s) = \begin{cases} \dfrac{1}{2}\,s^2\,(1-t)\,, & 0 \le s \le t \le 1, \\ \dfrac{1}{2}\,t\,(s\,(2-s)-t)\,, & 0 \le t < s \le 1. \end{cases}$$

Analogously, it is not difficult to verify that

$$\tilde{v}^1(t,s) = \begin{cases} \dfrac{1}{2}\,(1-t)\,, & 0 \le s \le t < 1, \\ \dfrac{1}{2}\,\dfrac{t}{s^2}\,(s\,(2-s)-t)\,, & 0 < t < s \le 1, \end{cases}$$

where $\tilde{v}^1(t,s)$ is the continuous extension of $\frac{\frac{\partial}{\partial t}g_0(t,s)}{\phi(s)}$ to $(0,1)\times[0,1]$.

Since

$$\frac{\partial}{\partial s}\tilde{v}^1(t,s) = \begin{cases} -\dfrac{1}{2}, & 0 \le s \le t \le 1, \\ \dfrac{t}{s^3}\,(t-s)\,, & 0 < t < s \le 1, \end{cases}$$

is negative for all $(t,s)\in(0,1)\times[0,1]$, we have that $\tilde{v}^1(t,s)$ is decreasing as a function of s for all $t\in(0,1)$. So,

$$k_1^1(t) = \min_{s\in I}\tilde{v}^1(t,s) = \tilde{v}^1(t,1) = \frac{1}{2}\,t\,(1-t)\,,$$

$$k_2^1(t) = \max_{s\in I}\tilde{v}^1(t,s) = \tilde{v}^1(t,0) = \frac{1}{2}\,(1-t)\,.$$

In this case, we have that

$$k_1^1 = k_1^1\left(\frac{1}{2}\right) = \frac{1}{6} \text{ and } k_2^1 = k_2^1(0) = \frac{1}{2}.$$

Therefore, the cone K that we use to localize the solution is given by

$$K = \left\{ u \in C^3(I) : \begin{array}{l} u(t) \geq 0,\, u'(t) \geq 0,\, u'''(t) \leq 0 \;\; t \in [0,1], \\ u(t) \geq \dfrac{1}{3}\, t^2\,(3-2\,t)\,\|u\|_\infty,\, u'(t) \geq t\,(1-t)\,\left\|u'\right\|_\infty \end{array} \right\}.$$

Thus, from Theorem 4.9, there is at least one nontrivial solution $u \in K$ of problem (4.3).

Finally, if we consider the following nonlinear problem for $M \in \mathbb{R}$:

$$\begin{cases} u^{(4)}(t) + M\,u(t) = (t^4+1)\left(e^{-\sqrt{(u(t))^2+(u'(t))^2+(u''(t))^2+(u'''(t))^2}} + \dfrac{1}{\ln(e+|u(t)|+|u'(t)|+|u''(t)|+|u'''(t)|)}\right), \;\; t \in [0,1], \\ u(0) = u'(0) = 0, \\ u'(1) = u'''(1) = 0. \end{cases} \tag{4.4}$$

As we have pointed out at the beginning of this section, we know that g_M satisfies the property (P_{g_1}) for all $M \in (-2.36^4, 4\pi^4]$ for suitable functions $k_1[M](t)$ and $k_2[M](t)$, $t \in [0,1]$, that depend on the value of the parameter M.

Also, $\frac{\partial}{\partial t} g_M$ satisfies the property (P_{g_1}) for all $M \in (-2.36^4, 2.22^4]$ for adequate functions $k_1^1[M](t)$ and $k_2^1[M](t)$, $t \in [0,1]$.

Let us denote $k_2[M] = \max_{t\in[0,1]} |k_2[M](t)|$ and $k_2^1[M] = \max_{t\in[0,1]} |k_2^1[M](t)|$.

In this case, we have three situations depending on the value of M. They are given by the following cases:

- If $M \in (-2.36^4, 4\pi^4]$, we consider the cone

$$K_0 = \left\{ u \in C(I) : \begin{array}{l} u(t) \geq 0 \;\; t \in [0,1], \\ u(t) \geq \dfrac{k_1[M](t)}{k_2[M]}\,\|u\|_\infty \end{array} \right\}.$$

- If $M \in (-2.36^4, 2.22^4]$, we consider the cone

$$K_1 = \left\{ u \in C^1(I) : \begin{array}{l} u(t) \geq 0,\, u'(t) \geq 0 \;\; t \in [0,1], \\ u(t) \geq \dfrac{k_1[M](t)}{k_2[M]}\,\|u\|_\infty,\, u'(t) \geq \dfrac{k_1^1[M](t)}{k_2^1[M]}\,\|u'\|_\infty \end{array} \right\}.$$

- If $M \in (-2.36^4, 0]$, we consider the cone

$$K_3 = \left\{ u \in C^3(I) : \begin{array}{l} u(t) \geq 0,\, u'(t) \geq 0,\, u'''(t) \leq 0 \;\; t \in [0,1], \\ u(t) \geq \dfrac{k_1[M](t)}{k_2[M]}\,\|u\|_\infty,\, u'(t) \geq \dfrac{k_1^1[M](t)}{k_2^1[M]}\,\|u'\|_\infty \end{array} \right\}.$$

Therefore, from Theorem 4.9, there is at least one nontrivial solution $u \in K_i$ for each $i \in \{0, 1, 3\}$ of problem (4.4) according to the values of the parameter M.

Notice that, if $M \in (-31, 2852, 0]$, we can ensure the existence of a solution of problem (4.4), which is positive and increasing on $(0, 1]$ and whose first derivative is concave in I.

Moreover, we can ensure the existence of a positive and increasing solution on $(0, 1]$ of problem (4.4), whenever $M \in (-31.2852, 24.2891]$.

Finally, if we look for positive solutions, not necessarily increasing, we can ensure their existence for all $M \in (-31.2852, 389.636]$.

Obviously, the more restrictive the imposed conditions on the solutions are, the smaller will be the interval of parameters M for which we can ensure the existence of solutions. In this sense, we point out that $\lambda_2' \approx 16 \cdot \lambda_2^1$.

Acknowledgements

The three authors were partially supported by Grant PID2020-113275GB-I00, funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe" of the "European Union", and by Xunta de Galicia (Spain), project ED431C 2023/12.

References

[1] P. Almenar, L. Jódar, *Solvability of a class of N-th order linear focal problems*, Math. Model. Anal.22 (2017), n4, 528–547.

[2] P. Almenar, L. Jódar, *The Sign of the Green Function of an n-th Order Linear Boundary Value Problem*, Mathematics 2020, 8, 673, 22pp.

[3] P. Almenar, L. Jódar, *Estimation of the smallest eigenvalue of an nth-order linear boundary value problem*, Math. Methods Appl. Sci.44 (2021), 6, 4491–4514.

[4] P. Almenar, L. Jódar, *The principal eigenvalue of some nth order linear boundary value problems*, Bound. Value Probl. (2021), 84, 16 pp.

[5] P. Almenar, L. Jódar, *New results on the sign of the Green function of a two-point n-th order linear boundary value problem*, Bound. Value Probl. (2022), 50, 22 pp.

[6] G. Butler, L. Erbe, *Integral comparison theorems and extremal points for linear differential equations*, J. Diff. Equ. (1983), 47, 214–226.

[7] A. Cabada, *Green's functions in the theory of ordinary differential equations*, Springer Briefs in Mathematics, Springer, New York, NY (2014).

[8] A. Cabada, L. López-Somoza *An explicit formula of the parameter dependence of the partial derivatives of the Green's functions related to arbitrary two-point boundary conditions.* arXiv preprint arXiv: 2405.17320.

[9] A. Cabada, L. López-Somoza, F. Minhós: *Existence, non-existence and multiplicity results for a third order eigenvalue three-point boundary value problem*, J. Nonlinear Sci. Appl., 10 (2017), 5445–5463.

[10] A. Cabada, L. Saavedra, *The eigenvalue characterization for the constant sign Green's functions of $(k, n-k)$ problems*, Bound. Value Probl., 44 (2016), 35pp.

[11] A. Cabada, L. Saavedra, *Disconjugacy characterization by means of spectral $(k; n-k)$ problems*, Appl. Math. Lett., 52 (2016), 21–29.

[12] A. Cabada, L. Saavedra, *Characterization of constant sign Green's fuction for a two-point boundary-value problem by means of spectral theory*, Electron. J. Differential Equations, 2017 (2017), 146, pp. 1–95.

[13] A. Cabada, L. Saavedra, *Existence of solutions for n^{th}-order nonlinear differential boundary value problems by means of fixed point theorems*, Nonlinear Anal. Real World Appl., 42 (2018), 180–206.

[14] W.A. Coppel, *Disconjugacy*, in: Lecture Notes in Mathematics, vol. 220, Springer-Verlag, Berlin, New York, 1971.

[15] C. De Coster, P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Mathematics in Science and Engineering 205, Elsevier B. V., Amsterdam, 2006.

[16] U. Elias, *Eventual disconjugacy of $y^{(n)} + \mu\, p(x)\, y = 0$ for every μ*, Arch. Math. (Brno), 40(2) (2004), 193–200.

[17] U. Elias *Green's functions for a nondisconjugate differential operator*, J. Diff. Equ. (1980), 37, 319–350.

[18] P. W. Eloe, J. Henderson, *Focal point characterizations and comparisons for right focal differential operators*, J. Math. Anal. Appl. (1994), 181, 22–34.

[19] P. W. Eloe, J. Ridenhour, *Sign Properties of Green's Functions for a Family of Two-Point Boundary Value Problems*, Proc. Amer. Math. Soc., 120, 2 (1994), 443–452.

[20] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, New Yourk, 2003.

[21] D. Guo, L. Lakshmikantham, *Nonlinear Problems in Abstract Cones, Academic Press, San Diego*, 1988.

[22] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[23] G. S. Ladde, V. Lakshmikantham, A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*; Pitman, Boston M. A. 1985.

[24] Z. Nehari *Disconjugate linear differential operators*, Trans. Am. Math. Soc. (1967), 129, 500–516.

[25] A. Peterson *Green's functions for focal type boundary value problems*, Rocky Mountain J. Math. (1979), 9, 721–732.

[26] A. Peterson *Focal Green's functions for fourth-order differential equations*, J. Math. Anal. Appl. (1980), 75, 602–610.

[27] W. Simons, *Some disconjugacy criteria for selfadjoint linear differential equations*, J. Math. Anal. Appl., 34 (2017), 445-463.

[28] P. J. Torres, *Existence of One-Signed Periodic Solutions of Some Second-Order Differential Equations via Krasnoselskii Fixed Point Theorem*, J. Diff.Equ., 190 (2) (2003), 643–662.

[29] A. Zettl, *A constructive characterization of disconjugacy*, Bull. Amer. Math. Soc., 81(1) (1975), 145–147.