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Differential problems with Stieltjes derivatives and applications

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Para Antonio,

el delirante delineante que, con su humor,
me introdujo en el mundo de las matemáticas:

¿Qué tienen en común un señor en una silla,
un ventilador roto y el número 150?
Pues, que entre que se sienta*, y* no vienta*,*
¡ciento cincuenta!

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Ignacio Márquez Albés

Contents

Summary

This Thesis, entitled *Differential problems with Stieltjes derivatives and applications*, is a collection of the research work developed by the author during his predoctoral stage. As the title suggests, this research revolved around the concept of Stieltjes derivative. Roughly speaking, this type of derivative is a modification of the usual derivative through a nondecreasing and left–continuous map. This change in the definition allows us to study several differential problems under the same framework.

In what follows, we present a summary of the five chapters conforming this manuscript in order to provide a more detailed explanation of its contents. According to the topics covered in the different chapters included in this manuscript, we can divide it into three different parts. The first part corresponds to Chapter 1 and it is focused on measure theory and, in particular, on some Stieltjes integrals, which are a fundamental tool for this Thesis. The second part of the manuscript consists of Chapters 2 and 3, where we turn our attention to the study of two different derivatives, the displacement derivative and the Stieltjes derivative, as well as their relations. Finally, Chapters 4 and 5 conform the third part of this work. Here, we study differential equations with Stieltjes derivatives in depth, while also featuring their applicability to different situations.

Chapter 1: Integration theory

In order to construct a self–contained work, Chapter 1 includes a collection of concepts and results which are fundamental for the work in the following chapters. In particular, this chapter focuses on integration theory.

First, we present the concepts of measurable and measure spaces following, primarily, [5, 73]. With this, we introduce the concept of integrals with respect to a measure, as well as the Radon–Nikodým Theorem. These two tools are the key for the construction of integrals in Chapter 2.

In Section 1.2 we turn our attention to the study of some Stieltjes integrals. Specifically, we explore the Lebesgue–Stieltjes and the Kurzweil–Stieltjes integrals on the real line. Following [5], we define the former as the integral with respect to a Lebesgue–Stieltjes measure, i.e. a Borel measure that assigns finite value to bounded sets. Later, using Carathéodory's Extension Theorem, we show that given a nondecreasing and left–continuous map, we can construct a Lebesgue–Stieltjes measure. Conversely, given a Lebesgue–Stieltjes measure, μ, we can construct a map satisfying those hypotheses by considering the map $g : \mathbb{R} \to \mathbb{R}$

defined as

$$g(x) = \begin{cases} -\mu([x,0)), & \text{if } x < 0. \\ 0, & \text{if } x = 0, \\ \mu([0,x)), & \text{if } x > 0. \end{cases}$$

In other words, there exists a bijection between the set of Lebesgue–Stieltjes measures and the set of nondecreasing and left–continuous functions up to a constant. This is particularly interesting for the construction of a measure in Chapter 2, as well as for the contents of Chapter 3.

On the other hand, the Kurzweil–Stieltjes integral is not defined as the integral with respect to a measure, but rather in a way that is reminiscent of the Riemann integral. In this case, the definition involves a function defining the integral, called integrator. This, together with the bijection mentioned above, allows us to establish some connections between the Kurzweil–Stieltjes and the Lebesgue–Stieltjes integral, following [65].

Chapter 2: The displacement derivative

In this chapter we focus on the construction of a new derivative on the real line following the work in [61]. The idea for this derivative comes from observing the definitions in [15, 54, 67] and noting that they all revolve around the same idea: measuring how far apart things are while preserving the sense of direction. With this idea in mind, we defined the concept of *displacement*. A displacement on a set X is a map $\Delta : X \times X \to \mathbb{R}$ satifying the following two properties:

(a) For all $x \in X$, $\Delta(x,x) = 0$.

(b) For all $x, y \in X$,

$$|\Delta(x,y)| = \sup\left\{ \liminf_{n\to\infty} |\Delta(x,z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(y,z_n)| = 0 \right\}.$$

Displacement spaces are proven to be a generalization of metric and pseudometric maps and, in a similar way to those, it is possible to endow the corresponding space with a topological structure defined in terms its displacement map. In Section 2.1 we explore the topological aspects related to this new concept.

Later, in Section 2.2 we focus on constructing some analytical structures on the real lines compatible with the concept of displacement. Specifically, we work under the assumption that there exists $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ satisying the following hypotheses:

(H1) For all $x \in [a,b]$, $\Delta(x,x) = 0$.

(H2) For all $x \in [a,b]$, the map $\Delta(x,\cdot)$ is nondecreasing and left–continuous.

(H3) There exists $\gamma : [a,b] \times [a,b] \to [1,+\infty)$ such that

  (i) For all $x, y, z, \overline{z} \in [a,b]$,

$$|\Delta(z,x) - \Delta(z,y)| \le \gamma(z,\overline{z})|\Delta(\overline{z},x) - \Delta(\overline{z},y)|.$$

(ii) For all $z \in [a, b]$,
$$\lim_{\overline{z}\to z} \gamma(z, \overline{z}) = \lim_{\overline{z}\to z} \gamma(\overline{z}, z) = 1.$$

(iii) For all $z \in [a, b]$, the maps $\gamma(z, \cdot), \gamma(\cdot, z) : [a, b] \to [1, +\infty)$ are bounded.

Observe that the hypotheses (H1)–(H3) do not match the ones considered in [61]. This is because, although it might not be clear directly from the hypotheses, these conditions are enough to ensure that the map Δ defines a displacement on $[a, b]$, which allows us to remove some of the hypotheses there considered.

In Section 2.2.1, we turn our attention to the construction of a measure in this context. We do this by noting that the hypotheses allow us to define "local" Lebesgue–Stieltjes measures which are absolutely continuous with respect to each other. This means that the hypotheses of the Radon–Nikodým Theorem are satisfied, which plays a fundamental role for this section. After that, in Section 2.2.2, we introduce and study the concept of displacement derivative, which is essentially defined as

$$f'_\Delta(x) = \lim_{y\to x} \frac{f(y) - f(x)}{\Delta(x, y)}.$$

Finally, in Section 2.2.3, we connect the concept of measure and derivative through the Fundamental Theorem of Calculus following similar reasonings to the ones in [54] for the Stieltjes derivative.

Chapter 3: The Stieltjes derivative

This third chapter is devoted to the study of the Stieltjes derivative on the real line. Roughly speaking, this derivative is computed as

$$f'_g(x) = \lim_{y\to x} \frac{f(y) - f(x)}{g(y) - g(x)},$$

for some nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$, which we call derivator. Naturally, this derivative is a particular case of the displacement derivative introduced in the previous chapter. However, the connection between these two concepts goes beyond that. In Section 3.1 we explore these relations, proving that both derivatives are equivalent using the bijection existing between Lebesgue–Stieltjes measures and nondecreasing and left–continuous functions. Later, in Section 3.2 we gather all the information available on this type of derivatives by adapting the results in Chapter 2 to this context, as well as including the results in [33, 54].

The final aspect that we cover in this chapter is the relations between differential equations with Stieltjes derivatives, also known as *Stieltjes differential equations*, of the form

$$x'_g(t) = f(t, x(t)), \quad t \in [a, b), \tag{1}$$

and other known differential problems, using the ideas in [33,49,54]. It follows from the definition of this derivative that ODEs are a particular case of (1). However, in Section 3.3.1 we show that, under the correct hypotheses, every differential problem with Stieltjes derivatives reduces to an ODE. In particular, this equivalence provides a way to obtaining solutions to equations of the form (1) provided we can solve the associated ODE. In a similar fashion, in Section 3.3.2 we have a look at impulsive differential problems of the form

$$\begin{aligned} x'(t) &= f(t, x(t)), && t \in [a, b) \setminus J, \\ x(t^+) &= x(t) + I_t(x(t)), && t \in J, \end{aligned}$$

for a countable set J. In this case, by choosing an adequate derivator, we can show that this problem is equivalent to a Stieltjes differential equation. Finally, in Section 3.3.3, we introduce the Hilger derivative and the differential problems with this type of derivative. Then, adapting some of the ideas in [79], we show that such problems can be studied as equations of the form (1) for a specific derivator.

Chapter 4: Stieltjes differential problems with a single derivator

Chapter 4 revolves around the study of existence and uniqueness of solution for different Stieltjes differential problems, as the title suggests. Observe that that same title remarks the usage of only one derivator. By this we mean that, even thought we might be looking at problems in $\mathbb{R}^n$, we are always considering the same Stieltjes derivative in all the components. In particular, given a nondecreasing and left–continuous function, $g : \mathbb{R} \to \mathbb{R}$, we will consider the following three types of differential problems with Stieltjes derivatives: initial value problems of the form

$$x'_g(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2}$$

with $f : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathbb{R}^n$; Stieltjes differential equations with functional arguments such as

$$x'_g(t) = f(t, x(t), x), \quad B(x(t_0), x) = 0, \tag{3}$$

with $f : [t_0, t_0 + T) \times \mathbb{R} \times X \to \mathbb{R}$ and $B : \mathbb{R} \times X \to \mathbb{R}$ for an some Banach space, X; and Stieltjes differential inclusions,

$$x'_g(t) \in F(t, x(t)), \quad x(t_0) = x_0, \tag{4}$$

with $F : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$.

Section 4.1 deals with initial value problems like (2) following [48, 49, 51, 53, 58, 59]. Specifically, in Section 4.1.1, we present methods to obtain explicit solutions for the linear equation in its homogeneous and nonhomogeneous formulation, as well as the initial value problem with separation of variables. Next, in Section 4.1.2 we adapt some well–known existence and uniqueness results for ODEs to the context of Stieltjes differential equations. In particular, we explore the differential problems under the Lipschitz condition,

$$\|f(t, x) - f(t, y)\| \le L(t)\|x - y\|, \quad g\text{–a.a. } t \in [t_0, t_0 + T), \quad x, y \in \mathbb{R}^n;$$

the Osgood condition,

$$\|f(t, x) - f(t, y)\| \le \omega(\|x - y\|) \quad g\text{–a.a. } t \in [t_0, t_0 + T), \quad x, y \in \mathbb{R}^n;$$

the Montel–Tonelli condition,

$$\|f(t,x)-f(t,y)\| \leq \varphi(t)\omega(\|x-y\|), \quad g\text{–a.a. } t \in [t_0, t_0+T), \quad x, y \in \mathbb{R}^n;$$

and the Perron condition,

$$\|f(t,x)-f(t,y)\| \leq \omega(t, \|x-y\|), \quad g\text{–a.a. } t \in [t_0, t_0+T), \quad x, y \in \mathbb{R}^n.$$

Later, we focus on the study of (2) through the method of lower and upper solutions in $\mathbb{R}$. This is done in Section 4.1.3, where we first look for extremal solutions between a pair of well–ordered lower and upper solutions, and next, we investigate necessary and sufficient conditions for the supremum of lower solutions and the infimum of upper solutions to be the extremal solutions of (2).

The study of (3) is carried out in Section 4.2. Here, using an existence results obtained for (2) through the method of lower and upper solutions, and Heikkilä's generalized iterative technique, we obtain a result ensuring the existence of the extremal solutions between a lower and an upper solution. From there, following the ideas of Biles and Binding in [9], we obtain a new result for (2) in the framework of lower and upper solutions which, in turn, can be used to prove a new result for (3).

Finally, Section 4.3 is dedicated to the research of Stieltjes differential inclusions. This is done by considering the closed–convex envelope of the map F in (4). In this context, we obtain an existence result that, to the best of our knowledge, provides new information even in the case when Stieltjes derivatives reduce to derivatives in the usual sense. To conclude, we turn to the initial value problem (2) again, and obtain a new existence result using the new result for the inclusion problem with Stieltjes derivatives and the *Krasovskij map* associated to the map f in (2), $\mathcal{K}f : [t_0, t_0+T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, defined as

$$\mathcal{K}f(t,x) = \bigcap_{r>0} \overline{\text{co}} f\left(t, B(x,r)\right), \quad (t,x) \in [t_0, t_0+T) \times \mathbb{R}^n.$$

Throughout this chapter, we illustrate the different results obtained for the considered problems with some analytical examples, as well as through a few real–world applications of the corresponding problems such as a model describing the motion of a vehicle impulsed by an electric engine, or a model for a silkworm population.

Chapter 5: Stieltjes differential problems with a several derivators

Similarly to Chapter 4, Chapter 5 addresses the existence and uniqueness of solution of differential problems with Stieltjes derivatives. In the case of Chapter 4 we considered only one derivator when we studied systems of differential equations, whereas in this new chapter we shall consider a different derivator for each of the equations in the system. To be precise, we take a map $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$, such that each g_i, $i = 1, 2, \ldots, n$, is a derivator, and then we consider systems of equations of the form

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad x_i(t_0) = x_{0,i}, \quad i = 1, 2, \ldots, n, \tag{5}$$

with $f_i : [t_0, t_0+T) \times \mathbb{R}^n \to \mathbb{R}$. Observe that, given the relations presented in Chapter 3, this new setting allows us to study more complex phenomena in which we might encounter processes marked by different impulses or time scales. This is can be observed in some of the examples included in this chapter to show the applicability of the results obtained in this context.

Clearly, initial value problems of the form (5) generalize the corresponding problems in Chapter 4. However, despite this connection between the problems, most of the concepts used before need to be adapted to this new framework. Hence, Section 5.1 is devoted to extending the concepts of continuity introduced in Chapter 3 so they are suitable for the study of (5). This is done by presenting some of the information available in [50, 60]. Later, in Section 5.2, with the adequate definitions and the idea of regarding (5) as a generalization of (2), we explore whether the existence and uniqueness results obtained in Section 4.1.2 in Chapter 4 can be generalized to this new setting or not. In particular, following [50, 60], we obtain analogous results under the Lipschitz, Osgood, Montel–Tonelli and Perron conditions that reduce to those in Chapter 4 when (5) yields (2).

Finally, in Section 5.3 we consider again the method of lower and upper solutions. However, in order to do so, we do not study problem (5) directly. Instead, we turn our attention to vectorial measure differential equations, that is, systems of integral equations of the form

$$x_i(t) = x_{0,i} + \int_{t_0}^{t} f_i(s, x(s))\,\mathrm{d}\,g_i(s), \quad i = 1, 2, \dots, n, \tag{6}$$

where we consider the integral in the Kurzweil–Stieltjes sense. Then, following [52], we obtain a result ensuring the existence of the extremal solutions of (6) between a lower and an upper solution. This result is later adapted for problem (5) using the relations between the Lebesgue–Stieltjes and Kurzweil–Stieltjes integrals introduced in Chapter 1. It is important to note that this yields an existence result between a given pair of lower and an upper solutions for a system of differential equations with Stieltjes derivatives, providing more information for (2) which was only studied through this method in the scalar case. We also obtain an existence result for vectorial measure differential equations with functional arguments which, once again, is then adapted to the context of Stieltjes differential equations with several derivators.

Introduction

Usual population models, such as the Malthusian or the logistic model, implicitly assume that the reproduction of a population happens so frequently in time that it has a continuous influence on the evolution of the population size. However, many species exhibit very short periods of reproduction due, for example, to rutting seasons or eggs hatching in a short lapse, which might not be properly represented by these models. One possible way to solve this is to introduce impulses into these models to include the sudden changes in the population. Nonetheless, this is not the only aspect that is not well represented with the classical population models. For instance, some species go through dormant states for some periods during which reproduction slows down or even stops completely. This, of course, affects the dynamic of the population size. For these cases, using equations in time scales has been proven very useful.

Sometimes, we may even encounter some species that show both cases of this "atypical" population behaviour, as it is the case for silkworms. These worms exhibit impulsive changes and two dormant states. Starting with the latter, we have that in the first dormant state, worms transform into moths inside chrysalides where the population is unable to interact; while the second one begins when the whole moth population dies after laying their eggs. This leads to a moment with sudden changes and, in a similar fashion, the hatching of the eggs, happens almost simultaneously, triggering another change that can be represented with impulses. Therefore, for a better representation of this population we should consider a model using impulses and time scales. In some papers like [2, 16, 36], it has been proven that, under certain circumstances, these problems can be studied as only one of the two types. That is, provided some conditions are met, impulsive differential equations can be transformed into equations in time scales and vice versa. Alternatively, as shown in [33, 54], it is possible to use differential equations with Stieltjes derivatives to study these problems together under more general conditions.

Roughly speaking, the Stieltjes derivative is defined as the derivative with respect to another function, called derivator. Specifically, given a nondecreasing and left–continuous map, $g : \mathbb{R} \to \mathbb{R}$, the Stieltjes derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $t \in \mathbb{R}$, denoted by $f'_g(t)$, is defined as

$$f'_g(t) = \lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)} \quad \text{if } g \text{ is continuous at } t,$$
$$f'_g(t) = \lim_{s \to t^+} \frac{f(s) - f(t)}{g(s) - g(t)} \quad \text{if } g \text{ is discontinuous at } t,$$

whenever the corresponding limit exists. This definition of derivative can be found on [54], although similar concepts have been studied before, see [8, 14, 20, 21, 28, 45, 70, 74, 83]. Naturally, this definition of derivative is an extension of the usual derivative. Following this idea, in [61], we find another of derivative, called displacement derivative, whose definition encompasses that of Stieltjes derivative and other interesting derivatives, such as the absolute derivative in [15]. The topic of derivatives in the real line is one of the focal points of this thesis, particularly, the Stieltjes derivative and the displacement derivative introduced in [61].

Differential problems involving Stieltjes derivatives are the other main aspect that concerns this work. This subject represents most of the work done by the author during his predoctoral stage, see [48–53, 58–60]. The applicability of this type of problem, as commented before, is that it offers a unified framework to study problems on time scales and impulsive problems. Of course, from the theoretical point of view, this setting is also interesting as it represents a generalization of some classical differential problems like ODEs or differential inclusions. However, it is important to keep in mind that working in a more general framework comes with some drawbacks. For instance, consider the linear initial value problem

$$x_g'(t) = \lambda x(t), \quad t \in [-1,1], \quad x(-1) = 1,$$

with $\lambda \in \mathbb{R}$, $\lambda > 0$, and $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} t, & \text{if } t \leq 0, \\ t+1, & \text{if } t > 0. \end{cases}$$

In this case, the solution of the problem is given by

$$x(t) = \begin{cases} e^{\lambda(t+1)}, & \text{if } -1 \leq t \leq 0, \\ e^{\lambda(t+1)}(\lambda+1), & \text{if } 0 < t \leq 1. \end{cases}$$

Indeed, to see this, it is enough to note that the Stieljtes derivative is computed as the usual derivative for $t \in [-1,1] \setminus \{0\}$, and at $t = 0$, we have that

$$x_g'(0) = \lim_{s \to 0^+} \frac{x(s) - x(0)}{g(s) - g(0)} = \lim_{s \to 0^+} \frac{e^{\lambda(s+1)}(\lambda+1) - e^{\lambda}}{s+1} = \lambda e^{\lambda} = \lambda x(0).$$

Observe that the natural extension of the solution of the linear problem does not solve it in this case. That is, the equality

$$x(t) = \exp\left(\int_{[-1,t)} \lambda \, \mathrm{d}\, g(s)\right),$$

does not hold for all $t \in [-1,1]$. In particular, for any $t \in (0,1]$ we have that

$$\begin{aligned} \exp\left(\int_{[-1,t)} \lambda \, \mathrm{d}\, g(s)\right) &= \exp\left(\int_{[-1,0)} \lambda \, \mathrm{d}\, g(s) + \int_{\{0\}} \lambda \, \mathrm{d}\, g(s) + \int_{(0,t]} \lambda \, \mathrm{d}\, g(s)\right) \\ &= \exp\left(\int_{[-1,0)} \lambda \, \mathrm{d}\, s + \lambda \mu_g(\{0\}) + \int_{(0,t]} \lambda \, \mathrm{d}\, s\right) = e^{2(\lambda+t)}, \end{aligned}$$

which does not coincide with the expression of $x(t)$ on that interval. The reason behind this are the discontinuity points of the derivator, which play a major role in the study of these problems. In this particular case, the difficulties coming from the discontinuities of g can be easily solved because we only need to worry about one conflicting point. However, given that derivators are nondecreasing, we can encounter up to a countable number of discontinuities, see [35]. Nevertheless, it is this type of points that brings the flavor to this setting, allowing interesting behaviour in the problems and models based on this derivative.

Throughout this thesis we will explore the adaptation of some well–know results for differential problems in this new setting, accounting for its particularities. Specifically, we shall develop new results for differential equations and differential inclusions with Stieltjes derivatives, while showing their application with a variety of technical and real–world examples.

Aims and objectives

The main goal of this thesis is to study some differential problems involving Stieltjes derivatives such as initial value problems of the form

$$x'_g(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

with $f : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathbb{R}^n$; Stieltjes differential equations with functional arguments such as

$$x'_g(t) = f(t, x(t), x), \quad B(x(t_0), x) = 0, \tag{2}$$

with $f : [t_0, t_0 + T) \times \mathbb{R} \times X \to \mathbb{R}$ and $B : \mathbb{R} \times X \to \mathbb{R}$ for an some Banach space, X; and Stieltjes differential inclusions,

$$x'_g(t) \in F(t, x(t)), \quad x(t_0) = x_0, \tag{3}$$

with $F : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$. Specifically, some of the objectives we set for this thesis are:

- To establish reasonable mathematical models in terms of differential equations with Stieltjes derivatives such as (1) or even (2). With this, we intend to show the applicability of this new type of problems, originally conceived as theoretical mathematical objects.

- To prove the existence of solution of problems such as (1), (2) and (3) under very general hypotheses. In particular, for equations such as (1) and (2), we aim to allow the function f to present discontinuities with respect to any of their variables.

- To combine known techniques for ordinary differential equations, such as the method of lower and upper solutions or iteration techniques, with this new setting in order to obtain new results for Stieltjes differential equations.

- To extend some results and applications for differential equations with Stieltjes derivatives to systems of differential equations with Stieltjes derivatives involving several different derivatives. Specifically, given n nondecreasing and left–continuous maps, $g_i : \mathbb{R} \to \mathbb{R}$, we want to study problems of the form

  $$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad x_i(t_0) = x_{0,i}, \quad i = 1, 2, \ldots, n,$$

  with $f_i : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, as well as analogous reformulations of (2) and (3) in this setting.

Chapter 1
Integration theory

In this first chapter we introduce some of the basic notions necessary for the work ahead. Particularly, we focus on integration theory. After all, integration is, as we will see, one of the main tools for this thesis. It will be used in Chapter 2 for the construction of a new measure centered around a specific type of topological space, as well as in its relations with a new type of derivative. Later, in Chapter 3, the integration theory will allow us to establish connections between the derivative introduced in Chapter 2 and the Stieltjes derivative, which is the main topic of this thesis. Finally, in the remaining chapters, the integrals arise naturally thanks to the connection between differential problems and integral problems. Therefore, it makes sense to study them in detail. We would like to note at this point that this chapter is, primarily, a gathering of results available in the literature.

We will mainly focus on measure theory. In particular, in Section 1.1 we make explicit the construction of integrals with respect to a measure, as well as we introduce the Radon–Nikodým Theorem. This section is a fundamental pillar for Chapter 2 in which we will construct a measure in the context of displacemet calculus. Later, we have a look at Stieltjes integrals. Especifically, we consider two types of Stieltjes integrals. First, we introduce the Lebesgue–Stieltjes integral as the integral with respect to a measure satisfying certain properties. This integral plays a fundamental role in the work ahead, from the construction to the already mentioned measure in Chapter 2, to the study of differential problems in Chapters 4 and 5. Later, we present the Kurzweil–Stieltjes integral. This kind of integral is not the integral with respect to a measure and it is, in some sense, constructed in a similar way to the Riemann integral. However, the interest of this integral for this work lies in its relation with the Lebesgue–Stieltjes integral.

1.1 Measure theory

In what follows, we gather the basic definitions and results for measure theory that will be needed in the work ahead, following, primarily, [5, 73]. We will introduce the basic notion of measurable and measure space, which is a measurable space with an assigned measure. From there, we will define the integral of a function with respect to a given measure. Later, we will explore different relations between measures defined on a given set, which will be fundamental for the integration theory in Chapter 2. In particular, we will focus on the Radon–Nidokým Theorem and its consequences.

Let us begin by recalling the basic structure regarding measure theory: measurable spaces. These spaces are the base upon which we define measures, and thus, measure spaces.

Definition 1.1. *Let X be a set and $\mathcal{P}(X)$ be the set of all subsets of X. A set $\mathcal{M} \subset \mathcal{P}(X)$ is a σ–*algebra on X *if the following properties are satisfied:*

(i) $X \in \mathcal{M}$.

(ii) *If* $A \in \mathcal{M}$*, then* $X \backslash A \in \mathcal{M}$.

(iii) *Given a family* $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$*, we have that* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

In that case, we say the pair $(X, \mathcal{M})$ *is a* measurable space*, and the elements of* $\mathcal{M}$*,* measurable sets.

Remark 1.2. It follows directly from the definition that the arbitrary intersection of σ–algebras is a σ–algebra.

Note that the concepts of σ–algebra and topology are quite similar. One of the main differences between them is that (ii) and (iii) in Definition 1.1 imply that σ–algebras are closed under countable intersections whereas topologies are not. However, topologies admit the arbitrary union of sets, whereas σ–algebras do not. In relation with the concept of topological spaces, we have the following definition.

Definition 1.3. *Let* (X, τ) *be a topological space and* $\mathcal{M}$ *be a* σ*–algebra on* X*. If* $\mathcal{M}$ *contains all the open sets of* X*, we say that* $\mathcal{M}$ *is* a τ–Borel σ–algebra *on* X*. The smallest* σ*–algebra containing all the open sets of* X *is called* the Borel σ–algebra*, and it is usually denoted by* $\mathcal{B}(\tau)$*. The elements of* $\mathcal{B}(\tau)$ *are called* Borel sets.

Remark 1.4. It is important to note the difference between *a* Borel σ–algebra and *the* Borel σ–algebra. Essentially, a Borel σ–algebra is a σ–algebra containing $\mathcal{B}(\tau)$, the Borel σ–algebra. Also, note that in what lies ahead, we will omit the symbol τ of the notation when it is clear which topology we are considering.

This type of σ–algebra is particularly interesting for the aim of this section: the definition of integrals with respect to measures. This is because the sets belonging to the σ–algebra of a given measurable space are the sets over which we will consider integrals. Therefore, if we construct a measure over a Borel σ–algebra, we are ensuring that we can consider integrals over the corresponding Borel sets. In the particular setting of the real line equipped with the usual topology, this means that we would be able to consider integrals over intervals, for example.

Next, we present the other fundamental concept of measure theory, that of a measure. Essentially, measures are a way of assigning values to the elements of a given σ–algebra. This value could represent different things. For example, in probability theory, measures are used to describe the likelihood of a given event, which is represented by a member of a given σ–algebra, as explained in [5].

Definition 1.5. *Let* X *be a set and* $\mathcal{M}$ *a* σ*–algebra over* X*. A* positive measure on X *is a set function* $\mu : \mathcal{M} \to [0, +\infty]$ *such that*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

for any disjoint countable collection of elements of $\mathcal{M}$*,* $\{A_n\}_{n=1}^{\infty}$*. Under these circumstances, the space* $(X, \mathcal{M}, \mu)$ *is called a* measure space.

Remark 1.6. Here, it is assumed that μ takes values in $[0, +\infty]$ and, therefore, the adjetive positive. When we consider $[-\infty, +\infty]$ we say that μ is a *signed measure*. In the work ahead, we will refer to positive measures as simply measures for simplicity. We will also sometimes refer to a measure space $(X, \mathcal{M}, \mu)$ as simply X when the σ–algebra, $\mathcal{M}$, and the measure, μ, play no major role, or are clear in the framework.

As it was mentioned before, it is interesting to be able to integrate over Borel sets. The following definition provides a class of measures for which this is possible to achieve. We also introduce some related concepts that will be used in the work ahead.

Definition 1.7. *Let (X, τ) be a topological space and $\mathcal{M}$ a σ–algebra on X. We shall say that a measure $\mu : \mathcal{M} \to [0, +\infty]$ is a τ–*Borel measure *if $\mathcal{M}$ is a τ–Borel σ–algebra.*

Assume that X is also Hausdorff. A τ–Borel measure, μ, is inner regular *if*

$$\mu(A) = \sup\{\mu(K) : K \subset A,\ K \in \mathcal{M},\ K \textit{ compact}\}, \quad A \in \mathcal{M}.$$

Similarly, μ is outer regular *if*

$$\mu(A) = \inf\{\mu(U) : A \subset U,\ U \textit{ open}\}, \quad A \in \mathcal{M}.$$

If μ is both inner and outer regular, it is said that μ is regular.

Next, we present some properties of measures over a given measurable space. Most of these results follow directly from the definition. However, their proofs can be found in [73, Theorem 1.19].

Proposition 1.8. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:*

(i) $\mu(\emptyset) = 0$.

(ii) *If $A, B \in \mathcal{M}$ are such that $A \subset B$, then $\mu(A) \le \mu(B)$.*

(iii) *If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ is such that $A_n \subset A_{n+1}$, $n = 1, 2, \dots$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(iv) *If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ is such that $\mu(A_1) < +\infty$ and $A_{n+1} \subset A_n$, $n = 1, 2, \dots$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

The next concept that we introduce is that of measurable functions. Following [5,73], we find two possible definitions for such concept. Here, we present [5, Definition 2.1.3] for its similarities with the concept of continuous maps between topological spaces.

Definition 1.9. *Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be measurable spaces. A map $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is said to be $(\mathcal{M}_X, \mathcal{M}_Y)$–*measurable*, or simply* measurable*, if $f^{-1}(U) \in \mathcal{M}_X$ for every $U \in \mathcal{M}_Y$.*

Remark 1.10. Note that the definition requires both spaces to be measurable spaces, not necessarily measure spaces. This means that the measurability of a function is determined by the σ–algebras on the spaces involved, not the measures on them. Moreover, in the work ahead we will consider maps whose codomain is not a measure space, but a topological space. In that case, it will be assumed that such space is a measurable space with the Borel σ–algebra generated by its topology.

Along those lines, we find [73, Definition 1.3 (c)], where the authors only define measurability for such kind of maps. In particular, the definition reads as follows: if $(X, \mathcal{M})$ is a measurable space and (Y, τ) is a topological space, a map $f : X \to Y$ is measurable if $f^{-1}(U) \in \mathcal{U}$ for every $U \in \tau$. This definition implies measurability in the sense of Definition 1.9 when we consider the corresponding Borel σ–algebra in the codomain, see [5, Proposition 2.1.1. (i)].

Note that with the definition of measurability in Definition 1.9, it is obvious that the composition of measurable functions yields a measurable function. Observe, however, that this is not true for [73, Definition 1.3 (c)]. Furthermore, it is clear from Definition 1.9 and the definition of continuous maps between topological spaces that given two topological spaces, (X_1, τ_1), (X_2, τ_2), every (τ_1, τ_2)–continuous function is $(\mathcal{B}(\tau_1), \mathcal{B}(\tau_2))$–measurable. Therefore, using that [73, Definition 1.3 (c)] implies Definition 1.9, we can obtain the following result.

Proposition 1.11. *Let $(X, \mathcal{M})$ be a measurable space and $(Y, \tau_Y), (Z, \tau_Z)$ be topological spaces. If $f : X \to Y$ is a $(\mathcal{M}, \mathcal{B}(\tau_Y))$–measurable and $g : Y \to Z$ is (τ_Y, τ_Z)–continuous, then the composition $g \circ f$ is $(\mathcal{M}, \mathcal{B}(\tau_Z))$–measurable.*

Remark 1.12. In particular, given a measurable space, X, and a measurable map, $f : X \to \mathbb{R}$, the maps $f^+(x) = \max\{f(x), 0\}$, $x \in X$, and $f^-(x) = -\min\{f(x), 0\}$, $x \in X$, are also measurable.

A particularly interesting type of measurable functions are simple functions. This kind of functions are quite useful because, as it will be shown later in Proposition 1.15, any positive measurable function can be approximated by a sequence of simple functions. This plays a fundamental role in the definition of the integral of positive functions with respect to a given measure.

Definition 1.13. *Let $(X, \mathcal{M})$ be a measurable space. A function $S : X \to \mathbb{R}$ is said to be* simple *if its range is a finite subset of $\mathbb{R}$. In that case, letting α_k, $k = 1, 2, \dots, n$, be the different values of S and $A_k = S^{-1}(\{\alpha_k\})$, $k = 1, 2, \dots, n$, we can write*

$$S(x) = \sum_{k=1}^{n} \alpha_k \chi_{A_k}(x), \quad x \in X,$$

where χ_{A_k} is the characteristic function of the set A_k.

Remark 1.14. It follows directly from the definition that S is measurable if and only if the sets A_k, $k = 1, 2, \dots, n$, are all measurable.

As mentioned before, it is possible to show that every positive measurable function can be approximated by positive simple measurable functions. In order to do so, we need to recall the partial ordering for functions. To that end, consider the partial ordering in $\mathbb{R}^n$ defined as follows: for $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$,

$$x \le y :\iff x_i \le y_i, \quad i \in \{1, 2, \dots, n\}.$$

This allows us to define the following partial ordering in the set of maps with values on $\mathbb{R}^n$: given two maps $f_1, f_2 : X \to \mathbb{R}^n$,

$$f_1 \le f_2 :\iff f_1(x) \le f_2(x), \quad x \in X. \tag{1.1}$$

With this notation, we introduce [73, Theorem 1.17].

Proposition 1.15. *Let X be a measure space and $f : X \to [0, +\infty]$ be a measurable function. Then there exists a sequence of simple measurable functions $S_n : X \to [0, +\infty)$, $n \in \mathbb{N}$, such that $0 \le S_1 \le S_2 \le \cdots \le f$ and*

$$S_n(x) \xrightarrow{n\to\infty} f(x), \quad x \in X.$$

Next, we defined the integral of nonnegative functions with respect to a measure. Note that Proposition 1.15 ensures that this concept is well–defined.

Definition 1.16. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $S : X \to [0, +\infty)$ be a measurable simple function writen in the form*

$$S(x) = \sum_{k=1}^{n} \alpha_k \chi_{A_k}(x), \quad x \in X.$$

The integral of S on E with respect to μ *is defined as*

$$\int_E S \,\mathrm{d}\,\mu := \sum_{k=1}^{n} \alpha_k \mu(A_k \cap E).$$

For a measurable map $f : X \to [0, +\infty]$, define the integral of f on E with respect to μ *as*

$$\int_E f \,\mathrm{d}\,\mu := \sup \left\{ \int_E S \,\mathrm{d}\,\mu : S \text{ simple, } 0 \le S \le f \right\}.$$

The following result collects some of the properties of the integral of nonnegative functions with respect to measures. These results follow from the definitions, but most of the proofs can be found in [73, Proposition 1.24].

Proposition 1.17. *Let $(X, \mathcal{M})$ be a measurable space, $\mu, \widetilde{\mu} : \mathcal{M} \to [0, +\infty]$ measures on X, $A, B \in \mathcal{M}$, $f, \widetilde{f} : X \to [0, +\infty]$ measurable functions and $\lambda, \widetilde{\lambda} \in [0, +\infty]$. Then:*

(a) *The map $\lambda f + \widetilde{\lambda}\,\widetilde{f}$ is also measurable and*

$$\int_A (\lambda f + \widetilde{\lambda}\,\widetilde{f}) \,\mathrm{d}\,\mu = \lambda \int_A f \,\mathrm{d}\,\mu + \widetilde{\lambda} \int_A \widetilde{f} \,\mathrm{d}\,\mu.$$

(b) *The map* $\lambda\mu + \widetilde{\lambda}\widetilde{\mu}$ *is also a measure and*

$$\int_A f \, \mathrm{d}(\lambda\mu + \widetilde{\lambda}\widetilde{\mu}) = \lambda \int_A f \, \mathrm{d}\,\mu + \widetilde{\lambda} \int_A f \, \mathrm{d}\,\widetilde{\mu}.$$

(c) *If* $f \leq \widetilde{f}$*, then* $\int_A f \, \mathrm{d}\,\mu \leq \int_A \widetilde{f} \, \mathrm{d}\,\mu$.

(d) *If* $A \subset B$*, then* $\int_A f \, \mathrm{d}\,\mu \leq \int_B f \, \mathrm{d}\,\mu$.

(e) *If* $f(x) = 0$ *for all* $x \in A$*, then* $\int_A f \, \mathrm{d}\,\mu = 0$*, even if* $\mu(A) = +\infty$.

(f) *If* $\mu(A) = 0$*, then* $\int_A f \, \mathrm{d}\,\mu = 0$*, even if* $f(x) = +\infty$ *for all* $x \in A$.

The next result, known as Lebesgue's Monotone Convergence Theorem, shows a useful property in relation with the integral of a sequence of nonnegative functions with respect to a measure and the integral of its pointwise limit.

Theorem 1.18 (Lebesgue's Monotone Convergence Theorem)**.** *Let* $(X, \mathcal{M}, \mu)$ *be a measurable space,* $E \in \mathcal{M}$ *and* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable functions on* E *such that the pointwise limit*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in E$*. If* $0 \leq f_n(x) \leq f_{n+1}(x) \leq \infty$ *for all* $x \in E$*,* $n \in \mathbb{N}$*, then* f *is measurable and*

$$\lim_{n\to\infty} \int_E f \, \mathrm{d}\,\mu = \int_E f \, \mathrm{d}\,\mu.$$

Finally, Definition 1.15 gives sense to the following result, [73, Theorem 1.29], which provides a way to obtain a measure from a given measure and a nonnegative measurable function.

Proposition 1.19. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space and* $f : X \to [0, +\infty]$ *be a measurable function. The map* $\overline{\mu} : \mathcal{M} \to [0, +\infty]$ *given by*

$$\overline{\mu}(A) = \int_A f \, \mathrm{d}\,\mu, \quad A \in \mathcal{M}$$

is also a measure on X.

Remark 1.20. In this case, given a measurable function $g : X \to [0, +\infty]$, its integral in the new measure can be computed as

$$\int_E g \, \mathrm{d}\,\overline{\mu} := \int_E gf \, \mathrm{d}\,\mu.$$

So far, only the integral of nonnegative functions has been defined. In order to define the integral of a more general set of functions with respect to a measure, it is necessary to work with the following kind of functions, called integrable functions.

Definition 1.21. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. A function $f : X \to \mathbb{R}$ is said to be μ–*integrable on *E, or* integrable with respect to μ, *if f is μ–measurable and*

$$\int_E |f| \, \mathrm{d}\,\mu < +\infty. \tag{1.2}$$

The set of integrable functions on the set E is denoted by $\mathcal{L}^1_\mu(E, \mathbb{R})$.

Remark 1.22. The notation $\mathcal{L}^1_\mu(E, \mathbb{R})$ denotes the set over which we consider the integral, E, as well as the codomain of the function, in this case $\mathbb{R}$, both of which may vary. This is why we make it explicit in the notation.

Note that the integral in (1.2) is one of those presented in Definition 1.16. Indeed, first of all, it is clear that $|f| \geq 0$. Moreover, Proposition 1.11 ensures that $|f|$ is measurable as it is the composition of a measurable function with a continuous function.

Definition 1.21 allows us to introduce the following concept. Observe that Remark 1.12 plays a fundamental role in this definition.

Definition 1.23. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$. Given $f \in \mathcal{L}^1(E, \mathbb{R})$, write $f = f^+ - f^-$ with*

$$f^+(x) = \max\{0, f(x)\}, \quad f^-(x) = -\min\{0, f(x)\}, \quad x \in X.$$

The integral of f on E with respect to μ *is defined as*

$$\int_E f \, \mathrm{d}\,\mu := \int_E f^+ \, \mathrm{d}\,\mu - \int_E f^- \, \mathrm{d}\,\mu. \tag{1.3}$$

Remark 1.24. Note that by definition $f^+, f^- \geq 0$ and they are measurable, so the corresponding integrals in (1.3) are well–defined. Moreover, due to the monotonicity of the integral, (c) in Proposition 1.17, each of those integrals is finite and so the integral of f is well–defined as well.

It is clear from the definition that all properties in Proposition 1.17, except (d), still hold when the functions considered are integrable instead of nonnegative. In particular, property (a) remains true even when $\lambda, \tilde{\lambda} \in \mathbb{R}$. Further properties can be obtained for the integral of integrable functions, as presented in [73, Theorem 1.33]

Proposition 1.25. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$ and $f \in \mathcal{L}^1_\mu(E, \mathbb{R})$. Then*

$$\left| \int_E f \, \mathrm{d}\,\mu \right| \leq \int_E |f| \, \mathrm{d}\,\mu.$$

We can now introduce the concept of Carathéodory function which, together with Proposition 1.19, will be necessary for the aims of Chapter 2. In order to do so, we require the following notion.

Definition 1.26. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $A \subset X$. A property $\mathcal{P}$ is said to be satisfied μ–*almost everywhere in *A (μ–a.e. in A) or that it holds for μ–*almost all *$x \in A$ (μ–a.a. $x \in A$) if there exists $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $\mathcal{P}$ holds in $A \backslash E$.*

We can now present the definition of Carathéodory functions with respect to a measure, as well as a result derived from [4, Chapter 1, Section 4] showing some of their properties.

Definition 1.27. *Let $I \subset \mathbb{R}$, $(I, \mathcal{M}, \mu)$ be a measurable space and $X \subset \mathbb{R}^m$. A function $f : I \times X \to \mathbb{R}^n$ is said to be μ–*Carathéodory *if the following properties are satisfied:*

(i) *The map $f(\cdot, x)$ is measurable for all $x \in X$.*

(ii) *The map $f(t, \cdot)$ is continuous for μ–a.a. $t \in I$.*

(iii) *For all $r > 0$ there exists $h_r \in \mathcal{L}^1_\mu(I, [0, +\infty))$ such that*

$$\|f(t, x)\| \leq h_r(t), \quad \mu\text{–a.a. } t \in I, \quad x \in X, \, \|x\| \leq r.$$

Proposition 1.28. *Let $(I, \mathcal{M}, \mu)$, $I \subset \mathbb{R}$, be a measurable space and $X \subset \mathbb{R}^m$. Suppose that $f : I \times X \to \mathbb{R}^n$ is a μ–Carathéodory function and $x : I \to X$ is a measurable function. Then the composition $f(\cdot, x(\cdot))$ is measurable. Moreover, if x is bounded then $f(\cdot, x(\cdot)) \in \mathcal{L}^1_\mu(I, \mathbb{R})$.*

Remark 1.29. Note that the last part of the result cannot be found in the mentioned result in [4], however it is a direct consequence of condition (iii) in Definition 1.27.

As shown in Definition 1.27, the sets of measure zero play an important part when working with measure spaces, as most of the time properties are only require to hold everywhere except in a null–measure set. In fact, such sets allow us to define an equivalence relation on the set of measure defined over a common σ–algebra. In order to establish a formal definition of this equivalence, we need the following definition.

Definition 1.30. *Let $(X, \mathcal{M})$ be a measurable space and $\mu_1, \mu_2 : \mathcal{M} \to [0, +\infty]$ be measures. The measure μ_1 is said to be* absolutely continuous with respect to μ_2*, and it is denoted by $\mu_1 \ll \mu_2$, if*

$$\mu_1(E) = 0 \text{ for every } E \in \mathcal{M} \text{ such that } \mu_2(E) = 0.$$

Note that $\ll$ defines a order relation on the set of all measures over a measurable space, instead of an equivalence relation. In order to obtain an equivalence relation between measures, and therefore and equivalence between the measure spaces, the measures need to be absolutely continuous one with respect to the other. This yields the following definition.

Definition 1.31. *Let $(X, \mathcal{M})$ be a measurable space and $\mu_1, \mu_2 : \mathcal{M} \to [0, +\infty]$ be measures. The measures μ_1 and μ_2 are* equivalent *if $\mu_1 \ll \mu_2 \ll \mu_1$.*

Definition 1.31, together with the Radon–Nikodým Theorem, are two fundamental tools for the integration theory presented in Chapter 2. In order to state the Radon–Nikodým Theorem, we need the following concept.

Definition 1.32. *Let $(X, \mathcal{M}, \mu)$ be a measurable space. The measure μ is said to be σ–*finite *if there exists a countable family $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$ such that $X = \bigcup_{n=1}^\infty E_n$ and $\mu(E_n) < +\infty$ for all $n = 1, 2, \ldots$*

We now have all the tools necessary for the statement of the Radon–Nikodým Theorem. Such result can be found in [73, Theorem 6.10], as part of the Lebesgue–Radon–Nikodým Theorem, the statement corresponding to the Radon–Nikodým Theorem is contained in part (b). Here, we present only the latter, with a slightly modified version that can be directly obtained from [73, Theorem 6.10]

Theorem 1.33 (Radon–Nikodým)**.** *Let $(X,\mathcal{M})$ be a measurable space and $\mu_1,\mu_2:\mathcal{M}\to[0,+\infty]$ be measures. Assume that μ_1 is σ–finite and $\mu_2\ll\mu_1$. Then there exists a measurable function $h:X\to[0,+\infty)$, known as the* Radon–Nikodým derivative of μ_2 with respect to μ_1*, such that*

$$\mu_2(A)=\int_A h\,\mathrm{d}\,\mu_1,\quad A\in\mathcal{M}.$$

Moreover, if there is another function $\overline{h}:X\to[0,+\infty)$ such that

$$\mu_2(A)=\int_A \overline{h}\,\mathrm{d}\,\mu_1,\quad A\in\mathcal{M},$$

then $h=\overline{h}$ for μ_1–a.a. $x\in X$.

Remark 1.34. When this last condition is satisfied, we say that h is *uniquely defined up to a set of μ_1–measure zero*, or simply *μ_1–uniquely defined*. It is important to note that these conditions does not appear explicitly in [73, Theorem 6.10 (b)]. However, the statement of such result involves the set $L^1_{\mu_1}(X)$, defined as the quotient space of $\mathcal{L}^1(X,[0,+\infty))$ under the following equivalence relation: for $f_1,f_2:X\to\mathbb{R}$,

$$f_1\sim f_2:\iff f_1(x)=f_2(x),\quad \mu_1\text{–a.a. } x\in X.$$

As a final note, we present the following corollary which is a direct consequence of Definition 1.31 and the Radon–Nikodým Theorem. This result provides an explicit relation between the value assigned to a set in equivalent measures, which will be key for the construction of a measure in Chapter 2.

Corollary 1.35. *Let $(X,\mathcal{M})$ be a measurable space and $\mu_1,\mu_2:\mathcal{M}\to[0,+\infty]$ be σ–finite measures. If μ_1 and μ_2 are equivalent then there exist $h_{1,2},h_{2,1}:X\to[0,+\infty)$ measurable functions such that*

$$\mu_1(A)=\int_A h_{1,2}\,\mathrm{d}\,\mu_2,\quad \mu_2(A)=\int_A h_{2,1}\,\mathrm{d}\,\mu_1,\quad A\in\mathcal{M}.$$

Remark 1.36. Note that such functions are both μ_1 and μ_2–uniquely defined.

Remark 1.37. Let $(X,\mathcal{M})$ be a measurable space and $\mu_1,\mu_2,\mu_3:\mathcal{M}\to[0,+\infty]$ be nonzero σ–finite equivalent measures. Let $h_{i,j}$, $i,j\in\{1,2,3\}$, denote the corresponding functions in Corollary 1.35. Then it follows that $h_{1,1}(x)=1$ for μ_1–a.a. $x\in X$, $h_{1,2}(x)\neq 0$ for μ_i–a.a. $x\in X$, $i\in\{1,2\}$, and

$$h_{1,3}(x)=h_{1,2}(x)h_{2,3}(x),\quad \mu_i\text{–a.a. } x\in X,\quad i\in\{1,2,3\}.$$

As a consequence, we obtain that $h_{1,2}(x)=1/h_{2,1}(x)$ for μ_i–a.a. $x\in X$, $i\in\{1,2\}$.

1.2 Stieltjes integrals

In this second part of the chapter, we turn our attention to the study of Stieltjes integrals. Stieltjes integrals are generalizations of the usual integrals in euclidean spaces. Many different definitions are available, such as the Riemann–Stieltjes integral –see [3]–, which generalizes the Riemann integral, the Lebesgue–Stieltjes integral, an extension of the Lebesgue integral, or the Kurzweil–Stieltjes integral, among others. In this section, we present these last two integrals, while studying their relations.

1.2.1 The Lebesgue–Stieltjes integral

The Lebesgue–Stieltjes integral is a generalization of the Lebesgue integral over the real line. This integral can be defined in several equivalent ways. Here, we present a definition from the measure point of view, a similar definition to that in [5].

Definition 1.38. *Let $X \subset \mathbb{R}$ and $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is a* Lebesgue–Stieltjes measure *if the following properties are satisfied:*

(i) $\mathcal{B}(\tau_u) \subset \mathcal{M}$*, where τ_u denotes the usual topology in X.*

(ii) *For every bounded set $A \in \mathcal{M}$, $\mu(A) < +\infty$.*

The Lebesgue–Stieltjes integral *is defined as the integral with respect to a Lebesgue–Stieltjes measure.*

Remark 1.39. Note that this is a generalization of Lebesgue's measure, m. Indeed, it is a known fact that the σ–algebra of Lebesgue measurable sets, $\mathcal{L}$, is a Borel σ–algebra. Furthermore, given a bounded set $A \subset \mathcal{L}$, there exists $M \in \mathbb{R}$ such that $A \subset [-M, M]$. Therefore, $0 \le m(A) \le m([-M, M]) = 2M < +\infty$, and so, m is a Lebesgue–Stieltjes measure.

Remark 1.40. Let μ be a Lebesgue–Stieltjes measure on $X \subset \mathbb{R}$. Write

$$X = \bigcup_{n=1}^{\infty} ([-n, n] \cap X),$$

and, since each $[-n, n] \cap X$, $n = 1, 2, \dots$, is bounded, we have that $\mu([-n, n] \cap X)$ is finite for every $n = 1, 2, \dots$. Hence, we have that every Lebesgue–Stieltjes measure is σ–finite. Furthermore, since Lebesgue–Stieltjes measures are, by definition, τ_u–Borel, it follows from [73, Theorem 2.18] that the restriction of a Lebesgue–Stieltjes measure to $\mathcal{B}(\tau_u)$ is regular, see Definition 1.7.

An alternative way of defining the Lebesgue–Stieltjes integral is by considering a function, usually called integrator, which defines a measure that is a Lebesgue–Stieltjes measure in the sense of Definition 1.38. Here, we will show that there exists a bijection between these two definitions when we work over a certain σ–algebra. We will first show how to construct a Lebesgue–Stieltjes measure in the sense of Definition 1.38 from a function. In order to do so, we will first recall the definition of an outer measure, as well as an important theorem yielding a method to construct a measure from a given outer measure.

Definition 1.41. *Let X be a set. An* outer measure *is a set map $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ that satisfies the following conditions:*

(i) $\mu^*(\emptyset) = 0$;

(ii) *if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$;*

(iii) *for any countable family $\{E_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)$,*

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

The interest of introducing this definition comes from being able to construct a measurable space from it. This is done using Carathéodory's Extension Theorem, which we present bellow. This result collects the information in [5, Theorems 1.3.3, 1.3.6], although some similar results can be found in [12, 66, 73, 76].

Theorem 1.42 (Carathéodory's Extension Theorem)**.** *Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$ be a family of sets satisfying the following properties:*

(i) *If $A, B \in \mathcal{C}$, then $A \cap B \in \mathcal{C}$.*

(ii) *For any $A \in \mathcal{C}$, there exists a finite pairwise disjoint family of sets of $\mathcal{C}$, $\{B_n\}_{n=1}^{k}$, such that*

$$X \setminus A = \bigcup_{n=1}^{k} B_n.$$

Given a map $\varphi : \mathcal{C} \to [0, +\infty]$ satisying:

(iii) $\varphi(\emptyset) = 0$;

(iv) *for any sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{C}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, we have that*

$$\varphi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \varphi(A_n);$$

the map $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ given by

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \varphi(A_n) : \{A_n\}_{n=1}^{\infty} \subset \mathcal{C},\ A \subset \bigcup_{n=1}^{\infty} A_n\right\}, \tag{1.4}$$

is an outer measure; the set

$$\mathcal{M}_{\mu^*} = \{A \in \mathcal{P}(X) : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)) \text{ for all } E \in \mathcal{P}(X)\}, \tag{1.5}$$

is a σ–algebra on X; and the restriction $\mu = \mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure on $\mathcal{M}_{\mu^*}$.

Furthermore, assume there exist $\{F_n\}_{n=1}^{\infty} \subset \mathcal{C}$ *such that*

$$X = \bigcup_{n=1}^{\infty} F_n, \quad \varphi(F_n) < +\infty, \quad n = 1, 2, \dots .$$

Then, if $\widetilde{\mu}$ *is a measure on the smallest* σ*–algebra containing* $\mathcal{C}$*,* $\sigma(\mathcal{C})$*, such that* $\widetilde{\mu} = \varphi$ *on* $\mathcal{C}$*, we have that* $\widetilde{\mu} = \mu^*$ *on* $\sigma(\mathcal{C})$*. In other words, the measure* μ *is unique on* $\sigma(\mathcal{C})$*.*

Remark 1.43. Note that Theorem 1.42 provides more information than we anticipated. In particular, this results shows how to construct an outer measure from a given map. Note, however, that if we consider φ to be an outer measure, the corresponding map μ^* yields the map φ. Hence, the result provides a way to obtain a measure from a given outer measure.

Remark 1.44. Note that, by construction, we have that $\mu^*(A) = \varphi(A)$ for all $A \in \mathcal{C}$. Furthermore, the set $\mathcal{M}_{\mu^*}$ contains the family $\mathcal{C}$ and, as a consequence, $\sigma(\mathcal{C}) \subset \mathcal{M}_{\mu^*}$.

Remark 1.45. Further properties can be obtained from [5, Theorem 1.3.2]. In particular, we have that $A \in \mathcal{M}_{\mu^*}$ for every $A \in \mathcal{P}(X)$ such that $\mu^*(A) = 0$. This ensures that the measures constructed through Theorem 1.42 are *complete*, that is, every subset of a measurable set with measure zero is measurable. Indeed, let $A \in \mathcal{M}_{\mu^*}$, $\mu(A) = 0$, and $E \subset A$. By the definition of outer measure, we have that

$$0 \le \mu^*(E) \le \mu^*(A) = \mu^*|_{\mathcal{M}_{\mu^*}}(A) = \mu(A) = 0.$$

Therefore, $\mu^*(E) = 0$, and so [5, Theorem 1.3.2] ensures that $E \in \mathcal{M}_{\mu^*}$.

In the next example we show how to construct a Lebesgue–Stieltjes measure from a given nondecreasing and left–continuous function using Carathéodory's Extension Theorem. This example is particularly interesting as most of the Lebesgue–Stieltjes measures that we consider in the work ahead are of this form. In what follows, we denote

$$g(x^-) = \lim_{y \to x^-} g(y), \quad g(x^+) = \lim_{y \to x^+} g(y).$$

Example 1.46. Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and consider the following family of subsets of $\mathbb{R}$,

$$\mathcal{C} = \{[a, b) : -\infty < a < b \le +\infty\} \cup \{(-\infty, a) : -\infty < a \le +\infty\} \cup \{\emptyset\},$$

and the map $\varphi : \mathcal{C} \to [0, +\infty]$ given by

$$\varphi(I) = \begin{cases} 0 & \text{if } I = \emptyset, \\ g(b) - g(a) & \text{if } I = [a, b),\ a, b \in \mathbb{R},\ a < b, \\ \lim\limits_{x \to +\infty} g(x) - g(a) & \text{if } I = [a, +\infty),\ a \in \mathbb{R}, \\ g(a) - \lim\limits_{x \to -\infty} g(x) & \text{if } I = (-\infty, a),\ a \in \mathbb{R}, \\ \lim\limits_{x \to +\infty} g(x) - \lim\limits_{x \to -\infty} g(x) & \text{if } I = \mathbb{R}. \end{cases} \tag{1.6}$$

First, let us show that condition (i) in Theorem 1.42 is satisfied. Let $A, B \in \mathcal{C}$. We distinguish some different cases. If $A = \emptyset$ or $B = \emptyset$, then $A \cap B = \emptyset \in \mathcal{C}$. Next, if $A = (-\infty, a)$ and $B = (-\infty, b)$, $-\infty < a, b \leq +\infty$, then $A \cap B = (-\infty, \min\{a, b\}) \in \mathcal{C}$. Now, if $A = [a, b)$, $-\infty < a < b \leq +\infty$, $B = (-\infty, c)$, $-\infty < c \leq +\infty$, then if $a < c$,

$$A \cap B = [a, \min\{b, c\});$$

otherwise $A \cap B = \emptyset$. In both cases, $A \cap B \in \mathcal{C}$. Finally, if $A = [a_1, b_1)$ and $B = [a_2, b_2)$ for some $-\infty < a_i < b_i \leq +\infty$, $i \in \{1, 2\}$, then if $\max\{a_1, a_2\} < \min\{b_1, b_2\}$,

$$A \cap B = [\max\{a_1, a_2\}, \min\{b_1, b_2\}),$$

otherwise, $A \cap B = \emptyset$. In either case, $A \cap B \in \mathcal{C}$, and so condition (i) in Theorem 1.42 is satisfied.

For condition (ii), take $A \in \mathcal{C}$. If $A = \emptyset$ or $A = \mathbb{R}$, then the condition is trivially satisfied. If $A = (-\infty, a)$ for some $a \in \mathbb{R}$, then $\mathbb{R} \setminus A = [a, +\infty) \in \mathcal{C}$, so the property clearly holds. Finally, consider $A = [a, b)$ for some $-\infty < a < b \leq +\infty$. If $b = +\infty$, then $\mathbb{R} \setminus A = (-\infty, a) \in \mathcal{C}$, so the condition is satisfied. Otherwise, $-\infty < a < b < +\infty$, and so, $\mathbb{R} \setminus A = (-\infty, a) \cup [b, +\infty)$. Note that $(-\infty, a), [b, +\infty) \in \mathcal{C}$. Moreover, since $a < b$, we have that $(-\infty, a) \cap [b, +\infty) = \emptyset$. In conclusion, condition (ii) is satisfied.

Now, given that condition (iii) is satisfied by definition, all that is left to do is to check that condition (iv) holds. Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{C}$ and denote $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$. If $A = \emptyset$, then the condition is trivially satisfied. We now consider all the other possible cases:

Case 1: $A = [a, b)$ for some $a, b \in \mathbb{R}$, $a < b$.
Under these circumstances, the sets A_n, $n = 1, 2, \ldots$, are either of the form $[c, d)$, $c, d \in \mathbb{R}$, $c < d$, or empty. First assume that there are finitely many sets of the form $[c, d)$, $c, d \in \mathbb{R}$, $c < d$, that cover A. Without loss of generality, we assume that these sets are the first k–th sets, that is, $A_n = [a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$, $a_n < b_n$, $n = 1, 2, \ldots, k$, and

$$A = \bigcup_{n=1}^{k} [a_n, b_n).$$

It is clear that, in this case, there exist $n_0, n_1 \in \{1, 2, \ldots, k\}$ such that $a = a_{n_0}$, $b = b_{n_1}$. Furthermore, given that the family $\{A_n\}_{n=1}^{k}$ is pairwise disjoint and A is a connected set, it follows that for each $n \in \{1, 2, \ldots, k\} \setminus \{n_1\}$, there exists $m_n \in \{1, 2, \ldots, k\} \setminus \{n_0\}$ such that $b_n = a_{m_n}$. Therefore, it follows that

$$\sum_{n=1}^{k} \varphi(A_n) = \sum_{n=1}^{k} (g(b_n) - g(a_n)) = g(b_{n_1}) - g(a_{n_0}) = g(b) - g(a) = \varphi(A). \tag{1.7}$$

Now, for the general case, we assume without loss of generality that, for all $n = 1, 2, \ldots$, $A_n = [a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$, $a_n < b_n$. Fix $k \in \mathbb{N}$. Since the sets A_n are pairwise disjoint, this implies the finite family $\{A_n\}_{n=1}^{k}$ does not cover the set A. Furthermore, note that the set $A \setminus \bigcup_{n=1}^{k} A_k$ has, at most, $k + 1$ different connected components, which are, in fact, intervals of the form $[c, d)$, $c, d \in \mathbb{R}$. Assume that there are $j \in \{1, 2, \ldots, k + 1\}$

different connected components, denoted by B_n, $n = 1, 2, \ldots, j$, and consider the family $\{C_n\}_{n=1}^{k+j}$ defined as

$$C_n = \begin{cases} A_n & \text{if } n \in \{1, 2, \ldots, k\}, \\ B_{n-k} & \text{if } n \in \{k+1, k+2, \ldots, k+j\}. \end{cases}$$

In that case, $\{C_n\}_{n=1}^{k+j}$ is a finite collection covering A, and thus, in the conditions of (1.7), which yields

$$\varphi(A) = \sum_{n=1}^{k+j} \varphi(C_n) \geq \sum_{n=1}^{k} \varphi(C_n) = \sum_{n=1}^{k} \varphi(A_n).$$

Since $k \in \mathbb{N}$ was arbitrarily chosen, we have that the previous inequality holds for all $k = 1, 2, \ldots$ so, by letting $k \to \infty$, we get

$$\varphi(A) \geq \sum_{n=1}^{\infty} \varphi(A_n).$$

Thus, all that is left to do is to show that the reverse inequality holds. In order to do so, we proceed as follows: since g is left–continuous, for each $0 < \varepsilon < b - a$ and $n \in \mathbb{N}$, there exists $a'_n < a_n$ such that

$$g(a_n) - g(a'_n) < \frac{\varepsilon}{2^n}.$$

Then, for a given $0 < \varepsilon < b-a$, the family $\{(a'_n, b_n)\}_{n=1}^{\infty}$ is an open cover of $[a, b-\varepsilon]$, so the Heine–Borel Theorem ensures the existence of a finite collection of intervals, $\{(a'_{n_l}, b_{n_l})\}_{l=1}^{N}$, covering $[a, b-\varepsilon]$. In particular, for any $0 < \varepsilon < b - a$, the collection $\{[a'_{n_l}, b_{n_l})\}_{l=1}^{N}$ also covers $[a, b-\varepsilon]$, which ensures that it covers $[a, b-\varepsilon)$. Therefore, for any $0 < \varepsilon < b - a$,

$$\begin{aligned} g(b-\varepsilon) - g(a) = \varphi([a, b-\varepsilon)) \leq \sum_{l=1}^{N} \varphi([a'_{n_l}, b_{n_l})) = \sum_{l=1}^{N} (g(b_{n_l}) - g(a'_{n_l})) \\ < \sum_{l=1}^{N} \left(g(b_{n_l}) - g(a_{n_l}) + \frac{\varepsilon}{2^{n_l}} \right) \leq \varepsilon + \sum_{n=1}^{\infty} (g(b_n) - g(a_n)). \end{aligned}$$

By letting $\varepsilon \to 0^+$, and since g is left–continuous, we obtain

$$\varphi(A) = g(b) - g(a) \leq \sum_{n=1}^{\infty} (g(b_n) - g(a_n)) = \sum_{n=1}^{\infty} \varphi(A_n),$$

which concludes the study of this case.

Case 2: $A = [a, +\infty)$ for some $a \in \mathbb{R}$.
In this case, the sets A_n are either empty, of the form $[c, d)$, $c, d \in \mathbb{R}$, $c < d$, or $[c, +\infty)$ for some $c \in \mathbb{R}$. First, let us consider the case where there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} = [c, +\infty)$ for some $c \in \mathbb{R}$. If $c = a$, then the property holds trivially. Otherwise, $a < c$. In that case, the sets A_n, $n = 1, 2, \ldots, n \neq n_0$, are of the form $[a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$,

$a_n < b_n$, or empty. Furthermore, given that the family $\{A_n\}_{n=1}^{\infty}$ is pairwise disjoint, the collection $\{B_n\}_{n=1}^{\infty}$ defined as

$$B_n = \begin{cases} A_n & \text{if } n \in \{1, 2, \dots, n_0\}, \\ A_{n+1} & \text{if } n \geq n_0. \end{cases} \tag{1.8}$$

is pairwise disjoint and $[a, c) = \bigcup_{k=1}^{\infty} B_n$. Therefore, Case 1 for $[a, c)$ and the collection $\{B_n\}_{n=1}^{\infty}$ yields

$$\begin{aligned}\sum_{k=1}^{\infty} \varphi(A_n) &= \varphi(A_{n_0}) + \sum_{k=1}^{\infty} \varphi(B_n) \\ &= \lim_{x \to +\infty} g(x) - g(c) + g(c) - g(a) = \lim_{x \to +\infty} g(x) - g(a) = \varphi(A).\end{aligned}$$

Assume now that the sets A_n, $n \in \mathbb{N}$, are of the form $[c, d)$, $c, d \in \mathbb{R}$, $c < d$, or empty. Without loss of generality, we assume that, for all $n = 1, 2, \dots$ $A_n = [a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$, $a_n < b_n$. Note that this implies that $\sup a_n = +\infty$, for otherwise, one of the sets A_n would be unbouded or the sets $\{A_n\}_{n=1}^{\infty}$ would not be pairwise disjoint. Define

$$\widetilde{a}_1 = a, \quad \widetilde{a}_k = \sup\{a_n : a_n \leq a + k,\ n = 1, 2, \dots\}, \quad k \in \{2, 3, \dots\}.$$

Observe that $\widetilde{a}_k$, $k = 2, 3, \dots$, are well–defined as the sets $\{a_n : a_n \leq a + k,\ n = 1, 2, \dots\}$, $k = 2, 3, \dots$, are nonempty. Furthermore, it follows that $\widetilde{a}_k \xrightarrow{k \to \infty} +\infty$. Moreover, $\widetilde{a}_k \in A$, $k = 1, 2, \dots$, therefore, there exists $n_k \in \mathbb{N}$ such that $\widetilde{a}_k \in [a_{n_k}, b_{n_k})$. Note that, since the sets A_n are pairwise disjoint, we must have that $\widetilde{a}_k = a_{n_k}$. Consider the collection $\{\widetilde{A}_k\}_{k=1}^{\infty}$ given by $\widetilde{A}_k = [\widetilde{a}_k, \widetilde{a}_{k+1})$. By construction, we have that each A_n, $n = 1, 2, \dots$, is contained in one, and only in one, $\widetilde{A}_k$, $k = 1, 2, \dots$

Now, for each $n = 1, 2, \dots$, consider

$$\Lambda_k = \{n = 1, 2, ... : [a_n, b_n) \subset \widetilde{A}_k\}.$$

Note that the family $\{\Lambda_n\}_{n \in \mathbb{N}}$ is a partition of $\mathbb{N}$. Thus,

$$\sum_{n=1}^{\infty} \varphi(A_n) = \sum_{n=1}^{\infty} (g(b_n) - g(a_n)) = \sum_{k=1}^{\infty} \left(\sum_{n \in \Lambda_k} (g(b_n) - g(a_n)) \right).$$

However, for each $k \in \mathbb{N}$, we have that $\widetilde{A}_k = \bigcup_{n \in \Lambda_k} [a_n, b_n)$, and so, Case 1 yields that

$$\begin{aligned}\sum_{k=1}^{\infty} \left(\sum_{n \in \Lambda_k} (g(b_n) - g(a_n)) \right) &= \sum_{k=1}^{\infty} \varphi(\widetilde{A}_k) = \sum_{k=1}^{\infty} (g(\widetilde{a}_{k+1}) - g(\widetilde{a}_k)) \\ &= \lim_{k \to \infty} g(\widetilde{a}_{k+1}) - g(a_1) = \lim_{x \to +\infty} g(x) - g(a) = \varphi(A),\end{aligned}$$

as the series $\sum_{k=1}^{\infty} (g(\widetilde{a}_{k+1}) - g(\widetilde{a}_k))$ is a telescoping series.

Case 3: $A = (-\infty, a)$ *for some* $a \in \mathbb{R}$.
This can be proved in an analogous way to Case 2. We provide a sketch of the proof but omit the details.

In this case, the sets A_n are either empty, of the form $[c, d)$, $c, d \in \mathbb{R}$, $c < d$, or $(-\infty, c)$ for some $c \in \mathbb{R}$. We distinguish two cases. First, we assume that there is $n_0 \in \mathbb{N}$ such that $A_{n_0} = (-\infty, c)$ for some $c \in \mathbb{R}$; which implies that the sets A_n, $n = 1, 2, \ldots, n \neq n_0$, are of the form $[a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$, $a_n < b_n$. In this case, we construct a collection similar to (1.8) and reason analogously. The other case we consider is the complementary, that is, we have that the sets A_n, $n \in \mathbb{N}$, are of the form $[c, d)$, $c, d \in \mathbb{R}$, or empty. Once again, we assume without loss of generality that $A_n = [a_n, b_n)$ for $a_n, b_n \in \mathbb{R}$, $a_n < b_n$, $n = 1, 2, \ldots$ Similarly to Case 2, we define

$$\widetilde{a}_1 = a, \quad \widetilde{a}_k = \inf\{a_n : a_n \geq a - k,\ n = 1, 2, \ldots\}, \quad k \in \{2, 3 \ldots\},$$

and construct a new family of sets, $\{\widehat{A}_n\}_{n=1}^{\infty}$, given by $\widehat{A}_n = [a_{k+1}, a_k)$ and we reason analogously from there.

Case 4: $A = \mathbb{R}$
If $A_n = \mathbb{R}$ for some $n \in \mathbb{N}$, then the condition is trivially satisfied. Otherwise, there exists at least one element of $\{A_n\}_{n=1}^{\infty}$ that is bounded from below, say $A_{n_0} = [a_{n_0}, b_{n_0})$, $a_{n_0} \in \mathbb{R}$, $a_{n_0} < b_{n_0} \leq +\infty$. Consider the collections $\{B_n\}_{n=1}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$, defined as

$$B_n = A_n \cap [a_{n_0}, +\infty), \quad C_n = A_n \cap (-\infty, a_{n_0}).$$

Clearly, both collections are pairwise disjoint and

$$\bigcup_{n=1}^{\infty} B_n = [a_{n_0}, +\infty), \quad \bigcup_{n=1}^{\infty} C_n = (-\infty, a_{n_0}).$$

Therefore, Cases 2 and 3 ensure that

$$\sum_{n=1}^{\infty} \varphi(B_n) = \varphi([a_{n_0}, +\infty)), \quad \sum_{n=1}^{\infty} \varphi(C_n) = \varphi((-\infty, a_{n_0})).$$

Furthermore, for all $n = 1, 2, \ldots$, B_n and C_n are disjoint and $A_n = B_n \cup C_n$. Therefore, Cases 1, 2 and 3 ensure that $\varphi(A_n) = \varphi(B_n) + \varphi(C_n)$ for all $n = 1, 2, \ldots$, so

$$\sum_{n=1}^{\infty} \varphi(A_n) = \sum_{n=1}^{\infty} \varphi(B_n) + \sum_{n=1}^{\infty} \varphi(C_n) = \lim_{x \to +\infty} g(x) - g(a_{n_0}) + g(a_{n_0}) - \lim_{x \to -\infty} g(x) = \varphi(A).$$

In conclusion, we have that the hypotheses of Theorem 1.42 are satisfied, so the map μ^* defined as (1.4) is an outer measure, the set $\mathcal{M}_{\mu^*}$ defined as in (1.5) is a σ–algebra and the restriction $\mu = \mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure. Let us show that it is a Lebesgue–Stieltjes measure.

First, we prove that $\mathcal{B}(\tau_u) \subset \mathcal{M}_{\mu^*}$. In order to do so, and given that τ_u is generated by open intervals, it is enough to show that every open interval belongs to $\mathcal{M}_{\mu^*}$. Let $a, b \in \mathbb{R}$,

$a < b$ and consider the collection $\{I_n\}_{n=1}^{\infty}$ given by $I_n = [a+1/n, b)$, $n = 1, 2, \ldots$. Clearly, we have that $\{I_n\}_{n=1}^{\infty} \subset \mathcal{C} \subset \mathcal{M}_{\mu^*}$. Therefore, since $\mathcal{M}_{\mu^*}$ is a σ–algebra, we have that

$$\bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right) = (a, b) \in \mathcal{M}_{\mu^*}.$$

Hence, every bounded open interval is a member of $\mathcal{M}_{\mu^*}$, which ensures that every open set in the usual topology lies in $\mathcal{M}_{\mu^*}$, and so, $\mathcal{B}(\tau_u) \in \mathcal{M}_{\mu^*}$.

Finally, let $A \in \mathcal{M}_{\mu^*}$ be a bounded set. Then, there is $M > 0$ such that $A \subset [-M, M)$ and, since μ is a measure, we have that

$$0 \leq \mu(A) \leq \mu([-M, M)) = \mu^*([-M, M)) = \varphi([-M, M)) = g(M) - g(-M),$$

which implies that $\mu(A)$ is finite.

Hence, the measure μ is a Lebesgue–Stieltjes measure. In the work ahead, we will denote this measure by μ_g and the σ–algebra in (1.5) by $\mathcal{LS}_g$. We will refer to measurability with respect to $\mathcal{LS}_g$ as *g–measurability* and we will denote the integral of a g–measurable function, f, with respect to the measure μ_g by

$$\int_E f(s) \, \mathrm{d}\, g(s), \quad E \in \mathcal{LS}_g.$$

Similarly, we will use the term *g–integrable functions* for μ_g–integrable functions and we will denote by $\mathcal{L}^1_g(X, \mathbb{R}) = \mathcal{L}^1_{\mu_g}(X, \mathbb{R})$.

As a final comment, we show explicitly the measure of some basic τ_u–Borel sets. Recall that we already know that

$$\mu_g([a, b)) = g(a) - g(b), \quad a, b \in \mathbb{R},\ a < b.$$

First, we show how to compute the measure of a singleton. Let $a \in \mathbb{R}$. Since $\{a\} \in \mathcal{B}(\tau)$, we have that $\{a\}$ is a g–measurable set and, using Proposition 1.8, we have that

$$\begin{aligned}\mu_g(\{a\}) &= \mu_g\left(\bigcap_{n=1}^{\infty}\left[a, a + \frac{1}{n}\right)\right) \\ &= \lim_{n\to\infty} \mu_g\left(\left[a, a + \frac{1}{n}\right)\right) = \lim_{n\to\infty}\left(g\left(a + \frac{1}{n}\right) - g(a)\right) = g(a^+) - g(a).\end{aligned}$$

Note that this means that there could be singletons with nonzero μ_g–measure, i.e. the measure μ_g can present *atoms*. As a direct consequence, we can compute the measure of any interval. Indeed, let $a, b \in \mathbb{R}$, $a < b$. Using the fact that μ_g is a measure, we have that

$$\mu_g([a, b]) = \mu_g([a, b) \cup \{b\}) = \mu_g([a, b)) + \mu_g(\{a\}) = g(b^+) - g(a).$$

Similarly, we have that $\mu_g([a, b)) = \mu_g((a, b) \cup \{a\}) = \mu_g((a, b)) + \mu_g(\{a\})$, from which

$$\mu_g((a, b)) = \mu_g([a, b)) - \mu_g(\{a\}) = g(b) - g(a^+).$$

Lastly, in a similar fashion,

$$\mu_g((a,b]) = \mu_g([a,b]) - \mu_g(\{a\}) = g(b^+) - g(a^+).$$

As a final comment, note that the map $g_c(t) = g(t) + c$, $c \in \mathbb{R}$, is another function in the hypotheses of this example, and thus, it defines a measure, which coincides with μ_g. To see that, it is enough to note that the map φ_c defined as (1.6) for g_c coincides with φ in (1.6).

Remark 1.47. Given $g : \mathbb{R} \to \mathbb{R}$ nondecreasing and left–continuous, and $X \subset \mathbb{R}$ g–measurable, we can construct a measure space on X by considering the restriction of μ_g to the σ–algebra

$$\mathcal{M}_X = \{A \cap X : A \in \mathcal{LS}_g\}.$$

Furthermore, given $a, b \in \mathbb{R}$, $a < b$, and a map $g : [a,b] \to \mathbb{R}$ which is nondecreasing and left–continuous, we can construct a Lebesgue–Stieltjes measure on $[a,b]$ as follows: consider $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ defined as

$$\widetilde{g}(x) = \begin{cases} g(a) & \text{if } x \le a, \\ g(x) & \text{if } a < x \le b, \\ g(b) & \text{if } x > b. \end{cases}$$

Then, the map $\widetilde{g}$ is in the hypotheses of Example 1.46, and so it defines a measurable space on $\mathbb{R}$. Now, by considering the corresponding restriction of the measure, we obtain a measure space on $[a,b]$.

Remark 1.48. It is possible to construct a Lebesgue–Stieltjes measure analogously if we consider the map g to be right–continuous instead, by doing the obvious changes. Furthermore, if we remove the continuity conditions on g and modify the map φ, we can still apply Theorem 1.42, see [5, Chapter 1, Section 2, Subsection 3].

Further generalizations can be done. For example, let $g \in BV([a,b], \mathbb{R})$, that is, the set of bounded variation functions on $[a,b]$ with values in $\mathbb{R}$. In that case, Jordan's Decomposition Theorem, [19, Proposition 4.4.2], guarantees the existence of two nondecreasing functions $g_1, g_2 : [a,b] \to \mathbb{R}$ such that $g = g_1 - g_2$. If in addition, g is left–continuous, then so are g_1 and g_2, see [19, Lemma 4.4.1]. Hence, given a left–continuous function of bounded variation, g, we can construct a signed measure, μ_g, in terms of its Jordan's decomposition functions, g_1 and g_2, as follows:

$$\mu_g(A) = \mu_{g_1}(A) - \mu_{g_2}(A), \quad A \in \mathcal{LS}_g := \mathcal{LS}_{g_1} \cap \mathcal{LS}_{g_2}.$$

Note that $\mathcal{B}(\tau_u) \subset \mathcal{LS}_g$ as Lebesgue–Stieltjes are always Borel. Furthermore, μ_g is σ–finite and it assigns finite value to bounded sets. Moreover, its restriction to $\mathcal{B}(\tau_u)$ is regular.

Observe that, for any $a \in \mathbb{R}$, we can write

$$[a, +\infty) = \bigcup_{n=1}^{\infty} [a+n-1, a+n), \quad (-\infty, a) = \bigcup_{n=1}^{\infty} [a-n, a-n+1),$$

and $\mathbb{R} = (-\infty, a) \cup [a, +\infty)$. Observe that this means that we can write any element of $\mathcal{C}$ in Example 1.46 as a countable pairwise disjoint union of intervals of the form $[c,d)$, $c, d \in \mathbb{R}$,

$c < d$. In particular, this means that the outer measure obtained in Example 1.46 can be computed as

$$\mu_g^*(A) = \inf\left\{ \sum_{n=1}^{\infty}(g(b_n)-g(a_n)) : A \subset \bigcup_{n=1}^{\infty}[a_n,b_n),\ \{[a_n,b_n)\}_{n=1}^{\infty} \subset \widetilde{\mathcal{C}} \right\}, \qquad (1.9)$$

with

$$\widetilde{\mathcal{C}} = \{[a,b) : a,b \in \mathbb{R},\ a<b,\}. \qquad (1.10)$$

Furthermore, we can restrict ourselves to an even smaller set to compute the outer measure. In order to show that property, we need following result.

Lemma 1.49. *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing and left–continuous functions and $\widetilde{\mathcal{C}}$ be as in* (1.10). *For every $\mathcal{U} = \{[a_n,b_n)\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{C}}$, there exists $\mathcal{V} = \{[c_n,d_n)\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{C}}$ such that the sets in $\mathcal{V}$ are pairwise disjoint and*

$$\bigcup_{n\in\mathbb{N}}[a_n,b_n) = \bigcup_{n\in\mathbb{N}}[c_n,d_n) \qquad \sum_{n\in\mathbb{N}}\varphi([c_n,d_n)) \le \sum_{n\in\mathbb{N}}\varphi([a_n,b_n)),$$

where φ is defined as in (1.6).

Proof. Let $U = \bigcup_{n\in\mathbb{N}}[a_n,b_n)$ and C_U be the set of all connected components of U with respect to the usual topology of $\mathbb{R}$. First, note that the set C_U is at most countable, as the set U is the countable union of connected sets. Secondly, observe that all the elements of C_U are connected subsets of $\mathbb{R}$. Thus, we have that they are intervals (including the whole $\mathbb{R}$) or singletons. However, an element of C_U cannot be a singleton. Indeed, suppose that $I \in C_U$ is a singleton, say $I=\{x\}$ for some $x\in U$. In that case, there exists $n_0\in\mathbb{N}$ such that $x\in[a_{n_0},b_{n_0})$, which is a contradiction since I is a connected component of U and $I\subset[a_{n_0},b_{n_0})$. Hence, all the elements of C_U intervals. Furthermore, we claim that $\sup I \notin I$ for any $I\in C_U$ bounded from above. Indeed, let $I\in C_U$ be bounded from above and suppose that $\sup I\in I\subset U$. In this conditions, there exists $n_1\in\mathbb{N}$ such that $\sup I\in[a_{n_1},b_{n_1})$. Hence, we have that the set $I\cup[a_{n_1},b_{n_1})$ is a connected set containing I, which is a contradiction with $I\in C_U$. Therefore, we have that C_U is an at most countable collection of pairwise disjoint sets of the form (a,b), $[a,b)$, $[a,+\infty)$, $(-\infty,b)$, $a,b\in\mathbb{R}$, or $C_U=\{\mathbb{R}\}$.

For each $I\in C_U$ and define $\mathcal{F}_I\subset\widetilde{\mathcal{C}}$ as

- if $I=(a,b)$, $a,b\in\mathbb{R}$, then $\mathcal{F}_I = \{[a+(b-a)/(n+1), a+(b-a)/n)\}_{n\in\mathbb{N}}$;
- if $I=[a,b)$, $a,b\in\mathbb{R}$, then $\mathcal{F}_I = \{[a+(n-1)(b-a)/n, a+n(b-a)/(n+1))\}_{n\in\mathbb{N}}$;
- if $I=[a,+\infty)$, $a\in\mathbb{R}$, then $\mathcal{F}_I = \{[a+n-1, a+n)\}_{n\in\mathbb{N}}$;
- if $I=(-\infty,b)$, $b\in\mathbb{R}$, then $\mathcal{F}_I = \{[b-n, b-n+1)\}_{n\in\mathbb{N}}$.
- if $I=\mathbb{R}$, then $\mathcal{F}_I = \{[n,n+1)\}_{n\in\mathbb{Z}}$.

This way, for each $I \in C_U$ we find a countable pairwise disjoint family in $\widetilde{\mathcal{C}}$, $\mathcal{F}_I$, such that $I = \bigcup_{J\in\mathcal{F}_I} J$. Furthermore, for each $I \in C_U$, given that I and $J \in \mathcal{F}_I$ are Borel sets (and thus, g–measurable), it follows from Remark 1.44 that

$$\sum_{J\in\mathcal{F}_I} \varphi(J) = \sum_{J\in\mathcal{F}_I} \mu_g^*(J) = \sum_{J\in\mathcal{F}_I} \mu_g(J) = \mu_g\left(\bigcup_{J\in\mathcal{F}_I} J\right) = \mu_g(I) = \mu_g^*(I) = \varphi(I). \quad (1.11)$$

Define $\mathcal{V} = \bigcup_{I\in C_U} \mathcal{F}_I$. First, observe that $\mathcal{V}$ is a countable set as it is defined as an at most countable union of countable sets. Furthermore, by definition, $\mathcal{F}_I \subset \widetilde{\mathcal{C}}$ for all $I \in C_U$, which guarantees that $\mathcal{V} \subset \widetilde{\mathcal{C}}$. Hence, we can write $\mathcal{V} = \{[c_n, d_n)\}_{n\in\mathbb{N}}$ for some $c_n, d_n \in \mathbb{R}$. Let us show that $\mathcal{V}$ satisfies the properties in the statement of the result.

First, note that the sets in $\mathcal{V}$ are pairwise disjoint. Indeed, let $[c_n, d_n)$ and $[c_m, d_m)$ be two elements of $\mathcal{V}$. If they belong to the same connected component, $I \in C_U$, then, by construction of $\mathcal{F}_I$, we have that $[c_n, d_n) \cap [c_m, d_m) = \emptyset$. Otherwise, $[c_n, d_n) \in I$ and $[c_m, d_m) \in I'$ for some $I, I' \in C_U$, $I \neq I'$. Then, the definition of connected component guarantees that $I \cap I' = \emptyset$, which yields $[c_n, d_n) \cap [c_m, d_m) = \emptyset$. Hence, the family $\mathcal{V}$ is pairwise disjoint. Furthermore,

$$U = \bigcup_{I\in C_U} I = \bigcup_{I\in C_U}\left(\bigcup_{J\in\mathcal{F}_I} J\right) = \bigcup_{V\in\mathcal{V}} V = \bigcup_{n\in\mathbb{N}} [c_n, d_n).$$

Finally, using (1.11) and, once again, the fact that U and each $I \in C_U$ are a Borel sets (and thus, g–measurable), it follows from Remark 1.44 that

$$\begin{aligned}\sum_{n\in\mathbb{N}} \varphi([c_n, d_n)) &= \sum_{V\in\mathcal{V}} \varphi(V) = \sum_{I\in C_U}\left(\sum_{J\in\mathcal{F}_I} \varphi(J)\right) = \sum_{I\in C_U} \varphi(I) = \sum_{I\in C_U} \mu_g^*(I)\\ &= \sum_{I\in C_U} \mu_g(I) = \mu_g\left(\bigcup_{I\in C_U} I\right) = \mu_g(U) = \mu_g^*(U) \le \sum_{n\in\mathbb{N}} \varphi([a_n, b_n)),\end{aligned}$$

where the last inequality follows from (1.9). □

As a consequence, we can obtain a new expression for the outer measure associated to a nondecreasing and left–continuous map in Example 1.46, as we anticipated.

Theorem 1.50. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous functions and μ_g^* be as in* (1.9). *Then,*

$$\mu_g^*(A) = \inf\left\{\sum_{n=1}^{\infty}(g(b_n) - g(a_n)) : A \subset \bigcup_{n=1}^{\infty} [a_n, b_n),\ \{[a_n, b_n)\}_{n=1}^{\infty} \subset \widetilde{\mathcal{C}} \textit{ pairwise disjoint}\right\}. \quad (1.12)$$

Proof. It is clear that the infimum in (1.9) is less or equal than the one in (1.12). On the other hand, Lemma 1.49 guarantees the reverse inequality. Indeed, given $\{[a_n, b_n)\}_{n\in\mathbb{N}} \in \widetilde{\mathcal{C}}$

such that $A \subset \bigcup_{n\in\mathbb{N}}[a_n, b_n)$, we can find $\{[c_n, d_n)\}_{n\in\mathbb{N}} \in \widetilde{\mathcal{C}}$ pairwise disjoint such that $A \subset \bigcup_{n\in\mathbb{N}}[c_n, d_n)$ and

$$\sum_{n\in\mathbb{N}}(g(d_n) - g(c_n)) = \sum_{n\in\mathbb{N}}\varphi([c_n, d_n)) \leq \sum_{n\in\mathbb{N}}\varphi([a_n, b_n)) = \sum_{n\in\mathbb{N}}(g(a_n) - g(b_n)).$$

This, of course, is enough to show that infimum in (1.12) is less or equal than the one in (1.9), finishing the proof. □

Before continuing to explore the bijection between the two possible definitions for the Lebesgue–Stieltjes integrals, we present some other interesting properties of the measures constructed as in Example 1.46. For example, we can obtain the following approximation result that will be useful in the work ahead. In order to simplify its statement we define

$$\mathcal{S}(\widetilde{\mathcal{C}}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \exists \, \{I_k\}_{k=1}^N \subset \widetilde{\mathcal{C}} \text{ such that } f = \sum_{k=1}^N \chi_{I_k} \right\}. \tag{1.13}$$

Proposition 1.51. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and E be a g–measurable set such that $\mu_g(E) < +\infty$. Then, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}(\widetilde{\mathcal{C}})$ such that*

$$\lim_{n\to\infty}\int_{\mathbb{R}} |s_n(x) - \chi_E(x)| \, \mathrm{d}\, g(x) = 0, \tag{1.14}$$

where $\chi_E : \mathbb{R} \to \mathbb{R}$ denotes the characteristic function of E. In particular, there exists a sequence in $\mathcal{S}(\widetilde{\mathcal{C}})$ that satisfies (1.14) *and converges to $\chi_E(x)$ for g–almost all $x \in \mathbb{R}$.*

Proof. Throughout this proof, given $A, B \subset \mathbb{R}$, we denote $A\triangle B = (A \setminus B) \cup (B \setminus A)$. We divide the proof into two steps.

Step 1: For each $\varepsilon > 0$, there exists $\{[a_k, b_k)\}_{k=1}^N \subset \widetilde{\mathcal{C}}$ such that $\mu_g(E\triangle \bigcup_{k=1}^N [a_k, b_k)) < \varepsilon$.
Let $\varepsilon > 0$. Then, it follows from (1.12) that there exists a family $\{[a_k, b_k)\}_{k\in\mathbb{N}} \subset \widetilde{\mathcal{C}}$, which is pairwise disjoint, and such that

$$\sum_{k\in\mathbb{N}}(g(b_k) - g(a_k)) < \mu_g^*(E) + \frac{\varepsilon}{2}.$$

Given that the sets E and $[a_k, b_k)$, $k \in \mathbb{N}$, are g–measurable, the inequality above is equivalent to

$$\sum_{k\in\mathbb{N}}\mu_g([a_k, b_k)) < \mu_g(E) + \frac{\varepsilon}{2}. \tag{1.15}$$

In particular, the hypotheses ensure that $\sum_{k\in\mathbb{N}}\mu_g([a_k, b_k)) < +\infty$. Therefore, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty}\mu_g([a_k, b_k)) < \frac{\varepsilon}{2}.$$

Let us show that the family $\{[a_k, b_k)\}_{k=1}^N$ satisfies the desire property.

First, observe that the sets $E \setminus \bigcup_{k=1}^{N}[a_k, b_k)$ and $\bigcup_{k=1}^{N}[a_k, b_k) \setminus E$ are g–measurable. Thus, given that $E \triangle \bigcup_{k=1}^{N}[a_k, b_k)$ is the disjoint union of those two sets, we have that it is g–measurable and

$$\mu_g\left(E \triangle \bigcup_{k=1}^{N}[a_k, b_k)\right) = \mu_g\left(E \setminus \bigcup_{k=1}^{N}[a_k, b_k)\right) + \mu_g\left(\bigcup_{k=1}^{N}[a_k, b_k) \setminus E\right). \tag{1.16}$$

Now, on the one hand, given that $\{[a_n, b_n)\}_{n\in\mathbb{N}}$ is a pairwise disjoint cover of E, we have that $E \setminus \bigcup_{k=1}^{N}[a_k, b_k) \subset \bigcup_{k=N+1}^{\infty}[a_k, b_k)$, so,

$$\mu_g\left(E \setminus \bigcup_{k=1}^{N}[a_k, b_k)\right) \leq \mu_g\left(\bigcup_{k=N+1}^{\infty}[a_k, b_k)\right) = \sum_{k=N+1}^{\infty}\mu_g([a_k, b_k)) < \frac{\varepsilon}{2}. \tag{1.17}$$

On the other hand, it is obvious that

$$\mu_g\left(\bigcup_{k=1}^{N}[a_k, b_k) \setminus E\right) \leq \mu_g\left(\bigcup_{k=1}^{\infty}[a_k, b_k) \setminus E\right).$$

Furthermore, we can write

$$\bigcup_{k\in\mathbb{N}}[a_k, b_k) = \left(\bigcup_{k\in\mathbb{N}}[a_k, b_k) \setminus E\right) \cup \left(\bigcup_{k\in\mathbb{N}}[a_k, b_k) \cap E\right) = \left(\bigcup_{k\in\mathbb{N}}[a_k, b_k) \setminus E\right) \cup E,$$

which, noting that this is a disjoint union, yields

$$\sum_{k\in\mathbb{N}}\mu_g([a_k, b_k)) = \mu_g\left(\bigcup_{k\in\mathbb{N}}[a_k, b_k)\right) = \mu_g\left(\bigcup_{k\in\mathbb{N}}[a_k, b_k) \setminus E\right) + \mu_g(E).$$

Hence, (1.15) ensures that $\mu_g(\bigcup_{k=1}^{\infty}[a_k, b_k) \setminus E) < \varepsilon/2$ which, in turn, proves that

$$\mu_g\left(\bigcup_{k=1}^{N}[a_k, b_k) \setminus E\right) < \frac{\varepsilon}{2}.$$

Now, the desired property follows from (1.16) and (1.17).

Step 2: There exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}(\widetilde{\mathcal{C}})$ satisfying (1.14).
For each $n \in \mathbb{N}$, there exists a pairwise disjoint family $\{I_k\}_{k=1}^{N_n} \subset \widetilde{\mathcal{C}}$ such that

$$\mu_g\left(E \triangle \bigcup_{k=1}^{N_n} I_k\right) < \frac{1}{n}.$$

Thus, consider the sequence $s_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, defined as

$$s_n(x) = \sum_{k=1}^{N_n}\chi_{I_k}(x), \quad x \in \mathbb{R}.$$

Of course, $\{s_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(\widetilde{\mathcal{C}})$ and thus, s_n is g–measurable for every $n \in \mathbb{N}$. Furthermore, denoting $A_n = E \triangle \bigcup_{k=1}^{N_n} I_k$, $n \in \mathbb{N}$, we have that, for each $n \in \mathbb{N}$,

$$|s_n(x) - \chi_E(x)| = \begin{cases} 1, & \text{if } x \in A_n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for each $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} |s_n(x) - \chi_E(x)| \,\mathrm{d}\, g(x) = \int_{A_n} 1 \,\mathrm{d}\, g(x) + \int_{\mathbb{R}\setminus A_n} 0 \,\mathrm{d}\, g(x) = \mu_g(A_n) < \frac{1}{n}.$$

Therefore, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |s_n(x) - \chi_E(x)| \,\mathrm{d}\, g(x) = 0.$$

Now, Theorems 3.11 and 3.12 in [73] ensure that there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that $s_{n_k}(x)$ converges to $\chi_E(x)$ for g–almost all $x \in \mathbb{R}$, which finishes the proof. □

Along the lines of Remark 1.48, consider two nondecreasing and left–continuous functions, $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, and the map $g : \mathbb{R} \to \mathbb{R}$ defined as $g(t) = g_1(t)+g_2(t)$, $t \in \mathbb{R}$. Clearly, the map g is also nondecreasing and left–continuous, so it defines a Lebesgue–Stieltjes measure. In the next result, we explore the relations between the measures defined by g_1, g_2 and g.

Proposition 1.52. *Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be two nondecreasing and left–continuous functions and define $g : \mathbb{R} \to \mathbb{R}$ as $g(t) = g_1(t) + g_2(t)$, $t \in \mathbb{R}$. Then*

$$\mu_g^*(E) = \mu_{g_1}^*(E) + \mu_{g_2}^*(E), \quad E \in \mathcal{P}(\mathbb{R}). \tag{1.18}$$

In particular, we have that every g_1 and g_2–measurable set is also g–measurable and

$$\mu_g(E) = \mu_{g_1}(E) + \mu_{g_2}(E), \quad E \in \mathcal{LS}_{g_1} \cap \mathcal{LS}_{g_2}.$$

Proof. Let $E \in \mathcal{P}(\mathbb{R})$. Then, using the characterization provided by (1.9), we have that

$$\begin{aligned}
\mu_g^*(E) &= \inf\left\{ \sum_{n=1}^{\infty} (g(b_n) - g(a_n)) : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n) \right\} \\
&= \inf\left\{ \sum_{n=1}^{\infty} [(g_1(b_n) - g_1(a_n)) + (g_2(b_n) - g_2(a_n))] : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n) \right\} \\
&= \inf\left\{ \sum_{n=1}^{\infty} (g_1(b_n) - g_1(a_n)) + \sum_{n=1}^{\infty} (g_2(b_n) - g_2(a_n)) : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n) \right\} \\
&\geq \sum_{i=1}^{2} \inf\left\{ \sum_{n=1}^{\infty} (g_i(b_n) - g_i(a_n)) : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n) \right\} = \mu_{g_1}^*(E) + \mu_{g_2}^*(E).
\end{aligned}$$

Therefore, it is enough to show that $\mu_g^*(E) \le \mu_{g_1}^*(E) + \mu_{g_2}^*(E)$ to obtain (1.18).

Let $\varepsilon > 0$. It follows from (1.12) and the definition of infimum of a set that there exist $\mathcal{R}_1 = \{[a_{1,n}, b_{1,n})\}_{n=1}^{\infty}$ and $\mathcal{R}_2 = \{[a_{2,m}, b_{2,m})\}_{m=1}^{\infty}$, each of them pairwise disjoint, such that

$$E \subset \bigcup_{n=1}^{\infty} [a_{1,n}, b_{1,n}), \qquad \sum_{n=1}^{\infty} (g_1(b_{1,n}) - g_1(a_{1,n})) \le \mu_{g_1}^*(E) + \frac{\varepsilon}{2},$$
$$E \subset \bigcup_{m=1}^{\infty} [a_{2,m}, b_{2,m}), \qquad \sum_{m=1}^{\infty} (g_2(b_{2,m}) - g_1(a_{2,m})) \le \mu_{g_2}^*(E) + \frac{\varepsilon}{2}.$$

Define

$$\mathcal{R} = \{[a_{1,n}, b_{1,n}) \cap [a_{2,m}, b_{2,m}) : n, m = 1, 2, \dots\} \setminus \{\emptyset\}.$$

Following the reasonings in Example 1.46 to check hypothesis (i) in Theorem 1.42, we have that the elements of $\mathcal{R}$ are of the form $[c, d)$, $c, d \in \mathbb{R}$, $c < d$, since, by construction, we removed those intersections that might be empty. Specifically, denoting

$$a_n^m = \max\{a_{1,n}, a_{2,m}\}, \quad b_n^m = \min\{b_{1,n}, b_{2,m}\}, \quad n, m = 1, 2, \dots,$$

we have that $\mathcal{R} = \{[a_n^m, b_n^m)\}_{(n,m)\in\mathcal{I}}$, where

$$\mathcal{I} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : \max\{a_{1,n}, a_{2,m}\} < \min\{b_{1,n}, b_{2,m}\}\}.$$

Of course, this means that $\mathcal{R}$ is countable. Furthermore, let $x \in E$. In that case, given that $\mathcal{R}_1$ and $\mathcal{R}_2$ are covers of E, there exists $n_0, m_0 \in \mathbb{N}$ such that $x \in [a_{1,n_0}, b_{1,n_0})$ and $x \in [a_{2,m_0}, b_{2,m_0})$, which means that

$$\max\{a_{1,n_0}, a_{2,m_0}\} \le x < \min\{b_{1,n_0}, b_{2,m_0}\},$$

and so $x \in [a_{n_0}^{m_0}, b_{n_0}^{m_0})$. This guarantees that

$$E \subset \bigcup_{(n,m)\in\mathcal{I}} [a_n^m, b_n^m),$$

or, in other words, $\mathcal{R}$ is a countable cover of E. Therefore,

$$\begin{aligned}
\mu_g^*(E) &\le \sum_{(n,m)\in\mathcal{I}} (g(b_n^m) - g(a_n^m)) \\
&= \sum_{(n,m)\in\mathcal{I}} ((g_1(b_n^m) - g_1(a_n^m)) + (g_2(b_n^m) - g_2(a_n^m))) \\
&= \sum_{(n,m)\in\mathcal{I}} (g_1(b_n^m) - g_1(a_n^m)) + \sum_{(n,m)\in\mathcal{I}} (g_2(b_n^m) - g_2(a_n^m)).
\end{aligned}$$

Observe that it is enough to show that

$$\sum_{(n,m)\in\mathcal{I}} (g_i(b_n^m) - g_i(a_n^m)) \le \sum_{k=1}^{\infty} (g_i(b_{i,k}) - g_i(a_{i,k})), \quad i = 1, 2, \tag{1.19}$$

since, in that case, we have that

$$\mu_g^*(E) \leq \sum_{(n,m)\in\mathcal{I}} (g_1(b_{1,n}) - g_1(a_{1,n})) + \sum_{(n,m)\in\mathcal{I}} (g_2(b_{2,m}) - g_2(a_{2,m}))$$
$$\leq \sum_{n=1}^{\infty}(g_1(b_{1,n}) - g_1(a_{1,n})) + \sum_{m=1}^{\infty}(g_2(b_{2,m}) - g_2(a_{2,m})) < \mu_{g_1}^*(E) + \mu_{g_2}^*(E) + \varepsilon,$$

which ensures that $\mu_g^*(E) \leq \mu_{g_1}^*(E) + \mu_{g_2}^*(E)$ as $\varepsilon > 0$ was arbitrarily fixed.

Let us show that (1.19) holds. For each $n \in \mathbb{N}$, define $\mathcal{I}_n = \{m \in \mathbb{N} : (n,m) \in \mathcal{I}\}$. Note that $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$, and so,

$$\sum_{(n,m)\in\mathcal{I}} (g_1(b_n^m) - g_1(a_n^m)) = \sum_{n=1}^{\infty} \left(\sum_{m\in\mathcal{I}_n} (g_1(b_n^m) - g_1(a_n^m)) \right). \tag{1.20}$$

On the other hand, by definition, we have that

$$\bigcup_{m\in\mathcal{I}_n} [a_n^m, b_n^m) = \bigcup_{m\in\mathcal{I}_n} ([a_{1,n}, b_{1,n}) \cap [a_{2,m}, b_{2,m})) \subset [a_{1,n}, b_{1,n}).$$

Furthermore, given that the family $\mathcal{R}_2$ is pairwise disjoint, it follows that $\{[a_n^m, b_n^m)\}_{m=1}^{\infty}$ is also pairwise disjoint. Hence, given that μ_{g_1} is a measure, we have that

$$\sum_{m\in\mathcal{I}_n} (g_1(b_n^m) - g_1(a_n^m)) = \sum_{m\in\mathcal{I}_n} \mu_{g_1}([a_n^m, b_n^m)) = \mu_{g_1}\left(\bigcup_{m\in\mathcal{I}_n} [a_n^m, b_n^m) \right)$$
$$\leq \mu_{g_1}([a_{1,n}, b_{1,n})) = g_1(b_{1,n}) - g_1(a_{1,n}).$$

As a consequence, it follows from (1.20) that (1.19) holds for $i = 1$. The case $i = 2$ is analogous and we omit it. Hence, we have that (1.18) holds.

The rest of the result now follows from Theorem 1.42. □

Remark 1.53. Observe that the information in Proposition 1.52 remains true when we consider g to be the sum of a finite number of nondecreasing and left–continuous functions, g_i, $i = 1, 2, \ldots, n$. In that case, we have that every g_i–measurable set, $i = 1, 2, \ldots, n$, is g–measurable and

$$\mu_g(E) = \sum_{i=1}^{n} \mu_{g_i}(E), \quad E \in \bigcap_{i=1}^{n} \mathcal{LS}_{g_i}.$$

Note that this conditions guarantees that every g_i–measurable map, $i = 1, 2, \ldots, n$, is also g–measurable.

The last properties regarding the measures constructed as in Example 1.46 are related to two interesting sets defined in terms of the function g: the set of discontinuities, i.e.,

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}; \tag{1.21}$$

and the set of points around which g is constant, namely,

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}. \tag{1.22}$$

First, note that, for every $t \in D_g$, it holds that $\mu_g(\{t\}) = g(t^+) - g(t) > 0$. In particular, the integral of a measurable function over a singleton formed by $t \in D_g$ is given by

$$\int_{\{t\}} f(s)\,\mathrm{d}\,g(s) = f(t)(g(t^+) - g(t)). \tag{1.23}$$

Also, note that, by definition, C_g is an open set in the usual topology of $\mathbb{R}$, which is second countable. Therefore, C_g can be uniquely rewritten as the countable union of disjoint open intervals, say

$$C_g = \bigcup_{n\in\mathbb{N}} (a_n, b_n). \tag{1.24}$$

With this notation, we define

$$N_g^- := \{a_n : n \in \mathbb{N}\} \setminus D_g, \quad N_g^+ := \{b_n : n \in \mathbb{N}\} \setminus D_g, \quad N_g = N_g^- \cup N_g^+. \tag{1.25}$$

The following result sums up some properties of these sets presented in [54] and in Froda's Theorem, [35].

Proposition 1.54. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function. Then:*

(i) *D_g is at most countable;*

(ii) $\mu_g(C_g) = \mu_g(N_g) = 0$.

Remark 1.55. As a consequence of Proposition 1.54, a property holds μ_g–almost everywhere in a set E if and only if it holds μ_g–almost everywhere in $E\setminus O_g$ where $O_g = C_g \cup N_g$.

Finally, we conclude this section by completing the mentioned bijection between the two definitions of Lesbesgue–Stieltjes integrals. The first implication has already been provided in Example 1.46, where we showed that given a nondecreasing and left–continuous function on $\mathbb{R}$, we can construct a Lebesgue–Stieltjes measure on $\mathbb{R}$ in the sense of Definition 1.38. Now we have a look at the converse, that is, we show that given a Lebesgue–Stieltjes measure on $\mathbb{R}$, we can construct a nondecreasing and left–continuous function defining the measure over the Borel sets. This is done by following the ideas in [5, Chapter 1, Section 2, Subsection 3].

Proposition 1.56. *Let $(\mathbb{R}, \mathcal{M}, \mu)$ be a measure space and μ be a Lebesgue–Stieltjes measure. Then, there exists a nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$\mu([a, b)) = g(b) - g(a), \quad a, b \in \mathbb{R},\ a < b. \tag{1.26}$$

Proof. Let $c \in \mathbb{R}$ and define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \begin{cases} -\mu([x, c)) & \text{if } x < c. \\ 0 & \text{if } x = c, \\ \mu([c, x)) & \text{if } x > c. \end{cases} \tag{1.27}$$

By construction, g is nondecreasing. Furthermore, g is left–continuous. Indeed, take $x \in \mathbb{R}$. Since g is nondecreasing, we know that the left–hand side limit of g at x exists, and moreover,

$$\lim_{y \to x^-} g(y) = \lim_{n \to \infty} g\left(x - \frac{1}{n}\right).$$

We study two possible cases separately. First, suppose that $x > c$. In that case, we know that $x - 1/n < c$ for $n \in \mathbb{N}$ large enough, so the definition of g and Theorem 1.8 ensure that

$$\lim_{n \to \infty} g\left(x - \frac{1}{n}\right) = \lim_{n \to \infty} \mu\left(\left[c, x - \frac{1}{n}\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} \left[c, x - \frac{1}{n}\right)\right) = \mu([c, x)) = g(x).$$

Now, if $x \leq c$, using Theorem 1.8 we have that

$$\lim_{n \to \infty} g\left(x - \frac{1}{n}\right) = \lim_{n \to \infty} \left(-\mu\left(\left[x - \frac{1}{n}, c\right)\right)\right) = -\mu\left(\bigcap_{n=1}^{\infty} \left[c, x - \frac{1}{n}\right)\right).$$

Now, the intersection in the previous equation equals $[c, x)$ if $x \neq c$, and $\emptyset$ if $x = c$. In either case, we have that the value is $g(x)$.

Hence, all that is left to do is to check that (1.26) holds. Fix $a, b \in \mathbb{R}$, $a < b$. Note that if $a = c$ or $b = c$, (1.26) holds trivially. We distinguish three possible cases. The first one, if $c < a < b$. In this case, we have that

$$\mu([c, b)) = \mu([c, a) \cup [a, b)) = \mu([c, a)) + \mu([a, b)),$$

from which we get that

$$\mu([a, b)) = \mu([c, b)) - \mu([c, a)) = g(b) - g(a).$$

Next, if $a < c < b$, we proceed similarly:

$$\mu([a, b)) = \mu([a, c) \cup [c, b)) = \mu([a, c)) + \mu([c, b)) = g(b) - g(a).$$

Finally, if $a < b < c$,

$$\mu([a, c)) = \mu([a, b) \cup [b, c)) = \mu([a, b)) + \mu([b, c)),$$

which yields

$$\mu([a, b)) = \mu([a, c)) - \mu([b, c)) = g(b) - g(a).$$

Therefore, condition (1.26) holds for g as in (1.27), which concludes the proof. □

Remark 1.57. Note that there exists infinitely many nondecreasing and left–continuous maps satisfying (1.26), as the family of functions $g_k(x) = g(x) + k$, $k \in \mathbb{R}$, with g as in (1.27) clearly satisfies equation (1.26).

Remark 1.58. It is possible to consider right–continuity instead of left–continuity by making the obvious changes. In fact, in probability theory this is the usual case, see [5] for example. Furthermore, it is possible to extend this result for a measure space $([a,b],\mathcal{M},\mu)$, $a,b\in\mathbb{R}$, $a<b$, by consider the map $g:\mathbb{R}\to\mathbb{R}$

$$g(t)=\begin{cases}0 & \text{if } t\leq a,\\ \mu([a,t)) & \text{if } a<t\leq b,\\ \mu([a,b]) & \text{if } t>b.\end{cases} \tag{1.28}$$

In that case, we have that

$$\mu([c,d))=g(d)-g(c),\quad \text{for all } [c,d)\subset[a,b).$$

Observe that given $(X,\mathcal{M},\mu)$ a Lebesgue–Stieltjes measure space, we can assign it a nondecreasing and left–continuous function g, to which we can repeat the procces in Example 1.46 to construct a new Lebesgue–Stieltjes measure μ_g. Noting that, in that case, we have that $\mu_g=\mu$ on the family $\mathcal{C}$ in Example 1.46, then Theorem 1.42 ensures that μ_g and μ must coincide in $\sigma(\mathcal{C})$, which is at least, $\mathcal{B}(\tau)$. Therefore, this shows that there exists a bijection between the Lebesgue–Stieltjes measures on $\mathcal{B}(\tau)$ and the set of nondecreasing and left–continuous functions (up to a constant).

Remark 1.59. It is possible to construct a bijection between the set of signed measures and the set of functions of bounded variation on an interval in a similar fashion as in the Lebesgue–Stieltjes case, as presented in [19, Proposition 4.4.3].

1.2.2 The Kurzweil–Stieltjes integral

The Kurzweil–Stieltjes integral is another generalization of the Lebesgue integral. This particular integral is constructed in a similar fashion to the Riemann integral, or more precisely, the Riemann–Stieltjes integral since it is defined using two functions. In here, we shall show that this integral is more general than that of Lebesgue–Stieltjes as well. In what follows, we define the Kurzweil–Stieltjes integral and we include some of its most important properties that can be found in [65, 82].

First, we introduce the basic definitions required to properly define this integral.

Definition 1.60. *Let $a,b\in\mathbb{R}$, $a\leq b$. A* partition of $[a,b]$ *is a set $P=\{x_0,x_1,x_2,\dots,x_n\}$ such that $a=x_0<x_1<\dots<x_n=b$. A* tagged partition of $[a,b]$ *is a pair $(P,\{t_k\}_{k=1}^n)$ such that $P=\{x_0,x_1,x_2,\dots,x_n\}$ is a partition of $[a,b]$ and $t_k\in[x_{k-1},x_k]$ for every $k\in\{1,2,\dots,n\}$. A* gauge on $[a,b]$ *is a function $\delta:[a,b]\to(0,+\infty)$. Given a gauge δ, a tagged partition $(P,\{t_k\}_{k=1}^n)$, $P=\{x_0,x_1,\dots,x_n\}$, is said to be* δ–fine, *if for each $k\in\{1,2,\dots,n\}$,*

$$[x_{k-1},x_k]\subset(t_k-\delta(t_k),t_k+\delta(t_k))$$

In order to simplify the definition of the Kurzweil–Stieltjes integral of a function, we will use the following notation: given $f,g:[a,b]\to\mathbb{R}$ and a tagged partition $(P,\{t_k\}_{k=1}^n)$, $P=\{x_0,x_1,x_2,\dots,x_n\}$, we denote

$$S(f,g,P,\{t_k\})=\sum_{k=1}^n f(t_k)(g(x_k)-g(x_{k-1})).$$

Definition 1.61. *Let $f, g : [a,b] \to \mathbb{R}$. We say that f is* Kurzweil–Stieltjes integrable on $[a,b]$ with respect to g *if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a gauge on $[a,b]$, δ_ε, such that for every δ_ε–fine tagged partition of $[a,b]$, $(P, \{t_k\}_{k=1}^n)$,*

$$|S(f,g,P,\{t_k\}) - I| < \varepsilon.$$

In that case, the number I is called the Kurzweil–Stieltjes integral of f on $[a,b]$ with respect to g*, and we denote it by*

$$^{(KS)}\!\!\int_a^b f(s)\,\mathrm{d}\,g(s).$$

Remark 1.62. Lemmas 6.2.3. and 6.2.4 in [65] ensure that the integral is well–defined and its value is uniquely determined.

Remark 1.63. The integral notation introduced above allows us to distinguish between the Lebesgue–Stieltjes integral and the Kurzweil–Stieltjes integral when both of them make sense. On that note, it is interesting to note that the Kurzweil–Stieltjes integrator needs not be monotonous or sideway continuous.

This integral satisfies some of the usual properties of integrals, such as lineality. In the next result, we gather the information of Theorems 6.2.7 and 6.2.10 in [65].

Theorem 1.64. *Let $a, b, c_1, c_2 \in \mathbb{R}$, $a \le b$, $c \in (a,b)$ and $f_1, f_2, g : [a,b] \to \mathbb{R}$. Then:*

(i) *If f_1 and f_2 are Kurzweil–Stieltjes integrable on $[a,b]$ with respect to g, then so is the function $c_1 f_1 + c_2 f_2$ and*

$$^{(KS)}\!\!\int_a^b (c_1 f_1 + c_2 f_2)(s)\,\mathrm{d}\,g(s) = c_1\,^{(KS)}\!\!\int_a^b f_1(s)\,\mathrm{d}\,g(s) + c_2\,^{(KS)}\!\!\int_a^b f_2(s)\,\mathrm{d}\,g(s).$$

(ii) *If g is Kurzweil–Stieltjes integrable on $[a,b]$ with respect to f_1 and f_2, then it is Kurzweil–Stieltjes integrable on $[a,b]$ with respect to $c_1 g_1 + c_2 g_2$ and*

$$^{(KS)}\!\!\int_a^b g(s)\,\mathrm{d}(c_1 f_1 + c_2 f_2)(s) = c_1\,^{(KS)}\!\!\int_a^b g(s)\,\mathrm{d}\,f_1(s) + c_2\,^{(KS)}\!\!\int_a^b g(s)\,\mathrm{d}\,f_2(s).$$

(iii) *If f_1 is Kurzweil–Stieltjes integrable with respect to g on $[a,c]$ and $[c,b]$, then it is Kurzweil–Stieltjes integrable on $[a,b]$ with respect to g and*

$$^{(KS)}\!\!\int_a^b f_1(s)\,\mathrm{d}\,g(s) = \,^{(KS)}\!\!\int_a^c f_1(s)\,\mathrm{d}\,g(s) + \,^{(KS)}\!\!\int_c^b f_1(s)\,\mathrm{d}\,g(s).$$

In terms of the indefinite Kurzweil–Stieltjes integral we have some interesting properties. First, is satisfies Hake's property, as shown in [65, Theorem 6.5.6].

Theorem 1.65. *Let $f, g : [a,b] \to \mathbb{R}$. Then:*

(i) *If the integral* ${}^{(KS)}\!\int_t^b f\,\mathrm{d}\,g$ *exists for every* $t \in (a,b]$*, and* $A \in \mathbb{R}$ *is such that*

$$\lim_{t\to a^+}\left({}^{(KS)}\!\int_t^b f\,\mathrm{d}\,g + f(a)(g(t)-g(a))\right) = A,$$

then ${}^{(KS)}\!\int_a^b f\,\mathrm{d}\,g$ *exists and equals* A*.*

(ii) *If the integral* ${}^{(KS)}\!\int_a^t f\,\mathrm{d}\,g$ *exists for every* $t \in [a,b)$*, and* $A \in \mathbb{R}$ *is such that*

$$\lim_{t\to b^-}\left({}^{(KS)}\!\int_a^t f\,\mathrm{d}\,g + f(b)(g(b)-g(t))\right) = A,$$

then ${}^{(KS)}\!\int_a^b f\,\mathrm{d}\,g$ *exists and equals* A*.*

When the integrator function is regulated, further properties of the indefinite integral can be obtained. First, let us formally introduce the definition of regulated functions as well as some notation.

Definition 1.66. *Let* $a,b \in \mathbb{R}$*,* $a < b$*. A function* $f : [a,b] \to \mathbb{R}^n$ *is said to be* regulated *if the limits*

$$\lim_{x\to a^+} f(x), \qquad \lim_{x\to b^-} f(x),$$

exist, and so do

$$\lim_{x\to c^-} f(x), \qquad \lim_{x\to c^+} f(x), \quad \textit{for all } c \in (a,b).$$

We denote by $G([a,b],\mathbb{R}^n)$ *the set of regulated functions defined on* $[a,b]$ *with values on* $\mathbb{R}^n$*.*

Remark 1.67. It follows from the results in [7] that the set $G([a,b],\mathbb{R}^n)$ is a Banach space when endowed with the sup–norm, $\|\cdot\|_\infty$, defined as

$$\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|, \quad f \in G([a,b],\mathbb{R}^n).$$

Before continuing with the properties of the Kurzweil–Stieltjes integral for regulated integrators, we introduce some notation. Let $f \in G([a,b],\mathbb{R})$, we denote $f(a^-) = f(a)$, $f(b^+) = f(b)$ and

$$\Delta^- f(t) = f(t) - f(t^-), \qquad \Delta^+ f(t) = f(t^+) - f(t), \quad t \in [a,b]. \tag{1.29}$$

We also denote

$$\Delta f(t) = \Delta^+ f(t) - \Delta^- f(t), \quad t \in [a,b].$$

It is important to note that if f is left–continuous at $t \in [a,b]$, then we have that $\Delta^- f(t) = 0$, and so

$$\Delta f(t) = \Delta^+ f(t) = f(t^+) - f(t). \tag{1.30}$$

With this notation, we have the following result.

Theorem 1.68. *Let $f : [a,b] \to \mathbb{R}$ and $g \in G([a,b],\mathbb{R})$ be such that f is Kurzweil–Stieltjes integrable on $[a,b]$ with respect to g. Then the function*

$$h(t) = {}^{(KS)}\!\!\int_a^t f \,\mathrm{d}\, g, \quad t \in [a,b],$$

is regulated and satisfies

$$h(t^+) = h(t) + f(t)\Delta^+ g(t), \quad t \in [a,b),$$
$$h(t^-) = h(t) - f(t)\Delta^- g(t), \quad t \in (a,b].$$

The next result, known as the substitution formula for the Kurzweil–Stieltjes integral, provides information about the integral when the integrator function is a Kurzweil–Stieltjes indefinite integral.

Theorem 1.69. *Let $f, g, h : [a,b] \to \mathbb{R}$ be such that f is bounded and g is Kurzweil–Stieltjes integrable on $[a,b]$ with respect to h. If one of the following integrals exists,*

$${}^{(KS)}\!\!\int_a^b f(x)\,\mathrm{d}\left({}^{(KS)}\!\!\int_a^x g \,\mathrm{d}\, h\right), \quad {}^{(KS)}\!\!\int_a^b f g \,\mathrm{d}\, h,$$

the other one also exists and they are equal.

Finally, we establish the relations between the Lebesgue–Stieltjes and the Kurzweil–Stieltjes integrals, following [65, Chapter 6, Section 12]. In particular, we first present Theorem 6.12.3 in [65], which shows that integrability in the sense of Lebesgue–Stieltjes implies integrability in the Kurzweil–Stieltjes sense and provides explicit relations between the two values.

Theorem 1.70. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and $f : [a,b] \to \mathbb{R}$. If the Lebesgue–Stieltjes integral $\int_{[a,b)} f \,\mathrm{d}\, g$ exists, then so does the Kurzweil–Stieltjes integral ${}^{(KS)}\!\int_a^b f \,\mathrm{d}\, g$ and*

$$\int_{[a,b)} f \,\mathrm{d}\, g = {}^{(KS)}\!\!\int_a^b f \,\mathrm{d}\, g. \tag{1.31}$$

The converse relation does not hold in general. That is, integrable functions in the Kurzweil–Stieltjes sense need not be Lebesgue–Stieltjes integrable. However, [65, Theorem 6.12.7] provides a sufficient condition for a Kurzweil–Stieltjes integrable function to be Lebesgue–Stieltjes integrable.

Theorem 1.71. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and $f : [a,b] \to \mathbb{R}$ be a nonnegative g–measurable function. If f is Kurzweil–Stieltjes integrable, then it is Lebesgue–Stieltjes integrable and the relation* (1.31) *holds.*

The previous results showed that the Kurzweil–Stieltjes integral is more general than the Lebesgue–Stieltjes version. However, in the work ahead, we will still use the measure formulation of the integral rather than the Kurzweil–Stieltjes one. This is done first in the construction of the Δ–measure and later, in the study of differential equations. This choice is made because it allows to establish easier connections with the results in classical analysis. Most of the results ahead can be adapted to fit the Kurzweil–Stieltjes context by implementing some changes.

Chapter 2

The displacement derivative

Derivatives are, in the classical sense of Newton [67], infinitesimal rates of change of one (dependent) variable with respect to another (independent) variable. Formally, the derivative of f with respect to x is

$$f'(x) := \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

The symbol Δ represents what we call the *variation*, that is, the change of magnitude underwent by a given variable. This variation is, in the classical setting, defined in the most simple possible way as $\Delta x = \widetilde{x} - x$, where x is the point at which we want to compute the derivative (the point of departure) and $\widetilde{x}$ another point which we assume close enough to x. From this, it follows naturally that the variation of the dependent variable has to be expressed as $\Delta f = f(\widetilde{x}) - f(x)$. This way, when $\widetilde{x}$ tends to x, that is, when Δx tends to zero, we have

$$f'(x) := \lim_{\widetilde{x} \to x} \frac{f(\widetilde{x}) - f(x)}{\widetilde{x} - x}.$$

Of course, this naive way of defining the variation is by no means the unique way of giving meaning to such expression. The intuitive idea of variation is naturally linked to the mathematical concept of distance. After all, in order to measure how much a quantity has varied it is enough to see *how far apart* the new point $\widetilde{x}$ is from the first x that is, we have to measure, in some sense, the distance between them. This manner of extending the notion of variation –and thus of derivative– has been accomplished in different ways. The most crude of these is what is called the *absolute derivative*.

Definition 2.1 ([15, expression (1)]). *Let (X, d_X) and (Y, d_Y) be two metric spaces and consider $f : X \to Y$ and $x \in X$. We say f is* absolutely differentiable at x *if and only if the following limit –called absolute derivative of f at x– exists:*

$$f^{|\prime|}(x) := \lim_{\widetilde{x} \to x} \frac{d_Y(f(x), f(\widetilde{x}))}{d_X(x, \widetilde{x})}.$$

In the case of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ we have that, as expected, $f^{|\prime|} = |f'|$ [15, Proposition 3.1]. Hence, this result conveys the true meaning of the absolute derivative –it is the absolute value of the derivative– and it extends the notion of derivative to the broader setting of metric spaces. Even so, this definition may seem somewhat unfulfilling as a generalization. For instance, in the case of the real line, it does not preserve the spirit of the intuitive notion of *'infinitesimal rates of change'*: changes of rate have, of necessity, to be allowed to be *negative*.

A more subtle extension of differentiability to the realm of metric spaces can be achieved through *mutational analysis* where the affine structure of differentials is changed by a family of functions, called *mutations*, that mimic the properties and behavior of derivatives. The interested reader is referred to [56] for more information on the subject.

The considerations above bring us to another possible extension of the notion of derivative: that of the *Stieltjes derivative*, also known as *g–derivative*. This derivative will be properly defined on Chapter 3 but, essentially, it is defined as

$$f'_g(x) = \lim_{\widetilde{x}\to x} \frac{f(\widetilde{x}) - f(x)}{g(\widetilde{x}) - g(x)} \text{ if } g \text{ is continuous at } x,$$
$$f'_g(x) = \lim_{\widetilde{x}\to x^+} \frac{f(\widetilde{x}) - f(x)}{g(\widetilde{x}) - g(x)} \text{ if } g \text{ is discontinuous at } x,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing and left–continuous function. Here, we have defined Δx through a rescaling of the abscissae axis by g. Observe that, although $\rho(x, \widetilde{x}) = |g(\widetilde{x}) - g(x)|$ is a pseudometric [33], $\Delta x = g(\widetilde{x}) - g(x)$ is allowed to change sign.

This generalization can be taken one step further. The definition of Δx does not have to depend on a rescaling, but its absolute value definitely has to suggest, in a broad sense, the notion, if not of distance, of being *far apart* or *close* as well as the *direction* –change of sign. This is, essentially, the reason why the notion of *displacement* was introduced in [61]. In this chapter, we will present the original concept of displacement space and the displacement derivative following that paper. First, we study displacement spaces as topological objects, and then we move on to the analytical part, where we revise the hypotheses required for this setting.

The rest of the chapter is structured as follows. First, in Section 2.1, we introduce the concept of displacement and we present some of its topological properties, while illustrating its applicability in the real world and its connections to other known topological spaces. Later, in Section 2.2.1 we turn our attention to the definition of a measure in terms of a displacement. After that, in Section 2.2.2 we define the displacement derivative, and finally, in Section 2.2.3, we study the relations between the displacement measure and the displacement derivative through the Fundamental Theorem of Calculus.

2.1 Displacement spaces

As mentioned before, the aim of this section is to illustrate the definition of displacement spaces from the topological point of view while establishing some connections with other known spaces and showing their applicability to real life situations following [61]. In order to do so, we first recall some simple examples of topological spaces. In particular, we start by recalling the definition of pseudometric and metric space. These objects are pairs formed by a set and a map that measures the distance between points of the set in a given way.

Definition 2.2. *Let X be a set and $\rho : X \times X \to [0, +\infty)$ satisfy the following conditions:*

(i) Indiscernibility of identicals: $\rho(x, x) = 0$ *for every* $x \in X$.

(ii) Symmetry: $\rho(x, y) = \rho(y, x)$ *for every* $x, y \in X$.

(iii) Triangular inequality: *for every* $x, y, z \in X$*,*

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z). \tag{2.1}$$

Then ρ *receives the name of* pseudometric map *and the pair* (X, ρ) *is a* pseudometric space. *If, in addition,* ρ *also satisfies*

(iv) Identity of indiscernibles: *if* $\rho(x,y) = 0$*, then* $x = y$*;*

then it is called a metric map *and the pair* (X, ρ) *is a* metric space.

Note that, although it is possible to construct a topology on X from the correspoding map, Definition 2.2 does not directly endow the space X of a topological structure. We will show later, in a more general setting how to do it. To that end, we introduced the concept of displacement presented in [61]. In a similar fashion to Definition 2.2, displacement spaces are formed by a set and a map that measures distances between its points, but in a much more relaxed sense as we will show later with some examples. In order to give sense to the following definition, it is important to note that all the limits occurring in this work will be, unless stated otherwise, in the usual sense, i.e. with respect to the usual topology.

Definition 2.3. *Let* X *be a set and* $\Delta : X \times X \to \mathbb{R}$ *satisfy the following properties:*

(a) Indiscernibility of identicals: $\Delta(x,x) = 0,\ x \in X.$

(b) Δ–generalized triangle inequality: *For all* $x, y \in X$*,*

$$|\Delta(x,y)| = \sup\left\{\liminf_{n\to\infty} |\Delta(x,z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(y,z_n)| = 0\right\}. \tag{2.2}$$

Then Δ *receives the name of* displacement *and the pair* (X, Δ) *is a* displacement space.

Remark 2.4. Note that condition (a) ensures that the supremum in condition (b) is well–defined, since for any $x, y \in X$, the set

$$\left\{\liminf_{n\to\infty} |\Delta(x,z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(y,z_n)| = 0\right\}$$

is non-empty as we can always consider the sequence $\{z_n\}_{n\in\mathbb{N}} = \{y\}_{n\in\mathbb{N}}$. Conversely, if condition (b) is well–defined and satisfied, we have that condition (a) is satisfied, since in that case we would have that for any $x \in X$,

$$0 \leq |\Delta(x,x)| = \sup\left\{\liminf_{n\to\infty} |\Delta(x,z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(x,z_n)| = 0\right\} = 0.$$

Moreover, observe that condition (b) is satisfied if for all $x, y \in X$,

$$|\Delta(x,y)| \geq \sup\left\{\liminf_{n\to\infty} |\Delta(x,z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(y,z_n)| = 0\right\},$$

as the reverse inequality always holds.

By looking at the definitions, we can see that every metric space is a pseudometric space, but for pseudometric spaces and displacement spaces the question becomes less trivial. The following lemma, not only gives a useful condition for (b) to be satisfied, but it also shows that (b) is, indeed, a generalization of the triangle inequatily (2.1), making clear that every pseudometric map is also a displacement.

Lemma 2.5. *Let X be a set and $\Delta : X \times X \to \mathbb{R}$ be a map. Assume that the following property holds:*

(b') *There exists a strictly increasing left–continuous map $\varphi : [0, +\infty) \to [0, +\infty)$, continuous at* 0*, satisfying $\varphi(0) = 0$ and such that, for $\psi(x, y) := \varphi(|\Delta(x, y)|)$,*

$$\psi(x, z) \leq \psi(x, y) + \psi(y, z), \quad x, y, z \in X. \tag{2.3}$$

Then Δ satisfies (b).

Proof. Fix $x, y \in X$ and let $\{z_n\}_{n\in\mathbb{N}} \subset X$ be such that $|\Delta(y, z_n)| \xrightarrow{n\to\infty} 0$. Then, condition (2.3) yields $\psi(x, z_n) - \psi(y, z_n) \leq \psi(x, y)$ for all $n \in \mathbb{N}$. Hence,

$$\psi(x, y) \geq \liminf_{n\to\infty}(\psi(x, z_n) - \psi(y, z_n)) \geq \liminf_{n\to\infty} \psi(x, z_n) - \limsup_{n\to\infty} \psi(y, z_n).$$

Since φ is continuous at 0 and $\varphi(0) = 0$, it follows that $\limsup_{n\to\infty} \psi(y, z_n) = 0$, so

$$\psi(x, y) \geq \liminf_{n\to\infty} \psi(x, z_n). \tag{2.4}$$

Let us show that

$$\liminf_{n\to\infty} \varphi(|\Delta(x, z_n)|) \geq \varphi\left(\liminf_{n\to\infty} |\Delta(x, z_n)|\right). \tag{2.5}$$

Indeed, the definition of $\liminf$ implies that for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|\Delta(x, z_n)| \geq \liminf_{n\to\infty} |\Delta(x, z_n)| - \varepsilon$. Thus, the fact that φ is a strictly increasing function implies that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then

$$\varphi\left(|\Delta(x, z_n)|\right) \geq \varphi\left(\liminf_{n\to\infty} |\Delta(x, z_n)| - \varepsilon\right).$$

Consequently, for any $\varepsilon > 0$ it holds that

$$\liminf_{n\to\infty} \varphi\left(|\Delta(x, z_n)|\right) \geq \varphi\left(\liminf_{n\to\infty} |\Delta(x, z_n)| - \varepsilon\right),$$

which, using the left–continuity of φ, leads to (2.5). Hence, it follows from (2.4) and (2.5) that

$$\varphi(|\Delta(x, y)|) \geq \liminf_{n\to\infty} \psi(x, z_n) = \liminf_{n\to\infty} \varphi(|\Delta(x, z_n)|) \geq \varphi\left(\liminf_{n\to\infty} |\Delta(x, z_n)|\right),$$

which, together with the fact that φ is strictly increasing, yields

$$|\Delta(x, y)| \geq \liminf_{n\to\infty} |\Delta(x, z_n)|.$$

Note that this holds for any $\{z_n\}_{n\in\mathbb{N}} \subset X$ such that $|\Delta(y, z_n)| \xrightarrow{n\to\infty} 0$, which concludes the proof. □

Remark 2.6. Lemma 2.5 shows that condition (b) is a way of avoiding the triangle inequality, but it is not the only one. For instance, in [72, Definition 3.1], the authors work in the context of *RS–generalized metric spaces*, (X, Δ), and use the following condition

(D_3') There exists $C > 0$ such that if $x, y \in X$ and

$$\lim_{n\to\infty} \Delta(x_n, x) = \lim_{n\to\infty} \Delta(x, x_n) = \lim_{n,m\to\infty} \Delta(x_n, x_m) = 0,$$

then

$$\Delta(x, y) \leq C \limsup \Delta(x_n, y).$$

More complicated conditions can be found in [56, (H3) Section 3.1, (H3') Section 4.1].

We now present a series of examples of displacement spaces that arise naturally from real world situations, where other spaces like metric spaces might not be adequate.

Example 2.7. Consider the sphere $\mathbb{S}^1$ and the map $\Delta : \mathbb{S}^1 \times \mathbb{S}^1 \to [0, 2\pi)$ defined as

$$\Delta(x, y) = \min\{\theta \in [0, +\infty) \ : \ xe^{i\theta} = y\}, \quad x, y \in \mathbb{S}^1.$$

In other words, $\Delta(x, y)$ is a displacement that measures the minimum counter-clockwise angle necessary to move from x to y. From a real life point of view, Δ can describe the way cars move in a roundabout. Suppose that a car enters the roundabout at a point x and wants to exit at a point y. In that case, circulation rules force the car to move in a given direction, which happens to be counter-clockwise in most of the countries around the World. In this case, drivers are assumed to take the exit y as soon as they reach it.

Let us show that Δ is a displacement. First, it is clear that (a) holds. For (b'), take $\varphi(r) = r$. Then, for $x, y, z \in \mathbb{S}^1$, if $\Delta(x, y) + \Delta(y, z) \geq 2\pi$, then (b') clearly holds. Otherwise, $\Delta(x, z) = \Delta(x, y) + \Delta(y, z)$, so (b') holds.

Note that we have just proven that the triangle inequality, (2.1), holds. However, this cannot be a pseudometric space as Δ is not symmetric.

Example 2.8. Let (X, E) be a complete weighted directed graph, i.e. $X = \{x_1, \dots, x_n\}$ is a finite set of $n \in \mathbb{N}$ vertices and $E \in \mathcal{M}_n(\mathbb{R})$ is a matrix with zeros in the diagonal and positive numbers elsewhere. The element $e_{j,k}$ of the matrix E denotes the weight of the directed edge from vertex x_j to vertex x_k. This kind of graph can represent, for instance, the time it takes to get from one point in a city to another by car, as Figure 2.1 illustrates.

Now, consider the set $\{x_1, \dots, x_4\}$ and the matrix E as given by Figure 2.1, that is,

$$E \equiv (e_{j,k})_{j,k=1}^4 := \begin{pmatrix} 0 & 9 & 4 & 10 \\ 10 & 0 & 14 & 8 \\ 7 & 9 & 0 & 5 \\ 11 & 6 & 7 & 0 \end{pmatrix},$$

and the map $\Delta(x_j, x_k) := e_{j,k}$. Note that the zeros on the diagonal of the matrix E ensure that (a) holds, that is, it takes no time to go from a point to itself. Moreover, it can be checked that Δ is subadditive –which is to be expected since, if we could get faster from a point to

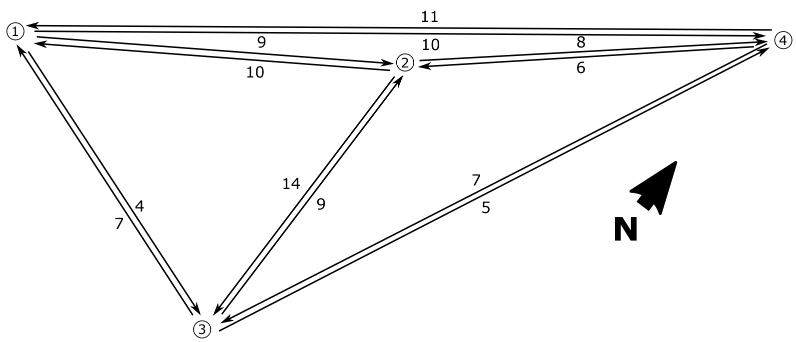


Figure 2.1: Graph indicating the time in minutes it takes to go from one place to another in Santiago de Compostela by car (using the least time consuming path) according to *Google Maps* –good traffic conditions assumed. The points are placed in their actual relative geometric positions, being **1**: Faculty of Mathematics (USC), **2**: Cathedral, **3**: Train station, **4**: Bus station. Most of the streets in Santiago are one way, which accounts for the differences in time depending on the direction of the displacement.

another through a third one *Google Maps* would have chosen that option. Hence, (b') holds for $\varphi(r) = r$, and so Δ is a displacement.

As a final remark, note that Δ fails to be symmetric, which once again proves that it cannot be a pseudometric.

We now present a characteristic type of displacement space that will be useful for the work ahead. This example will be fundamental to studying the relations between the derivative introduced later in this chapter and the Stieltjes derivative in Chapter 3.

Example 2.9. Let $X \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, left–continuous map and consider the map $\Delta : X \times X \to \mathbb{R}$ defined as

$$\Delta(x, y) = g(y) - g(x), \quad x, y \in X.$$

It follows from the definition that (a) and (b') with $\varphi(t) = t$ are satisfied. Therefore, Δ is a displacement, known as the *Stieltjes displacement* defined by g, and the topological space is called *Stieltjes topological space*. Note that for the particular case $g = \mathrm{Id}$, Δ is called the *usual displacement in* $\mathbb{R}$ and the associated topology is known as the *usual topology of* $\mathbb{R}$, denoted by τ_u.

Note that the map $\rho(x, y) = |\Delta(x, y)| = |g(y) - g(x)|$, $x, y \in \mathbb{R}$, is a pseudometric, known as the *Stieltjes pseudometric* defined by g. Furthermore, for $g = \mathrm{Id}$, ρ is a metric map known as the *usual metric of* $\mathbb{R}$.

The following result gives a useful characterization for a displacement to be a Stieltjes displacement.

Lemma 2.10. *Let X be a set and (X, Δ) be a displacement space. Then*

$$\Delta(x, y) = g(y) - g(x), \quad x, y \in X,$$

for some $g : X \to \mathbb{R}$*, if and only if, for every* $x, y, z \in X$*,*

1. $\Delta(x, y) = -\Delta(y, x)$,
2. $\Delta(x, z) = \Delta(x, y) + \Delta(y, z)$.

In particular, if $X \subset \mathbb{R}$*,* Δ *is a Stieltjes displacement if and only if* Δ *satisfies conditions* 1*,* 2 *and*

3. *There exists* $x_0 \in X$ *such that* $\Delta(x_0, \cdot)$ *is nondecreasing and left–continuous.*

Proof. To show the necessity, first assume that $\Delta(x, y) = g(y) - g(x)$ for some function $g : X \to \mathbb{R}$. Then for any $x, y, z \in X$ we have that

$$\Delta(x, y) = g(y) - g(x) = -(g(x) - g(y)) = -\Delta(y, x),$$

and

$$\Delta(x, y) + \Delta(y, z) = g(y) - g(x) + g(z) - g(y) = g(z) - g(x) = \Delta(x, z).$$

Conversely, assume that 1 and 2 hold. Take $x_0 \in X$ and define $g(x) = \Delta(x_0, x)$ for $x \in X$. Then,

$$\Delta(x, y) = \Delta(x, x_0) + \Delta(x_0, y) = -\Delta(x_0, x) + g(y) = g(y) - g(x).$$

The other equivalence now follows inmediately from condition 3. □

We can use this result to show that there exist displacement spaces on subsets of the real line that are not Stieltjes displacement spaces. Consider the following example.

Example 2.11. Let $X = [0, 1]$ and $\Delta : X \times X \to [0, +\infty)$ be given by

$$\Delta(x, y) = e^{y^2 - x^2} - e^{x - y}. \tag{2.6}$$

Clearly, condition (a) is satisfied. This can be observed in Figure 2.2. Now, for condition (b), fix $x, y \in X$. Since $\Delta(x, \cdot)$ is a continuous function, condition (2.2) is equivalent to

$$|\Delta(x, y)| = \sup\left\{\lim_{n\to\infty} |\Delta(x, z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(y, z_n)| = 0\right\}. \tag{2.7}$$

Let $\{z_n\}_{n\in\mathbb{N}} \subset X$ be a sequence such that $|\Delta(y, z_n)| \xrightarrow{n\to\infty} 0$, that is,

$$\lim_{n\to\infty} \left(e^{z_n^2 - y^2} - e^{y - z_n}\right) = 0.$$

Given that the sequence $\{e^{z_n + y^2}\}_{n\in\mathbb{N}}$ is bounded, we have that $e^{z_n^2 + z_n}$ converges to $e^{y^2 - y}$, from which we get that

$$\lim_{n\to\infty} \left(z_n^2 + z_n\right) = y^2 + y. \tag{2.8}$$

Define

$$h(t) = \frac{-1 + \sqrt{1 + 4t}}{2}, \quad t \in [0, +\infty).$$

For any $a \in [0,1]$ we have that $h(a^2+a)=a$. Therefore, applying h to both sides of (2.8) and noting that h is a continuous function, we obtain

$$y = h(y^2+y) = h\left(\lim_{n\to\infty} z_n^2 + z_n\right) = \lim_{n\to\infty} h(z_n^2+z_n) = \lim_{n\to\infty} z_n.$$

That is, if $\{z_n\}_{n\in\mathbb{N}} \subset X$ is such that $|\Delta(y,z_n)| \xrightarrow{n\to\infty} 0$, it follows that z_n converges to y as $n\to\infty$. Hence (2.7) is trivially satisfied, and thus Δ is a displacement. However, Δ is not a Stieltjes displacement as

$$\Delta(1/2,0) = e^{-1/4} - e^{1/2} \neq e^{-1/2} - e^{1/4} = -\Delta(0,1/2).$$

Note that this also shows that the symmetry property does not hold, therefore (X,Δ) cannot be a pseudometric or metric space.

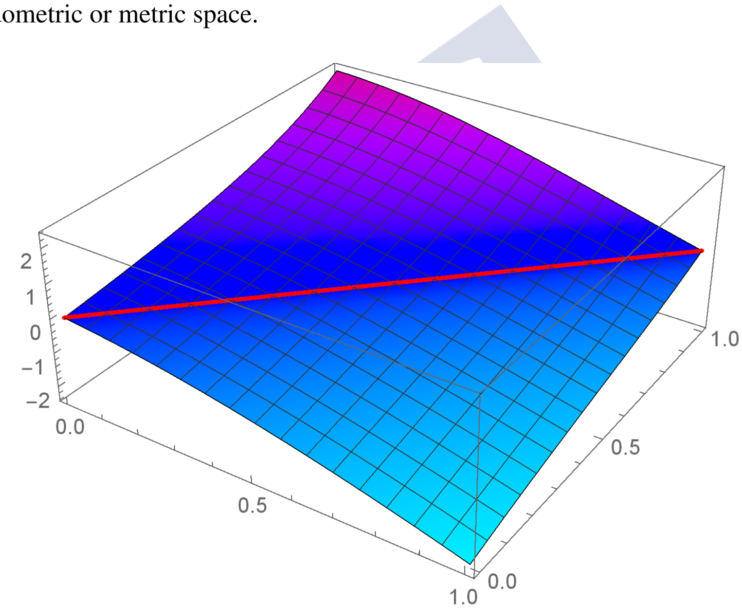


Figure 2.2: Representation of $z=\Delta(x,y)$ for Δ as in (2.6). Observe that, along the line $x=y$, the function remains constant and equal to 0.

We now turn our attention to the study of displacement spaces as topological objects, for which we need to define a topology. The topology inherent to displacement spaces –and therefore pseudometric and metric spaces– is defined in terms of a particular type of set, called *open ball*.

Definition 2.12. *Let X be a set and $\Delta : X\times X \to \mathbb{R}$ be a map. Given $x\in X$ and $r>0$, we define the set $B_\Delta(x,r)$, as*

$$B_\Delta(x,r) := \{y \in X \ : \ |\Delta(x,y)| < r\}.$$

We also define

$$\tau_\Delta := \left\{U \subset X : \forall\, x \in U, \exists\, r \in \mathbb{R}^+ \text{ such that } B_\Delta(x,r) \subset U\right\}.$$

If the pair (X,Δ) *is a displacement space, the set* $B_\Delta(x,r)$ *is known as the* open ball of center x and radius r *and* τ_Δ *is a topology on* X *called* Δ–topology.

Remark 2.13. It is important to note that if (X,Δ) is a displacement space, τ_Δ is indeed a topology on X. First of all, it is clear that $\emptyset \in \tau_\Delta$ as it trivially satisfies the condition in the definition of τ_Δ. Similarly, by Definition 2.12, we have that $B_\Delta(x,r) \subset X$ for all $x \in X$, $r > 0$, and so $X \in \tau_\Delta$. Now, let $U, V \in \tau_\Delta$ and denote $W = U \cap V$. If $x \in W$, then x lays in U and in V. Therefore, there exist $r_U, r_V > 0$, such that

$$B_\Delta(x, r_U) \subset U, \quad B_\Delta(x, r_V) \subset V.$$

Taking $r = \min\{r_U, r_V\}$, we have that $B(x,r) \subset B_\Delta(x,r_U) \cap B_\Delta(x,r_V) \subset W$, and so $W \in \tau_\Delta$. Lastly, let $\{U_j\}_{j\in\mathcal{I}} \subset \tau_\Delta$ and let $\mathcal{U} = \bigcup_{j\in\mathcal{I}} U_j$. If $x \in \mathcal{U}_j$, then there exists j_0 such that $x \in U_{j_0}$. Definition 2.12 guarantees the existence of r_{j_0} such that $B_\Delta(x, r_{j_0}) \subset U_{j_0} \subset \mathcal{U}$. That is, $\mathcal{U} \in \tau_\Delta$, and so τ_Δ is a topology on X.

Note that if X is a pseudometric space, the topology in Definition 2.12 yields the usual pseudometric topology. In those topologies, it is a known fact that open balls are open sets. However, that result is not obvious in the context of displacement topologies. The following result shows that that statement is true, while also providing a characterization for a map to satisfy condition (b) in Definition 2.3.

Lemma 2.14. *Let* X *be a set and* $\Delta : X \times X \to \mathbb{R}$ *be a map. Then* Δ *satisfies condition* (b) *if and only if*

$$B_\Delta(x,r) \in \tau_\Delta \quad \text{for all } x \in X \text{ and } r > 0, \tag{2.9}$$

where $B_\Delta(x,r)$ *and* τ_Δ *are as in* Definition 2.12.

Proof. Assume first that Δ satisfies (b) and let $x \in X$ and $r > 0$ be fixed. If $B_\Delta(x,r) = \emptyset$ then $B_\Delta(x,r) \in \tau_\Delta$ trivially. Otherwise, $B_\Delta(x,r) \neq \emptyset$. In that case, let us show that for every $y \in B_\Delta(x,r)$, there exists $\varepsilon \in \mathbb{R}^+$ such that $B_\Delta(y,\varepsilon) \subset B_\Delta(x,r)$. Assume this is not the case. Then, there exists $y \in B_\Delta(x,r)$ and $\{z_n\}_{n\in\mathbb{N}} \subset X$ such that, for all $n \in \mathbb{N}$,

$$|\Delta(y, z_n)| < 1/n, \quad |\Delta(x, z_n)| \geq r.$$

Hence, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that $|\Delta(y,z_n)| \xrightarrow{n\to\infty} 0$ and, by condition (b),

$$\liminf_{n\to\infty} |\Delta(x, z_n)| \geq r > |\Delta(x,y)|,$$

which contradicts the definition of supremum in (2.2).

Conversely, assume that (2.9) holds. Fix $x, y \in X$, define $r := |\Delta(x,y)|$ and $\varepsilon > 0$. Clearly, $y \in B_\Delta(x, r+\varepsilon)$, so there exists $\delta_\varepsilon > 0$ such that $B_\Delta(y,\delta_\varepsilon) \subset B_\Delta(x, r+\varepsilon)$. Hence, if $\{z_n\}_{n\in\mathbb{N}} \subset X$ is such that $|\Delta(y,z_n)| \xrightarrow{n\to\infty} 0$, there exists $N \in \mathbb{N}$ such that $|\Delta(y,z_n)| < \delta_\varepsilon$ for every $n \geq N$, so $|\Delta(x,z_n)| < r+\varepsilon$ for every $n \geq N$. Thus, $\liminf_{n\to\infty} |\Delta(x,z_n)| \leq r+\varepsilon$. Since ε was arbitrarily fixed, we get that $\liminf_{n\to\infty} |\Delta(x,z_n)| \leq r$, which concludes the proof. □

Remark 2.15. Note that hypothesis (a) is not necessary for the Lemma 2.14 or the definition of the topology itself. At the end of the day, this hypothesis only guarantees that the open balls are non-empty. With respect to hypothesis (b), this result shows that it is the minimal condition required for open balls to be open.

In the general setting of displacement spaces, it is hard to obtain further topological properties. However, in the particular setting of the real line, more properties can be deduced. In the following we show that, under certain hypotheses, the open sets in the topology in Definition 2.12 can be expressed as countable union of τ_u–Borel sets. At this point, we assume that the reader is familiar with the axioms of countability, Lindelöf spaces as well as the Sorgenfrey topology on the real line (see, for example, [39]).

Lemma 2.16. *Let* $(\mathbb{R}, \Delta)$ *be a displacement space satisfying the following properties:*

(i) *For each* $x \in \mathbb{R}$*, the map* $\Delta(x, \cdot)$ *is nondecreasing.*

(ii) *The set* $J_\Delta = \{x \in \mathbb{R} : \Delta(x, x^-) < 0 < \Delta(x, x^+)\}$ *is at most countable.*

Then every open set can be expressed as a countable union of intervals and singletons which are also open in the Δ*–topology. In particular, every open set in the* Δ*–topology is a Borel set with respect to the usual topology in* $\mathbb{R}$*.*

Proof. First of all, note that given $x \in \mathbb{R}$ and $r > 0$, we can express $B_\Delta(x, r)$ as follows:

$$B_\Delta(x, r) = \{y \in \mathbb{R} : |\Delta(x, y)| < r\} = \{y \in \mathbb{R} : -r < \Delta_x(y) < r\} = \Delta_x^{-1}((-r, r)),$$

where $\Delta_x : \mathbb{R} \to \mathbb{R}$ is defined as $\Delta_x(t) = \Delta(x, t)$, $t \in \mathbb{R}$. Moreover, since Δ_x is nondecreasing, $B_\Delta(x, r)$ is a connected set. In particular, we have that $B_\Delta(x, r)$ is a singleton if and only if $x \in J_\Delta$ and $r < \min\{|\Delta(x, x^-)|, |\Delta(x, x^+)|\}$; otherwise it is a nongenerate interval (not necessarily open in the usual topology) with extremal points

$$a = \inf\{t \in \mathbb{R} : -r < \Delta_x(t)\}, \quad b = \sup\{t \in \mathbb{R} : \Delta_x(t) < r\}.$$

Let $U \in \tau_\Delta$. Then, by the definition of open set, $U = \bigcup_{x \in U} B_\Delta(x, r_x)$ and so, since each $B_\Delta(x, r_x)$ is a connected set, we can write

$$U = \bigcup_{i \in \mathcal{I}} (a_i, b_i) \cup \bigcup_{j \in \mathcal{J}} [a_j, b_j) \cup \bigcup_{k \in \mathcal{K}} (a_k, b_k] \cup \bigcup_{l \in \mathcal{L}} [a_l, b_l] \cup \bigcup_{x \in U \cap J_\Delta} \{x\}, \tag{2.10}$$

for some sets of indices $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$, where each of the sets occurring in (2.10) is an open ball of τ_Δ. Note that since J_Δ is at most countable, it is enough to show that each of the sets formed by a union of intervals can be reduced to a countable union of them.

The set $A = \bigcup_{i \in \mathcal{I}} (a_i, b_i)$ is an open set in $(\mathbb{R}, \tau_u)$ and therefore second countable, which implies that A is Lindelöf and, hence, there exists a countable subcover of A, i.e. $A = \bigcup_{n \in \mathbb{N}} (a_{i_n}, b_{i_n})$ for some set of indices $\{i_n\}_{n \in \mathbb{N}}$. Similarly, the set $B = \bigcup_{j \in \mathcal{J}} [a_j, b_j)$ is an open set in the Sorgenfrey line, which is hereditarily Lindelöf [39, p. 79], and so $B = \bigcup_{n \in \mathbb{N}} [a_{j_n}, b_{j_n})$ for some set of indices $\{j_n\}_{n \in \mathbb{N}}$. Using a similar argument, the set $C = \bigcup_{k \in \mathcal{K}} (a_k, b_k]$ can be expressed as $\bigcup_{n \in \mathbb{N}} (a_{k_n}, b_{k_n}]$. Finally, the set $D = \bigcup_{l \in \mathcal{L}} [a_l, b_l]$ can be decomposed as $D = \bigcup_{l\in\mathcal{L}}[a_l, b_l) \cup \bigcup_{l\in\mathcal{L}}(a_l, b_l]$, and once again, arguing as for the sets B and C, we obtain that

$$D = \bigcup_{l_n\in\mathbb{N}} [a_{l_n}, b_{l_n}) \cup \bigcup_{m_n\in\mathbb{N}} (a_{m_n}, b_{m_n}],$$

for some sets of indices $\{l_n\}_{n\in\mathbb{N}}$, $\{m_n\}_{n\in\mathbb{N}}$. However, by the definition of D, we have that $a_l, b_l \in D$ for all $l \in \mathcal{L}$, so

$$D = \bigcup_{l_n\in\mathbb{N}} [a_{l_n}, b_{l_n}] \cup \bigcup_{l'_n\in\mathbb{N}} [a_{m_n}, b_{m_n}],$$

which is clearly countable. Therefore, U is the countable union of intervals and singletons which are open with respect to τ_Δ. □

Remark 2.17. This last proof relies heavily on the fact that the real number system, with its usual order, is bounded complete, that is, that every bounded set has an infimum and a supremum. Observe also that the interaction between the topologies τ_u and τ_Δ plays a mayor role in the proof. Furthermore, the relations between the topologies also have consequences in the implications that follow from the result. Essentially, Lemma 2.16 shows that, for the real line, every τ_u–Borel σ–algebra is, in particular, a τ_Δ–Borel σ–algebra. Hence, the integration theory that will follow will be valid for the open sets of τ_Δ when the hypotheses of Lemma 2.16 are satisfied.

Remark 2.18. Observe that it is enough for condition (ii) in Lemma 2.16 to be satisfied that for every $x \in \mathbb{R}$, the map $\Delta(x, \cdot)$ is continuous at x from, at least, one side. In particular, it follows that every Stieltjes displacement satisfies the hypotheses of Lemma 2.16. Therefore, every open set in a Stieltjes topological space can be expressed as a countable union of intervals. This information was already available in [33, Proposition 2.1]. However, the proof does not show that such intervals are elements of the corresponding topology, thus failing to show that the topology is second countable. This has no impact on the rest of the results in that article, as the authors only use the fact that open sets in the corresponding topology are Borel sets in the usual sense. A similar thing occurs regarding [61, Lemma 5], where the authors claimed that Δ–topology is second countable. This is done by showing a open set can be expressed as a countable union of open balls. Nevertheless, its proof presents two problems. First of all, the proof does not take into account the possibility of an open ball being a singleton which makes the result invalid as, without condition (ii) in Lemma 2.16, this could lead to an uncountable set. Secondly, the results fails to show that open balls can be covered by a given family of countable sets, thus failing to show that the topology is second countable. Once again, this has no major impact on the rest of the results of the paper as it is only used that open sets in the Δ–topology are Borel sets, which happens, unknowingly, under the conditions of Lemma 2.16.

Next, we aim to study the maps between two displacement spaces. It is known that between metric spaces, continuity can be determined in terms of open balls. In the next result we show that this is also the case in the setting of displacement spaces.

Lemma 2.19. *Let* (X, Δ_1) *and* (Y, Δ_2) *be displacement spaces. A map* $f : X \to Y$ *is continuous if and only if*

$$\forall\, x \in X_1,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } f(y) \in B_{\Delta_2}(f(x), \varepsilon)\ \ \forall\, y \in B_{\Delta_1}(x, \delta). \tag{2.11}$$

Proof. First, assume that f is continuous and fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Since $U = B_{\Delta_2}(f(x), \varepsilon)$ belongs to τ_{Δ_2}, we have that $f^{-1}(U) \in \tau_{\Delta_1}$. Moreover, $x \in f^{-1}(U)$ and so, there exists $\delta > 0$ such that $B_{\Delta_1}(x, \delta) \subset f^{-1}(U)$. Hence,

$$f(B_{\Delta_1}(x, \delta)) \subset f(f^{-1}(U)) \subset U,$$

that is, there exists $\delta > 0$ such that if $|\Delta_1(x, y)| < \delta$ implies that $|\Delta_2(f(x), f(y))| < \varepsilon$.

Conversely, let $U \in \tau_{\Delta_2}$, $y \in f^{-1}(U)$ and $x = f(y)$. Since $U \in \tau_{\Delta_2}$, there exists $\varepsilon_x > 0$ such that $B_{\Delta_2}(x, \varepsilon_x) \subset U$. Now, condition (2.11) guarantees the existence of $\delta_y > 0$ such that

$$f(z) \in B_{\Delta_2}(x, \varepsilon_x),\ \ \forall z \in B_{\Delta_1}(y, \delta_y).$$

Note that $B_{\Delta_1}(y, \delta_y) \subset f^{-1}(U)$ as $f(z) \in U$ for any $z \in B_{\Delta_1}(y, \delta_y)$. Since $y \in f^{-1}(U)$ was arbitrary, $f^{-1}(U)$ is open and so f is continuous. □

A particularly interesting type of maps between displacement spaces are the homeomorphisms. Essentially, these maps allow us to compare the corresponding sets together with their topologies. In a similar fashion to metric spaces, in the context of displacement spaces we have a useful tool to check whether two displacement topologies on a same space are homeomorphic. We first introduce the concept of topologically equivalent displacements.

Definition 2.20. *Let* X *be a set. Two displacements on* X*,* Δ_1, Δ_2*, are said to be* topologically equivalent *if for every* $x \in X$ *and* $r > 0$*, there exist* $r_1, r_2 > 0$ *such that*

$$B_{\Delta_1}(x, r_1) \subset B_{\Delta_2}(x, r), \quad B_{\Delta_2}(x, r_2) \subset B_{\Delta_1}(x, r).$$

Remark 2.21. Note that this is an equivalence relationship.

The next result follows from the definition and Lemma 2.19 and it gives an alternative condition for displacements to be equivalent.

Proposition 2.22. *Let* X *be a set. Two displacements on* X*,* Δ_1 *and* Δ_2*, are equivalent if and only if the identity map,* $\mathrm{Id} : X \to X$*, is* $(\tau_{\Delta_1}, \tau_{\Delta_2})$ *and* $(\tau_{\Delta_2}, \tau_{\Delta_1})$*–continuous.*

Remark 2.23. Note that this result shows that two displacement spaces that are equivalent are homeomorphic.

Next, we include a result that provides a sufficient condition for two displacement spaces to be equivalent.

Proposition 2.24. *Let* X *be a set and* Δ_1, Δ_2 *be displacements on* X*. Suppose that for every* $x \in X$*, there exist* $\alpha, \beta > 0$ *such that*

$$\alpha|\Delta_1(x, y)| \leq |\Delta_2(x, y)| \leq \beta|\Delta_1(x, y)|, \quad y \in X. \tag{2.12}$$

Then Δ_1 *and* Δ_2 *are equivalent.*

Proof. Let $x \in X$ and $r > 0$ be fixed. Take $r_1 = r/\beta$ and $y \in B_{\Delta_1}(x, r_1)$. Then

$$|\Delta_2(x,y)| \leq \beta |\Delta_1(x,y)| < r,$$

and so $y \in B_{\Delta_2}(x,y)$. Hence, $B_{\Delta_1}(x,r_1) \subset B_{\Delta_2}(x,r)$. Similarly, taking $r_2 = \alpha r$ and $z \in B_{\Delta_2}(x, r_2)$, we have that

$$\alpha |\Delta_1(x,z)| \leq |\Delta_2(x,z)| < r_2.$$

Hence $|\Delta_1(x,z)| < r$, and thus $B_{\Delta_2}(x,r_2) \subset B_{\Delta_1}(x,r)$, which concludes the proof. □

Remark 2.25. Note that this condition is, even in the particular setting of metric spaces, only a sufficient condition. Indeed, consider the maps $\Delta_1 : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$ and , $\Delta_2 : \mathbb{R}\times\mathbb{R} \to [0,1)$ defined as

$$\Delta_1(x,y) = |y-x|, \quad \Delta_2(x,y) = \frac{|y-x|}{1+|y-x|}, \quad x,y \in \mathbb{R}.$$

The maps Δ_1 and Δ_2 are metrics on $\mathbb{R}$, and so, they are displacements. Furthermore, it is easy to check that for every $x \in \mathbb{R}$ and $r > 0$,

$$B_{\Delta_1}(x,r) \subset B_{\Delta_2}(x,r), \quad B_{\Delta_2}(x,(r+1)/r) \subset B_{\Delta_1}(x,r).$$

Therefore, they are equivalent. However, condition (2.12) cannot hold as one of the maps is bounded and the other one is not.

Lastly, we have a look at the set of bounded continuous functions between displacement spaces. In particular, we wonder whether such set can be endowed of a Banach space structure. Such result has only been obtained for the particular case where the codomain is $\mathbb{R}^n$ with the usual topology. In order to show such property, we introduce the following definition.

Definition 2.26. *Let $a,b \in \mathbb{R}$, $a < b$, and $([a,b],\Delta)$ be a displacement space. A map $f : [a,b] \to \mathbb{R}^n$ is said to be* continuous with respect to Δ*, or simply* Δ–continuous*, if $f : ([a,b],\Delta) \to (\mathbb{R}^n, \tau_u)$ is continuous. We denote*

$$\mathcal{BC}_\Delta([a,b],\mathbb{R}^n) = \{f : [a,b] \to \mathbb{R}^n : f \text{ is continuous with respect to } \Delta \text{ and bounded}\}.$$

We also recall that the set of bounded functions defined on a given interval, $[a,b]$, with values on $\mathbb{R}^n$, denoted by $\mathcal{B}([a,b],\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|, \quad f \in \mathcal{B}([a,b],\mathbb{R}^n), \tag{2.13}$$

where $\|\cdot\|$ denotes the usual norm in $\mathbb{R}^n$. This information, together with the next result, will be key to prove that $(\mathcal{BC}_\Delta([a,b]),\mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space.

Proposition 2.27. *Let $(X, \|\cdot\|)$ be a Banach space and $Y \subset X$. If Y is closed in X, then $(Y, \|\cdot\|)$ is a Banach space.*

Proposition 2.28. *Let $a,b \in \mathbb{R}$, $a < b$, and $([a,b],\Delta)$ be a displacement space. The set $\mathcal{BC}_\Delta([a,b],\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_\infty$ in* (2.13) *is a Banach space.*

Proof. Proposition 2.27 ensures that it is enough to show that $\mathcal{BC}_\Delta([a,b],\mathbb{R}^n)$ is a closed subspace of the the Banach space $\mathcal{B}([a,b],\mathbb{R}^n)$.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{BC}_\Delta([a,b],\mathbb{R}^n)$ converging to a function f. Then, $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, and so f is bounded. Thus, all that is left to prove is that f is Δ-continuous.

Fix $\varepsilon > 0$ and $t_0 \in [a,b]$. Since $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f there exists $n_0 \in \mathbb{N}$ such that

$$\|f_{n_0}(s) - f(s)\| < \frac{\varepsilon}{3}, \quad s \in [a,b].$$

On the other hand, since f_{n_0} is Δ–continuous at t_0 there exists $\delta > 0$ such that

$$\|f(t_0) - f(t)\| < \frac{\varepsilon}{3}, \quad t \in B_\Delta(t_0,\delta).$$

Therefore, for any $t \in B_\Delta(t_0,\delta)$ we have that

$$\|f(t_0) - f(t)\| \le \|f(t_0) - f_{n_0}(t_0)\| + \|f_{n_0}(t_0) - f_{n_0}(t)\| + \|f_{n_0}(t) - f(t)\| \le 3\frac{\varepsilon}{3} = \varepsilon,$$

that is, f is Δ–continuous at t_0. Since the point $t_0 \in [a,b]$ was arbitrarily chosen, f is Δ–continuous. □

2.2 Displacement calculus

Let us recall the aim of the rest of this chapter: to construct some analytical structure over closed intervals of the real line which is compatible with the displacement structure introduced in Section 2.1. In particular, we will focus on the construction of a measurable space in Section 2.2.1 and in the definition of a displacement derivative and its relation with the mentioned measure in Sections 2.2.2 and 2.2.3, respectively. The results ahead are slight modifications of those in [61]. Here, we have consider the minimal hypotheses there considered for the results there to remain true, which we present now: consider an interval, $[a,b] \subset \mathbb{R}$, and assume that there exists a nonzero map $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ satisfying the following hypotheses:

(H1) For all $x \in [a,b]$, $\Delta(x,x) = 0$.

(H2) For all $x \in [a,b]$, the map $\Delta_x(\cdot) := \Delta(x,\cdot)$ is nondecreasing and left–continuous.

(H3) There exists $\gamma : [a,b] \times [a,b] \to [1,+\infty)$ such that

(i) For all $x, y, z, \overline{z} \in [a,b]$,

$$|\Delta(z,x) - \Delta(z,y)| \le \gamma(z,\overline{z})|\Delta(\overline{z},x) - \Delta(\overline{z},y)|.$$

(ii) For all $z \in [a,b]$,

$$\lim_{\overline{z}\to z} \gamma(z,\overline{z}) = \lim_{\overline{z}\to z} \gamma(\overline{z},z) = 1.$$

(iii) For all $z \in [a,b]$, the maps $\gamma(z,\cdot), \gamma(\cdot,z) : [a,b] \to [1,+\infty)$ are bounded.

Remark 2.29. Note that (H2) ensures that, in order to confirm that (H3, i) holds, it is enough to check that there exists $\gamma : [a,b]\times[a,b]\to[1,+\infty)$ such that for every $z,\overline{z}\in[a,b]$, we have that

$$\Delta(z,y)-\Delta(z,x)\le\gamma(z,\overline{z})(\Delta(\overline{z},y)-\Delta(\overline{z},x)),\quad x,y\in[a,b],\ x<y.$$

As we mentioned before, this set of hypotheses does not match those considered in [61]. Let us discuss the differences between them. The first difference that we are going to consider is the continuity hypotheses on the map Δ. In [61], the map Δ_x, $x\in[a,b]$, is only continuous from the left at x. Thus, the continuity hypothesis here considered is more general. However, as it is proved in [61, Proposition 2], the converse relation also holds provided that (H3, i) is satisfied. Therefore, the continuity hypotheses considered in both cases are equivalent in their respective settings.

Naturally, we need to discuss the other difference between the hypotheses (H1)–(H3) and the ones considered in [61]. In that work, the authors also imposed condition (b) in Definition 2.3, ensuring that the map considered is a displacement, thus justifying the name of the derivative there defined. Of course, hypotheses (H1)–(H3) do not include that condition. However, this does not mean that this setting represents a more general context than that in [61], as we show in the next result.

Proposition 2.30. *Let* $[a,b]\subset\mathbb{R}$ *and* $\Delta:[a,b]\times[a,b]\to\mathbb{R}$ *satisfy hypotheses* (H1) *and* (H3, i). *Then,* $([a,b],\Delta)$ *is a displacement space.*

Proof. In order to show the result, we need to show that conditions (a) and (b) in Definition 2.3 are satisfied. Of course, hypothesis (H1) is the same as condition (a) in Definition 2.3. Therefore, we need to show that (2.2) holds for each $x,y\in[a,b]$. In order to do so, we will show that for each $x,y\in[a,b]$ and $\{z_n\}_{n\in\mathbb{N}}\subset[a,b]$ such that $|\Delta(y,z_n)|\to0$ as $n\to\infty$ the following limit exists and

$$\lim_{n\to\infty}|\Delta(x,z_n)|=|\Delta(x,y)|.$$

Observe that this is enough to show that (2.2) holds as, in that case, we are considering the supremum over the singleton $\{|\Delta(x,y)|\}$.

Let $x,y\in[a,b]$ and $\varepsilon>0$ be fixed and consider a sequence $\{z_n\}_{n\in\mathbb{N}}\subset[a,b]$ such that $|\Delta(y,z_n)|\to0$ as $n\to\infty$. In that case, we have that $\Delta(y,z_n)\to\Delta(y,y)=0$ as $n\to\infty$, so there exists $N\in\mathbb{N}$ such that if $n\ge N$, then

$$|\Delta(y,z_n)-\Delta(y,y)|<\frac{\varepsilon}{\gamma(x,y)}.$$

Thus, (H3,i) ensures that for $n\ge N$,

$$||\Delta(x,z_n)|-|\Delta(x,y)||\le|\Delta(x,z_n)-\Delta(x,y)|\le\gamma(x,y)|\Delta(y,z_n)-\Delta(y,y)|<\varepsilon,$$

which concludes the proof. □

In conclusion, the set of hypotheses here presented is a simplified but equivalent version of those in [61]. Note that, of course, there exist maps satisfying all of them. One of the simplest examples that we can consider is the family of Stieltjes displacements in Example 2.9. Furthermore, there exist maps satisfying (H1)–(H3) that are not Stieltjes displacements as we show in the following example.

Example 2.31. Let $a, b \in \mathbb{R}$, $a < b$, $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and consider the map $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ defined as

$$\Delta(x, y) = e^{|x|}(g(y) - g(x)), \quad x, y \in [a, b]. \tag{2.14}$$

It is clear from the definition that hypotheses (H1) and (H2) are satisfied. Furthermore, for (H3) it is enough to take $\gamma : [a, b] \times [a, b] \to [1, +\infty)$ defined as

$$\gamma(z, \overline{z}) = \max\{1, e^{|z|-|\overline{z}|}\}, \quad z, \overline{z} \in [a, b].$$

Observe that conditions (H3,ii) and (H3,iii) are clearly satisfied, so we only need to show that (H3, i) holds. Let $z, \overline{z} \in [a, b]$. For any $x, y \in [a, b]$, we have that

$$\begin{aligned} |\Delta(z, x) - \Delta(z, y)| &= e^{|z|}|g(y) - g(x)| = e^{|z|-|\overline{z}|+|\overline{z}|}|g(y) - g(x)| \\ &\le \gamma(z, \overline{z})e^{|\overline{z}|}|g(y) - g(x)| = \gamma(z, \overline{z})|\Delta(\overline{z}, x) - \Delta(\overline{z}, y)|. \end{aligned}$$

In other words, the map Δ satisfies (H1)–(H3). Note, however, that it is not, in general, a Stieltjes displacement. For example, take $a = -2$, $b = 2$ and $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} t, & \text{if } t \le 0, \\ t + 1, & \text{if } t > 0. \end{cases} \tag{2.15}$$

Then, we have that

$$\Delta(0, 1) = e^0(g(1) - g(0)) = 2 \ne 2e = -e^1(g(0) - g(1)) = -\Delta(1, 0),$$

which is enough to ensure that Δ is not a Stieltjes displacement, see Lemma 2.10. This might seem odd considering how close (2.14) is to the definition of the Stieltjes displacement defined by g. Essentially, for each $x \in [a, b]$, Δ generates a nondecreasing and left–continuous map, which can be regarding as a rescaling of the map g. In Figure 2.3 we compare the map Δ and the corresponding Stieltjes displacement for g as in (2.15).

Given that the hypotheses (H1)–(H3) might be complicated to check, we include the following result that gives a sufficient condition for a map $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ to satisfy all of them.

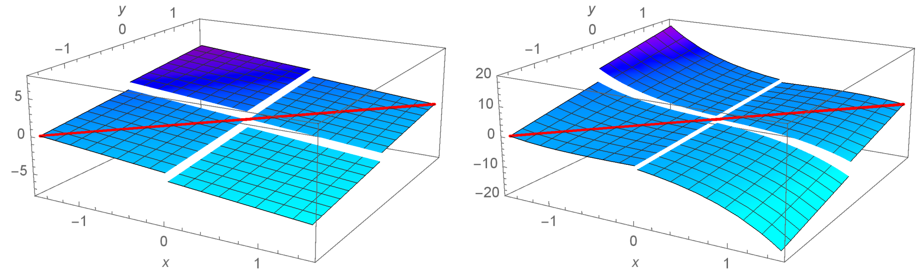
Proposition 2.32. *Let $\Delta : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ be a map and let us denote by $D_2\Delta$ its partial derivative with respect to its second variable. If $D_2\Delta$ exists, is continuous on $[a, b] \times [a, b]$, and there exists $r > 0$ such that*

$$D_2\Delta(x, y) \ge r \quad \textit{for all } (x, y) \in [a, b] \times [a, b],$$

then Δ satisfies (H2) *and* (H3).

Proof. The assumptions imply that for each $x \in [a, b]$, the mapping $\Delta(x, \cdot)$ is increasing and continuous, which is more than (H2). Now fix $z, \overline{z} \in [a, b]$. For $a \le x < y \le b$, the generalized mean value theorem guarantees the existence of $\xi \in (x, y)$ such that

$$\frac{\Delta(z, y) - \Delta(z, x)}{\Delta(\overline{z}, y) - \Delta(\overline{z}, x)} = \frac{D_2\Delta(z, \xi)}{D_2\Delta(\overline{z}, \xi)},$$



(a) Representation of $z = \Delta_g(x, y)$ where Δ_g is the Stieltjes displacement defined by g in (2.15).

(b) Representation of $z = \Delta(x, y)$ where Δ is the displacement defined in (2.14) for g as in (2.15).

Figure 2.3: Figures comparing the differences between a Stieltjes displacement and the corresponding modification in (2.14). The red line representes the value of the displacements along the line $x = y$, showing that (H1) holds in both cases.

which implies (H3, i) for

$$\gamma(z, \overline{z}) = \max \left\{ 1, \max_{a \le \xi \le b} \frac{D_2\Delta(z, \xi)}{D_2\Delta(\overline{z}, \xi)} \right\}. \tag{2.16}$$

The function γ is well–defined and bounded as the three variable mapping

$$(z, \overline{z}, \xi) \in [a, b] \times [a, b] \times [a, b] \longmapsto \frac{D_2\Delta(z, \xi)}{D_2\Delta(\overline{z}, \xi)}$$

is continuous on a compact domain. In particular, (H3, iii) holds.

Finally, (H3, ii) is a consequence of the fact that $D_2\Delta$ is continuous on $[a, b] \times [a, b]$, and, therefore, uniformly continuous on $[a, b] \times [a, b]$. Indeed, let $z \in [a, b]$ be fixed. For a given $\varepsilon > 0$, we can find $\delta > 0$ such that for each $\overline{z} \in [a, b]$, $|z - \overline{z}| < \delta$, we have

$$|D_2\Delta(z, \xi) - D_2\Delta(\overline{z}, \xi)| < r\,\varepsilon \quad \text{for every } \xi \in [a, b].$$

Therefore, if $|z - \overline{z}| < \delta$ then we have

$$\left| \frac{D_2\Delta(z, \xi)}{D_2\Delta(\overline{z}, \xi)} - 1 \right| = \frac{|D_2\Delta(z, \xi) - D_2\Delta(\overline{z}, \xi)|}{D_2\Delta(\overline{z}, \xi)} < \varepsilon \quad \text{for every } \xi \in [a, b].$$

We have just proven that

$$\lim_{\overline{z} \to z} \frac{D_2\Delta(z, \xi)}{D_2\Delta(\overline{z}, \xi)} = 1 \quad \text{uniformly in } \xi \in [a, b].$$

Now, for each $\overline{z} \in [a,b]$ there exists $\xi_{\overline{z}}$ such that

$$\gamma(z,\overline{z}) = \max\left\{1, \frac{D_2\Delta(z,\xi_{\overline{z}})}{D_2\Delta(\overline{z},\xi_{\overline{z}})}\right\}.$$

Hence $\gamma(z,\overline{z}) \to 1$ as $\overline{z} \to z$. Similarly, $\gamma(z,\overline{z}) \to 1$ as $z \to \overline{z}$. □

Example 2.33. Consider the map $\Delta : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$\Delta(x,y) = e^{y^2-x^2} - e^{x-y}, \quad x,y \in [0,1].$$

Recall that this is a displacement that is not a Stieltjes displacement, as shown in Example 2.11. Noting that condition (a) in Definition 2.3 coincides with (H1), it follows that (H1) is satisfied. Moreover, Δ clearly has continuous partial derivatives, and

$$D_2\Delta(x,y) = 2ye^{y^2-x^2} + e^{x-y} \geq e^{x-y} \geq e^{-1} \quad \text{on } [0,1] \times [0,1].$$

Hence, Δ satisfies (H1)–(H3) for γ defined as in (2.16).

2.2.1 The displacement measure

We now turn our attention to the definition of a measure based on a map Δ satisfying hypotheses (H1)–(H3). Such measure will be known as the displacement measure and it is defined in terms of "local" Lebesgue–Stieltjes measures, as we will show now.

Hypothesis (H2) guarantees that the map Δ_z is nondecreasing and left–continuous for each $z \in [a,b]$. Hence, it defines a Lebesgue–Stieltjes measure on $[a,b]$, see Example 1.46. In the following, μ_z will denote the Lebesgue–Stieltjes measure associated to Δ_z and $\mathcal{M}_z$ the Lebesgue–Stieltjes σ–algebra associated to such measure. Define

$$\mathcal{M} := \bigcap_{z\in[a,b]} \mathcal{M}_z. \tag{2.17}$$

Observe that $\mathcal{M}$ is a σ–algebra as pointed out in Remark 1.2. Moreover, since Lebesgue–Stieljes measures are, by definition, τ_u–Borel measures, it follows that $\mathcal{B}(\tau_u) \subset \mathcal{M}$.

Note that hypothesis (H3) allows us to understand the relationship between the different possible measures depending on $z \in [a,b]$. Specifically, it explains the relation of such measures over sets of the form $[c,d) \subset [a,b]$, which characterize the measures on $\mathcal{M}$. In particular, we have that given $z, \overline{z} \in [a,b]$, we have that $\mu_z(I) \leq \gamma(z,\overline{z})\mu_{\overline{z}}(I)$ for any interval I and, as a consequence,

$$\mu_z(A) \leq \gamma(z,\overline{z})\mu_{\overline{z}}(A), \quad \text{for all } A \in \mathcal{M}. \tag{2.18}$$

That is, $\mu_{\overline{z}}|_{\mathcal{M}} \ll \mu_z|_{\mathcal{M}} \ll \mu_{\overline{z}}|_{\mathcal{M}}$ for all $z, \overline{z} \in [a,b]$. With this idea in mind, from now on, we assume that μ_z is defined over $\mathcal{M}$ instead of $\mathcal{M}_z$ for every $z \in [a,b]$. As a consequence, we obtain that μ_z and $\mu_{\overline{z}}$ are equivalent in the sense of Definition 1.31 for all $z, \overline{z} \in [a,b]$.

In order to continue our journey to define the displacement measure we need the following definitions. These definitions will be consistent with the measure we aim to construct.

Definition 2.34. *Let* $[a,b] \subset \mathbb{R}$, $A \subset [a,b]$ *and* $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *We say that a function* $f : [a,b] \to \mathbb{R}$ *is* Δ–measurable *if* $f : ([a,b],\mathcal{M}) \to (\mathbb{R},\mathcal{B}(\tau_u))$ *is measurable in the sense of* Definition 1.9. *Similarly, we say that a property* $\mathcal{P}$ *is satisfied* Δ–almost everywhere in A, *shortened as* Δ*–a.e. in* A, *or that it holds for* Δ–almost all $x \in A$, *or simply* Δ*–a.a.* $x \in A$, *if there exists* $z \in [a,b]$ *such that* $\mathcal{P}$ *is satisfied* μ_z*–almost everywhere in* A *in the sense of* Definition 1.26.

Remark 2.35. Note that by construction of $\mathcal{M}$, $f : [a,b] \to \mathbb{R}$ is Δ–measurable if and only if $f : ([a,b],\mathcal{M}_z) \to (\mathbb{R},\mathcal{B}(\tau_u))$ is measurable for every $z \in [a,b]$. Similarly, given the equivalence existing between the different measures μ_z, $z \in [a,b]$, a property $\mathcal{P}$ is satisfied Δ–almost everywhere in A if and only if $\mathcal{P}$ is satisfied μ_z–almost everywhere in A for all $z \in [a,b]$.

With these definitions, we can make explicit the relation between μ_z and $\mu_{\overline{z}}$, $z, \overline{z} \in [a,b]$, using Corollary 1.35. Since μ_z and $\mu_{\overline{z}}$, $z, \overline{z} \in [a,b]$, are equivalent, we know that there exists two Δ–measurable functions, $h_{z,\overline{z}}, h_{\overline{z},z} : [a,b] \to [0,\infty)$, such that

$$\mu_z(A) = \int_A h_{z,\overline{z}} \,\mathrm{d}\,\mu_{\overline{z}}, \quad \mu_{\overline{z}}(A) = \int_A h_{\overline{z},z} \,\mathrm{d}\,\mu_z, \quad \text{for all } A \in \mathcal{M}. \tag{2.19}$$

In what follows, we denote by $h_{x,y}$ the Radon–Nikodým derivative of μ_x with respecto to μ_y, $x, y \in [a,b]$.

Proposition 2.36. *Let* $[a,b] \subset \mathbb{R}$, $z, \overline{z} \in [a,b]$ *and* $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Denote by* $h_{z,\overline{z}}$ *the function in* (2.19). *Then*

$$\frac{1}{\gamma(\overline{z},z)} \le h_{z,\overline{z}}(t) \le \gamma(z,\overline{z}), \quad \Delta\text{–}a.a.\ t \in [a,b].$$

Proof. First, assume that $h_{z,\overline{z}}(t) \le \gamma(z,\overline{z})$, does not hold for Δ–a.a. $t \in [a,b]$. Then, there would exist $A \in \mathcal{M}$ such that $\mu_{\overline{z}}(A) > 0$ and

$$h_{z,\overline{z}}(t) > \gamma(z,\overline{z}), \quad \text{for all } t \in A.$$

Hence,

$$\mu_z(A) = \int_A h_{z,\overline{z}}(s) \,\mathrm{d}\,\mu_{\overline{z}}(s) > \int_A \gamma(z,\overline{z}) \,\mathrm{d}\,\mu_z(s) = \gamma(z,\overline{z})\mu_{\overline{z}}(A),$$

which contradicts (2.18). Therefore,

$$h_{z,\overline{z}}(t) \le \gamma(z,\overline{z}), \quad \Delta\text{–a.a. } t \in [a,b]. \tag{2.20}$$

For the other inequality, take $h_{\overline{z},z}$ as in (2.19). Using (2.20), we have that

$$\gamma(\overline{z},z) \ge h_{\overline{z},z}(t) = \frac{1}{h_{z,\overline{z}}(t)}, \quad \Delta\text{–a.a. } t \in [a,b],$$

where the last equality follows from Remark 1.37. □

Observe that Proposition 2.36 ensures that, for $z, \overline{z} \in [a, b]$ fixed,

$$\frac{1}{\gamma(\overline{z}, z)} \leq h_{z,\overline{z}}(t) \leq \gamma(z, \overline{z}), \quad \text{for all } t \in [a, b] \setminus A_z,$$

for some $A \in \mathcal{M}$ such that with $\mu_z(A_z) = 0$. Let us define $\widetilde{h}_{z,\overline{z}} : [a, b] \to [0, +\infty)$ as

$$\widetilde{h}_{z,\overline{z}}(t) = \begin{cases} h_{z,\overline{z}}(t), & \text{if} \quad t \in [a, b] \setminus A_z, \\ 1, & \text{if} \quad t \in A_z. \end{cases}$$

Then, it follows that

$$\frac{1}{\gamma(\overline{z}, z)} \leq \widetilde{h}_{z,\overline{z}}(t) \leq \gamma(z, \overline{z}), \quad \text{for all } t \in [a, b]. \tag{2.21}$$

Moreover, $\widetilde{h}_{z,\overline{z}} = h_{z,\overline{z}}$ Δ–a.e. in $[a, b]$, and in particular, μ_z–a.e. in $[a, b]$. Hence, we have that

$$\mu_z(A) = \int_A \widetilde{h}_{z,\overline{z}} \, \mathrm{d}\, \mu_{\overline{z}}, \quad \text{for all } A \in \mathcal{M}.$$

Thus, we can assume without loss of generality that the functions in (2.19) satisfy (2.21). Given this consideration, we can obtain the following result.

Proposition 2.37. *Let* $[a, b] \subset \mathbb{R}$, $\overline{z} \in [a, b]$ *and* $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Then*

$$\lim_{z \to \overline{z}} h_{z,\overline{z}}(t) = 1, \quad \textit{for all } t \in [a, b]. \tag{2.22}$$

Proof. Fix $\overline{z}, t \in [a, b]$. It follows from (2.21) that

$$\frac{1}{\gamma(\overline{z}, z)} \leq h_{z,\overline{z}}(t) \leq \gamma(z, \overline{z}), \quad \text{for all } z \in [a, b]. \tag{2.23}$$

Hence it is enough to consider the limit when $z \to \overline{z}$ in the previous inequalities, together with hypothesis (H3, ii), to obtain the result. □

Remark 2.38. Note that, given $\overline{z} \in [a, b]$, we also have that there exist $m_{\overline{z}}, M_{\overline{z}} > 0$ such that

$$m_{\overline{z}} \leq h_{t,\overline{z}}(t) \leq M_{\overline{z}}, \quad \text{for all } t \in [a, b].$$

Indeed, fix $\overline{z}, t \in [a, b]$. Then, (2.23) with $z = t$ yields

$$\frac{1}{\gamma(\overline{z}, t)} \leq h_{t,\overline{z}}(t) \leq \gamma(t, \overline{z}).$$

Now the result follows from (H3, iii).

We finally have all the tools necessary for the definition of the measure introduced in [61]. To that extent, we will first prove that the next family of set maps is well–defined.

Proposition 2.39. *Let* $[a,b] \subset \mathbb{R}$ *and* $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Consider* $\alpha : [a,b] \to [a,b]$ *a* Δ*–measurable map and* $z \in [a,b]$. *Then the map* $\mu_\alpha : \mathcal{M} \to [0,+\infty]$ *given by*

$$\mu_\alpha(A) = \int_A h_{\alpha(t),z}(t)\,\mathrm{d}\,\mu_z(t), \quad A \in \mathcal{M}, \tag{2.24}$$

is well–defined. Moreover, the definition of μ_α *is independent of the choice of* $z \in [a,b]$.

Proof. In order to show that the map is well–defined, it is enough to show that $h_{\alpha(\cdot),z}(\cdot)$ is μ_z–measurable. To that end, define the map $h_z : [a,b] \times [a,b] \to [0,+\infty)$ given by

$$h_z(t,x) = h_{x,z}(t), \quad t \in [a,b].$$

We will show that h_z is a μ_z–Carathéodory function in the sense of Definition 1.27, as this will ensure that the composition $f(\cdot,\alpha(\cdot))$ is μ_z–measurable (see Proposition 1.28).

Note that condition (i) in Definition 1.27 is trivially satisfied as $h_z(\cdot,x) = h_{x,z}(\cdot)$ is μ_z–measurable by definition (see Corollary 1.35). As for condition (ii), let $x \in [a,b]$. It follows directly from Remark 1.37 and (2.22) that

$$\lim_{\overline{z}\to x} h_{\overline{z},z}(t) = \lim_{\overline{z}\to x} h_{\overline{z},x}(t) h_{x,z}(t) = h_{x,z}(t), \quad \text{for } \Delta\text{–a.a. } t \in [a,b],$$

that is, $h_z(t,\cdot)$ is continuous on $[a,b]$ for Δ–a.a. $t \in [a,b]$, and therefore, for μ_z–a.a. $t \in [a,b]$. Finally, (H3, iii) guarantees the existence of $M > 0$ such that $|\gamma(x,z)| < M$ for all $x \in [a,b]$. Hence, it follows from (2.21) that

$$|h_z(t,x)| \le M, \quad \text{for } \Delta\text{–a.a. } t \in [a,b], \text{ for all } x \in [a,b].$$

It is clear that $M \in \mathcal{L}^1_{\mu_z}([a,b],[0,+\infty))$, so it follows from Remark 2.35 that (iii) holds, i.e., the map h_z is μ_z–Carathéodory.

Now, to show that the definition of μ_α is independent of the choice of the point $z \in [a,b]$ in (2.24), let $z, \overline{z} \in [a,b]$, $z \neq \overline{z}$. Remark 1.37 ensures that $h_{\alpha(s),\overline{z}}(s) = h_{\alpha(s),z}(s) h_{z,\overline{z}}(s)$ for Δ–a.a. $s \in [a,b]$. Hence,

$$\int_A h_{\alpha(s),z}(s)\,\mathrm{d}\,\mu_z(s) = \int_A h_{\alpha(s),z}(s) h_{z,\overline{z}}(s)\,\mathrm{d}\,\mu_{\overline{z}}(s) = \int_A h_{\alpha(s),\overline{z}}(s)\,\mathrm{d}\,\mu_{\overline{z}}(s). \qquad \square$$

This result, together with Proposition 1.19 ensure that μ_α is in fact a measure.

Definition 2.40. *Let* $[a,b] \subset \mathbb{R}$ *and* $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Consider a* Δ*–measurable map,* $\alpha : [a,b] \to [a,b]$, *and* $z \in [a,b]$. *The map* $\mu_\alpha : \mathcal{M} \to [0,+\infty)$ *given by*

$$\mu_\alpha(A) = \int_A h_{\alpha(t),z}(t)\,\mathrm{d}\,\mu_z(t), \quad A \in \mathcal{M},$$

is a measure, which we call Δ_α–measure. *In the particular case where* α *is the identity map, it will be called the* displacement measure*, or simply* Δ–measure*, and it will be denoted by* $\mu \equiv \mu_{\mathrm{Id}}$.

Remark 2.41. Let $g : [a, b] \to \mathbb{R}$ be a nondecreasing left–continuous function and Δ be the Stieltjes displacement associated with g as in Example 2.9. Then, it follows that $\mu_z = \mu_g$ for all $z \in [a, b]$. As a consequence, we have that $h_{z,\overline{z}} = 1$ for all $z, \overline{z} \in [a, b]$ and, therefore, the Δ–measure coincides with the Lebesgue–Stieltjes measure defined by g.

Observe that the notation and nomenclature used in Definition 2.34 so far is consistent with the corresponding definitions for the Δ–measure, μ, as we anticipated. Indeed, for example, f is Δ*–measurable* if and only if it is μ*–measurable*, as μ and μ_z, $z \in [a, b]$, are defined over the same σ–algebra, $\mathcal{M}$. Justifying the equivalence for properties holding μ and μ_z everywhere is not as immediate from the definition, but it can be obtained as a consequence from the next result.

Proposition 2.42. *Let* $[a, b] \subset \mathbb{R}$ *and* $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Then* μ *and* μ_z*,* $z \in [a, b]$*, are equivalent in the sense of* Definition 1.31.

Proof. It follows directly from the definition that $\mu \ll \mu_z$. Hence, all that is left to do is show that $\mu_z \ll \mu$ for all $z \in [a, b]$. Fix $z \in [a, b]$. Hypothesis (H3, iii) ensures that there exists $K > 0$ such that

$$\gamma(z, x) < K, \quad x \in [a, b].$$

Hence, (2.21) implies that

$$\mu(A) = \int_A h_{\alpha(s),z}(s) \, \mathrm{d}\, \mu_z(s) \geq \int_A \frac{1}{\gamma(z, \alpha(s))} \, \mathrm{d}\, \mu_z(s) \geq \int_A \frac{1}{K} \, \mathrm{d}\, \mu_z(s) = \frac{\mu_z(A)}{K}.$$

Thus, if $\mu(A) = 0$ then $\mu_z(A) = 0$, i.e. $\mu_z \ll \mu$ for all $z \in [a, b]$. □

Finally, by defining the corresponding concepts of integrability of functions, it is possible to establish the following relationship.

Proposition 2.43. *Let* $[a, b] \subset \mathbb{R}$ *and* $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Given* $X \in \mathcal{M}$*, define the set of* Δ–integrable functions over X *as*

$$\mathcal{L}^1_\Delta(X, \mathbb{R}) := \left\{ f : X \to \mathbb{R} : f \text{ is } \Delta\text{–measurable}, \int_X |f| \, \mathrm{d}\, \mu_z < +\infty, \text{ for some } z \in [a, b] \right\}.$$

Then $\mathcal{L}^1_\mu(X, \mathbb{R}) = \mathcal{L}^1_\Delta(X, \mathbb{R})$.

Proof. First of all, recall that the concepts of μ–measurablility and Δ–measurability are equivalent, so all that is left to do is prove that the finite character of the integrals is equivalent.

Let $f \in \mathcal{L}^1_\Delta(X, \mathbb{R})$ and $z \in [a, b]$ be such that $\int_X |f| \, \mathrm{d}\, \mu_z$ is finite. Hypothesis (H3, iii) guarantees the existence of $M > 0$ such that $|\gamma(x, z)| < M$ for all $x \in [a, b]$. Thus, using (2.21) we have that

$$\begin{aligned} \int_X |f| \, \mathrm{d}\, \mu &= \int |f(s) h_{s,z}(s)| \, \mathrm{d}\, \mu_z(s) \\ &\leq \int |f(s)\gamma(s, z)| \, \mathrm{d}\, \mu_z(s) \leq M \int |f(s)| \, \mathrm{d}\, \mu_z(s) < +\infty, \end{aligned}$$

that is, $f \in \mathcal{L}^1_\mu(X,\mathbb{R})$.

Conversely, let $f \in \mathcal{L}^1_\mu(X,\mathbb{R})$ and $z \in [a,b]$. Again, hypothesis (H3, iii) implies that there exists $K > 0$ such that $|\gamma(z,x)| < K$ for all $x \in [a,b]$. Hence

$$\int_X |f| \,\mathrm{d}\,\mu = \int |f(s)h_{s,z}(s)| \,\mathrm{d}\,\mu_z(s) \geq \int \left|\frac{f(s)}{\gamma(z,s)}\right| \mathrm{d}\,\mu_z(s) \geq \frac{1}{K}\int |f(s)| \,\mathrm{d}\,\mu_z(s),$$

so $\int_X |f| \,\mathrm{d}\,\mu_z < +\infty$, which concludes the proof. □

Remark 2.44. Note that in the last segment of the proof, $z \in [a,b]$ was arbitrarily chosen. That is, if f is μ–integrable, it is μ_z–integrable for all $z \in [a,b]$. The other implication also holds, as presented in the proof above. Hence, we can rewrite $\mathcal{L}^1_\mu(X,\mathbb{R})$, or equivalently $\mathcal{L}^1_\Delta(X,\mathbb{R})$, as the set

$$\left\{ f : X \to \mathbb{R} : f \text{ is } \Delta\text{–measurable}, \int_X |f| \,\mathrm{d}\,\mu_z < +\infty, \text{ for all } z \in [a,b] \right\}.$$

As a final comment for this section, observe that $\mu : \mathcal{M} \to [0,+\infty]$ is a Borel measure that assigns finite measure to bounded sets, that is, it is a Lebesgue–Stieltjes measure (see Definition 1.38). Therefore, it can be represented over $\mathcal{B}(\tau_u)$ by a nondecreasing and left–continuous function g as defined in (1.28). This relationship is key for the study of μ over some interesting sets related to the map Δ. These sets will be fundamental in the definition of the displacement derivative. Let us define the sets C_Δ and D_Δ as

$$C_\Delta := \{x \in (a,b) : \Delta(x,\cdot) = 0 \text{ in } (x-\varepsilon, x+\varepsilon) \subset (a,b) \text{ for some } \varepsilon > 0\}, \tag{2.25}$$
$$D_\Delta := \{x \in [a,b) : \Delta(x,x^+) \neq 0\}. \tag{2.26}$$

Remark 2.45. It follows from the definition that $D_\Delta = D_{\Delta_z}$, $z \in [a,b]$, where D_{Δ_z} should be understood in the sense of (1.21). Indeed, first, let us show that $D_{\Delta_z} \subset D_\Delta$ for all $z \in [a,b]$ by proving that if $x \notin D_\Delta$ then $x \notin D_{\Delta_z}$, $z \in [a,b]$. Fix $z \in [a,b]$ and $x \notin D_\Delta$. Then we must have that $\Delta(x,x^+) = 0$, so given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\Delta(x,y)| < \frac{\varepsilon}{\gamma(z,x)}, \quad \text{for all } y \in [a,b],\ 0 < y - x < \delta.$$

Hence, if $y \in [a,b]$ is such that $0 < y - x < \delta$, then

$$|\Delta(z,x) - \Delta(z,y)| \leq \gamma(z,x)|\Delta(x,x) - \Delta(x,y)| < \varepsilon.$$

That is, Δ_z is continuous from the right at x, so $x \notin D_{\Delta_z}$. In other words, $D_{\Delta_z} \subset D_\Delta$ for all $z \in [a,b]$.

Conversely, suppose that $x \notin D_{\Delta_z}$, $z \in [a,b]$. In particular, we have that $x \notin D_{\Delta_x}$, which implies that $\Delta(x,x^+) = 0$. Thus, $x \notin D_\Delta$. In conclusion, we have that $D_\Delta \subset D_{\Delta_z}$ for all $z \in [a,b]$, which is enough to guarantee that $D_\Delta = D_{\Delta_z}$, $z \in [a,b]$.

Proposition 2.46. *Let g be as in* (1.28) *for the Δ–measure μ and let D_g and C_g be as in* (1.21) *and* (1.22)*, respectively. Then $D_\Delta = D_g \cap [a,b)$ and $C_\Delta = C_g \cap (a,b)$.*

Proof. For the equality $D_\Delta = D_g \cap [a,b)$, it is enough to note that for any $t \in [a,b)$ we have that

$$\begin{aligned} g(t^+) - g(t) &= \lim_{s\to t^+} \int_{[t,s)} h_{r,t}(r) \, \mathrm{d}\, \mu_t(r) \\ &= \int_{\{t\}} h_{r,t}(r) \, \mathrm{d}\, \mu_t(r) = h_{t,t}(t)(\Delta_t(t^+) - \Delta_t(t)) = \Delta_t(t^+), \end{aligned}$$

where the last part follows from (1.23).

Now, in order to see that $C_\Delta = C_g \cap (a,b)$, let $t \in C_\Delta$. Then $\Delta_t(\cdot) = 0$ on $(t-\varepsilon, t+\varepsilon) \cap (a,b)$ for some $\varepsilon \in \mathbb{R}^+$. Let $r, s \in (t-\varepsilon, t+\varepsilon) \cap (a,b)$, $r < s$. Then, using Remark 2.38,

$$0 \le g(s) - g(r) = \mu([r,s)) = \int_{[r,s)} h_{x,t}(x) \, \mathrm{d}\, \mu_t(x) \le M_t \mu_t([r,s)) = 0,$$

since $[r,s) \subset (t-\varepsilon, t+\varepsilon)$. Thus g is constant on $(t-\varepsilon, t+\varepsilon) \cap (a,b)$. Now, it follows that $t \in C_g$ noting that g is constant outside of (a,b). Conversely, if $t \in C_g \cap (a,b)$, then g is constant on $(t-\varepsilon, t+\varepsilon)$ for some $\varepsilon \in \mathbb{R}^+$. Let $r, s \in (t-\varepsilon, t+\varepsilon) \cap [a,b)$, $r < s$. Then, Remark 2.38 implies that

$$0 \le m_t(\Delta_t(s) - \Delta_t(r)) = m_t \mu_t([r,s)) \le \int_{[r,s)} h_{x,t}(x) \, \mathrm{d}\, \mu_t(x) = g(s) - g(r) = 0.$$

That is, Δ_t is constant on $(t-\varepsilon, t+\varepsilon) \cap (a,b)$, and since $\Delta_t(t) = 0$, it follows that $t \in C_\Delta$. □

The first consequence of Proposition 2.46 is that C_Δ is an open set in the usual topology. Moreover, it can be uniquely expressed as the countable union of open intervals as in (1.24). For those intervals, define

$$N_\Delta := \{a_n, b_n : n \in \mathbb{N}\} \backslash D_\Delta. \tag{2.27}$$

Note that $N_\Delta = N_g \cap (a,b)$. Hence, the following result follows directly from Proposition 1.54.

Corollary 2.47. *Let C_Δ, D_Δ and N_Δ be as in* (2.25)*,* (2.26) *and* (2.27)*, respectively. Then*

(i) *D_Δ is at most countable;*

(ii) $\mu(C_\Delta) = \mu(N_\Delta) = 0$.

Remark 2.48. As a consequence of Corollary 2.47, a property holds μ–a.e. in E if it holds on $E \setminus O_\Delta$ with

$$O_\Delta := C_\Delta \cup N_\Delta.$$

Moreover, if $x \notin O_\Delta \cup D_\Delta$ we have that $\Delta_x(y) \neq 0$ for all $y \in [a,b]$, $x \neq y$.

2.2.2 The displacement derivative

We now introduce the concept of derivative of a function defined over an interval $[a, b]$ with respect to a map satisfying (H1)–(H3). We chose this setting because it allows us to have some properties on the derivative, such as linearity, which will be helpful in order to study the relationship between the displacement derivative and its integral.

In order to properly define this derivative, we will need the following auxiliary functions. Given a function $f : [a, b] \to \mathbb{R}$ and a point $x \in [a, b]$, we define the function

$$F_x(\cdot) = \frac{f(\cdot) - f(x)}{\Delta(x, \cdot)},$$

which we will assume to be defined in a neighbourhood of x in which the expression makes sense, namely, where $\Delta_x(\cdot) \neq 0$. Note that if $x \in C_\Delta$, there exists $\varepsilon_x > 0$ such that the expression does not make sense for any neighbourhood $(x - \varepsilon, x + \varepsilon) \cap [a, b]$, $\varepsilon \in (0, \varepsilon_x)$. With this notation, we present the following definition.

Definition 2.49. *Let* $[a, b] \subset \mathbb{R}$, $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy* (H1)–(H3) *and* $f : [a, b] \to \mathbb{R}$ *be a function. The* displacement derivative *or* Δ–derivative *of* f *at* $x \in [a, b] \setminus C_\Delta$ *is defined as*

$$f'_\Delta(x) = \begin{cases} \displaystyle\lim_{y \to x} F_x(y), & x \notin D_\Delta, \\ \displaystyle\lim_{y \to x^+} F_x(y), & x \in D_\Delta, \end{cases}$$

provided that the corresponding limit exists. In that case, we say that f *is* Δ–differentiable at x. *If* f *is* Δ*–differentiable at every* $x \in [a, b] \setminus C_\Delta$*, we say that* f *is* Δ–differentiable in $[a, b]$.

Remark 2.50. In the work ahead, we will simply write

$$f'_\Delta(x) = \begin{cases} \displaystyle\lim_{y \to x} \frac{f(y) - f(x)}{\Delta(x, y)}, & x \notin D_\Delta, \\ \displaystyle\lim_{y \to x^+} \frac{f(y) - f(x)}{\Delta(x, y)}, & x \in D_\Delta. \end{cases}$$

However, those limits should be understood in the sense of Definition 2.49, which are well–defined when x is an accumulation point of the domain of the function F_x. This explains why the points of C_Δ are excluded in Definition 2.49. Moreover, these limits should be properly understood at some other conflicting points. For example, imagine $\Delta_x(\cdot) = 0$ in some $(x - \delta, x)$, $\delta > 0$, and $\Delta_x(\cdot) \neq 0$ on $(x, x + \varepsilon)$ for every $\varepsilon > 0$. Then

$$\lim_{y \to x} F_x(y) = \lim_{y \to x^+} F_x(y),$$

since F_x is not the defined at the right of x. The same thing occurs if $x = a$ and $\Delta_x(\cdot) \neq 0$ on $(a, a + \varepsilon)$ for every $\varepsilon > 0$. Similarly, if $\Delta_x(\cdot) = 0$ in some $(x, x + \delta)$, $\delta > 0$, or $x = b$, and $\Delta_x(\cdot) \neq 0$ on $(x - \varepsilon, x)$ for every $\varepsilon > 0$. In that case, the function F_x is not defined at the left of x, so

$$\lim_{y \to x} F_x(y) = \lim_{y \to x^-} F_x(y).$$

Remark 2.51. Note that given a point $x \in D_\Delta$, the Δ–derivative of a function f exists if and only if the limit of f from the right of x, $f(x^+)$, exists. In that case,

$$f'_\Delta(x) = \frac{f(x^+) - f(x)}{\Delta(x, x^+)}.$$

Remark 2.52. Considering that Proposition 2.30 ensures that $([a,b], \Delta)$ is a displacement space, one might think that the natural choice for the definition of the derivative would be by taking the limit in the τ_Δ topology instead of in the usual topology of the real line. However, if $x \notin D_\Delta$, x is a continuity point of Δ_x, and it is easy to see that such limit can be translated into a limit in the usual topology, which is far more convenient for the theory constructed there. It is at this point that the importance of hypothesis (a) in Definition 2.3 arises as commented in Remark 2.15. Without that hypothesis we would not be able to ensure that the balls of any center and radii are nonempty, so the limits in the Δ–topology might not be well-defined in that case.

The following results include some of the basic properties we anticipated earlier. This properties are analogous to the properties of the usual derivative.

Proposition 2.53. *Let $[a,b] \subset \mathbb{R}$, $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ satisfy hypotheses* (H1)–(H3) *and $f : [a,b] \to \mathbb{R}$ be Δ–differentiable at $x \in [a,b] \setminus D_\Delta$. If*

$$\Delta_x(y) < 0, \quad \textit{for all } y \in [a,b] \textit{ such that } y < x, \tag{2.28}$$

f is continuous from the left at x in the usual sense. Similarly, if

$$\Delta_x(y) > 0, \quad \textit{for all } y \in [a,b] \textit{ such that } y > x, \tag{2.29}$$

f is continuous from the right at x in the usual sense.

Proof. Let $x \in [a,b] \setminus D_\Delta$ and assume that (2.29) holds. This condition together with the existence of $f'_\Delta(x)$ ensures that

$$f'_\Delta(x) = \lim_{y \to x^-} \frac{f(y) - f(x)}{\Delta(x, y)}.$$

Therefore, we have that

$$\lim_{y \to x^-} (f(y) - f(x)) = \lim_{y \to x^-} \left(\frac{f(y) - f(x)}{\Delta(x,y)} \Delta(x,y) \right) = f'_\Delta(x) \lim_{y \to x^-} \Delta(x,y) = 0,$$

since Δ_x is continuous at x. Thus, f is continuous from the left at x. The proof for the other case is analogous and we omit it. □

Proposition 2.54. *Let $[a,b] \subset \mathbb{R}$, $x \in [a,b]$ and $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ satisfy* (H1)–(H3). *If $f_1, f_2 : [a,b] \to \mathbb{R}$ are Δ–differentiable at x, then:*

(i) *The function $\lambda_1 f_1 + \lambda_2 f_2$ is Δ–differentiable at x for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_\Delta(x) = \lambda_1 (f_1)'_\Delta(x) + \lambda_2 (f_2)'_\Delta(x). \tag{2.30}$$

(ii) *The product* $f_1 f_2$ *is* Δ*–differentiable at* x *and*

$$(f_1 f_2)'_\Delta(x) = (f_1)'_\Delta(x) f_2(x) + (f_2)'_\Delta(x) f_1(x) + (f_1)'_\Delta(x)(f_2)'_\Delta(x)\Delta(x,x^+). \tag{2.31}$$

(iii) *If* $f_2(x)(f_2(x) + (f_2)'_\Delta(x)\Delta(x,x^+)) \neq 0$*, the quotient* f_1/f_2 *is* Δ*–differentiable at* x *and*

$$\left(\frac{f_1}{f_2}\right)'_\Delta(x) = \frac{(f_1)'_\Delta(x) f_2(x) - f_1(x)(f_2)'_\Delta(x)}{f_2(x)(f_2(x) + (f_2)'_\Delta(x)\Delta(x,x^+))}. \tag{2.32}$$

Proof. Let $x \in [a,b]$ be such that $(f_1)'_\Delta(x)$ and $(f_2)'_\Delta(x)$ exist. Then $x \notin C_\Delta$, and so we have that (2.29) and/or (2.28) hold.

(i) Fix $\lambda_1, \lambda_2 \in \mathbb{R}$. Assume that (2.29) holds. In that case,

$$\begin{aligned} &\lim_{y\to x^+} \frac{\lambda_1 f_1(y) + \lambda_2 f_2(y) - (\lambda_1 f_1(x) + \lambda_2 f_2(x))}{\Delta(x,y)} \\ &= \lim_{y\to x^+} \lambda_1 \frac{f_1(y) - f_1(x)}{\Delta(x,y)} + \lim_{y\to x^+} \lambda_2 \frac{f_2(y) - f_2(x)}{\Delta(x,y)}, \end{aligned}$$

that is,

$$\lim_{y\to x^+} \frac{\lambda_1 f_1(y) + \lambda_2 f_2(y) - (\lambda_1 f_1(x) + \lambda_2 f_2(x))}{\Delta(x,y)} = \lambda_1 (f_1)'_\Delta(x) + \lambda_2 (f_2)'_\Delta(x). \tag{2.33}$$

If $x \in D_\Delta$ or $\Delta_x(\cdot) = 0$ on some $[x-\delta, x]$, $\delta > 0$, then the limit in (2.33) is $(\lambda_1 f_1 + \lambda_2 f_2)'_\Delta(x)$ (see Remark 2.50) and the proof is complete. Otherwise, $x \notin D_\Delta$ and (2.28) holds. In that case,

$$\begin{aligned} &\lim_{y\to x^-} \frac{\lambda_1 f_1(y) + \lambda_2 f_2(y) - (\lambda_1 f_1(x) + \lambda_2 f_2(x))}{\Delta(x,y)} \\ &= \lim_{y\to x^-} \lambda_1 \frac{f_1(y) - f_1(x)}{\Delta(x,y)} + \lim_{y\to x^-} \lambda_2 \frac{f_2(y) - f_2(x)}{\Delta(x,y)}, \end{aligned}$$

which again, yields

$$\lim_{y\to x^-} \frac{\lambda_1 f_1(y) + \lambda_2 f_2(y) - (\lambda_1 f_1(x) + \lambda_2 f_2(x))}{\Delta(x,y)} = \lambda_1 (f_1)'_\Delta(x) + \lambda_2 (f_2)'_\Delta(x). \tag{2.34}$$

since $x \notin D_\Delta$. Therefore, $\lambda_1 f_1 + \lambda_2 f_2$ is Δ–differentiable at x and (2.30) holds.

Finally, assume that $x \notin D_\Delta$ and (2.28) holds but (2.29) does not. Then, we have that (2.34) holds (repeating the same argument) and $\Delta_x(\cdot) = 0$ on some $[x, x+\delta]$, $\delta > 0$. Then, Remark 2.50 ensures that $(\lambda_1 f_1 + \lambda_2 f_2)'_\Delta(x)$ exists and so (2.30) holds.

(ii) First, observe that we can rewrite $f_1 f_2(y) - f_1 f_2(x)$ as

$$\frac{(f_1(y) - f_1(x))(f_2(x) + f_2(y)) + (f_2(y) - f_2(x))(f_1(x) + f_1(y))}{2}, \quad y \in [a,b]. \tag{2.35}$$

Once again, assume that (2.29) holds. Then, it follows from (2.35) that

$$\lim_{y\to x^+} \frac{f_1f_2(y)-f_1f_2(x)}{\Delta(x,y)}$$

exists and equals

$$\frac{(f_1)'_\Delta(x)(f_2(x)+f_2(x^+))+(f_2)'_\Delta(x)(f_1(x)+f_1(x^+))}{2}. \tag{2.36}$$

Now, if $x\in D_\Delta$, it follows from Remark 2.51 that

$$f_i(x^+)=(f_i)'_\Delta(x)\Delta(x,x^+)+f_i(x), \quad i=1,2. \tag{2.37}$$

Thus, (2.36) yields that

$$\lim_{y\to x^+} \frac{f_1f_2(y)-f_1f_2(x)}{\Delta(x,y)} = (f_1)'_\Delta(x)f_2(x)+(f_2)'_\Delta(x)f_1(x)+(f_1)'_\Delta(x)(f_2)'_\Delta(x)\Delta(x,x^+). \tag{2.38}$$

On the other hand, if $x\notin D_\Delta$, it follows from Proposition 2.53 and (2.36) that

$$\lim_{y\to x^+} \frac{f_1f_2(y)-f_1f_2(x)}{\Delta(x,y)} = (f_1)'_\Delta(x)f_2(x)+(f_2)'_\Delta(x)f_1(x),$$

which matches (2.38) since $\Delta(x,x^+)=0$. In other words, (2.38) holds in both cases. Hence, if $x\in D_\Delta$ or $\Delta_x(\cdot)=0$ on some $[x-\delta,x]$, $\delta>0$, then the limit in (2.38) is $(f_1f_2)'_\Delta(x)$ and the proof is complete. Otherwise, $x\notin D_\Delta$ and (2.28) holds. In that case, we obtain from (2.35) that

$$\lim_{y\to x^-} \frac{f_1f_2(y)-f_1f_2(x)}{\Delta(x,y)}$$

exists and equals

$$\frac{(f_1)'_\Delta(x)(f_2(x)+f_2(x^-))+(f_2)'_\Delta(x)(f_1(x)+f_1(x^-))}{2}.$$

However, Proposition 2.53 ensures that

$$\lim_{y\to x^-} \frac{f_1f_2(y)-f_1f_2(x)}{\Delta(x,y)} = (f_1)'_\Delta(x)f_2(x)+(f_2)'_\Delta(x)f_1(x). \tag{2.39}$$

Therefore, f_1/f_2 is Δ–differentiable at x and (2.31) holds.

Finally, assume that $x\notin D_\Delta$ and (2.28) holds but (2.29) does not. Then, repeating the same arguments, we have that (2.39) holds and $\Delta_x(\cdot)=0$ on some $[x,x+\delta]$, $\delta>0$. Then, Remark 2.50 ensures that $(f_1f_2)'_\Delta(x)$ exists and so (2.31) holds.

(iii) First, note that the extra hypothesis guarantees that $f_2(x)\neq 0$. Furthermore, we also have that $f_2(x)+(f_2)'_\Delta(x)\Delta(x,x^+)\neq 0$ which, provided that $x\in D_\Delta$, ensures that $f_2(x^+)\neq 0$, see (2.37).

Assume that (2.29) holds. Since $f_2(x) \neq 0$, it follows from Proposition 2.53 (if $x \notin D_\Delta$) and the definition of limit from the right (if $x \in D_\Delta$) that there exists $\varepsilon > 0$ such that f_2 does not vanish in $[x, x+\varepsilon) \cap [a,b]$. Hence, the following expression is well–defined for any $y \in [x, x+\varepsilon) \cap [a,b]$,

$$\frac{f_1(y)}{f_2(y)} - \frac{f_1(x)}{f_2(x)} = \frac{f_1(y)f_2(x) - f_1(x)f_2(y)}{f_2(x)f_2(y)},$$

which we can rewrite as

$$\frac{f_1(y)}{f_2(y)} - \frac{f_1(x)}{f_2(x)} = \frac{(f_1(y) - f_1(x))f_2(x) + f_1(x)(f_2(x) - f_2(y))}{f_2(x)f_2(y)}. \tag{2.40}$$

Taking the corresponding limit from the right, we have that

$$\lim_{y \to x^+} \frac{(f_1/f_2)(y) - (f_1/f_2)(x)}{\Delta(x,y)} = \frac{(f_1)'_\Delta(x)f_2(x) - f_1(x)(f_2)'_\Delta(x)}{f_2(x)f_2(x^+)}. \tag{2.41}$$

Now, if $x \in D_\Delta$, it follows from (2.37) that

$$\lim_{y \to x^+} \frac{(f_1/f_2)(y) - (f_1/f_2)(x)}{\Delta(x,y)} = \frac{(f_1)'_\Delta(x)f_2(x) - f_1(x)(f_2)'_\Delta(x)}{f_2(x)(f_2(x) + (f_2)'_\Delta(x)\Delta(x,x^+))}. \tag{2.42}$$

On the other hand, if $x \notin D_\Delta$, it follows from Proposition 2.53 and (2.41) that

$$\lim_{y \to x^+} \frac{(f_1/f_2)(y) - (f_1/f_2)(x)}{\Delta(x,y)} = \frac{(f_1)'_\Delta(x)f_2(x) - f_1(x)(f_2)'_\Delta(x)}{(f_2(x))^2},$$

which matches (2.42). That is, (2.42) holds in both cases. Hence, if $x \in D_\Delta$ or $\Delta_x(\cdot) = 0$ on some $[x-\delta, x]$, $\delta > 0$, then the limit in (2.42) is $(f_1/f_2)'_\Delta(x)$ and the proof is complete. Otherwise, $x \notin D_\Delta$ and (2.28) holds. In that case, given that $f_2(x) \neq 0$, it follows from Proposition 2.53 that there exists $\varepsilon' > 0$ such that f_2 does not vanish in $(x - \varepsilon', x] \cap [a,b]$. Hence, (2.40) is valid for all $y \in (x-\varepsilon', x] \cap [a,b]$. As a consequence, we obtain that

$$\lim_{y \to x^-} \frac{(f_1/f_2)(y) - (f_1/f_2)(x)}{\Delta(x,y)}$$

exists and equals

$$\frac{(f_1)'_\Delta(x)f_2(x) - f_1(x)(f_2)'_\Delta(x)}{f_2(x)f_2(x^-)}.$$

However, Proposition 2.53 ensures that

$$\lim_{y \to x^-} \frac{(f_1/f_2)(y) - (f_1/f_2)(x)}{\Delta(x,y)} = \frac{(f_1)'_\Delta(x)f_2(x) - f_1(x)(f_2)'_\Delta(x)}{(f_2(x))^2}. \tag{2.43}$$

Therefore, f_1/f_2 is Δ–differentiable at x and (2.32) holds.

Finally, assume that $x \notin D_\Delta$ and (2.28) holds but (2.29) does not. Then, repeating the same arguments, we have that (2.43) holds and $\Delta_x(\cdot) = 0$ on some $[x, x+\delta]$, $\delta > 0$. Then, Remark 2.50 ensures that $(f_1/f_2)'_\Delta(x)$ exists and so (2.32) holds. □

Remark 2.55. Observe that all the formulas in Proposition 2.54 yield the well–known formulas for the usual derivative when we consider the corresponding displacement, namely, $\Delta(x, y) = y - x$; $x, y \in \mathbb{R}$.

Proposition 2.56. *Let* $[a, b] \subset \mathbb{R}$ *and* $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Let* f *be a real–valued function defined on a neighborhood of* $x \in [a, b] \setminus D_\Delta$ *denoted by* U_x*, and* h *be another function defined on a neighborhood of* $f(x)$*, denoted by* $V_{f(x)}$.

1. *If* $h'(f(x))$ *and* $f'_\Delta(x)$ *exist, then so does* $(h \circ f)'_\Delta(x)$ *and*

$$(h \circ f)'_\Delta(x) = h'(f(x))f'_\Delta(x). \tag{2.44}$$

2. *If* $h'_\Delta(f(x))$, $f'_\Delta(x)$ *and* $(\Delta_{f(x)})'(f(x))$ *exist and*

$$\Delta(f(x), y) \neq 0, \quad \textit{for } y \in [a, b],\ y \neq f(x), \tag{2.45}$$

then $(h \circ f)'_\Delta(x)$ *also exists and*

$$(h \circ f)'_\Delta(x) = h'_\Delta(f(x))f'_\Delta(x)(\Delta_{f(x)})'(f(x)). \tag{2.46}$$

Proof. First, in order to prove (2.44), define $Q : V_{f(x)} \to \mathbb{R}$ as

$$Q(s) = \begin{cases} \dfrac{h(s) - h(f(x))}{s - f(x)}, & \text{if } s \neq f(x), \\ h'(f(x)), & \text{if } s = f(x). \end{cases}$$

Observe that, by construction, $\lim_{s \to f(x)} Q(s) = h'(f(x))$. Furthermore, we have that, for each $y \in U_x$ such that $f(y) \in V_{f(x)}$ and $\Delta(x, y) \neq 0$,

$$\frac{h(f(y)) - h(f(x))}{\Delta(x, y)} = Q(f(y))\frac{f(y) - f(x)}{\Delta(x, y)}. \tag{2.47}$$

On the other hand, since $f'_\Delta(x)$ exists, we have that (2.28) and/or (2.29) hold. Assume that (2.29) holds. In that case, Proposition 2.53 ensures that f is continuous from the right at x. Therefore, using (2.47) we have that

$$\begin{aligned} \lim_{y \to x^+} \frac{h(f(y)) - h(f(x))}{\Delta(x, y)} &= \lim_{y \to x^+} \left(Q(f(y))\frac{f(y) - f(x)}{\Delta(x, y)} \right) \\ &= \lim_{s \to f(x)} Q(s) \lim_{y \to x^+} \frac{f(y) - f(x)}{\Delta(x, y)} = h'(f(x))f'_\Delta(x). \end{aligned}$$

This finishes the proof of (2.44) if $\Delta_x(\cdot) = 0$ on some $[x - \delta, x]$, $\delta > 0$. Otherwise, we have that (2.28) holds. In that case, Proposition 2.53 ensures that f is continuous from the left at x, and so (2.47) yields

$$\begin{aligned} \lim_{y \to x^-} \frac{h(f(y)) - h(f(x))}{\Delta(x, y)} &= \lim_{y \to x^-} \left(Q(f(y))\frac{f(y) - f(x)}{\Delta(x, y)} \right) \\ &= \lim_{s \to f(x)} Q(s) \lim_{y \to x^-} \frac{f(y) - f(x)}{\Delta(x, y)} = h'(f(x))f'_\Delta(x). \end{aligned}$$

Note that this is enough to obtain (2.44) even in the remain case, i.e. when (2.28) holds and (2.29) does not.

Now, for (2.46), define $\widehat{Q} : V_{f(x)} \to \mathbb{R}$ as

$$\widehat{Q}(s) = \begin{cases} \dfrac{h(s) - h(f(x))}{\Delta(f(x), s)} \dfrac{\Delta(f(x), s)}{s - f(x)}, & \text{if } s \neq f(x), \\ h'_{\Delta}(f(x))(\Delta_{f(x)})'(f(x)), & \text{if } s = f(x). \end{cases}$$

Observe that (2.45) ensures that $\widehat{Q}$ is well–defined. Furthermore, hypothesis (H1) and the existence of $h'_{\Delta}(f(x))$ and $(\Delta_{f(x)})'(f(x))$ are enough to see that

$$\lim_{s \to f(x)} \widehat{Q}(s) = h'_{\Delta}(f(x))(\Delta_{f(x)})'(f(x)).$$

Moreover, for any $y \in U_x$ such that $f(y) \in V_{f(x)}$ and $\Delta(x, y) \neq 0$, we have that

$$\frac{h(f(y)) - h(f(x))}{\Delta(x, y)} = \widehat{Q}(f(y)) \frac{f(y) - f(x)}{\Delta(x, y)}. \tag{2.48}$$

Once again, since $f'_{\Delta}(x)$ exists, we have that (2.28) and/or (2.29) hold. Assume that (2.29) holds. In that case, Proposition 2.53 ensures that f is continuous from the right at x, and so, using (2.48) we have that

$$\begin{aligned} \lim_{y \to x^+} \frac{h(f(y)) - h(f(x))}{\Delta(x, y)} &= \lim_{y \to x^+} \left(\widehat{Q}(f(y)) \frac{f(y) - f(x)}{\Delta(x, y)} \right) \\ &= \lim_{s \to f(x)} \widehat{Q}(s) \lim_{y \to x^+} \frac{f(y) - f(x)}{\Delta(x, y)} \\ &= h'_{\Delta}(f(x))(\Delta_{f(x)})'(f(x)) f'_{\Delta}(x), \end{aligned}$$

which finishes the proof if $\Delta_x(\cdot) = 0$ on some $[x - \delta, x]$, $\delta > 0$. Otherwise, we have that (2.28) holds and, repeating the arguments in the previous case, we have that

$$\begin{aligned} \lim_{y \to x^-} \frac{h(f(y)) - h(f(x))}{\Delta(x, y)} &= \lim_{y \to x^-} \left(\widehat{Q}(f(y)) \frac{f(y) - f(x)}{\Delta(x, y)} \right) \\ &= \lim_{s \to f(x)} \widehat{Q}(s) \lim_{y \to x^-} \frac{f(y) - f(x)}{\Delta(x, y)} \\ &= h'_{\Delta}(f(x))(\Delta_{f(x)})'(f(x)) f'_{\Delta}(x). \end{aligned}$$

Again, observe that this enough to obtain (2.46) even in the case that (2.28) holds and (2.29) does not, which finishes the proof. □

Proposition 2.57. *Let* $[a, b] \subset \mathbb{R}$ *and* $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy hypotheses* (H1)–(H3). *Let* $f : [a, b] \to \mathbb{R}$ *be nondecreasing in a neighborhood of* $x \in [a, b]$. *If* f *is* Δ*–differentiable at* x, *then* $f'_{\Delta}(x) \geq 0$.

Proof. Let us denote by J the neighborhood of x where f is nondecreasing. Since $f'_\Delta(x)$ exists, we have that $x \notin C_\Delta$ so (2.29) and/or (2.28) hold.

First, assume that (2.29) holds. Then, since f is nondecreasing in J, $f(y) - f(x) \geq 0$ for all $y \in J$ such that $y > x$, and so

$$\frac{f(y) - f(x)}{\Delta(x, y)} \geq 0, \quad \text{for all } y \in J,\ y > x.$$

Therefore, by taking the limit from the right, we have that

$$\lim_{y \to x^+} \frac{f(y) - f(x)}{\Delta(x, y)} \geq 0.$$

If $x \in D_\Delta$ or $\Delta_x(\cdot) = 0$ on some $[x - \delta, x]$, $\delta > 0$, then this limit is equal to $f'_\Delta(x)$ and we are done.

On the other hand, assume that (2.28) holds. We have that $f(y) - f(x) \leq 0$ for all $y \in J$, $y < x$. Thus, it follows that

$$\frac{f(y) - f(x)}{\Delta(x, y)} \geq 0, \quad \text{for all } y \in J,\ y < x,$$

which yields

$$\lim_{y \to x^-} \frac{f(y) - f(x)}{\Delta(x, y)} \geq 0.$$

Note that this ensures that $f'_\Delta(x) \geq 0$ for all the possible cases left. □

2.2.3 The Fundamental Theorem of Calculus

We now turn our focus into making explicit the relationship between the Δ–derivative of a function and its integral with respect to the Δ–measure. In particular, our first goal now is to show that, for $f \in \mathcal{L}^1_\mu([a, b], \mathbb{R})$ and $F(x) = \int_{[a,x)} f(s)\,\mathrm{d}\,\mu$, the equality $F'_\Delta(x) = f(x)$ holds for μ–a.a. $x \in [a, b]$.

In order to do so, we will need to guarantee the differentiability of monotonic functions. For that matter, we will use the following two results that can be found in [54], see Lemmas 4.2 and 4.3.

Proposition 2.58. *Let $g : [a, b] \to \mathbb{R}$ be a nondecreasing and left–continuous function, $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ and S be a nonempty subset of $\{1, 2, \dots, n\}$. If $f : [a, b] \to \mathbb{R}$ is such that $f(a) \leq f(b)$ and*

$$\frac{f(x_k) - f(x_{k-1})}{g(x_k) - g(x_{k-1})} < -\alpha \quad \textit{for each } k \in S,$$

then

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| > |f(b) - f(a)| + \alpha L$$

where $L = \sum_{k \in S} (g(x_k) - g(x_{k-1}))$. The same result is true if $f(a) \geq f(b)$ and

$$\frac{f(x_k) - f(x_{k-1})}{g(x_k) - g(x_{k-1})} > \alpha \quad \textit{for each } k \in S.$$

The next result is a generalization of Botsko's Covering Lemma.

Proposition 2.59. *Let $g : [a, b] \to \mathbb{R}$ be a nondecreasing left–continuous function, $\varepsilon > 0$ and $H \subset (a, b)$ be such that $\mu_g^*(H)$. Then:*

1. *If $\mathcal{I}$ is any collection of open subintervals of $[a, b]$ that covers H, then there exists a finite disjoint collection of intervals in $\mathcal{I}$, $\{I_1, I_2, \dots, I_N\}$, such that*
$$\sum_{k=1}^{N} \mu_g(I_k) > \frac{\varepsilon}{3}.$$
2. *If P is a finite subset of $[a, b] \setminus D_g$ and $\mathcal{I}$ is any collection of open subintervals of $[a, b]$ that covers $H \setminus P$, then there exists a finite disjoint collection of intervals in $\mathcal{I}$, $\{I_1, I_2, \dots, I_N\}$, such that*
$$\sum_{k=1}^{N} \mu_g(I_k) > \frac{\varepsilon}{4}.$$

We now have all the tools necessary to prove that monotonic functions are Δ–differentiable except in a null–measure set with respect to the Δ–measure. We will only prove this result for nondecreasing functions as the other case follows from this one.

Proposition 2.60. *Let $[a, b] \subset \mathbb{R}$ and $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ satisfy hypotheses* (H1)–(H3). *If $f : [a, b] \to \mathbb{R}$ is a nondecreasing function, then there exists $N \subset [a, b]$ such that $\mu(N) = 0$ and*
$$f'_\Delta(x) \text{ exists for all } x \in [a, b) \setminus N.$$

Proof. We divide the proof in three parts.

Part 1: reducing the problem.
First of all, note that f is regulated since it is nondecreasing. Hence, it follows from Remark 2.51 that $f'_\Delta(x)$ exists for all $x \in [a, b) \cap D_\Delta$. Moreover, since either $a \in D_\Delta$ or $\mu(\{a\}) = 0$, it is enough to show that $f'_\Delta(x)$ exists for all $x \in (a, b) \setminus (D_\Delta \cup O_\Delta)$ according to Remark 2.48.

Let $x \in (a, b) \setminus (D_\Delta \cup O_\Delta)$. Then we have that $\Delta_x(y) \neq 0$ for every $y \neq x$ (see Remark 2.48). Thus we can define the Dini upper and lower Δ–derivatives as
$$\overline{f'_\Delta}(x) := \limsup_{y \to x} \frac{f(y) - f(x)}{\Delta(x, y)}, \quad \underline{f'_\Delta}(x) := \liminf_{y \to x} \frac{f(y) - f(x)}{\Delta(x, y)}.$$

Furthermore, since f is nondecreasing, it has a countable number of discontinuity points, so
$$\mu(\{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : f \text{ is discontinuous at } x\}) = 0.$$

Thus, it is enough to show that the sets
$$F := \{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : f \text{ continuous at } x,\ \overline{f'_\Delta}(x) > \underline{f^\Delta}(x)\},$$
$$E := \{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : \overline{f'_\Delta}(x) = +\infty\},$$

both have Δ–measure zero (and therefore μ–measure zero).

Part 2: F has Δ–measure zero.
Fix $z \in (a,b) \setminus (D_\Delta \cup O_\Delta)$ and define the Dini upper and lower Δ_z–derivatives as

$$\overline{f'_{\Delta_z}}(x) := \limsup_{y\to x} \frac{f(y)-f(x)}{\Delta_z(y)-\Delta_z(x)}, \quad \underline{f'_{\Delta_z}}(x) := \liminf_{y\to x} \frac{f(y)-f(x)}{\Delta_z(y)-\Delta_z(x)},$$

respectively. Note that the bound functions in (H3) yield that

$$\begin{aligned}\overline{f'_{\Delta}}(x) &= \limsup_{y\to x} \frac{f(y)-f(x)}{\Delta_x(y)-\Delta_x(x)} = \limsup_{y\to x} \frac{f(y)-f(x)}{\Delta_x(y)-\Delta_x(x)}\frac{\Delta_z(y)-\Delta_z(x)}{\Delta_z(y)-\Delta_z(x)} \\ &= \overline{f'_{\Delta_z}}(x) \limsup_{y\to x} \frac{\Delta_x(y)-\Delta_x(x)}{\Delta_z(y)-\Delta_z(x)} \le \overline{f'_{\Delta_z}}(x)\gamma(z,x),\end{aligned}$$

and, analogously,

$$\begin{aligned}\underline{f'_{\Delta}}(x) &= \liminf_{y\to x} \frac{f(y)-f(x)}{\Delta_x(y)-\Delta_x(x)} = \liminf_{y\to x} \frac{f(y)-f(x)}{\Delta_x(y)-\Delta_x(x)}\frac{\Delta_z(y)-\Delta_z(x)}{\Delta_z(y)-\Delta_z(x)} \\ &= \underline{f'_{\Delta_z}}(x) \liminf_{y\to x} \frac{\Delta_x(y)-\Delta_x(x)}{\Delta_z(y)-\Delta_z(x)} \ge \underline{f'_{\Delta_z}}(x)\frac{1}{\gamma(z,x)}.\end{aligned}$$

Hence, F is a subset of

$$F_z := \left\{(a,b)\setminus(D_\Delta\cup O_\Delta) : f \text{ continuous at } x,\ \overline{f'_{\Delta_z}}(x)\gamma(z,x)\gamma(x,z) > \underline{f'_{\Delta_z}}(x)\right\}.$$

Now, since $\gamma : [a,b]\times[a,b] \to [1,+\infty)$, it is clear that $F_z \subset \bigcup_{n\in\mathbb{N}} F_n$ with

$$F_n := \left\{(a,b)\setminus(D_\Delta\cup O_\Delta) : f \text{ continuous at } x,\ \overline{f'_{\Delta_z}}(x)n > \underline{f'_{\Delta_z}}(x)\right\},$$

so it is suffices to show that $\mu_z(F_n) = 0$ for all $n \in \mathbb{N}$.

By contradiction, assume that there exists $n_0 \in \mathbb{N}$ such that $\mu_z(F_{n_0}) > 0$. In that case, we rewrite F_{n_0} as the countable union of sets $F_{n_0,r,s}$ with $r,s \in \mathbb{Q}$, $r > s > 0$, given by

$$F_{n_0,r,s} := \left\{x \in F_{n_0} : \overline{f'_{\Delta_z}}(x)n_0 > r > s > \underline{f'_{\Delta_z}}(x)\right\}.$$

Thus there exist $r_0, s_0 \in \mathbb{Q}$, $r_0 > s_0 > 0$, and $\varepsilon > 0$ such that $\mu_z(F_{n_0,r_0,s_0}) = \varepsilon$. Now take

$$\alpha = \frac{r_0-s_0}{2n_0}, \quad \beta = \frac{r_0+s_0}{2n_0}$$

and define $h(x) = f(x) - \beta\Delta_z(x)$ and

$$H := \left\{x \in F_{n_0} : \overline{h'_{\Delta_z}}(x) > \alpha,\ \underline{h'_{\Delta_z}}(x) < -\alpha\right\}.$$

Note that $F_{n_0,r_0,s_0} = H$. Moreover, h is of bounded variation as it is the difference of two nondecreasing functions. Hence, the set

$$V(h) := \left\{ \sum_P |h(x_k) - h(x_{k-1})| : P \text{ is a partition of } [a,b],\ P \cap D_\Delta \subset \{a,b\} \right\}$$

is bounded from above. Let $T := \sup V(h)$. Since $\alpha, \varepsilon > 0$, there exists a partition of $[a,b]$, $P = \{x_0, x_1, \dots, x_{n-1}\}$, such that $x_k \notin D_\Delta$ for any $k \in \{1, 2, \dots, n-1\}$ and

$$\sum_{k=1}^{n} |h(x_k) - h(x_{k-1})| > T - \frac{\alpha\varepsilon}{4}.$$

For a given $x \in H \setminus P$, we have that $x \in (x_{k-1}, x_k)$ for some $k \in \{1, 2, \dots, n\}$. Note that $x \notin D_\Delta$, so both Δ_z and h are continuous at x (see Remark 2.45). Moreover, since $\overline{h'_{\Delta_z}}(x) > \alpha$ and $\underline{h'_{\Delta_z}}(x) < -\alpha$, we can choose $a_x, b_x \in (x_{k-1}, x_k) \setminus D_\Delta$ such that $a_x < x < b_x$ and

$$\begin{aligned} \frac{h(b_x) - h(a_x)}{\Delta_z(b_x) - \Delta_z(a_x)} &< -\alpha \quad \text{if} \quad h(x_{k-1}) \geq h(x_k), \\ \frac{h(b_x) - h(a_x)}{\Delta_z(b_x) - \Delta_z(a_x)} &> \alpha \quad \text{if} \quad h(x_{k-1}) < h(x_k). \end{aligned}$$

Note that $\mu_z(a_x, b_x) = \Delta_z(b_x) - \Delta_z(a_x)$. By doing this, we obtain a collection of open subintervals of (a,b), $\mathcal{I} = \{(a_x, b_x) : x \in H \setminus \{x_1, x_2, \dots, x_{n-1}\}\}$, that covers the sets $H \setminus \{x_1, x_2, \dots, x_{n-1}\}$ and $\{x_1, x_2, \dots, x_{n-1}\} \cap D_\Delta = \emptyset$, which ensures that $\{x_1, x_2, \dots, x_{n-1}\} \cap D_{\Delta_z} = \emptyset$ (see Remark 2.45). Then, Proposition 2.59 ensures the existence of a finite disjoint subcollection $\{I_1, I_2, \dots, I_N\}$ of $\mathcal{I}$ such that

$$\sum_{k=1}^{N} \mu_z(I_k) > \frac{\varepsilon}{4}.$$

Now let $Q = \{y_0, y_1, \dots, y_q\}$ be the partition of $[a,b]$ determined by the points of P and the endpoints of the intervals $I_1, I_2, \dots, I_N$. For each $[x_{k-1}, x_k]$ containing at least one of the intervals in $\{I_1, I_2, \dots, I_N\}$, Proposition 2.58 yields that

$$\sum_{[y_{i-1}, y_i] \subset [x_{k-1}, x_k]} |h(y_i) - h(y_{i-1})| > |h(x_k) - h(x_{k-1})| + \alpha L_k,$$

where the summation is taken over the closed intervals determined by Q contained in $[x_{k-1}, x_k]$ and L_k is the sum of the Δ_z–measures of those intervals $I_1, I_2, \dots, I_N$ contained in $[x_{k-1}, x_k]$. By taking the previous inequality and summing over k, we obtain

$$\sum_{k=1}^{q} |h(y_k) - h(y_{k-1})| > \sum_{k=1}^{n} |h(y_k) - h(y_{k-1})| + \alpha \sum_{k=1}^{N} \mu_z(I_k) > T,$$

which contradicts the definition of T.

Part 3: E has Δ–measure zero.
Fix $z \in [a,b]$. Then for all $x \in (a,b) \setminus (D_\Delta \cup O_\Delta)$ we have that $\overline{f'_\Delta}(x) \leq \overline{f'_{\Delta_z}}(x)\gamma(z,x)$. Hence $E \subset E_z$ with

$$E_z := \left\{ x \in (a,b) \setminus (D_\Delta \cup O_\Delta) : \overline{f'_{\Delta_z}}(x) = +\infty \right\}.$$

Thus, it is enough to show that $\mu_z(E_z) = 0$. Suppose this is not the case. Then there is $\varepsilon > 0$ such that $\mu_z(E_z) = \varepsilon$. Let $M > 0$ be such that $M > 3(f(b) - f(a))/\varepsilon$. If $x \in E_z$ then $\overline{f'_{\Delta_z}}(x) > M$ and there exist $a_x, b_x \in (a,b) \setminus D_\Delta$ such that $a_x < x < b_x$ and

$$\frac{f(b_x) - f(a_x)}{\Delta_z(b_x) - \Delta_z(a_x)} > M.$$

Thus, $\{(a_x, b_x) : x \in E_z\}$ covers E_z. Proposition 2.59 guarantees the existence of a finite disjoint subcollection $\{I_1, I_2, \ldots, I_N\}$ such that

$$\sum_{k=1}^{N} \mu_z(I_k) > \frac{\varepsilon}{3}.$$

Let $I_k = (a_k, b_k)$ for each k. Then $\mu_z(I_k) = \Delta_z(b_k) - \Delta_z(a_k)$ as each $I_k \subset (a,b) \setminus (D_\Delta \cup O_\Delta)$. Now, since f is nondecreasing we have

$$f(b) - f(a) \geq \sum_{k=1}^{N} (f(b_k) - f(a_k)) > M \sum_{k=1}^{N} (\Delta_z(b_k) - \Delta_z(a_k)) > f(b) - f(a),$$

which is a contradiction. □

Finally, a key result for the proof of the Fundamental Theorem of Calculus is Fubini's Theorem on almost everywhere differentiation of series for Δ–derivatives. We now state and prove such result in the context of Δ–derivatives following the ideas in [80] but using Proposition 2.60 instead of the classical Lebesgue Differentiation Theorem.

Proposition 2.61. *Let $[a,b] \subset \mathbb{R}$ and $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ satisfy* (H1)–(H3). *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of real–valued nondecreasing functions defined on $[a,b]$. If the series*

$$f_0(x) = \sum_{n=1}^{\infty} f_n(x) \quad \textit{converges for all } x \in [a,b],$$

then

$$(f_0)'_\Delta(x) = \sum_{n=1}^{\infty} (f_n)'_\Delta(x) \quad \textit{for } \mu\textit{–a.a. } x \in [a,b].$$

Proof. Without loss of generality, we can assume that $f_n \geq 0$ for all $n \in \mathbb{N}$, as otherwise it would be enough to consider $\widetilde{f_n}(x) = f_n(x) - f_n(a)$, $n \in \mathbb{N}$.

First, note that f_n, $n = 0, 1, 2, \dots$, are nondecreasing functions. Hence, there exist E_n, $n = 0, 1, 2, \dots$, such that f_n is Δ–differentiable in $[a, b] \setminus E_n$, and $\mu(E_n) = 0$ for all $n = 0, 1, 2, \dots$ Therefore, if we take $E = \bigcup_{n=0}^{\infty} E_n$ we have that

$$f_n \text{ is } \Delta\text{–differentiable in } [a, b] \setminus E, \quad \mu(E) = 0, \quad n = 0, 1, 2, \dots.$$

For each $k \in \mathbb{N}$, define

$$S_k(x) = \sum_{n=1}^{k} f_n(x), \quad r_k(x) = f_0(x) - S_k(x), \quad x \in [a, b].$$

It follows from Proposition 2.54 that S_k and r_k, $k \in \mathbb{N}$, are Δ–differentiable in $[a, b] \setminus E$ and

$$(S_k)'_\Delta(x) = \sum_{n=1}^{k} (f_n)'_\Delta(x), \quad (r_k)'_\Delta(x) = (f_0)'_\Delta(x) - (S_k)'_\Delta(x), \quad x \in [a, b] \setminus E, \quad k \in \mathbb{N}.$$

Now, r_k, $k \in \mathbb{N}$, and f_n, $n \in \mathbb{N}$, are nondecreasing in $[a, b] \setminus E$, so Proposition 2.57 ensures that $(r_k)'_\Delta(x), (f_n)'_\Delta(x) \geq 0$ for all $x \in [a, b] \setminus E$, $k, n \in \mathbb{N}$. Hence, for any $k \in \mathbb{N}$ we have that

$$(S_k)'_\Delta(x) = \sum_{n=1}^{k} (f_n)'_\Delta(x) \leq \sum_{n=1}^{k+1} (f_n)'_\Delta(x) = (S_{k+1})'_\Delta(x) \leq (f_0)'_\Delta(x), \quad x \in [a, b] \setminus E.$$

Thus,

$$\sum_{n=1}^{\infty} (f_n)'_\Delta(x) = \lim_{k \to \infty} (S_k)'_\Delta(x) \leq (f_0)'_\Delta(x) < +\infty, \quad x \in [a, b] \setminus E.$$

Therefore, it is enough to show that

$$\lim_{k \to \infty} (S_k)'_\Delta(x) = (f_0)'_\Delta(x), \quad \mu\text{–a.a. } x \in [a, b].$$

Since we know that $(S_k)'_\Delta(x) \leq (f_0)'_\Delta(x)$ for all $x \in [a, b] \setminus E$, $k \in \mathbb{N}$, it is enough to find a subsequence that converges to $(f_0)'_\Delta(x)$ for μ–a.a. $x \in [a, b]$.

We know that $S_k(b)$ converges to $f_0(b)$, which is finite since f_0 is nondecreasing. Hence, we can find $k_i \in \mathbb{N}$ such that $1 \leq k_i \leq k_{i+1}$, $i \in \mathbb{N}$, and

$$\sum_{i=1}^{\infty} (f(b) - S_{k_i}(b)) < +\infty.$$

In that case, we have that

$$0 \leq f(x) - S_{k_i}(x) = r_{k_i}(x) \leq r_{k_i}(b) = f(b) - S_{k_i}(x), \quad x \in [a, b], \quad i \in \mathbb{N}$$

from which we get

$$\sum_{i=1}^{\infty} (f(x) - S_{k_i}(x)) \leq \sum_{i=1}^{\infty} (f(b) - S_{k_i}(b)) < +\infty, \quad x \in [a, b].$$

Now, the sequence $r_{k_i} = f - S_{k_i}$ is a sequence of nondecreasing functions such that $\sum_{i=1}^{\infty} r_{k_i}(x)$ converges for all $x \in [a,b]$, so we can repeat the arguments at the beginning of the proof to obtain that

$$\sum_{i=1}^{\infty} (r_{k_i})'_{\Delta}(x) < +\infty, \quad \mu\text{–a.a. } x \in [a,b],$$

from which we get that

$$0 = \lim_{i\to\infty} (r_{k_i})'_{\Delta}(x) = \lim_{i\to\infty} f'_{\Delta}(x) - (S_{k_i})'_{\Delta}(x), \quad \mu\text{–a.a. } x \in [a,b],$$

which concludes the proof. □

We now have all the necessary tools to state and prove the Fundamental Theorem of Calculus for Δ–derivatives. For these results we follow the ideas in [54, 80].

Theorem 2.62 (Fundamental Theorem of Calculus for the Δ–derivative)**.** *Let* $[a,b] \subset \mathbb{R}$ *and* $\Delta : [a,b]\times[a,b] \to \mathbb{R}$ *satisfy hypotheses* (H1)–(H3). *Let* $f \in \mathcal{L}^1_{\mu}([a,b),\mathbb{R})$ *and* $F : [a,b] \to \mathbb{R}$ *be given by*

$$F(x) = \int_{[a,x)} f(s)\,\mathrm{d}\,\mu.$$

Then $F'_{\Delta}(x) = f(x)$ *for* μ*–a.a.* $x \in [a,b)$.

Proof. We consider two cases separately: when Δ is a Stieltjes displacement and the general case.

Case 1: Stieltjes displacements.
Let Δ be a Stieltjes displacement and $g : [a,b] \to \mathbb{R}$ be the nondecreasing left–continuous that defines it, see Example 2.9. As pointed out in Remark 2.41, we have that $\mu = \mu_g$. We study different cases of functions until we arrive to the general case.

Case 1.1: Functions that are the restriction to $[a,b)$ *of a function in* $\mathcal{S}(\widetilde{\mathcal{C}})$.
By definition, the functions in $\mathcal{S}(\widetilde{\mathcal{C}})$ are linear combinations of functions of the form $\chi_{[\alpha,\beta)} : \mathbb{R} \to \mathbb{R}$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, see (1.13). Thus, it is clear from the linearity of the Δ–derivative, Proposition 2.54, that it is enough to show that the result holds for one of those functions. Furthermore, without loss of generality, we can assume that $a \le \alpha < \beta \le b$.

Assume that $f(x) = \chi_{[\alpha,\beta)}(x)$ with $a \le \alpha < \beta \le b$. In that case, we have that $F : [a,b] \to \mathbb{R}$ is given by

$$F(x) = \begin{cases} 0 & \text{if } a \le x \le \alpha, \\ g(x) - g(\alpha) & \text{if } \alpha < x \le \beta, \\ g(\beta) - g(\alpha) & \text{if } \beta < x \le b. \end{cases}$$

Observe that F is nondecreasing. Thus, Proposition 2.60 guarantees the existence of a set $N \subset [a,b)$ such that $\mu_g(N) = 0$ and $F'_{\Delta}(x)$ exists for all $x \in [a,b) \setminus N$. Without loss of generality, we can assume that $N_\Delta = N_g \subset N$, which guarantees that the derivative in $[a,b)\setminus N$ can be computed as the right–hand side limit.

Let $x \in [a,b)\setminus N$. If $x \notin [\alpha,\beta]$, it follows that $F'_\Delta(x) = 0$ as the function is constant on a neighbourhood of x. Therefore, $F'_\Delta(x) = \chi_{(\alpha,\beta)}(x) = f(x)$. On the other hand, if $x \in (\alpha,\beta)$,

$$F'_\Delta(x) = \lim_{y\to x^+} \frac{F(y)-F(x)}{\Delta(x,y)} = \lim_{y\to x^+} \frac{g(y)-g(\alpha)-(g(x)-g(\alpha))}{g(y)-g(x)} = 1 = \chi_{(\alpha,\beta)}(x).$$

Thus, all that it is left to do is study what happens at $x=\alpha$ and $x=\beta$. In either of those cases, we have two posibilities: $x \in D_\Delta$ or $x \notin D_\Delta$. If $x \notin D_\Delta$, then its g–measure is equal to zero, and so we are done. Otherwise, $x \in D_\Delta$ and, in that case,

$$F'_\Delta(x) = \frac{F(x^+)-F(x)}{\Delta(x,x^+)} = \frac{g(x^+)-g(\alpha)-(g(x)-g(\alpha))}{g(x^+)-g(x)} = 1 = \chi_{(\alpha,\beta)}(x),$$

which finished the proof for this case.

Case 1.2: Simple functions in $[a,b)$.
Given that simple functions are linear combinations of functions of the form χ_E for some g–measurable set $E \subset [a,b)$, it is enough to show that the result holds for one of those functions.

Let $E \subset [a,b)$ be a g–measurable set. Given that E is bounded, it follows that $\mu_g(E) < +\infty$. Therefore, Proposition 1.51 guarantees that there exists $\{s_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}(\widetilde{\mathcal{C}})$ converging to $\chi_E(x)$ for g–almost all $x \in \mathbb{R}$ and satisfying (1.14). Define

$$F_n(x) = \int_{[a,x)} s_n(r)\,\mathrm{d}\,\mu_g(r), \quad x \in [a,b].$$

Observe that it follows from (1.14) that, for each $x \in [a,b]$,

$$0 \le \lim_{n\to\infty} \int_{[a,x)} |s_n(r)-\chi_E(r)|\,\mathrm{d}\,\mu_g(r) \le \lim_{n\to\infty} \int_{\mathbb{R}} |s_n(r)-\chi_E(r)|\,\mathrm{d}\,\mu_g(r) = 0.$$

Therefore, we have that

$$F(x) = \lim_{n\to\infty} F_n(x) = F_1(x) + \sum_{k=2}^{\infty} (F_k(x)-F_{k-1}(x)), \quad x \in [a,b].$$

Since each addend is a nondecreasing function, we can apply Case 1.1 and Proposition 2.61 to deduce that, for μ_g–a.a. $x \in [a,b)$, we have that

$$\begin{aligned} F'_\Delta(x) &= (F_1)'_\Delta(x) + \sum_{k=2}^{\infty}((F_k)'_\Delta(x)-(F_{k-1})'_\Delta(x)) \\ &= s_1(x) + \sum_{k=2}^{\infty}(s_k(x)-s_{k-1}(x)) = \lim_{n\to\infty} s_n(x) = \chi_E(x). \end{aligned} \tag{2.49}$$

Case 1.3: Nonnegative integrable functions in $[a,b)$.
Let $f \in \mathcal{L}^1_g([a,b),[0,+\infty))$. In that case, Proposition 1.15 guarantees the existence of a sequence of nonnegative simple functions, $\{f_n\}_{n\in\mathbb{N}}$, such that $0 \le f_1 \le f_2 \le \cdots \le f$, and $f_n(x)$ converges to $f(x)$ for all $x \in [a,b)$. Define

$$F_n(x) = \int_{[a,x)} f_n(r)\,\mathrm{d}\,\mu_g(r), \quad x \in [a,b].$$

It follows from Lebesgue's Monotone Convergence Theorem (Theorem 1.18) that

$$F(x) = \lim_{n\to\infty} F_n(x) = F_1(x) + \sum_{k=2}^{\infty}(F_k(x) - F_{k-1}(x)), \quad x \in [a,b].$$

Hence, repeating the arguments for (2.49) using Case 1.2 instead of Case 1.1, we obtain that $F'_{\Delta}(x) = f(x)$ for μ_g–a.a. $x \in [a,b)$.

Case 1.4: General case.
For a general function $f \in \mathcal{L}^1_{\Delta}([a,b),\mathbb{R})$ we have that $f = f_1 - f_2$, where each of the functions is in the conditions of Case 1.3. Hence, by linearity of the Lebesgue–Stieltjes integral and the Δ–derivative, we have that $F'_{\Delta}(x) = f(x)$ for μ_g–a.a. $x \in [a,b)$.

Case 2: General displacement.
Without loss of generality, we assume that $f \ge 0$, as the general case can be reduced to the difference of two such functions. In that case, the function F is nondecreasing, so Proposition 2.60 ensures that there exists $A_0 \subset [a,b)$ such that $F'_{\Delta}(x)$ exists for all $x \in [a,b) \setminus A_0$.

Let $r : \mathbb{N} \to \mathbb{Q} \cap [a,b]$ be a bijection. For each $n \in \mathbb{N}$, denote $r_n = r(n)$ and define $H_n : [a,b] \to \mathbb{R}$ as

$$H_n(t) = \int_{[a,t)} f(s) h_{s,r_n}(s)\,\mathrm{d}\,\mu_{r_n}, \quad t \in [a,b]. \tag{2.50}$$

Note that, with this definition, we have that

$$\begin{aligned}\frac{F(y)-F(x)}{\Delta(x,y)} &= \frac{\displaystyle\int_{[a,y)} f(s)\,\mathrm{d}\,\mu - \int_{[a,x)} f(s)\,\mathrm{d}\,\mu}{\Delta(x,y)} \\ &= \frac{\displaystyle\int_{[a,y)} f(s)h_{s,r_n}(s)\,\mathrm{d}\,\mu_{r_n} - \int_{[a,x)} f(s)h_{s,r_n}(s)\,\mathrm{d}\,\mu_{r_n}}{\Delta(x,y)} \\ &= \frac{H_n(y)-H_n(x)}{\Delta(x,y)},\end{aligned} \tag{2.51}$$

for any $n \in \mathbb{N}$ and $x, y \in [a,b]$ such that $\Delta(x,y) \neq 0$.

Observe that Case 1 ensures that, for each $n \in \mathbb{N}$, there exists $A_n \subset [a,b)$ such that $\mu_{r_n}(A_n) = 0$, $(H_n)'_{\Delta_{r_n}}(x)$ exists for all $x \in [a,b) \setminus A_n$ and

$$(H_n)'_{\Delta_{r_n}}(x) = f(x)h_{x,r_n}(x), \quad x \in [a,b) \setminus A_n. \tag{2.52}$$

Consider the set $A=\bigcup_{n=0}^{\infty}A_n$. We claim that $\mu(A)=0$. Indeed, Proposition 2.42 guarantees that $\mu\ll\mu_{r_n}$, $n\in\mathbb{N}$, and so, $\mu(A_n)=0$ for all $n\in\mathbb{N}$. Now, since A is a countable union of sets of μ–measure zero, we have that $\mu(A)=0$. Therefore, it is enough to show that $F'_\Delta(x)=f(x)$ for all $x\in[a,b)\setminus A$ to conclude the proof of the result.

Let $x\in[a,b)\setminus A$. Given that $\mathbb{Q}\cap[a,b]$ is dense in $[a,b]$, there exists $\{z_k\}_{k\in\mathbb{N}}\subset\mathbb{Q}\cap[a,b]$ such that $z_k\to x$ as $k\to\infty$. For simplicity, for each $k\in\mathbb{N}$, we denote $\Delta_k(\cdot)=\Delta(z_k,\cdot)$, and by H_k the corresponding function in (2.50) for z_k, $k\in\mathbb{N}$. Moreover, for such x we know that $F'_\Delta(x)$ and $(H_k)'_{\Delta_k}(x)$, $k\in\mathbb{N}$, exist. In particular, this implies that $x\notin C_\Delta$. Hence, we have that (2.28) and/or (2.29) holds. This, together with (H3,i), yields that

$$\Delta_k(y)-\Delta_k(x)<0,\quad \text{for all } y\in[a,b] \text{ such that } y<x,\quad k\in\mathbb{N}, \tag{2.53}$$

and

$$\Delta_k(y)-\Delta_k(x)>0,\quad \text{for all } y\in[a,b] \text{ such that } y>x,\quad k\in\mathbb{N}. \tag{2.54}$$

First, we assume that (2.29) holds or $x\in D_\Delta$. In that case, hypothesis (H3, i) yields that for any $k\in\mathbb{N}$ and $y\in[a,b)$, $y>x$,

$$\frac{H_k(y)-H_k(x)}{\Delta_k(y)-\Delta_k(x)}\frac{1}{\gamma(x,z_k)}\le\frac{H_k(y)-H_k(x)}{\Delta(x,y)}\le\frac{H_k(y)-H_k(x)}{\Delta_k(y)-\Delta_k(x)}\gamma(z_k,x).$$

Thus, by letting $y\to x^+$, it follows from (2.51) and (2.52) that

$$f(x)h_{x,z_k}(x)\frac{1}{\gamma(x,z_k)}\le\lim_{y\to x^+}\frac{F(y)-F(x)}{\Delta(x,y)}\le(H_k)'_{\Delta_k}(x)\gamma(z_k,x),\quad k\in\mathbb{N}. \tag{2.55}$$

Note that it is at this point that we make use of condition (2.54), as it ensures that the limit

$$\lim_{y\to x^+}\frac{H_k(y)-H_k(x)}{\Delta_k(y)-\Delta_k(x)}$$

is well–defined, which then coincides with $(H_k)'_{\Delta_k}(x)$ as the derivative exists. Given that (2.55) holds for all $k\in\mathbb{N}$, by taking the limit as $k\to\infty$, and bearing in mind (H3, ii) and Proposition (2.22), we obtain that

$$\lim_{y\to x^+}\frac{F(y)-F(x)}{\Delta(x,y)}=f(x).$$

Now, given that Δ–derivative of F at x exists, we have that the previous limit equals $F'_\Delta(x)$, which concludes the proof for this case.

Now, if (2.28) holds and $x\notin D_\Delta$, we proceed analogously. In this case, (H3, i) yields that for any $k\in\mathbb{N}$ and $y\in[a,b)$, $y<x$,

$$\frac{H_k(y)-H_k(x)}{\Delta_k(y)-\Delta_k(x)}\gamma(x,z_k)\le\frac{H_k(y)-H_k(x)}{\Delta(x,y)}\le\frac{H_k(y)-H_k(x)}{\Delta_k(y)-\Delta_k(x)}\frac{1}{\gamma(z_k,x)}.$$

Now, letting $y\to x^-$, it follows from (2.53), (2.51) and (2.52) that

$$f(x)h_{x,z_k}(x)\frac{1}{\gamma(x,z_k)}\le\lim_{y\to x^-}\frac{F(y)-F(x)}{\Delta(x,y)}\le(H_k)'_{\Delta_k}(x)\gamma(z_k,x),\quad k\in\mathbb{N}.$$

Again, by taking the limit as $k \to \infty$, we obtain that

$$\lim_{y \to x^-} \frac{F(y) - F(x)}{\Delta(x, y)} = f(x),$$

which finishes the proof as the previous limit equals $F'_\Delta(x)$ since the derivative exists. □

Remark 2.63. The proof of Theorem 2.62 is a modification of the one for [61, Theorem 1] as it presents some limitations. For example, in the proof of [61, Theorem 1] the authors make use of the fact that the set $M_0(\Delta)$ is dense in the corresponding space of integrable functions. Nevertheless, it is not mentioned or proved that this is the case. Another limitation of the proof provided in [61] appears in the study of Case 1, where the authors make use of the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral. This is done over a function that depends on the point at which they are considering the derivative. The mentioned function is differentiable except in a set of measure zero with respect to the corresponding measure. However, given that the function depends on the point at which we compute the derivative, the corresponding set of measure zero depends on the mentioned point, which represents a problem as the union of these sets of measure zero over the interval $[a, b)$ may result in a set of positive measure. This turns the proof of that case invalid.

Note that Theorem 2.62 contains [54, Theorem 2.4] as a particular case. Furthermore, the proof presented above is an improvement on that of Theorem 2.4 in [54]. Indeed, in the mentioned paper, the authors make use of the fact that integrable functions can be regarded as the pointwise limit of a sequence of step functions whose discontinuity points do not belong to D_g. Nevertheless, such property is not proved to be true, but rather, the authors refer to [80], where it appears as an exercise for the reader. Here, we have avoided such property and, instead, we have proved some other approximation result (Proposition 1.51) to obtain the result.

Now, we move on to investigate when the reciprocal condition is satisfied. That is, when the Δ–integral of a Δ–derivative of a function yields the original function. To that end, we introduce the following concept that is somehow analogous to the absolute continuity in the usual sense.

Definition 2.64. *Let* $[a, b] \subset \mathbb{R}$ *and* $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *Let* $x \in [a, b]$ *and* $F : [a, b] \to \mathbb{R}$. *We shall say that* F *is* Δ–absolutely continuous on $[a, b]$ *if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for every open pairwise disjoint family of subintervals* $\{(a_n, b_n)\}_{n=1}^m$ *verifying*

$$\sum_{n=1}^{m} (\Delta_x(b_n) - \Delta_x(a_n)) < \delta$$

we have that

$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

A map $F : [a, b] \to \mathbb{R}^n$ *is* Δ*–absolutely continuous if each of its components is a* Δ*–absolutely continuous function. We denote by* $\mathcal{AC}_\Delta([a, b], \mathbb{R}^n)$ *the set of* Δ*–absolutely continuous functions on* $[a, b]$ *with values on* $\mathbb{R}^n$.

Remark 2.65. Note that this property does not depend on the point x chosen, which justifies the name. Indeed, let $x \in [a,b]$, $\varepsilon > 0$ and F be a Δ–absolutely continuous function on $[a,b]$. Then, there exists $\widetilde{\delta} > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ we have that

$$\sum_{n=1}^{m} (\Delta_x(b_n) - \Delta_x(a_n)) < \widetilde{\delta} \implies \sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

Now, hypothesis (H3) ensures that

$$\Delta_y(b_n) - \Delta_y(a_n) \leq \gamma(y,x)(\Delta_x(b_n) - \Delta_x(a_n)), \quad n = 1, 2, \ldots, m,$$

for all $y \in [a,b]$. Hence, taking $\delta = \gamma(y,x)\widetilde{\delta}$, the statement follows.

In the following, we study some properties that are common to Δ–absolutely continuous functions. The result that we are about to prove are known for the usual concept of absolute continuity and can be derived from this more general setting. First, we study the implications that Δ–absolute continuity has over the usual continuity.

Proposition 2.66. *Let $[a,b] \subset \mathbb{R}$, $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ be a map satisfying* (H1)–(H3) *and $F : [a,b] \to \mathbb{R}$ be a Δ–absolutely continuous function on $[a,b]$. Then F is left–continuous everywhere. Moreover, F is continuous where Δ is continuous.*

Proof. Fix $x \in (a,b]$ and $\varepsilon > 0$. Let $\delta > 0$ be given by the definition of Δ–absolute continuity of F. Since $\Delta_x(\cdot)$ is left–continuous at x, there exists $\widetilde{\delta} > 0$ such that if $0 < x - t < \widetilde{\delta}$ then,

$$\Delta_x(x) - \Delta_x(t) < \delta \implies |F(x) - F(t)| < \varepsilon,$$

so F is continuous from the left at x. The proof in the case Δ_x is right–continuous at $x \in [a,b)$ is analogous, and we omit it. □

Now, we look at the composition of functions. In particular, we look at the composition of a Lipschitz function with a Δ–absolutely continuous function. It is well known that when working in the usual setting, this composition yields an absolutely continuous function. This remains true in this context.

Proposition 2.67. *Let $[a,b] \subset \mathbb{R}$ and $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ be a map satisfying* (H1)–(H3). *Let $f_1 : [a,b] \to [c,d]$ be a Δ–absolutely continuous function on $[a,b]$ and let $f_2 : [c,d] \to \mathbb{R}$ satisfy a Lipschitz condition on $[c,d]$. Then $f_2 \circ f_1$ is Δ–absolutely continuous on $[a,b]$.*

Proof. Let $L > 0$ be a Lipschitz constant for f_2 on $[c,d]$ and fix $x \in [a,b]$. For each $\varepsilon > 0$ take $\delta > 0$ in Definition 2.64 with ε replaced by ε/L. Now, for an open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ such that

$$\sum_{n=1}^{m} (\Delta_x(b_n) - \Delta_x(a_n)) < \delta,$$

we have that

$$\sum_{n=1}^{m} |f_2(f_1(b_n)) - f_2(f_1(a_n))| \le L \sum_{n=1}^{m} |f_1(b_n) - f_1(a_n)| < \varepsilon,$$

that is, $f_2 \circ f_1$ is Δ_x–absolutely continuous. □

Further properties can be obtained for Δ–absolutely continuous functions. In particular, we can prove that such functions are always of bounded variation.

Proposition 2.68. *Let* $[a, b] \subset \mathbb{R}$, $\Delta : [a, b] \times [a, b] \to \mathbb{R}$ *be a map satisfying* (H1)–(H3) *and* $F : [a, b] \to \mathbb{R}$ *be a* Δ*–absolutely continuous function on* $[a, b]$. *Then* F *is of bounded variation.*

Proof. To prove this result we will use the following remark: if for any $[\alpha, \beta] \subset (a, b)$ there exists $c > 0$ such that the total variation of F on $[\alpha, \beta]$ is bounded from above by c, then F has bounded variation on $[a, b]$. Indeed, assume that for any $[\alpha, \beta] \subset (a, b)$ there exists $c > 0$ such that the total variation of F on $[\alpha, \beta]$ is bounded from above by c. Then for each $x \in (a, b)$,

$$|F(x)| \le \left| F(x) - F\left(\frac{a+b}{2}\right)\right| + \left|F\left(\frac{a+b}{2}\right)\right| \le c + \left|F\left(\frac{a+b}{2}\right)\right|.$$

Hence, $|F|$ is bounded on $[a, b]$. Let $K > 0$ be one of its bounds. For any partition $\{x_0, x_1, \dots, x_n\}$ of $[a, b]$, we have that

$$\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| = |F(x_1) - F(a)| + \sum_{k=2}^{n-1} |F(x_k) - F(x_{k-1})| + |F(b) - F(x_{n-1})|,$$

which is bounded from above by $4K + c$, and so our claim holds.

Now, to prove that F has bounded variation on $[a, b]$, fix $x \in [a, b]$ and take $\varepsilon = 1$ in the definition of Δ–absolute continuity. Then, there exists $\delta > 0$ such that for any family $\{(a_n, b_n)\}_{n=1}^{m}$ of pairwise disjoint open subintervals of $[a, b]$,

$$\sum_{n=1}^{m} (\Delta_x(b_n) - \Delta_x(a_n)) < \delta \implies \sum_{n=1}^{m} |F(b_n) - F(a_n)| < 1.$$

Consider a partition $\{y_0, y_1, \dots, y_n\}$ of $[\Delta_x(a), \Delta_x(b)]$ such that $0 < y_k - y_{k-1} < \delta$, $k = 1, 2, \dots, n$. Define $I_k = \Delta_x^{-1}([y_{k-1}, y_k))$, $k = 1, 2, \dots, n$. Since Δ_x is nondecreasing, the sets I_k are empty or they are intervals not necessarily open nor close. In any case,

$$[a, b] = \bigcup_{k=1}^{m} I_k,$$

and so it is enough to show that F has bounded variation on the closure of each I_k. We assume the nontrivial case, that is, $\overline{I_k} = [a_k, b_k]$, $a_k < b_k$. If $[\alpha, \beta] \subset (a_k, b_k)$ and $\{t_0, t_1, \dots, t_m\}$ is a partition of $[\alpha, \beta]$, then

$$\sum_{i=1}^{m} (\Delta_x(t_i) - \Delta_x(t_{i-1})) = \Delta_x(\beta) - \Delta_x(\alpha) \le y_k - y_{k-1} < \delta,$$

which implies that $\sum_{i=1}^{m} |F(x_i) - F(x_{i-1})| < 1$. Now, our previous claim implies that F has bounded variation on each $\overline{I}_k$, and therefore F has bounded variation on $[a,b]$. □

As a consequence of Propositions 2.66 and 2.68 we have that given a Δ–absolutely continuous function F, there exist two nondecreasing and left–continuous functions, F_1, F_2, such that $F = F_1 - F_2$. As commented on Remark 1.48, this defines a signed measure for F over $\mathcal{B}(\tau_u)$, μ_F, given by

$$\mu_F(E) = \mu_1(E) - \mu_2(E), \quad E \in \mathcal{B}([a,b],\tau_u). \tag{2.56}$$

Lemma 2.69. *Let $[a,b] \subset \mathbb{R}$, $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ be a map satisfying* (H1)–(H3)*, $F : [a,b] \to \mathbb{R}$ be a Δ–absolutely continuous function on $[a,b]$ and μ_F be the signed measure defined in* (2.56)*. Then $\mu_F \ll \mu$.*

Proof. Let $x \in [a,b]$ and $\varepsilon > 0$. Take $\delta > 0$ given by the definition of Δ–absolute continuity with ε replaced by $\varepsilon/2$. Fix an open set $V \subset (a,b)$ such that $\mu_x(V) < \delta$. Without loss of generality, we can assume that $V = \bigcup_{n\in\mathbb{N}}(a_n,b_n)$ for a pairwise disjoint family of open intervals. For each $n \in \mathbb{N}$, take $a'_n \in (a_n,b_n)$. Then, for each $m \in \mathbb{N}$ we have

$$\sum_{n=1}^{m}(\Delta_x(b_n) - \Delta_x(a'_n)) = \mu_x\left(\bigcup_{n=1}^{m}[a'_n,b_n)\right) \leq \mu_x(V) < \delta,$$

and so $\sum_{n=1}^{m} |F(b_n) - F(a'_n)| < \varepsilon/2$. By letting a'_n tend to a_n, we obtain

$$\sum_{n=1}^{m} |F(b_n) - F(a_n^+)| \leq \varepsilon/2, \quad \text{for each fixed } m \in \mathbb{N}.$$

Thus, if $\mu(V) < \delta$ we have that

$$|\mu_F(V)| = \left|\sum_{n=1}^{\infty} \mu_F(a_n,b_n)\right| \leq \sum_{n=1}^{\infty} |F(b_n) - F(a_n^+)| < \varepsilon.$$

Let $E \in \mathcal{B}([a,b],\tau_u)$ be such that $\mu_x(E) = 0$. By outer regularity, there exist open sets $V_n \subset [a,b]$, $n \in \mathbb{N}$, such that $E \subset V_n$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \mu_x(V_n) = \mu_x(E), \quad \lim_{n\to\infty} \mu_i(V_n) = \mu_i(E), \quad i = 1,2.$$

Now, by the first part of the proof, we know that $\mu_F(V_n) \xrightarrow{n\to\infty} \mu_F(E) = 0$ since $\mu_x(V_n)$ converges to $\mu_x(E) = 0$, so

$$\mu_F(E) = \mu_1(E) - \mu_2(E) = \lim_{n\to\infty} \mu_F(V_n) = 0.$$

Hence, $\mu_F \ll \mu_x$, and since $\mu_x \ll \mu$, the result follows. □

As we mentioned before, we aim to prove that, for a certain set of functions, we can recover a function as the integral of its derivative. In the usual setting, this is only true if and only if the function is absolutely continuous. For our setting, this will also be true with corresponding changes in concepts. We divide the proof of such result in the following more general lemma and the complementary theorem.

Lemma 2.70. *Let* $[a,b] \subset \mathbb{R}$, $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *be a map satisfying* (H1)–(H3) *and* $f \in \mathcal{L}^1_\mu([a,b),\mathbb{R})$. *Consider* $F : [a,b] \to \mathbb{R}$ *given by*

$$F(x) = \int_{[a,x)} f \, \mathrm{d}\,\mu.$$

Then F *is* Δ*–absolutely continuous on* $[a,b]$.

Proof. Note that it is enough to consider the case $f \geq 0$, as the general case can be expressed as a difference of two functions of this type.

Fix $\varepsilon > 0$ and $x \in [a,b]$. Hypothesis (H3, iii) implies that there exists $K > 0$ such that $|\gamma(t,x)| < K$ for all $t \in [a,b]$. Now, Remark 2.44 ensures that $f \in \mathcal{L}^1_{\mu_x}([a,b),\mathbb{R})$ so there exists $\delta > 0$ such that

$$\int_E f \, \mathrm{d}\,\mu_x < \frac{\varepsilon}{K}, \quad \text{for all } E \in \mathcal{M} \text{ such that } \mu_x(E) < \delta.$$

Recall that (2.21) holds μ–almost everywhere (see Remark 2.35 and Proposition 2.42), so

$$\int_E f(s) \, \mathrm{d}\,\mu_x(s) = \int_E f(s) h_{x,s}(s) \, \mathrm{d}\,\mu(s) \geq \int_E \frac{f(s)}{\gamma(s,x)} \, \mathrm{d}\,\mu(s) > \frac{1}{K} \int_E f(s) \, \mathrm{d}\,\mu(s).$$

Thus, we have that

$$\int_E f \, \mathrm{d}\,\mu < \varepsilon, \quad \text{for all } E \in \mathcal{M} \text{ such that } \mu_x(E) < \delta.$$

Consider a family of intervals in the conditions of the definition of Δ–absolute continuity, $\{(a_n,b_n)\}_{n=1}^m$, such that

$$\mu_x\left(\bigcup_{n=1}^m [a_n,b_n)\right) = \sum_{n=1}^m \mu_x([a_n,b_n)) = \sum_{n=1}^m (\Delta_x(b_n) - \Delta_x(a_n)) < \delta.$$

In that case,

$$\sum_{n=1}^m |F(b_n) - F(a_n)| = \sum_{n=1}^m (F(b_n) - F(a_n)) = \sum_{n=1}^m \int_{[a_n,b_n)} f \, \mathrm{d}\,\mu = \int_E f \, \mathrm{d}\,\mu < \varepsilon,$$

that is, F satisfies the definition of Δ–absolute continuity on $[a,b]$. □

We can finally establish the other version of the Fundamental Theorem of Calculus that also provides a characterization of Δ–absolutely continuous functions.

Theorem 2.71 (Fundamental Theorem of Calculus for the Δ–integral)**.** *Let* $[a,b] \subset \mathbb{R}$ *and* $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3). *A function* $F : [a,b] \to \mathbb{R}$ *is* Δ*–absolutely continuous on* $[a,b]$ *if and only if the following conditions are fulfilled:*

(i) *there exists* $F'_\Delta(x)$ *for* μ*–a.a.* $x \in [a,b)$*;*

(ii) $F'_{\Delta} \in \mathcal{L}^1_{\mu}([a,b),\mathbb{R})$;

(iii) *for each* $x \in [a,b]$,

$$F(x) = F(a) + \int_{[a,x)} F'_{\Delta} \, \mathrm{d}\,\mu.$$

Proof. Lemma 2.70 ensures that the three conditions are sufficient for F to be Δ–absolutely continuous. For the converse, consider μ_F to be the Lebesgue–Stieltjes measure defined by F in (2.56) and let $z \in [a,b]$ be fixed. Lemma 2.69 and the Radon–Nykodym Theorem, Theorem 1.33, guarantee that there exists a measurable function $l : ([a,b),\mathcal{B}(\tau_u)) \to (\mathbb{R},\mathcal{B}(\tau_u))$ such that

$$\mu_F(E) = \int_E l \, \mathrm{d}\,\mu, \quad \text{for any Borel set } E \subset [a,b).$$

In particular,

$$F(x) - F(a) = \mu_F([a,x)) = \int_{[a,x)} l(s) \, \mathrm{d}\,\mu(s).$$

Now, Theorem 2.62 ensures that $F'_{\Delta}(s) = l(s)$ for μ–almost all $s \in [a,b)$, and so the result follows. □

Chapter 3

The Stieltjes derivative

In this chapter we focus on the study of Stieltjes derivatives, one of the main tools for the work ahead: the study of Stieltjes differential equations, i.e. differential equations with Stieltjes derivatives. These derivates are, essentially, a modification of the usual derivative where we are considering the derivatives of function with respect to another function. This is done, as we mentioned in Chapter 2, by considering a rescaling the abscissae axis in terms of a function. Hence, it is easy to understand that Stieltjes derivatives can be regarded as a particular case of Δ–derivatives.

In what follows, we will explore in depth the relations between Stieltjes and displacement derivatives. This is done in in Section 3.1. Later, in Section 3.2, we have a look at the properties of Stieltjes derivatives, combining those that can be derived from its relation with Δ–derivatives, as well as some other that are specific to this context. Finally, in Section 3.3 we start to look at Stieltjes differential equations. In particular, we establish relations with some other known types of differential problems such as ordinary differential equations, impulsive differential equations or equations in time scales. This is done in order to showcase the interest of studying this derivative as well as the differential problems associate to it.

3.1 Stieltjes derivatives and displacement derivatives

As presented before, Stieltjes derivatives are a generalization of usual derivatives defined in terms of a nondecreasing and left–continuous function $g:\mathbb{R}\to\mathbb{R}$, which can also be thought of as a particular case of Δ–derivatives for the corresponding Stieltjes displacement defined by g, see Example 2.9. Hence, with the notation introduced in (1.21) and (1.22), we have the following definition.

Definition 3.1. *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing and left–continuous function and consider a map $f:[a,b]\to\mathbb{R}$. We define the* Stieltjes derivative*, or g*–derivative*, of f at a point $x\in[a,b]\setminus C_g$ as*

$$f'_g(x)=\begin{cases}\displaystyle\lim_{y\to x}\frac{f(y)-f(x)}{g(y)-g(x)}, & x\notin D_g,\\[2ex] \displaystyle\lim_{y\to x^+}\frac{f(y)-f(x)}{g(y)-g(x)}, & x\in D_g,\end{cases}$$

provided the corresponding limits exist. In that case, we say that f is g–differentiable at *x. If f is g–differentiable at every $x\in[a,b]\setminus C_g$, we say that f is g*–differentiable in *$[a,b]$.*

Remark 3.2. These limits should be understood in a sense analogous to those in Definition 2.49 and Remark 2.50, that is, in terms of the corresponding functions,

$$F_x(\cdot) = \frac{f(\cdot) - f(x)}{g(\cdot) - g(x)}, \quad x \in [a, b].$$

The expressions in Definition 3.1 are a simplifed way to write this.

Remark 3.3. In order to see Definition 3.1 as a particular case of a Δ–derivative in Definition 2.49, we need to reflect on the sets C_g and C_Δ. First of all, it is clear that a nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$ defines a Stieltjes displacement on $[a, b]$, Δ_g, by considering the corresponding restriction, $g|_{[a,b]}$. Hence, we can consider the sets C_g as in (1.22) and

$$\begin{aligned} C_{\Delta_g} &= \{x \in (a, b) : \Delta_g(x, \cdot) = 0 \text{ in } (x - \varepsilon, x + \varepsilon) \subset (a, b) \text{ for some } \varepsilon > 0\} \\ &= \{x \in (a, b) : g \text{ is constant on } (x - \varepsilon, x + \varepsilon) \subset (a, b) \text{ for some } \varepsilon > 0\}, \end{aligned}$$

It is clear by definition –or as a consequence of Proposition 2.46– that $C_{\Delta_g} = C_g \cap (a, b)$. This means that Definition 3.1 might be excluding more points than the corresponding Δ_g–derivative. In particular, the points that can be excluded from Definition 3.1 and not in Definition 2.49 are the extremal points of the interval, a and b. However, if that is the case, that means that g is constant in a neighbourhood of such points, which implies that the corresponding displacement map, Δ_g, is null on the induced neighbourhood in $[a, b]$. Hence, those points will be not considered in the definition of Δ_g–derivative, as pointed out by Remark 2.50. As a consequence, we obtain that Definition 3.1 can be regarded as a particular case of Definition 2.49.

Remark 3.4. Recalling Remark 2.51 we have that the g–derivative of a function f exists at a point $x \in D_g$ if and only if the limit of f from the right of x, $f(x^+)$, exists. In that case,

$$f'_g(x) = \frac{f(x^+) - f(x)}{\Delta g(x)},$$

where $\Delta g(x)$ is computed as in (1.30).

An interesting thing about Stieltjes derivatives is that, despite appearing as a particular case of Δ–derivatives, they can be proven to be equivalent to this kind of derivatives. This might be obvious in some cases. For example, consider the displacement in Example 2.31. In that case, it follows from the definitions that, for any $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, $f'_\Delta(x)$ exists, if and only if, $f'_g(x)$ exists, and in that case, we have that

$$f'_\Delta(x) = e^{-x} f'_g(x).$$

However, for a general displacement this equivalence might be less obvious. In fact, as we will see in the following result, the equivalence between displacement derivatives and Stieltjes derivatives is given by the nondecreasing and left–continuous function associated to the displacement measure.

Proposition 3.5. *Let* $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3) *and* $g : [a,b] \to \mathbb{R}$ *be the nondecreasing left–continuous function in* (1.28) *associated to the* Δ*–measure in* Definition 2.40. *Then,*

$$\lim_{s \to t} \frac{g(s) - g(t)}{\Delta(t,s)} = 1, \quad \textit{for all } t \in [a,b] \setminus C_\Delta.$$

Proof. Fix $t \in [a,b] \setminus C_\Delta$. It follows from (2.23) that $1/\gamma(t,r) \leq h_{r,t}(r) \leq \gamma(r,t)$ for all $r \in [a,b]$. As a consequence, we have that

$$\frac{1}{\sup\limits_{r \in [t,s)} \gamma(t,r)} \leq h_{r,t}(r) \leq \sup_{r \in [t,s)} \gamma(r,t). \tag{3.1}$$

Since $t \notin C_\Delta$ we have that (2.29) and/or (2.28) hold. First assume that (2.29) holds. In that case, for any $s \in [a,b]$, $s > t$, we have that

$$\frac{g(s) - g(t)}{\Delta(t,s)} = \frac{\int_{[a,s)} h_{r,t}(r) \, \mathrm{d}\, \mu_t - \int_{[a,t)} h_{r,t}(r) \, \mathrm{d}\, \mu_t}{\Delta(t,s)} = \frac{\int_{[t,s)} h_{r,t}(r) \, \mathrm{d}\, \mu_t}{\Delta(t,s)}.$$

Now, for any $s > t$ it follows from (3.1), that

$$\frac{1}{\sup\limits_{r \in [t,s)} \gamma(t,r)} \Delta(t,s) \leq \int_{[s,t)} h_{r,t}(r) \, \mathrm{d}\, \mu_t \leq \sup_{r \in [t,s)} \gamma(r,t) \Delta(t,s),$$

since $\mu_t([t,s)) = \Delta(t,s)$. Equivalently,

$$\frac{1}{\sup\limits_{r \in [t,s)} \gamma(t,r)} \leq \frac{\int_{[t,s)} h_{r,t}(r) \, \mathrm{d}\, \mu_t}{\Delta(t,s)} = \frac{g(s) - g(t)}{\Delta(t,s)} \leq \sup_{r \in [t,s)} \gamma(r,t), \quad s > t,$$

and so, allowing $s \to t^+$ and bearing (H3, ii) in mind, we obtain

$$1 = \lim_{s \to t^+} \frac{1}{\sup\limits_{r \in [t,s)} \gamma(t,r)} \leq \lim_{s \to t^+} \frac{g(s) - g(t)}{\Delta(t,s)} \leq \lim_{s \to t^+} \sup_{r \in [t,s)} \gamma(r,t) = 1.$$

If $\Delta_x(\cdot) = 0$ on some $[x - \delta, x]$, $\delta > 0$, then the proof is complete. Otherwise, condition (2.28) holds.

If condition (2.28) holds, we have that for any $s \in [a,b]$, $s < t$,

$$\frac{g(s) - g(t)}{\Delta(t,s)} = \frac{\int_{[a,s)} h_{r,t}(r) \, \mathrm{d}\, \mu_t - \int_{[a,t)} h_{r,t}(r) \, \mathrm{d}\, \mu_t}{\Delta(t,s)} = -\frac{\int_{[s,t)} h_{r,t}(r) \, \mathrm{d}\, \mu_t}{\Delta(t,s)}. \tag{3.2}$$

Once again, it follows from (3.1), that for any $s < t$,

$$-\sup_{r \in [t,s)} \gamma(r,t)(-\Delta(t,s)) \leq -\int_{[s,t)} h_{r,t}(r) \, \mathrm{d}\, \mu_t \leq -\frac{1}{\sup\limits_{r \in [t,s)} \gamma(t,r)} (-\Delta(t,s)),$$

since $\mu_t([s,t)) = -\Delta(t,s)$. Equivalently,

$$\sup_{r\in[t,s)} \gamma(r,t) \leq -\frac{\int_{[s,t)} h_{r,t}(r)\,\mathrm{d}\,\mu_t}{\Delta(t,s)} = \frac{g(s)-g(t)}{\Delta(t,s)} \leq \frac{1}{\sup_{r\in[t,s)} \gamma(t,r)}, \quad s<t,$$

which, allowing $s \to t^-$ yields

$$\lim_{s\to t^-} \frac{g(s)-g(t)}{\Delta(t,s)} = 1.$$

Now the result follows, since either we have that $\Delta_x(\cdot) = 0$ on some $[x, x+\delta]$, $\delta > 0$, or (2.29) holds. □

Remark 3.6. It follows directly from Proposition 3.5 that

$$\lim_{s\to t} \frac{\Delta(t,s)}{g(s)-g(t)} = 1, \quad \text{for all } t \in [a,b] \setminus C_\Delta.$$

Proposition 3.5 is enough to show the equivalence between the two derivatives. We first include the main result for said equivalence, and then we discuss some technical details that are left out in the result.

Proposition 3.7. *Let* $\Delta : [a,b]\times[a,b] \to \mathbb{R}$ *satisfy* (H1)–(H3) *and* $g : [a,b] \to \mathbb{R}$ *be the non-decreasing left–continuous function in* (1.28) *associated to the* Δ*–measure in* Definition 2.40. *For a function* $f : [a,b] \to \mathbb{R}$ *and* $t \in [a,b] \setminus C_\Delta$*, the following statements are equivalent:*

(i) *there exists* $\lim_{s\to t} \dfrac{f(s)-f(t)}{\Delta(t,s)}$ *and equals* $L \in \mathbb{R}$;

(ii) *there exists* $\lim_{s\to t} \dfrac{f(s)-f(t)}{g(t)-g(s)}$ *and equals* $L \in \mathbb{R}$.

Proof. Given $t \in [a,b] \setminus C_g$, it is enough to note that

$$\lim_{s\to t} \frac{f(s)-f(t)}{\Delta(t,s)} = \lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)} \frac{g(s)-g(t)}{\Delta(t,s)},$$

$$\lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)} = \lim_{s\to t} \frac{f(s)-f(t)}{\Delta(t,s)} \frac{\Delta(t,s)}{g(s)-g(t)},$$

and apply Proposition 3.5 or Remark 3.6. □

Remark 3.8. Once again, the limits here should be understood in the sense of Definitions 2.49 and 3.1, respectively. It is also important to note that the result remains true if we replace all the limits in the statement of the proposition with lateral limits.

Proposition 3.7 is almost enough to ensure the equivalence between the two derivatives. However, there are two technical aspects that need to be discussed. The first one concerns Definition 3.1, where g is required to be defined over all $\mathbb{R}$, whereas in (1.28), g is only defined on $[a,b]$. To solve this, it is enough to consider the function $\widetilde{g}:\mathbb{R}\to\mathbb{R}$ defined as

$$\widetilde{g}(t)=\begin{cases} g(a) & \text{if } t<a,\\ g(t) & \text{if } a\le t\le b,\\ g(b) & \text{if } t>b.\end{cases} \tag{3.3}$$

In that case, for any $t\in[a,b]$, we have that the limits in (ii) in Proposition 3.7 for g and $\widetilde{g}$ yield the same result –even at a and b, thanks to Remark 3.8.

The other technicality that requires some discussion is the fact that the points that are excluded in Definition 3.1 are those of C_g, while Proposition 3.7 only works for points that do not belong to C_Δ. This is not an issue thanks to Proposition 2.46 and Remark 3.3. Hence, we have the following result.

Theorem 3.9. *Let $\Delta:[a,b]\times[a,b]\to\mathbb{R}$ satisfy* (H1)–(H3) *and $g:[a,b]\to\mathbb{R}$ be the nondecreasing left–continuous function in* (1.28) *associated to the Δ–measure in* Definition 2.40. *Then, given $f:[a,b]\to\mathbb{R}$ and $t\in[a,b]\setminus C_\Delta$,*

$$f'_\Delta(t) \text{ exists if and only if } f'_g(t) \text{ exists.}$$

Therefore, we have that both derivatives are, indeed, equivalent. However, obtaining the corresponding function g that is left–continuous and nondecreasing might be hard. Indeed, although (1.28) gives an explicit expression of the function, it is defined in terms of the Δ–measure, which depends on the functions in (2.19) given by the Radon–Nidokým Theorem, Theorem 1.33. The following result gives a simple way to obtain, under certain hypotheses, the corresponding function.

Proposition 3.10. *Let $\Delta:[a,b]\times[a,b]\to\mathbb{R}$ satisfy* (H1)–(H3) *and $g:[a,b]\to\mathbb{R}$ be the nondecreasing left–continuous function in* (1.28) *associated to the Δ–measure in* Definition 2.40. *Let us denote by $D_2\Delta$ the partial derivative of Δ with respect to its second variable. Suppose that $D_2\Delta$ exists, is positive on $[a,b]\times[a,b]$ and $D_2\Delta\in\mathcal{L}^1_m([a,b]\times[a,b],[0,+\infty))$ where m denotes Lebesgue's measure. Then*

1. *For all $t\in[a,b]$, $g'(t)$ exists and $g'(t)=D_2\Delta(t,t)$. In particular, g is increasing.*
2. *Given $s\in[a,b]$,*

$$g'(t)=h_{t,s}(t)D_2\Delta(s,t),\quad \text{for a.a. } t\in[a,b]$$

where $h_{t,s}$ is the corresponding function in (2.19).

Proof. Fix $t\in[a,b]$. Since $D_2\Delta(t,t)>0$, the function Δ_t is increasing in a neighborhood of t. Thus, $t\notin C_\Delta$, and so, if we apply Proposition 3.5, we have that

$$D_2\Delta(t,t)=\lim_{s\to t}\frac{\Delta(t,s)}{s-t}=\lim_{s\to t}\frac{\Delta(t,s)}{s-t}\lim_{s\to t}\frac{g(s)-g(t)}{\Delta(t,s)}=\lim_{s\to t}\frac{g(s)-g(t)}{s-t}.$$

Hence, $g'(t)$ exists and equals $D_2\Delta(t,t)$. The rest of 1 now follows.

Now for 2, take $t, s \in [a,b]$. Assume that $s < t$. Then, by the definition of the Δ–measure, $\mu([s,t)) = \int_{[s,t)} h_{r,s}(r)\,\mathrm{d}\,\mu_s$. Since μ_s is the Lebesgue–Stieltjes measure induced by Δ_s, which is differentiable, we have that

$$\mu([s,t)) = \int_s^t h_{r,s}(r)(\Delta_s)'(r)\,\mathrm{d}\,r = \int_s^t h_{r,s}(r)D_2\Delta(s,r)\,\mathrm{d}\,r.$$

On the other hand, by construction we have that $\mu([s,t)) = g(t) - g(s)$. Hence, by the Fundamental Theorem of Calculus, we have that

$$\int_s^t h_{r,s}(r)D_2\Delta(s,r)\,\mathrm{d}\,r = \mu([s,t)) = \int_s^t g'(r)\,\mathrm{d}\,r,$$

and so $h_{t,s}(t)D_2\Delta(s,t) = g'(t)$ for almost all $t \in [a,b]$.

If $t < s$, the reasoning is analogous and we omit it. □

Remark 3.11. Combining 1 and 2 in Proposition 3.10 we can obtain an explicit expression for $h_{s,t}$, namely

$$h_{t,s}(t) = \frac{D_2\Delta(t,t)}{D_2\Delta(s,t)}, \quad \text{for a.a. } t \in [a,b].$$

Example 3.12. Consider the displacement $\Delta : [0,1]\times[0,1] \to \mathbb{R}$ in Example 2.33, given by

$$\Delta(x,y) = e^{y^2-x^2} - e^{x-y}.$$

In this case, we have that

$$D_2\Delta(x,y) = 2ye^{y^2-x^2} + e^{x-y}, \quad (x,y) \in [0,1]\times[0,1]$$

which is continuous and positive. Hence, taking $a = 0$ we can compute $g : [0,1] \to \mathbb{R}$ in (1.28) as

$$g(t) = \int_0^t D_2\Delta(s,s)\,\mathrm{d}\,s = \int_0^t (2s+1)\,\mathrm{d}\,s = t^2 + t.$$

3.2 Interesting results for Stieltjes derivatives

In this section we gather some existing results that are useful for the topic of Stieltjes derivatives. Some of these results are direct adaptations of those proved in the previous chapter, but most of them can be found in different articles such as [33, 54]. However, the study of this type of derivatives is not limited to these papers. Previous work on the topic exists, such as [37], where the author presents similar results, even more general at times, for derivatives such as the one in Definition 3.1 where g is of bounded variation. For the aim of this work, we will limit ourselves to the framework previously presented.

We start by stating the basic properties of the Stieltjes derivatives, such as linearity or the product and quotient rule. This result is a direct translation of Proposition 2.54.

Proposition 3.13. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function, $x \in \mathbb{R}$ and f_1, f_2 be two real–valued functions defined on a neighborhood of x. If f_1 and f_2 are g–differentiable at x, then:*

(i) *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g–differentiable at x for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(x) = \lambda_1 (f_1)'_g(x) + \lambda_2 (f_2)'_g(x).$$

(ii) *The product $f_1 f_2$ is g–differentiable at x and*

$$(f_1 f_2)'_g(x) = (f_1)'_g(x) f_2(x) + (f_2)'_g(x) f_1(x) + (f_1)'_g(x)(f_2)'_g(x)\Delta g(x).$$

(iii) *If $f_2(x)(f_2(x) + (f_2)'_g(x)\Delta g(x)) \neq 0$, the quotient f_1/f_2 is g–differentiable at x and*

$$\left(\frac{f_1}{f_2}\right)'_g(x) = \frac{(f_1)'_g(x) f_2(x) - f_1(x)(f_2)'_g(x)}{f_2(x)(f_2(x) + (f_2)'_g(x)\Delta g(x))}.$$

Remark 3.14. Observe that the formulas here presented do not match those in [54], where they were stated without proof. Let us illustrate that the formulas there presented are not correct with some examples.

First we show that the formula for the product of two functions in [54],

$$(f_1 f_2)'_g(x) = (f_1)'_g(x) f_2(x^+) + (f_2)'_g(x) f_1(x^+), \tag{3.4}$$

is not correct. Consider the map $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} x+1 & \text{if } x \leq 0, \\ x+2 & \text{if } x > 0, \end{cases}$$

and $f_1, f_2 : [-1/2, 1] \to \mathbb{R}$ defined as

$$f_1(x) = \begin{cases} x+1 & \text{if } -\dfrac{1}{2} \leq x \leq 0, \\ x+2 & \text{if } 0 < x \leq 1, \end{cases} \qquad f_2(x) = \begin{cases} \dfrac{1}{x+1} & \text{if } -\dfrac{1}{2} \leq x \leq 0, \\ \dfrac{1}{x+2} & \text{if } 0 < x \leq 1. \end{cases}$$

First of all, observe that $f_1 f_2(x) = 1$, $x \in [-1/2, 1]$, from which it follows that

$$(f_1 f_2)'_g(x) = 0, \quad x \in [-1/2, 1]. \tag{3.5}$$

Similarly, by definition, we have that $f_1 = g|_{[-1/2,1]}$, which yields $(f_1)'_g(x) = 1$ for all $x \in [-1/2, 1]$. Furthermore, according to Remark 3.4,

$$(f_2)'_g(0) = \frac{f_2(0^+) - f_2(0)}{\Delta g(0)} = \frac{1/2 - 1}{1} = -\frac{1}{2}.$$

With this information, we can evaluate the right–hand side of (3.4) at 0,

$$(f_1)'_g(0) f_2(0^+) + (f_2)'_g(0) f_1(0^+) = 1 \cdot \frac{1}{2} - \frac{1}{2} \cdot 2 = -\frac{1}{2},$$

which contradicts (3.4) and (3.5). Note, however, that (ii) in Proposition 3.13 holds, as

$$(f_1)'_g(0)f_2(0) + (f_2)'_g(0)f_1(0) + (f_1)'_g(0)(f_2)'_g(0)\Delta g(0) = 1 \cdot 1 - \frac{1}{2} \cdot 1 - 1 \cdot \frac{1}{2} \cdot 1 = 0.$$

Now, for the quotient formula in [54],

$$\left(\frac{f_1}{f_2}\right)'_g(x) = \frac{(f_1)'_g(x)f_2(x) - (f_2)'_g(x)f_1(x)}{f_2(x)f_2(x^+)}, \tag{3.6}$$

take $g, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 0 & \text{if } 0 < x \leq 1, \\ x & \text{if } x > 1, \end{cases} \qquad f_1(x) = x + 2, \qquad f_2(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ t + 2 & \text{if } x > 0. \end{cases} \tag{3.7}$$

For this choice of functions, we have that $f_1/f_2 : \mathbb{R} \to \mathbb{R}$ is defined as

$$\frac{f_1}{f_2}(x) = \begin{cases} x + 2 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We will show that (3.6) fails at $x = 0$. Observe that, given that $g(x) = g(0)$ for $x \in (0, 1)$, the derivatives of functions at 0 are computed as the limit from the left, as pointed out by Remark 3.2. In particular, we have that

$$\begin{aligned} \left(\frac{f_1}{f_2}\right)'_g(0) &= \lim_{s \to 0^-} \frac{f_1/f_2(s) - f_1/f_2(0)}{g(s) - g(0)} = \lim_{s \to 0^-} \frac{s + 2 - 2}{s - 0} = 1, \\ (f_1)'_g(0) &= \lim_{s \to 0^-} \frac{f_1(s) - f_1(0)}{g(s) - g(0)} = \lim_{s \to 0^-} \frac{s + 2 - 2}{s - 0} = 1, \\ (f_2)'_g(0) &= \lim_{s \to 0^-} \frac{f_2(s) - f_2(0)}{g(s) - g(0)} = \lim_{s \to 0^-} \frac{1 - 1}{s - 0} = 0. \end{aligned}$$

Therefore, evaluating the right–hand side of (3.6) at $x = 0$,

$$\frac{(f_1)'_g(0)f_2(0) - (f_2)'_g(0)f_1(0)}{f_2(0)f_2(0^+)} = \frac{1 \cdot 1 - 0 \cdot 2}{1 \cdot 2} = \frac{1}{2},$$

and so (3.6) does not hold. Once again, observe that the corresponding formula in Proposition 3.13 holds, since

$$\frac{(f_1)'_g(0)f_2(0) - (f_2)'_g(0)f_1(0)}{f_2(0)(f_2(0) + (f_2)'_g(0)\Delta g(0)} = \frac{1 \cdot 1 - 0 \cdot 2}{1 \cdot (1 + 0 \cdot 1)} = 1 = \left(\frac{f_1}{f_2}\right)'_g(0).$$

Moreover, in [75, Lemma 13], we can find an expression for the product rule at the discontinuity points of the derivator. Specifically, we have that

$$(f_1 f_2)'_g(x) = (f_1)'_g(x)f_2(x^+) + f_1(x)(f_2)'_g(x), \quad x \in D_g.$$

This formula matches the one in Proposition 3.13, statement (ii), since for $x \in D_g$, Remark 3.4 ensures that $f_2(x^+) = f_2(x) + (f_2)'_g(x)\Delta g(x)$. Nevertheless, this expression is not valid in general. For example, considering $g, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined as in (3.7), we have that $f_1 \cdot f_2 : \mathbb{R} \to \mathbb{R}$ is defined as

$$f_1 \cdot f_2(x) = \begin{cases} x+2 & \text{if } x \leq 0, \\ (x+2)^2 & \text{if } x > 0, \end{cases}$$

and as a consequence,

$$(f_1 \cdot f_2)'_g(0) = \lim_{s \to 0^-} \frac{f_1 \cdot f_2(s) - f_1 \cdot f_2(0)}{g(s) - g(0)} = \lim_{s \to 0^-} \frac{s+2-2}{s-0} = 1.$$

However,

$$(f_1)'_g(0) f_2(0^+) + f_1(0)(f_2)'_g(0) = 1 \cdot 2 + 2 \cdot 0 = 2 \neq 1,$$

showing that the expression in [75, Lemma 13] is not valid for a generic point in $\mathbb{R} \setminus C_g$.

Similarly, we have the chain rule for Stieltjes derivatives. As presented in the more general case of Δ–derivatives, Proposition 2.56, we have two versions of the result. A more general version of the chain rule for g–derivatives can be found in [34, Proposition 5.4] where the authors established a similar result considering Stieltjes derivatives with respect to two different derivators that are not required to be monotonic.

Proposition 3.15. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function. Let f be a real–valued function defined on a neighborhood of $x \notin D_\Delta$ and h another function defined on a neighborhood of $f(x)$.*

1. *If $h'(f(x))$ and $f'_g(x)$ exist, then so does $(h \circ f)'_g(x)$ and*

$$(h \circ f)'_g(x) = h'(f(x)) f'_g(x).$$

2. *If $h'_g(f(x))$, $f'_g(x)$ and $g'(f(x))$ exist and*

$$g(y) \neq g(f(x)), \quad \text{for } y \in \mathbb{R},\, y \neq f(x), \tag{3.8}$$

then $(h \circ f)'_g(x)$ also exists and

$$(h \circ f)'_g(x) = h'_g(f(x)) f'_g(x) g'(f(x)). \tag{3.9}$$

Remark 3.16. Observe that a similar formulation for the chain rule of g–derivatives can be found in [54, Theorem 2.3]. Nevertheless, condition (3.8) is not included in its statement. Let us show that (3.9) needs not be true if condition (3.8) is not satisfied. Consider the maps $g, f, h : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x^2, & \text{if } x > 0, \end{cases} \qquad f(x) = -x^2, \qquad h(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ x^2, & \text{if } x > 0. \end{cases}$$

By definition,

$$f'_g(0) = \lim_{s\to 0^+} \frac{f(s)-f(0)}{g(s)-g(0)} = \lim_{s\to 0^+} \frac{-s^2-0}{s^2-0} = -1,$$
$$h'_g(f(0)) = \lim_{s\to 0^+} \frac{h(s)-h(0)}{g(s)-g(0)} = \lim_{s\to 0^+} \frac{s^2-0}{s^2-0} = 1.$$

Furthermore, observe that $(h \circ f)(x) = -f(x)$, $x \in \mathbb{R}$, so $(h \circ f)'_g(0) = 1$. However, standard computations show that $g'(0) = 0$, which means that (3.9) does not hold at $x = 0$. This is because condition (3.8) fails at that point.

We now introduce the concept of g–continuity. This concept is the adaptation of the Δ–continuity presented in Definition 2.26 for the particular case of Stieltjes displacements.

Definition 3.17. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function. A function $f : A \subset \mathbb{R} \to \mathbb{R}^n$ is g–*continuous at a point *$t \in A$, or* continuous with respect to *g* at *t, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(t) - f(s)\| < \varepsilon, \quad \textit{for all } s \in A,\ |g(t) - g(s)| < \delta.$$

*If it is g–continuous at every point $t \in A$, we say that f is g–*continuous on *A.*

Remark 3.18. Although this definition is presented in a closer way to that of the usual continuity in $\mathbb{R}^n$, it is in fact the same as in Definition 2.26 for $\Delta(x, y) = g(y) - g(x)$. To see that, it is enough to consider Lemma 2.19. We chose this formulation for the definition of g–continuity as this is the definition that was introduced in [33].

We denote by $\mathcal{C}_g([a, b], \mathbb{R}^n)$ the set of g–continuous functions from $[a, b]$ with values in $\mathbb{R}^n$, and following the notation introduced in Definition 2.26, we denote by $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ the set of g–continuous functions from $[a, b]$ with values in $\mathbb{R}^n$ which are also bounded. Recall that $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ is a Banach space with the sup–norm, see Proposition 2.28.

It is important to note that those two sets are not necessarily the same, despite being the same for the particular case of $g = \mathrm{Id}$. To show that, consider the following example that can be found in [33].

Example 3.19. Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = \begin{cases} t, & \text{if } t \le 0, \\ t+1, & \text{if } t > 0. \end{cases}$$

Consider the function $f : [-1, 1] \to \mathbb{R}$ given by

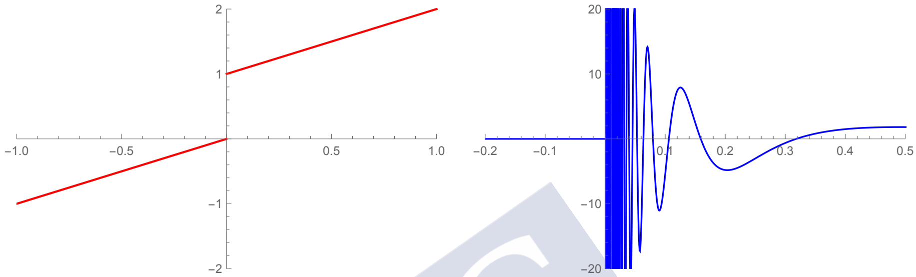
$$f(t) = \begin{cases} 0, & \text{if } t \le 0, \\ \dfrac{\sin(1/t)}{t}, & \text{if } t > 0. \end{cases}$$

It is clear that f is not bounded. Let us show that f is g–continuous on $[-1, 1]$. Note that in this case, g–continuity at points $t \neq 0$ is the same as usual continuity, and so it follows that f is g–continuous at every $t \neq 0$.

To prove that f is g–continuous at $t_0 = 0$, we notice that, for any $\varepsilon > 0$, it suffices to take $\delta \in (0, 1)$ because in that case the relation

$$|g(0) - g(t)| < \delta$$

implies that $t < 0$, and therefore $|f(0) - f(t)| = 0 < \varepsilon$ as desired.



(a) Graph of g (b) Graph of f

Figure 3.1: Graphs of g and f around the point $t_0 = 0$.

Remark 3.20. Example 3.19 shows that g–continuous functions on closed intervals need not be bounded. This is because closed intervals are not necessarily compact in the corresponding topology defined by g, τ_g. Indeed, let us show that the closed interval $[-1, 1]$ is not compact in τ_g for g as in Example 3.19.

The map $g : \mathbb{R} \to \mathbb{R}$ is trivially g–continuous. That is, the map $g : (\mathbb{R}, \tau_g) \to (\mathbb{R}, \tau_u)$ is continuous, see Remark 3.18. If $[-1, 1]$ was compact in the topology τ_g, its image under the continuous map g should be compact in τ_u. However, $g([-1, 1]) = [-1, 0] \cup (1, 2]$, which cannot be compact in τ_u as it is not closed in that topology. Thus, $[-1, 1]$ is not compact in the topology τ_g.

Next, we present some useful properties of g–continuous functions that follow directly from the definition. In particular, the following result, [33, Proposition 3.2], gives us some information about the continuity in the usual sense of such functions.

Proposition 3.21. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function, and $f : [a, b] \to \mathbb{R}^n$ be g–continuous on $[a, b]$. Then:*

1. *f is continuous from the left at every $t \in (a, b]$;*
2. *if g is continuous at $t \in [a, b)$, then so is f;*
3. *if g is constant on some $[c, d] \subset [a, b]$, then so is f.*

In particular, g–continuous functions on $[a, b]$ are continuous on $[a, b]$ when g is continuous on $[a, b)$.

Some useful properties of continuous functions are still true in the context of g–continuity. In particular, the next result, [33, Corollary 3.5], ensures that g–continuous functions are measurable in the corresponding sense.

Proposition 3.22. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function, $A \subset \mathbb{R}$ a Borel set and $f : A \subset \mathbb{R} \to \mathbb{R}^n$ be a map. If f is g–continuous, it is Borel–measurable and, as a consequence, g–measurable.*

A question that arises naturally is whether g–differentiable functions are g–continuous. One might be tempted to think so, considering that when we take $g = \mathrm{Id}$ the result is true. However, this is not the case as presented in the next example.

Example 3.23. Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function and $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} t, & \text{if } t < 0, \\ t+1, & \text{if } t \geq 0. \end{cases}$$

Clearly, $f(0^+)$ exists. Hence Remark 2.51 ensures that $f'_g(0)$ exists. However, f cannot be g–continuous at 0 since it is not left–continuous at that point, which is a necessary condition according to statement 1 in Proposition 3.21.

Interestingly, absolutely continuous functions in the sense of Definition 2.64 are continuous with respect to g. First, let us rewrite such definition in the specific context of Stieltjes displacements.

Definition 3.24. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function and consider a map $F : [a, b] \to \mathbb{R}$. We shall say that F is g–*absolutely continuous on $[a, b]$*, or* absolutely continuous on $[a, b]$ with respect to g*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ verifying*

$$\sum_{n=1}^{m} (g(b_n) - g(a_n)) < \delta,$$

we have that

$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

A map $F : [a, b] \to \mathbb{R}^n$ is g–absolutely continuous if each of its components is a g–absolutely continuous function. We shall denote by $\mathcal{AC}_g([a, b], \mathbb{R}^n)$ the set of g–absolutely continuous functions on $[a, b]$ with values on $\mathbb{R}^n$.

Remark 3.25. It follows from the definition that g–absolutely continuous functions on $[a, b]$ are g–continuous on $[a, b]$.

Given that the definition of g–absolute continuity is a particular case of Definition 2.64, we can recover some important properties for this type of functions. An explicit proof of the following results can be found in [33, 54].

We start by stating the two versions of the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integrals. The first version can be obtained by putting Theorem 2.62 and Lemma 2.70 together.

Theorem 3.26 (Fundamental Theorem of Calculus for the Stieltjes derivative)**.** *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing left–continuous function and $f\in\mathcal{L}^1_g([a,b),\mathbb{R})$. Then the function $F:[a,b]\to\mathbb{R}$ defined as*

$$F(t)=\int_{[a,t)} f(s)\,\mathrm{d}\,\mu,$$

is well–defined, g–absolutely continuous on $[a,b]$ and

$$F'_g(t)=f(t),\quad \textit{for g–a.a. } t\in[a,b).$$

We can also adapt Theorem 2.71 to the context of Stieltjes derivatives to obtain the following result.

Theorem 3.27 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral)**.** *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing left–continuous function. A function $F:[a,b]\to\mathbb{R}$ is g–absolutely continuous on $[a,b]$ if and only if the following conditions are fulfilled:*

(i) *there exists $F'_g(t)$ for g–a.a. $t\in[a,b)$;*

(ii) *$F'_g\in\mathcal{L}^1_g([a,b),\mathbb{R})$;*

(iii) *for each $t\in[a,b]$,*

$$F(t)=F(a)+\int_{[a,t)} F'_g(s)\,\mathrm{d}\,g(s).$$

In what follows, we explore further properties of g–absolutely continuous functions. The next property, [33, Proposition 5.3], is a particular case of Proposition 2.67.

Proposition 3.28. *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing left–continuous function, $f_1:[a,b]\to[c,d]$ be a g–absolutely continuous function and $f_2:[c,d]\to\mathbb{R}$ satisfy a Lipschitz condition on $[c,d]$. Then $f_2\circ f_1$ is g–absolutely continuous on $[a,b]$.*

Proposition 3.28 can be used to ensure that the product of g–absolutely continuous functions is also g–absolutely continuous. Furthermore, the same result ensures that the pointwise maximum and minimum of g–absolutely continuous functions is g–absolutely continuous, as we show in the next proposition.

Proposition 3.29. *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing left–continuous function and f_1,f_2 be g–absolutely continuous on $[a,b]$. Then, the maps $F,F_{\max},F_{\min}:[a,b]\to\mathbb{R}$ defined as*

$$F(t)=f_1(t)f_2(t),\quad F_{\max}(t)=\max\{f_1(t),f_2(t)\},\quad F_{\min}(t)=\min\{f_1(t),f_2(t)\},$$

are g–absolutely continuous on $[a,b]$.

Proof. Rewrite the functions $F,F_{\max},F_{\min}$ as

$$\begin{aligned}
F(t)&=\frac{(f_1(t)+f_2(t))^2-f_1(t)^2-f_2(t)^2}{2}, && t\in[a,b],\\
F_{\max}(t)&=\frac{(f_1(t)+f_2(t))+|f_1(t)-f_2(t)|}{2}, && t\in[a,b],\\
F_{\min}(t)&=\frac{(f_1(t)-f_2(t))-|f_1(t)-f_2(t)|}{2}, && t\in[a,b].
\end{aligned}$$

Now, Proposition 3.28 ensures that the composition of a g–absolutely continuous map on $[a, b]$ with the maps $h_1(t) = t^2$ and $h_2(t) = |t|$ are g–absolutely continuous, from which the result follows. □

Finally, we state a direct adaptation of Proposition 2.68. This is a simplified version of [54, Proposition 5.3], as this result also contains other properties of g–absolutely continuous function. However, such properties can be obtained as a consequence of Remark 3.25 and Proposition 3.21, and we omit them.

Proposition 3.30. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and $F : [a, b] \to \mathbb{R}$ be a g–absolutely continuous function. Then F has bounded variation.*

Given that every function with bounded variation is also bounded, it follows from this result together with Remark 3.25, that $\mathcal{AC}_g([a, b], \mathbb{R}^n)$ is a subset of $\mathcal{BC}_g([a, b], \mathbb{R}^n)$. With this relation in mind, we include the following result, [33, Proposition 5.6], that gives a sufficient condition for a subset of $\mathcal{AC}_g([a, b], \mathbb{R}^n)$ to be relatively compact on $\mathcal{BC}_g([a, b], \mathbb{R}^n)$.

Proposition 3.31. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left–continuous function and $\mathcal{S}$ be a subset of $\mathcal{AC}_g([a, b], \mathbb{R}^n)$. Assume that the following conditions are satisfied:*

(i) *The set $\{F(a) : F \in \mathcal{S}\}$ is bounded.*

(ii) *There exists $h \in \mathcal{L}^1_g([a, b), [0, +\infty))$ such that*

$$\|F'_g(t)\| \leq h(t), \quad g\text{–a.a. } t \in [a, b), \ F \in \mathcal{S}.$$

Then $\mathcal{S}$ is a relatively compact subset of $\mathcal{BC}_g([a, b], \mathbb{R}^n)$.

As a direct application of Proposition 3.31, we can obtain a fixed–point theorem that will be useful later on in the study of differential equations with functional arguments.To that end, we include the following result, [40, Theorem 1.2.2], which is necessary for the proof of Proposition 3.33.

Theorem 3.32. *Let X be a metric space equipped with a partial ordering, $\leq$, such that for each $x \in X$, the sets*

$$\{y \in X : y \leq x\}, \quad \{y \in X : y \geq x\},$$

are closed, $Y \subset X$, $[a, b]$ a nonempty order interval in Y and $G : [a, b] \to [a, b]$ a nondecreasing mapping. If $\{Gx_n\}_{n=0}^{\infty}$ converges in Y whenever $\{x_n\}_{n=0}^{\infty}$ is a monotone sequence in $[a, b]$, then G has the least fixed point in $[a, b]$, x_*, and the greatest one, x^*; and they satisfy

$$x_* = \min\{y \in [a, b] : Gy \leq y\}, \quad x^* = \max\{y \in [a, b] : Gy \geq y\}.$$

Proposition 3.33. *Let $\alpha, \beta \in \mathcal{AC}_g([a, b], \mathbb{R})$ be such that $\alpha(t) \leq \beta(t)$, $t \in [a, b]$. Denote*

$$[\alpha, \beta]_{\mathcal{AC}_g([a,b],\mathbb{R})} = \{\gamma \in \mathcal{AC}_g([a, b], \mathbb{R}) : \alpha(t) \leq \gamma(t) \leq \beta(t), \ t \in [a, b]\}.$$

and let $G : [\alpha, \beta]_{\mathcal{AC}_g([a,b],\mathbb{R})} \to [\alpha, \beta]_{\mathcal{AC}_g([a,b],\mathbb{R})}$ be a nondecreasing map. If there exists $h \in \mathcal{L}^1_g([a, b), [0, +\infty))$ such that for all $\gamma \in [\alpha, \beta]_{\mathcal{AC}_g([a,b],\mathbb{R})}$,

$$|(G\gamma)'_g(t)| \leq h(t) \quad g\text{–a.a. } t \in [a, b),$$

then G has the least fixed point in $[\alpha,\beta]_{\mathcal{AC}_g([a,b],\mathbb{R})}$, γ_*, and the greatest one, γ^*, and they satisfy the following equalities:

$$\gamma_* = \min\{\gamma \in [\alpha,\beta]_{\mathcal{AC}_g([a,b],\mathbb{R})} : G\gamma \le \gamma\}, \quad \gamma^* = \max\{\gamma \in [\alpha,\beta]_{\mathcal{AC}_g([a,b],\mathbb{R})} : G\gamma \ge \gamma\}.$$

Proof. Denote $I = [a,b]$. In order to apply Theorem 3.32, we consider $Y = \mathcal{AC}_g(I,\mathbb{R}^n)$ and the ordered metric space $X = \mathcal{BC}_g(I,\mathbb{R}^n)$ equipped with the supremum norm and the usual partial ordering. It suffices to show that if $\{\gamma_k\}_{k\in\mathbb{N}}$ is a monotone sequence in $[\alpha,\beta]_{\mathcal{AC}_g(I,\mathbb{R})}$, then the sequence $\{G\gamma_k\}_{k\in\mathbb{N}}$ converges in Y.

Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a monotone sequence. The hypotheses on G ensure that $\{G\gamma_k\}_{k\in\mathbb{N}}$ is also a monotone sequence. Moreover, $\{G\gamma_k\}_{k\in\mathbb{N}}$ is a relatively compact subset of X by Proposition 3.31. Indeed, the set $\{G\gamma_k(a) : k \in \mathbb{N}\}$ is bounded because $\alpha \le G\gamma_k \le \beta$ for all $k \in \mathbb{N}$. Moreover, there exists $h \in \mathcal{L}^1_g([a,b),[0,+\infty))$ such that

$$|(G\gamma_k)'_g(t)| \le h(t) \quad g\text{–a.a. } t \in [a,b), \quad k \in \mathbb{N}.$$

Hence, $\{G\gamma_k\}_{k\in\mathbb{N}}$ is relatively compact and so, $\{G\gamma_k\}_{k\in\mathbb{N}}$ has a subsequence convergent in X to a function, say γ. Since $\{G\gamma_k\}_{k\in\mathbb{N}}$ is a monotone sequence, the whole sequence converges to γ. To finish the proof, we need to show that $\gamma \in Y$.

Define

$$H(t) = \int_{[a,t)} h(s)\,\mathrm{d}\,g(s), \quad t \in I.$$

Note that H is a nondecreasing function and, moreover, Theorem 3.26 ensures that H is g–absolutely continuous on I. Since $G\gamma_k \in \mathcal{AC}_g(I,\mathbb{R})$, $k \in \mathbb{N}$, for all $s,t \in I$, $s < t$, we have

$$|G\gamma_k(t) - G\gamma_k(s)| = \left|\int_{[s,t)} (G\gamma_k)'_g(r)\,\mathrm{d}\,g(r)\right| \le \int_{[s,t)} h(r)\,\mathrm{d}\,g(r) = H(t) - H(s), \quad k \in \mathbb{N}.$$

Since this holds for all $k \in \mathbb{N}$, we obtain $|\gamma(t) - \gamma(s)| \le H(t) - H(s)$. Now, given $\varepsilon > 0$, we take $\delta > 0$ given by the definition of g–absolute continuity of H. Hence, for every family $\{(a_n,b_n)\}_{n=1}^m$ of open disjoint subintervals of I such that $\sum_{n=1}^m [g(b_n) - g(a_n)] < \delta$, it holds that

$$\sum_{n=1}^m |\gamma(b_n) - \gamma(a_n)| \le \sum_{n=1}^m H(b_n) - H(a_n) = \sum_{n=1}^m |H(b_n) - H(a_n)| < \varepsilon,$$

and so, $\gamma \in Y$ by definition, which concludes the proof. □

Finally, we include some technical lemmas. The first one can be deduce from the Fundamental Theorem of Calculus, and it reads as follows.

Lemma 3.34. *Let $M : [a,b) \to [0,\infty)$ be a g-integrable function. If $F \subset [a,b)$ is a set of positive g–measure, then there exists $F_1 \subset F$ such that $\mu_g(F_1) = \mu_g(F)$ and for all $s \in F_1$,*

$$\lim_{t\to s^+} \frac{g(t) - g(s)}{\mu_g([s,t)\cap F)} = 1, \quad \lim_{t\to s^+} \frac{1}{\mu_g([s,t)\cap F)} \int_{[s,t)\setminus F} M(r)\,\mathrm{d}\,g(r) = 0.$$

Proof. Let $G : [a,b] \to \mathbb{R}$ be the map given by

$$G(t) = \int_{[a,t)} \chi_F(s) \, \mathrm{d}\, g(s).$$

Clearly $\chi_F \in \mathcal{L}^1_g([a,b),[0,1])$, so Theorem 3.26 ensures that $G \in \mathcal{AC}_g([a,b],\mathbb{R})$. Hence, there exists $F_0 \subset F$ such that $\mu_g(F \setminus F_0) = 0$, $G'_g(s)$ exists for all $s \in F_0$ and

$$G'_g(s) = \chi_F(s) = 1, \quad s \in F_0.$$

Thus, for $s \in F_0$ we have that

$$1 = \lim_{t \to s^+} \frac{G(t) - G(s)}{g(t) - g(s)} = \lim_{t \to s^+} \frac{\int_{[s,t)} \chi_F(r) \, \mathrm{d}\, g(r)}{g(t) - g(s)} = \lim_{t \to s^+} \frac{\mu_g([s,t) \cap F)}{g(t) - g(s)}.$$

For the other equality, consider the map $H : [a,b] \to \mathbb{R}$ defined as

$$H(t) = \int_{[a,t)} M(s) \cdot \chi_{I \setminus F} \, \mathrm{d}\, g(s).$$

Again, we have that $H \in \mathcal{AC}_g([a,b],\mathbb{R})$ since $M_0 = M \cdot \chi_{I \setminus F} \in \mathcal{L}^1_g([a,b),[0,+\infty))$. Thus there exists $F_1 \subset F_0$ such that $\mu_g(F_0 \setminus F_1) = 0$, $H'_g(s)$ exists for all $s \in F_1$ and

$$H'_g(s) = M(s) \cdot \chi_{I \setminus F}(s) = 0, \quad s \in F_1.$$

Hence, for every $s \in F_1$ we have that

$$0 = \lim_{t \to s^+} \frac{H(t) - H(s)}{g(t) - g(s)} = \lim_{t \to s^+} \frac{\int_{[s,t)} M_0(r) \, \mathrm{d}\, g(r)}{g(t) - g(s)}.$$

Now, since $F_1 \subset F_0$, it follows that

$$0 = \lim_{t \to s^+} \frac{\int_{[s,t)} M_0(\tau) \, \mathrm{d}\, g(\tau)}{g(t) - g(s)} \cdot \lim_{t \to s^+} \frac{g(t) - g(s)}{\mu_g([s,t) \cap F)} = \lim_{t \to s^+} \frac{\int_{[s,t)} M_0(\tau) \, \mathrm{d}\, g(\tau)}{\mu_g([s,t) \cap F)},$$

which concludes the proof. □

The following lemma is an adaptation of [80, Lemma 6.92]. In order to prove it, we need the following definition.

Definition 3.35. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function and $A \subset \mathbb{R}$. We define the g–*diameter *of A, and we denote it by $g\text{–}\operatorname{diam}(A)$, as*

$$g\text{–}\operatorname{diam}(A) = \sup\{|g(s) - g(t)| : s,t \in A\}.$$

Remark 3.36. In the particular case of $g = \mathrm{Id}$, we have that the definition of g–diameter of a set matches the definition of diameter of a set. Furtermore, in that case, we have that for any interval I, $\operatorname{diam}(I) = \sup I - \inf I$.

Lemma 3.37. *Let $\Phi : [a,b] \to \mathbb{R}$ be a map such that $\Phi'_g(t)$ exists for all $t \in E \subset [a,b] \setminus D_g$. If $m(\Phi(E)) = 0$, where m denotes Lebesgue's measure, then*

$$\Phi'_g(t) = 0, \quad g\text{–a.a. } t \in E.$$

Proof. Without loss of generality, we can assume that $E \cap (C_g \cup N_g) = \emptyset$. Define the sets

$$B_n = \left\{ t \in E : |\Phi(s) - \Phi(t)| > \frac{|g(s) - g(t)|}{n},\ s \in [a,b],\ 0 < s - t < \frac{1}{n} \right\}, \quad n \in \mathbb{N},$$

and $B = \{t \in E : \Phi'_g \neq 0\}$. Then $B = \bigcup_{n \in \mathbb{N}} B_n$. Indeed, if $t \in B$, then there exists $c > 0$ such that

$$c = |\Phi'_g(t)| = \lim_{s \to t^+} \left| \frac{\Phi(s) - \Phi(t)}{g(s) - g(t)} \right|.$$

Let $\delta > 0$ be such that for $0 < s - t < \delta$,

$$\left| \frac{\Phi(s) - \Phi(t)}{g(s) - g(t)} \right| > \frac{c}{2}.$$

Let $N \in \mathbb{N}$ be such that $1/N < \min\{\delta, c/2\}$. Then, for all $n \geq N$, if $0 < s - t < 1/n$, it holds that

$$\left| \frac{\Phi(s) - \Phi(t)}{g(s) - g(t)} \right| < \frac{c}{2} < \frac{1}{n},$$

and so $t \in B_n$ for some $n \in \mathbb{N}$. Conversely, if $t \in B_n$ for some $n \in \mathbb{N}$, then $t \in B$ as

$$|\Phi'_g(t)| = \lim_{s \to t^+} \left| \frac{\Phi(s) - \Phi(t)}{g(s) - g(t)} \right| > \frac{1}{n} > 0.$$

Hence, it is enough to show that $\mu_g(B_n) = 0$ for all $n \in \mathbb{N}$. Moreover, since each B_n can be covered by finitely many intervals of length less than $1/n$, it suffices to show that $\mu_g(J \cap B_n) = 0$ for every such interval J. Therefore, if we denote $A = J \cap B_n$, we need to show that $\mu_g(A) = 0$.

Let $\varepsilon > 0$. Since $m(\Phi(A)) = 0$ and the measure m is outer regular, there exists a family $\{J_k\}_{k=1}^{\infty}$ of open intervals such that

$$\Phi(A) \subset \bigcup_{k=1}^{\infty} J_k, \quad \sum_{k=1}^{\infty} \operatorname{diam}(J_k) < \frac{\varepsilon}{n}.$$

Let us denote $A_k = A \cap \Phi^{-1}(J_k)$. Then $A = \bigcup_{k=1}^{\infty} A_k$. Moreover, we claim that

$$g\text{–}\operatorname{diam}(A_k) \leq n \cdot \operatorname{diam}(\Phi(A_k)).$$

Indeed, the definition of B_n yields that for each pair $s, t \in A_k$, $s < t$,

$$0 < g(s) - g(t) = |g(s) - g(t)| < n|\Phi(s) - \Phi(t)| \leq n \cdot \operatorname{diam}(\Phi(A_k)),$$

and so, the inequality follows. Thus, if we prove that $\mu_g(A_k) \le g-\operatorname{diam}(A_k)$ we are done, since

$$\mu_g(A_k) \le \sum_{k=1}^{\infty} \mu_g(A_k) \le \sum_{k=1}^{\infty} g-\operatorname{diam}(A_k) \le n \sum_{k=1}^{\infty} \operatorname{diam}(\Phi(A_k)) \le n \sum_{k=1}^{\infty} \operatorname{diam}(J_k) < \varepsilon.$$

To show that $\mu_g(A_k) \le g-\operatorname{diam}(A_k)$, let us denote $a_k = \inf A_k$ and $b_k = \sup A_k$. We distinguish two cases: $a_k \in A_k$ or $a_k \notin A_k$.

Assume first that $a_k \in A_k$, then by definition of b_k, one can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A_k such that $\{x_n\}$ is nondecreasing and x_n converges to b_k. Hence,

$$g-\operatorname{diam}(A_k) = \sup\{g(t) - g(s) : t, s \in A_k,\ s < t\} \ge g(x_n) - g(a_k), \quad n \in \mathbb{N},$$

and so $g-\operatorname{diam}(A_k) \ge g(b_k) - g(a_k)$. Now, if $b_k \in A_k$, then $b_k \notin D_g$ and $A_k \subset [a_k, b_k]$ so

$$\mu_g(A_k) \le \mu_g([a_k, b_k]) = g(b_k^+) - g(a_k) = g(b_k) - g(a_k) \le g-\operatorname{diam}(A_k).$$

Otherwise $b_k \notin A_k$, so $A_k \subset [a_k, b_k)$ and

$$\mu_g(A_k) \le \mu_g([a_k, b_k)) = g(b_k) - g(a_k) \le g-\operatorname{diam}(A_k).$$

Assume now that $a_k \notin A_k$, then $a_k < s < t \le b_k$ and so $g(t) - g(s) \le g(b_k) - g(a_k^+)$. Therefore $g-\operatorname{diam}(A_k) \le g(b_k) - g(a_k^+)$. Moreover, $g-\operatorname{diam}(A_k) = g(b_k) - g(a_k^+)$. Indeed, let $\varepsilon' > 0$. Since g is left-continuous at b_k, there exists $\delta_1 > 0$ such that if $0 \le b_k - t < \delta_1$, then $g(b_k) - g(t) < \varepsilon'/2$. Since $b_k = \sup A_k$, there exists $t_0 \in A_k$ such that $0 \le b_k - t_0 < \delta_1$ and so $g(b_k) - g(t_0) < \varepsilon'/2$. On the other hand, by definition of $g(a_k^+)$, there exists $\delta_2 > 0$ such that if $0 < s - a_k < \delta_2$, then $g(s) - g(a_k^+) < \varepsilon'/2$. Since $a_k = \inf(A_k)$, there exists $s \in A_k$ such that $0 < s - a_k < \min\{\delta_2, t_0 - a_k\}$. Hence, there exist $s < t$, $s, t \in A_k$, such that

$$g(t) - g(s) > g(b_k) - \frac{\varepsilon'}{2} - g(a_k^+) - \frac{\varepsilon'}{2} = g(b_k) - g(a_k^+) - \varepsilon'.$$

Therefore $g-\operatorname{diam}(A_k) = g(b_k) - g(a_k^+)$. Again, if $b_k \in A_k$, then $b_k \notin D_g$ and $A_k \subset (a_k, b_k]$, so

$$\mu_g(A_k) \le \mu_g((a_k, b_k]) = g(b_k^+) - g(a_k^+) = g(b_k) - g(a_k^+) = g-\operatorname{diam}(A_k).$$

Otherwise, $b_k \notin A_k$, so $A_k \subset (a_k, b_k)$ and

$$\mu_g(A_k) \le \mu_g((a_k, b_k)) = g(b_k) - g(a_k^+) = g-\operatorname{diam}(A_k).$$ □

3.3 Stieltjes differential equations and their relations with other problems

In this section, and for the rest of this thesis, we focus on the study of differential equations with Stieltjes derivatives, or *Stieltjes differential equations*. In particular, given a nondecreasing and left–continuous function, $g : \mathbb{R} \to \mathbb{R}$, we will look at problems of the form

$$x'_g(t) = f(t, x(t)), \quad g\text{–a.a. } t \in [a, b) \tag{3.10}$$

with $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b) \times \mathbb{R}^n \to \mathbb{R}^n$.

Given the relation introduced at the beginning of the chapter between Δ–derivatives and Stieltjes derivatives, it is clear that Stieltjes equations can be used to study differential problems with Δ–derivatives. In what follows, following [33, 49, 54], we present some other problems that are equivalent to differential problems with Stieltjes derivatives, providing a better understanding of the utility of this type of problems. These equivalences are shown in the context of $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, but they can be extended to more complex spaces with the obvious changes. Moreover, we will establish such equivalences by showing that pointwise solutions of an equation of the form of (3.10) are pointwise solutions of the other problems, and vice versa.

3.3.1 Ordinary differential equations

First, we study the relation between differential equations with Stieltjes derivatives of the form

$$x_g'(t) = f(t, x(t)), \quad g\text{–a.a. } t \in I = [a, b), \tag{3.11}$$

for some $g : \mathbb{R} \to \mathbb{R}$ nondecreasing and left–continuous function, $a, b \in \mathbb{R}$, $a < b$, and $f : I \times \mathbb{R} \to \mathbb{R}$; and ordinary differential equations, namely

$$y'(t) = F(t, y(t)), \quad m\text{–a.a. } t \in J = [\widetilde{a}, \widetilde{b}), \tag{3.12}$$

for some $\widetilde{a}, \widetilde{b} \in \mathbb{R}$, $\widetilde{a} < \widetilde{b}$, and $F : J \times \mathbb{R} \to \mathbb{R}$. Here m denotes the Lebesgue measure.

It is obvious that every equation of the form of (3.12) defines a differential equation with Stieltjes derivates of the form of (3.11) since Stieltjes derivatives yield the usual derivative when we consider $g = \mathrm{Id}$. However, the interest of this section is to show that, under certain hypotheses, differential equations with Stieltjes derivatives can be studied as ordinary differential equations in the sense that there is a way to transform solutions of one of the problems into solutions of the other problem.

A simple way to establish this relation is to assume that $g \in \mathcal{C}^1(I, \mathbb{R})$ and $g' > 0$. In that case, for any $x : I \to \mathbb{R}$ such that $x_g'(t)$ exists, it follows that

$$\lim_{s \to t} \frac{x(s) - x(t)}{s - t} = \lim_{s \to t} \frac{x(s) - x(t)}{g(s) - g(t)} \lim_{s \to t} \frac{g(s) - g(t)}{s - t},$$

that is, $x'(t)$ exists and $x'(t) = x_g'(t) g'(t)$. Therefore, if x solves (3.11) on I then x solves (3.12) with $J = I$ and

$$F(t, x) = f(t, x) g'(t), \quad (t, x) \in J \times \mathbb{R}.$$

Conversely, assume that y solves (3.12) and let $c \in \mathcal{C}(J, \mathbb{R})$, $c > 0$. Then, we consider

$$g(t) = \int_{t_0}^{t} c(s) \mathrm{d}s, \quad t \in J.$$

In that case, we have that

$$\lim_{s \to t} \frac{y(s) - y(t)}{g(s) - g(t)} = \lim_{s \to t} \frac{y(s) - y(t)}{s - t} \lim_{s \to t} \frac{s - t}{g(s) - g(t)},$$

that is $x'_g(t)$ exists and $y'_g(t) = y'(t)/g'(t) = y'(t)/c(t)$. Thus, taking $I = J$ and

$$f(t,x) = F(t,x)c(t), \quad (t,x) \in I \times \mathbb{R},$$

we have that y solves (3.11). For the particular selection of $c = 1$ we obtain that $g(x) = x$, which yields the relation explained earlier.

In what follows, we present some of the results obtained in [49] where the authors proved that the same relation holds under more relaxed conditions on the derivator g. Moreover, thanks to these results, we obtain a way of solving differential equations with Stieltjes derivatives under the correct set of hypotheses.

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and continuous function. Without loss of generality, we assume that $g(\mathbb{R}) = \mathbb{R}$. If this is not, it suffices to consider the nondecreasing continuous map $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ given by

$$\widetilde{g}(t) = \begin{cases} g(a) + (x-a), & \text{if } x < a, \\ g(x), & \text{if } a \leq x < b, \\ g(b) + (x-b), & \text{if } x \geq b, \end{cases}$$

and note that $g = \widetilde{g}$ on $[a,b)$. In that case, equation (3.11) and equation (3.11) with g replaced by $\widetilde{g}$ are exactly the same.

In this context, we define the pseudo–inverse of g as the function $\gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(x) = \min\{t \in \mathbb{R} \,:\, g(t) = x\}, \quad x \in \mathbb{R}. \tag{3.13}$$

This definition is good. To prove it, just notice that g is continuous, nondecreasing and $g(\pm\infty) = \pm\infty$, which implies that

$$g^{-1}(\{x\}) = \{t \in \mathbb{R} \,:\, g(t) = x\} \tag{3.14}$$

is a compact interval –or even a singleton if $x \notin C_g \cup N_g$, since $D_g = \emptyset$, see Remark 2.48.

The most important properties of γ are gathered in the following statement. Some of them can be deduced from [44, Theorem 1.8].

Proposition 3.38. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and continuous function such that $g(\mathbb{R}) = \mathbb{R}$. If $\gamma : \mathbb{R} \to \mathbb{R}$ is defined as in* (3.13)*, then the following properties hold:*

1. *For all $x \in \mathbb{R}$, $g(\gamma(x)) = x$.*
2. *For all $t \in \mathbb{R}$, $\gamma(g(t)) \leq t$.*
3. *For all $t \in \mathbb{R}$, $t \notin C_g \cup N_g^+$, $\gamma(g(t)) = t$.*
4. *The map γ is strictly increasing.*
5. *The map γ is left–continuous everywhere and continuous at every $x \in \mathbb{R}$, $x \notin g(C_g)$.*

Proof. Properties 1 and 2 are a direct consequence of the definition (3.13) since

$$g(\gamma(x))) = g(\min\{t \in \mathbb{R} \,:\, g(t) = x\}) = x, \quad \text{for all } x \in \mathbb{R},$$

and

$$\gamma(g(t)) = \min\{s \in \mathbb{R} \,:\, g(s) = g(t)\} \leq t, \quad \text{for all } t \in \mathbb{R}.$$

To prove 3, note that for $t \in \mathbb{R} \setminus (C_g \cup N_g)$ the set (3.14) for $x = g(t)$ is the singleton $\{t\}$. Now, if $t \in N_g^-$, we have that $t = a_{n_0}$ for some $n_0 \in \mathbb{N}$. In that case, it is clear that $g(t) = g(s)$ for all $s \in (t, b_{n_0})$ and $g(s) < g(t)$ for all $s < t$, as any other case would lead to a contradiction with the definition of C_g. Thus, we have that

$$\gamma(g(t)) = \min\{s \in \mathbb{R} : g(t) = g(s)\} = \min(a_{n_0}, b_{n_0}) = a_{n_0} = t.$$

For statement 4, fix $x, y \in \mathbb{R}$, $x < y$. If $t \in g^{-1}(\{x\})$ and $s \in g^{-1}(\{y\})$ then $t < s$, for otherwise we would have $x = g(t) \geq g(s) = y$, which would be contradiction. Hence,

$$\gamma(x) = \min g^{-1}(\{x\}) < \min g^{-1}(\{y\}) = \gamma(y).$$

Finally, we prove 5. First, property 4 ensures that

$$\gamma(x^-) \leq \gamma(x) \leq \gamma(x^+) \quad \text{for all } x \in \mathbb{R}. \tag{3.15}$$

Assume, reasoning by contradiction, that $\gamma(x^-) < \gamma(x)$ for some $x \in \mathbb{R}$. Since γ is strictly increasing, we can fix τ such that

$$\gamma(y) < \tau < \gamma(x) \quad \text{for all } y < x. \tag{3.16}$$

Now we deduce from the monotonicity of g and property 1 that

$$y = g(\gamma(y)) \leq g(\tau) \leq g(\gamma(x)) = x \quad \text{for all } y < x,$$

which implies that $g(\tau) = x$. Now property 2 yields $\gamma(x) = \gamma(g(\tau)) \leq \tau$, which is a contradiction with (3.16). Hence γ is left–continuous everywhere.

We shall prove that γ is right–continuous at every $x \in \mathbb{R} \setminus g(C_g)$ from (3.15) and a similar contradiction argument. Assume that for one of those $x \in \mathbb{R}$ we can find τ such that

$$\gamma(x) < \tau < \gamma(y) \quad \text{for all } y > x.$$

Since g is nondecreasing we have

$$x = g(\gamma(x)) \leq g(\tau) \leq g(\gamma(y)) = y \quad \text{for all } y > x,$$

and therefore $x = g(\gamma(x)) = g(\tau)$. Since $\gamma(x) < \tau$, for any $t \in (\gamma(x), \tau)$ we have $g(t) = x$ and $t \in C_g$, which is a contradiction with the choice of x. □

We now have the necessary tools to reduce a Stieltjes differential equation to an ODE in a more general context than that of derivators belonging to $\mathcal{C}^1([a, b), \mathbb{R})$.

Theorem 3.39. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(\mathbb{R}) = \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}$ be defined as in* (3.13)*, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$. If $y : [g(a), g(b)) \longrightarrow \mathbb{R}$ is a solution of*

$$y'(s) = f(\gamma(s), y(s)), \quad s \in [g(a), g(b)) \setminus (E \cup g(C_g)),$$

for some $E \subset [g(a), g(b))$, then $x : [a, b) \to \mathbb{R}$ given by $x(t) = y(g(t))$ solves

$$x'_g(t) = f(t, x(t)), \quad t \in [a, b) \setminus (g^{-1}(E) \cup C_g \cup N_g).$$

In particular, $x = y \circ g$ solves the problem g–almost everywhere in $[a, b)$ provided that $g^{-1}(E)$ be a null g–measure subset of $[a, b)$.

Proof. Fix $t \in [a, b) \setminus (g^{-1}(E) \cup C_g \cup N_g)$. Then assertion 3 in Proposition 3.38 yields that $\gamma(g(t)) = t$ and, moreover $g(t) \in [g(a), g(b)) \setminus (E \cup g(C_g))$. Hence $y'(g(t))$ exists and

$$y'(g(t)) = f(\gamma(g(t)), y(g(t))) = f(t, x(t)).$$

Thus, it is enough to show that $x'_g(t)$ exists and equals $y'(g(t))$.

Fix $\varepsilon > 0$. Since $y'(g(t))$ exists, there exists $\widetilde{\delta} > 0$ such that

$$\left[z \in [g(a), g(b)],\ 0 < |z - g(t)| < \widetilde{\delta}\right] \implies \left|\frac{y(z) - y(g(t))}{z - g(t)} - y'(g(t))\right| < \varepsilon.$$

On the other hand, since g is continuous at t, there exists $\delta > 0$ such that

$$[s \in [a, b],\ |s - t| < \delta] \implies |g(s) - g(t)| < \widetilde{\delta}.$$

Now, Remark 2.48 ensures that if $0 < |s - t| < \delta$ then $0 < |g(s) - g(t)| < \widetilde{\delta}$. Hence, it follows that

$$[s \in [a, b],\ 0 < |s - t| < \delta] \implies \left|\frac{y(g(s)) - y(g(t))}{g(s) - g(t)} - y'(g(t))\right| < \varepsilon,$$

that is, $x'_g(t)$ exists and $x'_g(t) = y'(g(t))$. □

Remark 3.40. This result provides a way of obtaining explicit solutions for the differential problem with Stieltjes derivatives, provided we are able to solve the corresponding ODE.

Remark 3.41. Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function such that $g \in \mathcal{C}^1([a, b), \mathbb{R})$, $g([a, b)) = [a, b)$ and $g' > 0$ in $[a, b)$. In this case, g is a bijection and so $\gamma = g^{-1}$. Under these assumptions, we have that Theorem 3.39 provides the same information as the equivalence presented at the beginning of this section. That is, x solves

$$x'(t) = f(t, x)g'(t), \quad t \in [a, b), \tag{3.17}$$

if and only if $x \circ g^{-1}$ solves

$$y'(s) = f(g^{-1}(s), y(s)), \quad t \in [a, b). \tag{3.18}$$

Indeed, if x solves (3.17), we have that $x'(s)$ exists for all $s \in [a, b)$. Thus, we have that $(x \circ g^{-1})'(t)$ exists for all $t \in [a, b)$ and

$$(x \circ g^{-1})'(t) = x'(g^{-1}(t))(g^{-1})'(t) = f(g^{-1}(t), x(g^{-1}(t)))g'(g^{-1}(t))\frac{1}{g'(g^{-1}(t))},$$

and so $x \circ g^{-1}$ solves (3.18). Conversely, if $x \circ g^{-1}$ solves (3.18), we have that $(x \circ g^{-1})'(t)$ exists for all $t \in [a, b)$. Now, since $x = x \circ g^{-1} \circ g$, it follows that $x'(t)$ exists for all $t \in [a, b)$ and

$$x'(t) = (x \circ g^{-1})'(g(t))g'(t) = f(g^{-1}(g(t)), x(g^{-1}(g(t))))g'(t) = f(t, x(t))g'(t),$$

that is, x solves (3.17).

Next we prove the converse result of Theorem 3.39, thus showing the equivalence between the problems with usual derivatives and Stieltjes derivatives.

Theorem 3.42. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(\mathbb{R}) = \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}$ be defined as in* (3.13)*, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$. If $x : [a, b) \to \mathbb{R}$ is a solution of*

$$x'_g(t) = f(t, x(t)), \quad t \in [a, b) \setminus (E \cup C_g \cup N_g),$$

for some $E \subset [a, b)$, then $y : [g(a), g(b)) \to \mathbb{R}$ given by $y(s) = x(\gamma(s))$ solves

$$y'(s) = f(\gamma(s), y(s)), \quad s \in [g(a), g(b)) \setminus g(E \cup C_g \cup N_g).$$

In particular, $y = x \circ \gamma$ solves m–almost everywhere in $[g(a), g(b))$ provided that $g(E)$ be a null m–measure subset of $[g(a), g(b))$.

Proof. Fix $s \in [g(a), g(b)] \setminus g(E \cup C_g \cup N_g)$. Then there exists $u \in [a, b) \setminus (E \cup C_g \cup N_g)$ such that $g(u) = s$. Moreover, Remark 2.48 ensures that $g^{-1}(\{s\}) = \{u\}$ so $\gamma(s) = u \in [a, b) \setminus (E \cup C_g \cup N_g)$ and $x'_g(\gamma(s))$ exists. Furthermore,

$$x'_g(\gamma(s)) = f(\gamma(s), x(\gamma(s))) = f(\gamma(s), y(s)),$$

so, it is enough to show that $y'(s)$ exists and equals $x'_g(\gamma(s))$.

Fix $\varepsilon > 0$. Since $x'_g(\gamma(s))$ exists, there exists $\widetilde{\delta} > 0$ such that

$$\left[z \in [a, b], \ 0 < |z - \gamma(s)| < \widetilde{\delta}\right] \implies \left| \frac{x(z) - x(\gamma(s))}{g(z) - g(\gamma(s))} - x'_g(\gamma(s)) \right| < \varepsilon.$$

On the other hand, $s \in [g(a), g(b)] \setminus g(E \cup C_g \cup N_g)$, so assertion 5 in Proposition 3.38 ensures that γ is continuous at s. Hence, there exists $\delta > 0$ such that

$$[r \in [g(a), g(b)], \ |r - s| < \delta] \implies |\gamma(r) - g(s)| < \widetilde{\delta}.$$

Now, assertion 3 in Proposition 3.38 guarantees that if $0 < |s - t| < \delta$ then $0 < |g(s) - g(t)| < \widetilde{\delta}$. Hence, it follows that

$$[r \in [g(a), g(b)], \ 0 < |r - s| < \delta] \implies \left| \frac{x(\gamma(r)) - x(\gamma(s))}{g(\gamma(r)) - g(\gamma(s))} - x'_g(\gamma(s)) \right| < \varepsilon.$$

The result now follows by property 1 in Proposition 3.38. □

As a final comment for this section, note that the results in here are not valid for discontinuous derivators. However, if its set of discontinuity points is a discrete set, then we can argue "piece–by–piece" to obtain the general solution. That is, we can use the results in this section to solve the problem in each of the subintervals generated by D_g.

3.3.2 Impulsive differential equations

Let $J = \{t_k \,:\, k \in \mathbb{N}\}$ be a countable subset of an interval $[a,b)$ such that its set of accumulation points, J', has Lebesgue measure equal to zero. We now study the relation between an impulsive ordinary differential equation of the form

$$x'(t) = f(t, x(t)), \quad m\text{–a.a. } t \in [a,b) \setminus J, \tag{3.19}$$

$$x(t^+) = x(t) + I_t(x(t)), \quad t \in J, \tag{3.20}$$

with $f : [a,b) \times \mathbb{R} \to \mathbb{R}$, $I_t : \mathbb{R} \to \mathbb{R}$ and m the Lebesgue measure; and differential equations with Stieltjes derivatives of the form

$$x_g'(t) = F(t, x(t)), \quad g\text{–a.a. } t \in [a,b), \tag{3.21}$$

for some $g : \mathbb{R} \to \mathbb{R}$ nondecreasing and left–continuous and $F : [a,b) \times \mathbb{R} \to \mathbb{R}$. In particular, we shall show that both problems are equivalent in the sense that the set of solutions is the same for an appropiate selection of g and F. This is the idea presented in [33, 54].

To this end, let us define $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(t) = t + \sum_{\{k \in \mathbb{N} \,:\, t_k < t\}} 2^{-k}, \quad t \in \mathbb{R}, \tag{3.22}$$

where we assume that the sum takes the value zero when $\{k \in \mathbb{N} \,:\, t_k < t\} = \emptyset$. First of all, note that since the series $\sum_{n=1}^{\infty} 2^{-n}$ converges absolutely to 1, any reordering of such series also converges to 1. Hence, g is well–defined. Moreover, g is increasing since given $t, s \in \mathbb{R}$, $t < s$, we have that $\{k \in \mathbb{N} \,:\, t_k < t\} \subset \{k \in \mathbb{N} \,:\, t_k < s\}$, and so

$$g(t) = t + \sum_{\{k \in \mathbb{N} \,:\, t_k < t\}} 2^{-k} < s + \sum_{\{k \in \mathbb{N} \,:\, t_k < s\}} 2^{-k} = g(s).$$

Similarly, we have that g is left–continuous everywhere. Indeed, fix $t \in \mathbb{R}$. For any $s \in \mathbb{R}$, $s < t$, we have that

$$g(t) - g(s) = t - s + \sum_{\{k \in \mathbb{N} \,:\, s \leq t_k < t\}} 2^{-k}.$$

Thus,

$$\lim_{s \to t^-} (g(t) - g(s)) = \lim_{s \to t^+} \left(t - s + \sum_{\{k \in \mathbb{N} \,:\, s \leq t_k < t\}} 2^{-k} \right) = 0,$$

i.e. g is left–continuous at t. Using a similar argument, we can prove that g is continuous on $\mathbb{R} \setminus J$. Fix $t \in \mathbb{R} \setminus J$. In that case, for any $s \in \mathbb{R}$, $s > t$, we have that

$$g(s) - g(t) = s - t + \sum_{\{k \in \mathbb{N} \,:\, t \leq t_k < s\}} 2^{-k} = s - t + \sum_{\{k \in \mathbb{N} \,:\, t < t_k < s\}} 2^{-k},$$

since $t \neq t_k$, $k \in \mathbb{N}$. Hence,

$$\lim_{s \to t^-} (g(s) - g(t)) = \lim_{s \to t^-} \left(s - t + \sum_{\{k \in \mathbb{N} \,:\, t < t_k < s\}} 2^{-k} \right) = 0,$$

or equivalently, g is continuous from the right at $t \in \mathbb{R} \setminus J$. Finally, at every $t_k \in J$, we have

$$g(t_k^+) - g(t_k) = 2^{-k} > 0,$$

and, since g is continuous on $\mathbb{R} \setminus J$, $D_g = J$.

Remark 3.43. Note that in particular g is continuous at every point of J'. Thus, $\mu_g(J') = 0$.

With this choice of g we can prove the equivalence of problems (3.19)–(3.20) and (3.21) for the right choice of F.

Theorem 3.44. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be given by* (3.22) *and* $F : [a, b) \times \mathbb{R} \to \mathbb{R}$ *given by*

$$F(t, x) = \begin{cases} f(t, x), & \text{if } t \notin J, \\ \dfrac{I_t(x)}{g(t^+) - g(t)}, & \text{if } t \in J. \end{cases} \tag{3.23}$$

Then x *solves* (3.19)–(3.20) *if and only if* x *solves* (3.21) *with* (3.23).

Proof. First of all, note that for any $t \notin J \cup J'$ we have that g is continuous and

$$\lim_{s \to t} \frac{x(s) - x(t)}{g(s) - g(t)} = \lim_{s \to t} \frac{x(s) - x(t)}{s - t}.$$

Thus, for such t we have that $x'_g(t)$ exists if and only if $x'(t)$ does; and moreover both derivatives are equal. Furthermore, our assumptions and Remark 3.43 ensure that x solves (3.19) if, and only if x solves

$$x'_g(t) = F(t, x), \quad g\text{–a.a. } t \in [a, b) \setminus J.$$

Thus, all that is left to show is that x satisfies (3.20) if and only if x solves (3.21) for all $t \in J$. First, assume that x satisfies (3.20). In that case, we know that $x(t^+)$ exists for all $t \in J$, and so Remark 2.51 ensures that $x'_g(t)$ exists and

$$x'_g(t) = \frac{x(t^+) - x(t)}{g(t^+) - g(t)} = \frac{x(t) + I_t(x(t)) - x(t)}{g(t^+) - g(t)} = F(t, x(t)).$$

Conversely, assume x solves (3.21) for all $t \in J$. Then Remark 2.51 guarantees that $x(t^+)$ exists and

$$x(t^+) = x(t) + x'_g(t)(g(t^+) - g(t)) = x(t) + \frac{I_t(x(t))}{g(t^+) - g(t)}(g(t^+) - g(t)) = x(t) + I_t(x(t)),$$

which concludes the proof. □

As a final comment, it is common to see other formulations of impulsive differential equations. Some other formulations are obtain by considering (3.20) with $x(t)$ replaced by $x(t^-)$. In this case, as long as we are working with g–absolutely continuous solutions, the same problem with Stieltjes derivatives works as g–absolutely continuous functions are left–continuous everywhere (see Proposition 3.21). This will be the case in the work ahead.

3.3.3 Equations on time scales

Finally, we have a look at equations on time scales as differential equations with Stieltjes derivatives. The calculus on time scales was initiated by Stefan Hilger in [42] as a theory that could unify discrete and continuous analysis. Thus, if we show the mentioned equivalence between problems, we can use differential equations with Stieltjes derivatives as a way to study discrete and continuous problems.

In order to stablish the relation between the different equations, we need to introduce some concepts about time scales necessary to properly define the equations on time scales. The following definitions and results can be found in [10].

Definition 3.45. *A* time scale *is a nonempty closed subset of* $\mathbb{R}$.

This is, of course, the basic definition of this section. It is important to note that, in what follows, we will assume time scales to have the topology that they inherit from the usual topology in the real line.

Definition 3.46. *Let* $\mathbb{T}$ *be a time scale. The operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ *given by*

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} : s > t\}, & \textit{if} \quad (t, +\infty) \cap \mathbb{T} \neq \emptyset, \\ \sup \mathbb{T}, & \textit{otherwise}, \end{cases} \tag{3.24}$$

$$\rho(t) = \begin{cases} \sup\{s \in \mathbb{T} : s < t\}, & \textit{if} \quad (-\infty, t) \cap \mathbb{T} \neq \emptyset, \\ \inf \mathbb{T}, & \textit{otherwise}, \end{cases}$$

are known as forward jump operator *and* backward jump operator*, respectively. Note that* $\sigma(\mathbb{T})$*,* $\rho(\mathbb{T}) \subset \mathbb{T}$ *since* $\mathbb{T}$ *is closed.*

Let $t \in \mathbb{T}$*. We say that* t *is* right–scattered *if* $\sigma(t) > t$*; and similarly,* t *is said to be* left–scattered *if* $\rho(t) < t$*. If* t *is both right and left–scattered, we say that* t *is an* isolated point*. The point* t *is* right–dense *if* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$*; and analogously,* t *is* left–dense *if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$*. If* t *is both right and left–dense, we say that* t *is* dense.

In order to define the Hilger derivative of a function, we need to define another important set. Given a time scale $\mathbb{T}$, we define the set $\mathbb{T}^*$ as

$$\mathbb{T}^* = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < +\infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = +\infty. \end{cases}$$

Essentially, this set is the same as $\mathbb{T}$ except when $\mathbb{T}$ has a left–scattered maximum. In that case, the maximum is removed from the set.

Definition 3.47. *Let* $\mathbb{T}$ *be a time scale,* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^*$*. We define the* Hilger derivative *of* f *at* t*, and we denote it by* $f^\Delta(t)$*, as the real number such that for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *satisfying that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|, \quad s \in (t - \delta, t + \delta) \cap \mathbb{T}, \tag{3.25}$$

provided that such number exists. In that case, we say that f *is* Hilger differentiable at t*. If* f *is Hilger differentiable at every* $t \in \mathbb{T}^*$*, we say that* f *is* Hilger differentiable on $\mathbb{T}^*$.

This derivative has analogous properties to those of other derivatives previously shown in this work. Such properties include linearity, the product rule or the quotient rule. For more information on such properties, see [10]. We now present some other properties found in that same source that are necessary to reduce the equations on time scales to differential equations with Stieltjes derivatives.

Proposition 3.48. *Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^*$. Then:*

1. *If f is Hilger differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is right–scattered, then f is Hilger differentiable at t and*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

3. *If t is right–dense, then f is differentiable at t if and only if the following limit exists and is finite:*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

In that case, $f^{\Delta}(t)$ takes that value.

We can finally justify the equivalence between the equations on time scales and the differential equations with Stieltjes derivatives. In particular, we shall show there exists a way of extending the solutions of the first–order dynamic equation

$$x^{\Delta}(t) = f(t, x(t)), \quad t \in \mathbb{T} \cap [a, b) \tag{3.26}$$

for some time scale $\mathbb{T}$, $a, b \in \mathbb{T}$, $a < b$, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; to a solution of the problem

$$y_g'(t) = f(t, y(t)), \quad g\text{–a.a. } t \in [a, b), \tag{3.27}$$

for a convenient choice of a nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$. To this end, we shall follow the ideas of [33, 54, 79]. In particular, in [79] we find the definition of the derivator $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) = \begin{cases} a & \text{if} \quad t \le a, \\ \inf\{s \in \mathbb{T} : s \ge t\} & \text{if} \quad a < t \le b, \\ b & \text{if} \quad t > b. \end{cases} \tag{3.28}$$

It follows from the definition that g is nondecreasing and left–continuous, see [79, Lemma 4]. Moreover, it is clear that $g(t) = t$ for all $t \in \mathbb{T}$ and $C_g = \mathbb{R} \setminus \mathbb{T}$.

The next result, an adaptation of [54, Theorem 3.1], shows how to transform a solution of equation (3.26) into a solution of (3.27) assuming that the solution satisfies certain properties.

Theorem 3.49. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $a < b$, $x : \mathbb{T} \to \mathbb{R}$ be continuous from the left at every right–scattered point of $\mathbb{T}$ and g be given by* (3.28)*. If x solves* (3.26)*, then the map $y : \mathbb{R} \to \mathbb{R}$ defined as $y = x \circ g$ solves* (3.27).

Proof. Let $t \in \mathbb{T} \cap [a, b)$. Since $b \in \mathbb{T}$, it follows that $t \in \mathbb{T}^*$. Now, either t is right–scattered or t is right–dense. We study these two cases separately.

First, we assume that t is right–scattered. In this case, we know that x is left–continuous at t, so statement 2 in Proposition 3.48 ensures that $x^\Delta(t)$ exists and

$$x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Now, for any $s \in (t, \sigma(t))$ we have that $g(s) = \sigma(t)$. Thus, since $g(t) = t$, we have that

$$x^\Delta(t) = \frac{x(g(s)) - x(g(t))}{g(s) - g(t)}, \quad s \in (t, \sigma(t)). \tag{3.29}$$

On the other hand, since t is right–scattered, we have that g is discontinuous at t. Indeed, first note that, since $t \in \mathbb{T}^*$, we must have that

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

as the other definition of σ in (3.24) can only happen when $t = \sup \mathbb{T}^*$, which would contradict $t \in \mathbb{T}^* = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] = \mathbb{T} \setminus (\rho(t), t]$. Thus, since t is right–scattered, we have that $\sigma(t) > t$, and so

$$\lim_{s \to t^+} g(s) = \lim_{s \to t^+} \inf\{r \in \mathbb{T} : r \geq s\} = \inf\{r \in \mathbb{T} : r > t\} = \sigma(t) > t = g(t), \tag{3.30}$$

so $t \in D_g$. Hence, it follows from (3.29) that $y'_g(t)$ exists and

$$y'_g(t) = \lim_{s \to t^+} \frac{y(s) - y(t)}{g(s) - g(t)} = \lim_{s \to t^+} \frac{x(g(s)) - x(g(t))}{g(s) - g(t)} = x^\Delta(t).$$

Now, assume that t is right–dense. Proposition 3.48 ensures that x is Hilger differentiable at t if and only if the limit

$$\lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t} \tag{3.31}$$

exists. In that case, $x^\Delta(t)$ takes the value of that limit. Moreover, since g is continuous at t (see [79, Lemma 4]), we know that y is g–differentiable at t if and only if there exists

$$\lim_{s \to t} \frac{y(s) - y(t)}{g(s) - g(t)}, \tag{3.32}$$

and in that case, $y'_g(t)$ takes that value. Hence, it is enough to show that (3.31) exists if and only if (3.32) exists, and that in that case, both of them take the same value.

First, assume that (3.31) exists. Since t is right–dense there exist sequences $\{t_n\}_{n \in \mathbb{N}}$ such that $\{t_n\}_{n \in \mathbb{N}}$ converges to t and $g(t_n) \neq t$, $n \in \mathbb{N}$. Indeed, we have that

$$t = g(t) = \inf\{r \in \mathbb{T} : r \geq t\} < \inf\{r \in \mathbb{T} : r \geq s\} = g(s), \quad s > t,$$

so we can at least construct one of those sequences. Now, take an arbitrary sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\{t_n\}_{n\in\mathbb{N}}$ converges to t and $g(t_n) \neq t$, $n \in \mathbb{N}$. Since g is continuous at t, it follows that $\{g(t_n)\}_{n\in\mathbb{N}}$ converges to $g(t) = t$. Now, we know that $g(t_n) \in \mathbb{T} \setminus \{t\}$, and so

$$\lim_{n\to\infty} \frac{y(t_n) - y(t)}{g(t_n) - g(t)} = \lim_{n\to\infty} \frac{x(g(t_n)) - x(g(t))}{g(t_n) - g(t)} = \lim_{\substack{s\to t\\ s\in\mathbb{T}}} \frac{x(s) - x(t)}{s - t}.$$

Since the sequence was arbitrarily chosen, it follows that (3.32) exists and coincides with (3.31).

Conversely, assume that (3.32) exists. In that case, consider a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{T}\setminus\{t\}$ such that $\{t_n\}_{n\in\mathbb{N}}$ converges to t. It follows that (3.31) exists since

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \lim_{n\to\infty} \frac{x(g(t_n)) - x(g(t))}{g(t_n) - g(t)} = \lim_{n\to\infty} \frac{y(t_n) - y(t)}{g(t_n) - g(t)} = \lim_{s\to t} \frac{y(s) - y(t)}{g(s) - g(t)}.$$

In conclusion, we have proven that $y_g'(t)$ exists for all $t \in \mathbb{T} \cap [a, b)$ and

$$y_g'(t) = x^{\Delta}(t) = f(t, x(t)) = f(t, x(g(t)) = f(t, y(t)).$$

To conclude this proof, it is enough to note that $[a, b) \setminus \mathbb{T} = [a, b) \cap C_g$, and as a consequence, it has g–measure zero. □

Remark 3.50. Note that within the proof of this result, we have proven that x is Hilger differentiable at a point $t \in \mathbb{T}^*$ if and only if x is g–differentiable at t, which is essentially the statement of [54, Theorem 3.1]. Note, however, that the statement of that result is presented for $\mathbb{T}$ instead. It is unclear whether the result is true for the whole $\mathbb{T}$ as the proof is based on Proposition 3.48, and it is never discussed for the points of $\mathbb{T} \setminus \mathbb{T}^*$.

Now, we can prove the converse result. That is, given a solution of (3.27) we can construct a solution of (3.26) by considering the restriction of the function.

Theorem 3.51. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $a < b$, $g : \mathbb{R} \to \mathbb{R}$ be given by* (3.28) *and $y : \mathbb{R} \to \mathbb{R}$ be a g–continuous function. If y solves*

$$y_g'(t) = f(t, y(t)), \quad t \in [a, b) \cap \mathbb{T}, \tag{3.33}$$

then $x = y|_{[a,b)\cap\mathbb{T}}$ solves (3.26)*.*

Proof. Let $t \in [a, b) \cap \mathbb{T}$. Recall that this implies that $t \in \mathbb{T}^*$. We distinguish two cases: t is right–scattered and t is right–dense.

First, assume that t is right–scattered. Recall that this implies that $t \in D_g$, and so

$$f(t, y(t)) = y_g'(t) = \frac{y(t^+) - y(t)}{g(t^+) - g(t)}.$$

Moreover, since t is right–scattered, we have that g is constant on $(t, \sigma(t))$. Therefore, since y is g–continuous, it follows that y is constant on $(t, \sigma(t))$. Thus, since y is left–continuous, it follows that $y(t^+) = y(\sigma(t))$ and as pointed out in (3.30) we have that $g(t^+) = \sigma(t)$. Moreover $g(t) = t$, so

$$f(t, y(t)) = \frac{y(t^+) - y(t)}{g(t^+) - g(t)} = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}.$$

Now, noting that $t, \sigma(t) \in \mathbb{T}$ and recalling Proposition 3.48 we obtain that

$$f(t, x(t)) = f(t, y(t)) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t} = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t} = x^{\Delta}(t).$$

On the other hand, if t is right–dense, we need to prove that $x^{\Delta}(t)$ exists and equals $f(t, x(t))$. Equivalently, Proposition 3.48 ensures that we can prove that the following limit exists,

$$\lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t}, \tag{3.34}$$

and equals $f(t, x(t))$.

Since t is right–dense, we have $g(s) > g(t)$ for $s \in [a, b] \cap \mathbb{T}$, $s > t$. Hence, for every $s \in (t, b] \cap \mathbb{T}$, we have

$$\frac{x(s) - x(t)}{s - t} = \frac{y(s) - y(t)}{g(s) - g(t)},$$

which implies that the following limit exists and

$$\lim_{\substack{s \to t^+ \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t} = \lim_{\substack{s \to t^+ \\ s \in \mathbb{T}}} \frac{y(s) - y(t)}{g(s) - g(t)} = y'_g(t) = f(t, y(t)) = f(t, x(t)).$$

If t is left–scattered, then this is equivalent to (3.34) and we are done. If, on the contrary, t is left–dense, it follows that g is continuous, and moreover we have $g(s) < g(t)$ for $s \in [a, b] \cap \mathbb{T}$, $s < t$. Hence, for every $s \in [a, t) \cap \mathbb{T}$, we have

$$\frac{x(s) - x(t)}{s - t} = \frac{y(s) - y(t)}{g(s) - g(t)},$$

which implies that the following limit exists and

$$\lim_{\substack{s \to t^- \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t} = \lim_{\substack{s \to t^- \\ s \in \mathbb{T}}} \frac{y(s) - y(t)}{g(s) - g(t)} = y'_g(t) = f(t, y(t)) = f(t, x(t)),$$

which concludes the proof. □

Remark 3.52. Note that this result does not establish the equivalence between problems (3.26) and (3.27). This is because in the latter, the equation is satisfied g–a.e. in $[a, b)$, which might exclude some points of $\mathbb{T}$. Therefore, the problems are equivalent provided (3.33) is satisfied.

As a final comment on this equivalence, note that we have always assumed some continuity hypotheses on Theorems 3.49 and 3.51. However, as in the case of impulsive differential equations, as long as we consider g–absolutely continuous solutions of differential equations with Stieltjes derivatives, these hypotheses are satisfied.

Chapter 4

Stieltjes differential problems with a single derivator

As pointed out in Chapter 3, differential problems with Stieltjes derivatives provide a unified framework for the study of impulsive differential equations and equations on time scales. This is done by considering the appropiate derivator in each case. In this chapter, we turn our attention to the study of differential problems with Stieltjes derivatives with a single derivator. That is to say that we are assuming that there is only one set of impulse points or time scale at play. Later, in Chapter 5 we will deal with impulsive differential equations with different sets of impulses and equations on several time scales by considering differential problems with Stieltjes derivatives with respect to a family of functions.

Here, given a nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$, we will consider three types of differential problems following the results obtained in [48,49,51,53,59]. First, we will focus on the study of initial value problems of the form

$$x_g'(t) = f(t, x(t)), \quad g\text{–a.a. } t \in [t_0, t_0 + T), \quad x(t_0) = x_0, \tag{4.1}$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $x_0 \in \mathbb{R}^n$ and $f : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathbb{R}^n$. Later, we will look at a more complex formulation of the initial value problem, where the function defining the problem is allowed to consider functional arguments. Specifically, we look at the problem

$$x_g'(t) = f(t, x(t), x), \quad g\text{–a.a. } t \in [t_0, t_0 + T), \quad B(x(t_0), x) = 0,$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $f : [t_0, t_0 + T) \times \mathbb{R} \times X \to \mathbb{R}$ and $B : \mathbb{R} \times X \to \mathbb{R}$ for an adequate Banach space, X. Finally, we also consider inclusions with Stieltjes derivatives, i.e. problems of the form

$$x_g'(t) \in F(t, x(t)) \quad g\text{–a.a. } t \in [t_0, t_0 + T), \quad x(t_0) = x_0,$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $x_0 \in \mathbb{R}^n$ and $F : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all subsets of $\mathbb{R}^n$.

Note that the problems that we are going to study in this chapter are defined on intervals of the form $[t_0, t_0 + T)$ rather than on the corresponding closed interval as they were studied in most of the mentioned papers. Let us reflect on why that is. At the extremal point $t_0 + T$ two things can occur: either $t_0 + T \notin D_g$ or $t_0 + T \in D_g$. If $t_0 + T \notin D_g$, the set $\{t_0 + T\}$ has g–measure zero, so solving the problem on $[t_0, t_0 + T)$ is equivalent to solving it on the closed interval. However, if $t_0 + T \in D_g$, the g–derivative of a function defined on $[t_0, t_0 + T]$ at the point $t_0 + T$ is not defined as $t_0 + T$ as it is not an accumulation point of the corresponding function in Remark 3.2. Thus, solving the problem is only possible in $[t_0, t_0+T)$ in this case, which explains why we consider the problems on that interval.

The chapter is structured as follows. In Section 4.1 we study initial value problems with Stieltjes derivatives. In particular, in Section 4.1.1 we obtain explicit solutions for some particular equations, such as the linear equation. Next, in Section 4.1.2, we move on to the study of existence and uniqueness of solution for initial value problems, exploring separately the conditions for existence and the conditions for uniqueness. These conditions are later combined to obtain results on the existence and uniqueness of solution. In Section 4.1.3 we restrict ourselves to the context of the real line, and explore the concepts of lower and upper solutions. After that, we study the existence of extremal solutions, first between a pair of well–ordered lower and upper solutions, and then, in general, where we look for conditions ensuring that the supremum of lower solutions and the infimum of upper solutions are the extremal solutions. In Section 4.2 we turn our attention to a more complex problem: Stieltjes differential problems with functional arguments. This is done in the context of the real line as the study of this type of problems is based on the results in Section 4.1.3. Finally, in Section 4.3, we focus on the study of differential inclusions involving g–derivatives, where we obtain some new results for these problems even in the context of the usual derivative. Throughout these sections, we include some examples of real world applications of the obtained results for the corresponding Stieltjes equations.

4.1 Initial value problem

In this first section we turn our attention to the study of initial value problems. Specifically, given a nondecreasing and left–continuous map $g:\mathbb{R}\to\mathbb{R}$, we will study the initial value problem

$$x'_g(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{4.2}$$

with $t_0, T\in\mathbb{R}$, $T>0$, $X\subset\mathbb{R}^n$, $x_0\in X$ and $f:[t_0,t_0+T)\times X\to\mathbb{R}^n$. For simplicity, we shall denote $I_\tau=[t_0,t_0+\tau)$, $\tau\in(0,T]$, and $I=[t_0,t_0+T)$. Similarly, we denote by $\overline{I}_\tau$ and $\overline{I}$ the closure in $(\mathbb{R},\tau_u)$ of I_τ and I, respectively. That is, $\overline{I}_\tau=[t_0,t_0+\tau]$, $\tau\in(0,T]$, and $\overline{I}=[t_0,t_0+T]$. With this notation, we introduce the definition of solution of (4.2).

Definition 4.1. *A* solution *of the initial value problem* (4.2) *on an interval I_τ, $\tau\in(0,T]$, is a function $x\in\mathcal{AC}_g(\overline{I}_\tau,\mathbb{R}^n)$ such that $x(t_0)=x_0$, $x(t)\in X$ for all $t\in I_\tau$ and*

$$x'_g(t) = f(t, x(t)), \quad g\text{–a.a. } t\in I_\tau.$$

If $\tau=T$, we say that x is a global solution *of* (4.2)*; otherwise, i.e. if $\tau\in(0,T)$, we say that x is a* local solution *of* (4.2)*.*

Remark 4.2. Note that the Fundamental Theorem of Calculus ensures that the derivative of a solution on I_τ exists for g–almost all $t\in I_\tau$, thus giving sense to Definition 4.1. Furthermore, it follows that x is a solution of (4.2) on I_τ, $\tau\in(0,T]$, if and only if $x\in\mathcal{AC}_g(\overline{I}_\tau,\mathbb{R}^n)$ and it solves

$$x(t) = x_0 + \int_{[t_0,t)} f(s, x(s))\,\mathrm{d}\,g(s), \quad t\in\overline{I}_\tau.$$

Moreover, a solution of (4.2) on I_τ, $\tau \in (0, T]$, satisfies

$$x(t^+) = x(t) + f(t, x(t))\Delta g(t), \quad \text{for all } t \in I_\tau \text{ such that } x'_g(t) \text{ exists.} \tag{4.3}$$

At this point, we need to make an important comment about Definition 4.1. Although we are talking about solving the problem on intervals of the form $[t_0, t_0 + \tau)$, $\tau \in (0, T]$, the solutions are still required to be defined on its closure, $[t_0, t_0 + \tau]$, which might seem odd. The main reason behind this is that the Fundamental Theorem of Calculus is presented for g–absolutely continuous functions defined on the closed intervals. There is a way to consider solutions defined on $[t_0, t_0 + \tau)$ by considering a new family of functions introduced in the next definition.

Definition 4.3. *Let $a, b \in \mathbb{R}$, $a < b$, and $g : \mathbb{R} \to \mathbb{R}^n$ be a nondecreasing and left–continuous. We say that a map $x : [a, b) \to \mathbb{R}$ admits a g*–absolutely continuous extension to $[a, b]$ *if the limit $\lim_{t\to b^-} x(t)$ exists and the map $\widetilde{x} : [a, b] \to \mathbb{R}^n$ defined as*

$$\widetilde{x}(t) = \begin{cases} x(t), & \text{if } t \in [a, b), \\ x(b^-), & \text{if } t = b, \end{cases} \tag{4.4}$$

is g–absolutely continuous on $[a, b]$. We denote the set of all functions on $[a, b)$ that admit a g–absolutely continuous extension to $[a, b]$ by $E\mathcal{AC}_g([a, b], \mathbb{R}^n)$.

Definition 4.3 allows us to consider a different definition of a solution of (4.2) similar to that of Definition 4.1. In this alternative definition, instead of requiring the function to belong to $\mathcal{AC}_g(\overline{I}_\tau, \mathbb{R}^n)$, the function lies in $E\mathcal{AC}_g(\overline{I}_\tau, \mathbb{R}^n)$. However, these two definitions are equivalent as there is a bijection between the set of g–absolutely continuous functions on a closed interval and the set of functions that admit a g–absolutely continuous extension to that same interval. We present this information in the following result.

Proposition 4.4. *Let $a, b \in \mathbb{R}$, $a < b$, and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function. Then, there exists a bijection between the sets $\mathcal{AC}_g([a, b], \mathbb{R}^n)$ and $E\mathcal{AC}_g([a, b], \mathbb{R}^n)$.*

Proof. Consider the map $F : \mathcal{AC}_g([a, b], \mathbb{R}^n) \to E\mathcal{AC}_g([a, b], \mathbb{R}^n)$ given by $F(x) = x|_{[a,b)}$. Note that F is well–defined. Indeed, let $x \in \mathcal{AC}_g([a, b], \mathbb{R}^n)$. Proposition 3.21 ensures that the limit $\lim_{t\to b^-} x(t)$ exists and equals $x(b)$. Therefore, the limit $\lim_{t\to b^-} x|_{[a,b)}(t)$ also exists and, furthermore, the corresponding extension of $x|_{[a,b)}$ defined as in (4.4) is the original map x, which is g–absolutely continuous on $[a, b]$. Thus T is well–defined.

Define the map $\widetilde{F} : E\mathcal{AC}_g([a, b], \mathbb{R}^n) \to \mathcal{AC}_g([a, b], \mathbb{R}^n)$ as $\widetilde{F}(x) = \widetilde{x}$ where $\widetilde{x}$ is the g–absolutely continuous extension of x to $[a, b]$ in (4.4). The previous argument shows that $\widetilde{F} \circ F(x) = x$, $x \in \mathcal{AC}_g([a, b], \mathbb{R}^n)$. Therefore, it is enough to show that $F \circ \widetilde{F}(x) = x$, $x \in E\mathcal{AC}_g([a, b], \mathbb{R}^n)$. To see that, it suffices to note that the restriction to $[a, b)$ of the extension in (4.4) yields the original function. Therefore, the map F is a bijection between the two sets. □

Proposition 4.4 shows that the two definitions of solutions are equivalent. More specifically, it provides a way to obtain a solution in the sense of Definition 4.1 from a solution defined strictly on $[t_0, t_0 + T)$, and vice versa. For simplicity, we will still use Definition 4.1 as our definition of solution for its direct connections with the Fundamental Theorem of Calculus and other interesting properties of g–absolutely continuous functions.

With a clear definition of solution, we now study some conditions to obtain solutions in a "classical" sense, i.e solutions whose derivative is also continuous. Naturally, here we shall consider continuity in the sense of Definition 3.17. In order to study these type of solutions, we introduce the following interesting concept introduced in [33, Definition 7.7].

Definition 4.5. *Let $f : C \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $s \in \mathbb{R}$, $y \in \mathbb{R}^n$ be such that $(t, x) \in C$. We say that f is* $(g \times \mathrm{Id})$–continuous at (t, x) *if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$[(s, y) \in C,\ |g(t) - g(s)| < \delta,\ \|x - y\| < \delta] \ \Rightarrow\ \|f(t, x) - f(s, y)\| < \varepsilon.$$

We say that f is $(g \times \mathrm{Id})$–continuous in C *if it is $(g \times \mathrm{Id})$–continuous at every $(t, x) \in C$.*

Notice that $(g \times \mathrm{Id})$–continuity reduces to continuity in the usual sense when g is the identity map. Furthermore, observe that $(g \times \mathrm{Id})$–continuity on a set $C \subset \mathbb{R}^{n+1}$ is equivalent to continuity with respect to the product topology in C, $\tau_g \times \tau_u$, where τ_g is the g–topology in $\mathbb{R}$ and τ_u is the usual topology in $\mathbb{R}^n$.

In order to show that $(g \times \mathrm{Id})$–continuity is enough to obtain solutions in an analogous sense to the classical one, we need the following result which is a slight modification of [59, Lemma 2.5] to obtain a more general result.

Lemma 4.6. *Let $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(g \times \mathrm{Id})$–continuous function. If $x : A \to B$ is g–continuous, then $f(\cdot, x(\cdot))$ is g–continuous on A. Moreover, if A is a Borel set, then $f(\cdot, x(\cdot))$ is Borel measurable, and as a consequence, g–measurable.*

Proof. Fix $t \in A$ and $\varepsilon > 0$. Denote $u = x(t)$. Since f is $(g \times \mathrm{Id})$–continuous at (t, u), there exists $\delta_1 > 0$ such that

$$\|f(t, u) - f(s, v)\| < \varepsilon, \quad \text{for all } (s, v) \in A \times B,\ |g(t) - g(s)| < \delta_1,\ \|u - v\| < \delta_1].$$

Now, since x is g–continuous at t there exists $\delta_2 > 0$ such that for all $s \in A$ such that $|g(t) - g(s)| < \delta_2$, then $\|u - x(s)\| < \delta_1$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\|f(t, x(t)) - f(s, x(s))\| < \varepsilon, \quad \text{for all } s \in A,\ |g(t) - g(s)| < \delta$$

Therefore, $f(\cdot, x(\cdot))$ is g–continuous. The rest of the result follows from Proposition 3.22 and the extra hypothesis. □

With this result in mind, we can prove that "everywhere" solutions have g–continuous derivatives when f is $(g \times \mathrm{Id})$–continuous. By an "everywhere" solution we mean a function that solves the initial value problem for all points of an interval except those of C_g, as the g–derivative is not defined in such points.

Corollary 4.7. *Let $\tau \in (0, T]$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(g \times \mathrm{Id})$–continuous function on $I_\tau \times \mathbb{R}^n$ and $x \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R}^n)$ be such that $x(t_0) = x_0$ and*

$$x'_g(t) = f(t, x(t)), \quad \textit{for all } t \in I_\tau \setminus C_g.$$

Then x'_g is g–continuous on $I_\tau \setminus C_g$.

Proof. Given that x is g–absolutely continuous on $\overline{I}_\tau$, we have that x is g–continuous on $\overline{I}_\tau$, see Remark 3.25. Thus Lemma 4.6 yields that $f(\cdot, x(\cdot))$ is g–continuous and, since x'_g is defined on $I_\tau \setminus C_g$, the result follows. □

Recall that for $g = \mathrm{Id}$, the concept of $(g \times \mathrm{Id})$–continuity yields the usual continuity in $\mathbb{R}^{n+1}$. Moreover, in that case we have that $C_g = \emptyset$, so Corollary 4.7 guarantees that the solutions of the initial value problem with the usual derivative have continuous derivatives as long as the function defining the problem is continuous. Therefore, Corollary 4.7 is an extension of a known fact for ordinary differential equations.

Note, however, that finding "everywhere" solutions is not a trivial task. The following result [33, Proposition 7.6] shows that, under certain hypothesis on the map f, a solution in the sense of Definition 4.1 is in fact an "everywhere" solution. For a proof on this result, see Proposition 5.33 where we prove a more general result.

Proposition 4.8. *Let $x : \overline{I}_\tau \longrightarrow \mathbb{R}$, $\tau \in (0, T]$, be a solution of* (4.2). *If $f(\cdot, x(\cdot))$ is g–continuous on I_τ, then*

$$x'_g(t) = f(t, x(t)) \quad \textit{for all } t \in I_\tau \setminus C_g.$$

Thus, the following result follows directly from Lemma 4.6, Corollary 4.7 and Proposition 4.8. Essentially, this result ensures that the "classical" concept of solution can be obtained from Definition 4.1 provided that f satisfies the corresponding continuity condition.

Theorem 4.9. *Let $\tau \in (0, T]$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(g \times \mathrm{Id})$–continuous function on $I_\tau \times \mathbb{R}^n$ and $x : \overline{I}_\tau \to \mathbb{R}^n$ be a solution of* (4.2). *Then*

$$x'_g(t) = f(t, x(t)) \quad \textit{for all } t \in I_\tau \setminus C_g,$$

and x'_g is g–continuous on $I_\tau \setminus C_g$.

4.1.1 Explicit solutions

In this part of the thesis, we shall show how to obtain explicit solutions for some particular inital value problems in the context of the real line. First, we have a look at the linear equation

$$x'_g(t) + d(t)x(t) = h(t), \quad x(t_0) = x_0,$$

in its homogeneous formulation, i.e. with $h = 0$, and later in the general case presented above. Later, we turn our attention to the study of the separable initial value problem

$$x'_g(t) = c(t)f(x(t)), \quad x(t_0) = x_0,$$

which represents a more general case of the homogeneous linear equation. For the linear problem, we are able obtain solutions under some conditions on the maps d and h. However, for the problem with separatable variables, we are required to consider some conditions on the derivator as well.

Homogeneous linear equation

We start our quest for explicit solutions of some initial value problems by focusing on the particular case of the homogeneous linear differential equation with Stieltjes derivatives. Specifically, given a nondecreasing and left–continuous function, $g:\mathbb{R}\to\mathbb{R}$, we will be looking for the solutions of the equation

$$x_g'(t)=c(t)x(t),\quad x(t_0)=x_0, \tag{4.5}$$

with $c: I\to\mathbb{R}$ and $x_0\in\mathbb{R}$. Note that we are working on the context of $\mathbb{R}$. We do not have any information about the more general setting of $\mathbb{R}^n$. Here, we present some of the results available in [33, 58].

Before introducing the explicit solution of (4.5), let us motivate its definition. A first reasonable guess for a solution for (4.5) on I would be to consider, under the assumption that $c\in\mathcal{L}_g^1(I,\mathbb{R})$, the map

$$x(t)=x_0\exp\left(\int_{[t_0,t)}c(s)\,\mathrm{d}\,g(s)\right),\quad t\in\overline{I}, \tag{4.6}$$

as this is the solution for $g=\mathrm{Id}$. Let us briefly explain why this cannot be, in general, a solution of (4.5).

By definition, we would have that $x\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ and $x(t_0)=x_0$. Furthermore, one can easily check using the chain rule, Proposition 3.15, that $x'(t)=c(t)x(t)$ for all $t\in I\setminus D_g$. However, at a point $t\in I\cap D_g$ we have that

$$\begin{aligned}
x_g'(t)&=\lim_{s\to t^+}x_0\frac{\exp\left(\int_{[t_0,s)}c(r)\,\mathrm{d}\,g(r)\right)-\exp\left(\int_{[t_0,t)}c(r)\,\mathrm{d}\,g(r)\right)}{g(s)-g(t)}\\
&=\lim_{s\to t^+}x_0\frac{\exp\left(\int_{[t_0,t)}c(r)\,\mathrm{d}\,g(r)\right)\left(\exp\left(\int_{[t,s)}c(r)\,\mathrm{d}\,g(r)\right)-1\right)}{g(s)-g(t)}\\
&=x(t)\lim_{s\to t^+}\frac{\exp\left(\int_{[t,s)}c(r)\,\mathrm{d}\,g(r)\right)-1}{g(s)-g(t)}.
\end{aligned}$$

Now, given that the exponential map is continuous, it follows that

$$\lim_{s\to t^+}\exp\left(\int_{[t,s)}c(r)\,\mathrm{d}\,g(r)\right)=\exp\left(\int_{\{t\}}c(r)\,\mathrm{d}\,g(r)\right)=\exp(c(t)\Delta g(t)).$$

Thus, for $t\in I\cap D_g$,

$$\lim_{s\to t^+}\frac{\exp\left(\int_{[t,s)}c(r)\,\mathrm{d}\,g(r)\right)-1}{g(s)-g(t)}=\frac{\exp(c(t)\Delta g(t))-1}{\Delta g(t)},$$

which is not, in general, equal to $c(t)$. Therefore, the map x in (4.6) cannot be a solution of (4.5). However, in that reasoning we can see that, by modifying the function c accordingly at the points of D_g, we can obtain a solution. The next result, which is an adjustment of [33, Lemma 6.2], introduces the corresponding changes on the map c.

Lemma 4.10. *Let $a, b \in \mathbb{R}$, $a < b$. Given a map $c \in \mathcal{L}^1_g([a,b),\mathbb{R})$ such that*

$$c(t)\Delta g(t) > -1, \quad t \in [a,b) \cap D_g, \tag{4.7}$$

and

$$\sum_{t\in[a,b)\cap D_g} \left|\log\left(1 + c(t)\Delta g(t)\right)\right| < \infty, \tag{4.8}$$

the map $\widetilde{c} : [a,b) \to \mathbb{R}$ defined as

$$\widetilde{c}(t) = \begin{cases} c(t) & \text{if } t \in [a,b)\backslash D_g, \\ \dfrac{\log\left(1 + c(t)\Delta g(t)\right)}{\Delta g(t)} & \text{if } t \in [a,b) \cap D_g, \end{cases} \tag{4.9}$$

belongs to $\mathcal{L}^1_g([a,b),\mathbb{R})$.

Proof. It is clear that $\widetilde{c}$ is g–measurable since $\widetilde{c} = c$ on $[a,b)\backslash D_g$ and D_g is countable. Moreover,

$$\begin{aligned} \int_{[a,b)} |\widetilde{c}(s)| \,\mathrm{d}\, g(s) &= \int_{[a,b)\backslash D_g} |\widetilde{c}(s)| \,\mathrm{d}\, g(s) + \int_{[a,b)\cap D_g} |\widetilde{c}(s)| \,\mathrm{d}\, g(s) \\ &= \int_{[a,b)\backslash D_g} |c(s)| \,\mathrm{d}\, g(s) + \sum_{t\in[a,b)\cap D_g} \int_{\{t\}} |\widetilde{c}(s)| \,\mathrm{d}\, g(s) \\ &= \int_{[a,b)\backslash D_g} |c(s)| \,\mathrm{d}\, g(s) + \sum_{t\in[a,b)\cap D_g} \left|\log\left(1 + c(t)\Delta g(t)\right)\right|. \end{aligned}$$

Since $c \in \mathcal{L}^1_g([a,b),\mathbb{R})$ and it satisfies (4.8), it follows that the integral must be finite. □

Lemma 4.10 allows us to establish the following definition of an exponential map that, as we will see in Theorem 4.13, solves (4.5). This definition was originally presented in [33, Definition 6.1].

Definition 4.11. *Let $a, b \in \mathbb{R}$, $a < b$, and $c \in \mathcal{L}^1_g([a,b),\mathbb{R})$ satisfy* (4.7) *and* (4.8). *We define the map $e_c(\cdot, a) : [a,b] \to (0,+\infty)$ as*

$$e_c(t,a) := \exp\left(\int_{[a,t)} \widetilde{c}(s) \,\mathrm{d}\, g(s)\right), \quad t \in [a,b], \tag{4.10}$$

where $\widetilde{c}$ is the modified function in (4.9).

Remark 4.12. Lemma 4.10 ensures that the map $e_c(\cdot, a)$ is well–defined. Moreover, for any $t \in [a, b) \cap D_g$ we have that

$$\begin{aligned}
e_c(t^+, a) &= \lim_{s \to t^+} \exp\left(\int_{[a,s)} \widetilde{c}(s)\, \mathrm{d}\, g(s) \right) \\
&= \lim_{s \to t^+} \exp\left(\int_{[a,t)} \widetilde{c}(s)\, \mathrm{d}\, g(s) + \int_{[t,s)} \widetilde{c}(s)\, \mathrm{d}\, g(s) \right) \\
&= \lim_{s \to t^+} \left(\exp\left(\int_{[a,t)} \widetilde{c}(s)\, \mathrm{d}\, g(s) \right) \exp\left(\int_{[t,s)} \widetilde{c}(s)\, \mathrm{d}\, g(s) \right) \right) \\
&= e_c(t, a) \exp\left(\int_{\{t\}} \widetilde{c}(s)\, \mathrm{d}\, g(s) \right) \\
&= e_c(t, a)(1 + c(t)\Delta g(t)).
\end{aligned}$$

Essentially, Remark 4.12 shows that the limitations that the map in (4.6) had at the discontinuity points are avoided with this new exponential map. In order to properly justify that this map solves the initial value problem (4.5), we include the following result, which is an improvement on [33, Lemma 6.3].

Theorem 4.13. *Let $c \in \mathcal{L}^1_g(I, \mathbb{R})$ satisfy* (4.7) *and* (4.8) *on the interval I. Then the function $x : \overline{I} \to \mathbb{R}$ defined as*

$$x(t) = x_0 e_c(t, t_0), \quad t \in \overline{I},$$

is g–absolutely continuous on $\overline{I}$ and solves the initial value problem (4.5).

Proof. First of all, it is clear that $x(t_0) = x_0$, and so the initial condition is satisfied. Moreover, it follows from Lemma 4.10 and Theorem 3.26 that the map $h(t) = \int_{[t_0,t)} \widetilde{c}(s)\, \mathrm{d}\, g(s)$, $t \in \overline{I}$, is g–absolutely continuous on $\overline{I}$, and

$$h'_g(t) = \widetilde{c}(t) \quad g\text{–a.a. } t \in I.$$

Thus, Proposition 3.28 guarantees that $x(t) = x_0 \exp(h(t))$ is g–absolutely continuous on $\overline{I}$. Now, applying the chain rule for g-derivatives, Proposition 3.15, we have that for g–almost all $t \in I \setminus D_g$,

$$x'_g(t) = x_0 e_c(t, t_0) h'_g(t) = x_0 e_c(t, t_0)\widetilde{c}(t) = x_0 e_c(t, t_0) c(t) = x(t)c(t). \tag{4.11}$$

On the other hand, for $t \in I \cap D_g$, Remarks 3.4 and 4.12 yield

$$\begin{aligned}
x'_g(t) &= \frac{x_0 e_c(t^+, t_0) - x_0 e_c(t, t_0)}{\Delta g(t)} \\
&= \frac{x_0 e_c(t, t_0)(1 + c(t)\Delta g(t)) - x_0 e_c(t, t_0)}{\Delta g(t)} = x_0 e_c(t, t_0) c(t).
\end{aligned}$$

Thus, $x'_g(t) = c(t)x(t)$ for g–a.a. $t \in I$. $\square$

Remark 4.14. Given that $x(t) = x_0 e_c(t, t_0)$ is g–absolutely continuous, we have that

$$x(t) = x_0 \left(1 + \int_{[t_0,t)} c(s) e_c(s, t_0) \, \mathrm{d}\, g(s) \right), \quad t \in \overline{I}.$$

For an application of this result, we suggest looking at Example 4.95, where we propose a model for silkworm populations. Although this example involves an initial value problem with a function with functional arguments, it is solved by looking at homogeneous linear equations, for which we use Theorem 4.13.

Given the properties of the exponential map, it is natural to question whether the multiplicative inverse of the exponential map in Definition 4.11 solves a different linear equation. If we think about the linear equation with the usual derivative, we have that such map solves the linear equation $x'(t) = -c(t)x(t)$. Therefore, it seems natural to check if this remains true for a general derivator. The following result answers that question.

Proposition 4.15. *Let $a, b \in \mathbb{R}$, $a < b$, and $c \in \mathcal{L}^1_g([a, b), \mathbb{R})$ satisfy conditions* (4.7) *and* (4.8). *Then the map $h : [a, b] \to \mathbb{R}$ defined as*

$$h(t) = (e_c(t, a))^{-1}, \quad t \in [a, b], \tag{4.12}$$

is well–defined, $h \in \mathcal{AC}_g([a, b], \mathbb{R})$ and

$$h'_g(t) = \frac{-c(t)}{e_c(t, a)(1 + c(t)\Delta g(t))}, \quad g\text{–a.a. } t \in [a, b). \tag{4.13}$$

Proof. Define $h_1(t) = e_c(t, a)$, $t \in [a, b]$. Since h_1 is g–absolutely continuous on $[a, b]$, it has bounded varition on that interval and thus, it is bounded on there. In particular, if we take

$$m := \exp \left(- \int_{[a,b)} |\widetilde{c}(s)| \, \mathrm{d}\, g(s) \right), \quad M := \exp \left(\int_{[a,b)} |\widetilde{c}(s)| \, \mathrm{d}\, g(s) \right),$$

where $\widetilde{c}$ is the modified function in (4.9), we have that $0 < m \le h(t) \le M < +\infty$, $t \in [a, b]$. Hence, taking $h_2(t) = 1/t$, $t \in [m, M]$, we can rewrite h as $h(t) = h_2 \circ h_1$, which shows that it is well–defined. Moreover, Proposition 3.28 ensures that $h \in \mathcal{AC}_g([a, b], \mathbb{R})$.

Let $t \in [a, b)$ be such that $h'_g(t)$ exists. If $t \notin D_g$, then by the chain rule we have that

$$h'_g(t) = h'_2(h_1(t))(h_2)'_g(t) = \frac{-1}{(e_c(t, a))^2} e_c(t, a) c(t) = \frac{-c(t)}{e_c(t, a)},$$

which coincides with (4.42) since $\Delta g(t) = 0$. On the other hand, if $t \in D_g$, using Remarks 3.4 and 4.12 we have that

$$h'_g(t) = \frac{(e_c(t^+, a))^{-1} - (e_c(t, a))^{-1}}{\Delta g(t)} = \frac{(1 + c(t)\Delta g(t))^{-1} - 1}{e_c(t, a)\Delta g(t)} = \frac{-c(t)}{e_c(t, a)(1 + c(t)\Delta g(t))}$$

which concludes the proof. □

Essentially, Theorem 4.15 shows that if $c \in \mathcal{L}^1_g(I,\mathbb{R})$ satisfies conditions (4.7) and (4.8) on I, then $(e_c(t,t_0))^{-1}$ solves the Stieltjes differential equation

$$x'_g(t) = -c(t)x(t),$$

except at the discontinuity points of the derivator. In order to obtain an equality at those points, we would have to modify the map c in an analogous way to (4.9), which would lead to asking for $-c$ to satisfy conditions (4.7) and (4.8) on I. However, in that case, we would be in the conditions of Theorem 4.13, so it would provide no new information. In relation with this problem, we find some results in [33], where the authors defined another exponential map that solves a similar problem. Here we have gathered all that information in the following result.

Proposition 4.16. *Let $c \in \mathcal{L}^1_g(I,\mathbb{R})$ satisfy*

$$c(t)\Delta g(t) \neq -1 \quad \textit{for every } t \in I \cap D_g,$$

and

$$\sum_{t \in I \cap D_g} \big|\log|1 + c(t)\Delta g(t)|\big| < +\infty.$$

Then the set $T_c^- = \{t \in I \cap D_g : 1 + c(t)\Delta g(t) < 0\}$ has finite cardinality. Furthermore, if $T_c^- = \{t_1, \dots, t_k\}$ for some $t_0 \leq t_1 < t_2 < \cdots < t_k < t_{k+1} = t_0 + T$, then the map $\widehat{c} : I \to \mathbb{R}$ defined as

$$\widehat{c}(t) = \begin{cases} c(t) & \text{if } t \in I \backslash D_g, \\ \dfrac{\log|1 + c(t)\Delta g(t)|}{\Delta g(t)} & \text{if } t \in I \cap D_g, \end{cases}$$

belongs to $\mathcal{L}^1_g(I,\mathbb{R})$; the map $\widehat{e}_c(\cdot,a) : \overline{I} \to \mathbb{R}\backslash\{0\}$ given by

$$\widehat{e}_c(t,a) = \begin{cases} \exp\left(\displaystyle\int_{[t_0,t)} \widehat{c}(s)\,\mathrm{d}\,g(s)\right) & \text{if } t_0 \leq t \leq t_1, \\ (-1)^j \exp\left(\displaystyle\int_{[t_0,t)} \widehat{c}(s)\,\mathrm{d}\,g(s)\right) & \text{if } t_j < t \leq t_{j+1},\ j = 1, \dots, k, \end{cases}$$

is g–absolutely continuous on $\overline{I}$ and $x(t) = x_0\widehat{e}_c^{\,-1}(t,t_0)$, $t \in \overline{I}$, solves the initial value problem

$$x'_g(t) = -c(t)x(t^+), \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_0.$$

Although Proposition 4.15 was interesting for the study of the corresponding linear problem, its applications go beyond that. In particular, we can derive a version of Gronwall's inequality in the context of Lebesgue–Stieltjes integrals. Naturally, in this context, the exponential map involved is that of Definition 4.11. However, as we will see later, we can obtain a different version of Gronwall's inequality involving the usual exponential map. Let us state and prove our first version of Gronwall's inequality for the Lebesgue–Stieltjes integral, which we will use later to prove the uniqueness of solution for the general inital value problem with Lipschitz functions.

Proposition 4.17. *Let* $a, b \in \mathbb{R}$*,* $a < b$*, and* $u, K, L : [a,b] \to [0,+\infty)$ *be functions such that* $L, K \cdot L, u \cdot L \in \mathcal{L}^1_g([a,b),[0,+\infty))$*. If*

$$u(t) \le K(t) + \int_{[a,t)} L(s)u(s)\,\mathrm{d}\,g(s), \quad t \in [a,b], \tag{4.14}$$

then

$$u(t) \le K(t) + \int_{[a,t)} K(s)L(s)\exp\left(\int_{[s,t)} \widetilde{L}(r)\,\mathrm{d}\,g(r)\right)\mathrm{d}\,g(s), \quad t \in [a,b], \tag{4.15}$$

where $\widetilde{L}$ *is the modified function in* (4.9)*. Moreover, if the map* $\varphi(t) = K(t)(1 + L(t)\Delta g(t))$ *is nondecreasing, then*

$$u(t) \le \varphi(t)e_L(t,a), \quad t \in [a,b]. \tag{4.16}$$

Proof. First of all, note that conditions (4.7) and (4.8) are trivially satisfied for non–negative functions. Hence, the map $\widetilde{L}$ is well–defined.

Define $U(t) = \int_{[a,t)} L(s)u(s)\,\mathrm{d}\,g(s)$, $t \in [a,b]$. It follows from the hypotheses and Theorem 3.26 that U is well–defined, $U \in \mathcal{AC}_g([a,b],\mathbb{R})$ and

$$U'_g(t) = L(t)u(t), \quad g\text{–a.a. } t \in [a,b).$$

Let $h : [a,b] \to \mathbb{R}$ be as in (4.12) for $c = L$ and define $v(t) = U(t)h(t)$, $t \in [a,b]$. It follows from Propositions 4.15 and 3.29 that $v \in \mathcal{AC}_g([a,b],\mathbb{R})$, so $v'_g(t)$ exists g–almost everywhere in $[a,b]$.

Given $t \in [a,b)$ such that $v'_g(t)$ exists, Propositions 3.13 and 4.15 yield

$$\begin{aligned} v'_g(t) &= U'_g(t)(h(t) + h'_g(t)\Delta g(t)) + h'_g(t)U(t) \\ &= u(t)L(t)\left(h(t) - \frac{L(t)h(t)}{1 + L(t)\Delta g(t)}\Delta g(t)\right) - \frac{L(t)h(t)}{1 + L(t)\Delta g(t)}U(t) \\ &= u(t)L(t)h(t)\frac{1}{1 + L(t)\Delta g(t)} - \frac{L(t)h(t)U(t)}{1 + L(t)\Delta g(t)} \\ &= \frac{L(t)h(t)}{1 + L(t)\Delta g(t)}\left(u(t) - \int_{[a,t)} L(s)u(s)\,\mathrm{d}\,g(s)\right). \end{aligned}$$

Thus, inequality (4.14) and the fact that $1 + L(t)\Delta g(t) \ge 1$ for all $t \in [a,b]$, ensure that

$$v'_g(t) \le \frac{K(t)L(t)h(t)}{1 + L(t)\Delta g(t)} \le K(t)L(t)h(t), \quad g\text{–a.a. } t \in [a,b).$$

Therefore, it follows from Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral, Theorem 3.27, that

$$v(t) = v(a) + \int_{[a,t)} v'_g(s)\,\mathrm{d}\,g(s) \le \int_{[a,t)} K(s)L(s)h(s)\,\mathrm{d}\,g(s), \quad t \in [a,b]$$

and, as a consequence, for all $t \in [a,b]$ we have

$$\begin{aligned}\int_{[a,t)} L(s)u(s)\,\mathrm{d}\,g(s) &= e_L(t,a)v(t)\\ &\le e_L(t,a)\int_{[a,t)} K(s)L(s)h(s)\,\mathrm{d}\,g(s)\\ &= e_L(t,a)\int_{[a,t)} K(s)L(s)\left(e_L(s,a)\right)^{-1}\mathrm{d}\,g(s)\\ &= \int_{[a,t)} K(s)L(s)\exp\left(\int_{[s,t)}\widetilde{L}(r)\,\mathrm{d}\,g(r)\right)\mathrm{d}\,g(s).\end{aligned}$$

Thus, it follows from (4.14) that

$$u(t) \le K(t) + \int_{[a,t)} K(s)L(s)\exp\left(\int_{[s,t)}\widetilde{L}(r)\,\mathrm{d}\,g(r)\right)\mathrm{d}\,g(s), \quad t\in[a,b];$$

that is, (4.15) holds.

To prove (4.16), for each $t \in (a,b]$, define

$$\psi_t(s) = \exp\left(\int_{[s,t)}\widetilde{L}(r)\,\mathrm{d}\,g(r)\right) = \frac{e_L(t,a)}{e_L(s,a)}, \quad s\in[a,t].$$

Then, it follows from (4.15) that for all $t \in [a,b]$,

$$\begin{aligned}u(t) &\le K(t) + \int_{[a,t)} K(s)L(s)\psi_t(s)\,\mathrm{d}\,g(s)\\ &\le K(t)(1+L(t)\Delta g(t)) + \int_{[a,t)} K(s)(1+L(s)\Delta g(s))\frac{L(s)\psi_t(s)}{1+L(s)\Delta g(s)}\,\mathrm{d}\,g(s).\end{aligned}$$

Now, since $\varphi(t) = K(t)(1+L(t)\Delta g(t))$ is nondecreasing, we have that

$$u(t) \le \varphi(t)\left(1+\int_{[a,t)}\frac{L(s)\psi_t(s)}{1+L(s)\Delta g(s)}\,\mathrm{d}\,g(s)\right), \quad t\in[a,b].$$

On the other hand, Proposition 4.15 ensures that for all $t \in [a,b]$, $\psi_t \in \mathcal{AC}_g([a,t],\mathbb{R})$ and

$$(\psi_t)'_g(s) = \frac{-L(s)}{1+L(s)\Delta g(s)}\psi_t(s) \quad g\text{–a.a. } s\in[a,t).$$

This fact, together with the Fundamental Theorem of Calculus, Theorem 3.27, yields that

$$u(t) \le \varphi(t)\left(1-\int_{[a,t)}(\psi_t)'_g(s)\,\mathrm{d}\,g(s)\right) = \varphi(t)\left(1-(\psi_t(t)-\psi_t(a))\right) = \varphi(t)\psi_t(a),$$

for all $t \in [a,b]$, from which the result follows. □

Remark 4.18. The bound (4.16) in Proposition 4.17 is sharp, that is, there exist functions for which the equality is attained. Indeed, let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and continuous function, $K : [a, b] \to [0, +\infty)$ be constant and $L \in \mathcal{L}^1_g([a, b), [0, +\infty))$. The map $x(t) = Ke_L(t, a)$, $t \in [a, b]$, is g–absolutely continuous on $[a, b]$. As a consequence, and with the aid of Theorem 4.13, we have that

$$x(t) = K + \int_{[a,t)} L(s)Ke_L(s, a)\,\mathrm{d}\,g(s), \quad t \in [a, b],$$

that is, (4.14) holds. Furthermore, that same expression shows that (4.16) also holds with the equality.

This type of inequalities for Stieltjes integrals already exist in the literature, see [41,57,62, 65]. Let us briefly discuss their relations with Proposition 4.17. First, looking at Corollary 3.3 in [57], we note that the hypotheses required there are stronger than the ones in our case. Furthermore, (4.16) gives a sharper bound than the one presented there. A similar conclusion can be obtained for Theorem 7.5.3 in [65] when all the integrals involved are defined. Now, for [41, 62], the authors obtained a Gronwall type inequality in the context of a certain family of linear operators. The operators there considered can be the Stieltjes integrals in this thesis. In that case, the authors impose some conditions on the discontinuities of the integrator, and moreover, the inequality is expressed using an unknown function introduced in [41], called Gronwall majorant. Hence, in the context of our work, the inequality in Proposition 4.17 provides more information.

Note that (4.16) in Proposition 4.17 becomes the usual Gronwall's inequality when the derivator g is the identity map. Furthermore, as we mentioned before, we can obtain a different Gronwall type inequality involving the usual exponential map, i.e. not involving the modified map in (4.9). However, the bound in Proposition 4.17 is sharper than the one in the following result.

Corollary 4.19. *Let $a, b \in \mathbb{R}$, $a < b$, and $u, K, L : [a, b] \to [0, +\infty)$ be functions such that $L, K \cdot L, u \cdot L \in \mathcal{L}^1_g([a, b), [0, +\infty))$. If* (4.14) *holds, then*

$$u(t) \leq K(t) + \int_{[a,t)} K(s)L(s) \exp\left(\int_{[s,t)} L(r)\,\mathrm{d}\,g(r)\right) \mathrm{d}\,g(s), \quad t \in [a, b].$$

Moreover, if the map $\varphi(t) = K(t)(1 + L(t)\Delta g(t))$ is nondecreasing, then

$$u(t) \leq \varphi(t) \exp\left(\int_{[a,t)} L(r)\,\mathrm{d}\,g(r)\right), \quad t \in [a, b].$$

Proof. Given the inequalities in Proposition 4.17, it is enough to show that $\widetilde{L}(t) \leq L(t)$ for all $t \in [a, b]$. The inequality is trivial for points of $[a, b] \setminus D_g$, since $\widetilde{L} = L$ on such set.

Now, for $t \in [a, b] \cap D_g$, since $\log(1 + s) \leq s$ for $s \in (-1, +\infty)$ and (4.7) holds, we have that

$$\widetilde{L}(t) = \frac{\log\big(1 + L(t)\Delta g(t)\big)}{\Delta g(t)} \leq \frac{L(t)\Delta g(t)}{\Delta g(t)} = L(t),$$

which concludes the proof. □

Nonhomogeneous linear equation

Once we have fully solved the homogeneous linear initial value problem, we can study the nonhomogeneous case. Let us begin by making explicit our formulation of the problem. Given a nondecreasing and left–continuous map, $g:\mathbb{R}\to\mathbb{R}$, we consider the initial value problem

$$x'_g(t)+d(t)x(t)=h(t),\quad x(t_0)=x_0, \tag{4.17}$$

with $x_0\in\mathbb{R}$ and $d,h:I\to\mathbb{R}$. Naturally, we still restrict ourselves to the context of the real line, as we have only solved the homogeneous case in this context.

This problem was studied, in some extent, in [33, Proposition 6.8] where the authors proved the existence of a unique solution for (4.17) under certain hypotheses. The proof of the mentioned result contains a way to obtain a solution of (4.17) provided that one can obtain a solution of an auxiliary problem, whose solution is found by making use of an incorrect version of the product rule for Stieltjes derivatives. Here, we propose an alternative method to obtain the solution of (4.17) making use of the right formulation of the product rule. Let us state [33, Proposition 6.8], as it will be of interest for the work ahead.

Proposition 4.20. *Let $h,d\in\mathcal{L}^1_g(I,\mathbb{R})$ be such that*

$$d(t)\Delta g(t)\neq 1\quad \textit{for every } t\in I\cap D_g, \tag{4.18}$$

and

$$\sum_{t\in I\cap D_g}\left|\log\left|1-d(t)\Delta g(t)\right|\right|<+\infty. \tag{4.19}$$

Then, there exists a unique function $x\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ solving the initial value problem (4.17).

In order to obtain an explicit solution on I for (4.17), we recreate the method of variation of constants in this context. Roughly speaking, this method revolves around the idea that the solution of a nonhomogeneous linear equation can be expressed as the sum of a solution of the homogeneous linear equation plus one particular solution of the nonhomogeneous one. In order to obtain the particular solution, we consider the family of functions solving the homogeneous equation, $x_C(t)=Ce_{-d}(t,t_0)$, $t\in\overline{I}$, $C\in\mathbb{R}$. From there, we make a guess that a particular solution is similar to that one, where we allow the constants to vary, i.e. we consider them as a function. Explicitly, we guess that the solution is of the form

$$x(t)=C(t)e_{-d}(t,t_0),\quad t\in\overline{I},$$

for some function $C:\overline{I}\to\mathbb{R}$. Then, we try our guess on the corresponding nonhomogeneous linear equation. In order to do so, we need to make use of the product rule for Stieltjes derivatives, statement (ii) in Proposition 3.13. Let $t\in I$ be such that $x'_g(t)$ exists. In that case,

$$\begin{aligned} x'_g(t)&=C'_g(t)e_{-d}(t,t_0)+C(t)e_{-d}(t,t_0)(-d(t))+C'_g(t)e_{-d}(t,t_0)(-d(t))\Delta g(t)\\ &=e_{-d}(t,t_0)(C'_g(t)(1-d(t)\Delta g(t))-C(t)d(t)). \end{aligned}$$

Hence, for such $t \in I$, it follows that $x_g'(t) + d(t)x(t) = e_{-d}(t,t_0)C_g'(t)(1 - d(t)\Delta g(t))$. Therefore, if x solves the nonhomogeneous linear equation, we must have that for such $t \in I$,

$$h(t) = e_{-d}(t,t_0)C_g'(t)(1 - d(t)\Delta g(t)). \tag{4.20}$$

Therefore, if we can find a function C satisfying (4.20), we obtain a particular solution of the nonhomogeneous linear equation and, as a consequence, the general solution of the same problem. After that, it is enough to impose the initial condition to obtain a solution of corresponding initial value problem. In the next result, we provide an explicit expression for a solution of (4.17), obtained using the method described above.

Proposition 4.21. *Let $d, h \in \mathcal{L}_g^1(I,\mathbb{R})$ be such that $-d$ satisfies conditions* (4.7) *and* (4.8) *on I in place of c. Then, the map $x : \overline{I} \to \mathbb{R}$ defined as*

$$x(t) = e_{-d}(t,t_0)\left(x_0 + \int_{[t_0,t)} \frac{h(s)}{e_{-d}(s,t_0)(1 - d(s)\Delta g(s))}\,\mathrm{d}\,g(s)\right), \quad t \in \overline{I},$$

is a solution of (4.17) *on I.*

Proof. First of all, note that the conditions on d guarantee that $e_{-d}(\cdot,t_0)$ is well–defined. Consider the maps $E(t) = e_{-d}(t,t_0)(1 - d(t)\Delta g(t))$, $H(t) = h(t)/E(t)$, $t \in I$. Since $E(t) = e_{-d}(t,t_0)$ for all $t \in I \setminus D_g$ and D_g is countable, E is g–measurable. Moreover, since $E > 0$ by definition, and h and E are g–measurable, it follows that H is g–measurable. Furthermore, H belongs to $\mathcal{L}_g^1(I,\mathbb{R})$. Indeed, given that $e_{-d}(\cdot,t)$ is bounded from below by some positive constant on I, it is enough to show that the map $\overline{h}(t) = h(t)/(1 - d(t)\Delta g(t))$, $t \in I$, is g–integrable on I. Observe that the set $A = \{t \in I : d(t)\Delta g(t) > 1/2\}$ has finite cardinality as

$$\sum_{t\in A}\frac{1}{2} < \sum_{t\in I\cap D_g} |d(t)\Delta g(t)| \le \int_I |d(s)|\,\mathrm{d}\,g(s) < +\infty.$$

As a consequence, we have that $|\overline{h}(t)| \le 2|h(t)|$ for all $t \in I \setminus A$, from which the g–integrability of $\overline{h}$ follows. Thus, $H \in \mathcal{L}_g^1(I,\mathbb{R})$. Now, it follows from Theorem 3.26 and Proposition 3.29 that x is g–absolutely continuous on $\overline{I}$. Hence, all that is left to do is to check that x solves (4.17).

By definition, we have that $x(t_0) = x_0$. Let $t \in I$ be such that $x_g'(t)$ exists. By Proposition 3.13 and Theorems 4.13 and 3.26, we have that

$$\begin{aligned} x_g'(t) &= -d(t)e_{-d}(t,t_0)\left(x_0 + \int_{[t_0,t)} H\,\mathrm{d}\,g\right) + e_{-d}(t,t_0)H(t) - e_{-d}(t,t_0)d(t)H(t)\Delta g(t) \\ &= -d(t)x(t) + e_{-d}(t,t_0)H(t)(1 - d(t)\Delta g(t)) = -d(t)x(t) + h(t), \end{aligned}$$

i.e. x solves (4.17). □

Remark 4.22. Note that Proposition 4.21 does not guarantee the uniqueness of solution. Nevertheless, this can be deduced from the Lipschitz uniqueness result presented later. Furthermore, observe that it is possible to recreate the reasoning here presented using [33, Lemma 6.5] under the corresponding assumptions (i.e., conditions (4.18) and (4.19)) to obtain the unique solution mentioned in Proposition 4.20. We have shown the result under the assumption that $-d$ satisfies (4.7) and (4.8) for simplicity.

Separation of variables

This last part is devoted to the explicit resolution of the separable initial value problem

$$x'_g(t) = c(t)f(x(t)), \quad x(t_0) = x_0, \tag{4.21}$$

with $t_0, x_0 \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$. Note that the particular case of $f(x) = x$ yields problem (4.5). With this idea in mind, and recalling that we had to redefine c at the discontinuity points of g, we first consider the particular case of (4.21) corresponding to a continuous derivator. In that case, we have the following result.

Theorem 4.23. *Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0, +\infty), [0, +\infty))$, i.e. c is g–integrable on compact subsets of $[t_0, +\infty)$ with respect to the usual topology. Assume there is some $R > 0$ such that f is continuous and positive on $J = (x_0 - R, x_0 + R)$. Define*

$$F(x) = \int_{x_0}^{x} \frac{\mathrm{d}\, r}{f(r)} \quad \textit{for every } x \in J.$$

If there exists $\tau > 0$ such that g is continuous on I_τ and

$$\int_{[t_0,t)} c(s)\,\mathrm{d}\,g(s) \in F(J), \quad t \in \overline{I}_\tau,$$

then a solution of (4.21) on I_τ is given by the map $x : \overline{I}_\tau \to \mathbb{R}$ defined as

$$x(t) = F^{-1}\left(\int_{[t_0,t)} c(s)\,\mathrm{d}\,g(s)\right) \quad t \in \overline{I}_\tau. \tag{4.22}$$

Proof. First of all, note that $x(t_0) = x_0$ so the initial condition is satisfied. Let us show that x is g–absolutely continuous on $\overline{I}_\tau$ and satisfies the equation. Define $C : \overline{I}_\tau \to \mathbb{R}$ as

$$C(t) = \int_{[t_0,t)} c(s)\,\mathrm{d}\,g(s), \quad t \in \overline{I}_\tau.$$

Theorem 3.26 ensures that C is g–absolutely continuous on $\overline{I}_\tau$. Thus, $C'_g(t)$ exists for g–a.a. $t \in I_\tau$ and

$$C'_g(t) = c(t), \quad g\text{–a.a. } t \in I_\tau.$$

On the other hand, since F^{-1} is locally Lipschitz and $x = F^{-1} \circ C$, we can deduce from Proposition 3.28 that x is g–absolutely continuous on $\overline{I}_\tau$. In particular, there exists $x'_g(t)$ for g–almost all $t \in I_\tau$. For such points, and since g is continuous, we can apply the chain rule, Proposition 3.15, to compute $x'_g(t)$ as

$$x'_g(t) = (F^{-1})'(C(t))C'_g(t) = \frac{1}{F'(F^{-1}(C(t)))}c(t) = \frac{1}{F'(x(t))}c(t) = f(x(t))c(t),$$

which concludes the result. $\square$

Remark 4.24. Formula (4.22) is equivalent to

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}\, r}{f(r)} = \int_{[t_0,t)} c(s)\,\mathrm{d}\, g(s).$$

Observe that this formula yields the same solution for (4.5) as Theorem 4.13 since $\widetilde{c} = c$ when g is continuous.

Example 4.25. We want to set up a simple model for the motion of a vehicle impulsed by an electric engine which we can turn on and off as often as we please.

Let $g(t)$ denote the number of seconds that the engine has been on until time t. This function is continuous, nondecreasing, and constant on the time intervals when the engine is turned off. We shall assume that $g(0) = 0$.

Let $s(t)$ denote the vehicle's speed after t seconds. For simplicity, we assume that speed increases on every time interval $[t, t+h]$, $h > 0$, at a rate proportional to the time the engine has been on during that time interval. Moreover, we consider that accelerating the vehicle is harder at very slow or at very high speeds, so we assume that, when all forces are factured in, the effective acceleration is given by a function f of $s(t)$. This leads to the following expression

$$s(t+h) - s(t) = f(s(t))(g(t+h) - g(t)),$$

which, considering the limit as $h \to 0^+$, yields the g–differential equation

$$s_g'(t) = f(s(t)), \quad t \geq 0, \quad s(0) = s_0. \tag{4.23}$$

We suggest using logistic–type functions like $f : \mathbb{R} \to \mathbb{R}$

$$f(s) = \alpha \max\{0, (s+\beta)(s_{\max} - s)\}, \quad s \in \mathbb{R}$$

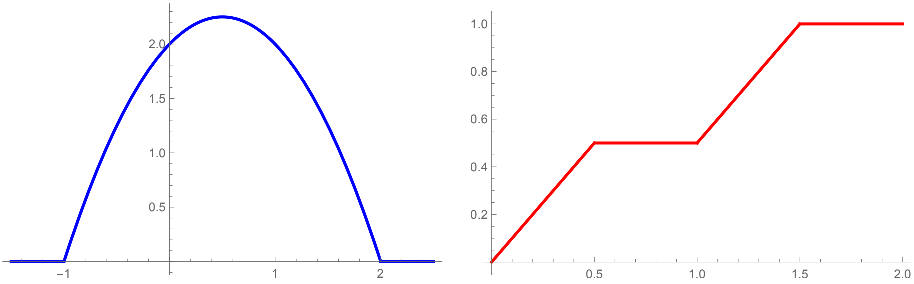
where α, β and $s_{\max}$ are some positive constants. Observe that $f(s) > 0$ for $s \in [0, s_{\max})$ and $f(s) = 0$ for $s \geq s_{\max}$, which means that the engine can accelerate the vehicle only when its speed belongs to the interval $[0, s_{\max})$. Also, notice that if $\beta < s_{\max}$, f attains a maximum at $(s_{\max} - \beta)/2$, which means that the engine is more efficient when the vehicle is moving at that specific value of speed. We shall assume that this is the case in what follows. Hence, we have that

$$f(s) = \begin{cases} \alpha(s+\beta)(s_{\max} - s), & \text{if } s \in [\beta, s_{\max}], \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f satisfies a Lipschitz condition on $\mathbb{R}$. As we will see later through Theorem 4.37, this condition is enough to ensure that (4.23) has, at most, one solution. We shall prove that this problem has a solution using Theorem 4.23. For simplicity, we consider the particular choice of $\beta = 1$ and $s_{\max} = 2$, obtaining

$$f(s) = \max\{0, (s+1)(2-s)\}, \quad s \in \mathbb{R}, \tag{4.24}$$

whose graph can be found in Figure 4.1, (a).



(a) Graph of f as in (4.24). (b) Graph of g as in (4.27).

Figure 4.1: Graphs of some functions defining problem (4.23).

Following Theorem 4.23, we have that the solution for $s_0 = 0$ is implicitly given by the formula

$$\int_0^{s(t)} \frac{\mathrm{d}\, r}{(r+1)(2-r)} = \int_{[0,t)} \mathrm{d}\, g(s) = g(t), \quad t \geq 0.$$

Elementary computations yield that, in that case, the solution is given by the expression

$$s(t) = \frac{2e^{3g(t)} - 2}{e^{3g(t)} + 2}, \quad t \geq 0. \tag{4.25}$$

Observe that this is the same solution obtained through Theorem 3.39 when the derivator g satisfies the corresponding hypothesis. Indeed, in that case, the corresponding initial value problem with the usual derivative is

$$y'(s) = f(y(s)), \quad y(0) = 0, \quad t \geq 0, \tag{4.26}$$

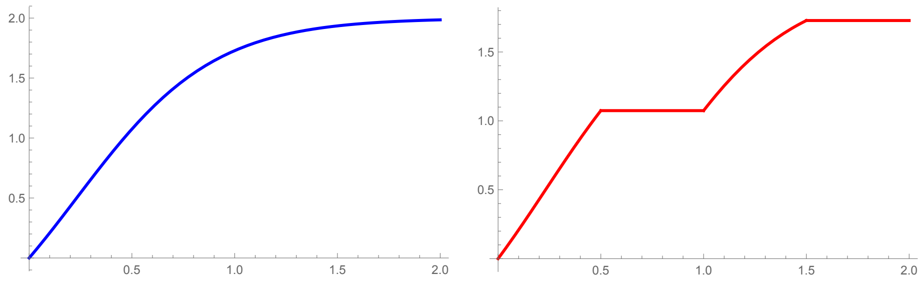
whose solution is given by

$$y(t) = \frac{2e^{3t} - 2}{e^{3t} + 2}, \quad t \geq 0,$$

and it is represented in Figure 4.2 (a). Then, according to Theorem 3.39, the solution of (4.23) is given by $x = y \circ g$, which matches (4.25).

Observe that (4.26) corresponds to the particular choice of the derivator $g = \mathrm{Id}$, where we would keep the engine on at all times. We can consider more interesting cases. For example, imagine that we turn the device off for $t \in [1/2, 1] \cup [3/2, 2]$. In that case, then we could consider a derivator $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ t, & \text{if } t \in [0, 1/2], \\ 1/2, & \text{if } t \in [1/2, 1], \\ t - 1/2, & \text{if } t \in [1, 3/2], \\ 1, & \text{if } t \in [3/2, 2], \\ t - 1, & \text{if } t \geq 2, \end{cases} \tag{4.27}$$



(a) Solution (4.25) on $[0,2]$ for $g = \mathrm{Id}$. (b) Solution (4.25) on $[0,2]$ for g as in (4.27).

Figure 4.2: Graphs of the solution of (4.25) on $[0,2]$ for f as in (4.24) for different derivators.

Note that, when $g'(t)$ exists, it only takes the values 0 and 1, representing the times at which the engine is off and on, respectively. This can be seen in Figure 4.1 (b). In Figure 4.2 we present the solution of (4.23) with f as in (4.24) for $g = \mathrm{Id}$ and g as in (4.27), where we can observe the effects of turning off the engine.

Theorem 4.23 is false, in general, when g is discontinuous. Indeed, observe that (4.22) does not give (4.10) when f is the identity and g has at least one discontinuity point in I_τ. At discontinuity points of g we cannot use Proposition 3.15 to compute derivatives of compositions by means of the chain rule, so we need an alternative approach and an alternative formula for the solutions.

For the rest of this section, we assume that g is discontinuous exactly at the points of a sequence $\{\tau_k\}_{k=1}^{\infty}$, where $t_0 < \tau_1 < \tau_2 < \cdots$. In other words, we assume D_g to be a discrete set , i.e. all its elements are isolated points. In this setting, we can obtain an explicit solution under certain hypotheses. We do not have a formula for the general case.

Solving (4.21) on the interval $[t_0, \tau_1)$ can be done with the aid of Theorem 4.23, because g is continuous on $[t_0, \tau_1)$. Furthermore, since solutions are continuous from the left everywhere, we get the solution on $[t_0, \tau_1)$ with the same formula. Specifically, under suitable conditions, a solution of (4.21) on the interval $[t_0, \tau_1)$ is implicitly given by the expression

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}\, r}{f(r)} = \int_{[t_0,t)} c(s)\,\mathrm{d}\, g(s), \quad t \in [t_0, \tau_1]. \tag{4.28}$$

Obtaining the solution formula on the right of τ_1 is a matter of induction. First, according to the definition of g–derivative at discontinuity points, the differential equation in (4.21) for $t = \tau_1$ reads simply as follows:

$$x(\tau_1^+) = x(\tau_1) + c(\tau_1) f(x(\tau_1))(g(\tau_1^+) - g(\tau_1)) \equiv x_1. \tag{4.29}$$

Therefore, we have to solve another initial value problem

$$x_g'(t) = c(t) f(x(t)),\ t \in (\tau_1, \tau_2],\ x(\tau_1^+) = x_1, \tag{4.30}$$

by means of (4.22), with obvious modifications: under suitable conditions –see Remark 4.26– a solution of (4.30) is defined by

$$\int_{x_1}^{x(t)} \frac{\mathrm{d}\, r}{f(r)} = \int_{(\tau_1,t)} c(s)\,\mathrm{d}\, g(s) \quad \text{for all } t \in (\tau_1,\tau_2], \tag{4.31}$$

where x_1 is defined in (4.29).

Remark 4.26. Formula (4.31) gives a solution of (4.30) provided that, for instance, f is continuous and positive on $J = (x_1 - R, x_1 + R)$, for some $R > 0$, and

$$\int_{(\tau_1,t)} c(s)\,\mathrm{d}\, g(s) \in F(J), \quad t \in (\tau_1,\tau_2],$$

where $F(x) = \int_{x_1}^{x} \mathrm{d}\, r/f(r)$, $x \in J$.

Summing up, a solution of (4.21) can be recursively computed as follows: define $x(t)$ on $[t_0,\tau_1]$ by means of (4.28); assume that we have defined $x(t)$ on $[t_0,\tau_k]$, for some $k \in \mathbb{N}$, then compute the number

$$x_k = x(\tau_k) + c(\tau_k) f(x(\tau_k))(g(\tau_k^+) - g(\tau_k)), \tag{4.32}$$

and define $x(t)$ implicitly on $(\tau_k,\tau_{k+1}]$ by the expression

$$\int_{x_k}^{x(t)} \frac{\mathrm{d}\, r}{f(r)} = \int_{(\tau_k,t)} c(s)\,\mathrm{d}\, g(s), \quad t \in (\tau_k,\tau_{k+1}]. \tag{4.33}$$

Let us illustrate this proccess with the following example where we present a model for a bacteria population in a semicontinuous cultivation proccess.

Example 4.27. Semicontinuous cultivation is a system to produce bacteria in which a portion of the culture medium is periodically removed and the remaining culture is used as the starting point for continuation of the culture. For the model we are about to consider we take into consideration the information in [71], where semicontinuous cultivation of the cyanobacteria *Spirulina plantensis* is studied. We highlight the following important feature of the system in [71]: illumination was controlled to have a 12 hours light/dark photoperiod, which resulted in two different reproduction phases every day.

Bearing the above considerations in mind, we shall set up a mathematical model for the production of *Spirulina plantensis* in a semicontinuous culture system. First, we consider days as our time units. We assign light periods to be the time intervals $[k, k+1/2)$, $k = 0,1,2,\ldots$, while the dark periods are represented by $[k+1/2, k+1)$, $k = 0,1,2,\ldots$. Second, we assume that half of the culture is removed every 10 days and immediately refilled with new nutrients so that the remaining bacteria start reproducing again. Finally, we take as derivator a nondecreasing function $g : \mathbb{R} \to \mathbb{R}$ continuous everywhere with the exception of the positive multiples of 10, and such that

$$g'(t) = \begin{cases} 1, & \text{if } t \in (k, k+1/2),\ k = 0,1,2,\ldots, \\ 1/2, & \text{if } t \in (k+1/2, k+1),\ k = 0,1,2,\ldots, \end{cases}$$

and $g((10k)^+) - g(10k) = 1$ for $k = 1, 2, 3, \dots$. A concise explicit expression for $g(t)$ can be obtained by defining first its values on the first day, namely

$$h(t) = \begin{cases} t, & \text{if } t \in [0, 1/2), \\ t/2 + 1/4, & \text{if } t \in (1/2, 1], \end{cases}$$

and then we can define the remaining values by "periodicity", and introducing jump discontinuities at relevant places, as

$$g(t) = \begin{cases} 0, & \text{if } t < 0, \\ h(t - [t]) + 3[t]/4 + [t/10], & \text{if } t \geq 0, \end{cases}$$

where $[\cdot]$ denotes the integer part. Observe that we should modify the values $g(10k)$, $k = 1, 2, \dots$, so that g be left–continuous, but we shall not do it to avoid technicalities. See Figure 4.3 for a plot of this function.

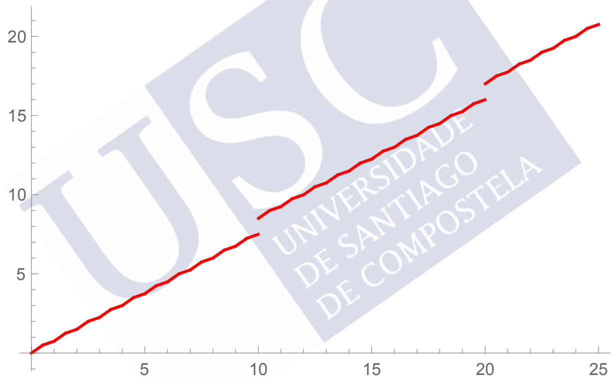


Figure 4.3: Graph of $g(t)$ for a semicontinuous bacteria culture. Observe different slopes for light and dark periods, and discontinuities at the renewal moments.

We are now ready to introduce a g–differential model for the biomass concentration $x(t)$, measured in grams per liter at time t, with a given initial concentration, $x(0) = x_0$. Biomass concentration should satisfy

$$x_g'(t) = f(t, x(t)) \quad t \geq 0, \quad x(0) = x_0, \tag{4.34}$$

where $f(t, x)$ is assumed to be logistic except at the renewal moments, i.e. at positive multiples of 10, when we remove half of the culture and immediately refill the flask with new nutrients. Specifically, we define

$$f(t, x) = \begin{cases} \alpha\, x(N - x), & \text{if } t \neq 10k, \; k = 1, 2, \dots, \\ -x/2, & \text{if } t = 10k, \; k = 1, 2, \dots, \end{cases}$$

where $\alpha > 0$ and $N > 0$ are biological parameters to be adjusted from experimental results.

Using the formulas (4.28) and (4.33), we compute the solution: for $t \in [0, 10]$ the solution is

$$x(t) = \frac{\frac{x_0 N}{N - x_0} e^{\alpha N g(t)}}{1 + \frac{x_0}{N - x_0} e^{\alpha N g(t)}}.$$

Assume we have computed $x(t)$ for all $t \in [0, 10k]$, for some $k = 1, 2, \ldots$, then we define

$$x_k = x(10k^+) = \frac{x(10k)}{2},$$

and the solution for $t \in (10k, 10k + 10]$ is given by

$$x(t) = \frac{\frac{x_k N}{N - x_k} e^{\alpha N [g(t) - g(10k^+)]}}{1 + \frac{x_k}{N - x_k} e^{\alpha N [g(t) - g(10k^+)]}}.$$

Observe that $g(10k^+) = 8.5k$ for all $k = 1, 2, \ldots$.

See Figure 4.4 for a plot of the solution corresponding to $x_0 = 0.4$ grams per liter, $\alpha = 0.1$ and $N = 1.5$. These choices yield a good approximation of the experimental results obtained in [71] for the cyanobacteria *Spirulina platensis*, see [71, Figure 2] .

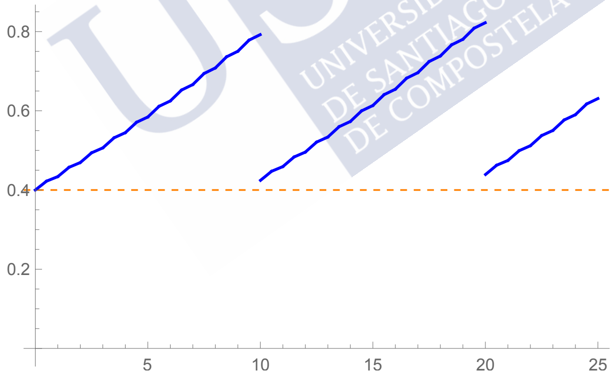


Figure 4.4: Biomass $x(t)$ grams per liter in a semicontinuous culture, with initial density of 0.4 g/L, and parameters $\alpha = 0.1$, $N = 1.5$.

4.1.2 Existence and uniqueness of solution

After investigating the explicit solutions of some initial value problems, we now turn our attention to the study of existence and uniqueness of solution for the initial value problem (4.2). In particular, we first have a look at the results guaranteeing the existence of solution. Later, we establish some conditions that ensure the uniqueness of solution, and finally, combining these two type of results, we obtain some result for the existence of a unique solution of the initial value problem. The results gathered in this section can be found in [33, 48, 58, 59].

For the aim of this section we will assume without loss of generality that the derivator is continuous at the initial point. We shall show that this is, indeed, the case in the context of (4.1) with $n = 1$, but the arguments can be extended to higher dimensions. To show that this is possible it is enough to show that there is an equivalent Stieltjes differential equation with a derivator that is continuous at t_0 that can be studied instead. The next result is a more detailed explanation of the one provided in [33].

Proposition 4.28. *Let $f : I\times\mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous function such that $t_0 \in D_g$. Consider $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ defined as*

$$\widetilde{g}(t) = \begin{cases} g(t_0^+) & \text{if } t \le t_0, \\ g(t) & \text{if } t > t_0, \end{cases}$$

and consider the initial value problem

$$y'_{\widetilde{g}}(t) = f(t, y(t)), \quad y(t_0) = x_0 + f(t_0, x_0)\Delta g(t_0). \tag{4.35}$$

If x is a solution of (4.2) *on I_τ, $\tau \in (0, T]$, then the map $y : \overline{I}_\tau \to \mathbb{R}$ defined as*

$$y(t) = \begin{cases} x_0 + x'_g(t_0)\Delta g(t_0) & \text{if } t = t_0, \\ x(t) & \text{if } t \in (t_0, t_0 + \tau], \end{cases} \tag{4.36}$$

is a solution of (4.35) *on I_τ. Conversely, if y is a solution of* (4.35) *on I_τ, $\tau \in (0, T]$, then the map $x : \overline{I}_\tau \to \mathbb{R}$ defined as*

$$x(t) = \begin{cases} x_0 & \text{if } t = t_0, \\ y(t) & \text{if } t \in (t_0, t_0 + \tau], \end{cases} \tag{4.37}$$

is a solution of (4.2) *on I_τ.*

Proof. First of all, note that by definition, $\widetilde{g}$ is left–continuous and nondecreasing, continuous at t_0 and $g(t) = \widetilde{g}(t)$ for all $t > t_0$.

Let x be a solution of (4.2) on I_τ, $\tau \in (0, T]$, and consider $y : \overline{I}_\tau \to \mathbb{R}$ given by (4.36). Since $x \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R})$ and $t_0 \in D_g$, $x'_g(t_0)$ exists and thus, y is well–defined. Moreover, y satisfies (4.35) for $\widetilde{g}$–a.a. $t \in I_\tau$ as $\mu_{\widetilde{g}}(\{t_0\}) = 0$. Therefore, all that is left to do is to show that $y \in \mathcal{AC}_{\widetilde{g}}(\overline{I}_\tau, \mathbb{R})$.

Since $y = x$ and $\widetilde{g} = g$ on $(t_0, t_0 + \tau)$ and $\{t_0\}$ has $\widetilde{g}$–measure zero, it follows that $y'_{\widetilde{g}}(t)$ exists for $\widetilde{g}$–a.a. $t \in I_\tau$. Furthermore,

$$\int_{I_\tau} |y'_{\widetilde{g}}(s)| \,\mathrm{d}\,\widetilde{g}(s) = \int_{(t_0,t_0+\tau)} |y'_{\widetilde{g}}(s)| \,\mathrm{d}\,\widetilde{g}(s) = \int_{(t_0,t_0+\tau)} |x'_g(s)| \,\mathrm{d}\,g(s) < +\infty,$$

so $y'_{\widetilde{g}} \in \mathcal{L}^1_{\widetilde{g}}(I_\tau, \mathbb{R})$. Finally, Theorem 3.27 ensures that for $t \in (t_0, t_0 + \tau]$,

$$\begin{aligned} x(t) &= x_0 + \int_{\{t_0\}} x'_g(s) \,\mathrm{d}\,g(s) + \int_{(t_0,t)} x'_g(s) \,\mathrm{d}\,g(s) \\ &= x_0 + x'_g(t_0)\Delta g(t_0) + \int_{(t_0,t)} y'_{\widetilde{g}}(s) \,\mathrm{d}\,\widetilde{g}(s), \end{aligned}$$

since $y = x$ on $(t_0, t_0 + \tau]$. Keeping in mind that $y(t_0) = x_0 + x'_g(t_0)\Delta g(t_0)$, we have just proven that

$$y(t) = y(t_0) + \int_{(t_0,t)} y'_{\widetilde{g}}(s)\,\mathrm{d}\,\widetilde{g}(s), \quad t \in \overline{I}_\tau,$$

so Theorem 3.27 guarantees that $y \in \mathcal{AC}_{\widetilde{g}}(\overline{I}_\tau, \mathbb{R})$.

Conversely, let y be solution of (4.35) on I_τ, $\tau \in (0, T]$, and consider the map $x : \overline{I}_\tau \to \mathbb{R}$ given by (4.37). By definition, $x(t_0) = x_0$ and

$$x'_g(t) = y'_{\widetilde{g}}(t) = f(t, y(t)) = f(t, x(t)), \quad g\text{–a.a. } t \in (t_0, t_0 + \tau).$$

Furthermore, since y is $\widetilde{g}$–absolutely continuous on $\overline{I}_\tau$ and $\widetilde{g}$ is continuous at t_0, it follows from Proposition 3.21 that $y(t_0^+) = y(t_0) = x_0 + f(t_0, x_0)\Delta g(t_0)$, so $x'_g(t_0)$ exists and

$$x'_g(t_0) = \frac{x(t_0^+) - x(t_0)}{\Delta g(t_0)} = \frac{y(t_0^+) - x(t_0)}{\Delta g(t_0)} = f(t_0, x_0), \tag{4.38}$$

i.e. x solves (4.2) for g–a.a. $t \in I_\tau$. Thus, all that is left to do is to show that $x \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R})$.

First, it follows directly from (4.38) together with the fact that $x = y$ and $g = \widetilde{g}$ on $(t_0, t_0 + \tau]$ that $x'_g(t)$ exists for g–a.a. $t \in I_\tau$. Moreover,

$$\begin{aligned}\int_{I_\tau} |x'_g(s)|\,\mathrm{d}\,\widetilde{g}(s) &= \int_{\{t_0\}} |x'_g(s)|\,\mathrm{d}\,g(s) + \int_{(t_0,t_0+\tau)} |x'_g(s)|\,\mathrm{d}\,g(s)\\ &= |x'_g(t_0)|\Delta g(t_0) + \int_{(t_0,t_0+\tau)} |y'_{\widetilde{g}}(s)|\,\mathrm{d}\,\widetilde{g}(s),\end{aligned}$$

which is a finite number as $y'_{\widetilde{g}} \in \mathcal{L}^1_{\widetilde{g}}(I_\tau, \mathbb{R})$. Thus, $x'_g \in \mathcal{L}^1_g(I_\tau, \mathbb{R})$. Finally, (4.38) and Theorem 3.27 ensure that, for $t \in (t_0, t_0 + \tau]$,

$$\begin{aligned} y(t) &= x_0 + f(t_0, x_0)\Delta g(t_0) + \int_{[t_0,t)} y'_{\widetilde{g}}(s)\,\mathrm{d}\,\widetilde{g}(s)\\ &= x_0 + \int_{\{t_0\}} x'_g(s)\,\mathrm{d}\,g(s) + \int_{(t_0,t)} x'_g(s)\,\mathrm{d}\,g(s) = x_0 + \int_{[t_0,t)} x'_g(s)\,\mathrm{d}\,g(s).\end{aligned}$$

Since $x = y$ on $(t_0, t_0 + \tau]$, it follows that

$$x(t) = x_0 + \int_{[t_0,t)} x'_g(s)\,\mathrm{d}\,g(s), \quad t \in \overline{I}_\tau,$$

which proves that x is g–absolutely continuous on $\overline{I}_\tau$, see Theorem 3.27. □

Existence of solution

Here we present some existence results for (4.2), following the ideas of the classical results for ODEs. In particular, most of the results here are based on the definition of g–Carathéodory function. To that end, we recall its definition, by presenting Definition 1.27 in the context of Lebesgue–Stieltjes measure spaces.

Definition 4.29. *Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing and left–continuous function, $J\subset\mathbb{R}$ and $X\subset\mathbb{R}^m$. A function $f:J\times X\to\mathbb{R}^n$ is said to be g–*Carathéodory *if the following properties are satisfied:*

(i) *$f(\cdot,x)$ is g–measurable for all $x\in X$.*

(ii) *$f(t,\cdot)$ is continuous for g–a.a. $t\in J$.*

(iii) *For all $r>0$ there exists $h_r\in\mathcal{L}^1_g(J,[0,+\infty))$ such that*

$$\|f(t,x)\|\le h_r(t),\quad g\text{–a.a. } t\in J,\quad x\in X,\ \|x\|\le r.$$

Remark 4.30. Note that problem (4.2) with a g–Carathéodory right–hand side is a particular case of a *measure differential equation* as named and studied in [64,78]. Indeed, let $\tau\in(0,T]$ and assume that f is a g–Carathéodory function on $I_\tau\times\mathbb{R}^n$. If x is a solution of (4.2) on I_τ, then, as pointed out in Remark 4.2, we have that

$$x(t)=x_0+\int_{[t_0,t)} f(s,x(s))\,\mathrm{d}\,g(s),\quad t\in I_\tau.$$

In particular, the integral also exists in the Kurzweil–Stieltjes sense and we have

$$x(t)=x_0+{}^{(KS)}\!\!\int_{t_0}^{t} f(s,x(s))\,\mathrm{d}\,g(s)\quad t\in I_\tau. \tag{4.39}$$

We have proven that x is a solution of the measure differential equation (4.39), for which important existence results are available in [64].

A converse result is true in a reasonably general situation. If a given function $x:I_\tau\longrightarrow\mathbb{R}$ solves the measure differential equation (4.39) and, in addition, the composition $f(\cdot,x(\cdot))$ is integrable in the Lebesgue–Stieltjes sense, then Theorem 3.26 guarantees that x solves (4.2) and $x\in\mathcal{AC}_g(\overline{I}_\tau,\mathbb{R}^n)$.

Our first existence result is a Peano–type result that can be found in [33, Theorem 7.5], which we include here with a more detailed proof.

Theorem 4.31. *Let $r>0$ and $f:I\times\overline{B(x_0,r)}\to\mathbb{R}^n$ be a g–Carathéodory function. Then there exists $\tau\in(0,T]$ such that* (4.2) *has a solution on I_τ.*

Proof. Take $R=r+\|x_0\|$. Since f is g–Carathéodory, there exists $h_R\in\mathcal{L}^1_g(I,[0,+\infty))$ such that

$$\|f(t,x)\|\le h_R(t)\quad g\text{–a.a. } t\in I,\quad x\in\mathbb{R}^n,\ \|x\|\le R.$$

In particular, we have that

$$\|f(t,x)\|\le h_R(t)\quad g\text{–a.a. } t\in I,\quad x\in\overline{B(x_0,r)}.$$

Fix $\tau\in(0,T]$ such that

$$\int_{[t_0,t_0+\tau)} h_R(s)\,\mathrm{d}\,g(s)\le r. \tag{4.40}$$

Define $X = \{x \in \mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n) : \|x(t) - x_0\| \le r \text{ for all } t \in \overline{I}_\tau\}$ and $F : X \to X$ as

$$Fx(t) = x_0 + \int_{[t_0,t)} f(s, x(s)) \, \mathrm{d}\, g(s), \quad t \in \overline{I}_\tau.$$

First note that X is a nonempty as the function

$$H(t) = x_0 + \int_{[t_0,t)} h_R(s) \, \mathrm{d}\, g(s), \quad t \in \overline{I}_\tau,$$

belongs to X. Moreover, by construction, X is a closed convex subset of $\mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$. Furthermore, Proposition 1.28 and Theorem 3.26 ensure that F is well–defined and $Fx \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R}^n)$ for all $x \in \mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$. Thus, it is enough to show that F is compact, as in that case Schauder's Fixed Point Theorem guarantees that F has at least one fixed point, or equivalently, (4.2) has a solution according to Remark 4.2.

Let $A \subset X$ be a bounded set. Let us show that $F(A)$ is a relatively compact subset of $\mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$. Indeed, first of all, note $\{x(t_0) : x \in F(A)\} = \{x_0\}$. Moreover, for g–a.a. $t \in I_\tau$ we have that

$$\|(Fx)'_g(t)\| = \left\| \left(x_0 + \int_{[t_0,t)} f(s, x(s)) \, \mathrm{d}\, g(s) \right)'_g (t) \right\| = \|f(t, x(t))\| \le h_R(t).$$

Hence, the hypotheses of Proposition 3.31 are satisfied. Thus, $F(A)$ is a relatively compact subset of $\mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$, which concludes the proof. □

A slight modification of this proof yields the existence of a global solution when we consider $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ bounded by an integrable function. The key to obtain this new result is to notice that the local character of the solution in Theorem 4.31 comes from (4.40).

Theorem 4.32. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following conditions:*

(i) *For each $x \in \mathcal{BC}_g(\overline{I}, \mathbb{R}^n)$, the map $f(\cdot, x(\cdot))$ is g–measurable.*

(ii) *There exists $h \in \mathcal{L}^1_g(I, [0, +\infty))$ such that*

$$\|f(t, x)\| \le h(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}^n.$$

Then (4.2) *has a solution on I.*

Proof. First of all, note that the hypotheses ensure that, for any $x \in \mathcal{BC}_g(\overline{I}, \mathbb{R}^n)$, we have that $f(\cdot, x(\cdot)) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$ as

$$\int_I \|f(s, x(s))\| \, \mathrm{d}\, g(s) \le \int_I h(s) \, \mathrm{d}\, g(s) < +\infty.$$

Thus, the map $F : \mathcal{BC}_g(\overline{I}, \mathbb{R}^n) \to \mathcal{BC}_g(\overline{I}, \mathbb{R}^n)$ defined as

$$Fx(t) = x_0 + \int_{[t_0,t)} f(s, x(s)) \, \mathrm{d}\, g(s),$$

is well–defined. The rest of the proof is analogous to that of Theorem 4.31, and we omit it. □

Remark 4.33. Observe that, in hypothesis (i) in Theorem 4.32, the compositions $f(\cdot, x(\cdot))$ are maps defined on I and not $\overline{I}$ despite x being defined on $\overline{I}$.

Remark 4.34. Propositions 1.28 and 3.22 ensure that condition (i) in Theorem 4.32 is satisfied if f is g–Carathéodory. Thus, Theorem 4.32 is a more general version of [48, Proposition 2.3] as in such result, condition (i) in Theorem 4.32 is replaced by the assumption of f being g–Carathéodory. Observe, however, that [48, Proposition 2.3] does not imply Theorem 4.32. That is, the conditions in Theorem 4.32 do not imply that f is g–Carathéodory. To see that, it is enough to consider $g : \mathbb{R} \to \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} t, & \text{if } t \leq 0, \\ t+1 & \text{if } t > 0, \end{cases} \qquad f(t,x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

It is easy to see that f is $\mathcal{L}^1_g([0,1),\mathbb{R})$–bounded by the constant function 1. Moreover $f(\cdot, x(\cdot))$ is g–measurable for each $x \in \mathcal{BC}_g([0,1],\mathbb{R})$. Indeed, given $x \in \mathcal{BC}_g([0,1],\mathbb{R})$, we have that x is g–measurable, as pointed out in Proposition 3.22. As a consequence, the set $A = x^{-1}((0,+\infty))$ is g–measurable, and so $f(\cdot, x(\cdot)) = \chi_A(\cdot)$ is g–measurable. However, f cannot be a g–Carathéodory function as $f(0,\cdot)$ is not continuous and $\mu_g(\{0\}) = 1$.

The next existence result is obtained through Proposition 4.31 and the following lemma. We introduce this result as a first approach to the Montel–Osgood–Tonelli condition, which is usually known as a uniqueness condition. However, this result will be fundamental to obtain an Montel–Osgood–Tonelli existence and uniqueness type result later in this work.

Lemma 4.35. *Let $X \subset \mathbb{R}^n$, $x_0 \in X$ and $f : I \times X \to \mathbb{R}^n$ satisfy the following conditions:*

(i) *For every $x \in X$, $f(\cdot, x)$ is g–measurable.*

(ii) $f(\cdot, x_0) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$.

(iii) *There exist $\varphi \in \mathcal{L}^1_g(I,[0,+\infty))$ and $\omega : [0,+\infty) \to [0,+\infty)$ nondecreasing, continuous at 0 with $\omega(0) = 0$, and such that*

$$\|f(t,x) - f(t,y)\| \leq \varphi(t)\omega(\|x-y\|), \quad g\text{–a.a. } t \in I, \quad x,y \in X.$$

Then f is g–Carathéodory.

Proof. First, note that condition (i) in the statement of the theorem corresponds to condition (i) from Definition 4.29. Furthermore, the continuity of the map $f(t,\cdot) : X \to \mathbb{R}^n$, for g–a.a. $t \in I$, follows from the continuity of ω at 0. Indeed, fix $t \in I$, and let $x \in X$ and $\varepsilon > 0$ be given. If $\varphi(t) = 0$, then the continuity is trivial, so we shall assume that $\varphi(t) \neq 0$. In that case, since ω is continuous at 0, there exists $\delta > 0$ such that if $0 < s < \delta$, then $\omega(s) < \varepsilon/\varphi(t)$. Hence, for all $y \in X$ such that $\|x - y\| < \delta$ we have

$$\|f(t,x) - f(t,y)\| \leq \varphi(t)\omega(\|x-y\|) < \varepsilon,$$

that is $f(t,\cdot)$ is continuous at x. Hence, all that is left to check is condition (iii) from Definition 4.29.

Consider $R > 0$ and let $h_R(t) = \varphi(t)\omega(R + \|x_0\|) + \|f(t, x_0)\|$, $t \in I$. Note that conditions (i) and (ii) ensure that $h_R \in \mathcal{L}^1_g(I)$, while using condition (iii) we get

$$\begin{aligned}\|f(t,x)\| &\le \|f(t,x) - f(t,x_0)\| + \|f(t,x_0)\| \le \varphi(t)\omega(\|x - x_0\|) + \|f(t,x_0)\| \\ &\le \varphi(t)\omega(\|x\| + \|x_0\|) + \|f(t,x_0)\| \le \varphi(t)\omega(R + \|x_0\|) + \|f(t,x_0)\| = h_R(t),\end{aligned}$$

for g–a.a. $t \in I$ and for all $x \in X$, $\|x\| \le R$, which concludes the proof. □

Essentially, Lemma 4.35 ensures that under the Montel–Osgood–Tonelli conditions, the initial value problem (4.2) has at least one solution. This is a direct consequence of our Peano–type result, Theorem 4.31. We formalize this information in the following result.

Theorem 4.36. *Let $r > 0$ and $f : I \times \overline{B(x_0, r)} \to \mathbb{R}^n$ satisfy the following conditions:*

(i) *For every $x \in \overline{B(x_0, r)}$, $f(\cdot, x)$ is g–measurable.*

(ii) $f(\cdot, x_0) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$.

(iii) *There exist $\varphi \in \mathcal{L}^1_g(I, [0, +\infty))$ and $\omega : [0, +\infty) \to [0, +\infty)$ nondecreasing, continuous at 0 with $\omega(0) = 0$, and such that*

$$\|f(t,x) - f(t,y)\| \le \varphi(t)\omega(\|x - y\|), \quad g\text{–a.a. } t \in I, \quad x, y \in \overline{B(x_0, r)}.$$

Then there exists $\tau \in (0, T]$ such that (4.2) *has a solution on I_τ.*

Uniqueness of solution

We now study some conditions guaranteeing the uniqueness of solution of the initial value problem (4.2). We follow analogous steps to those in the classical analysis, exploring more complex conditions in our results. The first uniqueness of solution result is a generalization of the Lipschitz uniqueness result. In this case, we obtain our result using the Gronwall–type inequality introduced earlier.

Theorem 4.37. *Let $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, and $f : I \times X \to \mathbb{R}^n$. If there exists $L \in \mathcal{L}^1_g(I_\tau, [0, +\infty))$ such that*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x - y\|, \quad g\text{–a.a. } t \in I_\tau, \quad x, y \in X,$$

then (4.2) *has at most one solution on I_τ.*

Proof. Suppose that $x_1, x_2 \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R}^n)$ are two solutions of (4.2) on I_τ. It follows from Theorem 3.27 that $f(\cdot, x_i(\cdot)) \in \mathcal{L}^1_g(I_\tau, \mathbb{R}^n)$, $i = 1, 2$. As a consequence, we have that the map $\|f(\cdot, x_1(\cdot)) - f(\cdot, x_2(\cdot))\|$ is g–integrable over I_τ.

Define $u(t) = \|x_1(t) - x_2(t)\|$, $t \in \overline{I}_\tau$. Clearly, u is nonnegative and bounded on $\overline{I}_\tau$ as x_1 and x_2 are bounded. Hence, it follows that $u \cdot L \in \mathcal{L}^1_g(I_\tau, [0, +\infty))$. Furthermore, the Fundamental Theorem of Calculus yields that for $t \in \overline{I}_\tau$,

$$\begin{aligned}u(t) &= \left\| \int_{[t_0,t)} f(s, x_1(s)) \,\mathrm{d}\, g(s) - \int_{[t_0,t)} f(s, x_2(s)) \,\mathrm{d}\, g(s) \right\| \\ &\le \int_{[t_0,t)} \|f(s, x_1(s)) - f(s, x_2(s))\| \,\mathrm{d}\, g(s) \le \int_{[t_0,t)} L(s)u(s) \,\mathrm{d}\, g(s).\end{aligned}$$

Hence, (4.14) holds with $K = 0$. As a consequence, (4.15) holds for $K = 0$, which implies that $u = 0$ on I_τ, or equivalently, $x_1 = x_2$ on that interval. □

The applications of Theorem 4.37 are well–known. For example, we can go back to the nonhomogeneous linear equation, (4.17). In that case, we have that $f(t,x) = h(t) - d(t)x(t)$, $(t,x) \in I \times \mathbb{R}$, and so for any $t \in I$,

$$|f(t,x) - f(t,y)| = |-d(t)x + d(t)y| = |d(t)||x-y|, \quad x, y \in \mathbb{R}.$$

Hence, if $d \in \mathcal{L}^1_g(I,\mathbb{R})$, we can ensure through Theorem 4.37 that (4.17) has at most one solution.

Of course, Theorem 4.37 also has its limitations. To solve these limitations, similarly to the classical results for ODEs, we can consider some moduli of continuity weaker than Lipschitz continuity; the so-called Osgood condition [68] (see also [1]). In order to obtain such an Osgood–type theorem for g–differential equations (4.2), we will need the following result, which is a particular case of the general Gronwall inequality, [77, Theorem 1.40]. To obtain this result, it is enough to note that the integrals involved in it exist in the sense of Kurzweil–Stieltjes integrals.

Lemma 4.38. *Let $h : [a,b] \to \mathbb{R}$ be a nondecreasing and left–continuous function and $\omega : [0,+\infty) \to [0,+\infty)$ be a continuous and nondecreasing function such that $\omega(0) = 0$, $\omega(s) > 0$ for $s > 0$. For a fixed $u_0 > 0$, define*

$$\Omega(r) = \int_{u_0}^{r} \frac{1}{\omega(s)}\,\mathrm{d}\,s, \quad r \in (0,+\infty).$$

and denote by

$$\alpha = \lim_{r\to 0^+} \Omega(r) \geq -\infty, \quad \beta = \lim_{r\to\infty} \Omega(r) \leq +\infty.$$

If $\psi : [a,b] \to [0,+\infty)$ is a h–measurable bounded function, and there exists $\kappa > 0$ such that

$$\psi(s) \leq \kappa + \int_{[a,s)} \omega(\psi(\tau))\,\mathrm{d}\,h(\tau), \quad s \in [a,b], \tag{4.41}$$

and $\Omega(\kappa) + h(b) - h(a) < \beta$, then

$$\psi(s) \leq \Omega^{-1}(\Omega(\kappa) + h(s) - h(a)), \quad s \in [a,b],$$

where $\Omega^{-1} : (\alpha,\beta) \to \mathbb{R}$ is the inverse function of Ω.

Now, with the aid of Lemma 4.38 and following the ideas of [77, Theorem 4.8], we can state and proof the Osgood Uniqueness Theorem for differential equations with Stieltjes derivatives. As we will see later, this result solves some of the limitations of Theorem 4.37.

Theorem 4.39. *Let $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$ and $\omega : [0,+\infty) \to [0,+\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$, $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *For g–a.a.* $t \in I_\tau$ *and for all* $x, y \in X$,

$$\|f(t,x) - f(t,y)\| \le \omega(\|x-y\|).$$

(ii) *For every* $u > 0$,

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^u \frac{1}{\omega(s)} \, \mathrm{d}\, s = +\infty.$$

Then (4.2) *has at most one solution on* I_τ.

Proof. Let x_1 and x_2 be two solutions of (4.2) on I_τ. Define $\psi(t) = \|x_1(t) - x_2(t)\|$, $t \in \overline{I}_\tau$. Since $\psi \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R})$, the map $\omega \circ \psi$ is bounded. Let $K > 0$ be an upper bound of $\omega \circ \psi$. Then for each $\sigma \in (0, \tau)$ we have

$$\int_{[t_0, t_0+\sigma)} \omega(\psi(s)) \, \mathrm{d}\, g(s) \le \int_{[t_0, t_0+\sigma)} K \, \mathrm{d}\, g(s) = K\mu_g([t_0, t_0+\sigma)) < \varepsilon(\sigma), \tag{4.42}$$

where $\varepsilon(\sigma) = K\mu_g([t_0, t_0+\sigma)) + \sigma > 0$.

Fixed an arbitrary $u_0 > 0$, consider the function

$$\Omega(r) = \int_{u_0}^r \frac{1}{\omega(s)} \, \mathrm{d}\, s, \quad r \in (0, +\infty).$$

Note that Ω is strictly increasing and $\Omega(u_0) = 0$. We claim that there is $0 < R < \tau$ such that

$$\Omega(\varepsilon(\delta)) + g(t_0+\tau) - g(t_0+\delta) < \beta := \lim_{r \to \infty} \Omega(r) \quad \text{for } \delta \in (0, R). \tag{4.43}$$

Indeed, first observe that for any $\delta \in (0, \tau)$

$$\Omega(\varepsilon(\delta)) + g(t_0+\tau) - g(t_0+\delta) < \Omega(\varepsilon(\delta)) + g(t_0+\tau) - g(t_0). \tag{4.44}$$

Without loss of generality, we assume that g is continuous at t_0, see Proposition 4.28. This ensures that $\lim_{\tau \to 0^+} \varepsilon(\sigma) = 0$, while condition (ii) implies that $\lim_{r \to 0^+} \Omega(r) = -\infty$. Thus

$$\lim_{\delta \to 0^+} \Omega(\varepsilon(\delta)) + g(t_0+\sigma) - g(t_0) = -\infty,$$

which together with (4.44) guarantees the existence of some $R > 0$ satisfying (4.43).

Using the Fundamental Theorem of Calculus, the inequality (4.42) and condition (i), for each $t \in \overline{I}_\tau$ we obtain

$$\begin{aligned}
\psi(t) &= \left\| \int_{[t_0, t)} \Big(f(s, x_1(s)) - f(s, x_2(s)) \Big) \, \mathrm{d}\, g(s) \right\| \\
&= \int_{[t_0, t_0+\delta)} \omega(\psi(s)) \, \mathrm{d}\, g(s) + \int_{[t_0+\delta, t)} \omega(\psi(s)) \, \mathrm{d}\, g(s) \\
&< \varepsilon(\delta) + \int_{[t_0+\delta, t)} \omega(\psi(s)) \, \mathrm{d}\, g(s)
\end{aligned}$$

for all $\delta \in (0, R)$. Therefore, the assumptions of Lemma 4.38 are satisfied and we have

$$\psi(t) \leq \Omega^{-1}(\Omega(\varepsilon(\delta)) + g(t) - g(t_0 + \delta)), \quad \delta \in (0, R), \quad t \in \overline{I}_\tau.$$

Applying Ω on both sides of the inequality, we obtain

$$\Omega(\psi(t)) - \Omega(\varepsilon(\delta)) \leq g(t) - g(t_0 + \delta) \leq g(t) - g(t_0), \quad \delta \in (0, R), \quad t \in \overline{I}_\tau.$$

Assume that $\psi \neq 0$ on $\overline{I}_\tau$. If that is the case, there is some $t^* \in \overline{I}_\tau$ such that $\psi(t^*) > 0$. Then for all $\delta \in (0, R)$ such that $\delta < t^* - t_0$ we have

$$\int_{\varepsilon(\delta)}^{\psi(t^*)} \frac{1}{\omega(s)} \, \mathrm{d}\, s = \Omega(\psi(t^*)) - \Omega(\varepsilon(\delta)) < g(t^*) - g(t_0),$$

and, by taking the limit as $\delta \to 0^+$, we obtain

$$\lim_{\delta \to 0^+} \int_{\varepsilon(\delta)}^{\psi(t^*)} \frac{1}{\omega(s)} \, \mathrm{d}\, s < g(t^*) - g(t_0) < +\infty,$$

which contradicts (ii). Hence we must have $\psi = 0$ on $\overline{I}_\tau$, i.e. $x_1 = x_2$ on that interval. □

Remark 4.40. The maps ω satisfying the conditions described in Theorem 4.39 are known as *Osgood moduli of continuity*. A classical example of these functions is the map ω given by

$$\omega(r) = \begin{cases} 0, & \text{if } r = 0, \\ r \log\left(\dfrac{1}{r}\right), & \text{if } 0 < r \leq \dfrac{1}{e}, \\ \dfrac{1}{e}, & \text{if } r > \dfrac{1}{e}. \end{cases}$$

It can be proven that ω does not satisfy a Lipschitz condition in any interval containing 0. This information can be used to show that Theorem 4.37 cannot be used to ensure the uniqueness of solution of (4.2) when we consider the function $f(t, x) = \omega(|x|)$, $(t, x) \in I \times \mathbb{R}$. Later, in Proposition 4.52, we study a family of Osgood moduli of continuity based on this example, where we also prove the limitations of Theorem 4.37 for this family.

Note that the initial value problem in Remark 4.40 does not depend on t as it is defined in terms of the Osgood moduli of continuity. It is possible to improve Theorem 4.39 to solve this issue, following the ideas of the so-called Montel-Tonelli uniqueness criterion (cf. [1, Theorem 1.5.1]). Note that in the proof of the following result we make use of some Kurzweil–Stieltjes integrals.

Theorem 4.41. *Let $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$ and $\omega : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$, $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *There exists $\varphi \in \mathcal{L}^1_g(I_\tau, [0, +\infty))$ such that*

$$\|f(t, x) - f(t, y)\| \leq \varphi(t)\omega(\|x - y\|), \quad g\text{–a.a. } t \in I_\tau, \quad x, y \in X. \tag{4.45}$$

(ii) *For every* $u > 0$,

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{u}\frac{1}{\omega(s)}\,\mathrm{d}\,s = +\infty.$$

Then (4.2) *has at most one solution on* I_τ.

Proof. Let x_1 and x_2 be two solutions of (4.2) on I_τ. Recall that Theorem 3.27 ensures that $f(\cdot, x_i(\cdot)) \in \mathcal{L}^1_g(I_\tau, \mathbb{R}^n)$, $i = 1, 2$. For each $t \in \overline{I}_\tau$, write $\psi(t) = \|x_1(t) - x_2(t)\|$. Note that $\omega\circ\psi$ is bounded by the same argument as in Theorem 4.39. Therefore,

$$|\omega(\psi(t))\varphi(t)| \le K|\varphi(t)|, \quad t \in \overline{I}_\tau,$$

where $K > 0$ is an upper bound of $\omega\circ\psi$. This ensures that $\varphi\cdot\omega\circ\psi \in \mathcal{L}^1_g(I_\tau, [0, +\infty))$.

Define $\overline{g} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \le t_0,\\ \displaystyle\int_{[t_0,t)} \varphi(s)\,\mathrm{d}\,g(s), & \text{if } t_0 < t \le t_0+\tau,\\ \displaystyle\int_{[t_0,t_0+\tau)} \varphi(s)\,\mathrm{d}\,g(s), & \text{if } t > t_0+\tau. \end{cases}$$

Observe that $\overline{g}$ is nondecreasing and left–continuous. Recalling the relation between the integrals of Lebesgue–Stieltjes and Kurzweil–Stieltjes, for each $t \in \overline{I}_\tau$ we have

$$\begin{aligned}\int_{[t_0,t)} \omega(\psi(s))\varphi(s)\,\mathrm{d}\,g(s) &= {}^{(KS)}\!\!\int_{t_0}^{t} \omega(\psi(s))\varphi(s)\,\mathrm{d}\,g(s)\\ &= {}^{(KS)}\!\!\int_{t_0}^{t} \omega(\psi(s))\,\mathrm{d}\,\overline{g}(s) = \int_{[t_0,t)} \omega(\psi(s))\,\mathrm{d}\,\overline{g}(s), \end{aligned} \tag{4.46}$$

where the second equality is a consequence of the substitution formula for Kurzweil–Stieltjes integral, Theorem 1.69. Recall that without loss of generality, we can assume that g is continuous at t_0, which implies that $\overline{g}$ is also continuous at t_0. This, together with equality above, implies that for each $\sigma \in (0, \tau)$ we have

$$\int_{[t_0,t_0+\sigma)} \omega(\psi(s))\,\mathrm{d}\,\overline{g}(s) \le \int_{[t_0,t_0+\sigma)} K\,\mathrm{d}\,\overline{g}(s) < K\mu_{\overline{g}}([t_0, t_0+\sigma)) + \sigma =: \varepsilon(\sigma). \tag{4.47}$$

Therefore, the result can be proved by reasoning as in the proof of Theorem 4.39 with the appropriate adjustments, i.e. replacing g by $\overline{g}$ accordingly. □

The generalization of Osgood's criterion can be taken a step further, following the ideas of Perron's theorem for ODEs. The main difference between this result and Theorem 4.41 lies in the fact that in the inequality (4.45) the variables are separatable in some sense. For Perron's criterion, we do not assume separability on the variables. However, more complex conditions on the corresponding map are necessary to guarantee uniqueness of solution.

Theorem 4.42. *Let* $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I\times X \to \mathbb{R}^n$ *and* $\omega : \overline{I}_\tau \times [0, +\infty) \to [0, +\infty)$ *be* $(g\times \mathrm{Id})$*–continuous. Assume that the following conditions are satisfied:*

(i) *For g–a.a. $t \in I_\tau$ and for all $x, y \in X$,*
$$\|f(t,x) - f(t,y)\| \le \omega(t, \|x-y\|).$$

(ii) *For each $r \in [0,+\infty)$, $\omega(\cdot, r) \in \mathcal{L}^1_g(I_\tau, [0,+\infty))$.*

(iii) *For every $t \in D_g \cap \overline{I}_\tau$, there exists $\delta_t > 0$ such that for all $s \in (t-\delta_t, t+\delta_t) \cap \overline{I}_\tau$, $\omega(s,\cdot)$ is nondecreasing.*

(iv) *The only g–absolutely continuous function on $\overline{I}_\tau$ satisfying*
$$z'_g(t) \le w(t, z(t)), \quad g\text{–a.a. } t \in I_\tau, \quad z(t_0) \le 0, \tag{4.48}$$
is the null function.

Then (4.2) *has at most one solution on I_τ.*

Proof. Let x_1 and x_2 be two solutions of (4.2) on I_τ. Recall that Theorem 3.27 guarantees that $f(\cdot, x_i(\cdot)) \in \mathcal{L}^1_g(I_\tau, \mathbb{R}^n)$, $i = 1, 2$. Define $\psi(t) = \|x_1(t) - x_2(t)\|$, $t \in \overline{I}_\tau$. We will show that ψ satisfies (4.48), which then implies $\psi \equiv 0$, i.e., $x_1 = x_2$.

Clearly, $\psi(t_0) = 0$ and, by Proposition 3.28, $\psi \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R})$. We claim that conditions (ii)-(iii) imply $\omega(\cdot, \psi(\cdot)) \in \mathcal{L}^1_g(I_\tau, [0,+\infty))$. First of all, note that Lemma 4.6 ensures that the composition is g–continuous, and in particular, g–measurable. Now, let
$$\mathcal{O} = \bigcup_{t \in \overline{I}_\tau \cap D_g} (t - \delta_t, t + \delta_t).$$

Note that $\mathcal{O}$ is open by definition, hence $\overline{I}_\tau \setminus \mathcal{O}$ is compact. The fact that g is continuous on $\overline{I}_\tau \setminus \mathcal{O}$ together with Proposition 3.21 yield that $\omega(\cdot, \psi(\cdot))$ is continuous on $\overline{I}_\tau \setminus \mathcal{O}$, therefore bounded. Let $M > 0$ be an upper bound of $\omega(\cdot, \psi(\cdot))$ on $\overline{I}_\tau \setminus \mathcal{O}$, and let $K > 0$ be an upper bound of ψ on $\overline{I}_\tau$. Then
$$\omega(s, \psi(s)) \le M + \omega(s, K), \quad s \in \overline{I}_\tau,$$
and so it follows that $\omega(\cdot, \psi(\cdot)) \in \mathcal{L}^1_g(I_\tau, [0,+\infty))$. Put
$$\Psi(t) = \int_{[t_0, t)} \omega(s, \psi(s)) \, \mathrm{d}\, g(s), \quad t \in \overline{I}_\tau. \tag{4.49}$$

Thus $\Psi \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R})$ and consequently it is g–differentiable for g–a.a. $t \in I_\tau$. Denote by E the set of points of I_τ where both derivatives ψ'_g and Ψ'_g exist. If $t \in D_g \cap E$, then by Remark 3.4 we have that
$$\begin{aligned}
\psi'_g(t) &= \frac{\|x_1(t^+) - x_2(t^+)\| - \|x_1(t) - x_2(t)\|}{\Delta g(t)} \\
&= \frac{\|x_1(t) - x_2(t) + \big(f(t, x_1(t)) - f(t, x_2(t))\big)\Delta g(t)\|}{\Delta g(t)} - \frac{\|x_1(t) - x_2(t)\|}{\Delta g(t)} \\
&\le \|f(t, x_1(t)) - f(t, x_2(t))\| \le \omega(t, \|x_1(t) - x_2(t)\|),
\end{aligned}$$

that is, $\psi_g'(t) \leq \omega(t, \psi(t))$ for $t \in D_g \cap E$. On the other hand, for $t \in E \setminus D_g$ we have

$$\begin{aligned}
\psi_g'(t) &= \lim_{s \to t+} \frac{\psi(s) - \psi(t)}{g(s) - g(t)} \\
&= \lim_{s \to t^+} \frac{\|x_1(s) - x_2(s)\| - \|x_1(t) - x_2(t)\|}{g(s) - g(t)} \\
&\leq \lim_{s \to t^+} \frac{\|x_1(s) - x_1(t) - \big(x_2(s) - x_2(t)\big)\|}{g(s) - g(t)} \\
&= \lim_{s \to t^+} \frac{1}{g(s) - g(t)} \left\| \int_{[t,s)} \big(f(r, x_1(r)) - f(r, x_2(r))\big) \, \mathrm{d}\, g(r) \right\| \\
&\leq \lim_{s \to t^+} \frac{1}{g(s) - g(t)} \int_{[t,s)} \omega(r, \psi(r)) \, \mathrm{d}\, g(r) \\
&= \lim_{s \to t+} \frac{\Psi(s) - \Psi(t)}{g(s) - g(t)} = \Psi_g'(t).
\end{aligned}$$

Theorem 3.26 ensures that $\Psi_g'(t) = \omega(t, \psi(t))$, so we conclude that $\psi_g'(t) \leq \omega(t, \psi(t))$ for every $t \in E$; proving that ψ satisfies (4.48). □

It is worth highlighting that the first appearance of the so-called Perron uniqueness criterion for ODEs dates back to 1925 and appears in [11], one year before Perron's work [69] came to light. However, Bompiani's result in [11] relies on the additional assumption of ω being monotone increasing in the second variable.

Condition (iii), as it reads in Theorem 4.42, resembles the notion of *monotonicity around a set* that appears in the theory of variational measures in connection with more general notions of differentiability, cf. [26]. Naturally, for g continuous, such a condition plays no role, and in the case when $g = \mathrm{Id}$ we retrieve the classical result for ODEs, see [1, Theorem 1.11.4] and [18]. There is a way to avoid condition (iii) in Theorem 4.42, as presented in the following result. Essentially, we impose some alternative conditions to ensure that the map in (4.49) is well–defined.

Theorem 4.43. *Let* $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$ *and* $\omega : \overline{I}_\tau \times [0, +\infty) \to [0, +\infty)$ *be g–Carathéodory function. If* ω *satisfies conditions* (i) *and* (iv) *in* Theorem 4.42*, then* (4.2) *has at most one solution on* I_τ.

Proof. Let x_1 and x_2 be two solutions of (4.2) on I_τ and define $\psi : \overline{I}_\tau \to \mathbb{R}$ as

$$\psi(t) = \|x_1(t) - x_2(t)\|, \quad t \in \overline{I}_\tau.$$

By definition, $\psi \in \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R})$. Therefore, it is g–measurable and bounded. Hence, Proposition 1.28 ensures that $\omega(\cdot, \psi(\cdot)) \in \mathcal{L}_g^1(I_\tau, [0, +\infty))$. Now, it is enough to define $\Psi : \overline{I}_\tau \to \mathbb{R}$ as in (4.49) and repeat the arguments in Theorem 4.42 to show that ψ satisfies (4.48) to obtain the result. □

Existence and uniqueness of solution

We now present some existence and uniqueness results for (4.2). Most of the results here are obtained by looking at the compatibilities between the hypotheses of the results ensuring existence of solution with the ones guaranteeing uniqueness. For example, we have the following result for the existence and uniqueness of a local solution which is a direct consequence of Theorem 4.31 together with Theorem 4.37.

Theorem 4.44. *Let $r > 0$ and $f : I \times \overline{B(x_0, r)} \to \mathbb{R}^n$ be a g–Carathéodory function. If there exists $L \in \mathcal{L}^1_g(I, [0, +\infty))$ such that*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x - y\|, \quad g\text{–a.a. } t \in I, \quad x, y \in \overline{B(x_0, r)},$$

then there exists $\tau \in (0, T]$ such that (4.2) has a unique solution on I_τ.

Remark 4.45. A similar result can be found in [33, Theorem 7.4], where instead of f being g–Carathéodory, f is assumed to satisfy the following conditions:

1. for every $x \in \overline{B(x_0, r)}$, $f(\cdot, x)$ is g–measurable;
2. $f(\cdot, x_0) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$.

Let us show that this set of hypotheses is equivalent to f being g–Carathéodory as long as the Lipschitz condition in Theorem 4.44 holds.

First assume that conditions 1 and 2 hold. Note that condition 1 matches condition (i) in Definition 4.29. Furthermore, the Lipschitz condition in Theorem 4.44 implies condition (ii) in Definition 4.29. Finally, for every $R > 0$ take $h_R(t) = L(t)R + \|f(t, x_0)\|$, which is g–integrable on I. Then for g–a.a. $t \in I$ and all $x \in \overline{B(x_0, r)}$, $\|x\| \le R$, we have

$$\|f(t,x)\| \le \|f(t,x) - f(t,x_0)\| + \|f(t,x_0)\| \le L(t)\|x - y\| + \|f(t,x_0)\| \le h_R(t).$$

Hence condition (iii) in Definition 4.29 is satisfied and f is g–Carathéodory.

Conversely, if f is g–Carathéodory, we just need to show that condition 2 holds. Given $x \in \overline{B(x_0, r)}$ we have that for g–a.a. $t \in I_\tau$

$$\|f(t,x_0)\| \le \|f(t,x_0) - f(t,x)\| + \|f(t,x)\| \le L(t)r + h_{\|x_0\|+r}(t),$$

with $h_{\|x_0\|+r}$ the corresponding function in condition (iii) in Definition 4.29. Thus $f(\cdot, x_0)$ is g–integrable on I, and so the two results are equivalent.

We can obtain a more general result than Theorem 4.44 by combining Theorems 4.36 and 4.41. Indeed, such combination leads to a local Montel–Osgood–Tonelli existence and uniqueness type result which, as we mentioned before, is based on a conditions more general than that of Lipschitz.

Theorem 4.46. *Let $r > 0$, $f : I \times \overline{B(x_0, r)} \to \mathbb{R}^n$ and $\omega : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *For every $x \in \overline{B(x_0, r)}$, $f(\cdot, x)$ is g–measurable.*

(ii) $f(\cdot, x_0) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$.

(iii) *For every* $u > 0$,

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{u} \frac{1}{\omega(s)} \,\mathrm{d}\, s = +\infty.$$

(iv) *There exist* $\varphi \in \mathcal{L}^1_g(I, [0 + \infty))$ *such that*

$$\|f(t,x) - f(t,y)\| \leq \varphi(t)\omega(\|x-y\|), \quad g\text{–a.a. } t \in I, \quad x, y \in \overline{B(x_0, r)}.$$

Then there exists $\tau \in (0, T]$ *such that* (4.2) *has a unique solution on* I_τ.

Remark 4.47. Note that this result yields [33, Theorem 7.4] for the particular choice of $\omega(r) = r$, $r \geq 0$.

So far, we have focused on the existence and uniqueness of solution for problems whose function is defined on a neighbourhood of the initial condition. All of these results have led to ensure the existence of a local solution. We can proceed analogously for problems defined on the whole space. For example, by combining Theorem 4.32 with Theorem 4.37, we can ensure the existence of a unique global solution under the assumption that a Lipschitz condition is satisfied.

Theorem 4.48. *Let* $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the following conditions:*

(i) *For each* $x \in \mathcal{BC}_g(\overline{I}, \mathbb{R}^n)$*, the map* $f(\cdot, x(\cdot))$ *is g–measurable.*

(ii) *There exists* $h \in \mathcal{L}^1_g(I, [0, +\infty))$ *such that*

$$\|f(t,x)\| \leq h(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}^n.$$

(iii) *There exists* $L \in \mathcal{L}^1_g(I, [0, +\infty))$ *such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \quad g\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then (4.2) *has a unique solution on* I.

Similarly to Theorem 4.44, in [33] we find a similar result to Theorem 4.48. Here we include the mentioned result, [33, Theorem 7.3], without its proof. For a proof of this result, we refer the reader to Theorem 5.58 in Chapter 5, where we proof an extension of this result in the context of differential equations with several Stieltjes derivatives.

Theorem 4.49. *Let* $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the following conditions:*

(i) *For every* $x \in \mathbb{R}^n$, $f(\cdot, x)$ *is g–measurable.*

(ii) $f(\cdot, x_0) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$.

(iii) *There exists* $L \in \mathcal{L}^1_g(I, [0, +\infty))$ *such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \quad g\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then (4.2) *has a unique solution on* I.

Note that in this case, the result in [33] deals with a more general case than Theorem 4.48 does, as no $\mathcal{L}^1_g(I,[0,+\infty))$–boundedness condition is required. However, we can obtain a more general result based on the Montel–Osgood–Tonelli conditions. To obtain such a result, we will need the following lemma that gives an a priori estimate for solutions of (4.2).

Lemma 4.50. *Let* $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\omega : [0,+\infty) \to [0,+\infty)$ *be a nondecreasing continuous function such that* $\omega(0) = 0$ *and* $\omega(s) > 0$ *for all* $s > 0$. *Assume that* g *is continuous at* t_0 *and that the following conditions are satisfied:*

(i) *For every* $x \in \mathbb{R}^n$, $f(\cdot,x)$ *is* g*–measurable.*

(ii) $f(\cdot,x_0) \in \mathcal{L}^1_g(I,\mathbb{R}^n)$.

(iii) *For every* $u > 0$,

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^u \frac{1}{\omega(s)}\,\mathrm{d}\,s = +\infty.$$

(iv) *There exist* $\varphi \in \mathcal{L}^1_g(I,[0,+\infty))$ *such that*

$$\|f(t,x) - f(t,y)\| \le \varphi(t)\omega(\|x-y\|), \quad g\text{–a.a. } t \in I, \quad x,y \in \mathbb{R}^n.$$

Then there exist $t_1 \in (t_0, t_0+T]$ *and a nondecreasing function* $h : [t_0,t_1] \to \mathbb{R}$ *such that for every solution of* (4.2), $x : I_\tau \to \mathbb{R}^n$, $\tau \in (0,T]$, *we have*

$$\|x(t) - x_0\| \le h(t), \quad t \in I_\tau \cap [t_0,t_1].$$

Proof. Define $\kappa : \overline{I} \to \mathbb{R}$ as

$$\kappa(t) = \int_{[t_0,t)} \|f(s,x_0)\|\,\mathrm{d}\,g(s), \quad t \in \overline{I}.$$

Now, if $x : \overline{I}_\tau \to \mathbb{R}^n$ is a solution of (4.2), then using condition (iv) we get

$$\|x(t) - x_0\| \le \kappa(t) + \int_{[t_0,t)} \omega(\|x(s)-x_0\|)\varphi(s)\,\mathrm{d}\,g(s), \quad t \in \overline{I}_\tau.$$

Define $\overline{g} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \le t_0, \\ \displaystyle\int_{[t_0,t)} \varphi(s)\,\mathrm{d}\,g(s), & \text{if } t_0 < t \le t_0+\tau, \\ \displaystyle\int_{[t_0,t_0+\tau)} \varphi(s)\,\mathrm{d}\,g(s), & \text{if } t > t_0+\tau. \end{cases}$$

Recalling the relation (4.46), it follows that

$$\|x(t) - x_0\| \le \kappa(t) + \int_{[t_0,t)} \omega(\|x(s)-x_0\|)\,\mathrm{d}\,\overline{g}(s), \quad t \in \overline{I}_\tau, \tag{4.50}$$

for every solution x of (4.2) defined on $\overline{I}_\tau$. In order to apply Lemma 4.38, fix an arbitrary $u_0 > 0$ and consider the function

$$\Omega(r) = \int_{u_0}^{r} \frac{1}{\omega(s)} \, \mathrm{d}\, s, \quad r \in (0, +\infty).$$

Since $\lim_{r \to 0^+} \Omega(r) = -\infty$, there exists $R > 0$ such that

$$\Omega(R) + \overline{g}(t_0 + T) - \overline{g}(t_0) < \beta := \lim_{r \to \infty} \Omega(r) \leq +\infty.$$

Since g is continuous at t_0, we can choose $t_1 \in (t_0, t_0 + T]$ such that $\kappa(t_1) \leq R$. The monotonicity of Ω then yields

$$\Omega(\kappa(t_1)) + \overline{g}(t_1) - \overline{g}(t_0) < \beta.$$

The inequality above together with (4.50) shows that the assumptions of Lemma 4.38 are satisfied on the interval $[t_0, t_1]$, therefore

$$\|x(t) - x_0\| \leq \Omega^{-1}(\Omega(\kappa(t_1)) + \overline{g}(t) - \overline{g}(t_0)) =: h(t), \quad t \in I_\tau \cap [t_0, t_1],$$

and $h : [t_0, t_1] \to \mathbb{R}$ is the desired monotone function. □

With the aid of Lemma 4.50 we can prove a Montel–Osgood–Tonelli existence and uniqueness type result. Note that this result is more general than Theorem 4.49 in its hypotheses, however, we can only ensure the existence of a unique local solution in return.

Theorem 4.51. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $\omega : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *For every $x \in \mathbb{R}^n$, $f(\cdot, x)$ is g–measurable.*

(ii) $f(\cdot, x_0) \in \mathcal{L}^1_g(I, \mathbb{R}^n)$.

(iii) *For every $u > 0$,*

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{u} \frac{1}{\omega(s)} \, \mathrm{d}\, s = +\infty.$$

(iv) *There exist $\varphi \in \mathcal{L}^1_g(I, [0, +\infty))$ such that*

$$\|f(t, x) - f(t, y)\| \leq \varphi(t)\omega(\|x - y\|), \quad g\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then there exists $\tau \in (0, T]$ such that (4.2) *has a unique solution on I_τ.*

Proof. Without loss of generality, we assume that g is continuous at t_0, see Proposition 4.28. Let $h : [t_0, t_1] \to \mathbb{R}$ be the function whose existence is guaranteed by Lemma 4.50. Denote $R := h(t_1)$ and define $\overline{g} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \displaystyle\int_{[t_0, t)} \varphi(s) \, \mathrm{d}\, g(s), & \text{if } t_0 < t \leq t_0 + \tau, \\ \displaystyle\int_{[t_0, t_0+\tau)} \varphi(s) \, \mathrm{d}\, g(s), & \text{if } t > t_0 + \tau. \end{cases}$$

Since g is continuous at t_0, we have that so is $\overline{g}$. Thus we can choose $\tau \in (0, T]$ such that $t_0 + \tau \leq t_1$ and

$$\omega(R)\mu_{\overline{g}}([t_0, t_0 + \tau)) + \int_{[t_0, t_0+\tau)} \|f(s, x_0)\| \,\mathrm{d}\, g(s) < R. \tag{4.51}$$

Consider $B = \left\{x \in \mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n) : \|x(t) - x_0\| \leq R,\, t \in \overline{I}_\tau\right\}$. Clearly, B is a closed and convex subset of $\mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$. Now let us define $F : B \to \mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$ by

$$Fx(t) = x_0 + \int_{[t_0, t)} f(s, x(s)) \,\mathrm{d}\, g(s), \quad t \in I_\tau.$$

We can prove in an analogous way to Theorem 4.36 that f is g–Carathéodory. Thus, Proposition 1.28 and Theorem 3.26 ensure that F is well-defined. Moreover, the continuity of ω together with condition (iv) implies that F is continuous. Further, recalling the relation (4.46), it follows from condition (iv) and (5.32) that, for $x \in B$, we have

$$\|Fx(t) - x_0\| \leq \int_{[t_0, t)} \omega(\|x(s) - x_0\|) \,\mathrm{d}\, \overline{g}(s) + \int_{[t_0, t)} \|f(s, x_0)\| \,\mathrm{d}\, g(s) < R,$$

for every $t \in \overline{I}_\tau$. That is, $F(B) \subset B$. It remains to verify that $F(B)$ is relatively compact in $\mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$. Firstly, note that for $x \in B$ we have

$$\|f(t, x(t))\| \leq \|f(t, x(t)) - f(t, x_0)\| + \|f(t, x_0)\| \leq \varphi(t)\omega(R) + \|f(t, x_0)\| =: M(t),$$

for g–a.a. $t \in I_\tau$. Observe that $F(B) \subset \mathcal{AC}_g(\overline{I}_\tau, \mathbb{R}^n)$ and

$$(Fx)'_g(t) = f(t, x(t)), \quad g\text{–a.a. } t \in I_\tau, \quad \text{for all } x \in B.$$

Since $M \in \mathcal{L}^1_g(I_\tau, [0, +\infty))$, it follows from Proposition 3.31 that $F(B)$ is relatively compact in $\mathcal{BC}_g(\overline{I}_\tau, \mathbb{R}^n)$. Thus, Schauder's Fixed Point Theorem guarantees the existence of solution of (4.2) on I_τ, while the uniqueness is a consequence of Theorem 4.39. □

To finish this section we illustrate the applicability of the Montel–Osgood–Tonelli result. To do so, we describe a family of initial value problems whose uniqueness of solution cannot be determined by means of a Lipschitz's uniqueness theorem, nonetheless each of these problems satisfies the Osgood condition. In particular, we shall describe a family of functions that are examples of Osgood moduli of continuity. To make our presentation more concise, let us define the family of maps $\mathrm{Exp}^{[k]} : \mathbb{R} \to \mathbb{R}$, $k = 0, 1, 2, \ldots$, as

$$\mathrm{Exp}^{[0]}(s) = s, \quad \mathrm{Exp}^{[k]}(s) = \exp(\mathrm{Exp}^{[k-1]}(s)), \quad k \in \mathbb{N}. \tag{4.52}$$

We also define the maps $\mathrm{Log}^{[k]} : (\mathrm{Exp}^{[k-1]}(0), +\infty) \to \mathbb{R}$, $k \in \mathbb{N}$, as

$$\mathrm{Log}^{[1]}(s) = \log(s), \quad \mathrm{Log}^{[k]}(s) = \log(\mathrm{Log}^{[k-1]}(s)), \quad k \in \mathbb{N},\ k \geq 2, \tag{4.53}$$

Clearly, $\mathrm{Log}^{[k]}$ and $\mathrm{Exp}^{[k]}$ define nondecreasing continuous functions. Moreover, we can easily show that, for every $k \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$\mathrm{Log}^{[j]}(\mathrm{Exp}^{[k]}(s)) = \mathrm{Exp}^{[k-j]}(s), \quad j = 1, \ldots, k. \tag{4.54}$$

Using mathematical induction we can also prove the following two identities which will be useful later on:

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\big(\operatorname{Log}^{[k]}(s)\big) = \left(s\prod_{j=1}^{k-1}\operatorname{Log}^{[j]}(s)\right)^{-1}, \tag{4.55}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\prod_{j=1}^{k}\operatorname{Log}^{[j]}(s)\right) = \frac{1}{s}\left(1+\sum_{l=2}^{k}\prod_{j=l}^{k}\operatorname{Log}^{[j]}(s)\right), \tag{4.56}$$

for every $k \in \mathbb{N},\ k \geq 2$, and $s \in (0,+\infty)$.

Proposition 4.52. *Let $\omega_k : [0,\infty) \to \mathbb{R}$, $k \in \mathbb{N}$, be given by*

$$\omega_k(s) = \begin{cases} 0, & \text{if} \quad s = 0, \\ s\displaystyle\prod_{j=1}^{k}\operatorname{Log}^{[j]}\left(\frac{1}{s}\right), & \text{if} \quad s \in \left(0, \dfrac{1}{e_k}\right), \\ \dfrac{1}{e_k^2}\displaystyle\prod_{j=1}^{k}\operatorname{Exp}^{[j]}(1), & \text{if} \quad s \geq \dfrac{1}{e_k}, \end{cases} \tag{4.57}$$

where $e_k := \operatorname{Exp}^{[k]}(1)$. Then, for all $k \in \mathbb{N}$ we have:

(a) *ω_k is well–defined, and $\omega_k(s) > 0$ for all $s > 0$.*

(b) *ω_k is continuous.*

(c) *$\omega_k'(s) > 0$ and $\omega_k''(s) < 0$ for all $s \in \big(0, e_k^{-1}\big)$; moreover,*

$$\lim_{s\to 0^+}\omega_k'(s) = +\infty.$$

(d) *ω_k is nondecreasing and subadditive.*

(e) *For all $u > 0$, we have that*

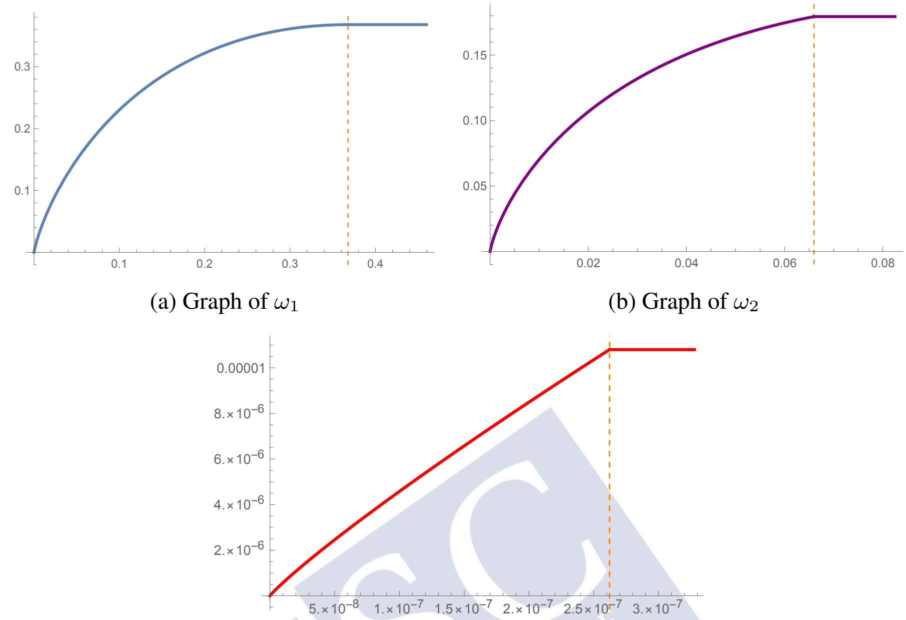
$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{u}\frac{1}{\omega_k(s)}\,\mathrm{d}\,s = +\infty.$$

Proof. (a) Fix $k \in \mathbb{N}$. In order to show that ω_k is well–defined it is enough to show that the arguments in the logarithms are adequate. Indeed, for all $j = 1,\dots,k$ and $s \in \big(0, e_k^{-1}\big)$ we have

$$\frac{1}{s} > e_k = \operatorname{Exp}^{[k]}(1) \geq \operatorname{Exp}^{[j-1]}(1) \geq \operatorname{Exp}^{[j-1]}(0).$$

Moreover, since $\operatorname{Log}^{[k]}$ is a nondecreasing function and having in mind (4.54), we get

$$\operatorname{Log}^{[j]}\left(\frac{1}{s}\right) \geq \operatorname{Log}^{[j]}\left(\operatorname{Exp}^{[j]}(1)\right) = 1 > 0, \quad j = 1,2,\dots,k. \tag{4.58}$$



(a) Graph of ω_1

(b) Graph of ω_2

(c) Graph of ω_3

Figure 4.5: Graphs of some ω_k. The dotted lines represent the equations $x = 1/e_k$.

Therefore, ω_k is well-defined and positive on $(0, \infty)$.

(b) The continuity of $\operatorname{Log}^{[j]}$ together with (4.54) implies that

$$\lim_{s \to \frac{1}{e_k}^-} \omega_k(s) = \frac{1}{e_k} \prod_{j=1}^{k} \operatorname{Log}^{[j]}(\operatorname{Exp}^{[k]}(1)) = \omega_k\left(\frac{1}{e_k}\right),$$

that is, ω_k is continuous at e_k^{-1}. We will prove that $\lim_{s \to 0^+} \omega_k(s) = 0$ by induction over k. For $k = 1$, using a change of variables and L'Hôpital's rule we get

$$\lim_{s \to 0^+} \omega_1(s) = \lim_{r \to \infty} \frac{\log(r)}{r} = \lim_{r \to \infty} \frac{1}{r} = 0.$$

Assuming that the assertion holds for $k \in \mathbb{N}$, that is, assuming

$$\lim_{r \to \infty} \frac{\prod_{j=1}^{k} \operatorname{Log}^{[j]}(r)}{r} = \lim_{s \to 0^+} \omega_k(s) = 0,$$

we will prove that $\lim_{s \to 0^+} \omega_{k+1}(s) = 0$.

First, note that $\lim_{r \to \infty} \operatorname{Log}^{[j]}(r) = +\infty$, $j \in \mathbb{N}$. Having in mind the identities (4.55) and (4.56), by a change of variables and using L'Hôpital's rule recursively we get

$$\begin{aligned}\lim_{s\to 0^+}\omega_{k+1}(s) &= \lim_{r\to\infty}\frac{\prod_{j=1}^{k+1}\mathrm{Log}^{[j]}(r)}{r}\\ &= \sum_{l=2}^{k}\lim_{r\to\infty}\frac{\prod_{j=l}^{k+1}\mathrm{Log}^{[j]}(r)}{r}\\ &= \sum_{l=2}^{k}\lim_{r\to\infty}\left(\frac{\prod_{j=1}^{k}\mathrm{Log}^{[j]}(r)}{r}\,\frac{\mathrm{Log}^{[k+1]}(r)}{\prod_{j=1}^{l-1}\mathrm{Log}^{[j]}(r)}\right).\end{aligned}$$

In view of the induction hypothesis, to conclude the proof of assertion (b) it suffices to observe that for each $l = 1, \ldots, k-1$, thanks to the L'Hôpital's rule and to the identities (4.55) and (4.56), we have

$$\lim_{r\to\infty}\frac{\mathrm{Log}^{[k+1]}(r)}{\prod_{j=1}^{l}\mathrm{Log}^{[j]}(r)} = 0.$$

(c) Firstly, note that for all $k \in \mathbb{N}$,

$$\omega_{k+1}(s) = \mathrm{Log}^{[k+1]}\left(\frac{1}{s}\right)\omega_k(s), \quad s \in \left(0, \frac{1}{e_{k+1}}\right),$$

thus by using (4.55) and (4.56) we can easily compute the derivatives on $\left(0, e_{k+1}^{-1}\right)$

$$\omega'_{k+1}(s) = \mathrm{Log}^{[k+1]}\left(\frac{1}{s}\right)\omega'_k(s) - 1, \quad \omega''_{k+1}(s) = \mathrm{Log}^{[k+1]}\left(\frac{1}{s}\right)\omega''_k(s) - \frac{\omega'_k(s)}{\omega_k(s)}. \tag{4.59}$$

A simple inductive argument then shows that $\lim_{s\to 0^+}\omega'_k(s) = +\infty$, $k \in \mathbb{N}$. Reasoning also by mathematical induction we will prove that

$$\omega'_k(s) > 0, \quad \omega''_k(s) < 0, \quad s \in \left(0, \frac{1}{e_k}\right) \text{ and } \omega'_k\left(\frac{1}{e_{k+1}}\right) > 1, \quad k \in \mathbb{N}. \tag{4.60}$$

For $k = 1$ we have $\omega'_1(s) = \log(1/s) - 1$ and $\omega''_1(s) = -1/s$. Clearly $\omega''_1(s) < 0$ for $s \in \left(0, e_1^{-1}\right)$ while $\omega'_1\left(e_2^{-1}\right) = e - 1$. Moreover, the fact that the $\log$ function is strictly monotone ensures that

$$\omega'_1(s) = \log\left(\frac{1}{s}\right) - 1 > \log(e) - 1 = 0.$$

Now assume that the (4.60) is true for some $k \in \mathbb{N}$, $k \geq 2$. Recall that the logarithms involved in (4.59) are nonnegative. Therefore, using the induction hypothesis and the fact that ω_k is positive, we get $\omega''_{k+1}(s) < 0$ for $s \in \left(0, e_{k+1}^{-1}\right)$. As a consequence, ω'_{k+1} is decreasing and we have

$$\omega'_{k+1}(s) \geq \lim_{t\to \frac{1}{e_{k+1}}^-}\omega'_{k+1}(t) = \omega'_k\left(\frac{1}{e_{k+1}}\right) - 1 > 0, \quad s \in \left(0, \frac{1}{e_{k+1}}\right),$$

where the last inequality follows from the induction hypothesis. Finally, using again the monotonicity of ω_k' we have

$$\omega_{k+1}'\left(\frac{1}{e_{k+2}}\right)=e\,\omega_k'\left(\frac{1}{e_{k+2}}\right)-1\geq e\,\omega_k'\left(\frac{1}{e_{k+1}}\right)-1>\mathrm{e}\,-1>1,$$

which concludes the proof of assertion (c).

(d) The monotonicity of ω_k now follows from assertion (c) together with the definition (4.57). On the other hand, since ω_k is continuous at 0 and $\omega_k''<0$ on $\left(0,e_k^{-1}\right)$, ω_k is concave on $\left[0,e_k^{-1}\right)$. Moreover, $\omega_k(0)=0$ implies that ω_k is subadditive on such an interval. This property can then be extended to the whole $[0,\infty)$ due the constant character of the function ω_k in the remaining interval $\left[e_k^{-1},\infty\right)$.

(e) For $k\in\mathbb{N}$ and $u\in\left(0,e_k^{-1}\right]$ we have

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_\varepsilon^u\frac{1}{\omega_k(s)}\,\mathrm{d}\,s&=\lim_{\varepsilon\to0^+}\int_\varepsilon^u\frac{1}{s\prod_{j=1}^k\operatorname{Log}^{[j]}\left(\frac{1}{s}\right)}\,\mathrm{d}\,s\\&=\lim_{\varepsilon\to0^+}-\int_\varepsilon^u\left(\frac{1}{s}\prod_{j=1}^k\operatorname{Log}^{[j]}\left(\frac{1}{s}\right)\right)^{-1}\left(\frac{-1}{s^2}\right)\mathrm{d}\,s\\&=\lim_{\varepsilon\to0^+}\left[-\operatorname{Log}^{[k+1]}\left(\frac{1}{s}\right)\right]_\varepsilon^u=+\infty.\end{aligned}$$

For $u>e_k^{-1}$ the assertion is a consequence of the additive property of the integral with respect to intervals. □

Essentially, Proposition 4.52 shows that the maps ω_k, $k\in\mathbb{N}$, are Osgood moduli of continuity. Furthermore, it provides us with some of the tools necessary for the following example, where we illustrate the potential of Montel–Osgood–Tonelli existence and uniqueness result for Stieltjes differential equations, while showing some of the limitations of the results involving Lipschitz conditions, Theorems 4.44 and 4.49.

Example 4.53. Consider a nondecreasing and left–continuous function, $g:\mathbb{R}\to\mathbb{R}$, and a g–integrable function, $\varphi:[0,1)\to[0,+\infty)$. For each $k\in\mathbb{N}$ we consider the initial value problem

$$x_g'(t)=f_k(t,x(t)),\quad g\text{–a.a. } t\in[0,1),\quad x(0)=x_0,\tag{4.61}$$

where $f_k:[0,1)\times\mathbb{R}\to\mathbb{R}$ is given by $f_k(t,x)=\varphi(t)\,\omega_k(|x|)$, $(t,x)\in[0,1)\times\mathbb{R}$, with ω_k as in (4.57). Note that f_k does not satisfy a Lipschitz's condition on the whole $\mathbb{R}$ as the derivative of ω_k is unbounded on any neighbourhood of 0, see Theorem 4.52 assertion (c), and therefore the hypotheses of Theorem 4.49 are not satisfied. Similarly, Theorem 4.44 cannot be applied for any ball around x_0 containing 0. In particular, if $x_0=0$ we can never assure the existence of a local solution for this problem through these results. However, the Montel–Osgood–Tonelli result allows us to show that problem (4.61) has a unique local solution regardless of the initial value.

Recalling that ω_k is subadditive, for every $x,y\in\mathbb{R}$ we have

$$\omega_k(|x|) \leq \omega_k(|x-y|+|y|) \leq \omega_k(|x-y|)+\omega_k(|y|),$$

and similarly $\omega_k(|y|) \leq \omega_k(|y-x|)+\omega_k(|x|)$. Hence

$$\big|\omega_k(|x|)-\omega_k(|y|)\big| \leq \omega_k(|x-y|), \quad x,y\in\mathbb{R}.$$

In summary, for all $x, y \in \mathbb{R}$ and $t \in [0,1)$

$$|f_k(t,x)-f_k(t,y)| = \varphi(t)|\omega_k(|x|)-\omega_k(|y|)| \leq \varphi(t)\omega_k(|x-y|).$$

Thus, the hypotheses of Theorem 4.51 are satisfied and, consequently, problem (4.61) has a unique local solution for each $k \in \mathbb{N}$.

4.1.3 Lower and upper solutions

We now study (4.2) through the method of lower and upper solutions. This will be done in the context of $n = 1$, $X = \mathbb{R}$, i.e.

$$x'_g(t) = f(t,x(t)), \quad x(t_0) = x_0, \tag{4.62}$$

with $f : I \times \mathbb{R} \to \mathbb{R}$. In this setting, we aim to obtain results that ensure the existence of solutions of problem (4.62) that lie between two well–ordered functions satisfying some conditions, following the results obtained in [48, 59]. We will also be interested in the existence of extremal solutions in this context, as investigated in [51]. For the general vectorial case, we have obtained some results in the more general case of Stieltjes differential equations with several derivators which are presented in Chapter 5.

We start by formalizing the basic definitions of this section: the concepts of lower and upper solution.

Definition 4.54. *Let $\tau \in (0,T]$. A* lower solution *of* (4.62) *on I_τ is a map $\alpha \in \mathcal{AC}_g(\overline{I}_\tau,\mathbb{R})$ such that $\alpha(t_0) \leq x_0$ and*

$$\alpha'_g(t) \leq f(t,\alpha(t)) \quad g\text{–a.a. } t \in I_\tau.$$

Similarly, an upper solution *of* (4.62) *on an I_τ is a function $\beta \in \mathcal{AC}_g(\overline{I}_\tau,\mathbb{R})$ such that $\beta(t_0) \geq x_0$ and*

$$\beta'_g(t) \geq f(t,\beta(t)) \quad g\text{–a.a. } t \in I_\tau.$$

Remark 4.55. Similarly to the concept of solution, we consider lower and upper solutions on an interval I_τ, $\tau \in (0,T]$, to be defined on the corresponding closed interval. This is done for consistency with the concept of solution. If one were to consider solutions defined on $E\mathcal{AC}_g(I,\mathbb{R})$, then the definitions of lower and upper solutions should be modified to lie on that space instead of on the space of g–absolutely continuous functions.

Before moving on to the study of existence of solution of (4.62), we need to make some observations. First of all, it is clear that every solution of (4.2) on I_τ, $\tau \in (0,T]$, is a lower and an upper solution of (4.2) on I_τ. Secondly, let α, β be a lower and an upper solution of (4.62) on I_τ, $\tau \in (0,T)$, respectively. Then for $t \in I_\tau \cap D_g$ we have

$$\alpha(t^+) \leq \alpha(t) + f(t,\beta(t))\Delta g(t), \quad \beta(t^+) \geq \beta(t) + f(t,\beta(t))\Delta g(t). \tag{4.63}$$

With this remark in mind, we can establish a useful differential inequality in the context of g-differential problems.

Theorem 4.56. *Let $\tau \in (0,T]$ and $f : I \times \mathbb{R} \to \mathbb{R}$ be such that for each $t \in I_\tau$, $f(t,\cdot)$ is nonincreasing and the mapping*

$$u \in \mathbb{R} \mapsto u + f(t,u)\Delta g(t) \quad \textit{is nondecreasing.} \tag{4.64}$$

Assume that α, β are a lower and an upper solution of (4.62) *on I_τ. If $x : I_\tau \to \mathbb{R}$ is a solution of* (4.62) *on I_τ, then*

$$\alpha(t) \leq x(t) \leq \beta(t), \quad t \in \overline{I}_\tau.$$

Proof. We will prove that $x \leq \beta$ on $\overline{I}_\tau$. First, note that $x(t_0) = x_0 \leq \beta(t_0)$, so we need to show that $x \leq \beta$ on $(t_0, t_0+\tau]$. By contradiction, assume that there exists $t^* \in (t_0, t_0+\tau]$ such that $x(t^*) > \beta(t^*)$ and let

$$t_1 = \inf\{s \in (t_0, t^*) : x > \beta \text{ in } (s, t^*)\}.$$

First, observe that the left–continuity of the functions –see Proposition 3.21– is enough to ensure that t_1 is well–defined. Furthermore, since $x > \beta$ in (t_1, t^*), we have $x(t_1^+) \geq \beta(t_1^+)$. Moreover, note that $x(t_1) \leq \beta(t_1)$. Indeed, if $x(t_1) > \beta(t_1)$, the fact that the functions are continous from the left would imply that $x > \beta$ in some interval (t_2, t_1), $t_2 \in (t_0, t_1)$, contradicting the definition of t_1.

Bearing (4.3) in mind and using (4.63) and (4.64) we obtain

$$x(t_1^+) = x(t_1) + f(t_1, x(t_1))\Delta g(t_1) \leq \beta(t_1) + f(t_1, \beta(t_1))\Delta g(t_1) \leq \beta(t_1^+) \leq x(t_1^+),$$

which yields $x(t_1^+) = \beta(t_1^+)$. From the monotonicity of $f(t,\cdot)$, it follows that

$$x(t^*) - x(t_1^+) = \int_{(t_1,t^*)} f(s, x(s))\, \mathrm{d}\, g(s) \leq \int_{(t_1,t^*)} f(s,\beta(s))\, \mathrm{d}\, g(s) \leq \beta(t^*) - \beta(t_1^+),$$

and consequently $x(t^*) \leq \beta(t^*)$, a contradiction. Thus we conclude that $x \leq \beta$ in I_τ. The inequality $\alpha \leq x$ can be obtained in analogous way, and we omit it. □

Note that it is enough to check that (4.64) holds for $t \in D_g$, as for the rest of the points it is trivially satisfied. With this idea in mind, we can give an idea of what it means. Essentially, this conditions guarantees that the inequalities between g–differentiable functions are preserved at a discontinuity points of the derivator. This condition is a fundamental requirement in the study of problem (4.62) through the method of lower and upper solutions and it is due to Monteiro and Slavík in the following result, [64, Theorem 4.4].

Theorem 4.57. *Let $a, b \in \mathbb{R}$, $a < b$, $g : [a, b] \to \mathbb{R}$ be a nondecreasing and left–continuous function, $B \subset \mathbb{R}$ be a closed set, $y_0 \in B$ and $f : [a, b] \times B \to \mathbb{R}$ satisfy the following conditions:*

(C1) *For every $y \in B$, the integral ${}^{(KS)}\!\int_a^b f(t, y)\,\mathrm{d}\,g(t)$ exists.*

(C2) *There exists $M : [a, b] \to \mathbb{R}$ which is Kurzweil–Stieltjes integrable with respect to g such that*

$$\left\| {}^{(KS)}\!\!\int_u^v f(t, y)\,\mathrm{d}\,g(t) \right\| \leq {}^{(KS)}\!\!\int_u^v M(t)\,\mathrm{d}\,g(t),$$

for every $[u, v] \subset [a, b]$ and $y \in B$.

(C3) *For each $t \in [a, b]$, the map $f(t, \cdot)$ is continuous on B.*

(C4) *f satisfies* (4.64) *on B.*

If the problem

$$y(t) = y_0 + {}^{(KS)}\!\!\int_a^t f(s, y(s))\,\mathrm{d}\,g(s), \quad t \in [a, b], \tag{4.65}$$

has a solution on $G([a, b], \mathbb{R})$, then there exist $y_*, y^* \in G([a, b], \mathbb{R})$ solutions of (4.65) *such that*

$$y_*(t) \leq y(t) \leq y^*(t), \quad t \in [a, b],$$

for any other function $y \in G([a, b], \mathbb{R})$ solving (4.65)*.*

Using a slight modification of [64, Example 4.2], we now illustrate the importance of the monotonicity condition in Theorem 4.56.

Example 4.58. Suppose that $g : [0, 2] \to \mathbb{R}$ and $f : [0, 2] \times \mathbb{R} \to \mathbb{R}$ are given by

$$g(t) = \begin{cases} t, & t \in [0, 1], \\ t + 1, & t \in (1, 2], \end{cases} \qquad f(t, y) = \begin{cases} 3y^{2/3}, & t \in [0, 1), \\ -\sin y, & t = 1, \\ 0, & t \in (1, 2]. \end{cases}$$

Note that g is left-continuous, but discontinuous from the right at $t = 1$. Furthermore, our choice of f is such that (4.64) holds, though the function $f(1, \cdot)$ is not even monotone.

Consider the initial value problem (4.62) with $x_0 = 0$. Clearly, on $[0, 1)$ the equation reduces to $x'(t) = 3x(t)^{2/3}$, while

$$0 = f(t, x(t)) = x'_g(t) = \lim_{s \to t} \frac{x(s) - x(t)}{s - t} = x'(t), \quad t \in (1, 2],$$

implying that each solution has to be constant on $(1, 2]$. Finally, since $\Delta g(1) = 1$ we get

$$x(1^+) - x(1) = x'_g(t) = f(1, x(1)) = -\sin x(1),$$

that is, $x(1^+) = x(1) - \sin x(1)$. It follows that all solutions have the form

$$z_\lambda(t) = \begin{cases} 0, & t \in [0, \lambda], \\ (t-\lambda)^3, & t \in (\lambda, 1], \\ z_\lambda(1) - \sin(z_\lambda(1)), & t \in (1, 2], \end{cases}$$

where $\lambda \in [0,1]$ is a parameter. In particular, we have that $\alpha = 0$ is a lower solution and $\beta = z_{1/3}$ is an upper solution. However, $\alpha(t) = \beta(t) = 0 < t^3 = z_0(t)$ for $t \in (0, 1/3]$. Furthermore, as presented in Figure 4.6, the inequality can be proven to hold on the whole interval $(0, 2]$.

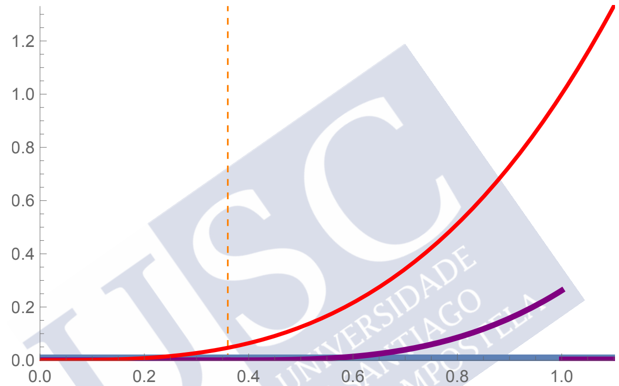


Figure 4.6: Graphs of α, β and z_0 in blue, purple and red, respectively. The dotted line marks the value $t = 1/3$.

From the inequalities in Theorem 4.56 we can obtain a result that is, in some sense, a generalization of Peano's uniqueness Theorem.

Corollary 4.59. *Let* $\tau \in (0, T]$ *and* $f : I \times \mathbb{R} \to \mathbb{R}$ *be a function satisfying* (4.64) *and such that for any* $t \in I_\tau$ *the map* $f(t, \cdot)$ *is nonincreasing. Then the problem* (4.62) *has at most one solution on* I_τ.

Proof. Let $x_1, x_2 : \overline{I}_\tau \to \mathbb{R}$ be two solutions of (4.62). Then we have that x_1 is both an upper and a lower solution of (4.62) on I_τ, so Theorem 4.56 ensures that

$$x_1(t) \le x_2(t) \le x_1(t), \quad t \in \overline{I}_\tau.$$

Thus, we have that $x_1 = x_2$ on $\overline{I}_\tau$. □

We now turn our attention to the study of extremal solutions on the whole interval between a given lower solution and an upper solution that are well–ordered in the sense that the lower solution lies below the upper solution, as in [48]. To that end, we introduce the following definition.

Definition 4.60. *Let α, β be a lower and an upper solution of* (4.62) *on I, respectively, such that $\alpha \le \beta$ on $\overline{I}$. A solution of* (4.62) *on I, x^*, is said to be the* greatest solution of (4.62) on I between α and β *if $\alpha(t) \le x^*(t) \le \beta(t)$, $t \in \overline{I}$, and*

$$x(t) \le x^*(t), \quad t \in \overline{I},$$

for every $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ solution of (4.62) *on I such that $\alpha(t) \le x(t) \le \beta(t)$, $t \in \overline{I}$. Similarly, a solution of* (4.62) *on I, x_*, is said to be the* least solution of (4.62) on I between α and β *if $\alpha(t) \le x_*(t) \le \beta(t)$, $t \in \overline{I}$, and*

$$x_*(t) \le x(t), \quad t \in \overline{I},$$

for every $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ solution of (4.62) *on I such that $\alpha(t) \le x(t) \le \beta(t)$, $t \in \overline{I}$. If problem* (4.62) *has the greatest and the least solution between α and β, we say that* (4.62) *has the extremal solutions between α and β.*

We can obtain a result on the existence of extremal solutions of (4.62) between a given pair lower and upper solutions which follows as a corollary of Theorem 4.57. In doing so, we give a partial answer on the affirmative to the following question posed at the end of [64]: "For classical ordinary differential equations, the existence of a lower solution α and an upper solution β, where $\alpha(t) \le \beta(t)$, $t \in \overline{I}$, guarantees the existence of a solution lying between α and β (...). Is there an analogue of this statement for measure differential equations?"

Theorem 4.61. *Let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a g–Carathéodory function and suppose that problem* (4.62) *has a lower solution on I, α, and an upper solution on I, β, such that $\alpha \le \beta$ on $\overline{I}$. If f satisfies* (4.64)*, then* (4.62) *has the extremal solutions on I between α and β.*

Proof. First of all, since $\alpha, \beta \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$, they are bounded. In particular, there exists $R > 0$ such that $|\alpha(t)|, |\beta(t)| < R$ for all $t \in \overline{I}$. Take $h = h_R$, the g–integrable function in part (iii) of Definition 4.29 for f and define $\overline{f} : I \times \mathbb{R} \to \mathbb{R}$ as

$$\overline{f}(t,x) = \begin{cases} f(t,\alpha(t)) & \text{if } x < \alpha(t), \\ f(t,x) & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t,\beta(t)) & \text{if } x > \beta(t). \end{cases}$$

It follows that

$$|\overline{f}(t,x)| \le h(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}.$$

Moreover, $\overline{f}$ is a g–Carathéodory function. Indeed, first of all note that condition (iii) in Definition 4.29 holds trivially at this point. Now, for condition (i) in Definition 4.29, fix $x \in \mathbb{R}$. Then

$$\overline{f}(t,x) = \overline{f}(t,\alpha(t))\chi_{A_1}(t) + \overline{f}(t,\beta(t))\chi_{A_2}(t) + \overline{f}(t,x)\chi_{A_3}(t),$$

where

$$\begin{aligned} A_1 &= \{t \in I : \alpha(t) > x\} = \alpha^{-1}((x,+\infty)), \\ A_2 &= \{t \in I : x > \beta(t)\} = \beta^{-1}((-\infty,x)), \\ A_3 &= \{t \in I : \alpha(t) \le x \le \beta(t)\} = \alpha^{-1}((-\infty,x]) \cap \beta^{-1}([x,+\infty)). \end{aligned}$$

Since each A_i, $i = 1, 2, 3$, is g–measurable, $\overline{f}(\cdot, x)$ is g–measurable for each $x \in \mathbb{R}$. Finally, for condition (ii) in Definition 4.29, fix $t \in I$. We have that

$$\overline{f}(t, x) = f(t, \max\{\alpha(t), \min\{x, \beta(t)\}\}), \quad x \in \mathbb{R},$$

and so $\overline{f}(t, \cdot)$ is continuous for g–a.a. $t \in I$, as it is the composition of continuous functions. Thus, $\overline{f}$ is g–Carathéodory.

Consider the modified problem

$$x_g'(t) = \overline{f}(t, x(t)) \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_0. \tag{4.66}$$

It follows from Theorem 4.32 that (4.66) has at least one solution. In particular, the measure differential equation

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} \overline{f}(s, x(s)) \, \mathrm{d}\, g(s), \quad t \in \overline{I}, \tag{4.67}$$

has at least one solution. Thus, Theorem 4.57 ensures the existence of x_*, x^*, solutions of (4.67), such that

$$x_*(t) \le y(t) \le x^*(t), \quad t \in \overline{I},$$

for any other function y solving (4.67). Note that Theorem 4.57 requires $\overline{f}(t, \cdot)$ to be continuous for all $t \in \overline{I}$, and not merely for g–a.a. $t \in I$. However, it suffices to redefine $\overline{f}(t, x) = 0$ at the exceptional g–null set and at $t_0 + T$ to have an equivalent problem with a nonlinearity which is continuous with respect to x for all $t \in I$.

Let y be a solution of (4.67). Then y is regulated, therefore Borel-measurable and, in particular, g–measurable, see Proposition 3.22. Thus, Proposition 1.28 guarantees that $f(\cdot, y(\cdot)) \in \mathcal{L}_g^1(I, \mathbb{R}^n)$. Hence, as pointed out in Remark 4.30, the integral in (4.67) exists in the Lebesgue-Stieltjes sense and y is a solution of (4.66). In particular, we have that x^* and x_* are solutions of (4.66).

Let x be a solution of (4.66). It is enough to show that

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in \overline{I}, \tag{4.68}$$

thus proving that x is a solution of (4.62) and, since x is arbitrarily chosen, that x_* and x^* are its extremal solutions between α and β. Reasoning by contradiction, assume that the first inequality in (4.68) does not hold. In that case, there exists $t_1 \in (t_0, t_0 + T]$ such that $\alpha(t_1) > x(t_1)$. By left–continuity of both functions at t_1, we deduce the existence of some $\varepsilon > 0$ such that $\alpha > x$ in $(t_1 - \varepsilon, t_1]$. Define

$$t_2 = \inf\{s \in (t_0, t_1) : \alpha > x \text{ in } (s, t_1)\}.$$

Notice that $\alpha(t_2) \le x(t_2)$ for, otherwise, left–continuity would imply $\alpha > x$ on some interval $(t_2 - \varepsilon', t_2)$, and this would be a contradiction with the definition of t_2. Now, since α is a lower solution, we have

$$\begin{aligned}\int_{[t_2,t_1)} \alpha'_g(s)\,\mathrm{d}\,g(s) &\le \int_{[t_2,t_1)} f(s,\alpha(s))\,\mathrm{d}\,g(s)\\ &= f(t_2,\alpha(t_2))\Delta g(t_2) + \int_{(t_2,t_1)} \overline{f}(s,x(s))\,\mathrm{d}\,g(s)\\ &= f(t_2,\alpha(t_2))\Delta g(t_2) + \int_{(t_2,t_1)} x'_g(s)\,\mathrm{d}\,g(s).\end{aligned}$$

Thus, the Fundamental Theorem of Calculus, Theorem 3.27, yields

$$\alpha(t_1) - \alpha(t_2) \le f(t_2,\alpha(t_2))\Delta g(t_2) + x(t_1) - x(t_2^+).$$

Then, (4.64) implies that

$$\begin{aligned}\alpha(t_1) &\le \alpha(t_2) + f(t_2,\alpha(t_2))\Delta g(t_2) + x(t_1) - x(t_2^+)\\ &\le x(t_2) + f(t_2,x(t_2))\Delta g(t_2) + x(t_1) - x(t_2^+) = x(t_1),\end{aligned}$$

where the last equality follows from (4.3). This contradicts the definition of t_1. Thus $\alpha \le x$ on $\overline{I}$. The proof of the second inequality in (4.68) is analogous and we omit it. □

Theorem 4.61 will be a fundamental tool for the study of functional differential problems in Section 4.2. In particular, we will obtain Theorem 4.94 reducing the problem there presented to one where we can apply Theorem 4.61. Later, in Example 4.95, we will apply Theorem 4.94 to the study of a model for a silkworm population. For an application of Theorem 4.61, the reader can just consider the functional argument to be a given constant and follow the example.

In what follows, we will focus on the study of existence of extremal solutions through the method of approximation by lower and upper solutions, following [47,51]. To that extent, we introduce the following definition.

Definition 4.62. *A solution of* (4.62) *on* I, x^*, *is said to be the* greatest solution *of* (4.62) *on* I *if*

$$x \le x^*, \quad \textit{for every } x \textit{ solution of } (4.62) \textit{ on } I.$$

Similarly, a solution of (4.62) *on* I, x_*, *is said to be the* least solution *of* (4.62) *on* I *if*

$$x_* \le x, \quad \textit{for every } x \textit{ solution of } (4.62) \textit{ on } I.$$

If problem (4.62) *has the greatest and the least solutions, we say that* (4.62) *has the extremal solutions.*

Our aim is to show that the infimum of upper solutions and the supremum of lower solutions are the least and the greatest solutions, respectively, provided some conditions are satisfied, some of which we introduce in the following definition.

Definition 4.63. *Let* $M \in \mathcal{L}^1_g(I,[0,+\infty))$ *be such that*

$$|f(t,x)| \leq M(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}. \tag{4.69}$$

We define the set of admissible lower solutions *of* (4.62) *on* I *as*

$$\mathcal{L} = \left\{ l \text{ lower solution of (4.62) on } I : |l'_g(t)| \leq M(t) \ g\text{-a.a. } t \in I \right\},$$

and the set of admissible upper solutions *of* (4.62) *on* I *as*

$$\mathcal{U} = \left\{ u \text{ upper solution of (4.62) on } I : |u'_g(t)| \leq M(t) \ g\text{-.a.a. } t \in I \right\}.$$

We define the maps $l_{\sup}, u_{\inf} : I \to \mathbb{R}$ *as*

$$l_{\sup}(t) = \sup\{l(t) : l \in \mathcal{L}\}, \quad u_{\inf}(t) = \inf\{u(t) : u \in \mathcal{U}\}.$$

Remark 4.64. Note that $l_{\sup}(t_0) = u_{\inf}(t_0) = x_0$ as the maps

$$l(t) = x_0 - \int_{[t_0,t)} M(s)\,\mathrm{d}\,g(s), \quad u(t) = x_0 + \int_{[t_0,t)} M(s)\,\mathrm{d}\,g(s), \quad t \in I,$$

belong to $\mathcal{L}$ and $\mathcal{U}$, respectively, and $l(t_0) = u(t_0) = x_0$.

With this notation, we aim to find some conditions guaranteeing that $u_{\inf}$ is the least solution of (4.62) and $l_{\inf}$, the greatest solution. The following result shows that we only need to study the behaviour of $u_{\inf}$ as the behaviour of $l_{\inf}$ can be deduced from it.

Theorem 4.65. *Consider problem* (4.62) *and* $M \in \mathcal{L}^1_g(I,[0,+\infty))$. *Define* $y_0 = -x_0$ *and* $\widetilde{f}(t,x) = -f(t,-x)$, $(t,x) \in I \times \mathbb{R}$. *Consider the initial value problem*

$$y'_g(t) = \widetilde{f}(t,y(t)), \quad g\text{–a.a. } t \in I, \quad y(t_0) = y_0, \tag{4.70}$$

and denote by $\widetilde{\mathcal{L}}$ *and* $\widetilde{\mathcal{U}}$ *the corresponding sets of admissible lower and upper solutions of problem* (4.70) *on* I *for* M, *respectively. Then, the map*

$$\begin{array}{rcl} \mathcal{T} : \mathcal{AC}_g(\overline{I},\mathbb{R}) & \longrightarrow & \mathcal{AC}_g(\overline{I},\mathbb{R}) \\ x & \longmapsto & -x \end{array}$$

satisfies that

$$\mathcal{T}(\mathcal{L}) = \widetilde{\mathcal{U}}, \quad \mathcal{T}(\mathcal{U}) = \widetilde{\mathcal{L}}, \tag{4.71}$$

$$\mathcal{T}(\widetilde{\mathcal{L}}) = \mathcal{U}, \quad \mathcal{T}(\widetilde{\mathcal{U}}) = \mathcal{L}, \tag{4.72}$$

with $\mathcal{L}$ *and* $\mathcal{U}$ *as in* Definition 4.63. *In particular,* $\mathcal{T}$ *maps solutions of* (4.62) *to solutions of* (4.70) *and vice versa.*

Proof. First of all, note that (4.71) and (4.72) are equivalent. Indeed, it suffices to note that $\mathcal{T}\circ\mathcal{T}=\mathrm{Id}$ and apply the $\mathcal{T}$ on both sides of the equalities. Thus, it is enough to show that (4.71) holds.

Now, observe that given $x\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ satisfying $|x_g'(t)|\leq M(t)$ for g–a.a. $t\in I$, we have that

$$|(\mathcal{T}x)_g'(t)|=|x_g'(t)|\leq M(t),\quad g\text{–a.a. } t\in I.$$

Thus, to show (4.71) it is enough to show that $\mathcal{T}$ maps bijectively the lower and upper solutions of (4.62) to the upper and lower solutions of (4.70), respectively.

Let $l\in\mathcal{AC}_g(\overline{I})$ be a lower solution of (4.62). Then $\mathcal{T}l(t_0)=-l(t_0)=-x_0=y_0$ and

$$(\mathcal{T}l)_g'(t)=-l_g'(t)\geq -f(t,l(t))=-f(t,-(\mathcal{T}l(t)))=\widetilde{f}(t,\mathcal{T}l(t)),\quad g\text{–a.a. } t\in I,$$

i.e. $\mathcal{T}l$ is an upper solution of (4.70), and so $\mathcal{T}(\mathcal{L})\subset\widetilde{\mathcal{U}}$. Conversely, let $\widetilde{u}\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ be an upper solution of (4.70). Then $\mathcal{T}\widetilde{u}(t_0)=-\widetilde{u}(t_0)=-y_0=x_0$ and

$$(\mathcal{T}\widetilde{u})_g'(t)=-\widetilde{u}_g'(t)\leq -\widetilde{f}(t,\widetilde{u}(t))=f(t,-\widetilde{u}(t))=f(t,\mathcal{T}\widetilde{u}(t)),\quad g\text{–a.a. } t\in I.$$

Thus $\mathcal{T}\widetilde{u}$ is a lower solution of (4.62), which shows that $\mathcal{T}(\mathcal{U})\subset\widetilde{\mathcal{L}}$. Now, applying $\mathcal{T}$ on both sides, we get that $\widetilde{\mathcal{U}}\subset\mathcal{T}(\mathcal{L})$, which proves the first equality in (4.71). The proof of the second equality in (4.71) is analogous and we omit it.

The rest of statement of the theorem now follows, as solutions are, simultaneously, lower and upper solutions. □

Remark 4.66. It follows directly from this result and the definition of $\mathcal{T}$ that $\mathcal{T}$ maps extremal solutions into extremal solutions, provided they exist. In particular, $\mathcal{T}$ maps the greatest solutions into the least solutions and vice versa.

Moreover, if we denote by $\widetilde{l}_{\sup}$ and $\widetilde{u}_{\inf}$ the corresponding functions in Definition 4.63 for (4.70), we have that $\mathcal{T}$ maps $\widetilde{l}_{\sup},\widetilde{u}_{\inf}$ onto $u_{\inf},l_{\sup}$, respectively. The converse mapping also holds. Hence, if we can find conditions that turn the infimum of upper solutions into the least solution, we can "translate" those conditions through the map $\mathcal{T}$ to obtain some conditions to ensure that the supremum of lower solutions is the greatest solution.

In order to show that $u_{\inf}$ can be turned into a solution, the first thing we need to show is that it is g–absolutely continuous. To obtain that property we will need the following lemma.

Lemma 4.67. *Consider* $\beta_1,\beta_2,...,\beta_k\in\mathcal{U}$. *If* $f:I\times\mathbb{R}\to\mathbb{R}$ *verifies* (4.64) *and* (4.69)*, then the map*

$$\beta_{\min}(t)=\min\{\beta_1(t),\beta_2(t),...,\beta_k(t)\},\quad t\in\overline{I},$$

belongs to $\mathcal{U}$.

Proof. To prove this, it suffices to show that given $\beta_1,\beta_2\in\mathcal{U}$, the map

$$\beta_{\min}(t)=\min\{\beta_1(t),\beta_2(t)\},\quad t\in\overline{I},$$

belongs to $\mathcal{U}$. First of all, note that $\beta_{\min}\in\mathcal{AC}_g(\overline{I},\mathbb{R})$, see Proposition 3.29. Moreover, $\beta_{\min}(t_0)\geq x_0$ trivially, and so, all that is left to prove is that

$$(\beta_{\min})_g'(t)\geq f(t,\beta_{\min}(t))\quad\text{and}\quad |(\beta_{\min})_g'(t)|\leq M(t),\quad g\text{–a.a. } t\in I.$$

Let $E = \{t \in I : \exists\, (\beta_1)'_g(t),\ (\beta_2)'_g(t),\ (\beta_{\min})'_g(t)\}$ and let $t_1 \in E$. Note that $t_1 \notin C_g$ since there exists g–derivatives at that point. We have distinguish possible cases.

Case 1: $\beta_1 \geq \beta_2$ *on a set* $S \subset \overline{I}$ *such that* $t_1 \in [S \cap (t_1, t_0 + T)]'$.
If $\beta_1(t_1) \geq \beta_2(t_1)$ then

$$
\begin{aligned}
(\beta_{\min})'_g(t_1) &= \lim_{t \to t_1^+} \frac{\beta_{\min}(t) - \beta_{\min}(t_1)}{g(t) - g(t_1)} = \lim_{\substack{t \to t_1^+ \\ t \in S \cap (t_1, 1)}} \frac{\beta_{\min}(t) - \beta_{\min}(t_1)}{g(t) - g(t_1)} \\
&= \lim_{t \to t_1^+} \frac{\beta_2(t) - \beta_2(t_1)}{g(t) - g(t_1)} = (\beta_2)'_g(t_1) \geq f(t_1, \beta_2(t_1)) = f(t_1, \beta_{\min}(t_1)).
\end{aligned}
$$

Otherwise, $\beta_2(t_1) > \beta_1(t_1)$, and so $t_1 \in D_g$ as

$$
\beta_1(t_1^+) = \lim_{\substack{t \to t_1^+ \\ t \in S \cap (t_1, 1)}} \beta_1(t) \geq \beta_2(t_1^+).
$$

Hence, using (4.64) we obtain

$$
\begin{aligned}
(\beta_{\min})'_g(t_1) &= \frac{\beta_2(t_1^+) - \beta_1(t_1)}{g(t_1^+) - g(t_1)} \\
&= \frac{\beta_2(t_1) + \Delta g(t_1)(\beta_2)'_g(t_1) - \beta_1(t_1)}{\Delta g(t_1)} \\
&\geq \frac{\beta_2(t_1) + \Delta g(t_1) f(t_1, \beta_2(t_1)) - \beta_1(t_1)}{\Delta g(t_1)} \\
&\geq \frac{\beta_1(t_1) + \Delta g(t_1) f(t_1, \beta_1(t_1)) - \beta_1(t_1)}{\Delta g(t_1)} = f(t_1, \beta_1(t_1)) = f(t_1, \beta_{\min}(t_1)).
\end{aligned}
$$

Thus $(\beta_{\min})'_g(t_1) \geq f(t, \beta_{\min}(t_1))$. Moreover, $|(\beta_{\min})'_g(t_1)| \leq M(t_1)$. Indeed, if $\beta_1(t_1) \geq \beta_2(t_1)$ then it is clear. If $\beta_1(t_1) < \beta_2(t_1)$ we have that

$$
(\beta_{\min})'_g(t_1) \geq f(t_1, \beta_{\min}(t_1)) \geq -M(t_1)
$$

and

$$
(\beta_{\min})'_g(t_1) = \frac{\beta_2(t_1^+) - \beta_1(t_1)}{\Delta g(t_1)} \leq \frac{\beta_1(t_1^+) - \beta_1(t_1)}{\Delta g(t_1)} = (\beta_1)'_g(t_1) \leq M(t_1),
$$

which concludes the proof for this case.

Case 2: $\beta_1 < \beta_2$ *on* $(t_1, t_1 + \delta)$ *for some* $\delta > 0$.
If $\beta_1(t_1) \leq \beta_2(t_1)$ then

$$
\begin{aligned}
(\beta_{\min})'_g(t_1) &= \lim_{t \to t_1^+} \frac{\beta_{\min}(t) - \beta_{\min}(t_1)}{g(t) - g(t_1)} \\
&= \lim_{t \to t_1^+} \frac{\beta_1(t) - \beta_1(t_1)}{g(t) - g(t_1)} = (\beta_1)'_g(t_1) \geq f(t_1, \beta_1(t_1)) = f(t_1, \beta_{\min}(t_1)).
\end{aligned}
$$

Otherwise, we have that $\beta_1(t_1) > \beta_2(t_1)$. Then $t_1 \in D_g$ since $\beta(t_1^+) < \beta_2(t_1^+)$. Once again, using (4.64) we obtain

$$\begin{aligned}(\beta_{\min})'_g(t_1) &= \frac{\beta_1(t_1^+) - \beta_2(t_1)}{\Delta g(t_1)} \\ &= \frac{\beta_1(t_1) + \Delta g(t_1)(\beta_1)'_g(t_1) - \beta_2(t_1)}{\Delta g(t_1)} \\ &\geq \frac{\beta_1(t_1) + \Delta g(t_1) f(t_1, \beta_1(t_1)) - \beta_2(t_1)}{\Delta g(t_1)} \\ &\geq \frac{\beta_2(t_1) + \Delta g(t_1) f(t_1, \beta_2(t_1)) - \beta_2(t_1)}{\Delta g(t_1)} = f(t_1, \beta_2(t_1)) = f(t_1, \beta_{\min}(t_1)).\end{aligned}$$

Thus, we just need to show that $|(\beta_{\min})'_g(t_1)| \leq M(t_1)$. Indeed, if $\beta_1(t_1) \leq \beta_2(t_1)$ then it is clear. If $\beta_1(t_1) > \beta_2(t_1)$ we have that

$$(\beta_{\min})'_g(t_1) \geq f(t_1, \beta_{\min}(t_1)) \geq -M(t_1)$$

and

$$(\beta_{\min})'_g(t_1) = \frac{\beta_1(t_1^+) - \beta_2(t_1)}{\Delta g(t_1)} \leq \frac{\beta_2(t_1^+) - \beta_2(t_1)}{\Delta g(t_1)} = (\beta_2)'_g(t_1) \leq M(t_1),$$

which concludes the proof. □

Remark 4.68. It follows from Theorem 4.65, that given $\alpha_1, \alpha_2, ..., \alpha_k \in \mathcal{L}$, the map

$$\alpha_{\max}(t) = \max\{\alpha_1(t), \alpha_2(t), ..., \alpha_k(t)\}, \quad t \in \overline{I},$$

belongs to $\mathcal{L}$, provided $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (4.64) and (4.69).

From this lemma we can deduce some properties of $u_{\inf}$. In particular, we can show that it is g–absolutely continuous and that the $\mathcal{L}^1_g(I, \mathbb{R})$–boundedness condition is satisfied. We also include another useful properties of this function.

Lemma 4.69. *Let* $f : I \times \mathbb{R} \to \mathbb{R}$ *satisfy* (4.64) *and* (4.69). *Then the following properties hold:*

(i) $u_{\inf}$ *is g–absolutely continuous on* $\overline{I}$*;*

(ii) $|(u_{\inf})'_g(t)| \leq M(t)$ *for g–a.a.* $t \in I$*;*

(iii) *there exists a nonincreasing sequence in* $\mathcal{U}$ *that converges uniformly on* $\overline{I}$ *to* $u_{\inf}$*.*

Proof. (i) Let $s, t \in \overline{I}$ be such that $s < t$. By definition of $u_{\inf}$, given $\varepsilon > 0$, there exist $u_1, u_2 \in \mathcal{U}$ such that

$$0 \leq u_1(t) - u_{\inf}(t) < \frac{\varepsilon}{2}, \quad 0 \leq u_2(s) - u_{\inf}(s) < \frac{\varepsilon}{2}.$$

Define $u(z) = \min\{u_1(z), u_2(z)\}$ for all $z \in \overline{I}$. By Lemma 4.67, $u \in \mathcal{U}$. Moreover, $0 \le u(t) - u_{\inf}(t) < \varepsilon/2$, $0 \le u(s) - u_{\inf}(s) < \varepsilon/2$. Hence,

$$\begin{aligned}|u_{\inf}(t) - u_{\inf}(s)| &\le |u_{\inf}(t) - u(t)| + |u(t) - u(s)| + |u(s) - u_{\inf}(s)| \\ &< \frac{\varepsilon}{2} + \left|\int_{[s,t)} M(r)\,\mathrm{d}\,g(r)\right| + \frac{\varepsilon}{2} = \int_{[s,t)} M(r)\,\mathrm{d}\,g(r) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$|u_{\inf}(t) - u_{\inf}(s)| \le \int_{[s,t)} M(r)\,\mathrm{d}\,g(r).$$

Define $F : \overline{I} \to \mathbb{R}$ as

$$F(t) = \int_{[t_0,t)} M(r)\,\mathrm{d}\,g(r).$$

Theorem 3.26 ensures that $F \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$. Hence, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals of $\overline{I}$, $\{(a_l, b_l)\}_{l=1}^m$, such that

$$\sum_{l=1}^m (g(b_l) - g(a_l)) < \delta \implies \sum_{l=1}^m |F(b_l) - F(a_l)| < \varepsilon.$$

Thus, if for every $\varepsilon > 0$ we take such $\delta > 0$, we have that

$$\sum_{l=1}^m |u_{\inf}(b_l) - u_{\inf}(a_l)| \le \sum_{l=1}^m |F(b_l) - F(a_l)| < \varepsilon,$$

for every open pairwise disjoint family of subintervals of $\overline{I}$, $\{(a_l, b_l)\}_{l=1}^m$, such that

$$\sum_{l=1}^m (g(b_l) - g(a_l)) < \delta,$$

which proves that $u_{\inf} \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$.

(ii) Let $s \in I$ be such that $(u_{\inf})'_g(s)$ exists. Define $\Phi_s : [s, t_0 + t] \to \mathbb{R}$ as

$$\Phi_s(t) = \int_{[s,t)} M(r)\,\mathrm{d}\,g(r).$$

We know that Φ_s is g–absolutely continuous so, by the Fundamental Theorem of Calculus, Theorem 3.27, we have that

$$|(u_{\inf})'_g(s)| = \lim_{t \to s^+} \frac{|u_{\inf}(t) - u_{\inf}(s)|}{g(t) - g(s)} \le \lim_{t \to s^+} \frac{\Phi_s(t) - \Phi_s(s)}{g(t) - g(s)} = (\Phi_s)'_g(s) = M(s),$$

Now, since $(u_{\inf})'_g(s)$ exists for g–a.a. $s \in I$, and the result follows.

(iii) Define $u_0 : \overline{I} \to \mathbb{R}$ as

$$u_0(t) = x_0 + \int_{[t_0,t)} M(\tau)\,\mathrm{d}\,g(\tau).$$

Note that $u_0 \in \mathcal{U}$. Assume that $u_1, u_2, ..., u_{k-1} \in \mathcal{U}$ have been defined. For every $i \in \{0, 1, ..., k-1\}$, choose $y_i \in \mathcal{U}$ satistying the following inequalities:

$$u_{\inf}\left(\frac{i}{k}\right) \le y_i\left(\frac{i}{k}\right) \le u_{\inf}\left(\frac{i}{k}\right) + \frac{1}{k}.$$

Define $u_k = \min\{u_{k-1}, y_0, ..., y_{k-1}\}$. Then $u_k \in \mathcal{U}$ by Lemma 4.67, and moreover, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is nonincreasing. Furthermore, $\{u_k\}_{k\in\mathbb{N}}$ verifies Proposition 3.31, since for each $k \in \mathbb{N}$, $|(u_k)'_g(t)| \le M(t)$ for g–a.a. $t \in I$ and $\{u_k(t_0) : k \in \mathbb{N}\} = \{x_0\}$ as $x_0 \le u_k(t_0) \le u_0(t_0) = x_0$. Hence, $\{u_k\}$ is a relatively compact set of $\mathcal{BC}_g(\overline{I}, \mathbb{R})$, and therefore there exists a subsequence $\{u_{k_j}\}$ that converges in $\mathcal{BC}_g(\overline{I}, \mathbb{R})$ to a limit, say v. Since $\{u_k\}$ is a monotone sequence, it also converges uniformly to v. Therefore, it is enough to show that $v = u_{\inf}$.

Since $u_k \ge u_{\inf}$ for all $k \in \mathbb{N}$, we have that $v \ge u_{\inf}$. Assume that $v \ne u_{\inf}$. Then, there exists $t_1 \in \overline{I}$ such that $v(t_1) > u_{\inf}(t_1)$. Both functions belong to $\mathcal{BC}_g(\overline{I}, \mathbb{R})$ so Proposition 3.21 ensures that they are left-continuous. Hence, there exist $c > 0$ and $\delta > 0$ such that

$$u_{\inf}(t) < v(t) - c, \quad \text{for all } t \in (t_1 - \delta, t_1].$$

Consider $k \in \mathbb{N}$ such that $1/k < \min\{c, \delta\}$. Then

$$u_{\inf}(t) < v(t) - c \le u_k(t) - c \le u_k(t) - 1/k, \quad \text{for all } t \in (t_1 - \delta, t_1]$$

and so, $u_{\inf}(t) + 1/k < u_k(t)$, for all $t \in (t_1 - 1/k, t_1]$. Now, for some $i = 0, 1, ..., k$, $i/n \in (t_1 - 1/k, t_1]$ and so $u_{\inf}(i/k) + 1/k < u_k(i/n)$, which is a contradiction. Therefore, $v = u_{\inf}$. □

Remark 4.70. The equivalence provided in Theorem 4.65 yields that $l_{\sup} \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$, $|(l_{\sup})'_g(t)| \le M(t)$ for g–a.a. $t \in I$ and that there exists nondecreasing sequence in $\mathcal{L}$ that converges uniformly to $l_{\sup}$.

The previous result provided some necessary conditions on f for $u_{\inf}$ to be a solution of problem (4.62). In the next result, we study how "far away" from being a solution $u_{\inf}$ is under those hypothesis.

Theorem 4.71. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ satisfy* (4.64) *and* (4.69) *and denote*

$$I_1 = \{t \in I : u_{\inf}(t) = u(t),\ u'_g(t) \ge f(t, u(t)) \text{ for some } u \in \mathcal{U}\} \cup D_g, \quad I_2 = I \setminus I_1.$$

Then

$$(u_{\inf})'_g(t) \ge f(t, u_{\inf}(t))\chi_{I_1}(t) + \liminf_{y \to (u_{\inf}(t))^+} f(t, y)\chi_{I_2}(t), \quad g\text{–a.a. } t \in I.$$

Proof. First, note that hypotheses guarantee that $u_{\inf} \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$, so $(u_{\inf})'_g$ exists g-almost everywhere in I.

Let $s \in I_1 \setminus D_g$ be such that $(u_{\inf})'_g(s)$ exists and let $u \in \mathcal{U}$ be the corresponding function to the definition of I_1 for that point. Then

$$(u_{\inf})'_g(s) = u'_g(s) \geq f(t, u(s)) = f(t, u_{\inf}(s)).$$

On the other hand, for $s \in D_g$, let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ be the sequence in Lemma 4.69, statement (iii). For all $k \in \mathbb{N}$, (4.63) and (4.64) yield

$$u_k(s^+) \geq u_k(s) + \Delta g(s) f(s, u_k(s)) \geq u_{\inf}(s) + \Delta g(s) f(s, u_{\inf}(s)).$$

Hence, since $\{u_k\}$ converges uniformly to $u_{\inf}$, it follows from the Moore–Osgood Theorem [38, Chapter VII, Theorem 2], that

$$u_{\inf}(s^+) \geq u_{\inf}(s) + \Delta g(s) f(s, u_{\inf}(s)),$$

or equivalently, $(u_{\inf})'_g(s) \geq f(s, u_{\inf}(s))$.

Finally, we study $(u_{\inf})'_g$ on I_2. To do so, consider $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ the same sequence as before. Since $|(u_n)'_g|$ is uniformly bounded on I by a function of $\mathcal{L}^1_g(I, \mathbb{R})$, we have that $\liminf_{n\to\infty}(u_k)'_g \in \mathcal{L}^1_g(I, \mathbb{R})$. Moreover, for $t, \tau \in I$, $\tau < t$, Fatou's Lemma yields

$$u_{\inf}(t) - u_{\inf}(\tau) = \liminf_{n\to\infty}(u_k(t) - u_k(\tau)) = \liminf_{n\to\infty} \int_{[\tau,t)} (u_k)'_g \, \mathrm{d}\, g \geq \int_{[\tau,t)} \liminf_{n\to\infty}(u_k)'_g \, \mathrm{d}\, g.$$

Now, since $u_{\inf} \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$, we have that

$$u_{\inf}(t) - u_{\inf}(\tau) = \int_{[\tau,t)} (u_{\inf})'_g(r) \, \mathrm{d}\, g(r),$$

from which we get

$$(u_{\inf})'_g(t) \geq \liminf_{n\to\infty}(u_k)'_g(t) \geq \liminf_{n\to\infty} f(t, u_k(t)) \quad g\text{–a.a. } t \in I.$$

Now, if $s \in I_2$ and $u_{\inf}(s) = u_k(s)$ for some $k \in \mathbb{N}$, the definition of I_2 implies that $s \notin D_g$ and either $(u_k)'_g(s)$ does not exist or $(u_k)'_g(s) < f(s, u_k(s))$. The set

$$\bigcup_{n\in\mathbb{N}} (\{t \in I \setminus D_g : \nexists\, (u_k)'_g(s)\} \cup \{t \in I \setminus D_g : (u_k)'_g(s) < f(s, u_k(s))\})$$

is a null-measure set with respect to the g-measure. Hence, for g–a.a. $t \in I_2$ we have that $u_{\inf}(t) < u_k(t)$ for all $k \in \mathbb{N}$ and so, since $\{u_k(t)\}$ is one of the infinitely many sequences that converges to $u_{\inf}(t)^+$, we have that

$$(u_{\inf})'_g(t) \geq \liminf_{n\to\infty} f(t, u_k(t)) \geq \liminf_{y\to(u_{\inf}(t))^+} f(t, y),$$

which concludes the proof. □

Remark 4.72. In this case, the counterpart of this result through the map $\mathcal{T}$ in Theorem 4.65 reads as follows: if $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (4.64) and (4.69) and we denote

$$I_3 = \{t \in I : l_{\sup}(t) = l(t),\, l_g'(t) \le f(t, l(t)) \text{ for some } l \in \mathcal{L}\} \cup D_g, \quad I_4 = I \setminus I_3,$$

we have that

$$(l_{\sup})_g'(t) \le f(t, l_{\inf}(t))\chi_{I_3}(t) + \limsup_{y \to (l_{\sup}(t))^-} f(t,y)\chi_{I_3}(t), \quad g\text{–a.a. } t \in I.$$

Corollary 4.73. *Let $t \in I \cap D_g$ and $f : I \times \mathbb{R} \to \mathbb{R}$ satisfy* (4.64) *and* (4.69)*. If $u_{\inf}$ is a upper solution of* (4.62) *on g–almost everywhere in $[t_0, t)$, then $u_{\inf}$ satisfies* (4.62) *at t, i.e.*

$$(u_{\inf})_g'(t) = f(t, u_{\inf}(t)). \tag{4.73}$$

Proof. Let $t \in I \cap D_g$. We know by Theorem 4.71 that $(u_{\inf})_g'(t) \ge f(t, u_{\inf}(t))$. Reasoning by contradiction, assume that $(u_{\inf})_g'(t) > f(t, u_{\inf}(t))$, or equivalently,

$$u_{\inf}(t^+) > u_{\inf}(t) + \Delta g(t) f(t, u_{\inf}(t)) =: a.$$

Take $z_0 \in (a, u_{\inf}(t^+))$ and define $u : \overline{I} \to \mathbb{R}$ as

$$u(s) = \begin{cases} u_{\inf}(s) & \text{if } s \in [t_0, t], \\ z_0 + \displaystyle\int_{(t_0, s)} M(r)\,\mathrm{d}\,g(r) & \text{if } s \in (t, t_0 + T]. \end{cases}$$

First, note that

$$u_g'(t) = \frac{u(t^+) - u(t)}{\Delta g(t)} = \frac{z_0 - u_{\inf}(t)}{\Delta g(t)} > \frac{a - u_{\inf}(t)}{\Delta g(t)} = f(t, u_{\inf}(t)) = f(t, u(t)).$$

Moreover, $|u_g'(s)| \le M(s)$ for g–a.a. $s \in I$ and $u \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ as it can be rewritten as the following sum of g–absolutely continuous functions on $\overline{I}$: $u = u_{\inf} \cdot \chi_{[t_0,t]} + h \cdot \chi_{(t, t_0+T]}$, with

$$h(s) = \begin{cases} z_0 - M(t)\Delta g(t) & \text{if } s \in [t_0, t), \\ z_0 - M(t)\Delta g(t) + \displaystyle\int_{[t,s)} M(r)\,\mathrm{d}\,g(r) & \text{if } s \in [t, t_0 + T]. \end{cases}$$

Hence, $u \in \mathcal{U}$ as $u_{\inf}$ is an upper solution on $[t_0, t]$. This is a contradiction as $u(t^+) = z_0 < u_{\inf}(t^+)$. □

Remark 4.74. The corresponding counterpart for $l_{\sup}$ reads as follows: for each $t \in I \cap D_g$, $l_{\sup}$, $(l_{\sup})_g'(t) = f(t, l_{\sup}(t))$ provided that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (4.64) and (4.69) and $l_{\sup}$ is a lower solution of (4.62) on $[t_0, t)$.

It follows from Theorem 4.71 that if $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (4.64), (4.69) and

$$\liminf_{y \to (u_{\inf}(t))^+} f(t,y) \ge f(t, u_{\inf}(t)), \quad g\text{-a.a. } t \in I \setminus D_g, \tag{4.74}$$

then $u_{\inf}$ is an upper solution. Furthermore, Corollary 4.73 ensures that, under these hypotheses, (4.73) holds for every $t \in I \cap D_g$. Similarly, it is enough for $l_{\sup}$ to be a lower solution that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (4.64), (4.69) and

$$\limsup_{y \to (l_{\sup}(t))^-} f(t,y) \le f(t, l_{\sup}(t)), \quad g\text{-a.a. } t \in I \setminus D_g. \tag{4.75}$$

Under these hypotheses, we also have that $(l_{\sup})'_g(t) = f(t, l_{\sup}(t))$ for every $t \in I \cap D_g$. Therefore, as long as (4.64) and (4.69) are satisfied, determining conditions so that $u_{\inf}$ and $l_{\sup}$ satisfy (4.62) on $I \setminus D_g$ is enough to guarantee that they are solutions on I.

Hence, we have obtained sufficient conditions on f for $u_{\inf}$ and $l_{\sup}$ to be upper and lower solutions, respectively. The interest of this type of results lies in the fact that if $u_{\inf}$ and $l_{\sup}$ are upper and lower solutions, respectively, by definition we have that they are the least upper solution and the greatest lower solution, respectively. From there, once we can ensure that they are solutions, we immediately obtain that they are the least and the greatest solution, respectively. Following this idea, we present the next result, from which we will derive some conditions that ensure that $u_{\inf}$ is a lower solution, which combined with the conditions that make $u_{\inf}$ an upper solution, yield conditions for it to be a solution.

Theorem 4.75. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ satisfy condition* (4.64) *and* (4.69). *Assume that*

$$(u_{\inf})'_g(t) \ge f(t, u_{\inf}(t)), \quad g\text{–a.a. } t \in I \setminus D_g,$$

and denote

$$J = \left\{ t \in I \setminus D_g : (u_{\inf})'_g(t) > \limsup_{y \to (u_{\inf}(t))^-} f(t,y) \right\}.$$

Then:

(a) *We can write $J = \bigcup_{k,m \in \mathbb{N}} J_{k,m}$ where, for each $k, m \in \mathbb{N}$,*

$$J_{k,m} = \left\{ t \in I \setminus D_g : (u_{\inf})'_g(t) - \frac{1}{k} > \sup \left\{ f(t,y) : u_{\inf}(t) - \frac{1}{m} < y < u_{\inf}(t) \right\} \right\}.$$

(b) *Each set $J_{k,m}$, $k, m \in \mathbb{N}$, contains no positive measure set.*

(c) *If $J_{k,m}$, $k, m \in \mathbb{N}$, is g–measurable then*

$$(u_{\inf})'_g(t) \le \limsup_{y \to (u_{\inf}(t))^-} f(t,y), \quad g\text{–a.a. } t \in I \setminus D_g.$$

Proof. For each $t \in J$ there exists $k \in \mathbb{N}$ such that

$$(u_{\inf})'_g(t) - \frac{1}{k} > \limsup_{y \to (u_{\inf}(t))^-} f(t,y) = \inf_{\varepsilon > 0} \left\{ \sup_{u_{\inf}(t) - \varepsilon < y < u_{\inf}(t)} f(t,y) \right\}.$$

Therefore, there exists $m \in \mathbb{N}$ such that

$$(u_{\inf})'_g(t) - \frac{1}{k} > \sup \left\{ f(t,y) : u_{\inf}(t) - \frac{1}{m} < y < u_{\inf}(t) \right\},$$

and so $t \in J_{k,m}$. Conversely, if $t \in J_{k,m}$ for some $k, m \in \mathbb{N}$, then

$$(u_{\inf})'_g(t) - \frac{1}{k} > \sup_{u_{\inf}(t)-1/m<y<u_{\inf}} f(t,y) \geq \limsup_{y \to (u_{\inf}(t))^-} f(t,y).$$

Hence, $t \in J$, which proves (a).

To prove (b) we reason by contradiction. Assume there exist $k, m \in \mathbb{N}$ such that $J_{k,m}$ contains a subset of positive g-measure, denoted again by $J_{k,m}$ for simplicity. Lemma 3.34 ensures the existence of $t_1 \in J_{k,m} \cap (t_0, t_0+T)$ and $\delta > 0$ such that for all $t \in (t_1, t_1+\delta)$,

$$\mu_g([t_1,t) \cap J_{k,m}) \geq \frac{1}{2}(g(t)-g(t_1)), \quad \int_{[t_1,t)\setminus J_{k,m}} M(s)dg(s) \leq \frac{1}{4k}\mu_g([t_1,t)\cap J_{k,m}).$$

Moreover, since $t_1 \notin D_g$, δ can be chosen so that $g(t) - g(t_1) < k/m$ for all $t \in (t_1, t_1+\delta)$.

Define $u \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ such that $u(t_0) = x_0$ and for all $t \in I$,

$$u'_g(t) = \begin{cases} (u_{\inf})'_g(t) & \text{if } t < t_1, \\ (u_{\inf})'_g(t) - \dfrac{1}{k} & \text{if } t \in [t_1, t_1+\delta] \cap J_{k,m}, \\ M(t) & \text{otherwise.} \end{cases}$$

First of all, Theorem 3.27 and Remark 4.64 yield that

$$u(t_1) = x_0 + \int_{[t_0,t_1)} u'_g(s)\,\mathrm{d}\,g(s) = x_0 + \int_{[t_0,t_1)} u'_{\inf}(s)\,\mathrm{d}\,g(s) = u_{\inf}(t_1).$$

Moreover, note that $|u'_g(t)| \leq M(t)$ for g–a.a. $t \in \overline{I}$. Indeed, the inequality is clear for $t \notin [t_1, t_1+\delta] \cap J_{k,m}$. For $t \in [t_1, t_1+\delta] \cap J_{k,m}$, $(u_{\inf})'_g(t) - 1/k \leq M(t) - 1/k < M(t)$ and

$$(u_{\inf})'_g(t) - \frac{1}{k} > \sup\left\{ f(t,y) : u_{\inf}(t) - \frac{1}{m} < y < u_{\inf}(t) \right\} \geq -M(t).$$

Now, the hypotheses and Corollary 4.73 ensure that $u_{\inf}$ is an upper solution. Thus, it follows that $u'_g(t) \geq f(t,u(t))$ for g–a.a. $t \in I \setminus ([t_1, t_1+\delta] \cap J_{k,m})$. Therefore, it is enough show that

$$u'_g(t) \geq f(t,u(t)), \quad g\text{–a.a. } t \in (t_1, t_1+\delta) \cap J_{k,m}, \tag{4.76}$$

as in that case we would have that $u \in \mathcal{U}$, which is a contradiction since $u_{\inf}(t) - u(t) > 0$

for all $t \in (t_1, t_1 + \delta)$. Indeed, for $t \in (t_1, t_1 + \delta)$ define $\psi(t) = u_{\inf}(t) - u(t)$. Then

$$
\begin{aligned}
\psi(t) &= \int_{[t_1,t)} \psi'_g(s)\,\mathrm{d}\,g(s) \\
&= \int_{[t_1,t)\cap J_{k,m}} \psi'_g(s)\,\mathrm{d}\,g(s) + \int_{[t_1,t)\setminus J_{k,m}} \psi'_g(s)\,\mathrm{d}\,g(s) \\
&= \int_{[t_1,t)\cap J_{k,m}} \frac{1}{k}\,\mathrm{d}\,g(s) + \int_{[t_1,t)\setminus J_{k,m}} ((u_{\inf})'_g(s) - M(s))\,\mathrm{d}\,g(s) \\
&= \frac{1}{k}\mu_g([t_1,t)\cap J_{k,m}) + \int_{[t_1,t)\setminus J_{k,m}} ((u_{\inf})'_g(s) - M(s))\,\mathrm{d}\,g(s) \qquad (4.77)\\
&\geq \frac{1}{k}\mu_g([t_1,t)\cap J_{k,m}) - 2\int_{[t_1,t)\setminus J_{k,m}} M(s)dg(s) \\
&\geq \frac{1}{k}\mu_g([t_1,t)\cap J_{k,m}) - \frac{1}{2k}\mu_g([t_1,t)\cap J_{k,m}) = \frac{1}{2k}\mu_g([t_1,t)\cap J_{k,m}) > 0,
\end{aligned}
$$

as we claimed.

Finally, let us prove (4.76). Starting from (4.77), we have that for $t \in (t_1, t_1 + \delta)$,

$$
\begin{aligned}
u_{\inf}(t) - u(t) &= \frac{1}{k}\mu_g([t_1,t)\cap J_{k,m}) + \int_{[t_1,t)\setminus J_{k,m}} ((u_{\inf})'_g(s) - M(s))\,\mathrm{d}\,g(s) \\
&\leq \frac{1}{k}\mu_g([t_1,t)\cap J_{k,m}) \leq \frac{1}{k}\mu_g([t_1,t)) = \frac{1}{k}(g(t) - g(t_1)) < \frac{1}{m}.
\end{aligned}
$$

This is, for $t \in (t_1, t_1 + \delta)$ it holds that $0 < u_{\inf}(t) - u(t) < 1/m$. Equivalently, for $t \in (t_1, t_1+\delta)$, $u_{\inf}(t) - 1/m < u(t) < u_{\inf}(t)$. Therefore, for g–a.a. $t \in (t_1, t_1+\delta)\cap J_{k,m}$,

$$
u'_g(t) = (u_{\inf})'_g(t) - \frac{1}{k} > \sup_{u_{\inf}(t)-1/m<y<u_{\inf}(t)} f(t,y) \geq f(t,u(t)),
$$

which concludes the proof of part (b).

Part (c) follows from parts (a) and (b) with the extra assumption. □

Remark 4.76. Combining condition (4.74) with Theorem 4.75, part (c), it is easy to see that $u_{\inf}$ is a solution of (4.2) if the sets $J_{k,m}$ are g–measurable and

$$
\limsup_{y\to(u_{\inf}(t))^-} f(t,y) \leq f(t,u_{\inf}(t)) \leq \liminf_{y\to(u_{\inf}(t))^+} f(t,y), \quad g\text{-a.a.} \quad t \in I\setminus D_g.
$$

However, since $u_{\inf}$ is unknown a priori, a reasonable sufficient condition to impose is

$$
\limsup_{y\to x^-} f(t,y) \leq f(t,x) \leq \liminf_{y\to x^+} f(t,y), \quad g\text{-a.a.} \quad t\in I\setminus D_g, \quad x\in\mathbb{R}. \qquad (4.78)
$$

We now present the corresponding adaptation of Theorem 4.75 for $l_{\sup}$ obtained thanks to equivalence presented in Theorem 4.65.

Theorem 4.77. *Let* $f : I \times \mathbb{R} \to \mathbb{R}$ *satisfy conditions* (4.64) *and* (4.69)*. Assume that*

$$(l_{\sup})'_g(t) \leq f(t, l_{\sup}(t)), \quad g\text{–a.a. } t \in I \setminus D_g,$$

and denote

$$\widehat{J} = \left\{ t \in I \setminus D_g : (l_{\sup})'_g(t) < \liminf_{y \to (l_{\sup}(t))^+} f(t,y) \right\}.$$

Then:

(a) *We can write* $\widehat{J} = \bigcup_{k,m \in \mathbb{N}} \widehat{J}_{k,m}$ *where, for each* $k, m \in \mathbb{N}$*,*

$$\widehat{J}_{k,m} = \left\{ t \in I \setminus D_g : (l_{\sup})'_g(t) + \frac{1}{k} < \inf \left\{ f(t,y) : l_{\sup}(t) < y < l_{\sup}(t) + \frac{1}{m} \right\} \right\}.$$

(b) *Each set* $\widehat{J}_{k,m}$*,* $k, m \in \mathbb{N}$*, contains no positive measure set.*

(c) *If* $\widehat{J}_{k,m}$*,* $k, m \in \mathbb{N}$*, is* g*–measurable then*

$$(l_{\sup})'_g(t) \geq \liminf_{y \to (l_{\sup}(t))^+} f(t,y), \quad g\text{–a.a. } t \in I \setminus D_g.$$

Remark 4.78. Combining condition (4.75) with Theorem 4.77, part (c), it is easy to see that $l_{\sup}$ is a solution of (4.2) if the sets $\widehat{J}_{k,m}$ are g–measurable and

$$\limsup_{y \to (l_{\sup}(t))^-} f(t,y) \leq f(t, l_{\sup}(t)) \leq \liminf_{y \to (l_{\sup}(t))^+} f(t,y), \quad g\text{-a.a.} \quad t \in I \setminus D_g.$$

However, since $l_{\sup}$ is unknown a priori, imposing (4.78) is enough. Obviously, the conditions on this remark are just the same as the ones obtained by "translating" Remark 4.76 through Theorem 4.65.

Condition (4.78) can be weakened by allowing those inequalities to fail over some curves that are usually called admissible curves. Let us illustrate this with the following result for $u_{\inf}$.

Theorem 4.79. *Let* $f : I \times \mathbb{R} \to \mathbb{R}$ *be a mapping satisfying* (4.64) *and* (4.69)*. Assume one of the following is satisfied:*

(i) f *satisfies* (4.78)*;*

(ii) *there exists a family of* g*-absolutely continuous functions,* $\gamma_k : [c_k, d_k] \subset I \to \mathbb{R}$*,* $k \in \mathbb{N}$*, such that for* g*–a.a.* $t \in I \setminus D_g$ *and all* $x \in \mathbb{R} \setminus \bigcup_{\{k \in \mathbb{N} : c_k \leq t \leq d_k\}} \{\gamma_k(t)\}$ *the inequality* (4.78) *holds, while for each* $k \in \mathbb{N}$ *and* g*–a.a.* $t \in [c_k, d_k] \setminus D_g$ *we have either* $(\gamma_k)'_g(t) = f(t, \gamma_k(t))$ *or*

$$(\gamma_k)'_g(t) \geq f(t, \gamma_k(t)) \quad \textit{whenever} \quad (\gamma_k)'_g(t) \geq \liminf_{y \to (\gamma_k(t))^+} f(t,y), \tag{4.79}$$

$$(\gamma_k)'_g(t) \leq f(t, \gamma_k(t)) \quad \textit{whenever} \quad (\gamma_k)'_g(t) \leq \limsup_{y \to (\gamma_k(t))^-} f(t,y). \tag{4.80}$$

Then:

(a) $(u_{\inf})'_g(t) = f(t, u_{\inf}(t))$ *for g–a.a.* $t \in I \setminus J$ *where* J *is as in* Theorem 4.75.

(b) $u_{\inf}$ *is the least solution of* (4.62) *provided that for all* $k, m \in \mathbb{N}$*, the set* $J_{k,m}$ *in* Theorem 4.75 *is g–measurable.*

Proof. We shall assume that (ii) holds, as the proof for (i) is analogous, but simpler. By Theorem 4.71 there exists $I_1 \subset I$ such that $D_g \subset I_1$ and

$$(u_{\inf})'_g(t) \geq f(t, u_{\inf}(t))\chi_{I_1}(t) + \liminf_{y \to (u_{\inf}(t))^+} f(t, y)\chi_{I \setminus I_1}(t), \quad g\text{-a.a. } t \in I. \tag{4.81}$$

Define $A_k = \{t \in [c_k, d_k] \setminus D_g : u_{\inf}(t) = \gamma_k(t)\}$, $k \in \mathbb{N}$. It follows from (4.74) and condition (ii) that

$$(u_{\inf})'_g(t) \geq f(t, u_{\inf}(t)) \text{ holds for } g\text{-a.a. } t \in I \setminus \bigcup_{k \in \mathbb{N}} A_k. \tag{4.82}$$

For each $k \in \mathbb{N}$, define $\Phi_k(t) = u_{\inf}(t) - \gamma_k(t)$, $t \in [c_k, d_k]$, and

$$E_k = \{t \in A_k : \exists\, (u_{\inf})'_g(t),\, (\gamma_k)'_g(t)\}.$$

For each $k \in \mathbb{N}$, apply Lemma 3.37 with $\Phi = \Phi_k$ and $E = E_k$. This yields that, for each $k \in \mathbb{N}$, $(u_{\inf})'_g(t) = (\gamma_k)'_g(t)$ for g–a.a. $t \in E_k$. Since $u_{\inf}, \gamma_k \in \mathcal{AC}_g([c_k, d_k], \mathbb{R})$, we have that $\mu_g(A_k \setminus E_k) = 0$. Hence $(u_{\inf})'_g(t) = (\gamma_k)'_g(t)$ for g–a.a. $t \in A_k$ so (4.82) yields

$$(u_{\inf})'_g(t) \geq f(t, u_{\inf}(t)), \quad g\text{-a.a. } t \in I \setminus \bigcup_{k \in \mathbb{N}} \Gamma_k,$$

where $\Gamma_k = \{t \in A_k : (\gamma_k)'_g(t) \neq f(t, \gamma_k(t))\}$. Let us show that, in fact, the inequality holds for g–a.a. $t \in I$.

Let $k \in \mathbb{N}$ be fixed and let $t_1 \in \Gamma_k$ be such that $(u_{\inf})'_g(t_1) = (\gamma_k)'_g(t_1)$. We study separately two cases: either

$$(\gamma_k)'_g(t_1) < \liminf_{y \to (\gamma_k(t_1))^+} f(t, y), \quad \text{or} \quad (\gamma_k)'_g(t_1) \geq \liminf_{y \to (\gamma_k(t_1))^+} f(t, y).$$

If $(\gamma_k)'_g(t_1) < \liminf_{y \to (\gamma_k(t_1))^+} f(t, y)$, then $(u_{\inf})'_g(t_1) < \liminf_{y \to (\gamma_k(t_1))^+} f(t, y)$. Hence, by (4.81), either t_1 belongs to a null-measure set or $t_1 \in I_1$, and so

$$(u_{\inf})'_g(t_1) \geq f(t_1, u_{\inf}(t_1)).$$

Otherwise, $(\gamma_k)'_g(t_1) \geq \liminf_{y \to (\gamma_k(t_1))^+} f(t, y)$ and so by (4.79) either t_1 belongs to a null-measure set or $(\gamma_k)'_g(t_1) \geq f(t_1, \gamma_k(t_1))$, and therefore $(u_{\inf})'_g(t_1) \geq f(t_1, u_{\inf}(t_1))$.

We have thus proven that $(u_{\inf})'_g(t) \geq f(t, u_{\inf}(t))$ for g–a.a. $t \in I$. Now, applying Theorem 4.75, for g–a.a. $t \in I \setminus J$ we have, either $t \in D_g$, and then Corollary 4.73 yields that $(u_{\inf})'(t) = f(t, u_{\inf}(t))$, or $t \notin D_g$ and

$$(u_{\inf})'_g(t) \leq \limsup_{y \to (u_{\inf}(t))^-} f(t, y). \tag{4.83}$$

In that case, (4.78) implies that $(u_{\inf})'_g(t) \le f(t, u_{\inf}(t))$ for g–a.a. $t \in (I \setminus J) \setminus \bigcup_{k\in\mathbb{N}} A_k$. Let us show that the inequality holds for g–a.a. $t \in I \setminus J$.

Let $k \in \mathbb{N}$ be fixed. Since $(u_{\inf})'_g = (\gamma_k)'_g$ g–almost everywhere in A_k, it suffices to see what happens at an arbitrary point $t_1 \in A_k$ such that $(u_{\inf})'_g(t_1) = (\gamma_k)'_g(t_1)$. Recall that $u_{\inf}(t_1) = \gamma_k(t_1)$ and $t_1 \notin D_g$. Now, if $(\gamma_k)'_g(t_1) > \limsup_{y\to(\gamma_k(t_1))^-} f(t, y)$, then

$$(u_{\inf})'_g(t_1) > \limsup_{y\to(u_{\inf}(t_1))^-} f(t, y),$$

hence, $t_1 \in J$. Otherwise, $(\gamma_k)'_g(t_1) \le \limsup_{y\to(\gamma_k(t_1))^-} f(t, y)$, by (4.80), either t_1 belong to a null-measure set or $(\gamma_k)'_g(t_1) \le f(t_1, \gamma_k(t_1))$, and therefore

$$(u_{\inf})'_g(t_1) \le f(t_1, u_{\inf}(t_1)).$$

Hence $(u_{\inf})'_g(t) \le f(t, u_{\inf}(t))$ for g–a.a. $t \in I \setminus J$, and so,

$$(u_{\inf})'_g(t) = f(t, u_{\inf}(t)), \quad g\text{-a.a. } t \in I \setminus J.$$

Part (b) follows from (a) with the extra assumption. □

Remark 4.80. The corresponding result for $l_{\sup}$ obtained through Theorem 4.65 reads as follows: if the hypotheses of Theorem 4.79 are satisfied, then

(a) $(l_{\sup})'_g(t) = f(t, l_{\sup}(t))$ for g–a.a. $t \in I \setminus \widehat{J}$ where $\widehat{J}$ is as in Theorem 4.77.

(b) $l_{\sup}$ is the greatest solution of (4.62) provided that for all $k, m \in \mathbb{N}$, the set $\widehat{J}_{k,m}$ in Theorem 4.77 is g–measurable.

Part (a) of Theorem 4.79 and its counterpart ensure that $u_{\inf}$ and $l_{\sup}$ are some sort of "weak" solution in an extremely weak sense, as countable union of sets having no positive g-measure may be rather big. Anyway, the measurability of the sets $J_{k,m}$ and $\widehat{J}_{k,m}$ is enough to turn $u_{\inf}$ and $l_{\sup}$, respectively, into solutions. With this idea in mind, we move on to the study of sufficient conditions for the sets $J_{k,m}$ and $\widehat{J}_{k,m}$ to be g–measurable. To that end, we will need the following lemmas.

Lemma 4.81. *Let $x : \overline{I} \to \mathbb{R}$ be a function which has bounded variation. Then there exists a sequence of step functions, $\{x_k\}_{k\in\mathbb{N}}$, converging uniformly to x on $\overline{I}$ such that for all $k \in \mathbb{N}$ and all $t \in \overline{I}$, $x_k(t) \in \mathbb{Q}$.*

Proof. We shall show that such sequence exists for a nondecreasing function, $y : \overline{I} \to \mathbb{R}$, as any function with bounded variation can be expressed as difference of two nondecreasing functions. Consider the sequence $y_k : \overline{I} \to \mathbb{R}$, $k \in \mathbb{N}$, given by

$$y_k(t) := \frac{1}{k}[k \cdot y(t)], \quad t \in \overline{I},$$

where $[z]$ denotes the integer part of z. First, note that $y_k(t) \in \mathbb{Q}$ for all $k \in \mathbb{N}$ and all $t \in \overline{I}$, and moreover, each y_k, $k \in \mathbb{N}$, is a step function since y is nondecreasing. Furthermore,

$$0 \le \|y - y_k\|_\infty = \sup_{t\in\overline{I}} \left| y(t) - \frac{1}{k}[k \cdot y(t)] \right| = \frac{1}{k} \sup_{t\in\overline{I}} |k \cdot y(t) - [k \cdot y(t)]| \le \frac{1}{k},$$

from which the result follows. □

Let $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $\{x_k\}_{k\in\mathbb{N}}$ be the corresponding sequence in Lemma 4.81. Define

$$D = \bigcup_{k\in\mathbb{N}} \{t \in I : x_k \text{ is not continuous at } t\}.$$

Note that D is a countable set as it is a countable union of countable sets. Given $\varepsilon > 0$, we define the set $\mathcal{S}_x$ as the set of step functions defined as follows: a function $v : (t_0, t_0+T) \to \mathbb{R}$ belongs to $\mathcal{S}_x$ if, and only if,

1. $x(t) - \varepsilon < v(t) < x(t)$ for all $t \in (t_0, t_0 + T)$;
2. $v(t) \in \mathbb{Q}$ for all $t \in (t_0, t_0 + T)$;
3. there exist $a_1 < a_2 < ... < a_m \in D$ such that

$$v \text{ is constant on } (t_0, a_1), (a_1, a_2), \ldots, (a_{m-1}, a_m), (a_m, t_0 + T).$$

Remark 4.82. The set $\mathcal{S}_x$ is nonempty. Indeed, since $\{x_k\}_{k\in\mathbb{N}}$ converges uniformly on $\overline{I}$ to x, there exists $k_0 \in \mathbb{N}$ such that

$$x(t) - \frac{\varepsilon}{3} < x_{k_0}(t) < x(t) + \frac{\varepsilon}{3}, \quad t \in \overline{I}.$$

Define $s(t) = x_{k_0}(t) - q$ for some $q \in (\varepsilon/3, 2\varepsilon/3) \cap \mathbb{Q}$. It is easy to see that $s \in \mathcal{S}_x$.

Lemma 4.83. *Let $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$, $\varepsilon > 0$ and $\mathcal{S}_x$ be as before. For all $t \in (t_0, t_0 + T)$, all $y \in (x(t) - \varepsilon, x(t))$ and all $\delta > 0$, there exists $s \in \mathcal{S}_x$ such that $y - \delta < s(t) < y$. Analogously, for all $t \in (t_0, t_0 + T)$, all $y \in (x(t) - \varepsilon, x(t))$ and all $\delta > 0$, there exists $s \in \mathcal{S}_x$ such that $y < s(t) < y + \delta$.*

Proof. We shall only prove the first part of the statement, as the second part is analogous. Fix $t \in (t_0, t_0 + T)$, $y \in (x(t) - \varepsilon, x(t))$ and $\delta > 0$. Take $\widetilde{\delta} \in (0, \delta]$ such that $x(t) - \varepsilon < y - \widetilde{\delta}$. Since $\{x_k\}_{k\in\mathbb{N}}$ converges uniformly on I to x and $y \in (x(t) - \varepsilon, x(t))$, we can find $j, k_0 \in \mathbb{N}$ big enough so that

$$x(t) - \frac{j-1}{j}\varepsilon < y - \widetilde{\delta} < y < x(t) - \frac{\varepsilon}{j} \quad \text{and} \quad x(r) - \frac{\varepsilon}{2j} < x_{k_0}(r) < x(r) + \frac{\varepsilon}{2j}, \quad r \in I.$$

The function $s(r) = x_{k_0}(r) - x_{k_0}(t) + q$ for some $q \in (y - \widetilde{\delta}, y) \cap \mathbb{Q}$ satisfies the statement of the lemma. Indeed, first $s \in \mathcal{S}_x$ since conditions 2 and 3 are trivially fulfilled and

$$\begin{aligned}
s(r) &= x_{k_0}(r) - x_{k_0}(t) + q < x(r) + \frac{\varepsilon}{2j} - x(t) + \frac{\varepsilon}{2j} + y = x(r) - x(t) + \frac{\varepsilon}{j} + y \\
&< x(r) - x(t) + \frac{\varepsilon}{j} + x(t) - \frac{\varepsilon}{j} = x(r); \\
s(r) &= x_{k_0}(r) - x_{k_0}(t) + q > x(r) - \frac{\varepsilon}{2j} - x(t) - \frac{\varepsilon}{2j} + y - \widetilde{\delta} = x(r) - x(t) - \frac{\varepsilon}{j} + y - \widetilde{\delta} \\
&> x(r) - x(t) - \frac{\varepsilon}{j} + x(t) - \frac{j-1}{j}\varepsilon = x(r) - \varepsilon.
\end{aligned}$$

Moreover, $s(t) = x_{k_0}(t) - x_{k_0}(t) + q = q \in (y - \widetilde{\delta}, y) \cap \mathbb{Q} \subset (y - \delta, y) \cap \mathbb{Q}$. □

We can now prove the following result that gives a sufficient condition for the sets $J_{k,m}$, $k, m \in \mathbb{N}$, in Theorem 4.75 to be g–measurable. We prove a more general result and obtain the corresponding conditions from it later.

Theorem 4.84. *Let $N \subset I$ be a g-null measure set and let $f : I \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, q)$ is g–measurable for each $q \in \mathbb{Q}$. If for all $t \in I \setminus N$ and all $x \in \mathbb{R}$,*

$$\max\left\{\liminf_{y\to x^-} f(t,y), \liminf_{y\to x^+} f(t,y)\right\} \geq f(t,x), \tag{4.84}$$

then, for all $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and all $\varepsilon > 0$, the mapping

$$\varphi_{x,\varepsilon}(t) = \sup\{f(t,y) : x(t) - \varepsilon < y < x(t)\}, \quad t \in I,$$

is g–measurable.

Proof. Fix $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $\varepsilon > 0$. Define $\mathcal{S}_x$ as before. Then $\mathcal{S}_x$ is a countable family of functions. Indeed, since D is countable, the set D^m is countable for each $m \in \mathbb{N}$. For each $\omega = (\omega_1, ..., \omega_m) \in D^m$, let us denote by $\mathcal{S}_\omega$ the set of step functions of $\mathcal{S}_x$ which are constant on the intervals whose extreme points are consecutives number of ω. It is easy to see that each $\mathcal{S}_\omega$ is countable, and so, $\mathcal{S}_x$ is countable as it can be written as

$$\mathcal{S}_x = \bigcup_{m\in\mathbb{N}} \left(\bigcup_{\omega\in D^m} \mathcal{S}_\omega\right).$$

Hence, given that $f(\cdot, s(\cdot))$ is g–measurable on $(t_0, t_0 + T)$ for $s \in \mathcal{S}$, it is enough to show that $\sigma = \sigma_0$ where

$$\sigma(t) := \sup_{y\in(x(t)-\varepsilon,x(t))} f(t,y), \quad \sigma_0(t) := \sup_{s\in\mathcal{S}} f(t,s(t)).$$

It is obvious that $\sigma(t) \geq \sigma_0(t)$ on $(t_0, t_0 + T)$. To prove that $\sigma_0 \geq \sigma$ on $(t_0, t_0 + T) \setminus N$, fix $t \in (t_0, t_0 + T) \setminus N$ and take a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $(x(t) - \varepsilon, x(t))$ such that $\lim_{k\to\infty} f(t, y_k) = \sigma(t)$. Our assumptions guarantee that for each n we have that either $\liminf_{y\to y_k^-} f(t,y) \geq f(t,y_k)$ or $\liminf_{y\to y_k^+} f(t,y) \geq f(t,y_k)$. Assume the first case holds as the other one is analogous. By definition, we have

$$f(t,y_k) \leq \liminf_{y\to y_k^-} f(t,y) = \lim_{r\to 0^+}\left(\inf_{y_k-r<z<y_k} f(t,z)\right).$$

Then, there exists $\delta > 0$ such that $\inf_{y_k-\delta<z<y_k} f(t,z) \geq f(t,y_k) - 1/n$. Hence for each $n \in \mathbb{N}$, by Lemma 4.83, there exists $s_k \in \mathcal{S}_x$ such that $y_k - \delta < s_k(t) < y_k$, and so

$$f(t,s_k(t)) \geq \inf_{y_k-\delta<z<y_k} f(t,z) \geq f(t,y_k) - \frac{1}{n}.$$

Therefore, $\sigma_0 := \sup_{s\in\mathcal{S}} f(t,s(t)) \geq f(t,s_k(t)) \geq f(t,y_k) - 1/n$. Since this holds for each $n \in \mathbb{N}$,

$$\sigma_0(t) \geq \lim_{k\to\infty}\left(f(t,y_k) - \frac{1}{n}\right) = \lim_{k\to\infty} f(t,y_k) = \sigma(t),$$

and so $\sigma = \sigma_0$ on $(t_0, t_0 + T) \setminus N$. □

Remark 4.85. Note that the sets $J_{k,m}$, $k,m \in \mathbb{N}$, in Theorem 4.75 can be rewitten as

$$J_{k,m} = \left\{ t \in I \setminus D_g : (u_{\inf})'_g(t) - \varphi_{u_{\inf},1/m}(t) - \frac{1}{k} > 0 \right\}, \quad k,m \in \mathbb{N}.$$

Thus, if (4.84) is satisfied, the maps $(u_{\inf})'_g(\cdot) - \varphi_{u_{\inf},1/m}(\cdot) - 1/k$, $k,m \in \mathbb{N}$, are g–measurable and, as a consequence, so are the sets $J_{k,m}$.

To obtain the corresponding condition for the sets $\widehat{J}_{k,m}$, $k,m \in \mathbb{N}$, in Theorem 4.77, we wil use Theorem 4.65 once again, although this time we will use the definition of the map $\widetilde{f}$.

Corollary 4.86. *Let $N \subset I$ be a g-null measure set and let $f : I \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, q)$ is g–measurable for each $q \in \mathbb{Q}$. If for all $t \in I \setminus N$ and all $x \in \mathbb{R}$,*

$$\min \left\{ \limsup_{y \to x^-} f(t,y), \limsup_{y \to x^+} f(t,y) \right\} \leq f(t,x), \tag{4.85}$$

then, for all $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and all $\varepsilon > 0$, the mapping

$$\psi_{x,\varepsilon}(t) = \inf\{f(t,y) : x(t) < y < x(t) + \varepsilon\}, \quad t \in I$$

is g–measurable.

Proof. Define $\widetilde{f}$ as in Theorem 4.65, i.e. $\widetilde{f}(t,x) = -f(t,-x)$, $(t,x) \in I \times \mathbb{R}$. Clearly, $\widetilde{f}(\cdot, q)$ is g–measurable for each $q \in \mathbb{Q}$. Moreover, f satisfying (4.85) is equivalent to $\widetilde{f}$ satisfying (4.84). Hence, Theorem 4.84 ensures that the map

$$\widetilde{\varphi}_{x,\varepsilon}(t) = \sup \left\{ \widetilde{f}(t,y) : x(t) - \varepsilon < y < x(t) \right\}, \quad t \in I,$$

is g–measurable for each $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $\varepsilon > 0$. The result now follows from noting that $\psi_{x,\varepsilon} = -\widetilde{\varphi}_{-x,\varepsilon}$ for each $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $\varepsilon > 0$. □

Remark 4.87. Note that the sets $\widehat{J}_{k,m}$, $k,m \in \mathbb{N}$, in Theorem 4.77 can be rewitten as

$$\left\{ t \in I \setminus D_g : (l_{\sup})'_g(t) - \psi_{l_{\sup},1/m} + \frac{1}{k} < 0 \right\}, \quad k,m \in \mathbb{N}.$$

Thus, if (4.85) is satisfied, the maps $(u_{\inf})'_g(\cdot) - \psi_{l_{\sup},1/m} + 1/k$, $k,m \in \mathbb{N}$, are g–measurable, and as a consequence, so are the sets $\widehat{J}_{k,m}$.

Thus, by gathering all the results obtained until this point, we have the following result for the existence of extremal solutions of problem (4.62).

Theorem 4.88. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ satisfy the hypotheses of* Theorem 4.79*. If $f(\cdot, q)$ is g–measurable for all $q \in \mathbb{Q}$, then:*

(1) *If* (4.84) *holds for g–a.a. $t \in I$ and all $x \in \mathbb{R}$, then $u_{\inf}$ is the least solution of problem* (4.62).

(2) *If* (4.85) *holds for g–a.a. $t \in I$ and all $x \in \mathbb{R}$, then $l_{\sup}$ is the greatest solution of problem* (4.62).

Next we illustrate the applicability of Theorem 4.88 in a family of examples with non–monotone discontinuities accumulating around the initial condition.

Example 4.89. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an arbitrary nondecreasing and left–continuous function and $\varphi : [0,1] \longrightarrow \mathbb{R}$ be a nondecreasing g–absolutely continuous function on $[0,1]$ such that $\varphi(0) = 0$. Note that such map exists. Indeed, it is enough to consider $c \in \mathcal{L}^1_g([0,1),\mathbb{R})$ such that $c(t) \geq 0$ for all $t \in [0,1)$ and

$$\varphi(t) = \int_{[0,t)} c(s)\, \mathrm{d}\, g(s), \quad t \in [0,1].$$

In particular, for each $\lambda > 0$, the map $\varphi_\lambda : [0,1] \to \mathbb{R}$ defined as

$$\varphi_\lambda(t) = \lambda(g(t) - g(0)), \quad t \in [0,1], \tag{4.86}$$

satisfies all the conditions stated above. Recall that, since g is nondecreasing, it can present infinitely many countable number of discontinuity points, which would result in the same property for the maps φ_λ, $\lambda > 0$, in (4.86). See Figure 4.7 to observe this behaviour.

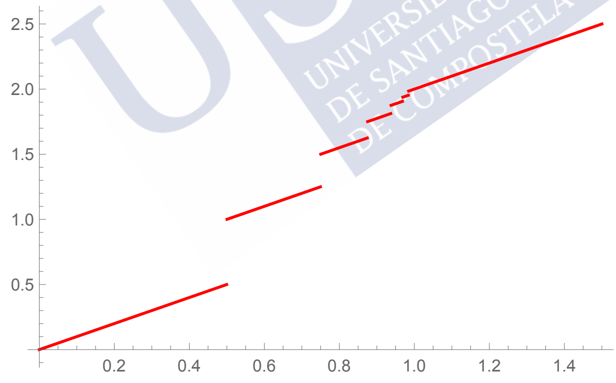


Figure 4.7: Example of a nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$ with infinitely many countable discontinuity points. In particular, this graph corresponds to φ_1 in (4.86) for such function g.

We shall prove by means of Theorem 4.88 that (4.62) has the extremal solutions for

$$f(t,x) = \begin{cases} 2 + \sin\left[\dfrac{1}{x + \varphi(t)}\right], & \text{if } t \in I \setminus D_g \text{ and } x > 0, \\ 2, & \text{otherwise,} \end{cases}$$

where $[z]$ denotes the integer part of z. We remark that f is discontinuous and non–monotone with respect to x on every neighborhood of the initial condition.

First, observe that $f(t,x) \in (1,3)$ for all $(t,x) \in I \times \mathbb{R}$, which implies (4.69); second, for each fixed $t \in I \cap D_g$ we have that $f(t,\cdot)$ is constantly equal to 2, which implies (4.64).

Now, for the last hypothesis in Theorem 4.79, since $\varphi(t) \geq 0$ for all $t \in I$, we deduce that discontinuities can only occur at points (t,x) such that $x = 0$ or

$$\frac{1}{x+\varphi(t)} = k \quad \text{for some } k \in \mathbb{N}.$$

Therefore, we define $\gamma_0(t) = 0$ for all $t \in I$ and, for each $k = 1, 2, \dots,$

$$\gamma_k(t) = \frac{1}{k} - \varphi(t), \quad t \in I.$$

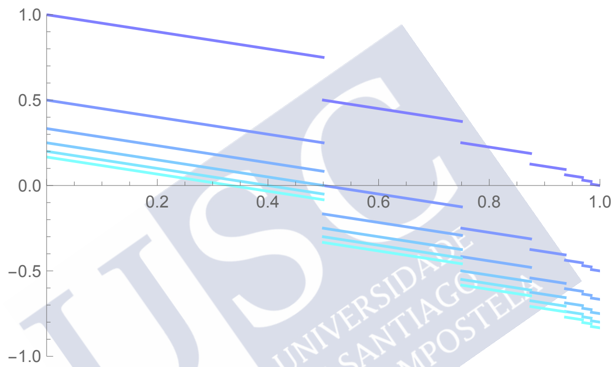


Figure 4.8: Representantion of some γ_k for g as in Figure 4.7 and φ as in (4.86) for $\lambda = 1/2$.

Notice that, for each fixed $t \in [0,1]$, the mapping $f(t,\cdot)$ is continuous on $\mathbb{R}\setminus\bigcup_{k=0}^{\infty}\{\gamma_k(t)\}$ (it might also be continuous at some points $x = \gamma_k(t)$, for some $k \in \mathbb{N}$, but this is not important). Therefore, for each fixed $t \in [0,1]$, the mapping $f(t,\cdot)$ satisfies (4.44) on $\mathbb{R} \setminus \bigcup_{k=0}^{\infty}\{\gamma_k(t)\}$. It remains to show that the curves γ_k, $k = 0, 1, \dots,$ either satisfy the differential equation, or they satisfy (4.79) and (4.80). Given $k = 0, 1, \dots,$ γ_k is nonincreasing and so, the definition of g–derivative yields that for g–a.a. $t \in [0,1]$,

$$(\gamma_k)'_g(t) \leq 0 < 1 \leq \min\left\{ f(t,\gamma_k(t)), \liminf_{y\to(\gamma_k(t))^+} f(t,y), \limsup_{y\to(\gamma_k(t))^-} f(t,y) \right\}.$$

Hence, we have that γ_k, $k = 0, 1, \dots,$ satisfies (4.80). Moreover,

$$(\gamma_k)'_g(t) \geq \limsup_{y\to(\gamma_k(t))^-} f(t,y), \quad k = 0, 1, \dots$$

only occurs for $t \in A$ with $\mu_g(A) = 0$. Therefore, condition (ii) in Theorem 4.79 is satisfied.

Finally, we check that $f(\cdot,q)$ is g–measurable for all $q \in \mathbb{Q}$ and that for g–a.a. $t \in I$ and all $x \in \mathbb{R}$ we have

$$\min\left\{ \limsup_{y\to x^-} f(t,y), \limsup_{y\to x^+} f(t,y) \right\} \leq f(t,x) \leq \max\left\{ \liminf_{y\to x^-} f(t,y), \liminf_{y\to x^+} f(t,y) \right\}.$$

The last part follows from the fact that, for each fixed $t \in [0,1]$, the mapping $f(t,\cdot)$ is continuous from the left at every $x \in \mathbb{R}$. Indeed, this is trivial if $t \in D_g$; otherwise, observe that $f(t,x) = 2$ for all $x \leq 0$, $f(t,x) = 2$ for $x > \gamma_1(t)$, and for $k = 1, 2, \dots$ we have

$$f(t,x) = 2 + \sin(k), \quad x \in (\gamma_{k+1}(t), \gamma_k(t)],\ x > 0.$$

To deduce that $f(\cdot,q)$ is g–measurable for each $q \in \mathbb{Q}$, just note that $f(\cdot,q)$ takes a finite number of different values on corresponding Borel–measurable subsets of $[0,1]$, hence $f(\cdot,q)$ is a Borel–measurable function, which implies that $f(\cdot,q)$ is g–measurable.

4.2 Stieltjes differential equations with functional arguments

In this section we have a look at Stieltjes differential equations that present some functional arguments. In particular, given a nondecreasing and left–continuous function, $g : \mathbb{R} \to \mathbb{R}$, we will consider the functional problem

$$x'_g(t) = f(t, x(t), x), \quad B(x(t_0), x) = 0, \tag{4.87}$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $f : [t_0, t_0+T) \times \mathbb{R} \times \mathcal{AC}_g([t_0,t_0+T],\mathbb{R}) \to \mathbb{R}$ and $B : \mathbb{R} \times \mathcal{AC}_g([t_0,t_0+T],\mathbb{R}) \to \mathbb{R}$. Observe that the conditions considered in this problem are more general than the ones before, as the map B allows us to consider, not only initial value problems, but more complex formulations such as boundary value problems, as studied in [75]. Furthermore, note that we are restricting ourselves to the context of the real line as we will study this problem using some of the results in Section 4.1.3. In particular, we will be looking for conditions to ensure the existence of extremal solutions between a pair of well–ordered lower and upper solutions, as in [48]. Later, we will obtain a new existence result for extremal solutions between a pair of well–ordered lower and upper solutions for problem (4.62) following the ideas of [9].

First, let us introduce the basic definitions for the study of this problem. As usual, we denote $I = [t_0, t_0+T)$ and $\overline{I} = [t_0, t_0+T]$.

Definition 4.90. *A* solution *of problem* (4.87) *on I is a function $x \in \mathcal{AC}_g(\overline{I},\mathbb{R}^n)$ such that $B(x(t_0),x) = 0$ and*

$$x'_g(t) = f(t, x(t), x), \quad g\text{–a.a. } t \in I.$$

A lower solution *of* (4.87) *on I is a function $\alpha \in \mathcal{AC}_g(\overline{I},\mathbb{R}^n)$ such that $B(\alpha(t_0),\alpha) \leq 0$ and*

$$\alpha'_g(t) \leq f(t, \alpha(t), \alpha), \quad g\text{–a.a. } t \in I.$$

Similarly, an upper solution *of problem* (4.87) *on I is a function $\beta \in \mathcal{AC}_g(\overline{I},\mathbb{R}^n)$ such that $B(\beta(t_0),\beta) \geq 0$ and*

$$\beta'_g(t) \geq f(t, \beta(t), \beta), \quad g\text{–a.a. } t \in I.$$

Remark 4.91. Similarly to (4.2) and (4.62), it is possible to consider solutions and lower and upper solutions to be defined strictly on I instead of $\overline{I}$. To do so, it is enough to ask for the maps to belong to $E\mathcal{AC}_g(\overline{I},\mathbb{R})$ instead of $\mathcal{AC}_g(\overline{I},\mathbb{R})$. We will still consider our definitions on the context of g–absolutely continuous maps for simplicity.

Note that these definitions yield Definitions 4.1 and 4.54 for problem (4.62) when f has no functional arguments and $B(s,y)=s-x_0$, $(s,y)\in I\times\mathcal{AC}_g(\overline{I},\mathbb{R})$. Similarly, we can extend the definition of extremal solutions between a lower and an upper solution for (4.62), Definition 4.60, as follows.

Definition 4.92. *Let α,β be a lower and an upper solution of* (4.87) *on I, respectively, such that $\alpha\le\beta$ on $\overline{I}$. A solution of* (4.87) *on I, x^*, is said to be the* greatest solution of (4.87) on I between α and β *if $\alpha(t)\le x^*(t)\le\beta(t)$, $t\in\overline{I}$, and*

$$x(t)\le x^*(t),\quad t\in\overline{I},$$

for every $x\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ solution of (4.87) *on I such that $\alpha(t)\le x(t)\le\beta(t)$, $t\in\overline{I}$. Similarly, a solution of* (4.87) *on I, x_*, is said to be the* least solution of (4.87) on I between α and β *if $\alpha(t)\le x_*(t)\le\beta(t)$, $t\in\overline{I}$, and*

$$x_*(t)\le x(t),\quad t\in\overline{I},$$

for every $x\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ solution of (4.62) *on I such that $\alpha(t)\le x(t)\le\beta(t)$, $t\in\overline{I}$. If* (4.87) *has the greatest and the least solution between α and β, we say that* (4.87) *has the extremal solutions between α and β.*

In order to obtain our result for the existence of extremal solutions for (4.87), we need the following version of Bolzano's theorem. This result can be found in [31, Lemma 2.3], and it will be used in the proof of Theorem 4.94.

Lemma 4.93. *Let $a,b\in\mathbb{R}$, $a\le b$, and let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be such that $h(a)\le 0\le h(b)$ and*

$$\liminf_{z\to x^-}h(z)\ge h(x)\ge\limsup_{z\to x^+}h(z),\quad x\in[a,b].\tag{4.88}$$

Then there exist $c_1,c_2\in[a,b]$ such that $h(c_1)=0=h(c_2)$. Moreover, if $h(c)=0$ for some $c\in[a,b]$ then $c_1\le c\le c_2$, i.e., c_1 and c_2 are, respectively, the least and the greatest of the zeros of h in $[a,b]$.

We are now in position to establish a new existence result based on the method of lower and upper solutions for (4.87) and Heikkilä's generalized iterative technique, see [40]. These ideas were followed in [30] in the case of the usual derivative, for which we use Proposition 3.33 and the notation there introduced.

Theorem 4.94. *Suppose that for $f:I\times\mathbb{R}\times\mathcal{AC}_g(\overline{I},\mathbb{R})\to\mathbb{R}$ the following conditions hold:*

(i) *There exist α,β lower and upper solutions, respectively, such that $\alpha\le\beta$ on $\overline{I}$.*

(ii) *There exists $h\in\mathcal{L}^1_g(I,[0,+\infty))$ such that*

$$|f(t,x,\gamma)|\le h(t),\quad g\text{–a.a. } t\in I,\quad x\in\mathbb{R},\quad \gamma\in[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}.$$

(iii) *For each $\gamma\in[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ the mapping*

$$(t,x)\in I\times\mathbb{R}\to f_\gamma(t,x):=f(t,x,\gamma)$$

is a g–Carathéodory function and satisfies (4.64).

(iv) *For g–a.a. $t \in I$ and all $x \in \mathbb{R}$, the map $f(t,x,\cdot)$ is nondecreasing in $[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$.*

Furthermore, assume that $B : \mathbb{R} \times \mathcal{AC}_g(\overline{I},\mathbb{R}) \to \mathbb{R}$ satisfies the following property:

(v) *For each $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ and $x \in \mathbb{R}$, $B(x,\cdot)$ is nonincreasing in $[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ and*

$$\liminf_{y\to x^-} B(y,\gamma) \geq B(x,\gamma) \geq \limsup_{y\to x^+} B(y,\gamma). \tag{4.89}$$

Then the problem (4.87) *has the extremal solutions on I between α and β.*

Proof. Consider the mapping $G : [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})} \to [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ defined as follows: for each $\gamma \in [\alpha,\beta]$, we define $G\gamma$ as the least solution between α and β of the initial value problem

$$x_g'(t) = f(t,x(t),\gamma), \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_\gamma, \tag{4.90}$$

where x_γ is the least solution in $[\alpha(t_0),\beta(t_0)]$ of the algebraic equation $B(x,\gamma) = 0$.

Claim 1 – G is well-defined. Let $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$. Hypothesis (v) ensures that $B(x,\cdot)$ is nonincreasing for every $x \in [\alpha(t_0),\beta(t_0)]$. Hence, for every $\alpha \leq \gamma \leq \beta$,

$$B\left(\beta(t_0),\gamma\right) \geq B(\beta(t_0),\beta) \geq 0 \geq B(\alpha(t_0),\alpha) \geq B\left(\alpha(t_0),\gamma\right).$$

Moreover, (4.108) ensures that the hypotheses of Lemma 4.93 for $h = B(\cdot,\gamma)$ are satisfied, from which we conclude that x_γ is well-defined. Now, the existence of the least solution of (4.109) between α and β is guaranteed by Theorem 4.61 as hypotheses (i), (iii) and (iv) are enough for the hypotheses of Theorem 4.61 to be satisfied. Thus, G is well–defined.

Claim 2 – G satisfies the conditions of Proposition 3.33. First, note that hypothesis (ii) guarantees that for every $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$,

$$|(G\gamma)_g'(t)| = |f(t,G\gamma(t),\gamma)| \leq h(t), \quad g\text{–a.a. } t \in I.$$

Therefore, all that is left to prove is that G is nondecreasing in $[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$.

Let γ_1, $\gamma_2 \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ be such that $\gamma_1 \leq \gamma_2$. By definition, x_{γ_i} is the least solution in $[\alpha(t_0),\beta(t_0)]$ of the equation $B(x,\gamma_i) = 0$, $i = 1,2$. Hence, if $x_{\gamma_1} = \alpha(t_0)$ then $x_{\gamma_1} \leq x_{\gamma_2}$. Otherwise, we have that $x_{\gamma_1} > \alpha(t_0)$. In this case, for every $x \in [\alpha(t_0),x_{\gamma_1})$ we have

$$B\left(x,\gamma_2\right) \leq B\left(x,\gamma_1\right) < 0,$$

where the last inequality follows from the fact that x_γ is the least of the zeros of the algebraic equations $B\left(x,\gamma_1\right) = 0$. Thus, we obtain that $x_{\gamma_1} \leq x_{\gamma_2}$ as well.

Then we can say that $G\gamma_2$ is an upper solution of the initial value problem

$$x_g'(t) = f(t,x(t),\gamma_1), \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_{\gamma_1}, \tag{4.91}$$

and $G\gamma_2 \geq \alpha$. Hence, applying Theorem 4.61 once again, we have that (4.91) has the least solution between α and $G\gamma_2$. Since the least solution of that problem between α and β is $G\gamma_1$, we have that $G\gamma_1 \leq G\gamma_2$.

Conclusion. By Proposition 3.33, G has the least fixed point, $\gamma_* \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$, and satisfies

$$\gamma_* = \min\left\{\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})} \ : \ G\gamma \leq \gamma\right\}. \tag{4.92}$$

Note that if $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ is a fixed point of G then γ is a solution of (4.87) on I between α and β. Now, if γ is a solution of (4.87) on I between α and β then $B(\gamma(t_0),\gamma) = 0$ and $\gamma(t_0) \in [\alpha(t_0),\beta(t_0)]$, which implies that $\gamma(t_0) \geq x_\gamma$ by the definition of x_γ. Therefore γ is an upper solution of (4.109) and hence $G\gamma \leq \gamma$. Thus, by (4.92), we conclude that $\gamma_* \leq \gamma$, i.e., γ_* is the least solution of (4.87) in $[\alpha,\beta]$.

To deduce the existence of the greatest solution of (4.87) in $[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ it suffices to redefine G in the obvious way. □

Theorem 4.98 yields a novel existence result for Stieltjes differential equations with functional arguments, whose applications are shown in the following example based on the reproduction cycle of a silkworm population, which is not properly represented by the usual population models. This example was first introduced in [48].

Example 4.95. Usual population models, such as the malthusian or logistic models, implicitly assume that the reproduction of a given species happens so frequently in the time interval considered that it has a continuous influence on the evolution of the population size. However, many species exhibit very short periods of reproduction –for example due to rutting seasons or eggs hatching in a short lapse of time– which might be more reasonably and easily modeled in terms of impulses. This kind of populations can only decrease between two consecutive moments of reproduction. On the other hand, some species go through dormant states for some periods during which the population size is unlikely to change in a noticeable manner.

Silkworm populations exhibit impulsive reproduction and two dormant states. In the first one, worms transform into moths inside chrysalides, and a second dormant state begins when the whole moth population dies and we have to wait until the next eggs eclosion, when we have a completely new silkworm population. Here, we introduce and explicitly solve a particular case of initial value problem with functional arguments whose solution $x(t)$ will give us the number of individuals in a silkworm colony at time $t > 0$. We assume that at the initial time $t = 0$ we have a number x_0 of newborns after a first eggs hatching.

The life cycle of silkworms has three well–known stages: worm, cocoon and moth. Moths lay eggs and die soon after, then eggs hatch and produce a completely new colony of silkworms. For the sake of a simpler mathematical notation, we assume that the worm stage has a duration of 30 days while the remaining stages are each 15 days long. Accordingly, the units for our t variable will be periods of 15 days, i.e. the time interval $[0,2]$ corresponds to worm stage, the interval $(2,3]$ corresponds to cocoon stage, and so on. We assume that this is a periodic process with period $T = 5$, and we classify times as follow:

STAGE	TIME INTERVALS
Worms	$(5k, 5k+2]$, $k = 0,1,2,\dots$
Cocoons	$(5k+2, 5k+3]$, $k = 0,1,2,\dots$
Moths	$(5k+3, 5k+4]$, $k = 0,1,2,\dots$
Eggs	$(5k+4, 5k+5]$, $k = 0,1,2,\dots$

Our first task is to device a nondecreasing and left–continuous derivator, $g : \mathbb{R} \longrightarrow \mathbb{R}$, which reflects how important times are depending on the corresponding stage. We propose that such a function g should

(a) be constant on cocoon and egg stages, during which the population size can be assumed to remain constant;

(b) have jump discontinuities at two types of events which we assume to happen so fast that we can model them as impulses, namely, the instants $5k + 4$ when all moths die and the instants $5k + 5$ when new silkworms are born;

(c) have greater slopes at the beginning of worm stages and at the end of moth stages. This choice gives greater importance to those periods when individuals are weaker and therefore the population size is more volatile.

As an instance, we consider a derivator $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \dfrac{1}{2}\sqrt{4t - t^2}, & \text{if } 0 < t \leq 2, \\ 1, & \text{if } 2 < t \leq 3, \\ 2 - \sqrt{6t - t^2 - 8}, & \text{if } 3 < t \leq 4, \\ 3, & \text{if } 4 < t \leq 5, \end{cases}$$

and $g(t) = 4 + g(t - 5)$ for every $t > 5$. See Figure 1 for a plot of our derivator g. Note that $\Delta g(4) = g(4^+) - g(4) = 1$, and so $x'_g(4) = x(4^+) - x(4)$. The same thing occurs at $t = 5$.

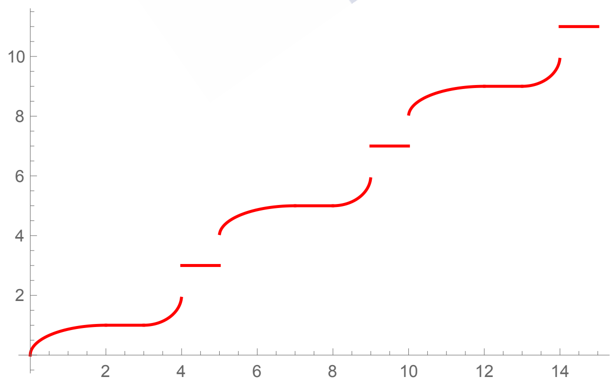


Figure 4.9: Graph of the silkworm derivator g.

In this case the set of negligible times is

$$C_g = \bigcup_{k=0}^{\infty} (5k + 2, 5k + 3) \cup (5k + 4, 5k + 5).$$

We are now ready to introduce a concise silkworm population model:

$$x'_g(t) = f(t, x(t), x), \quad t > 0, \quad x(0) = x_0 > 0, \tag{4.93}$$

where $f : [0, \infty) \setminus C_g \times \mathbb{R} \times L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathbb{R}$ is defined by

$$f(t, x, \varphi) = \begin{cases} -c\,x, & \text{if } t \in (5k, 5k+4),\ k = 0, 1, 2, ..., \\ -x, & \text{if } t = 5k+4,\ k = 0, 1, 2, ..., \\ \lambda \displaystyle\int_{t-5}^{t-1} \varphi(s)\,\mathrm{d}\,s, & \text{if } t = 5(k+1),\ k = 0, 1, 2, ..., \end{cases}$$

where $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ denotes the set of locally integrable functions with respect to the Lebesgue measure, m, and $c > 0$ and $\lambda > 0$ are parameters. In this case, the constant c represents the mortality rate and λ is a proportionality factor informing us how many times we have to multiply the average number of individuals we have in a season to know how many eggs will succesfully hatch in the next. It is important to note that the map f is not defined in C_g, nor is it necessary as it is a set with g–measure zero.

Observe that the differential equation in (4.93) is just $x'_g(t) = -cx(t)$ with the exception of the moments $t = 5k + 4$, where the moths die, and $t = 5k + 5$, when the eggs hatch. In the latter, this introduces a functional dependence in the system.

Problem (4.93) can be explicitly solved. Indeed, we have

$$x'_g(t) = -c\,x(t), \quad t \in [0, 4), \quad x(0) = x_0,$$

so, according to Proposition 4.13, the solution on the interval $[0, 4)$ is given by

$$x(t) = x_0 e_{-c}(t, 0) = x_0 \exp\left(\int_{[0,t)} -c\,\mathrm{d}\,g(s)\right) = x_0 e^{-cg(t)}, \quad t \in [0, 4),$$

and the same expression is valid for $t = 4$ because g–absolutely continuous functions are continuous from the left. At $t = 4$, (4.93) reduces to $x'_g(4) = -x(4)$, and therefore the definition of g–derivative yields

$$x(4^+) = 0, \tag{4.94}$$

i.e., the whole population disappears at $t = 4$. Since the solution is g–absolutely continuous on $[0, 5]$, and g is constant on $(4, 5]$, we know from Proposition 3.21 that x must be constant on $(4, 5]$, hence (4.94) implies that $x = 0$ on $(4, 5]$.

In order to carry on with the resolution of (4.93) for times $t > 5$ we have to find out the value $x(5^+)$ and then repeat the previous arguments on the interval $[5, 10)$ using $x(5^+)$ as a new initial condition, in a similar fashion to the method suggested in the method of separation of variables in Section 4.1.1. The differential equation (4.93) at $t = 5$ and the definition of g–derivative at discontinuity points give us

$$x(5^+) = x'_g(5) = f(5, x(5), x) = \lambda \int_0^4 x(s)\,\mathrm{d}\,s.$$

Therefore, following the suggested argument, we obtain that

$$x(t)=\begin{cases} x_0\, e^{-cg(t)}, & \text{if } 0\leqslant t\leqslant 4,\\ \left(\lambda\displaystyle\int_{5(k-1)}^{5k-1} x(s)\,\mathrm{d}\,s\right) e^{-c[g(t)-g(5k^+)]}, & \text{if } 5k<t\leqslant 5k+4,\ k=1,2,3,\dots,\\ 0, & \text{otherwise.}\end{cases}$$

See Figure 2 for a plot of the solution in the first three seasons for $x_0 = 8$, $c = 1$ and $\lambda = 1.1$ Note that this solution is always greater than 1 except where it vanishes. Indeed, for $t \in [0, 4]$ we have that

$$x(t) \geq x(4) = 8e^{-1} > 1.$$

Since $g(t) - g(5k^+)$ for $t \in (5k, 5k + 4]$, $k = 1, 2, 3, \ldots$, behaves identically as $g(t)$ for $t \in (0, 4]$ and noting that

$$1.1\cdot\int_{5(k-1)}^{5k-1} x(s)\,\mathrm{d}\,s > \int_{5(k-1)}^{5k-1} \mathrm{d}\,s > 1,$$

it follows that $x(t) \geq 1$ for all $t \in [0, \infty) \setminus C_g$.

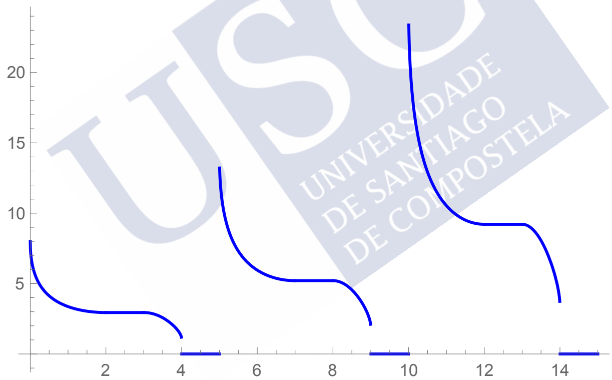


Figure 4.10: Solution of (4.93) for $x_0 = 8$, $c = 1$ and $\lambda = 1.1$

Observe that we were able to obtain an explicit expression for a solution of the g–differential equation (4.93), due to the simplicity of the model. In what follows, we present a more complicated and detailed model reflecting some new information about the population. For this new model, we will use Theorem 4.94 to ensure the existence of solution.

Let us consider the following functional and discontinuous model for the evolution of a silkworm population:

$$x_g'(t) = f(t, x(t), x), \quad t \in [0, 15), \quad x(0) = x(13), \tag{4.95}$$

with $x_0 > 0$ and $f : [0, 15) \setminus C_g \times \mathbb{R} \times \mathcal{AC}_g([0, 15], \mathbb{R}) \longrightarrow \mathbb{R}$ is defined by

$$f(t,x,\varphi)=\begin{cases} -c(\varphi(t))\,x^2, & \text{if } t\in(5k,5k+4),\ k=0,1,2,\\ -x, & \text{if } t=5k+4,\ k=0,1,2,\\ \lambda\left[\displaystyle\int_{t-5}^{t-1}\varphi(s)\,\mathrm{d}\,s\right], & \text{if } t=5(k+1),\ k=0,1,\end{cases}$$

where $c : [0,\infty) \longrightarrow [0,\infty)$ is monotone nonincreasing and $[\cdot]$ denotes the integer part. Observe that the g–differential equation in (4.95) is, for $t \in [0,4) \cup (5,9) \cup (10,14)$,

$$x_g'(t) = -c(x(t))x^2(t), \tag{4.96}$$

which means that population decays at a rate proportional to the usual competitive term $x^2(t)$. This introduces in our model intraspecific competition for resources. Moreover, we allow the proportionality "constant" to be a nondecreasing (and not necessarily continuous) function $-c(x(t))$.

We assume the number of eggs hatching at $t = 5$ and $t = 10$ to be proportional to the integer part of the mean value of the population in the preceding season. This is reasonable if, for instance, our silkworm population corresponds to a farm where it is kept under control from one to another hatching without exact computation of the number of individuals.

Finally, our generalized initial condition can be regarded as a control condition, as we are imposing that the number of individuals of the population at the initial time must be equal to the number of moths laying eggs in the third generation. In this case, we can rewrite the initial functional condition as $B(x(0), x) = 0$ for

$$B(s,\varphi) = s - \varphi(13) \quad \text{for all } (s,\varphi) \in \mathbb{R} \times \mathcal{AC}_g([0,15],\mathbb{R}).$$

Obviously, $\alpha(t) = 0$ for $t \in [0,15]$ is a lower solution of (4.95). In turn, the function $\beta(t)$ defined as the solution of (4.93) for $c = 1$, $x_0 = 8$ and $\lambda = 1.1$, and whose graph features in Figure 4.10, is an upper solution of (4.95) as long as we assume that

$$c(z) \geq 1 \quad \text{for all } z \in [0,\infty). \tag{4.97}$$

It is easy to check that the remaining conditions in Theorem 4.94 are satisfied and therefore we can conclude that the discontinuous and functional problem (4.95) has the extremal solutions between α and β. In particular, such solutions are nonnegative. Alternatively, it is possible to show the existence of solution in a similar fashion to (4.93), by noting that (4.96) is an example of separation of variables.

As anticipated before, we can obtain a new existence result for (4.62) from Theorem 4.94. In this case, the function f in (4.62) will be allowed to be discontinuous and it is not necessarily monotone. To obtain this result we follow the ideas of Biles and Binding in [9]. In particular, we first prove a similar result to [9, Lemma 1] in the context of g–differential equations.

Proposition 4.96. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) *$f(\cdot, x(\cdot))$ is g–measurable for every $x \in \mathcal{AC}_g(\overline{I},\mathbb{R})$.*

(ii) *For g–a.a. $t \in I$ and all $x \in \mathbb{R}$,*

$$\limsup_{y\to x^-} f(t,y) \leq f(t,x) \leq \liminf_{y\to x^+} f(t,y). \tag{4.98}$$

(iii) *There exists $h \in \mathcal{L}^1_g(I,[0,+\infty))$ such that*

$$|f(t,x)| \leq h(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}.$$

Consider the map $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$F(t, x, y) = \begin{cases} \sup\{f(t, z) : x \le z \le y\} & \text{if } x \le y, \\ \inf\{f(t, z) : y \le z \le x\} & \text{if } y \le x. \end{cases} \tag{4.99}$$

Then the following holds:

1. $F(t, x, x) = f(t, x)$ *for all* $t \in I$ *and all* $x \in \mathbb{R}$.
2. *For g–a.a.* $t \in I$ *and all* $x \in \mathbb{R}$*, the map* $F(t, x, \cdot)$ *is nondecreasing.*
3. *For each* $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$*, the map* $k_x(t, s) := F(t, s, x(t))$ *satisfies:*
   - (a) *for g–a.a.* $t \in I$*, the map* $k_x(t, \cdot)$ *is continuous;*
   - (b) *for all* $s \in \mathbb{R}$*, the map* $k_x(\cdot, s)$ *is g–measurable;*
   - (c) *there exists* $H \in \mathcal{L}^1_g(I, [0, +\infty))$ *such that*

$$|k_x(t, s)| \le H(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}.$$

Proof. By its definition, it is clear that F fulfills 1, 2 and 3 (c) for $H(t) = h(t)$. Let us see that 3 (a) holds. First note that for g–a.a. $t \in I$ and all $y \in \mathbb{R}$, the map $F(t, \cdot, y)$ is nonincreasing. As a consequence, it is enough to show that

$$\limsup_{z \to x^-} F(t, z, y) \le F(t, x, y) \le \liminf_{z \to x^+} F(t, z, y), \quad x \in \mathbb{R}. \tag{4.100}$$

Let $x, y \in \mathbb{R}$ and $t \in I$ be such that (4.98) holds. If $x \le y$, it follows from condition (ii) that

$$\begin{aligned} \limsup_{z \to x^-} F(t, z, y) &= \limsup_{z \to x^-} \left(\sup\{f(t, s) : z \le s \le y\} \right) \\ &= \limsup_{z \to x^-} \left(\max\{\sup\{f(t, s) : z \le s < x\}, \sup\{f(t, s) : x \le s \le y\}\} \right) \\ &\le \max\{f(t, x), \sup\{f(t, s) : x \le s \le y\}\} = F(t, x, y). \end{aligned}$$

Similarly, if $x > y$, then for all $z \in (y, x)$ we have $F(t, z, y) = \inf\{f(t, s) : y \le s \le z\}$. Hence, using a similar argument, we have that

$$\begin{aligned} \limsup_{z \to x^-} F(t, z, y) &= \limsup_{z \to x^-} \left(\inf\{f(t, s) : y \le s \le z\} \right) \\ &\le \inf\{f(t, s) : y \le s \le x\}\} = F(t, x, y). \end{aligned}$$

For the second inequality in (4.100) it is enough to define

$$\widetilde{f}(t, x) = -f(t, -x), \quad (t, x) \in I \times \mathbb{R}, \tag{4.101}$$

and let $\widetilde{F}$ be the corresponding function in (4.99) for $\widetilde{f}$. With this notation, the first inequality in (4.100) for $\widetilde{F}$ is equivalent to the second inequality in (4.100) for F as

$$F(t, x, y) = -\widetilde{F}(t, -x, -y), \quad (t, x, y) \in I \times \mathbb{R} \times \mathbb{R}.$$

Finally, in order to prove 3 (b), fix $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $s \in \mathbb{R}$. We shall show that the sets

$$\Gamma_{x,s,a} = \{t \in I : k_x(t,s) \le a\}, \quad a \in \mathbb{R},$$

are g–measurable. For a given $a \in \mathbb{R}$, we can rewrite $\Gamma_{x,s,a}$ as

$$\begin{aligned}\Gamma_{x,s,a} &= (\Gamma_{x,s,a} \cap \{t \in I : x(t) \le s\}) \cup (\Gamma_{x,s,a} \cap \{t \in I : x(t) \ge s\}) \\ &= \{t \in I : \iota_{x,s}(t) \le a\} \cup \{t \in I : \sigma_{x,s}(t) \le a\},\end{aligned}$$

where $\iota_{x,s}, \sigma_{x,s} : I \to \mathbb{R}$ are defined as

$$\iota_{x,s}(t) = \inf\{f(t,z) : x(t) \le z \le s\}, \quad \sigma_{x,s}(t) = \sup\{f(t,z) : s \le z \le x(t)\}. \tag{4.102}$$

Hence, it is enough to show that $\iota_{x,s}$ and $\sigma_{x,s}$ are g–measurable to prove that $\Gamma_{x,s,a}$ is a g–measurable set for each $a \in \mathbb{R}$. Moreover, if we denote by $\widetilde{\iota}_{y,r}$ the corresponding function in (4.102) for $\widetilde{f}$ as in (4.101), $y \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $r \in \mathbb{R}$, we have that

$$\sigma_{x,s}(t) = -\widetilde{\iota}_{-x,-s}(t), \quad t \in I,$$

so it is enough to show that $\iota_{x,s}$ is g–measurable, as $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and $s \in \mathbb{R}$ are arbitrarily chosen.

To show that $\iota_{x,s}$ is g–measurable, fix $\varepsilon > 0$ and $t \in I$ so that condition (ii) holds. By definition of $\iota_{x,s}$, there exists $v \in [x(t), s]$ such that $f(t,v) < \iota_{x,s}(t) + \varepsilon$. We claim that there exists $\rho \in \mathbb{Q}$, $\rho \ge 0$, such that if we define $r_{x,s,\rho}(t) = \min\{x(t) + \rho, s\}$, $t \in I$, then

$$f(t, r_{x,s,\rho}(t)) < f(t,v) + \varepsilon. \tag{4.103}$$

Indeed, if $v = x(t)$ the inequality holds trivially for $\rho = 0$; so let us assume that $v \in (x(t), s]$. In that case, by condition (ii),

$$f(t,v) \ge \limsup_{y \to v^-} f(t,y) = \lim_{y \to v^-} \left(\sup_{y \le z < v} f(t,z) \right),$$

and so there exists y_0 such that $\sup_{y_0 < z < v} f(t,z) < f(t,v) + \varepsilon$. Hence it is enough to choose $\rho \ge 0$ such that $x(t) + \rho \in (y_0, v)$ to obtain (4.103). Therefore

$$f(t, r_{x,s,\rho}(t)) < f(t,v) + \varepsilon < \iota(t) + 2\varepsilon,$$

and so, $\iota_{x,s}(t) = \inf\{f(t, r_{x,s,\rho}(t)) : \rho \in \mathbb{Q}, \rho \ge 0\}$. Note that $r_{x,s,\rho} \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ by Proposition 3.28. Hence, by (i), ι is the infimum of countable many g–measurable functions and therefore $\iota_{x,s}$ is g–measurable, which finishes the proof. □

Remark 4.97. Hypotheses (i)–(iii) are an adaptation to the context of g–derivatives of those in [9, Lemma 1] . Note that (ii) does not imply that f is continuous nor monotonic.

Let us introduce the following result ensuring the existence of extremal solutions of problem (4.62) between two fixed lower and upper solutions, obtained from Theorem 4.94 and Proposition 4.96.

Theorem 4.98. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ satisfy* (4.64) *and conditions* (i)–(iii) *in* Proposition 4.96. *Assume that problem* (4.62) *has a lower solution on I, α, and an upper solution on I, β, such that $\alpha \leq \beta$ on $\overline{I}$. Then, the problem* (4.62) *has the extremal solutions on I between α and β.*

Proof. In order to prove this result, define $\widehat{F} : I \times \mathbb{R} \times \mathcal{AC}_g(\overline{I}, \mathbb{R}) \to \mathbb{R}$ as

$$\widehat{F}(t, x, \gamma) = F(t, x, \gamma(t)),$$

with F defined as in Proposition 4.96; and consider the problem

$$x'_g(t) = \widehat{F}(t, x(t), x), \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_0, \tag{4.104}$$

First, note that (4.104) is equivalent to (4.62) since

$$\widehat{F}(t, x(t), x) = F(t, x(t), x(t)) = f(t, x(t)), \quad t \in I, \quad x \in \mathcal{AC}_g(\overline{I}, \mathbb{R}). \tag{4.105}$$

Moreover, (4.104) is a problem in the setting of Theorem 4.94 with $B(s, \gamma) = s - x_0$, $(s, \gamma) \in \mathbb{R} \times \mathcal{AC}_g(\overline{I}, \mathbb{R})$. Hence, it suffices to show that $\widehat{F}$ satisfies hypotheses (i)–(v) in Theorem 4.94.

First of all, given our assumptions and the fact that (4.105) holds, it follows that condition (i) in Theorem 4.94 is satified. Moreover, it is clear that condition (iii) in Proposition 4.96 guarantees that condition (ii) in Theorem 4.94 is satisfied. Condition (v) in Theorem 4.94 is satisfied trivially for $B(s, \gamma) = s - x_0$; and condition (iv) in Theorem 4.94 follows from statement 2 in Proposition 4.96. It only remains to check condition (iii) in Theorem 4.94, to which the rest of this proof is devoted. By the third statement in Proposition 4.96 we know that for each $\gamma \in [\alpha, \beta]_{\mathcal{AC}_g(\overline{I}, \mathbb{R})}$, $\widehat{F}_\gamma(\cdot, \cdot) = \widehat{F}(\cdot, \cdot, \gamma)$ is g–Carathéodory. Therefore, it is enough to show that $\widehat{F}_\gamma$ satisfies (4.64). Recall that it suffices to show that such property holds for $t \in I \cap D_g$.

Let $\gamma \in [\alpha, \beta]_{\mathcal{AC}_g(\overline{I}, \mathbb{R})}$, $t \in I \cap D_g$ and $a, b \in \mathbb{R}$ be such that $a < b$. If $a < b < \gamma(t)$, we need to show that

$$a + \Delta g(t) \sup_{a \leq \delta \leq \gamma(t)} f(t, \delta) \leq b + \Delta g(t) \sup_{b \leq \delta \leq \gamma(t)} f(t, \delta). \tag{4.106}$$

On the one hand, (4.64) yields that for every $c \in [a, b]$,

$$b + \Delta g(t) \sup_{b \leq \delta \leq \gamma(t)} f(t, \delta) \geq b + \Delta g(t) f(t, b) \geq c + \Delta g(t) f(t, c) \geq a + \Delta g(t) f(t, c),$$

and therefore,

$$b + \Delta g(t) \sup_{b \leq \delta \leq \gamma(t)} f(t, \delta) \geq a + \Delta g(t) \sup_{a \leq \delta \leq b} f(t, \delta).$$

On the other hand, it is obvious that

$$b + \Delta g(t) \sup_{b \leq \delta \leq \gamma(t)} f(t, \delta) > a + \Delta g(t) \sup_{b \leq \delta \leq \gamma(t)} f(t, \delta).$$

Thus, we have that

$$a + \Delta g(t) \sup_{a \le \delta \le \gamma(t)} f(t,\delta) = a + \Delta g(t) \max \left\{ \sup_{a \le \delta \le b} f(t,\delta), \sup_{b \le \delta \le \gamma(t)} f(t,\delta) \right\}$$
$$\le b + \sup_{b \le \delta \le \gamma(t)} f(t,\delta),$$

which proves (4.106). The case $\gamma(t) < a < b$ is analogous and we omit. Hence, we just need to show that if $a \le \gamma(t) \le b$, then

$$a + \Delta g(t) \sup_{a \le \delta \le \gamma(t)} f(t,\delta) \le b + \Delta g(t) \inf_{\gamma(t) \le \delta \le b} f(t,\delta). \tag{4.107}$$

By the definition of supremum, for each $\varepsilon > 0$ there is $c \in [a, \gamma(t)]$ such that

$$a + \Delta g(t) \sup_{a \le \delta \le \gamma(t)} f(t,\delta) \le a + \Delta g(t) f(t,c) + \varepsilon \le c + \Delta g(t) f(t,c) + \varepsilon.$$

Hence, for any $\delta \in [\gamma(t), b]$, condition (4.64) yields

$$a + \Delta g(t) \sup_{a \le \delta \le \gamma(t)} f(t,\delta) \le \delta + \Delta g(t) f(t,\delta) + \varepsilon \le b + \Delta g(t) f(t,\delta) + \varepsilon.$$

Hence, for all $\varepsilon > 0$ we have

$$a + \Delta g(t) \sup_{a \le \delta \le \gamma(t)} f(t,\delta) \le b + \Delta g(t) \inf_{\gamma(t) \le \delta \le b} f(t,\delta) + \varepsilon,$$

which implies (4.107). □

Remark 4.99. For the particular case where

$$\alpha(t) = x_0 - \int_{[t_0,t)} h(s)\,\mathrm{d}\,g(s), \quad \beta(t) = x_0 + \int_{[t_0,t)} h(s)\,\mathrm{d}\,g(s), \quad t \in I,$$

we obtain [48, Theorem 4.2]. Note that the proof of both results is almost identical.

Note that Theorems 4.61 and 4.98 are not comparable, in the sense that one of them does not imply the other one. With this idea in mind, and since Theorem 4.61 was fundamental in the proof of Theorem 4.94, we reformulate this result applying Theorem 4.98 instead.

Theorem 4.100. *Suppose that for $f : I \times \mathbb{R} \times \mathcal{AC}_g(\overline{I}, \mathbb{R}) \to \mathbb{R}$ the following conditions hold:*

(i) *There exist α, β lower and upper solutions, respectively, such that $\alpha \le \beta$ on $\overline{I}$.*

(ii) *There exists $h \in \mathcal{L}^1_g(I, [0, +\infty))$ such that*

$$|f(t,x,\gamma)| \le h(t), \quad g\text{–a.a. } t \in I, \quad x \in \mathbb{R}, \quad \gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}.$$

(iii) *For each* $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ *the mapping* $f_\gamma(t,x) := f(t,x,\gamma)$, $(t,x) \in I\times\mathbb{R}$, *satisfies* (4.64)*,* (4.98) *and* $f_\gamma(\cdot,x(\cdot))$ *is g–measurable for every* $x \in \mathcal{AC}_g(\overline{I},\mathbb{R})$*.*

(iv) *For g–a.a.* $t \in I$ *and all* $x \in \mathbb{R}$ *the map* $f(t,x,\cdot)$ *is nondecreasing in* $[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ *.*

Furthermore, assume that $B : \mathbb{R}\times\mathcal{AC}_g(\overline{I},\mathbb{R}) \to \mathbb{R}$ *satisfies the following property:*

(v) *For all* $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ *and all* $x \in \mathbb{R}$, $B(x,\cdot)$ *is nonincreasing in* $[\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ *and*

$$\liminf_{y\to x^-} B(y,\gamma) \geq B(x,\gamma) \geq \limsup_{y\to x^+} B(y,\gamma). \tag{4.108}$$

Then the problem (4.87) *has the extremal solutions on* I *between* α *and* β*.*

Proof. Consider the mapping $G : [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})} \to [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$ defined as in the proof of Theorem 4.94, i.e for each $\gamma \in [\alpha,\beta]$, define $G\gamma$ as the least solution between α and β of the initial value problem

$$x_g'(t) = f(t,x(t),\gamma), \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_\gamma, \tag{4.109}$$

where x_γ is the least solution in $[\alpha(t_0),\beta(t_0)]$ of the algebraic equation $B(x,\gamma) = 0$. Let us show that G is well–defined under the hypotheses of Theorem 4.100.

Let $\gamma \in [\alpha,\beta]_{\mathcal{AC}_g(\overline{I},\mathbb{R})}$. Hypothesis (v) ensures that, for every $x \in [\alpha(t_0),\beta(t_0)]$, $B(x,\cdot)$ is nonincreasing . Hence, for every $\alpha \leq \gamma \leq \beta$ we have

$$B\left(\beta(t_0),\gamma\right) \geq B(\beta(t_0),\beta) \geq 0 \geq B(\alpha(t_0),\alpha) \geq B\left(\alpha(t_0),\gamma\right).$$

Moreover, (4.108) ensures that the hypotheses of Lemma 4.93 for $h = B(\cdot,\gamma)$ are satisfied, from which we conclude that x_γ is well-defined. Now, the existence of the least solution of (4.109) between α and β is guaranteed by Theorem 4.98 as hypotheses (i)–(iii) ensure that the hypotheses of such result are satisfied. Thus, G is well–defined.

The arguments in the proof of Theorem 4.94 remain true, which concludes the proof. □

Note that, given the relations between g–differential equations and other differential problems, the main results in [30, 55] for discontinuous equations with impulses or in [13] for difference equations can be seen as particular cases of Theorems 4.94 and 4.100.

4.3 Stieltjes differential inclusions

In this final section of the chapter we look at differential inclusion problems with Stieltjes derivatives, also known as Stieltjes differential inclusions following [53]. Specifically, we consider a nondecreasing and left–continuous map, $g : \mathbb{R} \to \mathbb{R}$, and we study the problem

$$x_g'(t) \in F(t,x(t)), \quad x(t_0) = x_0, \tag{4.110}$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $x_0 \in \mathbb{R}^n$ and $F : [t_0,t_0+T)\times\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all subsets of $\mathbb{R}^n$. Every multivalued mapping considered here shall be assumed to be strict, i.e. to assume nonempty values. Once again, we denote $I = [t_0, t_0 + T)$ and $\overline{I} = [t_0, t_0 + T]$.

In this setting, we will provide some results on the existence of solution for the inclusion problem (4.110), and from there, obtain some existence results for problems of the form (4.2). Hence, we start with the following fundamental definition.

Definition 4.101. *A solution of* (4.110) *on I is a map $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R}^n)$ such that $x(t_0) = x_0$ and*

$$x'_g(t) \in F(t, x(t)), \quad g\text{–a.a. } t \in I.$$

Remark 4.102. Once again, in a similar fashion to (4.2) and (4.87), given that we are talking about solutions on I instead of $\overline{I}$, it would be more adequate to ask for functions defined stricly on I and not $\overline{I}$. As usual, the way to work around this is to consider functions in $E\mathcal{AC}_g(\overline{I}, \mathbb{R}^n)$ instead of $\mathcal{AC}_g(\overline{I}, \mathbb{R}^n)$. However, we will still consider Definition 4.101 for convenience. Recall that both definitions are equivalent, as shown in Proposition 4.4.

We shall prove an existence result for problem (4.110) by generalizing the ideas of [22], who studied the particular case $g = \mathrm{Id}$ and assumed upper semicontinuity of $F(t, \cdot)$ on $\mathbb{R}^n$ for a.a. $t \in \overline{I}$. Here, besides considering derivatives in a wider sense, we shall show that upper semicontinuity may fail at many points provided that they belong to the graph of another multivalued mapping satisfying some technical conditions with respect to F.

For the proper study of (4.110), we will need to introduce some tools for measure spaces. Given measure space, $(X, \mathcal{M}, \mu)$, we consider the space of integrable functions on X together with the map $\|\cdot\|_1 : \mathcal{L}^1_\mu(X, \mathbb{R}^n) \to [0, +\infty)$ given by

$$\|f\|_1 = \int_X \|f\| \,\mathrm{d}\,\mu.$$

The pair $(\mathcal{L}^1_\mu(X, \mathbb{R}^n), \|\cdot\|_1)$ fails to be a Banach space, however, by considering the set $L^1_\mu(X, \mathbb{R}^n)$, of classes of integrable functions where the ones that agree on μ–almost everywhere in X are identified, we obtain that $(L^1_\mu(X, \mathbb{R}^n), \|\cdot\|_1)$ is a Banach space. For more information on this, we refer to [73, Chapter 3]. Therefore, we can obtain some interesting results for measurable spaces. For example, combining [25, Theorem IV.8.9] and [25, Chapter V, Section 3, Corollary 14] applied to this context yields the following result that will be used later on the study of differential inclusions.

Theorem 4.103. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1_\mu(X, \mathbb{R}^n)$. Assume the following conditions are satisfied:*

1. *$\{x_k\}_{k\in\mathbb{N}}$ is bounded;*
2. *for each family of measurable sets, $\{E_m\}_{m\in\mathbb{N}}$, such that $E_{m+1} \subset E_m$, $m \in \mathbb{N}$, and $\bigcap_{m\in\mathbb{N}} E_m = \emptyset$, we have that*

$$\lim_{m\to\infty} \int_{E_m} x_k \,\mathrm{d}\,\mu = 0 \quad \textit{uniformly for } k \in \mathbb{N}.$$

Then, there exists a subsequence, $\{x_{k_l}\}_{l\in\mathbb{N}}$*, and* $y \in \mathcal{L}^1_\mu(X,\mathbb{R}^n)$ *such that*

$$\lim_{l\to+\infty}\int_X x_{k_l}\,\mathrm{d}\,\mu = \int_X y\,\mathrm{d}\,\mu.$$

Moreover, there exists a convex combination of $\{x_{k_l}\}_{l\in\mathbb{N}}$ *converging to* y *on* $(L^1_\mu(X,\mathbb{R}^n), \|\cdot\|_1)$.

Theorem 4.103 allows us to obtain a result providing some information about pointwise limit of sequences of g–absolutely continuous functions and their g–derivatives. In order to properly formulate such result, we need to introduce the concept of convex hull of a given set.

Definition 4.104. *Let* $X \subset \mathbb{R}^n$*. We define the* convex hull of X*, denote* $\mathrm{co}(X)$*, as the smallest convex set containing* X*. We define the* closed convex hull of X*, and we denote it by* $\overline{\mathrm{co}}(X)$*, as the closure of* $\mathrm{co}(X)$.

We can now obtain a result that generalizes [22, Theorem 4.1], which provides information on the limit of a sequence of absolutely continuous functions. Here, we extend that information to the context of g–absolutely continuity, showing that the limit function is g–absolutely continuous and its g–derivative on a set involving the Stieltjes derivatives of the functions in the sequence.

Lemma 4.105. *Let* $x_k : \overline{I} \to \mathbb{R}^n$*,* $k \in \mathbb{N}$*, be a sequence of functions in* $\mathcal{AC}_g(\overline{I},\mathbb{R}^n)$*. Assume there exists* $L \in \mathcal{L}^1_g(I,[0,+\infty))$ *such that for every* $k \in \mathbb{N}$ *we have*

$$\|(x_k)'_g(t)\| \le L(t), \quad g\text{–a.a. } t \in I. \tag{4.111}$$

If $x_k(t)$ *converges to* $x(t)$ *for all* $t \in \overline{I}$ *for some* $x : \overline{I} \to \mathbb{R}^n$*, then* $x \in \mathcal{AC}_g(\overline{I},\mathbb{R}^n)$ *and*

$$x'_g(t) \in \bigcap_{i=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k=i}^{\infty} \{(x_k)'_g(t)\} \quad \text{for } g\text{–a.a. } t \in I.$$

Proof. Consider the set $\mathcal{S} = \{(x_k)'_g : k \in \mathbb{N}\} \subset \mathcal{L}^1_g(I,\mathbb{R}^n)$, which is bounded thanks to (4.111). Moreover, $\mathcal{S}$ satisfies condition 2 in Theorem 4.103. Indeed, let us denote by $\mathcal{LS}_g(\overline{I})$ the Lebesgue–Stieltjes σ–algebra on $\overline{I}$ induced by g and define

$$\nu(E) = \int_E L(s)\,\mathrm{d}\,g(s), \quad E \in \mathcal{LS}_g(\overline{I}).$$

Then Proposition 1.19 ensures that ν is a measure, and so given any sequence of g–measurable sets $\{E_m\}_{m\in\mathbb{N}} \subset I$ in the hypotheses of condition 2 in Theorem 4.103, we have that

$$\lim_{m\to\infty}\int_{E_m} L(s)\,\mathrm{d}\,g(s) = \lim_{m\to\infty}\nu(E_m) = \lim_{m\to\infty}\nu\left(\bigcap_{k=1}^{m} E_k\right) = \nu(\emptyset) = 0.$$

Now, (4.111) implies that condition 2 in Theorem 4.103 holds. Hence, for every $t \in \overline{I}$,

$$\begin{aligned} x(t) &= \lim_{l\to\infty} x_{k_l}(t) \\ &= \lim_{l\to\infty}\left(x_{k_l}(t_0) + \int_{[t_0,t)} (x_{k_l})'_g(s)\,\mathrm{d}\,g(s)\right) = x(t_0) + \int_{[t_0,t)} y(s)\,\mathrm{d}\,g(s). \end{aligned}$$

Therefore, Theorem 3.26 shows that $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R}^n)$ and $x'_g(t) = y(t)$ for g–a.a. $t \in I$. Moreover, Theorem 4.103 also ensures the existence of a sequence of convex combinations of $(x_{k_l})'_g$, which we denote by $\{y_j\}_{j\in\mathbb{N}}$, converging in $L^1_g(I, \mathbb{R}^n)$ to y. Hence, there exists a subsequence $\{y_{j_q}\}_{q\in\mathbb{N}}$ such that $y_{j_q}(t) \to y(t)$ for g–a.a. $t \in I$. For every $q \in \mathbb{N}$ we have

$$y_{j_q}(t) \in \operatorname{co} \bigcup_{l=1}^{\infty} \{(x_{k_l})'_g(t)\} \subset \operatorname{co} \bigcup_{k=1}^{\infty} \{(x_k)'_g(t)\}, \quad g\text{–a.a. } t \in I,$$

so, by letting q tend to infinity, we get

$$x'_g(t) = y(t) \in \overline{\operatorname{co}} \bigcup_{k=1}^{\infty} \{(x_k)'_g(t)\} \quad g\text{–a.a. } t \in I.$$

Moreover, since for each fixed $i \in \mathbb{N}$ the sequence $\{x_k\}_{k=i}^{\infty}$ also converges to x, a repetition of the previous arguments shows that

$$x'_g(t) \in \overline{\operatorname{co}} \bigcup_{k=i}^{\infty} \{(x_k)'_g(t)\} \quad g\text{–a.a. } t \in I.$$

Hence,

$$x'_g(t) \in \bigcap_{i=1}^{\infty} \overline{\operatorname{co}} \bigcup_{k=i}^{\infty} \{(x_k)'_g(t)\} \quad g\text{–a.a. } t \in I,$$

which concludes the proof. □

To obtain our existence result for (4.110), we shall base our arguments on the concept of upper semicontinuity and on the following new concept of contingent derivative for multivalued mappings, introduced in [53] and based on the ideas of [6]. Let us start by presenting the definition of upper semicontinuity, [24, Definition 1.1].

Definition 4.106. *Let X, Y be Banach spaces, $\Omega \subset X$, $\Omega \neq \emptyset$ and $\mathcal{P}(Y)$ be the set of all possible subsets of Y. A map $F : \Omega \to \mathcal{P}(Y) \setminus \{\emptyset\}$ is said to be* upper semicontinuous*, and we shorten it as* u.s.c.*, if $F^{-1}(A)$ is closed in Ω for every closed set $A \subset Y$.*

Remark 4.107. In [24, Definition 1.2] we find a similar definition. The map F is said to be *ε–δ–upper semicontinuous*, and we shorten it as *ε–δ–u.s.c.*, if for every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(x) \subset F(x_0) + B_Y(0, \varepsilon), \quad x \in B_X(x_0, \delta) \cap \Omega,$$

where $B_X(z, \rho)$ and $B_Y(z, \rho)$ denote the open balls of center z and radius ρ in the the topologies of the normed spaces X and Y, respectively. It is shown in [24, Proposition 1.1] that every u.s.c. map is ε–δ–u.s.c., and that the converse holds provided that F has compact values in Y.

Next, we introduce the concept of contingent g–derivative of a multivalued mapping. The following definition is somehow based on the analytical description of the contingent derivative in the usual sense, see [6, Proposition 2, p. 177]. In particular, this definition coincides with that for the usual case through the mentioned result.

Definition 4.108. *Let $K : I_K \to \mathcal{P}(\mathbb{R}^n)$ with $I_K \subset \mathbb{R}$, $I_K \neq \emptyset$. The* contingent g–derivative of K at a point $(t,x) \in \operatorname{graph}(K)$ *is the set denoted by $\mathrm{D}_g\, K(t,x)$ defined as follows: we say that $v \in \mathrm{D}_g\, K(t,x)$ if there exist $\{h_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that*

1. *For every $k \in \mathbb{N}$, $h_k > 0$ and $t + h_k \in I_K$.*
2. *The sequence $\{h_k\}_{k\in\mathbb{N}}$ converges to 0 as $k \to \infty$.*
3. *For all $k \in \mathbb{N}$, $x_k \in K(t + h_k)$ and*

$$\lim_{k\to\infty} \frac{x_k - x}{g(t+h_k) - g(t)} = v.$$

Remark 4.109. In the conditions of Definition 4.108, if $t \notin D_g$, then $\{x_k\}_{k\in\mathbb{N}}$ necessarily converges to x. Indeed, in that case, $g(t+h_k)$ converges to $g(t)$ as g is continuous at that point. Hence, we have that

$$0 \leq \|x_k - x\| = \left\| \frac{x_k - x}{g(t+h_k) - g(t)} \right\| (g(t+h_k) - g(t)) \xrightarrow{k\to\infty} 0,$$

which implies that $\{x_k\}_{k\in\mathbb{N}}$ converges to x as $k \to \infty$.

For a better understanding of Definition 4.108, let us consider $K(t) = \{\gamma(t)\}$ for some $\gamma : I_\gamma \to \mathbb{R}^n$. Furthermore, assume that γ is g–differentiable from the right at a point $t_0 \in I_\gamma$. That is, the limit

$$\lim_{s\to t_0^+} \frac{\gamma(s) - \gamma(t_0)}{g(s) - g(t_0)}, \tag{4.112}$$

exists. In that case, and denoting by $\gamma_g'(t_0^+)$ the value of the limit (4.112), we have that $\mathrm{D}_g\, K(t_0, \gamma(t_0)) = \{\gamma_g'(t_0^+)\}$.

We now have all the necessary tools to prove the following existence result for (4.110), following the ideas of [17]. To the best of our knowledge, this result is new even in the particular case of $g = \mathrm{Id}$, i.e. when g–derivatives reduce to derivatives in the usual sense.

Theorem 4.110. *Let $F : I \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ satisfy the following conditions:*

(i) *For all $t \in I \cap D_g$, $F(t,\cdot)$ assumes convex and compact values in $\mathbb{R}^n$ and it is upper semicontinuous on $\mathbb{R}^n$.*

(ii) *For g–a.a. $t \in I \setminus D_g$, $F(t,x)$ is convex and compact in $\mathbb{R}^n$ for every $x \in \mathbb{R}^n \setminus K(t)$, and $F(t,\cdot)$ is upper semicontinuous on $\mathbb{R}^n \setminus K(t)$, where the set $K(t)$ is either empty, or there exist $K_p : I_p \subset I \setminus D_g \longrightarrow \mathcal{P}(\mathbb{R}^n)$, $p \in \mathbb{N}$, such that*

$$K(t) = \bigcup_{\{p\,:\,t\in I_p\}} K_p(t),$$

and if $x \in K_p(t)$ for some $p \in \mathbb{N}$, then

$$\bigcap_{r>0}^{\infty} \overline{\mathrm{co}} F(t, B(x,r)) \bigcap \mathrm{D}_g\, K_p(t,x) \subset F(t,x). \tag{4.113}$$

(iii) *For all $x \in \mathbb{R}^n$, there exists a g–measurable* selection *of $F(\cdot, x)$, that is, there exists a g–measurable function $f_x : I \to \mathbb{R}^n$ such that $f_x(t) \in F(t,x)$ for g–a.a. $t \in I$;*

(iv) *There exists $M \in \mathcal{L}^1_g(I, [0,+\infty))$ such that for g–a.a. $t \in I$ and all $x \in \mathbb{R}^n$,*

$$\|y\| \leq M(t) \quad \text{for any } y \in F(t,x).$$

Then (4.110) *has at least a solution.*

Proof. First of all, note that (iv) ensures that the g–measurable selections in (iii) are g–integrable on I. Indeed, let $x \in \mathbb{R}^n$ be fixed and f_x a selection in the conditions of (iii). Then $f_x(t) \in F(t,x)$ for g–a.a. $t \in I$, so (iv) implies that

$$\int_{[t_0,t_0+T)} \|f_x(t)\| \,\mathrm{d}\, g(t) \leq \int_{[t_0,t_0+T)} M(t)\,\mathrm{d}\, g(t) < +\infty.$$

Consider the sequence $x_k : [t_0, t_0+T] \to \mathbb{R}^n$, $k \in \mathbb{N}$, defined as follows: for each $k \in \mathbb{N}$, denote $t_{k,j} = t_0 + jT/k$, $j = 0,1,2,\dots,k$. Define $x_k(t_0) = x_0$,

$$x_k(t) = x_k(t_{k,j}) + \int_{[t_{k,j},t)} f_{k,j}(s)\,\mathrm{d}\, g(s), \quad t \in (t_{k,j}, t_{k,j+1}], \quad j = 0,1,2,\dots,k-1,$$

where $f_{k,j}$ is a g–measurable selection of $F(\cdot, x_k(t_{k,j}))$, $j = 0,1,2,\dots,k-1$, whose existence is guaranteed by (iii). Then, if we define $f_k : [t_0, t_0+T) \to \mathbb{R}^n$ as

$$f_k(t) = f_{k,j}(t), \quad t \in [t_{k,j}, t_{k,j+1}), \quad j = 0,1,2,\dots,k-1,$$

it follows that

$$x_k(t) = x_0 + \int_{[t_0,t)} f_k(s)\,\mathrm{d}\, g(s), \quad t \in \overline{I}.$$

Moreover, $f_k \in \mathcal{L}^1_g(I, \mathbb{R}^n)$ as pointed out at the beginning of the proof, and so $\{x_k\}_{k\in\mathbb{N}}$ is well–defined. For each $k \in \mathbb{N}$, Theorem 3.26 implies that $x_k \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and

$$(x_k)'_g(t) = f_k(t), \quad g\text{–a.a. } t \in I.$$

Hence, we have that for each $k \in \mathbb{N}$, $\|(x_k)'_g(t)\| \leq M(t)$ for g–a.a. $t \in I$. Moreover, $\{x_k(t_0) : k \in \mathbb{N}\} = \{x_0\}$, so Proposition 3.31 guarantees that $\{x_k\}_{k\in\mathbb{N}}$ is a relatively compact subset of $\mathcal{BC}_g(\overline{I}, \mathbb{R})$. Therefore, there exists a subsequence converging to a function in $\mathcal{BC}_g(\overline{I}, \mathbb{R})$, say x. Clearly, $x(t_0) = x_0$. Moreover, applying Lemma 4.105 to such subsequence, we deduce that $x \in \mathcal{AC}_g(\overline{I}, \mathbb{R})$ and there exists $E \subset I$ such that $\mu_g(E) = 0$ and

$$x'_g(t) \in \bigcap_{j=1}^{\infty} \overline{\text{co}} \bigcup_{k=j}^{\infty} \{(x_k)'_g(t)\} = \bigcap_{j=1}^{\infty} \overline{\text{co}} \bigcup_{k=j}^{\infty} \{f_k(t)\}, \quad t \in I \setminus E.$$

Let us prove that

$$x'_g(t) \in \bigcap_{r>0}^{\infty} \overline{\text{co}} F(t, B(x(t), r)), \quad t \in I \setminus E. \tag{4.114}$$

Fix $t \in I \setminus E$. For each $k \in \mathbb{N}$, take $i_k \in \{0, 1, 2, \dots, k-1\}$ such that $t \in [t_{k,i_k}, t_{k,i_k+1})$. Hence,

$$x'_g(t) \in \bigcap_{j=1}^{\infty} \overline{\text{co}} \bigcup_{k=j}^{\infty} \{f_k(t)\} \subset \bigcap_{j=1}^{\infty} \overline{\text{co}} \bigcup_{k=j}^{\infty} F(t, x_k(t_{k,i_k})). \tag{4.115}$$

Moreover, note that t_{k,i_k} converges to t from the left as $k \to \infty$, as

$$0 \leq t - t_{k,i_k} < t_{k,i_k+1} - t_{k,i_k} = \frac{T}{k}.$$

Therefore, since each x_k, $k \in \mathbb{N}$, is left–continuous –see Proposition 3.21– and $x_k(t)$ converges to $x(t)$, it follows that $x_k(t_{k,i_k})$ converges to $x(t)$. Hence, for every $r > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\|x_k(t_{k,i_k}) - x(t)\| < r, \quad \text{for all } k \geq k_0,$$

and therefore (4.115) yields

$$x'_g(t) \in \overline{\text{co}} \bigcup_{k=k_0}^{\infty} F(t, x_k(t_{k,i_k})) \subset \overline{\text{co}} F(t, B(x(t), r)).$$

This implies (4.114) because $r > 0$ was arbitrary.

Now we can prove that $x'_g(t) \in F(t, x(t))$ for g–a.a. $t \in I \setminus E$. We start by removing some inconvenient g–null measure sets from $I \setminus E$. For each $p \in \mathbb{N}$, define the set

$$A_p = \{t \in I \setminus (E \cup D_g) : x(t) \in K_p(t),\, x(s) \notin K_p(s),\, s \in (t, t + \varepsilon_t) \text{ for some } \varepsilon_t > 0\}.$$

For every $t \in A_p$ take the greatest ε_t possible in the conditions of the definition of A_p. The infinite sum

$$\sum_{t \in A_p} \varepsilon_t$$

is convergent as the sum of any finite subset of its terms is bounded by the length of the interval I. Therefore, only a countable number of ε_t can be positive. Hence, $\mu_g(A_p) = 0$ because A_p is countable and contains no discontinuity points of g.

We consider the g–null measure set $\widehat{E} = E \cup A_1 \cup A_2 \cup \cdots$ and, without loss of generality, we assume that condition (ii) is satisfied for all $t \in I \setminus (\widehat{E} \cup D_g)$. Fix $t \in I \setminus \widehat{E}$ and we consider two cases to prove that $x'_g(t) \in F(t, x(t))$.

Case 1: $t \in I \setminus (\widehat{E} \cup D_g)$ and $x(t) \in K_p(t)$ for some $p \in \mathbb{N}$.
Since $t \notin A_p$, we can find $h_k > 0$, $k \in \mathbb{N}$, such that $h_k \to 0$ as $k \to \infty$ and

$$x(t + h_k) \in K_p(t + h_k), \quad k \in \mathbb{N}.$$

Thus, the definition of contingent g–derivative ensures that

$$x'_g(t) = \lim_{k \to \infty} \frac{x(t + h_k) - x(t)}{g(t + h_k) - g(t)} \in \mathrm{D}_g\, K_p(t, x(t)).$$

Now condition (ii) implies that $x'_g(t) \in F(t, x(t))$.

Case 2. $t \in I \cap D_g$ *or* $t \in I \setminus (\widehat{E} \cup D_g)$ *and* $x(t) \notin K(t)$.
In either of these two situations, condition (i) ensures that $F(t, x(t))$ is convex and compact and $F(t, \cdot)$ is u.s.c. at $x(t)$.

Reasoning by contradiction, assume that $x_g'(t) \notin F(t, x(t))$. On the one hand, since $F(t, x(t))$ is a compact subset of $\mathbb{R}^n$, we can find $\varepsilon_0 > 0$ such that

$$x_g'(t) \notin F(t, x(t)) + B(0, \varepsilon_0). \tag{4.116}$$

Indeed, consider the map $\varphi(y) = \|x_g'(t) - y\|$, $y \in \mathbb{R}^n$. Clearly, φ is continuous on the whole $\mathbb{R}^n$ and since $F(t, x(t))$ is compact, φ attains a minimum in $F(t, x(t))$. That is, there exists $y_0 \in F(t, x(t))$ such that $\|x_g'(t) - y\| = \varphi(y) \geq \varphi(y_0)$ for all $y \in F(t, x(t))$ and $\varphi(y_0) > 0$ because $x_g'(t) \neq y_0$. Therefore (4.116) holds for $\varepsilon_0 = \varphi(y_0)$.

On the other hand, Remark 4.107 ensures that $F(t, \cdot)$ is ε–δ–u.s.c., so there exists $\delta > 0$ such that

$$F(t, y) \subset F(t, x(t)) + B(0, \varepsilon_0/2), \quad y \in \mathbb{R}^n, \ \|y - x(t)\| < \delta. \tag{4.117}$$

Take $k_0 \in \mathbb{N}$ such that

$$\|x_k(t_{k,i_k}) - x(t)\| < \delta \quad k \geq k_0.$$

Then (4.117) yields

$$F(t, x_k(t_{k,i_k})) \subset F(t, x(t)) + B(0, \varepsilon_0/2) \subset F(t, x(t)) + \overline{B(0, \varepsilon_0/2)}, \quad k \geq k_0.$$

Now, since $F(t, x(t)) + \overline{B(0, \varepsilon_0/2)}$ is a convex and closed set, it follows that

$$\overline{\text{co}} \bigcup_{k=k_0}^{\infty} F(t, x_k(t_{k,i_k})) \subset F(t, x(t)) + \overline{B(0, \varepsilon_0/2)} \subset F(t, x(t)) + B(0, \varepsilon_0).$$

We deduce from (4.115) that $x_g'(t) \in F(t, x(t)) + B(0, \varepsilon_0)$, which contradicts (4.116). □

Remark 4.111. It is important to note that the result of Theorem 4.110 may fail if (4.113) is not satisfied just at one point. Let us show that (4.110) with $g = \text{Id}$, $t_0 = 0$, $T = 1$, $x_0 = 0$, and

$$F(t, x) = \begin{cases} \{1\}, & \text{if } x < 0, \\ \{1/2\}, & \text{if } x = 0, \\ \{-1\}, & \text{if } x > 0, \end{cases}$$

has no solution.

Observe that $F(t, \cdot)$ assumes convex and compact values and it is u.s.c. on $\mathbb{R} \setminus \{0\}$. In this case we should take $K(t) = \{0\}$ for all $t \in [0, 1]$, and we have

$$\bigcap_{r>0} \overline{\text{co}} F(t, (-r, r)) \bigcap \text{D}_g \, K(t, 0) = [-1, 1] \bigcap \{0\} = \{0\} \not\subset F(t, 0).$$

Our next example shows that Theorem 4.110 is so general that it can be applied in cases where the nonlinear part is not u.s.c. or convex and compact valued on dense subsets of $\mathbb{R}^n$. To that end, we shall construct an ill–behaved multivalued mapping using a real valued function which is discontinuous at every rational number. Once again we consider the particular case of $g = \text{Id}$ to highlight that Theorem 4.110 is new even in the classical setting of usual derivatives.

Example 4.112. First, consider a bijection $r:\mathbb{N}\longrightarrow\mathbb{Q}$, denote $r_p=r(p)$ for each $p\in\mathbb{N}$, and define

$$\varphi(x)=\sum_{r_p<x}2^{-p},\quad x\in\mathbb{R}.$$

Obviously, $0<\varphi(x)<1$ for all $x\in\mathbb{R}$ and φ is increasing. Moreover, φ is continuous at every irrational and discontinuous at every rational. More precisely, for each $p\in\mathbb{N}$ we have

$$\varphi(r_p^-)=\varphi(r_p)<\varphi(r_p)+2^{-p}=\varphi(r_p^+).$$

This implies that we can find $\delta_p>0$ such that

$$|y-r_p|<\delta_p\Rightarrow\varphi(y)>\frac{\varphi(r_p)}{2}. \tag{4.118}$$

Now fix $\lambda\in(0,1)$ and define $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R}^n)$ as

$$F(t,x)=\begin{cases}[\lambda\varphi(x),\varphi(x)], & \text{if } t\in A,\ x\notin\mathbb{Q},\\ (\lambda\varphi(x),\varphi(x))\cap\mathbb{Q}, & \text{if } t\in A,\ x\in\mathbb{Q},\\ \left[-\varphi(x-t),-\lambda\varphi(x-t)\right], & \text{if } t\in B,\ x-t\notin\mathbb{Q},\\ (-\varphi(x-t),-\lambda\varphi(x-t))\cap\mathbb{Q}, & \text{if } t\in B,\ x-t\in\mathbb{Q},\end{cases}$$

where

$$A=\bigcup_{l=1}^{\infty}\left(\frac{1}{2l},\frac{1}{2l-1}\right),\quad B=\bigcup_{l=1}^{\infty}\left(\frac{1}{2l+1},\frac{1}{2l}\right).$$

Note that F is not explicitely defined for $t\in C=\{(2l)^{-1}:l\in\mathbb{N}\}\cup\{0\}$, nor it is necessary as it is countable, and thus $m(C)=0$. We shall show that the hypotheses of Theorem 4.110 are satisfied regardless of the values of F on $C\times\mathbb{R}$. Also note that $F(t,x)$ is neither convex nor compact if $(t,x)\in A\times\mathbb{Q}$ or $(t,x-t)\in B\times\mathbb{Q}$. Moreover, if $t\in A$, then $F(t,\cdot)$ is not u.s.c. at rational numbers because φ jumps upwards at rationals.

Clearly, (iv) is satisfied with $M(t)=1$, $t\in[0,1]$. Condition (iii) is easy to check: for each fixed $x\in\mathbb{R}\setminus\mathbb{Q}$, we can take the selection

$$f_x(t)=\begin{cases}\varphi(x), & \text{if } t\in A,\\ -\varphi(x-t), & \text{if } t\in B.\end{cases}$$

Observe that $f_x(t)$ may not be an element of $F(t,x)$ on $C_x=\{t\in B:x-t\in\mathbb{Q}\}=B\cap(x-\mathbb{Q})$, but this does not matter because C_x is a countable set. For the case $x\in\mathbb{Q}$, just take any $q\in(\lambda\varphi(x),\varphi(x))\cap\mathbb{Q}$ and consider

$$f_x(t)=\begin{cases}q, & \text{if } t\in A,\\ -\varphi(x-t), & \text{if } t\in B.\end{cases}$$

In any case, f_x is piecewise monotone, hence measurable.

Now for conditions (i) and (ii). Define

$$K(t)=\bigcup_{p=1}^{\infty}K_p(t),\quad t\in A\cup B,$$

where, for each $p \in \mathbb{N}$,

$$K_p(t) = \begin{cases} \{r_p\}, & \text{if } t \in A, \\ \{t + r_p\}, & \text{if } t \in B. \end{cases}$$

Clearly, $\mathrm{D}_g\, K_p(t, r_p) = \{0\}$ for all $t \in A$, and

$$\mathrm{D}_g\, K_p(t, t + r_p) = \{1\} \quad \text{for all } t \in B.$$

Moreover, for a.a. $t \in \overline{I}$ and every $x \in \mathbb{R} \setminus K(t)$, the set $F(t, x)$ is closed and convex. Let us prove that for a.a. $t \in \overline{I}$, the multivalued mapping $F(t, \cdot)$ is u.s.c. at every fixed $x \in \mathbb{R} \setminus K(t)$. If $t \in A$, then φ is continuous at x, so for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|y - x| < \delta \Rightarrow |\varphi(y) - \varphi(x)| < \varepsilon \Rightarrow |\mu\varphi(y) - \mu\varphi(x)| < \varepsilon \quad \text{for any } \mu \in [-1, 1].$$

Hence, if $t \in A$, and $|y - x| < \delta$, then

$$F(t, y) \subset [\lambda\varphi(y), \varphi(y)] \subset (\lambda\varphi(x) - \varepsilon, \varphi(x) + \varepsilon) = F(t, x) + (-\varepsilon, \varepsilon).$$

The proof is similar for $t \in B$, the only difference is that we have to use that φ is continuous at $x - t$, because $x \notin K_p(t) = \{t + r_p\}$ for any p.

Finally, we have to check that (4.113) holds. If $t \in A$, then for each fixed $p \in \mathbb{N}$, there exists $\delta_p > 0$ such that (4.118) holds if $|y - r_p| < \delta_p$. Then, if $|y - r_p| < \delta_p$ and $z \in F(t, y) \subset [\lambda\varphi(y), \varphi(y)]$, then

$$z \geq \lambda\varphi(y) > \lambda\varphi(r_p)/2.$$

Hence

$$\overline{\mathrm{co}} F(t, (r_p - \delta_p, r_p + \delta_p)) \subset [\lambda\varphi(r_p)/2, \infty),$$

which implies that

$$\bigcap_{r>0} \overline{\mathrm{co}} F(t, (r_p - r, r_p + r)) \subset [\lambda\varphi(r_p)/2, \infty) \subset (0, \infty).$$

Then the intersection in (4.113) is empty and therefore condition (4.113) holds. Checking condition (4.113) for $t \in B$ and $x = t + r_p$ for some p is easier. Clearly,

$$\bigcap_{r>0} \overline{\mathrm{co}} F(t, (t + r_p - r, t + r_p + r)) \subset (-\infty, 0],$$

and $\mathrm{D}_g\, K_p(t, t + r_p) = \{1\}$.

As a corollary of Theorem 4.110 we obtain the following existence principle for (4.2). To do so, we look at the inclusion problem

$$x_g'(t) \in \mathcal{K}f(t, x(t)), \quad g\text{–a.a. } t \in I, \quad x(t_0) = x_0, \tag{4.119}$$

where $\mathcal{K}f : I \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is the *Krasovskij map* defined as

$$\mathcal{K}f(t, x) = \bigcap_{r>0} \overline{\mathrm{co}} f\,(t, B(x, r))\,, \quad (t, x) \in I \times \mathbb{R}^n. \tag{4.120}$$

This definition ensures $f(t,x) \in \mathcal{K}f(t,x)$ for all $(t,x) \in I \times \mathbb{R}^n$, so every solution of (4.2) is a solution of (4.119). Thus, to obtain an existence result for (4.2) it is enough to impose conditions to guarantee the existence of solution of problem (4.119) and that every solution of the inclusion problem is a solution of the differential problem. Thus, we present the following result.

Theorem 4.113. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following conditions:*

(i) *For all $x \in \mathbb{R}^n$, the map $f(\cdot, x)$ is g–measurable.*

(ii) *There exists $L \in \mathcal{L}^1_g(I, [0, +\infty))$ and $N \subset I$, $\mu_g(N) = 0$, such that*

$$\|f(t,x)\| \leq L(t), \quad t \in I \setminus N, \quad x \in \mathbb{R}^n.$$

(iii) *For all $t \in I \backslash (N \cup D_g)$, $f(t,\cdot)$ is continuous on $\mathbb{R}^n \backslash K(t)$, where $K(t) = \bigcup_{p=1}^{\infty} K_p(t)$, and for each $p \in \mathbb{N}$ and $x \in K_p(t)$, we have*

$$\bigcap_{r>0}^{\infty} \overline{\text{co}} f(t, B(x,r)) \bigcap \mathrm{D}_g\, K_p(t,x) \subset \{f(t,x)\}. \tag{4.121}$$

(iv) *For all $t \in D_g$, $f(t,\cdot)$ is continuous on $\mathbb{R}^n$.*

Then, the initial value problem (4.2) *has at least one solution.*

To conclude this section, we present the following example in which it is possible to apply the previous result to obtain the existence of solution for the corresponding initial value problem.

Example 4.114. Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left–continuous map, φ as in Example 4.112 and $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\psi^{-1}(\mathbb{Q})$ is countable. Define the map $f : I \times \mathbb{R} \to \mathbb{R}$ as

$$f(t,x) = \varphi\left(\psi\left(\alpha x + \beta g(t)\right) \chi_{I \backslash D_g}(t) + t\chi_{D_g}(t)\right),$$

with $\alpha > 0$, $\beta \in (-\infty, -\alpha) \cup (0, +\infty)$. Note that $0 < f(t,x) < 1$ for all $(t,x) \in I \times \mathbb{R}$, so

$$\bigcap_{r>0}^{\infty} \overline{\text{co}} f(t, B(x,r)) \subset [0,1]$$

for all $(t,x) \in I \times \mathbb{R}$. Consider the initial value problem

$$x'_g(t) = f(t,x), \quad t \in I, \quad x(t_0) = x_0.$$

Let us show that the previous problem has at least a solution by proving that the hypotheses of Theorem 4.113 are satisfied.

Conditions (ii) and (iv) follow directly from the definition of f. For condition (i), fix $x \in \mathbb{R}$. The map

$$t \in I \mapsto \varphi\left(\psi\left(\alpha x + \beta g(t)\right) \chi_{I \backslash D_g}(t) + t\chi_{D_g}(t)\right)$$

is Borel–measurable as it is the composition of Borel–measurable maps. Hence, it is g–measurable.

Finally, for condition (iii), write $\psi^{-1}(\mathbb{Q}) = \{s_p : p \in \mathbb{N}\}$ and define $\gamma_p : I \to \mathbb{R}$, $p \in \mathbb{N}$, as

$$\gamma_p(t) = \frac{s_p - \beta g(t)}{\alpha},$$

$K_p : I \to \mathcal{P}(\mathbb{R})$, $p \in \mathbb{N}$, as $K_p(t) = \{\gamma_p(t)\}$ and $K(t) = \bigcup_{p=1}^{\infty} K_p(t)$. Note that for each $p \in \mathbb{N}$, γ_p is g–differentiable everywhere in I and $\gamma_g'(t) = -\beta/\alpha$, $t \in I$. Therefore, $D_g K_p(t, \gamma_p(t)) = \{-\beta/\alpha\}$, $p \in \mathbb{N}$, so $D_g K_p(t, \gamma_p(t)) \subset (-\infty, 0) \cup (1, +\infty)$, depending on the value of β.

Fix $t \in I \setminus D_g$. If $x \in K_p(t)$ for some $p \in \mathbb{N}$, then the intersection in (4.121) is empty, so the condition is trivially satisfied. On the other hand, if $x \in \mathbb{R} \setminus K(t)$, it follows that $\psi(\alpha x + \beta g(t)) \notin \mathbb{Q}$ since

$$\alpha x + \beta g(t) \neq s_p, \quad \text{for all } p \in \mathbb{N}.$$

Hence φ is continuous at $\psi(\alpha x + \beta g(t))$ and so $f(t, \cdot)$ is continuous at x as it is the composition of continuous functions. That is, the hypotheses of Theorem 4.113 are satisfied and so the initial value problem has at least a solution.

Chapter 5
Stieltjes differential problems with several derivators

In this chapter we have a look at multidimensional differential problems with Stieltjes derivatives, where each of the components is differentiated with respect to a different derivator. The interest of this kind of problems comes from two places. On the one hand, the theoretical interest of these problems is clear, as it provides a more general setting that the equations studied on Chapter 4. On the other hand, from the point of view of applications, the ability to consider different derivators allows us to study more precise models as different phenomena evolve in different ways and speed, as we show in some of the examples presented throughout this chapter.

In particular, in this chapter we will consider a map $\boldsymbol{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$, such that for each $i \in \{1, 2, \ldots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and we will look at problems of the form

$$x'_{\boldsymbol{g}}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{5.1}$$

with $t_0, T \in \mathbb{R}, T > 0$, $x_0 \in \mathbb{R}^n$ and $f : [t_0, t_0+T) \times \mathbb{R}^n \to \mathbb{R}^n$, and other formulations similar to those in Chapter 4. We will refer to this type of problems as $\boldsymbol{g}$–differential equations. Denoting $x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$ and $f = (f_1, f_2, \ldots, f_n)$, (5.1) is to be understood as the following system of differential equations:

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad x_i(t_0) = x_{0,i}, \quad i = 1, 2, \ldots, n,$$

where the derivative sign is to be interpreted as the Stieltjes derivative with respect to the corresponding function g_i. Note that if $\boldsymbol{g} = (g, g, \ldots, g)$ for some nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$, (5.1) becomes (4.1). Observe that, similarly to (4.1) and all the other problems considered in Chapter 4, we do not include the point $t_0 + T$ when studying problems of the form (5.1) as the same argument there presented still holds, that is, we cannot consider the derivative of a function with respect to g_i, $i \in \{1, 2, \ldots, n\}$, if $t_0 + T$ is a discontinuity point of such g_i.

In our quest to obtain results for (5.1), we mainly follow the results in Chapter 4 adapted to this context. This was partially done in [50]. However, we will also study vectorial measure differential equations of the form of

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f(s, x(s))\, \mathrm{d}\boldsymbol{g}(s), \quad t \in [t_0, t_0 + T], \tag{5.2}$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $x_0 \in \mathbb{R}^n$ and $f : [t_0, t_0 + T] \times \mathbb{R}^n \to \mathbb{R}^n$ as done in [52]. This equation should be understood as a “component-by-component” integration process in the Kurzweil–Stieltjes sense, that is, equation (5.2) corresponds to a systems of n scalar equations, each of which reads as follows

$$x_i(t) = x_{0,i} + {}^{(KS)}\!\!\int_{t_0}^{t} f_i(s, x(s))\, \mathrm{d}\, g_i(s), \quad t \in [t_0, t_0 + T], \quad i \in \{1, \dots, n\}. \tag{5.3}$$

The interest of studying this type of problems lies in the fact that there is a way of translating existence results for (5.2) into existence results of problems of the form of (5.1). It is important to note that if $\boldsymbol{g} = (g, g, \dots, g)$ for some nondecreasing and left–continuous map g, then (5.2) becomes a measure differential equation introduced in [27].

Remark 5.1. In the work ahead, the integral notation introduced above will also apply to Lebesgue–Stieltjes integrals. Such notation can be justified by regarding (5.2) as a particular case of the following vectorial equation

$$y(t) = y_0 + {}^{(KS)}\!\!\int_{t_0}^{t} \mathrm{d}[G(s)]\, f(s, y(s)), \quad t \in [t_0, t_0 + T], \tag{5.4}$$

where for each $t \in [t_0, t_0 + T]$, $G(t) \in \mathbb{R}^{n \times n}$ is the diagonal matrix

$$G(t) = \begin{bmatrix} g_1(t) & 0 & \dots & 0 \\ 0 & g_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n(t) \end{bmatrix}.$$

In the case when $f(t, x(t)) = x(t)$, equation (5.4) becomes a particular case of the so-called generalized linear differential equation; a branch of Kurzweil equations theory which has been extensively investigated in [78, 82].

The rest of the chapter is structured as follows. First, in Section 5.1 we turn our attention to the study of some types of continuity of functions that can be defined in terms of the map $\boldsymbol{g}$, where we study deeply some of the concepts introduced in [50]. Later, in Section 5.2, we study some conditions ensuring the existence and uniqueness of solution of (5.1), while also considering conditions to obtain solutions in a "classical" sense. This is done in a similar fashion to Chapter 4 and following [60]. Finally, in Section 5.3, we study measure differential equations of the form (5.2) through the method of lower and upper solutions as in [52]. From there, we derive some results for (5.1) using lower and upper solutions, generalizing some of the results obtained in Section 4.1.3 in Chapter 4.

5.1 Continuity with respect to several derivators

Similarly to Chapter 4, the main focus of this chapter is to obtain results that ensure the existence and uniqueness of solutions of the corresponding differential equation defined by a map $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, such that for each $i \in \{1, 2 \dots, n\}$, the map $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous. In particular, we will be looking for solutions lying in a specific set: the set of $\boldsymbol{g}$–absolutely continuous functions, which is a generalization of the set $\mathcal{AC}_g([a,b],\mathbb{R}^n)$ introduced in Definition 3.24. In order to formalize this definition, we will start by exploring some generalizations of certain concepts presented in Chapter 3. We will start by having a look at the concept of $\boldsymbol{g}$–continuity. A similar concept was first introduced in [50]. Here, we present another definition and, later, we discuss their relations. In what follows, we shall assume that $\mathbb{R}^n$ is endowed with the maximum norm, i.e. given $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, we consider the norm

$$\|x\| = \max\{|x_1|,|x_2|,\dots,|x_n|\}.$$

We do this for simplicity, but the theory that follows remains true when we consider the the usual norm in $\mathbb{R}^n$, as they are equivalent.

Definition 5.2. *Let* $\boldsymbol{g}:\mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{g}=(g_1,g_2,\dots,g_n)$, *be such that for each* $i\in\{1,2,\dots,n\}$, *the map* $g_i:\mathbb{R}\to\mathbb{R}$ *is nondecreasing and left–continuous. A function* $f:A\subset\mathbb{R}\to\mathbb{R}^n$, $f=(f_1,f_2,\dots,f_n)$, *is* $\boldsymbol{g}$–continuous at point $t\in A$, *or* continuous with respect to $\boldsymbol{g}$ at t *if* f_i *is* g_i*–continuous at* t *for each* $i\in\{1,2,\dots,n\}$. *If it is* $\boldsymbol{g}$*–continuous at every point* $t\in A$, *we say that* f *is* $\boldsymbol{g}$–continuous on A.

Remark 5.3. It is clear that Definition 5.2 is a generalization of Definition 3.17. It is enough to consider $\boldsymbol{g}=(g,g,\dots,g)$ for some nondecreasing and left–continuous function $g:\mathbb{R}\to\mathbb{R}$.

Before reflecting on the relation between Definition 5.2 and the corresponding definition in [50], let us introduce some notation. Let us denote by $\mathcal{C}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)$ the set of $\boldsymbol{g}$–continuous functions on $[a,b]$ with values in $\mathbb{R}^n$ and by $\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)$ the set of $\boldsymbol{g}$–continuous functions on $[a,b]$ with values in $\mathbb{R}^n$ which are also bounded. The set $\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)$ is a real vector space with the usual operations. Moreover, given its definition, we can rewrite $\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)$ as

$$\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)=\prod_{i=1}^{n}\mathcal{BC}_{g_i}([a,b],\mathbb{R}),$$

which means that it can be regarded as a product of Banach spaces, see Proposition 2.28. Hence, it is possible to endow $\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)$ with a Banach structure given by the norm

$$\|f\|_{\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)}=\max_{i=1,2,\dots,n}\{\|f_i\|_\infty\},\quad f=(f_1,f_2,\dots,f_n)\in\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n).$$

It follows from the definition that $\|\cdot\|_{\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)}$ and $\|\cdot\|_\infty$ are equivalent norms in $\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)$, from which we obtain that $(\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n),\|\cdot\|_\infty)$ is a Banach space equivalent to $(\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n),\|\cdot\|_{\mathcal{BC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n)})$.

Now, we compare Definition 5.2 with [50, Definition 3.1], a similar definition of continuity with respect to a map $\boldsymbol{g}:\mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{g}=(g_1,g_2,\dots,g_n)$, such that for each $i\in\{1,2,\dots,n\}$, $g_i:\mathbb{R}\to\mathbb{R}$ is nondecreasing and left–continuous. In order to do so, we include the mentioned definition.

Definition 5.4. *Let* $\boldsymbol{g}:\mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{g}=(g_1,g_2,\dots,g_n)$, *be such that for each* $i\in\{1,2,\dots,n\}$, *the map* $g_i:\mathbb{R}\to\mathbb{R}$ *is nondecreasing and left–continuous. A function* $f:A\subset\mathbb{R}\to\mathbb{R}^n$,

$f = (f_1, f_2, \dots, f_n)$, is $\vec{g}$–continuous at point *$t \in A$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(t) - f(s)\| < \varepsilon, \quad \textit{for all } s \in A,\ \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta.$$

If it is $\vec{g}$–continuous at every point $t \in A$, we say that f is $\vec{g}$–continuous on *A.*

In [50], the authors claimed that Definitions 5.2 and 5.4 are equivalent. However this is not the case, as shown in the next example. This misinformation does not affect the existence and uniqueness results obtained in such paper. However, it does have some implications when it comes to the study of solutions in the classical sense. This will be discussed in Section 5.2.

Example 5.5. Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be given by $g_1(t) = 0$ and

$$g_2(t) = \begin{cases} t, & \text{if } t \leq 0, \\ t+1, & \text{if } t > 0. \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}^2$, $f = (f_1, f_2)$, given by

$$f(t) = \begin{cases} (t, 0), & \text{if } t \leq 0, \\ \left(t+1, \dfrac{\sin(1/t)}{t}\right), & \text{if } t > 0. \end{cases}$$

Note that f_1 cannot be g_1–continuous as it is not constant. Therefore, f cannot be $\boldsymbol{g}$–continuous. However, f is $\vec{g}$–continuous. Indeed, first of all, note that

$$\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| = |g_2(t) - g_2(s)|, \quad s, t \in \mathbb{R},$$

so showing that f is $\vec{g}$–continuous is equivalent to showing that f is g_2–continuous in the sense of Definition 3.17. As a consequence, and given the definition of $\|\cdot\|$ in $\mathbb{R}^n$ we are considering, it suffices to show that f_1 and f_2 are g_2–continuous. Now, f_1 is trivially g_2–continuous as $f_1 = g_2$; and Example 3.19 shows that f_2 is g_2–continuous. Thus f is $\vec{g}$–continuous. Therefore, Definitions 5.2 and 5.4 cannot be equivalent.

It is interesting to mention that the converse implication does hold, that is, every continuous function in the sense of Definition 5.2 is also continuous in the sense of Definition 5.4. We gather this information in the following result, which can also be found in [50].

Proposition 5.6. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and consider $f : A \subset \mathbb{R} \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$. If f is $\boldsymbol{g}$–continuous, then f is $\vec{g}$–continuous.*

Proof. Fix $\varepsilon > 0$ and $t \in A$. Given $i \in \{1, 2, \dots, n\}$, we have that f_i is g_i–continuous at t, so there exists $\delta_i > 0$ such that

$$|f_i(t) - f_i(s)| < \varepsilon, \quad \text{for all } s \in A,\ |g_i(t) - g_i(s)| < \delta_i.$$

Take $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$. Then, if $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta$, we have that $|g_i(t) - g_i(s)| < \delta_i$ for all $i \in \{1, 2, \dots, n\}$, which implies that

$$\|f(t) - f(s)\| = \max_{i=1,\dots,n} \{|f_i(t) - f_i(s)|\} < \varepsilon,$$

that is, f is $\vec{g}$–continuous at t. □

As we have seen, the concepts of $\boldsymbol{g}$ and $\vec{g}$–continuity are not equivalent in general. However, in the light of Proposition 5.6, we can find a necessary and sufficient condition for the two types of continuity to be the same. The following result provides that information by exploring the converse implication to the one in Proposition 5.6.

Proposition 5.7. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous. Then, the following are equivalent:*

(i) *Every $\vec{g}$–continuous map is $\boldsymbol{g}$–continuous.*

(ii) *For each $j, k \in \{1, 2, \dots, n\}$, the map $g_k : \mathbb{R} \to \mathbb{R}$ is g_j–continuous.*

Proof. First assume that (i) holds. Fix $k \in \{1, 2, \dots, n\}$ and define $G : \mathbb{R} \to \mathbb{R}^n$ as

$$G(t) = (g_k(t), g_k(t), \dots, g_k(t)), \quad t \in \mathbb{R}.$$

To show that (ii) holds, it is enough to show that G is $\vec{g}$–continuous, as in that case, (i) would ensure that it is $\boldsymbol{g}$–continuous or, equivalently, g_k is g_j–continuous for each $j \in \{1, 2, \dots, n\}$. Note that this is straightforward. Indeed, let $t \in \mathbb{R}$, $\varepsilon > 0$ and take $\delta = \varepsilon$. Then, if $s \in \mathbb{R}$ is such that $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta$, it follows that

$$\|G(t) - G(s)\| = |g_k(t) - g_k(s)| \le \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta = \varepsilon.$$

In other words, G is $\vec{g}$–continuous at t, which finishes this implication.

Conversely, assume that (ii) holds and let $f : A \subset \mathbb{R} \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a $\vec{g}$–continuous map. Fix $i \in \{1, 2, \dots, n\}$, $t \in A$ and $\varepsilon > 0$. Since f is $\vec{g}$–continuous, there exists $\gamma > 0$ such that

$$\|f(t) - f(s)\| < \varepsilon, \quad \text{for all } s \in A \text{ such that } \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \gamma.$$

On the other hand, for each $k \in \{1, 2, \dots, n\}$, the map g_k is g_i–continuous and so, there exists $\delta_k > 0$ such that

$$|g_k(t) - g_k(s)| < \gamma, \quad \text{for all } s \in \mathbb{R} \text{ such that } |g_i(t) - g_i(s)| < \delta_k.$$

Therefore, taking $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$, for any $s \in A$ such that $|g_i(t) - g_i(s)| < \delta$, we have that $|g_k(t) - g_k(s)| \le \gamma$ for all $k \in \{1, 2, \dots, n\}$. Therefore, if $s \in A$ is such that $|g_i(t) - g_i(s)| < \delta$, we obtain that $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \gamma$, which ensures that

$$|f_i(t) - f_i(s)| \le \|f(t) - f(s)\| < \varepsilon.$$

That is, f_i is g_i–continuous. Now, since $i \in \{1, 2, \dots, n\}$ was arbitrarily fixed, we have that f is $\boldsymbol{g}$–continuous. □

Remark 5.8. As a consequence of Proposition 3.21, a necessary condition for (ii) in Proposition 5.7 to be satisfied is that

$$C_{g_j} = C_{g_k}, \quad D_{g_j} = D_{g_k}, \quad j, k = 1, 2, \dots, n.$$

Interestingly enough, it is possible to establish some other connections between the continuity in the sense of Definition 5.4 and other types of continuity in this thesis. In particular, we can show that there exists an equivalence between $\vec{g}$–continuity and continuity in the sense of Definition 3.17 for an adequate choice of a nondecreasing and left–continuous function.

Proposition 5.9. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and consider the map $\widehat{g} : \mathbb{R} \to \mathbb{R}$ defined as*

$$\widehat{g}(t) = g_1(t) + g_2(t) + \cdots + g_n(t), \quad t \in \mathbb{R}. \tag{5.5}$$

Then, $f : A \subset \mathbb{R} \to \mathbb{R}^n$ is $\vec{g}$–continuous at $t \in A$ if and only if f is $\widehat{g}$–continuous at t.

Proof. First, observe that for all $s \in A$,

$$\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| \leq |\widehat{g}(t) - \widehat{g}(s)| \leq n\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\|. \tag{5.6}$$

Indeed, it is clear that the first inequality holds as

$$|\widehat{g}(t) - \widehat{g}(s)| = \left|\sum_{i=1}^{n} g_i(t) - \sum_{i=1}^{n} g_i(s)\right| \leq \sum_{i=1}^{n} |g_i(t) - g_i(s)| \leq n\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\|,$$

for any $s \in A$. Now, for the other inequality we consider two cases. If $t \geq s$, we have that

$$\begin{aligned}|\widehat{g}(t) - \widehat{g}(s)| &= \widehat{g}(t) - \widehat{g}(s) = \sum_{i=1}^{n}(g_i(t) - g_i(s)) \\ &\geq \max_{i=1,\dots,n}\{g_i(t) - g_i(s)\} = \max_{i=1,\dots,n}\{|g_i(t) - g_i(s)|\} = \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\|.\end{aligned}$$

On the other hand, if $t < s$,

$$\begin{aligned}|\widehat{g}(t) - \widehat{g}(s)| &= \widehat{g}(s) - \widehat{g}(t) = \sum_{i=1}^{n}(g_i(s) - g_i(t)) \\ &\geq \max_{i=1,\dots,n}\{g_i(s) - g_i(t)\} = \max_{i=1,\dots,n}\{|g_i(t) - g_i(s)|\} = \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\|.\end{aligned}$$

Hence, we have proven that (5.6) holds. Now, the equivalence between the two types of continuity follows. □

Remark 5.10. Observe that Proposition 5.9 can be used for an alternative argument for Example 5.5. Furthermore, the same result can be used to justify that $\boldsymbol{g}$ and $\vec{g}$–continuity reduce to g–continuity when we consider $\boldsymbol{g} = (g, g, \dots g)$ for some nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$.

Thanks to Propositions 5.6 and 5.9 and Definition 5.2, we are able to deduce some properties of $\boldsymbol{g}$–continuous functions from Propositions 3.21 and 3.22. We gather such information in the following two results.

Proposition 5.11. *Let* $\boldsymbol{g}:\mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{g}=(g_1,g_2,\dots,g_n)$, *be a map such that for each* $i\in\{1,2,\dots,n\}$, *the function* $g_i:\mathbb{R}\to\mathbb{R}$ *is nondecreasing and left–continuous; and* $f:[a,b]\to\mathbb{R}^n$, $f=(f_1,f_2,\dots,f_n)$, *be* $\boldsymbol{g}$*–continuous on* $[a,b]$. *Then, for each* $i\in\{1,2,\dots,n\}$ *the following properties hold:*

1. f_i *is continuous from the left at every* $t\in(a,b]$*;*
2. *if* g_i *is continuous at* $t\in[a,b)$*, then so is* f_i*;*
3. *if* g_i *is constant on some* $[c,d]\subset[a,b]$*, then so is* f_i.

In particular, the map f *is continuous from the left at every* $t\in(a,b]$. *Moreover,* f *is continuous on* $[a,b]$ *when* $\boldsymbol{g}$ *is continuous on* $[a,b)$.

Proof. Given that f_i is g_i–continuous for each $i\in\{1,2,\dots,n\}$, it is enough to apply Proposition 3.21 to each of those maps to obtain the stated properties. □

Remark 5.12. Note that some of the properties stated in Proposition 5.11 can be deduced from the fact that f is $\widehat{g}$–continuous on $[a,b]$ for $\widehat{g}$ as in (5.5). In particular, it follows from Proposition 3.21 that if $f:[a,b]\to\mathbb{R}^n$ is $\widehat{g}$–continuous, then the following hold:

1. The map f is continuous from the left at every $t\in(a,b]$.
2. If $\widehat{g}$ is continuous at $t\in[a,b)$, then so is f. Equivalently, if each g_i, $i=1,2,\dots,n$, is continuous at $t\in[a,b)$, then so is f.
3. If $\widehat{g}$ is constant on some $[c,d]\subset[a,b]$, then so is f. Equivalently, if g_i, $i=1,2,\dots,n$, is constant on some $[c,d]\subset[a,b]$, then so is f.

Observe that Proposition 5.9 guarantees that such properties hold for $\vec{g}$–continuous functions.

The next property is the corresponding extension of Proposition 3.22 for this new context. Observe, however, that we state this result in the context of $\vec{g}$–continuity rather than $\boldsymbol{g}$–continuity as it represents a more general framework as pointed out in Proposition 5.6.

Proposition 5.13. *Let* $\boldsymbol{g}:\mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{g}=(g_1,g_2,\dots,g_n)$, *be a map such that for each* $i\in\{1,2,\dots,n\}$, *the function* $g_i:\mathbb{R}\to\mathbb{R}$ *is nondecreasing and left–continuous;* $A\subset\mathbb{R}$ *be a Borel set and* $f:A\subset\mathbb{R}\to\mathbb{R}^n$ *be* $\vec{g}$*–continuous on* A. *Then,* f *is Borel measurable.*

Proof. Proposition 5.9 ensures that f is $\widehat{g}$–continuous on A for $\widehat{g}$ as in (5.5). Therefore, Proposition 3.22 ensures that f is Borel measurable. □

Remark 5.14. Given that Lebesgue–Stieltjes measures are Borel measures, we have that every Borel measurable map is Lebesgue–Stieltjes measurable. In particular, we have that if A is a Borel set and $f:A\subset\mathbb{R}\to\mathbb{R}^n$ is $\vec{g}$–continuous on A, then f is g_j–measurable for all $j\in\{1,2,\dots,n\}$.

Another interesting concept of continuity introduced in Chapter 4 was the concept of $(g\times\mathrm{Id})$–continuity in Definition 4.5. The next definition is an adaptation of such definition to the context of $\boldsymbol{g}$–differential equations, thus providing a more general definition.

Definition 5.15. *Let* $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, *be such that for each* $i \in \{1, 2, \dots, n\}$, *the function* $g_i : \mathbb{R} \to \mathbb{R}$ *is nondecreasing and left–continuous;* $f : A \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, *and* $(t, x) \in A$. *We say that* f *is* $(\boldsymbol{g} \times \mathrm{Id})$–continuous at (t, x) *if for each* $i \in \{1, 2, \dots, n\}$, *the map* f_i *is* $(g_i \times \mathrm{Id})$*–continuous at* (t, x). *We say that* f *is* $(\boldsymbol{g} \times \mathrm{Id})$–continuous in A *if it is* $(\boldsymbol{g} \times \mathrm{Id})$*–continuous at every* $(t, x) \in A$.

Similarly to Definition 5.2, we can find a similar definition to Definition 5.15 in [50], based on the the ideas of Definition 5.4. As a consequence, we have that Definition 5.15 does not match [50, Definition 4.7], the corresponding definition of "product" continuity. Such definition reads as follows.

Definition 5.16. *Let* $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, *be such that for each* $i \in \{1, 2, \dots, n\}$, *the function* $g_i : \mathbb{R} \to \mathbb{R}$ *is nondecreasing and left–continuous;* $f : A \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, *and* $(t, x) \in A$. *We say that* f *is* $(\vec{g} \times \mathrm{Id})$*–continuous at* (t, x) *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$[(s, y) \in A,\ \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta,\ \|x - y\| < \delta] \ \Rightarrow\ \|f(t, x) - f(s, y)\| < \varepsilon.$$

We say that f *is* $(\vec{g} \times \mathrm{Id})$*–continuous in* A *if it is* $(\vec{g} \times \mathrm{Id})$*–continuous at every* $(t, x) \in A$.

Let us discuss the relations between Definitions 5.15 and 5.16. First, in an analogous fashion to Proposition 5.6, we have the following result showing that every $(\boldsymbol{g} \times \mathrm{Id})$–continuous function is also $(\vec{g} \times \mathrm{Id})$–continuous.

Proposition 5.17. *Let* $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, *be a map such that for each* $i \in \{1, 2, \dots, n\}$, *the function* $g_i : \mathbb{R} \to \mathbb{R}$ *is nondecreasing and left–continuous; and* $f : A \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$. *If* f *is* $(\boldsymbol{g} \times \mathrm{Id})$*–continuous, then* f *is* $(\vec{g} \times \mathrm{Id})$*–continuous.*

Proof. Fix $\varepsilon > 0$ and $(t, x) \in A$. Given $i \in \{1, 2, \dots, n\}$, we have that f_i is $(g_i \times \mathrm{Id})$–continuous at t, so there exists $\delta_i > 0$ such that

$$|f_i(t, x) - f_i(s, y)| < \varepsilon, \quad \text{for all } (s, y) \in A \text{ such that } |g_i(t) - g_i(s)| < \delta_i,\ \|x - y\| < \delta_i.$$

Take $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$. Then, if $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta$ and $\|x - y\| < \delta_i$, we have that $|g_i(t) - g_i(s)| < \delta_i$ and $\|x - y\| < \delta_i$ for all $i \in \{1, 2, \dots, n\}$, which implies that

$$\|f(t, x) - f(s, y)\| = \max_{i=1,\dots,n} \{|f_i(t, x) - f_i(s, y)|\} < \varepsilon,$$

that is, f is $\vec{g}$–continuous at t. □

Observe that Proposition 5.17 only guarantees the implication in one direction. This is because the converse implication is not true in general. To see that, it is enough to consider a simple modification of Example 5.5 as in the following example. Essentially, we reduce the corresponding "product" continuities to the corresponding $\boldsymbol{g}$ and $\vec{g}$–continuities.

Example 5.18. Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be as in Example 5.5 and consider the function $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_1, f_2)$, given by

$$f(t,(x,y)) = \begin{cases} (t,0), & \text{if } t \leq 0, \\ \left(t+1, \dfrac{\sin(1/t)}{t}\right), & \text{if } t > 0. \end{cases}$$

Note that, since f does not depend on $(x,y) \in \mathbb{R}^2$, $(\vec{g} \times \mathrm{Id})$ and $(\boldsymbol{g} \times \mathrm{Id})$–continuity reduce to $\vec{g}$ and $\boldsymbol{g}$–continuity, respectively. Thus, f is $(\vec{g} \times \mathrm{Id})$–continuous but not $(\boldsymbol{g} \times \mathrm{Id})$–continuous, see Example 5.5.

Once again, we can still ensure the equivalence between these two concepts of continuity in an analogous fashion to Proposition 5.7. Observe that the condition necessary in the next result for $(\boldsymbol{g} \times \mathrm{Id})$–continuity to be equivalent to $(\vec{g} \times \mathrm{Id})$–continuity is the same as the one required in Proposition 5.7.

Proposition 5.19. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous. Then, the following are equivalent:*

(i) *Every $(\vec{g} \times \mathrm{Id})$–continuous map is $(\boldsymbol{g} \times \mathrm{Id})$–continuous.*

(ii) *For each $j, k \in \{1, 2, \dots, n\}$, the map $g_k : \mathbb{R} \to \mathbb{R}$ is g_j–continuous.*

Proof. First assume that (i) holds. Fix $k \in \{1, 2, \dots, n\}$ and define $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$G(t,x) = (g_k(t), g_k(t), \dots, g_k(t)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

Now, noting that G does not depend on the variable x, it follows from the proof of Proposition 5.7 that it is $(\vec{g} \times \mathrm{Id})$–continuous, which implies that G is $(\boldsymbol{g} \times \mathrm{Id})$–continuous. Therefore, $G(\cdot, x_0)$, $x_0 \in \mathbb{R}^n$, is $\boldsymbol{g}$–continuous, which is equivalent to (ii).

Conversely, assume that (ii) holds and let $f : A \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a $(\vec{g} \times \mathrm{Id})$–continuous map. Fix $i \in \{1, 2, \dots, n\}$, $(t,x) \in A$ and $\varepsilon > 0$. Since f is $(\vec{g} \times \mathrm{Id})$–continuous, there exists $\gamma > 0$ such that

$$\|f(t,x) - f(s,y)\| < \varepsilon, \quad \text{for all } (s,y) \in A \text{ such that } \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \gamma,\ \|x - y\| < \gamma.$$

On the other hand, for each $k \in \{1, 2, \dots, n\}$, the map g_k is g_i–continuous and so, there exists $\delta_k > 0$ such that

$$|g_k(t) - g_k(s)| < \gamma, \quad \text{for all } s \in \mathbb{R} \text{ such that } |g_i(t) - g_i(s)| < \delta_k.$$

Therefore, taking $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$, for any $(s,y) \in A$ such that $|g_i(t) - g_i(s)| < \delta$, we have that $|g_k(t) - g_k(s)| \leq \gamma$ for all $k \in \{1, 2, \dots, n\}$. Therefore, if $(s,y) \in A$ is such that $|g_i(t) - g_i(s)| < \delta$ and $\|x - y\| < \gamma$, we obtain that $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \gamma$ and $\|x - y\| < \gamma$, which ensures that

$$|f_i(t,x) - f_i(s,y)| \leq \|f(t,x) - f(s,y)\| < \varepsilon.$$

That is, f_i is g_i–continuous. Now, since $i \in \{1, 2, \dots, n\}$ was arbitrarily fixed, we have that f is $\boldsymbol{g}$–continuous. □

In a similar fashion to Proposition 5.9, we can establish a connection between Definitions 5.16 and 4.5, its counterpart for the context of Chapter 4. In order to obtain such relation we need to consider the map $\widehat{g}$ introduced in (5.5) and consider the inequality (5.6).

Proposition 5.20. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and consider the map $\widehat{g} : \mathbb{R} \to \mathbb{R}$ defined as in* (5.5)*. Then, $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is $(\vec{g} \times \mathrm{Id})$–continuous at $(t, x) \in A \times B$ if and only if f is $(\widehat{g} \times \mathrm{Id})$–continuous at (t, x).*

Proof. Let $(t, x) \in A \times B$. First, assume that f is $(\vec{g} \times \mathrm{Id})$–continuous at (t, x). In that case, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(t, x) - f(s, y)\| < \varepsilon, \quad \text{for all } (s, v) \in A \times B,\ \|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta,\ \|x - y\| < \delta.$$

Note that if $s \in A$ is such that $|\widehat{g}(t) - \widehat{g}(s)| < \delta$ then (5.5) ensures that $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta$. Hence, given $\varepsilon > 0$ and $(s, y) \in A \times B$ such that $|\widehat{g}(t) - \widehat{g}(s)| < \delta$ and $\|x - y\| < \delta$, we have that $\|f(t, x) - f(s, y)\| < \varepsilon$. That is, f is $(\widehat{g} \times \mathrm{Id})$–continuous at (t, x).

Conversely, assume that f is $(\widehat{g} \times \mathrm{Id})$–continuous at (t, x) and let $\varepsilon > 0$. Under this conditions, we know that there exists $\widehat{\delta} > 0$ such that

$$\|f(t, x) - f(s, y)\| < \varepsilon, \quad \text{for all } (s, v) \in A \times B,\ |\widehat{g}(t) - \widehat{g}(s)| < \widehat{\delta},\ \|x - y\| < \widehat{\delta}.$$

Now, take $\delta = \widehat{\delta}/n$. If $(s, y) \in A \times B$ is such that $\|\boldsymbol{g}(t) - \boldsymbol{g}(s)\| < \delta$ and $\|x - y\| < \delta$, then (5.5) yields that

$$|\widehat{g}(t) - \widehat{g}(s)| < \widehat{\delta}, \quad \|x - y\| < \widehat{\delta},$$

which guarantees that $\|f(t, x) - f(s, y)\| < \varepsilon$. In other words, we have that f is $(\widehat{g} \times \mathrm{Id})$–continuous at (t, x). □

Later we will present the corresponding adaptations of the results regarding "everywhere" solutions in [33] to the context of $\boldsymbol{g}$–differential equations, following [50]. To that end, we will still use Definitions 5.2 and 5.16 instead of Definitions 5.4 and 5.15, as those are the concepts that are well–behaved when it comes to the composition of functions, as shown in the next result.

Lemma 5.21. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be such that for each $i \in \{1, 2, \dots, n\}$, $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(\vec{g} \times \mathrm{Id})$–continuous function on $A \times B$. If $x : A \to B$ is $\vec{g}$–continuous on A, then the composition $f(\cdot, x(\cdot))$ is $\vec{g}$–continuous on A. Moreover, if A is a Borel set, $f(\cdot, x(\cdot))$ is Borel measurable, and, in particular, g_i–measurable, $i \in \{1, 2, \dots, n\}$.*

Proof. The result follows directly from Propositions 5.9 and 5.20 and Lemma 4.6. □

Remark 5.22. It follows from Lemma 5.21 that the composition of a $\boldsymbol{g}$–continuous map with a $(\boldsymbol{g} \times \mathrm{Id})$–continuous one is $\vec{g}$–continuous. However, we cannot assure that the composition is $\boldsymbol{g}$–continuous. Indeed, to see that this is not the case, consider $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ nondecresing and left–continuous, I an interval and $t_0 \in I$ such that $\Delta g_1(t_0) = 0$ and $\Delta g_2(t_0) > 0$. Take $x : I \to \mathbb{R}^2$ given by $x(t) = (g_1(t), g_2(t))$ and $f : I \times \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_1, f_2)$, defined as

$$f_1(t, (x, y)) = g_1(t) - y, \quad f_2(t, (x, y)) = g_2(t) - x, \quad (t, (x, y)) \in I \times \mathbb{R}^2.$$

It is clear that x is $\boldsymbol{g}$–continuous at t_0. On the other hand, the composition $f_1(\cdot, x(\cdot))$ is not continuous at t_0, as

$$\lim_{t\to t_0^+} f_1(t,x(t)) = \lim_{t\to t_0^+} g_1(t) - g_2(t) = g_1(t_0) - g_2(t_0^+) < g_1(t_0) - g_2(t_0) = f_1(t_0, x(t_0)).$$

As a consequence, $f_1(\cdot, x(\cdot))$ cannot be g_1–continuous at t_0 as it is not continuous at t_0 and g_1 is, see Proposition 3.21. Therefore, the map $f(\cdot, x(\cdot))$ is not $\boldsymbol{g}$–continuous at t_0. However, the map f is $(\boldsymbol{g} \times \mathrm{Id})$–continuous at $(t_0, (g_1(t_0), g_2(t_0)))$. Indeed, we shall only show that f_1 is g_1–continuous at such point as the case for f_2 is analogous.

Let $\varepsilon > 0$ and take $0 < \delta < \varepsilon/2$. Denote $u_0 = (g_1(t_0), g_2(t_0))$. If $(t, (x, y)) \in I \times \mathbb{R}^2$ is such that $|g_1(t_0) - g_1(t)| < \delta$ and $\|u_0 - (x, y)\| < \delta$, then

$$\begin{aligned} |f_1(t_0,u_0) - f_1(t,(x,y))| &= |g_1(t_0) - g_2(t_0) - g_1(t) + y| \\ &\le |g_1(t_0) - g_1(t)| + |y - g_2(t_0)| \\ &\le |g_1(t_0) - g_1(t)| + \|u_0 - (x,y)\| < 2\delta < \varepsilon. \end{aligned}$$

In other words, the map f_1 is $(g_1 \times \mathrm{Id})$–continuous at $(t_0, (g_1(t_0), g_2(t_0)))$. A similar argument shows that f_2 is $(g_2 \times \mathrm{Id})$–continuous at that point, which proves that f is $(\boldsymbol{g} \times \mathrm{Id})$–continuous.

Finally, we turn our attention to the corresponding adaptation of the concept of absolutely continuity with respect to g. The next definition is presented following the ideas of Definitions 5.2 and 3.24.

Definition 5.23. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and $F : [a, b] \to \mathbb{R}^n$, $F = (F_1, F_2, \dots, F_n)$. We say that F is $\boldsymbol{g}$–*absolutely continuous on $[a, b]$, *or* absolutely continuous on $[a, b]$ with respect to $\boldsymbol{g}$, *if for each $i \in \{1, 2, \dots, n\}$, the map F_i is g_i–absolutely continuous on $[a, b]$. We denote by $\mathcal{AC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$ the set of $\boldsymbol{g}$–absolutely continuous functions on $[a, b]$ with values in $\mathbb{R}^n$.*

Remark 5.24. Note that by definition we have that

$$\mathcal{AC}_{\boldsymbol{g}}([a,b],\mathbb{R}^n) = \prod_{i=1}^{n} \mathcal{AC}_{g_i}([a,b],\mathbb{R}).$$

Moreover, it follows from this expression that if $\boldsymbol{g} = (g, g, \dots, g)$ for some nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$, Definition 5.23 yields Definition 3.24. It can also be derived from this characterization as a cartesian product and Remark 3.25 that $\boldsymbol{g}$–absolute continuity implies $\boldsymbol{g}$–continuity, and therefore, $\vec{g}$–continuity or $\widehat{g}$–continuity for $\widehat{g}$ as in (5.5).

Remark 5.25. Observe that every $\boldsymbol{g}$–absolutely continuous map is also $\widehat{g}$–absolutely continuous for $\widehat{g} : \mathbb{R} \to \mathbb{R}$ as in (5.5). Indeed, let $x \in \mathcal{AC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$, $i \in \{1, 2, \dots, n\}$ and $\varepsilon > 0$. By definition, $x_i \in \mathcal{AC}_{g_i}([a, b], \mathbb{R})$, so there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals of $[a, b]$, $\{(a_k, b_k)\}_{k=1}^m$, verifying

$$\sum_{k=1}^{m} (g_i(b_k) - g_i(a_k)) < \delta \tag{5.7}$$

implies that

$$\sum_{k=1}^{m} |x_i(b_k) - x_i(a_k)| < \varepsilon. \tag{5.8}$$

Hence, if $\{(a_k, b_k)\}_{k=1}^m$ is an open pairwise disjoint family of subintervals of $[a, b]$ such that

$$\sum_{k=1}^{m} (\widehat{g}(b_k) - \widehat{g}(a_k)) < \delta,$$

it follows that (5.7) holds, which implies (5.8). That is, $x_i \in \mathcal{AC}_{\widehat{g}}([a, b], \mathbb{R})$. Now, since $i \in \{1, 2, \dots, n\}$ was arbitrarily chosen, we have that $x \in \mathcal{AC}_{\widehat{g}}([a, b], \mathbb{R}^n)$.

Let $F : [a, b] \to \mathbb{R}$, $F = (F_1, F_2, \dots, F_n)$, be a $\boldsymbol{g}$–absolutely continuous function on $[a, b]$. Then, the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral, Theorem 3.27, yields that for each $i \in \{1, 2, \dots, n\}$, $(F_i)'_{g_i}(t)$ exists for g_i–a.a. $t \in [a, b)$ and $(F_i)'_{g_i} \in \mathcal{L}^1_{g_i}([a, b), \mathbb{R})$. Moreover, we have that

$$F_i(t) = F_i(a) + \int_{[a,t)} (F_i)'_{g_i}(s)\, \mathrm{d}\, g_i(s), \quad t \in [a, b], \quad i = 1, 2, \dots, n,$$

or, using the "component-by-component" integral expression introduced before,

$$F(t) = F(a) + \int_{[a,t)} F'_{\boldsymbol{g}}(s)\, \mathrm{d}\, \boldsymbol{g}(s), \quad t \in [a, b].$$

Finally, we can obtain from Proposition 3.31 a useful result about relatively compact subsets of $\mathcal{AC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$. Note that, as usual, $\mathcal{AC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$ is a subset of $\mathcal{BC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$.

Proposition 5.26. *Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, be a map such that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and $\mathcal{S}$ be a subset of $\mathcal{AC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$. Assume that for each $i \in \{1, 2, \dots, n\}$ the following conditions are satisfied:*

(i) *The set $\{F_i(a) : F = (F_1, F_2, \dots, F_n) \in \mathcal{S}\}$ is bounded.*

(ii) *There exists $h_i \in \mathcal{L}^1_{g_i}([a, b), [0, +\infty))$ such that*

$$|(F_i)'_{g_i}(t)| \le h_i(t), \quad g_i\text{–a.a. } t \in [a, b), \quad \text{for all } F = (F_1, F_2, \dots, F_n) \in \mathcal{S}.$$

Then $\mathcal{S}$ is a relatively compact subset of $\mathcal{BC}_{\boldsymbol{g}}([a, b], \mathbb{R}^n)$.

5.2 Initial value problem

In this part of the chapter, we shall focus on the study of initial value problems in the context of $\boldsymbol{g}$–differential equations. Specifically, given $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, such that each map $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, is nondecreasing and left–continuous; we will study the existence and uniqueness of solution of

$$x'_{\boldsymbol{g}}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{5.9}$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $X \subset \mathbb{R}^n$, $x_0 \in X$ and $f : [t_0, t_0 + T) \times X \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$. To that end, we introduce the concept of solution that is fundamental for the aims of this section. As usual, in what follows we denote by $I_\tau = [t_0, t_0 + \tau)$, $\tau \in (0, T]$, and $I = [t_0, t_0 + T)$. Similarly, we denote $\overline{I}_\tau = [t_0, t_0 + \tau]$, $\tau \in (0, T]$, and $\overline{I} = [t_0, t_0 + T]$.

Definition 5.27. *A* solution *of* (5.9) *on an interval I_τ, $\tau \in (0, T]$, is a map $x \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$, $x = (x_1, x_2, \ldots, x_n)$, such that $x(t_0) = x_0$, $x(t) \in X$ for all $t \in I_\tau$ and*

$$x'_{g_i}(t) = f_i(t, x(t)), \quad g_i\text{–a.a. } t \in I_\tau, \quad i \in \{1, 2, \ldots, n\}. \tag{5.10}$$

If $\tau = T$, we say that x is a global solution *of* (5.9)*; otherwise, i.e. if $\tau \in (0, T)$, we say that x is a* local solution *of* (5.9).

Remark 5.28. In the work ahead, we will say that properties are satisfied $\boldsymbol{g}$–almost everywhere in a set or for $\boldsymbol{g}$–almost all points of a set, meaning that they are satisfied with almost everywhere for the corresponding measure, μ_{g_i}. That is, we will consider the properties in a "component–by–component" fashion. Therefore, we will write expressions like (5.10) as

$$x'_{\boldsymbol{g}}(t) = f(t, x(t)), \quad \boldsymbol{g}\text{–a.a. } t \in I_\tau.$$

Remark 5.29. Analogously to the discussion about Definition 4.1, one might be interested in the reason for considering the solutions on an interval I_τ, $\tau \in (0, T]$, to be defined over the corresponding closed interval, $\overline{I}_\tau$. The justification for that remains the same as for the solutions (4.2): the Fundamental Theorem of Calculus. This result allows us to characterize the solutions of (5.9) in an analogous fashion to Remark 4.2. That is, x is a solution of (5.9) on I_τ, $\tau \in (0, T]$, if and only if $x \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$ and it solves

$$x(t) = x_0 + \int_{[t_0, t)} f(s, x(s)) \, \mathrm{d}\boldsymbol{g}(s), \quad t \in I_\tau. \tag{5.11}$$

Observe, however, that this can still be done by requiring the solutions to belong to the corresponding set generalizing the one in Definition 4.3. In this case, it would be for functions in the set $E\mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$ defined as the following cartesian product:

$$E\mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n) = \prod_{i=1}^{n} E\mathcal{AC}_{g_i}(\overline{I}_\tau, \mathbb{R}).$$

It follows from Proposition 4.4 that there exists a bijection between $E\mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$ and $\mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$. Hence, we will restrict ourselves to Definition 5.27 for simplicity.

Following the steps of Chapter 4, we study "everywhere" solutions of problem (5.9). In particular, we generalize Corollary 4.7, showing that "everywhere" solutions of (5.9) have $\vec{g}$–continuous derivatives provided that the map f is $(\vec{g} \times \mathrm{Id})$–continuous.

Proposition 5.30. *Let $\tau \in (0,T]$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a $(\vec{g} \times \mathrm{Id})$–continuous function on $I_\tau \times \mathbb{R}^n$ and $x \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$, be such that $x(t_0) = x_0$ and*

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad \text{for all } t \in I_\tau \setminus C_{g_i}, \quad i = 1, 2, \dots, n.$$

Then, $(x_i)'_{g_i}$ is $\vec{g}$–continuous on $I_\tau \setminus C_{g_i}$.

Proof. Given that x is $\boldsymbol{g}$–absolutely continuous on $\overline{I}_\tau$, we have that x is $\boldsymbol{g}$–continuous on $\overline{I}_\tau$, see Remark 5.24. Hence, it is $\vec{g}$–continuous on $\overline{I}_\tau$. Thus Lemma 5.21 yields that $f(\cdot, x(\cdot))$ is $\vec{g}$–continuous. Hence it follows from the definition of $\vec{g}$–continuity that, for each $i \in \{1, 2, \dots, n\}$, the map $f_i(\cdot, x(\cdot))$ is $\vec{g}$–continuous, and since $(x_i)'_{g_i}$ is defined on $I_\tau \setminus C_{g_i}$, the result follows. □

Remark 5.31. If $\boldsymbol{g} = (g, g, \dots, g)$ for some nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$, Proposition 5.30 yields Corollary 4.7.

It follows from Proposition 5.9 that, under the hypotheses of Proposition 5.30, $(x_i)'_{g_i}$ is $\widehat{g}$–continuous on $I_\tau \setminus C_{g_i}$ for $\widehat{g}$ as in (5.5). Furthermore, notice that if we replace the concept of $\vec{g}$–continuity for $\boldsymbol{g}$–continuity in the hypotheses of Proposition 5.30 we still obtain the same result. As proven in Remark 5.22, we cannot ensure that the corresponding g_i–continuity without further modifications to the hypotheses. To that end, we introduce the following result.

Proposition 5.32. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$ and $x \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$, $\tau \in (0, T]$, be such that $x(t_0) = x_0$ and*

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad \text{for all } t \in I_\tau \setminus C_{g_i}, \quad i = 1, 2, \dots, n.$$

If there exists $i \in \{1, 2, \dots, n\}$ such that $f_i(\cdot, x(\cdot))$ is g_i–continuous on I_τ, then $(x_i)'_{g_i}$ is g_i–continuous on $I_\tau \setminus C_{g_i}$. In particular, if $f(\cdot, x(\cdot))$ is $\boldsymbol{g}$–continuous on I_τ, then $(x_i)'_{g_i}$ is g_i–continuous on $I_\tau \setminus C_{g_i}$ for all $i \in \{1, 2, \dots, n\}$.

Proof. The hypotheses ensure that $f_i(\cdot, x(\cdot))$ is g_i–continuous on I_τ. Now, it is enough to note that $(x_i)'_{g_i}$ is defined on $I_\tau \setminus C_{g_i}$, to finish the proof. □

Following the steps of Chapter 4, we now generalize Proposition 4.8, a result that showed that under certain hypothesis on the map defining the problem, a solution can be turned into an "everywhere" solution. Here, we prove that under the corresponding condition of f for this context, a solution in the sense Definition 5.27 is an "everywhere" solution. Observe, however, that in order to do so, we need more than the corresponding continuity one.

Proposition 5.33. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a map and $x : \overline{I}_\tau \longrightarrow \mathbb{R}$, $x = (x_1, x_2, \dots, x_n)$, $\tau \in (0, T]$, be a solution of* (5.9)*. Then:*

(i) *If $f(\cdot, x(\cdot))$ is $\vec{g}$–continuous on I_τ, then*

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in (I_\tau \setminus (D_{\widehat{g}} \cup C_{g_i})) \cup D_{g_i}, \quad i = 1, 2, \dots, n,$$

for $\widehat{g}$ as in (5.5)*.*

(ii) *If $f_i(\cdot,x(\cdot))$ is g_i–continuous for some $i\in\{1,2,\dots,n\}$, then*

$$(x_i)'_{g_i}(t)=f_i(t,x(t))\quad \textit{for all } t\in I_\tau\setminus C_{g_i}.$$

In particular, if $f(\cdot,x(\cdot))$ is $\boldsymbol{g}$–continuous on I_τ, then

$$(x_i)'_{g_i}(t)=f_i(t,x(t))\quad \textit{for all } t\in I_\tau\setminus C_{g_i},\quad i=1,2,\dots,n.$$

Proof. Fix $i\in\{1,2,\dots,n\}$. Since x is a solution of (5.9), we have that $x_i\in\mathcal{AC}_{g_i}(\overline{I}_\tau,\mathbb{R}^n)$ and

$$(x_i)'_{g_i}(t)=f_i(t,x(t)),\quad g_i\text{–a.a. } t\in I_\tau. \tag{5.12}$$

It follows from (5.12) that in particular, $(x_i)'_{g_i}(t)=f_i(t,x(t))$ for all $t\in D_{g_i}$, so it is enough to show that the equality holds for $t\in I_\tau\setminus(C_{g_i}\cup D_{\widehat{g}})$ to prove (i) and for $t\in I_\tau\setminus(C_{g_i}\cup D_{g_i})$ to prove (ii). We will first prove the (ii) and then, by making small modifications to that proof, we will obtain (i).

Fix $t\in I_\tau\setminus(C_{g_i}\cup D_{g_i})$. Since g_i is not constant on any neighbourhood of t, we may have $g_i(s)<g_i(t)$ for all $s<t$, $g_i(s)>g_i(t)$ for all $s>t$, or both. If $g_i(s)<g_i(t)$ for all $s<t$ and $t>t_0$, then for all $s\in[t_0,t)$,

$$\begin{aligned}(g_i(t)-g_i(s))\inf_{s\le r<t} f_i(r,x(r))&\le\int_{[s,t)} f_i(r,x(r))\,\mathrm{d}\,g_i(r)\\&\le(g_i(t)-g_i(s))\sup_{s\le r<t} f_i(r,x(r)).\end{aligned}$$

Now, the Fundamental Theorem of Calculus ensures that

$$\int_{[s,t)} f(r,x(r))\,\mathrm{d}\,g_i(r)=x_i(t)-x_i(s),\quad s\in[t_0,t).$$

Hence,

$$\inf_{s\le r<t} f_i(r,x(r))\le\frac{x_i(s)-x_i(t)}{g_i(s)-g_i(t)}\le\sup_{s\le r<t} f_i(r,x(r)). \tag{5.13}$$

On the other hand, if $g_i(s)>g_i(t)$ for all $s>t$, then for all $s\in[t,t_0+\tau)$ we have

$$\begin{aligned}(g_i(s)-g_i(t))\inf_{t\le r<s} f_i(r,x(r))&\le\int_{[t,s)} f_i(r,x(r))\,\mathrm{d}\,g_i(r)\\&\le(g_i(s)-g_i(t))\sup_{t\le r<s} f_i(r,x(r)).\end{aligned}$$

Reasoning analogously to the previous case, we obtain

$$\inf_{t\le r<s} f_i(r,x(r))\le\frac{x_i(s)-x_i(t)}{g_i(s)-g_i(t)}\le\sup_{t\le r<s} f_i(r,x(r)). \tag{5.14}$$

Now, for (ii), assume that $f_i(\cdot,x(\cdot))$ is g_i–continuous on $\overline{I}_\tau$. In that case, since g_i is continuous at t, we have that $f_i(\cdot,x(\cdot))$ is continuous at t. Therefore, if t is such that $g_i(s)<g_i(t)$ for all $s<t$, (5.13) implies that the following limit exists and

$$\lim_{s\to t^-}\frac{x_i(s)-x_i(t)}{g_i(s)-g_i(t)}=f_i(t,x(t)). \tag{5.15}$$

If $g_i(s) = g_i(t)$ on some $[t, t+\delta]$, $\delta > 0$, then the limit in (5.15) is $(x_i)'_{g_i}(t)$ and the proof is complete. Similarly, if t is such that $g_i(s) > g_i(t)$ for all $s > t$, the continuity of $f_i(\cdot, x(\cdot))$ at t and (5.14) ensure that the following limit exists and

$$\lim_{s \to t^+} \frac{x_i(s) - x_i(t)}{g_i(s) - g_i(t)} = f_i(t, x(t)). \tag{5.16}$$

This covers all of the remaining cases, so the proof is finished for $f_i(\cdot, x(\cdot))$ g_i–continuous, and subsequently, for $f(\cdot, x(\cdot))$ $\boldsymbol{g}$–continuous.

On the other hand, if $t \in I_\tau \setminus (C_{g_i} \cup D_{\widehat{g}})$, we have that $t \in I_\tau \setminus (C_{g_i} \cup D_{g_i})$. Hence, (5.13) holds if t is such that $g_i(s) < g_i(t)$ for all $s < t$, and (5.14), if $g_i(s) > g_i(t)$ for all $s > t$. Therefore, for (i), $f(\cdot, x(\cdot))$ is $\vec{g}$–continuous, so $f(\cdot, x(\cdot))$ is $\widehat{g}$–continuous at t. As a consequence, $f(\cdot, x(\cdot))$ is continuous at t as $\widehat{g}$ is continuous at that point. Hence, by making analogous reasonings, we can obtain (5.15) and (5.16) to finish the proof. □

Remark 5.34. Notice that if $\boldsymbol{g} = (g, g, \dots, g)$ for some nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$, then Proposition 5.33 reduces to Proposition 4.8. In particular, we have that the proof of Proposition 5.33 is a more complex version of that for Proposition 4.8 in [33]. Observe that we can find a similar result to Proposition 5.33 in [50, Proposition 4.6]. However, such result is not correct due to Definitions 5.2 and 5.15 not being equivalent to Definitions 5.4 and 5.16, respectively. The correct formulation of the result is the one here proposed in Proposition 5.33.

As we have pointed out, Propositions 5.30 and 5.33 generalize the corresponding results in Chapter 4, Propositions 4.7 and 4.8, which lead to the conclusion of Theorem 4.9. However, we cannot directly establish an analogous result to Theorem 4.9 because of the limitations that the definitions present. Here, we propose a result that can be regarded as a generalization of Theorem 4.9, as it reduces to such result when we consider $\boldsymbol{g} = (g, g, \dots, g)$ for a nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$. Essentially, the following result ensures that the “classical” concept of solution can be obtained from Definition 5.27 provided that f satisfies the corresponding continuity condition and the solution satisfies the equation in some problematic points.

Theorem 5.35. *Let $\tau \in (0, T]$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(\vec{g} \times \mathrm{Id})$–continuous function on $I_\tau \times \mathbb{R}^n$ and $x : \overline{I}_\tau \to \mathbb{R}^n$ be a solution of* (5.9)*. If there exists $i \in \{1, 2, \dots, n\}$ such that*

$$(x_i)'_{g_i}(t) = f(t, x(t)), \quad \textit{for all } t \in D_{\widehat{g}} \setminus D_{g_i}, \tag{5.17}$$

then

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \textit{for all } t \in I_\tau \setminus C_{g_i}, \tag{5.18}$$

and $(x_i)'_{g_i}$ is $\vec{g}$–continuous on $I_\tau \setminus C_{g_i}$. In particular, if (5.17) *holds for all $i \in \{1, 2, \dots, n\}$, then* (5.18) *holds for all $i \in \{1, 2, \dots, n\}$ and each $(x_i)'_{g_i}$, $i \in \{1, 2, \dots, n\}$, is $\vec{g}$–continuous on $I_\tau \setminus C_{g_i}$.*

Proof. By definition, $x \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$, and therefore x is $\vec{g}$–continuous. Hence, given that f is $(\vec{g} \times \mathrm{Id})$–continuous on $\overline{I} \times \mathbb{R}^n$, Proposition 5.33 ensures that

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in (I_\tau \setminus (D_{\widehat{g}} \cup C_{g_i})) \cup D_{g_i}, \quad i = 1, 2, \dots, n.$$

Hence, given $i \in \{1, 2, \ldots, n\}$ such that (5.17) holds, we have that $(x_i)'_{g_i}(t) = f_i(t, x(t))$ for all $t \in I_\tau \setminus C_{g_i}$. Now, Proposition 5.30 ensures that $(x_i)'_{g_i}$ is $\vec{g}$–continuous in $I_\tau \setminus C_{g_i}$, which finishes the proof. □

Remark 5.36. Note that condition (5.17) becomes vacuous when $\boldsymbol{g} = (g, g, \ldots, g)$ for a nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$ as $\widehat{g} = ng$, from which we have $D_{\widehat{g}} = D_g$. Furthermore, in that case, $\vec{g}$ and $(\vec{g} \times \mathrm{Id})$–continuity reduce to g and $(g \times \mathrm{Id})$–continuity, respectively. In other words, Theorem 5.35 generalizes Theorem 4.9.

In [50, Theorem 4.8] we find a similar result to Theorem 5.35. In that case, the authors assumed the following extra hypothesis: for each $i \in \{1, 2, \ldots, n\}$, there exists a g_i–integrable map $h_i : I \to [0, +\infty)$ such that

$$|f_i(t, x)| \leq h_i(t), \quad g_i\text{–a.a. } t \in I, \quad x \in \overline{B(x_0, r)}. \tag{5.19}$$

With that extra assumption, the authors claimed that they can ensure the existence of a local solution. This part can be proven to be true using a later result, Theorem 5.39. However, the statement of [50, Theorem 4.8] goes beyond that, ensuring that such local solution satisfies (5.18) and that the solution is $\boldsymbol{g}$–continuous. The reasoning used there follows the arguments used to obtain Theorem 5.35. In particular, this argument presents the limitations commented in Remark 5.34, invalidating the proof there included. Furthermore, it is easy to see that the extra hypothesis, (5.19), is not enough to guarantee that the composition $f(\cdot, x(\cdot))$ is $\boldsymbol{g}$–continuous. It is enough to consider the example included in Remark 5.22 for two bounded derivators. Therefore, the result is not correct.

It is interesting, however, to consider the ideas of [50, Theorem 4.8]. In particular, the idea of finding "everywhere" solutions of (5.9) that are $\boldsymbol{g}$–continuous. To that end, we have Proposition 5.32 providing some conditions ensuring that that is the case. Combining that result with statement (ii) in Proposition 5.33, we can obtain the following result.

Theorem 5.37. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, be a map and $x : \overline{I}_\tau \to \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, be a solution of* (5.9)*. If $f_i(\cdot, x(\cdot))$ is g_i–continuous on I_τ for some $i \in \{1, 2, \ldots, n\}$, then*

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in I_\tau \setminus C_{g_i},$$

and $(x_i)'_{g_i}$ is g_i–continuous on $I_\tau \setminus C_{g_i}$. In particular, if $f(\cdot, x(\cdot))$ is $\boldsymbol{g}$–continuous on I_τ, then

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in I_\tau \setminus C_{g_i}, \quad i = 1, 2, \ldots, n,$$

and $(x_i)'_{g_i}$ is g_i–continuous on $I_\tau \setminus C_{g_i}$ for all $i \in \{1, 2, \ldots, n\}$.

5.2.1 Existence and uniqueness of solution

We now present some results guaranteeing the existence and uniqueness of solution of (5.9) in a similar fashion to the results in Chapter 4. That is, first, we study under which conditions we can ensure the existence of solution and later, we look for conditions guaranteeing uniqueness of solution. Finally, combining the previous results, we obtain results on the existence of a unique solution of problem (5.9) in [50, 60]. Before doing so, it is important that we make the following remark about the continuity of the derivators at the initial point.

Remark 5.38. When studying the existence and uniqueness of solutions of (5.9), we can assume without loss of generality that $\boldsymbol{g}$ is continuous at t_0. To see this, it is enough to consider the "component-by-component" formulation of the problem and apply a similar argument to that in Proposition 4.28.

As a final comment, note that given $\boldsymbol{g}:\mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{g}=(g_1,g_2,\dots,g_n)$, such that each map $g_i:\mathbb{R}\to\mathbb{R}$, $i=1,2,\dots,n$, is nondecreasing and left–continuous; we have that each g_i, $i=1,2,\dots,n$, defines a Lebesgue–Stieltjes measure, μ_{g_i}, $i=1,2,\dots,n$, over its corresponding σ–algebra, $\mathcal{LS}_{g_i}$, $i=1,2,\dots,n$. Furthermore, the map $\widehat{g}$ in (5.5) also defines a σ–algebra, $\mathcal{LS}_{\widehat{g}}$. It follows from Remark 1.53 that $\bigcap_{i=1}^n \mathcal{LS}_{g_i}\subset\mathcal{LS}_{\widehat{g}}$. Therefore, as long as we can ensure that a function is g_i–measurable for all $i\in\{1,2,\dots,n\}$, we do not need to worry about the $\widehat{g}$–measurability of such function.

Existence of solution

Our first existence result is, as it happened in Chapter 4, a Peano–type existence result. This result is a generalization of Theorem 4.31 and its proof is analogous to that of Theorem 4.31 in the more general context of (5.9).

Theorem 5.39. *Let $r>0$ and $f:I\times\overline{B(x_0,r)}\to\mathbb{R}^n$, $f=(f_1,f_2,\dots,f_n)$, be such that for every $i=1,2,\dots,n$, f_i is g_i–Carathéodory. Then there exists $\tau\in(0,T]$ such that* (5.9) *has a solution on I_τ.*

Proof. Let $R=r+\|x_0\|$. By hypothesis, for each $i\in\{1,2,\dots,n\}$, f_i is g_i–Carathéodory on $I\times\overline{B(x_0,r)}$. Therefore, for each $i\in\{1,2,\dots,n\}$, there exists $h_{R,i}\in\mathcal{L}^1_{g_i}(I,[0,+\infty))$ such that

$$|f_i(t,x)|\le h_{R,i}(t),\quad g_i\text{–a.a. } t\in I,\quad x\in\overline{B(x_0,r)},\ \|x\|\le R.$$

In particular, we have that, for each $i\in\{1,2,\dots,n\}$,

$$|f_i(t,x)|\le h_{R,i}(t),\quad g_i\text{–a.a. } t\in I,\quad x\in\overline{B(x_0,r)}.$$

We can assume, without loss of generality, that $\boldsymbol{g}$ is continuous at t_0, see Remark 5.38. Therefore, we can find $\tau\in(0,T]$ such that

$$\max_{i=1,\dots,n}\left\{\int_{[t_0,t_0+\tau)} h_{R,i}(s)\,\mathrm{d}\,g_i(s)\right\}\le r. \tag{5.20}$$

Define $X=\{x\in\mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}):\|x(t)-x_0\|\le r \text{ for all } t\in\overline{I}_\tau\}$ and $F:X\to X$ given by

$$Fx(t)=x_0+\int_{[t_0,t)} f(s,x(s))\,\mathrm{d}\,\boldsymbol{g}(s),\quad t\in\overline{I}_\tau.$$

It is clear that X is nonempty as the map $H=(H_1,H_2,\dots,H_n)$ given by

$$H_i(t)=x_{0,i}+\int_{[t_0,t)} h_{R,i}(s)\,\mathrm{d}\,g(s),\quad i=1,2,\dots,n,$$

belongs to X. Moreover, X is a closed convex subset set of $\mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R})$ by construction. Furthermore, the hypotheses and the selection of τ guarantee that F is well–defined. Indeed, given $x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R})$, we have that x is g_i–measurable for all $i \in \{1, 2, \ldots, n\}$, see Proposition 5.13 and Remark 5.14. Now, Proposition 1.28 ensures that $f_i(\cdot, x(\cdot))$ is g_i–measurable for each $i \in \{1, 2, \ldots, n\}$, and since $x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$, the same result ensures that $f_i(\cdot, x(\cdot)) \in \mathcal{L}_{g_i}(I_\tau, \mathbb{R})$, $i = 1, 2, \ldots, n$. Hence, the integrals in the definition of F are well–defined. Now, the selection of τ ensures that F maps X to itself. Thus, it is enough to show that F is compact to obtain the result as a consequence of Schauder's Fixed Point Theorem.

Let $A \subset X$ be a bounded set. Let us show that $F(A)$ is relatively compact subset of $\mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$ using Proposition 5.26. First, it is clear that $\{x_i(t_0) : x \in F(A)\} = \{x_{0,i}\}$ for each $i \in \{1, 2, \ldots, n\}$. Moreover, for any $x \in F(A)$, Theorem 3.26 ensures that

$$|(F_i x)'_{g_i}(t)| = |f_i(t, x(t))| \leq h_{R,i}(t), \quad g_i\text{–a.a. } t \in I_\tau, \quad i = 1, 2, \ldots, n.$$

Hence, we have that $F(A)$ is relatively compact, which proves that F is compact. Now the result follows. □

Next, we present the generalization of Theorem 4.32, thus guaranteeing the existence of a global solution. Similar to the proof of such result, the proof of the following result is a slight modification of that of Theorem 5.39, and it is based on the fact that the local character of the solution in Theorem 4.32 is due to condition (5.20). Therefore, it is enough to impose a boundedness condition in terms of the corresponding integrable functions.

Theorem 5.40. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, be such that, for each $i \in \{1, 2, \ldots, n\}$, the following conditions are satisfied:*

(i) *For each $x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$, the map $f_i(\cdot, x(\cdot))$ is g_i–measurable.*

(ii) *There exists $h_i \in \mathcal{L}^1_{g_i}(I, [0, +\infty))$ such that*

$$|f_i(t, x)| \leq h_i(t), \quad g_i\text{–a.a. } t \in I, \quad x \in \mathbb{R}^n.$$

Then (5.9) *has a solution on I.*

Proof. First of all, note that the hypotheses imply that for each $i \in \{1, 2, \ldots, n\}$ and $x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$, the map $f_i(\cdot, x(\cdot)) \in \mathcal{L}^1_{g_i}(I, \mathbb{R})$ since

$$\int_I |f_i(s, x(s))| \, \mathrm{d}\, g_i(s) \leq \int_I h_i(s) \, \mathrm{d}\, g_i(s) < +\infty.$$

Thus, the map $F : \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n) \to \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$ defined as

$$Fx(t) = x_0 + \int_{[t_0, t)} f(s, x(s)) \, \mathrm{d}\, \boldsymbol{g}(s), \quad t \in \overline{I},$$

is well–defined. The rest of the proof follows the steps of the proof of Theorem 5.39, and we omit it. □

Remark 5.41. Observe that it is enough for f_i to be g_i–Carathéodory on $I\times\mathbb{R}$, $i=1,2,\dots,n$, for condition (i) in Theorem 5.40 to be satisfied. Indeed, let $i\in\{1,2,\dots,n\}$ and consider $x\in\mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$. It follows from Propositions 5.6 and 5.13 and Remark 5.14 that x is g_i–measurable. Hence, Proposition 1.28 ensures that $f_i(\cdot,x(\cdot))$ is g_i–measurable.

Our last existence result is an Osgood–type existence result. The proof of this result is somehow similar to that Theorem 4.36, in the sense that it can be obtained as a consequence of the corresponding Peano existence result. Note that if the hypotheses of Theorem 4.36 are satisfied, then the hypotheses of the following result are also satified for $\boldsymbol{g}=(g,g,\dots,g)$ and we obtain the same information on the existence of solution. Hence, this result is an extension of Theorem 4.36.

Theorem 5.42. *Let $r>0$ and $f:I\times\overline{B(x_0,r)}\to\mathbb{R}^n$ $f=(f_1,f_2,\dots,f_n)$, be such that, for each $i\in\{1,2,\dots,n\}$, the following conditions are satisfied:*

(i) *For each $x\in\overline{B(x_0,r)}$, $f_i(\cdot,x)$ is g_i–measurable.*

(ii) $f_i(\cdot,x_0)\in\mathcal{L}^1_{g_i}(I,\mathbb{R})$.

(iii) *There exist $\varphi_i\in\mathcal{L}^1_{g_i}(I,[0,+\infty))$ and $\omega_i:[0,+\infty)\to[0,+\infty)$ nondecreasing, continuous at 0 with $\omega_i(0)=0$ and such that*

$$|f_i(t,x)-f_i(t,y)|\le\varphi_i(t)\omega_i(\|x-y\|),\quad g_i\text{–a.a. } t\in I,\quad x,y\in\overline{B(x_0,r)}.$$

Then there exists $\tau\in(0,T]$ such that (5.9) *has a solution on I_τ.*

Proof. The hypotheses guarantee that each map f_i, $i=1,2,\dots,n$, satisfies the hypotheses of Lemma 4.35 for the corresponding map g_i. Hence, for each $i\in\{1,2,\dots,n\}$, the map f_i is g_i–Carathéodory and so, the result follows from Theorem 5.39. □

Uniqueness of solution

We now focus on results ensuring the uniqueness of solution. The first of such results is analogous to the Lipschitz uniqueness result, which in the context of (4.2), is Theorem 4.37. The following result provides a more general result and, as much as the proof of Theorem 4.37, its proof is based on our version of Gronwall's inequality for Stieltjes integrals, Proposition 4.17.

Theorem 5.43. *Let $\tau\in(0,T]$, $X\subset\mathbb{R}^n$, $x_0\in X$ and $f:I\times X\to\mathbb{R}^n$, $f=(f_1,f_2,\dots,f_n)$. If there exists $L:I_\tau\to[0,+\infty)$ such that, for each $i\in\{1,2,\dots,n\}$, $L\in\mathcal{L}^1_{g_i}(I_\tau,[0+\infty))$ and*

$$|f_i(t,x)-f_i(t,y)|\le L(t)\|x-y\|,\quad g_i\text{–a.a. } t\in I_\tau,\quad x,y\in X,$$

then (5.9) *has at most one solution on I_τ.*

Proof. Suppose that $x,y\in\mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}^n)$ are two solutions of (5.9) on I_τ. It follows from Theorem 3.27 that $f_i(\cdot,x(\cdot)),f_i(\cdot,y(\cdot))\in\mathcal{L}^1_{g_i}(I_\tau,\mathbb{R})$ for all $i\in\{1,2,\dots,n\}$. As a consequence, we have that $|f_i(\cdot,x(\cdot))-f_i(\cdot,y(\cdot))|$ is g_i–integrable on I_τ for all $i\in\{1,2,\dots,n\}$.

Define $u(t)=\|x(t)-y(t)\|$, $t\in\overline{I}_\tau$. Since each of the components of $x-y$ is Borel measurable –see Remark 5.14– and u is the pointwise maximum of Borel measurable maps, we have that u is Borel measurable. Therefore, we have that u is g_i–measurable for all $i \in \{1, 2, \dots, n\}$. Observe that this ensures that the map $u \cdot L$ is g_i–measurable for all $i \in \{1, 2, \dots, n\}$. As a consequence, we have that it is $\widehat{g}$–measurable for $\widehat{g}$ as in (5.5). Moreover, u is nonnegative and bounded on $\overline{I}_\tau$ as x and y are bounded. Hence, we have that $u, u \cdot L \in \mathcal{L}^1_{g_i}(I_\tau, [0, +\infty))$ for each $i \in \{1, 2, \dots, n\}$. Therefore, Proposition 1.17, statement (b) ensures that $u, u \cdot L \in \mathcal{L}^1_{\widehat{g}}(I_\tau, [0, +\infty))$. Now, the Fundamental Theorem of Calculus yields that for a given $t \in \overline{I}_\tau$,

$$\begin{aligned} u(t) &= |x_{j_0}(t) - y_{j_0}(t)| && \text{for some } j_0 \in \{1, 2, \dots, n\} \\ &= \left| \int_{[t_0, t)} (f_{j_0}(s, x(s)) - f_{j_0}(s, y(s)))\, \mathrm{d}\, g_{j_0}(s) \right| \\ &\leq \int_{[t_0, t)} |f_{j_0}(s, x(s)) - f_{j_0}(s, y(s))|\, \mathrm{d}\, g_{j_0}(s) \\ &\leq \int_{[t_0, t)} L(s) \|x(s) - y(s)\|\, \mathrm{d}\, g_{j_0}(s) \\ &\leq \sum_{i=1}^{n} \int_{[t_0, t)} L(s) \|x(s) - y(s)\|\, \mathrm{d}\, g_i(s) \\ &= \int_{[t_0, t)} L(s) u(s)\, \mathrm{d}\, \widehat{g}(s) && \text{by Proposition 1.17, (b).} \end{aligned}$$

Hence, (4.14) holds for $\widehat{g}$ with $K = 0$ which implies that $u = 0$ on $\overline{I}_\tau$, i.e. $x = y$ on that interval. □

Remark 5.44. Note that, for the particular case of $\boldsymbol{g} = (g, g, \dots, g)$ for a nondecreasing and left–continuous function $g : \mathbb{R} \to \mathbb{R}$, the conditions on the integrability of L are the same the ones in Theorem 4.37. Furthermore, in that case, the Lipschitz condition in Theorem 5.43 reads as

$$|f_i(t, x) - f_i(t, y)| \leq L(t) \|x - y\|, \quad g\text{–a.a. } t \in I_\tau, \quad x, y \in X, \quad i = 1, 2, \dots, n.$$

Now, since we are considering the $\max$–norm in $\mathbb{R}^n$, we have that this implies that the Lipschitz condition in Theorem 4.37 is satisfied. Hence, Theorem 5.43 is a generalization of Theorem 4.37.

The next existence result is a version of Osgood's uniqueness result adapted to the context of (5.9). Its proof is a slight modification of Theorem 4.39 using, once again, the map $\widehat{g} : \mathbb{R} \to \mathbb{R}$ in (5.5).

Theorem 5.45. *Let $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *For each $i \in \{1, 2, \dots, n\}$,*

$$|f_i(t, x) - f_i(t, y)| \leq \omega(\|x - y\|), \quad g_i\text{–a.a. } t \in I_\tau, \quad x, y \in X.$$

(ii) *For every* $u > 0$,

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{u}\frac{1}{\omega(s)}\,\mathrm{d}\,s=+\infty.$$

Then (5.9) *has at most one solution on* I_τ.

Proof. Let x and y be two solutions of (5.9) on I_τ, $\widehat{g}:\mathbb{R}\to\mathbb{R}$ be as in (5.5) and $u_0 > 0$. Define $\psi:\overline{I}_\tau\to[0,+\infty)$ as $\psi(t)=\|x(t)-y(t)\|$ and $\Omega:(0,+\infty)\to(0,+\infty)$ as

$$\Omega(r)=\int_{u_0}^{r}\frac{1}{\omega(s)}\,\mathrm{d}\,s,\quad r\in(0,+\infty).$$

First, note that, by an analogous argument to the one in the proof of Theorem 5.43, we have that $|f_i(\cdot,x(\cdot))-f_i(\cdot,y(\cdot))|$ is g_i–integrable on I_τ for each $i\in\{1,2,\dots,n\}$. Similarly, ψ is Borel measurable and, since ω is continuous, the composition $\omega\circ\psi$ is Borel measurable. This ensures that $\omega\circ\psi$ is g_i–measurable for all $i\in\{1,2,\dots,n\}$. Furthermore, it is also $\widehat{g}$–measurable for $\widehat{g}$ as in (5.5). Moreover, $\omega\circ\psi$ is integrable on I_τ with respect to $\widehat{g}$ and g_i, $i\in\{1,2,\dots,n\}$. Indeed, since $x,y\in\mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R})$, it follows that $x-y$ is bounded, and thus so is ψ. As a consequence, the map $\omega\circ\psi$ is bounded as well. This ensures that the map $\omega\circ\psi$ is g_i–integrable on I_τ for all $i\in\{1,2,\dots,n\}$, from which we have that $\omega\circ\psi\in\mathcal{L}^1_{\widehat{g}}(I_\tau,[0,+\infty))$, see Proposition 1.17, statement (b).

Now, let $K>0$ be an upper bound of $\omega\circ\psi$. Then, for each $\sigma\in(0,\tau)$,

$$\int_{[t_0,t_0+\sigma)}\omega(\psi(s))\,\mathrm{d}\,\widehat{g}(s)\le\int_{[t_0,t_0+\sigma)}K\,\mathrm{d}\,\widehat{g}(s)=K\mu_{\widehat{g}}([t_0,t_0+\sigma))<\widehat{\varepsilon}(\sigma),$$

where $\widehat{\varepsilon}(\sigma)=K\mu_{\widehat{g}}([t_0,t_0+\sigma))+\sigma>0$. Noting that $\widehat{\varepsilon}$ and ω are in the same circumstances as ε and ω in the proof of Theorem 4.39 with $\widehat{g}$ in place of g, we obtain that (4.43) holds in this setting. That is, there exists $0<R<\tau$ such that

$$\Omega(\widehat{\varepsilon}(\delta))+\widehat{g}(t_0+\tau)-\widehat{g}(t_0+\delta)<\beta:=\lim_{r\to\infty}\Omega(r)\quad\text{for }\ \delta\in(0,R).$$

Now, Theorem 3.27 yields that for each $t\in\overline{I}_\tau$,

$$\begin{aligned}
\psi(t)&=|x_{j_0}(t)-y_{j_0}(t)| && \text{for some } j_0\in\{1,2,\dots,n\}\\
&\le\int_{[t_0,t)}|f_{j_0}(s,x(s))-f_{j_0}(s,y(s))|\,\mathrm{d}\,g_{j_0}(s)\\
&\le\int_{[t_0,t)}\omega(\|x(s)-y(s)\|)\,\mathrm{d}\,g_{j_0}(s)\\
&\le\sum_{i=1}^{n}\int_{[t_0,t)}\omega(\|x(s)-y(s)\|)\,\mathrm{d}\,g_i(s)\\
&=\int_{[t_0,t)}\omega(\|x(s)-y(s)\|)\,\mathrm{d}\,\widehat{g}(s) && \text{by Proposition 1.17, (b).}\\
&=\int_{[t_0,t_0+\delta)}\omega(\psi(s))\,\mathrm{d}\,\widehat{g}(s)+\int_{[t_0+\delta,t)}\omega(\psi(s))\,\mathrm{d}\,\widehat{g}(s)\\
&<\widehat{\varepsilon}(\delta)+\int_{[t_0+\delta,t)}\omega(\psi(s))\,\mathrm{d}\,\widehat{g}(s)
\end{aligned}$$

for all $\delta \in (0,R)$. Therefore, the assumptions of Lemma 4.38 are satisfied and we have

$$\psi(t) \le \Omega^{-1}(\Omega(\varepsilon(\delta)) + \widehat{g}(t) - \widehat{g}(t_0+\delta)), \quad \delta \in (0,R), \quad t \in \overline{I}_\tau.$$

Applying Ω on both sides of the inequality, we obtain

$$\Omega(\psi(t)) - \Omega(\varepsilon(\delta)) \le \widehat{g}(t) - \widehat{g}(t_0+\delta) \le \widehat{g}(t) - \widehat{g}(t_0), \quad \delta \in (0,R), \quad t \in \overline{I}_\tau.$$

Assume that $\psi \neq 0$ on $\overline{I}_\tau$. If that is the case, there is some $t^* \in \overline{I}_\tau$ such that $\psi(t^*) > 0$. Then for all $\delta \in (0,R)$ such that $\delta < t^* - t_0$ we have

$$\int_{\varepsilon(\delta)}^{\psi(t^*)} \frac{1}{\omega(s)}\,\mathrm{d}\,s = \Omega(\psi(t^*)) - \Omega(\varepsilon(\delta)) < \widehat{g}(t^*) - \widehat{g}(t_0),$$

and, by taking the limit as $\delta \to 0^+$, we obtain

$$\lim_{\delta\to 0^+} \int_{\varepsilon(\delta)}^{\psi(t^*)} \frac{1}{\omega(s)}\,\mathrm{d}\,s < \widehat{g}(t^*) - \widehat{g}(t_0) < +\infty,$$

which contradicts (ii). Hence we must have $\psi = 0$ on $\overline{I}_\tau$, i.e. $x = y$ on that interval. □

Remark 5.46. Using analogous arguments to the ones in Remark 5.44 we can show that the Osgood conditions in Theorem 5.45 imply the Osgood condition in Theorem 4.39 when $\boldsymbol{g} = (g, g, \dots, g)$ for a nondecreasing and left–continuous map g.

Following the results in Chapter 4, we now present the corresponding version of Montel–Tonelli's uniqueness result for (5.9). Analogously to the proof of Theorem 4.41, the proof of the following result reduces to the proof of the Osgood–type result, Theorem 5.45, after using a transformation to a Kurzweil–Stieltjes integral.

Theorem 5.47. *Let $\tau \in (0,T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0,+\infty) \to [0,+\infty)$ be a nondecreasing and continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *There exists a map $\varphi : I_\tau \to [0,+\infty)$ satisfying that for each $i \in \{1,2,\dots,n\}$, $\varphi \in \mathcal{L}^1_{g_i}(I_\tau,[0,+\infty))$ and*

$$|f_i(t,x) - f_i(t,y)| \le \varphi(t)\omega(\|x-y\|), \quad g_i\text{–a.a. } t \in I_\tau, \quad x,y \in X.$$

(ii) *For every $u > 0$,*

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^u \frac{1}{\omega(s)}\,\mathrm{d}\,s = +\infty.$$

Then (5.9) *has at most one solution on I_τ.*

Proof. Let x and y be two solutions of (5.9) on I_τ. Define $\psi : \overline{I}_\tau \to [0,+\infty)$ as

$$\psi(t) = \|x(t) - y(t)\|, \quad t \in \overline{I}_\tau.$$

We can show, using the same arguments as in Theorem 5.45, that $\omega \circ \psi$ is bounded and Borel measurable. Therefore, given $K > 0$ an upper bound of $\omega \circ \psi$, we have that

$$|\omega(\psi(t))\varphi(t)| \leq K|\varphi(t)|, \quad t \in \overline{I}_\tau.$$

This ensures that $\varphi \cdot \omega \circ \psi \in \mathcal{L}^1_{g_i}(I_\tau, [0, +\infty))$ for all $i \in \{1, 2, \ldots, n\}$. Hence, it follows from Proposition 1.17, statement (b) that $\varphi \cdot \omega \circ \psi$ is $\widehat{g}$–integrable on $\overline{I}_\tau$ for $\widehat{g}$ as in (5.5).

Define $\overline{g} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \displaystyle\int_{[t_0,t)} \varphi(s) \, \mathrm{d}\,\widehat{g}(s), & \text{if } t_0 < t \leq t_0 + \tau, \\ \displaystyle\int_{[t_0,t_0+\tau)} \varphi(s) \, \mathrm{d}\,\widehat{g}(s), & \text{if } t > t_0 + \tau. \end{cases}$$

Observe that Proposition 1.17, statement (b) ensures that $\overline{g}$ is well–defined. Furthermore, by definition, $\overline{g}$ is nondecreasing and left–continuous everywhere. Recalling the relation between the integrals of Lebesgue–Stieltjes and Kurzweil–Stieltjes, for each $t \in \overline{I}_\tau$ we have

$$\begin{aligned} \int_{[t_0,t)} \omega(\psi(s))\varphi(s) \, \mathrm{d}\,\widehat{g}(s) &= {}^{(KS)}\!\!\int_{t_0}^{t} \omega(\psi(s))\varphi(s) \, \mathrm{d}\,\widehat{g}(s) \\ &= {}^{(KS)}\!\!\int_{t_0}^{t} \omega(\psi(s)) \, \mathrm{d}\,\overline{g}(s) = \int_{[t_0,t)} \omega(\psi(s)) \, \mathrm{d}\,\overline{g}(s), \end{aligned} \tag{5.21}$$

where the second equality is a consequence of the substitution formula for Kurzweil–Stieltjes integral, Theorem 1.69. Observe that the integral in (5.21) is well–defined in the Lebesgue–Stieltjes sense as the map $\omega \circ \psi$ is Borel measurable. Recall that we can assume, without loss of generality, that $\boldsymbol{g}$ is continuous at t_0, which implies that $\widehat{g}$ is also continuous at t_0. As a consequence, we have that $\overline{g}$ is also continuous at t_0. This, together with (5.21), implies that for each $\sigma \in (0, \tau)$,

$$\int_{[t_0,t_0+\sigma)} \omega(\psi(s)) \, \mathrm{d}\,\overline{g}(s) \leq \int_{[t_0,t_0+\sigma)} K \, \mathrm{d}\,\overline{g}(s) < K\mu_{\overline{g}}([t_0, t_0 + \sigma)) + \sigma =: \overline{\varepsilon}(\sigma).$$

Therefore, the result can be proved by reasoning as in the proof of Theorem 5.45 with the appropriate adjustments, i.e. replacing $\widehat{g}$ by $\overline{g}$ and $\widehat{\varepsilon}(\cdot)$ by $\overline{\varepsilon}(\cdot)$ accordingly. □

Remark 5.48. Once again, note that the arguments in Remark 5.44 adapted to this context show that Theorem 5.47 generalizes Theorem 4.41 in the sense that if the hypotheses of the latter are satisfied, the hypotheses of the former are also satisfied.

The last uniqueness results are some Perron–type result. First, we present two possible generalizations of Theorem 4.42, one of the uniqueness result that generalized Perron's ideas. The first of these results yields a new result even in the context of (4.2) as we will see later.

Theorem 5.49. *Let $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, and $\omega : \overline{I}_\tau \times [0, +\infty) \to [0, +\infty)$ be $(g_i \times \mathrm{Id})$–continuous for all $i \in \{1, 2, \ldots, n\}$. Assume that the following conditions are satisfied:*

(i) *For each* $i \in \{1, 2, \dots, n\}$,

$$|f_i(t,x) - f_i(t,y)| \le \omega(t, |x_i - y_i|), \quad g_i\text{–a.a. } t \in I_\tau, \quad x, y \in X.$$

(ii) *For each* $i \in \{1, 2, \dots, n\}$ *and* $r \in [0, +\infty)$, $\omega(\cdot, r) \in \mathcal{L}^1_{g_i}(I_\tau, [0 + \infty))$.

(iii) *For every* $t \in \bigcup_{i=1}^n D_{g_i} \cap \overline{I}_\tau$, *there exists* $\delta_t > 0$ *such that for all* $s \in (t - \delta_t, t + \delta_t) \cap \overline{I}_\tau$, $\omega(s, \cdot)$ *is nondecreasing.*

(iv) *For each* $i \in \{1, 2, \dots, n\}$, *the only* g_i*–absolutely continuous function on* $\overline{I}_\tau$ *satisfying*

$$z'_{g_i}(t) \le w(t, z(t)), \quad g_i\text{–a.a. } t \in I_\tau, \quad z(t_0) \le 0, \tag{5.22}$$

is the null function.

Then (5.9) *has at most one solution on* I_τ.

Proof. Let x and y be two solutions of (5.9) on I_τ. For each $i \in \{1, 2, \dots, n\}$, define $\psi_i(t) = |x_i(t) - y_i(t)|$, $t \in \overline{I}_\tau$. We will show that for each $i \in \{1, 2, \dots, n\}$, ψ_i satisfies (5.22), which then implies $x_i = y_i$, and thus, $x = y$.

Fix $i \in \{1, 2, \dots, n\}$. Clearly, $\psi_i(t_0) = 0$ and, by Proposition 3.28, $\psi_i \in \mathcal{AC}_{g_i}(\overline{I}_\tau, \mathbb{R})$. We claim that conditions (ii)-(iii) imply $\omega(\cdot, \psi_i(\cdot)) \in \mathcal{L}^1_{g_i}(I_\tau, [0, +\infty))$. First, Lemma 4.6 ensures that the composition is g_i–continuous, and in particular, g_i–measurable. Now, let

$$\mathcal{O} = \bigcup_{t \in \overline{I}_\tau \cap D_{g_i}} (t - \delta_t, t + \delta_t).$$

Note that $\mathcal{O}$ is open by definition, hence $\overline{I}_\tau \setminus \mathcal{O}$ is compact. The fact that g_i is continuous on $\overline{I}_\tau \setminus \mathcal{O}$, together with Proposition 3.21, yields that $\omega(\cdot, \psi_i(\cdot))$ is continuous on $\overline{I}_\tau \setminus \mathcal{O}$, therefore bounded. Let $M_i > 0$ be an upper bound of $\omega(\cdot, \psi_i(\cdot))$ on $\overline{I}_\tau \setminus \mathcal{O}$, and let $K_i > 0$ be an upper bound of ψ_i on $\overline{I}_\tau$. Then $\omega(s, \psi_i(s)) \le M_i + \omega(s, K_i)$, $s \in \overline{I}_\tau$, and so it follows that $\omega(\cdot, \psi_i(\cdot)) \in \mathcal{L}^1_{g_i}(I_\tau, [0, +\infty))$.

Define $\Psi_i : \overline{I}_\tau \to \mathbb{R}$ as

$$\Psi_i(t) = \int_{[t_0, t)} \omega(s, \psi_i(s)) \, \mathrm{d}\, g_i(s), \quad t \in \overline{I}_\tau. \tag{5.23}$$

By definition, $\Psi_i \in \mathcal{AC}_{g_i}(\overline{I}_\tau, \mathbb{R})$ and consequently it is g_i–differentiable for g_i–a.a. $t \in I_\tau$. Denote by E the set of points of I_τ where both derivatives ψ'_{g_i} and $(\Psi_i)'_{g_i}$ exist. If $t \in D_{g_i} \cap E$, then, by Remark 3.4, we have that

$$\begin{aligned}
(\psi_i)'_{g_i}(t) &= \frac{|x_i(t^+) - y_i(t^+)| - |x_i(t) - y_i(t)|}{\Delta g_i(t)} \\
&= \frac{|x_i(t) - y_i(t) + \big(f_i(t, x(t)) - f_i(t, y(t))\big)\Delta g_i(t)|}{\Delta g_i(t)} - \frac{|x_i(t) - y_i(t)|}{\Delta g_i(t)} \\
&\le |f_i(t, x(t)) - f_i(t, y(t))| \le \omega(t, |x_i(t) - y_i(t)|),
\end{aligned}$$

that is, $(\psi_i)'_{g_i}(t) \le \omega(t, \psi_i(t))$ for $t \in D_g \cap E$. On the other hand, for $t \in E \setminus D_{g_i}$, Theorem 3.27 yields that

$$\begin{aligned}
(\psi_i)'_{g_i}(t) &= \lim_{s \to t^+} \frac{\psi_i(s) - \psi_i(t)}{g_i(s) - g_i(t)} \\
&= \lim_{s \to t^+} \frac{|x_i(s) - y_i(s)| - |x_i(t) - y_i(t)|}{g_i(s) - g_i(t)} \\
&\le \lim_{s \to t^+} \frac{\left|x_i(s) - x_i(t) - \big(y_i(s) - y_i(t)\big)\right|}{g_i(s) - g_i(t)} \\
&= \lim_{s \to t^+} \frac{1}{g_i(s) - g_i(t)} \left| \int_{[t,s)} \big(f_i(r, x(r)) - f_i(r, y(r))\big) \, \mathrm{d}\, g_i(r) \right| \\
&\le \lim_{s \to t^+} \frac{1}{g_i(s) - g_i(t)} \int_{[t,s)} \omega(\tau, \psi_i(\tau)) \, \mathrm{d}\, g_i(\tau) \\
&= \lim_{s \to t+} \frac{\Psi_i(s) - \Psi_i(t)}{g_i(s) - g_i(t)} = (\Psi_i)'_{g_i}(t).
\end{aligned}$$

Noting that $(\Psi_i)'_{g_i}(t) = \omega(t, \psi_i(t))$ –see Theorem 3.26– we conclude that

$$(\psi_i)'_{g_i}(t) \le \omega(t, \psi_i(t)), \quad t \in E;$$

proving that ψ_i satisfies (5.22). □

Let us discuss the relations between Theorems 5.49 and 4.42 under the circumstances for which both results make sense i.e., when $\boldsymbol{g} = (g, g, \dots, g)$ for a nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$. First, note that condition (i) in Theorem 5.49 does not translate directly into condition (i) in Theorem 4.42. However, if

$$|f_i(t, x) - f_i(t, y)| \le \omega(t, |x_i - y_i|), \quad g\text{–a.a. } t \in I_\tau, \quad x, y \in X,$$

for all $i \in \{1, 2, \dots, n\}$, then we have that

$$\|f(t, x) - f(t, y)\| \le \max_{i = 1, 2, \dots, n} \{\omega(t, |x_i - y_i|)\}, \quad g\text{–a.a. } t \in I_\tau, \quad x, y \in X.$$

This does not imply, in general, condition (i) in Theorem 4.42. Conversely, it is clear that condition (i) in Theorem 4.42 does not imply, in general, condition (i) in Theorem 5.49. Hence, the two results are not comparable. For a direct generalization of Theorem 4.42 we include the following result which is based on the map $\widehat{g} : \mathbb{R} \to \mathbb{R}$ in (5.5).

Theorem 5.50. *Let $\tau \in (0, T]$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $f : I \times X \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\widehat{g} : \mathbb{R} \to \mathbb{R}$ be as in* (5.5). *Assume there exists a map $\omega : \overline{I}_\tau \times [0, +\infty) \to [0, +\infty)$ which is $(\widehat{g} \times \mathrm{Id})$–continuous map and satisfies the following conditions:*

(i) *For $\widehat{g}$–a.a. $t \in I_\tau$ and all $x, y \in X$,*

$$\|f(t, x) - f(t, y)\| \le \omega(t, \|x - y\|).$$

(ii) *For each* $i \in \{1, 2, \dots, n\}$ *and* $r \in [0, +\infty)$, $\omega(\cdot, r) \in \mathcal{L}^1_{g_i}(I_\tau, [0 + \infty))$.

(iii) *For every* $t \in D_{\widehat{g}} \cap \overline{I}_\tau$, *there exists* $\delta_t > 0$ *such that for all* $s \in (t - \delta_t, t + \delta_t) \cap \overline{I}_\tau$, $\omega(s, \cdot)$ *is nondecreasing.*

(iv) *The only* $\widehat{g}$*–absolutely continuous function on* $\overline{I}_\tau$ *satisfying*

$$z'_{\widehat{g}}(t) \le w(t, z(t)), \quad \widehat{g}\text{–a.a. } t \in I_\tau, \quad z(t_0) \le 0, \tag{5.24}$$

is the null function.

Then (5.9) *has at most one solution on* I_τ.

Proof. Let x and y be two solutions of (5.9) on I_τ and define $\psi(t) = \|x(t) - y(t)\|$, $t \in \overline{I}_\tau$. We will show that $\psi \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R})$ and it satisfies (5.24), which implies that $x = y$ on $\overline{I}_\tau$. Observe that, by definition, $\psi(t_0) = 0$, so in order to show that ψ satisfies (5.24), we only need to prove the differential inequality.

First, note that by definition we have that $x - y \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau, \mathbb{R}^n)$. Therefore, Remark 5.25 ensures that $x - y \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R}^n)$. Now, given that ψ is the pointwise maximum of $\widehat{g}$–absolutely continuous functions, we have that $\psi \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R}^n)$, see Proposition 3.29.

Now, let us show that ψ satisfies (5.24). Since $\psi \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R}^n)$, we have that ψ is $\widehat{g}$–continuous. Then, Lemma 4.6 ensures that $\omega(\cdot, \psi(\cdot))$ is $\widehat{g}$–continuous and Borel measurable. Now, repeating the arguments in Theorem 5.49, we can show that

$$\omega(s, \psi_i(s)) \le M + \omega(s, K), \quad s \in \overline{I}_\tau,$$

for some constants $M, K > 0$. This, once again, implies that $\omega(\cdot, \psi(\cdot))$ is g_i–integrable on I_τ for each $i \in \{1, 2, \dots, n\}$, and as a consequence, $\omega(\cdot, \psi(\cdot)) \in \mathcal{L}^1_{\widehat{g}}(I_\tau, [0, +\infty))$, see Proposition 1.17, statement (b).

Define $\Psi : \overline{I}_\tau \to \mathbb{R}$ as

$$\Psi(t) = \int_{[t_0, t)} \omega(s, \psi(s)) \, \mathrm{d}\,\widehat{g}(s), \quad t \in \overline{I}_\tau. \tag{5.25}$$

By definition, $\Psi \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R})$ and consequently it is $\widehat{g}$–differentiable for $\widehat{g}$–a.a. $t \in I_\tau$. Let us denote by E the set of points of I_τ where both derivatives $\psi'_{\widehat{g}}$ and $\Psi'_{\widehat{g}}$ exist. We shall show that (5.24) holds for all the points of E.

Let $t \in D_{\widehat{g}} \cap E$. By definition of the norm in $\mathbb{R}^n$, $\|x(t^+) - y(t^+)\| = |x_{i_0}(t^+) - y_{i_0}(t)|$ for some $i_0 \in \{1, 2, \dots, n\}$. Hence, since x, y are solutions of (5.9),

$$\begin{aligned}
\|x(t^+) - y(t^+)\| &= |x_{i_0}(t) - y_{i_0}(t) + (f_{i_0}(t, x(t)) - f_{i_0}(t, y(t)))\Delta g_{i_0}(t)| \\
&\le |x_{i_0}(t) - y_{i_0}(t)| + |f_{i_0}(t, x(t)) - f_{i_0}(t, y(t))|\Delta g_{i_0}(t) \\
&\le \|x(t) - y(t)\| + \|f(t, x(t)) - f(t, y(t))\|\Delta g_{i_0}(t) \\
&\le \|x(t) - y(t)\| + \|f(t, x(t)) - f(t, y(t))\|\Delta \widehat{g}(t).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\psi'_{\widehat{g}}(t) &= \frac{\|x(t^+)-y(t^+)\| - \|x(t)-y(t)\|}{\Delta\widehat{g}(t)}\\ &\le \frac{\|x(t)-y(t)\| + \|f(t,x(t))-f(t,y(t))\|\Delta\widehat{g}(t) - \|x(t)-y(t)\|}{\Delta\widehat{g}(t)}\\ &= \|f(t,x(t))-f(t,y(t))\| \le \omega(t,\psi(t)).\end{aligned}$$

On the other hand, if $t \in E \setminus D_{\widehat{g}}$,

$$\begin{aligned}\psi'_{\widehat{g}}(t) &= \lim_{s\to t^+} \frac{\|x(s)-y(t)\| - \|x(t)-y(t)\|}{\widehat{g}(s)-\widehat{g}(t)}\\ &\le \lim_{s\to t^+} \frac{\|x(s)-y(s)-\big(x(t)-y(t)\big)\|}{\widehat{g}(s)-\widehat{g}(t)}\\ &= \lim_{s\to t^+} \frac{1}{\widehat{g}(s)-\widehat{g}(t)} \left\| \int_{[t,s)} \big(f(r,x(r))-f(r,y(r))\big)\,\mathrm{d}\,\boldsymbol{g}(r) \right\|\\ &= \lim_{s\to t^+} \frac{1}{\widehat{g}(s)-\widehat{g}(t)} \max_{i=1,\dots,n} \left\{ \left| \int_{[t,s)} \big(f_i(r,x(r))-f_i(r,y(r))\big)\,\mathrm{d}\,g_i(r) \right| \right\}\\ &\le \lim_{s\to t^+} \frac{1}{\widehat{g}(s)-\widehat{g}(t)} \max_{i=1,\dots,n} \left\{ \int_{[t,s)} |f_i(r,x(r))-f_i(r,y(r))|\,\mathrm{d}\,g_i(r) \right\}.\end{aligned}$$

Now, for each $r \in I_\tau$, the hypotheses ensure that

$$|f_i(r,x(r))-f_i(r,y(r))| \le \|f(r,x(r))-f(r,y(r))| \le \omega(r,\psi(r)), \quad i=1,2,\dots,n.$$

As a consequence,

$$\begin{aligned}(\psi)'_{\widehat{g}}(t) &\le \lim_{s\to t^+} \frac{1}{\widehat{g}(s)-\widehat{g}(t)} \max_{i=1,\dots,n} \left\{ \int_{[t,s)} \omega(r,\psi(r))\,\mathrm{d}\,g_i(r) \right\}\\ &\le \lim_{s\to t^+} \frac{1}{\widehat{g}(s)-\widehat{g}(t)} \sum_{i=1}^{n} \int_{[t,s)} \omega(r,\psi(r))\,\mathrm{d}\,g_i(r)\\ &= \lim_{s\to t^+} \frac{1}{\widehat{g}(s)-\widehat{g}(t)} \int_{[t,s)} \omega(r,\psi(r))\,\mathrm{d}\,\widehat{g}(r)\\ &= \lim_{s\to t+} \frac{\Psi(s)-\Psi(t)}{\widehat{g}(s)-\widehat{g}(t)} = \Psi'_{\widehat{g}}(t).\end{aligned}$$

Theorem 3.26 ensures that $\Psi'_{\widehat{g}}(t) = \omega(t,\psi(t))$, which concludes the proof. □

Observe that conditions (iii) in Theorems 5.49 and 5.50 are the same. To see that, it is enough to note that

$$D_{\widehat{g}} = \bigcup_{i=1}^{n} D_{g_i}. \tag{5.26}$$

Similarly to Theorem 4.42, there is a way to avoid these conditions in a similar fashion to Theorem 4.43. In the following, we present two generalizations of Theorem 4.43 by making the corresponding changes to the Theorems 5.49 and 5.50.

Theorem 5.51. *Let* $\tau \in (0,T]$*,* $X \subset \mathbb{R}^n$*,* $x_0 \in X$*,* $f : I \times X \to \mathbb{R}^n$*, and* $\widehat{g} : \mathbb{R} \to \mathbb{R}$ *be as in* (5.5)*. Assume that there exists a* $\widehat{g}$*–Carathéodory map,* $\omega : \overline{I}_\tau \times [0,+\infty) \to [0,+\infty)$*, satisfying conditions* (i) *and* (iv) *in* Theorem 5.50*. Then* (5.9) *has at most one solution on* I_τ*.*

Proof. Let x and y be two solutions of (5.9) on I_τ and define $\psi(t) = \|x(t)-y(t)\|$, $t \in \overline{I}_\tau$. As in the proof of Theorem 5.50, we have that $\psi \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R})$. Therefore, ψ is $\widehat{g}$–measurable and bounded. Thus, Proposition 1.28 ensures that $\omega(\cdot, \psi(\cdot)) \in \mathcal{L}^1_{\widehat{g}}(I_\tau, [0,+\infty))$. Now, it is enough to define $\Psi : \overline{I}_\tau \to \mathbb{R}$ as in (5.25) and repeat the same arguments as in the proof of Theorem 5.50 to show that ψ satisfies (5.24). □

Theorem 5.51 is the most direct generalization of Theorem 4.43 in the same way that Theorem 5.50 generalizes Theorem 4.42. A different type of generalization can be obtained using Theorem 5.49. In that case, we obtain the following result, which provides new information even in the context of (4.2) in an analogous sense to Theorem 5.49.

Theorem 5.52. *Let* $\tau \in (0,T]$*,* $X \subset \mathbb{R}^n$*,* $x_0 \in X$ *and* $f : I \times X \to \mathbb{R}^n$*,* $f = (f_1, f_2, \dots, f_n)$*. Assume that there exists* $\omega : \overline{I}_\tau \times [0,+\infty) \to [0,+\infty)$ *satisfying conditions* (i) *and* (iv) *in* Theorem 5.49 *and such that* ω *is* g_i*–Carathéodory for each* $i \in \{1,2,\dots,n\}$*. Then* (5.9) *has at most one solution defined on* I_τ*.*

Proof. Let x and y be solutions of (5.9) on I_τ and $i \in \{1,2,\dots,n\}$. Define $\psi_i : \overline{I}_\tau \to \mathbb{R}$ as

$$\psi_i(t) = |x_i(t) - y_i(t)|, \quad t \in \overline{I}_\tau.$$

It follows from Proposition 3.28 that $\psi_i \in \mathcal{AC}_{g_i}(\overline{I}_\tau, \mathbb{R})$. Therefore, ψ_i is g_i–measurable and bounded. Thus, Proposition 1.28 ensures that $\omega(\cdot, \psi_i(\cdot))$ is g_i–integrable on I_τ. Now, it suffices to define $\Psi_i : \overline{I}_\tau \to \mathbb{R}$ as in (5.23) and repeat the same arguments as in the proof of Theorem 5.49 to show that ψ_i satisfies (5.22). □

Existence and uniqueness of solution

We finally present some existence and uniqueness results for problem (5.9). We start by presenting some of the results that can be obtained directly by combining the previous results. Our first result gives a sufficient condition for (5.9) to have a unique local solution. This result is obtained from combining Theorems 5.39 and 5.43.

Theorem 5.53. *Let* $r > 0$ *and* $f : I \times \overline{B(x_0, r)} \to \mathbb{R}^n$*,* $f = (f_1, f_2, \dots, f_n)$*, be such that for each* $i \in \{1,2,\dots,n\}$*, the map* f_i *is* g_i*–Carathéodory. If there exists a function* $L : I \to [0,\infty)$*, such that for every* $i = 1,2,\dots,n$*,* $L \in \mathcal{L}^1_{g_i}(I, [0,+\infty))$ *and*

$$|f_i(t,x) - f_i(t,y)| \le L(t)\|x-y\|, \quad g_i\text{–a.a. } t \in I, \quad x, y \in \overline{B(x_0, r)},$$

then there exists $\tau \in (0,T]$ *such that* (5.9) *has a unique solution on* I_τ*.*

Remark 5.54. At this point, we need to talk about [50, Theorem 4.4]. In this result, the condition of f_i being g_i–Carathéodory, $i \in \{1, 2, \dots, n\}$, is replaced with the following conditions:

1. for every $i = 1, 2, \dots, n$, and $x \in \overline{B(x_0, r)}$, the map $f_i(\cdot, x)$ is g_i–measurable;
2. for every $i = 1, 2, \dots, n$, $f_i(\cdot, x_0) \in \mathcal{L}^1_{g_i}(I, \mathbb{R})$;

However, as it is shown in Remark 4.45, this is equivalent to f_i being g_i–Carathéodory, $i = 1, 2, \dots, n$, under the Lipschitz condition of Theorem 5.53. Observe, nevertheless, that [50, Theorem 4.4] does not explicitly reflect on the $\widehat{g}$–measurability of the map L in the Lipschitz condition. Here, we have shown that it can be deduced from the hypothesis of the result, improving, in a sense, the proof of the result.

We can obtain another result guaranteeing the existence and uniqueness of a local solution of (5.9) by imposing conditions that ensure that the hypotheses of Theorems 5.42 and 5.47 are satisfied. This leads to the following Montel–Osgood–Tonelli existence and uniqueness result for (5.9).

Theorem 5.55. *Let $r > 0$, $f : I \times \overline{B(x_0, r)} \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *For each $x \in \overline{B(x_0, r)}$, $f_i(\cdot, x)$ is g_i–measurable.*

(ii) $f_i(\cdot, x_0) \in \mathcal{L}^1_{g_i}(I, \mathbb{R})$.

(iii) *For every $u > 0$,*

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^u \frac{1}{\omega(s)} \, \mathrm{d}\, s = +\infty.$$

(iv) *There exists a map $\varphi : I \to [0, +\infty)$ satisfying that for each $i \in \{1, 2, \dots, n\}$, $\varphi \in \mathcal{L}^1_{g_i}(I, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)| \le \varphi(t)\omega(\|x - y\|), \quad g_i\text{–a.a. } t \in I, \quad x, y \in \overline{B(x_0, r)}.$$

Then there exists $\tau \in (0, T]$ such that (5.9) *has a unique solution on I_τ.*

Remark 5.56. Note that for the particular choice of $\omega(r) = r$, $r \ge 0$, we obtain [50, Theorem 4.4]. Therefore, this theorem is a more general result than Theorem 5.53. Furthermore, if $\boldsymbol{g} = (g, g, \dots, g)$ for a nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$, then the hypotheses of Theorem 5.55 are the same as those of Theorem 4.46, thus providing a more general result.

We now turn our attention to the study of global solutions. In this framework, we can obtain a simple result by gathering the hypotheses of Theorems 5.40 and 5.43. Given that such results can be regarded as the extensions of Theorems 4.32 and 4.37 to the context of (5.9), respectively, the next result is a generalization of 4.48 to the context of $\boldsymbol{g}$–differential equations.

Theorem 5.57. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, satisfy the following conditions:*

(i) *For each $i \in \{1, 2, \dots, n\}$ and $x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$, the map $f_i(\cdot, x(\cdot))$ is g_i–measurable.*

(ii) *For each $i \in \{1, 2, \dots, n\}$, there exists $h_i \in \mathcal{L}^1_{g_i}(I, [0, +\infty))$ such that*

$$|f_i(t, x)| \le h_i(t), \quad g_i\text{–a.a. } t \in I, \quad x \in \mathbb{R}^n.$$

(iii) *There exists a map $L : I \to [0, +\infty)$, satisfying that, for each $i \in \{1, 2, \dots, n\}$, $L \in \mathcal{L}^1_{g_i}(I, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)\| \le L(t)\|x - y\|, \quad g_i\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then (5.9) *has a unique solution defined on I.*

Although this result provides some sufficient conditions on the existence of a unique solution of problem (5.9), there is a more general version of such result that can be found in [50]. This result is a generalization of Theorem 4.49 and it is proven in a fashion similar to that of the original paper, [33].

Theorem 5.58. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, satisfy the following conditions:*

(i) *For every $i = 1, 2, \dots, n$ and $x \in \mathbb{R}^n$, the map $f_i(\cdot, x)$ is g_i–measurable.*

(ii) *For every $i = 1, 2, \dots, n$, $f_i(\cdot, x_0) \in \mathcal{L}^1_{g_i}(I, \mathbb{R})$.*

(iii) *There exists a map $L : I \to [0, +\infty)$, satisfying that, for each $i \in \{1, 2, \dots, n\}$, $L \in \mathcal{L}^1_{g_i}(I, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)| \le L(t)\|x - y\|, \quad g_i\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then (5.9) *has a unique solution on I.*

Proof. Define $F : \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n) \to \mathcal{BC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$ as the map given by

$$Fx(t) = x_0 + \int_{[t_0, t)} f(s, x(s)) \, \mathrm{d}\boldsymbol{g}(s), \quad t \in \overline{I}.$$

Note that hypotheses (i)-(iii) imply that for each $i = 1, 2, \dots, n$, f_i is g_i–Carathéodory, and so the corresponding F is well-defined (see the proof of Theorem 5.39). Hence, noting that x is a solution of (5.9) if and only if it is a fixed point of F, proving the result is equivalent to showing that F has a unique fixed point.

Let L be the function given by hypothesis (iii). Observe that the hypotheses and Proposition 1.17, statement (b), ensure that $L \in \mathcal{L}^1_{\widehat{g}}(I, [0, +\infty))$ for $\widehat{g}$ as in (5.5). Let us show that L satisfies (4.7) and (4.8) for $\widehat{g}$. Indeed, first of all note that (4.7) is trivially satisfied as L is nonnegative. Now, for (4.8), first note that for each $i = 1, 2, \dots, n$,

$$\sum_{t \in I \cap D_{g_i}} L(s)\Delta g_i(t) = \int_{I \cap D_{g_i}} L(s) \, \mathrm{d}\, g_i(s) \le \int_I L(s) \, \mathrm{d}\, g_i(s). \tag{5.27}$$

Hence, since $L \in \mathcal{L}^1_{g_i}(I,[0,+\infty))$ for each $i \in \{1,2,\dots,n\}$, we obtain that the sum in (5.27) is finite for every $i \in \{1,2,\dots,n\}$. Therefore, recalling the relation in (5.26) and given that $\log(1+s)$ is subadditive for $s \geq 0$ and $\log(1+s) \leq s$ for every $s \geq -1$, we have that

$$\begin{aligned}\sum_{t\in I\cap D_{\widehat{g}}} |\log(1+L(t)\Delta\widehat{g}(t))| &= \sum_{t\in I\cap D_{\widehat{g}}} \log(1+L(t)\Delta\widehat{g}(t))\\ &\leq \sum_{t\in I\cap D_{\widehat{g}}} \log(n+L(t)\Delta\widehat{g}(t))\\ &\leq \sum_{t\in I\cap D_{\widehat{g}}} \log\left(\sum_{i=1}^n (1+L(t)\Delta g_i(t))\right)\\ &\leq \sum_{t\in I\cap D_{\widehat{g}}} \left(\sum_{i=1}^n \log(1+L(t)\Delta g_i(t))\right)\\ &\leq \sum_{i=1}^n \left(\sum_{t\in I\cap D_{g_i}} \log(1+L(t)\Delta g_i(t))\right).\end{aligned}$$

Hence, the sum is finite and (4.8) is satisfied. Thus, the map $e_L(\cdot,t_0)$ is well–defined so we can define the following norm on $\mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$:

$$\|x\|_L = \sup_{t\in\overline{I}} \frac{\|x(t)\|}{e_L(t,t_0)}, \quad x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n).$$

Note that, since $e_L(\cdot,t_0)$ is nondecreasing and $e_L(t_0,t_0)=1$, the norms $\|\cdot\|_L$ and $\|\cdot\|_0$ are equivalent. as

$$\|x\|_L = \sup_{t\in\overline{I}} \frac{\|x(t)\|}{e_L(t,t_0)} \leq \sup_{t\in\overline{I}} \|x(t)\| = \|x\|_0 \leq e_L(t_0+T,t_0)\|x\|_L.$$

Therefore, $(\mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n), \|\cdot\|_L)$ is a Banach space. As a consequence, it suffices to show that F is a contraction to finish the proof via Banach's contraction principle.

To prove that F is a contraction we fix $x,y \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$. In that case, we have that $x-y \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$ and therefore, each of its components is Borel measurable, see Remark 5.14. Therefore, since the map $\|x-y\|$ is the pointwise maximum of Borel measurable maps, we have that it is Borel measurable. As a consequence, $\|x-y\|$ is $\widehat{g}$ and g_i–measurable for each $i \in \{1,2,\dots,n\}$. Furthermore, given that $x-y \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$, it follows that it is integrable with respect to the measures defined by $\widehat{g}$ and g_i for each $i \in \{1,2,\dots,n\}$. Now, observe that for each $t \in \overline{I}$,

$$\begin{aligned}\|Fx(t)-Fy(t)\| &= \left\| \int_{[t_0,t)} (f(s,x(s))-f(s,y(s)))\,\mathrm{d}\,\boldsymbol{g}(s)\right\|\\ &= \left| \int_{[t_0,t)} (f_{j_0}(s,x(s))-f_{j_0}(s,y(s)))\,\mathrm{d}\,g_{j_0}(s)\right|,\end{aligned}$$

for some $j_0 \in \{1, 2, \dots, n\}$ and therefore, condition (iii) yields

$$\begin{aligned}
\|Fx(t) - Fy(t)\| &\leq \int_{[t_0,t)} L(s)\|x(s) - y(s)\| \, \mathrm{d}\, g_{j_0}(s) \\
&\leq \sum_{i=1}^{n} \int_{[t_0,t)} L(s)\|x(s) - y(s)\| \, \mathrm{d}\, g_i(s) \\
&= \int_{[t_0,t)} L(s)\|x(s) - y(s)\| \, \mathrm{d}\, \widehat{g}(s) && \text{by Proposition 1.17, (b)} \\
&\leq \|x - y\|_L \int_{[t_0,t)} L(s) e_L(s, t_0) \, \mathrm{d}\, \widehat{g}(s) \\
&= \|x - y\|_L (e_L(t, t_0) - 1) && \text{by Lemma 4.13.}
\end{aligned}$$

Hence,

$$\|Fx - Fy\|_L = \sup_{t \in \overline{I}} \frac{\|Fx(t) - Fy(t)\|}{e_L(t, t_0)} \leq (1 - (e_L(t_0 + T, t_0))^{-1})\|x - y\|_L.$$

It is clear that $1 - (e_L(t_0 + T, t_0))^{-1} < 1$, and so F is a contraction. Thus, it has a unique fixed point, i.e., problem (5.9) has a unique solution. □

Remark 5.59. The proof of Theorem 5.58 is a modification of the one for [50, Theorem 4.3]. Essentially, the only difference between the two results is the fact that in Theorem 5.58 there is no discussion about the $\widehat{g}$–measurability of the maps involved in the proof, whereas here we have properly justified it.

Following [50], we illustrate the applicability of Theorem 5.58 in the following example, while also showing the real world interest of $\boldsymbol{g}$–differential equations. In particular, we consider a model for a fish population subject to harvesting in the form of fishing with open and closed seasons.

Example 5.60. Let $p(t)$ be the number of a fish species at time t and let $h(t)$ be the number of fished individuals of p at time t. We would like to propose a mathematical model for both, p and h, under the assumptions that population p only reproduces during certain periods of time and that fishing is allowed during some seasons of the year – open seasons– and forbidden at any other time –closed seasons. These fishing restrictions are imposed in order to protect the animals during their reproductive cycle and their newborns. Hence, if we identify each year with and interval of the form $(4k - 4, 4k]$, $k \in \mathbb{N}$, we can suppose that at times $t = 4k - 1$, $k \in \mathbb{N}$, new fishes are born. Then, considering the intervals $(4k - 2, 4k]$, $k \in \mathbb{N}$, as an estimation for the closed season could be reasonable.

Prior to proposing a first model for this situation, we need to choose two left–continuous and nondecreasing functions, $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, as derivators for the corresponding system of equations. If g_1 is the derivator of p, we need for g_1 to present jump discontinuities at times $t = 4k - 1$, $k \in \mathbb{N}$, since we assume that fish are born so fast that it can be represented by an impulse. On the other hand for the derivator for h, g_2, we need jump discontinuities at times $t = 4k - 2$, $k \in \mathbb{N}$, when the fishing stops completely, and at $t = 4k$, $k \in \mathbb{N}$, where fishing resumes. We also require g_2 to be constant on $(4k-2,4k]$, $k\in\mathbb{N}$, as there are no changes due to closed season. Further conditions could be asked for g_1 and g_2. For example, a greater slope of the derivator represents a greater importance of the proccess happening during that period of time. For the fish population, it could represent a greater death rate right after they are born. For both p and h, a greater slope at the beginnig of open season could be understood as a bigger number of fished individuals due to the size of the population being bigger. Bearing this in mind, one possible option could be

$$g_1(t)=\begin{cases}0, & \text{if } t\le 0,\\ \sqrt{t(6-t)}, & \text{if } t\in[0,3],\\ 4+\sqrt{1-(t-4)^2}, & \text{if } t\in(3,4],\end{cases}\qquad g_2(t)=\begin{cases}0, & \text{if } t\le 0,\\ \sqrt{t(4-t)}, & \text{if } t\in[0,2],\\ 3, & \text{if } t\in(2,4],\end{cases}$$

and $g_1(t)=5+g_1(t-4)$, $g_2(t)=4+g_2(t-4)$ for $t>4$. Note that both g_1 and g_2 have jump discontinuities of length 1 at their corresponding discontinuity points.

We know want to establish a system of differential equations representing the behaviour of the fish population, p, and the fishing, h. In order to do so, we include a list of important features that the model should present:

(i) Naturally, when a new generation of fish is born, the fish population increases, which requires that $p'_{g_1}>0$ at those times. Between two consecutives new generations of fish, the fish population must decay as a result of individuals dying by natural causes or by fishing. This is represented in the model by imposing $p'_{g_1}<0$ at the corresponding times.

(ii) Similarly, the fishing should also decay from the during open seasons as a consequence of the decay of resources available caused by the fishing itself and the natural deaths of fish. This translates into imposing $h'_{g_2}<0$ during open seasons. Once closed seasons start, the fishing stops immediately, for which we will impose a collapse condition on the model. After those points, the fishing remains constant as a consequence of g_2 being constant during closed seasons. Analogously, at the beginning of open seasons h should increase as a result of fishing being allowed again. This is translated into the model by imposing $h'_{g_2}>0$ at those points.

Let $I=[0,a)$, $0<a<+\infty$. For $t\in I$, we consider the following Stieltjes differential system:

$$\begin{cases} p'_{g_1}(t) &=& f_1(t,p(t),h(t)), \quad p(0)=p_0,\\ h'_{g_2}(t) &=& f_2(t,p(t),h(t)), \quad h(0)=h_0,\end{cases} \tag{5.28}$$

where $f_1,f_2:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given by

$$f_1(t,p,h)=\begin{cases} c_p p, & \text{if } t=4k-1, k\in\mathbb{N},\\ -\alpha p-\beta p h, & \text{otherwise,}\end{cases}$$

$$f_2(t,p,h)=\begin{cases} -h, & \text{if } t=4k-2, k\in\mathbb{N},\\ c_h p-h, & \text{if } t=4k, k\in\mathbb{N},\\ -\gamma p, & \text{otherwise.}\end{cases}$$

Here, the parameter $\alpha>0$ represent the death rate of the fish population caused by the environment they live in, $\beta>0$ is the fishing rate and $\gamma>0$, a decay rate for the fishing.

Finally, $0 < c_p, c_h < 1$ are two proportionality constants: one related to the number of newborns of species p and the other one, related to the number of fished individuals at the beginning of open season. This can easily be seen by considering the system at those points. For example, for $t = 4k-1$, $k \in \mathbb{N}$,

$$c_p p(4k-1) = p'_{g_1}(4k-1) = p(4k-1^+) - p(4k-1)$$

from which we obtain that $p(4k-1^+) = (1+c_p)p(4k+1)$.

Let us denote $\boldsymbol{g} = (g_1, g_2)$, $f = (f_1, f_2)$, $x(t) = (p(t), h(t))$ and $x_0 = (p_0, h_0)$. Then (5.28) can be regarded as the $\boldsymbol{g}$–differential equation

$$x'_{\boldsymbol{g}}(t) = f(t, x(t)), \quad t \in I, \quad x(0) = x_0. \tag{5.29}$$

Noting that (5.29) is an autonomous system with f a polynomial function, it is inmediate that hypotheses of Theorem 5.58 are satisfied. Therefore, (5.28) has a unique solution on I.

An adaptation of the Montel–Osgood–Tonelli existence and uniqueness result for (4.2), Theorem 4.51, can be obtained in the more general context of (5.9). Such result can be obtained by making small modifications to the proof of Theorem 4.51. This requires us to introduce the following lemma, which is, in some sense, a generalization of Lemma 4.50.

Lemma 5.61. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that $\boldsymbol{g}$ is continuous at t_0 and that the following conditions are satisfied:*

(i) *For every $i = 1, 2, \dots, n$ and $x \in \mathbb{R}^n$, the map $f_i(\cdot, x)$ is g_i–measurable.*

(ii) *For each $i \in \{1, 2, \dots, n\}$, $f_i(\cdot, x_0) \in \mathcal{L}^1_{g_i}(I, \mathbb{R})$.*

(iii) *For every $u > 0$,*

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^u \frac{1}{\omega(s)} \, \mathrm{d}\, s = +\infty.$$

(iv) *There exists a map $\varphi : I \to [0, +\infty)$ satisfying that for each $i \in \{1, 2, \dots, n\}$, $\varphi \in \mathcal{L}^1_{g_i}(I, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)| \le \varphi(t)\omega(\|x - y\|), \quad g_i\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then there exist $t_1 \in (t_0, t_0 + T]$ and a nondecreasing function $h : [t_0, t_1] \to \mathbb{R}$ such that for every solution of (5.9)*, $x : I_\tau \to \mathbb{R}^n$, $\tau \in (0, T]$, we have*

$$\|x(t) - x_0\| \le h(t), \quad t \in I_\tau \cap [t_0, t_1].$$

Proof. Define $\kappa : \overline{I} \to \mathbb{R}$ as

$$\kappa(t) = \sum_{i=1}^n \int_{[t_0, t)} |f_i(s, x_0)| \, \mathrm{d}\, g_i(s), \quad t \in \overline{I}.$$

Note that hypothesis (ii) ensures that κ is well–defined. Let $x : \overline{I}_\tau \to \mathbb{R}^n$ be a solution of (5.9). Remark 5.25 ensures that $x - x_0 \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R}^n)$. Now, Proposition 3.29 guarantees that $\|x - x_0\| \in \mathcal{AC}_{\widehat{g}}(\overline{I}_\tau, \mathbb{R}^n)$, and so, it is Borel measurable. Given that ω is continuous, it follows that $\omega(\|x - x_0\|)$ is Borel measurable and thus, $\widehat{g}$ and g_i–measurable for each $i \in \{1, 2, \dots, n\}$.

Now, for each $t \in \overline{I}_\tau$, there exists $j_t \in \{1, 2, \dots, n\}$ such that

$$\|x(t) - x_0\| = |x_{j_t}(t) - x_{0,j_t}(t)| = \left| \int_{[t_0,t)} f_{j_t}(s, x(s))\, \mathrm{d}\, g_{j_t}(s) \right|.$$

Hence, condition (iv) yields that for each $t \in \overline{I}_\tau$,

$$\begin{aligned}
\|x(t) - x_0\| &\leq \int_{[t_0,t)} |f_{j_t}(s, x(s))|\, \mathrm{d}\, g_{j_t}(s) \\
&\leq \int_{[t_0,t)} |f_{j_t}(s, x_0)|\, \mathrm{d}\, g_{j_t}(s) + \int_{[t_0,t)} |f_{j_t}(s, x(s)) - f_{j_t}(s, x_0)|\, \mathrm{d}\, g_{j_t}(s) \\
&\leq \sum_{i=1}^{n} \int_{[t_0,t)} |f_i(s, x_0)|\, \mathrm{d}\, g_i(s) + \int_{[t_0,t)} \varphi(s)\omega(\|x(s) - x_0\|)\, \mathrm{d}\, g_{j_t}(s) \\
&\leq \kappa(t) + \sum_{i=1}^{n} \int_{[t_0,t)} \varphi(s)\omega(\|x(s) - x_0\|)\, \mathrm{d}\, g_i(s) \\
&= \kappa(t) + \int_{[t_0,t)} \varphi(s)\omega(\|x(s) - x_0\|)\, \mathrm{d}\, \widehat{g}(s).
\end{aligned}$$

Define $\overline{g} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \displaystyle\int_{[t_0,t)} \varphi(s)\, \mathrm{d}\, \widehat{g}(s), & \text{if } t_0 < t \leq t_0 + \tau, \\ \displaystyle\int_{[t_0,t_0+\tau)} \varphi(s)\, \mathrm{d}\, \widehat{g}(s), & \text{if } t > t_0 + \tau. \end{cases}$$

Recalling the relation (5.21), it follows that

$$\|x(t) - x_0\| \leq \kappa(t) + \int_{[t_0,t)} \omega(\|x(s) - x_0\|)\, \mathrm{d}\, \overline{g}(s), \quad t \in \overline{I}_\tau, \tag{5.30}$$

for every solution x of (5.9) defined on $\overline{I}_\tau$. In order to apply Lemma 4.38, fix an arbitrary $u_0 > 0$ and consider the function

$$\Omega(r) = \int_{u_0}^{r} \frac{1}{\omega(s)}\, \mathrm{d}\, s, \quad r \in (0, +\infty).$$

Since $\lim_{r \to 0^+} \Omega(r) = -\infty$, there exists $R > 0$ such that

$$\Omega(R) + \overline{g}(t_0 + T) - \overline{g}(t_0) < \beta := \lim_{r \to \infty} \Omega(r) \leq +\infty.$$

Since $\boldsymbol{g}$ is continuous at t_0, we have that, for each $i \in \{1,2,\dots,n\}$, g_i is continuous at t_0 as well. Then, we can choose $t_1 \in (t_0, t_0+T]$ such that $\kappa(t_1) \leq R$. The monotonicity of Ω then yields

$$\Omega(\kappa(t_1)) + \overline{g}(t_1) - \overline{g}(t_0) < \beta.$$

The inequality above together with (5.30) shows that the assumptions of Lemma 4.38 are satisfied on the interval $[t_0, t_1]$, therefore

$$\|x(t) - x_0\| \leq \Omega^{-1}(\Omega(\kappa(t_1)) + \overline{g}(t) - \overline{g}(t_0)) =: h(t), \quad t \in I_\tau \cap [t_0, t_1],$$

and $h : [t_0, t_1] \to \mathbb{R}$ is the desired monotone function. □

Now, we can state and prove the following Montel–Osgood–Tonelli existence and uniqueness of solution result for problem (5.9). It is important to note that this result reduces to Theorem 4.51 when (5.9) reduces to (4.2). Observe that, although we are imposing global conditions on $\mathbb{R}^n$, we can only ensure the existence of a unique local solution as in Theorem 4.51. In that sense, Theorem 5.62 is only a partial improvement to Theorem 5.58.

Theorem 5.62. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0,+\infty) \to [0,+\infty)$ be a nondecreasing and continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$. Assume that the following conditions are satisfied:*

(i) *For every $i \in \{1,2,\dots,n\}$ and $x \in \mathbb{R}^n$, the map $f_i(\cdot, x)$ is g_i–measurable.*

(ii) *For every $i \in \{1,2,\dots,n\}$, $f_i(\cdot, x_0) \in \mathcal{L}^1_{g_i}(I, \mathbb{R})$.*

(iii) *For every $u > 0$,*

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^u \frac{1}{\omega(s)} \, \mathrm{d}\, s = +\infty.$$

(iv) *There exists a map $\varphi : I \to [0,+\infty)$ satisfying that for each $i \in \{1,2,\dots,n\}$, $\varphi \in \mathcal{L}^1_{g_i}(I, [0,+\infty))$ and*

$$|f_i(t,x) - f_i(t,y)| \leq \varphi(t)\omega(\|x-y\|), \quad g_i\text{–a.a. } t \in I, \quad x, y \in \mathbb{R}^n. \tag{5.31}$$

Then there exists $\tau \in (0, T]$ such that (5.9) *has a unique solution on I_τ.*

Proof. Without loss of generality, we assume that $\boldsymbol{g}$ is continuous at t_0, see Remark 5.38. Let $h : [t_0, t_1] \to \mathbb{R}$ be the function whose existence is guaranteed by Lemma 5.61 and denote $R = h(t_1)$. Define $\overline{g} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \displaystyle\int_{[t_0,t)} \varphi(s) \, \mathrm{d}\, \widehat{g}(s), & \text{if } t_0 < t \leq t_0 + \tau, \\ \displaystyle\int_{[t_0,t_0+\tau)} \varphi(s) \, \mathrm{d}\, \widehat{g}(s), & \text{if } t > t_0 + \tau. \end{cases}$$

Since $\boldsymbol{g}$ is continuous at t_0, we have that so are g_i, $i \in \{1,2,\dots,n\}$ and $\widehat{g}$. As a consequence, $\overline{g}$ is continuous at t_0. Thus, we can choose $\tau \in (0,T]$ such that $t_0+\tau \le t_1$ and

$$\omega(R)\mu_{\overline{g}}([t_0,t_0+\tau)) + \sum_{i=1}^{n} \int_{[t_0,\tau)} |f_i(s,x_0)| \,\mathrm{d}\, g_i(s) < R. \tag{5.32}$$

Consider $B = \left\{x \in \mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}^n) : \|x(t)-x_0\| \le R,\ t \in \overline{I}_\tau\right\}$. Clearly, B is a closed and convex subset of $\mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}^n)$. Now let us define $F : B \to \mathcal{BC}_g(\overline{I}_\tau,\mathbb{R}^n)$ by

$$Fx(t) = x_0 + \int_{[t_0,t)} f(s,x(s)) \,\mathrm{d}\, \boldsymbol{g}(s), \quad t \in I_\tau.$$

It follows from Lemma 4.35 that each f_i is g_i–Carathéodory. Furthermore, given $x \in B$, we have that x is Borel measurable and therefore, g_i–measurable for each $i \in \{1,2,\dots,n\}$. Thus Proposition 1.28 and Theorem 3.26 ensure that F is well-defined. Moreover, the continuity of ω together with condition (iv) implies that F is continuous. Furthermore, following a similar argument as the one used to obtain (5.30), and using the relation (5.21), it follows from condition (iv) and (5.32) that for $x \in B$,

$$\|Fx(t)-x_0\| \le \int_{[t_0,t)} \omega(\|x(s)-x_0\|) \,\mathrm{d}\, \overline{g}(s) + \sum_{i=1}^{n} \int_{[t_0,\tau)} |f_i(s,x_0)| \,\mathrm{d}\, g_i(s) < R,$$

for every $t \in \overline{I}_\tau$. That is, $F(B) \subset B$. It remains to verify that $F(B)$ is relatively compact in $\mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}^n)$. Firstly, note that for $x \in B$ and $i \in \{1,2,\dots,n\}$ we have

$$|f_i(t,x(t))| \le |f_i(t,x(t)) - f_i(t,x_0)| + |f_i(t,x_0)| \le \varphi(t)\omega(R) + |f_i(t,x_0)| =: M_i(t),$$

for g_i–a.a. $t \in I_\tau$. Observe that $F(B) \subset \mathcal{AC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}^n)$ and

$$((Fx)_i)'_{g_i}(t) = f_i(t,x(t)), \quad g_i\text{–a.a. } t \in I_\tau, \quad x \in B, \quad i = 1,2,\dots,n.$$

Since $M_i \in \mathcal{L}^1_{g_i}(I_\tau,[0,+\infty))$ for each $i \in \{1,2,\dots,n\}$, it follows from Proposition 5.26 that $F(B)$ is relatively compact in $\mathcal{BC}_{\boldsymbol{g}}(\overline{I}_\tau,\mathbb{R}^n)$. Thus, Schauder's Fixed Point Theorem guarantees the existence of solution of (5.9) on I_τ, while the uniqueness is a consequence of Theorem 5.45. □

For an application of Theorem 5.62, it is enough to consider a slight modification of Example 4.53 requiring the integrable map to be integrable with respect to all the different derivators.

Example 5.63. Let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1,g_2,\dots,g_n)$, be such that for each $i \in \{1,2,\dots,n\}$, the map $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous; and $\varphi : [0,1) \to [0,+\infty)$ be a Borel measurable map such that $\varphi \in \mathcal{L}^1_{g_i}([0,1),[0,+\infty))$ for each $i \in \{1,2,\dots,n\}$. For each $k \in \mathbb{N}$ we consider the initial value problem

$$x'_{\boldsymbol{g}}(t) = f_k(t,x(t)), \quad g\text{–a.a. } t \in [0,1), \quad x(0) = x_0, \tag{5.33}$$

where $f_k : [0,1) \times \mathbb{R}^n \to \mathbb{R}^n$, $f_k = (f_{k,1}, f_{k,2}, \dots, f_{k,n})$, is given by

$$f_{k,i}(t,x) = \varphi(t)\,\omega_k(\|x\|), \quad (t,x) \in [0,1) \times \mathbb{R}, \quad i \in \{1,2,\dots,n\},$$

with ω_k as in (4.57). Similarly to Example 4.53, we have that each $f_{k,i}$, $i \in \{1,2,\dots,n\}$, does not satisfy a Lipschitz's condition on the whole $\mathbb{R}$ as the derivative of ω_k is unbounded on any neighbourhood of 0, see Theorem 4.52 assertion (c), and therefore the hypotheses of Theorem 5.58 are not satisfied. Similarly, Theorem 5.53 cannot be applied for any ball around x_0 containing 0. In particular, if $x_0 = 0$ we can never assure the existence of a local solution for this problem through this results. However, the Montel–Osgood–Tonelli result, Theorem 5.62, yields that (5.33) has a unique solution. To see that this is the case, it is enough to repeat the arguments in Example 4.53 noting that φ is $\widehat{g}$–measurable as it is Borel measurable.

5.3 Vectorial measure differential equations

Measure differential equations, in the sense introduced in [27], are integral equations of the form

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f(s,x(s))\,\mathrm{d}\,g(s), \tag{5.34}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing and left–continuous function and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In Theorem 4.57, we presented an existence result for such problem. Herein we propose a more general version of such an equation where not only f can be a vectorial function but also the integrator g. More precisely, we are interested on equations of the form

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f(s,x(s))\,\mathrm{d}\boldsymbol{g}(s), \tag{5.35}$$

with $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, such that each map $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,n$, is nondecreasing and left–continuous; $t_0 \in \mathbb{R}$, $T > 0$, $x_0 \in \mathbb{R}^n$ and $f : [t_0, t_0+T] \times \mathbb{R}^n \to \mathbb{R}^n$.

As usual, we start by introducing the basic definitions for problem (5.35). In what follows, we shall denote $I = [t_0, t_0+T)$ and $\overline{I} = [t_0, t_0+T]$.

Definition 5.64. *A* solution *of* (5.35) *on* $\overline{I}$ *is a map* $x \in G(\overline{I}, \mathbb{R}^n)$ *such that*

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f(s,x(s),x)\,\mathrm{d}\boldsymbol{g}(s), \quad t \in \overline{I}.$$

A solution of (5.35) *on* $\overline{I}$*,* x^* *is said to be the* greatest solution of (5.35) *on* $\overline{I}$ *if for any other solution on* $\overline{I}$*,* y*, we have that* $y \le x^*$*. Similarly, a solution of* (5.35) *on* $\overline{I}$*,* x_**, is said to be the* least solution of (5.35) *on* $\overline{I}$ *if for any other solution on* $\overline{I}$*,* y*, we have that* $x_* \le y$*.*

Remark 5.65. Note that these definitions provide the corresponding definitions for (5.34) as well by considering the case $n = 1$. Also, observe that as a consequence of Theorem 1.68, we have that the solutions of (5.35) share, somehow, the discontinuity points of $\boldsymbol{g}$.

In a similar fashion, we can define the concepts of lower and upper solution of (5.35), as well as those of extremal solutions between a well–ordered pair of lower and upper solutions. In order to do so, we introduce the following notation: given $\alpha, \beta : \overline{I} \to \mathbb{R}^n$ such that $\alpha \leq \beta$, we define the functional interval

$$[\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)} = \{\gamma \in G(\overline{I}, \mathbb{R}^n) \,:\, \alpha \leq \gamma \leq \beta\}.$$

With this notation, we introduce the following definition.

Definition 5.66. *A* lower solution *of* (5.35) *on* $\overline{I}$ *is a function* $\alpha \in G(\overline{I}, \mathbb{R}^n)$ *such that* $\alpha(t_0) \leq x_0$ *and*

$$\alpha(v) - \alpha(u) \leq {}^{(KS)}\!\!\int_u^v f(s, \alpha(s))\,\mathrm{d}\boldsymbol{g}(s), \quad [u, v] \subset \overline{I}.$$

Symmetrically, an upper solution *of* (5.35) *on* $\overline{I}$ *is a map* $\beta \in G(\overline{I}, \mathbb{R}^n)$ *such that* $x_0 \leq \beta(t_0)$ *and*

$$\beta(v) - \beta(u) \geq {}^{(KS)}\!\!\int_u^v f(s, \beta(s))\,\mathrm{d}\boldsymbol{g}(s), \quad [u, v] \subset \overline{I}.$$

Let α, β *be a lower and an upper solution of* (5.35) *on* $\overline{I}$*, respectively, such that* $\alpha \leq \beta$*. A solution of* (5.35) *on* $\overline{I}$*,* x^**, is said to be the* greatest solution of (5.35) on $\overline{I}$ between α and β *on* $\overline{I}$ *if* $x^* \in [\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)}$ *and for any other solution ,* $y \in [\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)}$*, we have that* $y \leq x^*$*. Similarly, a solution of* (5.35) *on* $\overline{I}$*,* x_**, is said to be the* least solution of (5.35) on $\overline{I}$ between α and β *if* $x_* \in [\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)}$ *and for any other solution,* $y \in [\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)}$*, we have that* $x_* \leq y$*.*

Remark 5.67. It is important to note that if $\alpha : \overline{I} \to \mathbb{R}^n$ is a lower solution of (5.35), then for all $i \in \{1, \dots, n\}$, the following inequalities hold:

$$\Delta^+\alpha_i(t) = \alpha_i(t^+) - \alpha_i(t) \leq f_i(t, \alpha(t))\Delta^+ g_i(t), \quad t \in \overline{I}, \tag{5.36}$$

$$\Delta^-\alpha_i(t) = \alpha_i(t) - \alpha_i(t^-) \leq f_i(t, \alpha(t))\Delta^- g_i(t), \quad t \in \overline{I}, \tag{5.37}$$

where we are using the notation introduced in (1.29). Similarly, if $\beta : \overline{I} \to \mathbb{R}^n$ is an upper solution of (5.35), then for all $i \in \{1, \dots, n\}$,

$$\Delta^+\beta_i(t) = \beta_i(t^+) - \beta_i(t) \geq f_i(t, \beta(t))\Delta^+ g_i(t), \quad t \in \overline{I}, \tag{5.38}$$

$$\Delta^-\beta_i(t) = \beta_i(t) - \beta_i(t^-) \geq f_i(t, \beta(t))\Delta^- g_i(t), \quad t \in \overline{I}. \tag{5.39}$$

In order to obtain an existence result for problem (5.35), we will first have a look at (5.34). In particular, we present a new existence result for this problem obtained in [52] that answered one of the questions posed in [64], namely, the existence of (extremal) solutions between given lower and upper solutions. Our result somehow generalizes what is available in the classical theory of ordinary differential equations (cf. [46]) as the function f is not required to be continuous with respect to the first variable. To that end, we recall some of the results in [64]. First, we start by recalling conditions (C1)–(C4) in Theorem 4.57 for a map $f : \overline{I} \times \mathbb{R} \to \mathbb{R}$:

(C1) For every $y \in \mathbb{R}$, the integral ${}^{(KS)}\!\int_{t_0}^{t_0+T} f(t,y)\,\mathrm{d}\,g(t)$ exists.

(C2) There exists $M : \overline{I} \to \mathbb{R}$, Kurzweil–Stieltjes integrable with respect to g, such that

$$\left\| {}^{(KS)}\!\int_u^v f(t,y)\,\mathrm{d}\,g(t) \right\| \leq {}^{(KS)}\!\int_u^v M(t)\,\mathrm{d}\,g(t),$$

for every $[u,v] \subset \overline{I}$ and $y \in \mathbb{R}$.

(C3) For each $t \in \overline{I}$, the map $f(t,\cdot)$ is continuous on $\mathbb{R}$.

(C4) For all $t \in \overline{I}$,

$$u \in \mathbb{R} \mapsto u + f(t,u)\Delta g(t) \quad \text{is nondecreasing.}$$

We now present a result that combines Lemma 3.1 and Theorem 3.2 from [64] that makes use of these conditions to obtain some information about (5.34).

Theorem 5.68. *Assume that $f : \overline{I} \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (C1)–(C3). *Then for each function $x \in G(I,\mathbb{R})$, the integral ${}^{(KS)}\!\int_{t_0}^{t_0+T} f(t,x(t))\,\mathrm{d}\,g(t)$ exists, and we have*

$$\left| {}^{(KS)}\!\int_u^v f(t,x(t))\,\mathrm{d}\,g(t) \right| \leq {}^{(KS)}\!\int_u^v M(t)\,\mathrm{d}\,g(t), \quad [u,v] \subset \overline{I}. \tag{5.40}$$

Moreover, under these hypotheses, equation (5.34) *has a solution on $\overline{I}$.*

Furthermore, combining Theorems 4.4 and 4.12 in [64], we deduce the following result about extremal solutions for problem (5.34).

Theorem 5.69. *Suppose that $f : \overline{I} \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (C1)–(C4). *If equation* (5.34) *has a solution on $\overline{I}$, then it has the greatest solution, x^*, and the least solution, x_*, on $\overline{I}$. Moreover, for each $t \in \overline{I}$ we have*

$$x^*(t) = \sup\{\alpha(t) : \alpha \text{ lower solution of (5.34) on } [t_0,t]\},$$
$$x_*(t) = \inf\{\beta(t) : \beta \text{ upper solution of (5.34) on } [t_0,t]\}.$$

Using these results, we can investigate the existence of extremal solutions for equation (5.34) provided a lower and an upper solutions are known and well-ordered. This is done in a similar fashion to Theorem 4.61 for problem (4.62). Essentially, the idea for this result is to modify our problem in order to obtain the existence of the extremal solutions for the modified problem and, later, show that those solutions are also solutions of the original problem.

Theorem 5.70. *Let $f : \overline{I} \times \mathbb{R} \to \mathbb{R}$ be a map. Suppose that* (5.34) *has a lower solution, α, and an upper solution, β, such that $\alpha \leq \beta$. Define*

$$E := \left[\inf_{s \in \overline{I}} \alpha(s), \sup_{s \in \overline{I}} \beta(s) \right].$$

Assume that the following conditions hold:

(i) *For every $y \in E$, the integral* ${}^{(KS)}\!\int_{t_0}^{t_0+T} f(t,y)\,\mathrm{d}\,g(t)$ *exists.*

(ii) *There exists $M : \overline{I} \to \mathbb{R}$, Kurzweil-Stieltjes integrable with respect to g, such that*

$$\left| {}^{(KS)}\!\!\int_u^v f(t,y)\,\mathrm{d}\,g(t) \right| \leq {}^{(KS)}\!\!\int_u^v M(t)\,\mathrm{d}\,g(t)$$

for every $y \in E$ and $[u,v] \subset \overline{I}$.

(iii) *For each $t \in \overline{I}$, the mapping $y \mapsto f(t,y)$ is continuous in E.*

(iv) *For each $t \in \overline{I}$ the mapping*

$$u \in [\alpha(t), \beta(t)] \mapsto u + f(t,u)\Delta^+ g(t)$$

is nondecreasing.

Then equation (5.34) *has extremal solutions between α and β. Moreover, the greatest solution between α and β, x^*, satisfies that for each $t \in \overline{I}$,*

$$x^*(t) = \sup\{l(t) : l \text{ lower solution of (5.34) between } \alpha \text{ and } \beta\}, \tag{5.41}$$

Similarly, the least solution between α and β, x_*, satisfies that for each $t \in \overline{I}$,

$$x_*(t) = \inf\{u(t) : u \text{ upper solution of (5.34) between } \alpha \text{ and } \beta\}. \tag{5.42}$$

Proof. Define $\widetilde{f} : \overline{I} \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{f}(t,x) = \begin{cases} f(t,\alpha(t)) & \text{if } x < \alpha(t), \\ f(t,x) & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t,\beta(t)) & \text{if } x > \beta(t), \end{cases}$$

and consider the modified problem

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^t \widetilde{f}(s,x(s))\,\mathrm{d}\,g(s), \quad t \in \overline{I}. \tag{5.43}$$

Clearly, (iii) ensures that $\widetilde{f}$ satisfies (C3). To show that $\widetilde{f}$ satisfies (C1) and (C2), let $y \in \mathbb{R}$ and put

$$m(t) = \max\{\min\{y, \beta(t)\}, \alpha(t)\}, \quad t \in I.$$

Thus, $m \in G(I,E)$ and $\widetilde{f}(t,y) = f(t,m(t))$ for every $t \in I$. Note that Theorem 5.68 and conditions (i)–(iii) imply that ${}^{(KS)}\!\int_{t_0}^{t_0+T} f(t,m(t))\,\mathrm{d}\,g(t)$ exists and (5.40) hold. In other words, for every $y \in \mathbb{R}$, the integral ${}^{(KS)}\!\int_{t_0}^{t_0+T} \widetilde{f}(t,y)\,\mathrm{d}\,g(t)$ exists and

$$\left| {}^{(KS)}\!\!\int_u^v \widetilde{f}(t,y)\,\mathrm{d}\,g(t) \right| \leq {}^{(KS)}\!\!\int_u^v M(t)\,\mathrm{d}\,g(t), \quad [u,v] \subset I.$$

In summary, $\widetilde{f}$ satisfies the conditions of Theorem 5.68 and we conclude that (5.43) has a solution on $\overline{I}$. The existence of the greatest solution of (5.43), x^*, and the least solution of (5.43), x_*, is then a consequence of Theorem 5.69 and assumption (iv).

It only remains to show that if $x : \overline{I} \to \mathbb{R}$ is an arbitrary solution of equation (5.43), then $x \in [\alpha, \beta]_{G(\overline{I},\mathbb{R})}$, thus proving that x is a solution of (5.34) and the functions x_* and x^* are the intended extremal solutions between α and β.

Let $x : \overline{I} \to \mathbb{R}$ be a solution of equation (5.43). We shall only prove that $\alpha \le x$ on $\overline{I}$ as the proof for $x \le \beta$ on $\overline{I}$ is analogous and we omit it. Reasoning by contradiction, assume that there exists some $t_1 \in (t_0, t_0 + T]$ such that $\alpha(t_1) > x(t_1)$. Let

$$t_2 = \sup\{t \in [t_0, t_1) : \alpha(t) \le x(t)\}.$$

By the definition of supremum, either $\alpha(t_2) \le x(t_2)$, which includes the case $t_2 = t_0$, or there exists a sequence of points $u_k \in (t_0, t_2), k \in \mathbb{N}$, such that $u_k \to t_2$ as $k \to +\infty$ and $\alpha(u_k) \le x(u_k)$ for each $k \in \mathbb{N}$. We claim that, in either case, $\alpha(t_2) \le x(t_2)$. Indeed, it is enough to show that this inequality holds for the second case. In that case, it follows that $\alpha(t_2^-) \le x(t_2^-)$. Thus, using (5.37) and the fact that g is left-continuous, we get $\Delta^- \alpha(t_2) \le 0$, that is,

$$\alpha(t_2) \le \alpha(t_2^-) \le x(t_2^-) = x(t_2).$$

Hence, $\alpha(t_2) \le x(t_2)$ as we claimed. This means that we must have that $t_2 < t_1$. We will show that this leads to a contradiction with the definition of t_1.

First, observe that

$$\begin{aligned} \alpha(t_1) - x(t_1) &= \alpha(t_1) - \alpha(t_2) + \alpha(t_2) - x(t_1) \\ &\le {}^{(KS)}\!\!\int_{t_2}^{t_1} f(s, \alpha(s))\, \mathrm{d}\, g(s) + \alpha(t_2) - x(t_1). \end{aligned} \tag{5.44}$$

The definition of t_2 implies that $x(t) < \alpha(t)$, $t \in (t_2, t_1]$; thus, by Theorem 1.65, we have

$${}^{(KS)}\!\!\int_{t_2}^{t_1} f(s, \alpha(s))\, \mathrm{d}\, g(s) = \lim_{\sigma \to t_2^+} {}^{(KS)}\!\!\int_{\sigma}^{t_1} f(s, \alpha(s))\, \mathrm{d}\, g(s) + f(t_2, \alpha(t_2))\Delta^+ g(t_2).$$

Now, by the definition of $\widetilde{f}$, we have that

$$\begin{aligned} \lim_{\sigma \to t_2^+} {}^{(KS)}\!\!\int_{\sigma}^{t_1} f(s, \alpha(s))\, \mathrm{d}\, g(s) &= \lim_{\sigma \to t_2^+} {}^{(KS)}\!\!\int_{\sigma}^{t_1} \widetilde{f}(s, x(s))\, \mathrm{d}\, g(s) \\ &= {}^{(KS)}\!\!\int_{t_2}^{t_1} \widetilde{f}(s, x(s))\, \mathrm{d}\, g(s) - \widetilde{f}(t_2, x(t_2))\Delta^+ g(t_2) \\ &= x(t_1) - x(t_2) - \widetilde{f}(t_2, x(t_2))\Delta^+ g(t_2), \end{aligned}$$

where the last equality follows from the fact that x is a solution of (5.43). Therefore,

$${}^{(KS)}\!\!\int_{t_2}^{t_1} f(s, \alpha(s))\, \mathrm{d}\, g(s) = x(t_1) - x(t_2) - \widetilde{f}(t_2, x(t_2))\Delta^+ g(t_2) + f(t_2, \alpha(t_2))\Delta^+ g(t_2).$$

Combining this equality with (5.44) we obtain

$$\alpha(t_1) - x(t_1) \leq \alpha(t_2) + f(t_2, \alpha(t_2))\Delta^+ g(t_2) - x(t_2) - \widetilde{f}(t_2, x(t_2))\Delta^+ g(t_2).$$

At this point we need to distinguish two cases regarding the value of $\widetilde{f}$ at the point t_2. If $\alpha(t_2) \leq x(t_2) \leq \beta(t_2)$, then $\widetilde{f}(t_2, x(t_2)) = f(t_2, x(t_2))$. Since condition (iv) implies that

$$\alpha(t_2) + f(t, \alpha(t_2))\Delta^+ g(t_2) \leq x(t_2) + f(t, x(t_2))\Delta^+ g(t_2),$$

we conclude that $\alpha(t_1) - x(t_1) \leq 0$, a contradiction. In the case when $x(t_2) > \beta(t_2)$, then $\widetilde{f}(t_2, x(t_2)) = f(t_2, \beta(t_2))$, which implies that

$$\alpha(t_1) - x(t_1) \leq \alpha(t_2) + f(t_2, \alpha(t_2))\Delta^+ g(t_2) - \beta(t_2) - f(t_2, \beta(t_2))\Delta^+ g(t_2).$$

The contradiction again follows from condition (iv), now considering that $\alpha(t_2) \leq \beta(t_2)$. In conclusion, we have that $\alpha \leq x$ on $\overline{I}$.

Finally, equalities (5.41) and (5.42) follow from Theorem 5.69. □

Our goal now is to extend the information provided by Theorem 5.69 to the more general context of (5.35). In order to do so, we need to establish some interesting properties of the space of regulated functions. In what follows, we present some of the results in [32, 43, 52]. The first of these results charaterizes the functions in such space.

Theorem 5.71. *The following two statements are equivalent:*

(i) $f \in G([a, b], \mathbb{R})$.

(ii) *For every* $\varepsilon > 0$ *there exists a partition of* $[a, b]$, $P = \{x_0, x_1, x_2, \dots, x_n\}$, *such that for each* $j \in \{1, \dots, n\}$,

$$|f(s) - f(t)| < \varepsilon, \quad s, t \in (x_{j-1}, x_j).$$

In the mentioned articles, we can also find a useful characterization of the equiregulated sets in such space. To that end, we first recall the definition of equiregulated set.

Definition 5.72. *A set* $\mathcal{A} \subset G([a, b], \mathbb{R})$ *is said to be equiregulated if for every* $\varepsilon > 0$ *and every* $t_0 \in [a, b]$ *there exists* $\delta > 0$ *such that:*

$$|x(t) - x(t_0^+)| < \varepsilon \quad \textit{for all} \ \ t_0 < t < t_0 + \delta, \quad x \in \mathcal{A},$$

$$|x(t) - x(t_0^-)| < \varepsilon \quad \textit{for all} \ \ t_0 - \delta < t < t_0, \quad x \in \mathcal{A}.$$

As anticipated, the following result provides a unequivocal way to describe equiregulated sets on $G([a, b], \mathbb{R}^n)$. This result will be fundamental to prove Proposition 5.76.

Lemma 5.73. *The following statements are equivalent:*

(i) $\mathcal{A} \subset G([a, b], \mathbb{R})$ *is equiregulated.*

(ii) *For every* $\varepsilon > 0$ *there exists a partition of* $[a, b]$*,* $P = \{x_0, x_1, x_2, \dots, x_n\}$*, such that for each* $j \in \{1, \dots, n\}$*,*

$$|f(s) - f(t)| < \varepsilon, \quad s, t \in (x_{j-1}, x_j), \quad f \in \mathcal{A}.$$

Equiregulated sets allow us to characterize relatively compact sets through the Arzelá-Ascoli theorem. Next we present the following formulation of such result in the context of the space of regulated functions.

Theorem 5.74. *A subset* $\mathcal{A}$ *of* $G([a, b], \mathbb{R})$ *is relatively compact if and only if it is equiregulated and the set* $\{f(t) : f \in \mathcal{A}\}$ *is bounded for each* $t \in [a, b]$*.*

Remark 5.75. At this point is worth mentioning one particular case in which the assumptions ensuring compactness are satisfied. Let $\mathcal{A} \subset G([a, b], \mathbb{R})$ be such that there exist $M > 0$ and nondecreasing function $h : [a, b] \to \mathbb{R}$ satisfying

$$|f(v) - f(u)| \leq h(v) - h(u) \quad \text{for } f \in \mathcal{A}, \ [u, v] \subset [a, b],$$

and $|f(a)| \leq M$ for all $f \in \mathcal{A}$. By Lemma 5.73 the set $\mathcal{A}$ is equiregulated, and obviously $\{f(t) : f \in \mathcal{A}\}$ is bounded for each $t \in I$. Thus, in this case, $\mathcal{A}$ is relatively compact.

With all these tools, we can finally show that the pointwise supremum of regulated functions on $[a, b]$ also lies in $G([a, b], \mathbb{R})$. This property will be fundamental for the proof of Theorem 5.79 as, in a similar way to Theorem 5.70, we will prove that the supremum of lower solutions is the least solution of (5.35). Therefore, we need to show that such function is regulated, which motivates the following result.

Proposition 5.76. *Let* $\mathcal{A}$ *be a relatively compact subset of* $G([a, b], \mathbb{R})$*. Then, the function* $\xi_{\text{sup}} : [a, b] \to \mathbb{R}$ *given by*

$$\xi_{\text{sup}}(t) = \sup\{f(t) : f \in \mathcal{A}\}, \quad t \in [a, b],$$

is regulated.

Proof. From Theorem 5.74, we know that $\{f(t) : f \in \mathcal{A}\}$ is bounded for each $t \in [a, b]$; therefore, the function ξ_{sup} is well–defined. Moreover, since $\mathcal{A}$ is equiregulated, Lemma 5.73 ensures that, given $\varepsilon > 0$ there exists a partition of $[a, b]$, $P = \{x_0, x_1, x_2, \dots, x_n\}$, such that for $j \in \{1, \dots, n\}$,

$$|f(s) - f(t)| < \varepsilon, \quad s, t \in (x_{j-1}, x_j), \quad f \in \mathcal{A}.$$

Fix an arbitrary $j \in \{1, \dots, n\}$. For $s, t \in (x_{j-1}, x_j)$ we get

$$f(s) - \varepsilon < f(t) < f(s) + \varepsilon \leq \xi(s) + \varepsilon \quad \text{for all } f \in \mathcal{A},$$

and, consequently,

$$\xi_{\text{sup}}(s) - \varepsilon \leq \xi_{\text{sup}}(t) \leq \xi_{\text{sup}}(s) + \varepsilon.$$

In summary, ξ_{sup} satisfies assumption (ii) of Theorem 5.71, hence ξ_{sup} is regulated. □

Remark 5.77. Notice that Proposition 5.76 suffices to show that the pointwise infimum of regulated functions over a relatively compact subset of $G([a,b],\mathbb{R})$ is regulated. Indeed, let $\mathcal{A}$ be a relatively compact subset of $G([a,b],\mathbb{R})$ and consider the map $\xi_{\inf} : [a,b] \to \mathbb{R}$ defined by

$$\xi_{\inf}(t) = \inf\{f(t) : f \in \mathcal{A}\}, \quad t \in [a,b].$$

First, note that the set $\mathcal{B} = \{-f : f \in \mathcal{A}\}$ is relatively compact on $G([a,b],\mathbb{R})$, see Proposition 5.73 and Lemma 5.74. Moreover, we can rewrite $\xi_{\inf}$ as

$$\xi_{\inf}(t) = -\sup\{h(t) : h \in \mathcal{B}\} := -\widetilde{\xi}_{\sup}, \quad t \in [a,b].$$

Now, Proposition 5.76 guarantees that $\widetilde{\xi}_{\sup}$ is regulated, from which it follows that so is $\xi_{\inf}$.

We now have all the tools necessary to extend the ideas of Theorem 5.70 to the more general context of (5.35). To do so, we will require the notion of quasimonotonicity that we introduce in the next definition. In particular, we will use the concept of a quasimonotone nondecreasing map.

Definition 5.78. *Let $f : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$. We say that f is a* quasimonotone nondecreasing map *in a set $E \subset \overline{I} \times \mathbb{R}^n$ if given $t \in \overline{I}$ and $x, y \in \mathbb{R}^n$, $x \leq y$, such that $(t,x),(t,y) \in E$, the equality $x_i = y_i$ for some $i \in \{1,\dots,n\}$, implies that $f_i(t,x) \leq f_i(t,y)$.*

With this notion, we can establish an existence result for (5.35) based on Theorem 5.70. In what follows, e_i, $i \in \{1,\dots,n\}$, denotes the i-th canonical vector in $\mathbb{R}^n$, that is, the vector whose i-th term is 1 and all others are zero.

Theorem 5.79. *Let $f : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a map. Suppose that* (5.35) *has a lower solution, α, and an upper solution, β, such that $\alpha \leq \beta$ and assume that f is quasimonotone nondecreasing in*

$$E = \{(t,x) \in I \times \mathbb{R}^n \,:\, \alpha(t) \leq x \leq \beta(t)\}.$$

Furthermore, assume that for each $i \in \{1, 2, \dots, n\}$, the following conditions hold:

(i) *For each $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, the integral ${}^{(KS)}\!\int_{t_0}^{t_0+T} f_i(t,\eta(t))\,\mathrm{d}\,g_i(t)$ exists.*

(ii) *There exists $M_i : \overline{I} \to \mathbb{R}$, Kurzweil-Stieltjes integrable with respect to g_i, such that*

$$\left| {}^{(KS)}\!\!\int_u^v f_i(t,\eta(t))\,\mathrm{d}\,g_i(t) \right| \leq {}^{(KS)}\!\!\int_u^v M_i(t)\,\mathrm{d}\,g_i(t)$$

for every $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and $[u,v] \subset \overline{I}$.

(iii) *For each $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t), \beta_i(t)] \mapsto f_i(t, \eta(t) + (u - \eta_i(t))e_i)$$

is continuous.

(iv) *For each* $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ *and* $t \in \overline{I}$, *the mapping*

$$u \in [\alpha_i(t),\beta_i(t)] \mapsto u + f_i(t,\eta(t) + (u-\eta_i(t))e_i)\Delta^+ g_i(t)$$

is nondecreasing.

Then equation (5.35) *has the extremal solutions in* $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$. *Moreover, for* $t \in \overline{I}$, *the greatest solution,* $x^* = (x_1^*,\dots,x_n^*)$, *is given by*

$$x_i^*(t) = \sup\{l_i(t) : (l_1,\dots,l_n) \text{ lower solution of } (5.35) \text{ in } [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}\}, \tag{5.45}$$

and the least solution $y_* = (y_{*,1},\dots,y_{*,n})$ *is given by*

$$x_{*i}(t) = \inf\{u_i(t) : (u_1,\dots,u_n) \text{ upper solution of } (5.35) \text{ in } [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}\}. \tag{5.46}$$

Proof. Let $L = (L_1,\dots,L_n)$ be an arbitrary lower solution of (5.35) in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and, for each $i \in \{1,\dots,n\}$, define

$$h_i(t) = {}^{(KS)}\!\!\int_{t_0}^{t} M_i(s)\,\mathrm{d}\,g_i(s), \quad t \in \overline{I},$$

where M_i is the corresponding function in condition (ii). Note that by definition, each h_i, $i \in \{1,\dots,n\}$, is nondecreasing and left-continuous.

Given $x \in G(\overline{I},\mathbb{R})$ and $\varepsilon > 0$, we shall denote by $P_{x,\varepsilon}$ the partition whose existence is guaranteed by Theorem 5.71. With this notation, we define the set of functions $\mathcal{A}$ as follows: $\eta \in \mathcal{A}$ if $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, and it satifies the following two conditions:

(a) For each $i \in \{1,2,\dots,n\}$ and $\varepsilon > 0$, $P_{\eta_i,\varepsilon} \subset P_{L_i,\varepsilon} \cup P_{h_i,\varepsilon}$.

(b) $\eta(t_0) \le x_0$ and for each $i \in \{1,2,\dots,n\}$,

$$\eta_i(v) - \eta_i(u) \le {}^{(KS)}\!\!\int_u^v M_i(s)\,\mathrm{d}\,g_i(s), \quad [u,v] \subset \overline{I}.$$

First observe that $L \in \mathcal{A}$. Indeed, condition (a) is trivially satisfied. As for condition (b), by definition of lower solution $L(t_0) \le x_0$ and

$$L(v) - L(u) \le {}^{(KS)}\!\!\int_u^v f(s,\alpha(s))\,\mathrm{d}\,\boldsymbol{g}(s), \quad [u,v] \subset \overline{I}.$$

In particular, for each $i \in \{1,2,\dots,n\}$, we have that for any $[u,v] \subset \overline{I}$,

$$\begin{aligned} L_i(v) - L_i(u) &\le {}^{(KS)}\!\!\int_u^v f_i(s,L(s))\,\mathrm{d}\,g_i(s) \\ &\le \left| {}^{(KS)}\!\!\int_u^v f_i(t,L(t))\,\mathrm{d}\,g_i(t) \right| \le {}^{(KS)}\!\!\int_u^v M_i(t)\,\mathrm{d}\,g_i(t). \end{aligned}$$

Moreover, every solution of (5.35) in $[\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)}$ belongs to $\mathcal{A}$. Indeed, let x be a solution of (5.35). Then $x(t_0) = x_0$ and

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f(s, x(s), x) \,\mathrm{d}\,\boldsymbol{g}(s), \quad t \in \overline{I}.$$

In particular, we have that for each $i \in \{1, 2, \ldots, n\}$ and $[u, v] \subset \overline{I}$,

$$x_i(v) - x_i(u) = {}^{(KS)}\!\!\int_{u}^{v} f_i(t, x(t)) \,\mathrm{d}\, g_i(t),$$

from which it follows that condition (b) is satisfied. Furthermore, the same relation, together with condition (ii), ensures that condition (a) is satisfied.

Define $\xi^* : \overline{I} \to \mathbb{R}^n$, $\xi^* = (\xi_1^*, \ldots, \xi_n^*)$, where each ξ_i^*, $i \in \{1, \ldots, n\}$, is given by

$$\xi_i^*(t) = \sup\{\eta_i(t) : \eta \in \mathcal{A},\ \eta \text{ is a lower solution of (5.35)}\}, \quad t \in \overline{I}. \tag{5.47}$$

Note that, for each $t \in \overline{I}$ and $i \in \{1, \ldots, n\}$, the set $\{\eta_i(t) : \eta \in \mathcal{A}\} \subset [\alpha_i(t), \beta_i(t)]$. Therefore, the supremum $\xi_i^*(t)$ is well-defined. Moreover, condition (a) in the definition of $\mathcal{A}$ ensures that $\{\eta_i : \eta \in \mathcal{A}\}$ is equiregulated, see Lemma 5.73. Thus, Theorem 5.74 together with Proposition 5.76 imply that ξ_i^*, $i = 1, 2, \ldots, n$, is regulated. As a consequence, we have that $\xi^* \in G(\overline{I}, \mathbb{R}^n)$.

Claim 1 – ξ^* is the greatest solution of (5.35) *in* $[\alpha, \beta]_{G(\overline{I},\mathbb{R}^n)}$. Fix $i \in \{1, \ldots, n\}$ and define $\Phi_i : \overline{I} \times \mathbb{R} \to \mathbb{R}$ as

$$\Phi_i(t, x) = f_i(t, \xi^*(t) + (x - \xi_i^*(t))e_i), \quad t \in \overline{I}, \quad x \in \mathbb{R}.$$

Since $\xi^* \leq \beta$, the quasimonotonicity of f yields

$${}^{(KS)}\!\!\int_{u}^{v} \Phi_i(s, \beta_i(s)) \,\mathrm{d}\, g_i(s) \leq {}^{(KS)}\!\!\int_{u}^{v} f_i(s, \beta(s)) \,\mathrm{d}\, g_i(s) \leq \beta_i(v) - \beta(u), \quad [u, v] \subset \overline{I},$$

which shows that $\beta_i : \overline{I} \to \mathbb{R}$ is an upper solution of the scalar problem

$$x(t) = x_{0,i} + {}^{(KS)}\!\!\int_{t_0}^{t} \Phi_i(s, x(s)) \,\mathrm{d}\, g_i(s), \quad t \in \overline{I}. \tag{5.48}$$

Using a similar argument, we can show that for any lower solution η of (5.35) such that $\eta \in \mathcal{A}$, the function η_i is a lower solution of (5.48) between α_i and β_i. Noting that Φ_i satisfies the conditions of Theorem 5.70, it follows that (5.48) has the greatest solution, x_i^*, between α_i and β_i, and by (5.41), we get that $\eta_i(t) \leq x_i^*(t)$, $t \in \overline{I}$, for any $\eta \in \mathcal{A}$ lower solution of (5.35). Since the argument is valid for each $i \in \{1, \ldots, n\}$, we construct a map $x^* : \overline{I} \to \mathbb{R}^n$, $x^* = (x_1^*, \ldots, x_n^*)$, which clearly satisfies that $\xi^* \leq x^*$. The quasimonotonicity of f yields that for each $i \in \{1, \ldots, n\}$,

$$x_i^*(v) - x_i^*(u) = {}^{(KS)}\!\!\int_{u}^{v} \Phi_i(s, x_i^*(s)) \,\mathrm{d}\, g_i(s) \leq {}^{(KS)}\!\!\int_{u}^{v} f_i(s, x^*(s)) \,\mathrm{d}\, g_i(s), \quad [u, v] \subset \overline{I}.$$

This is, x^* is a lower solution of (5.35) in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, and for each $i \in \{1,\dots,n\}$,

$$\begin{aligned}|x_i^*(v) - x_i^*(u)| &= \left| {}^{(KS)}\!\!\int_u^v \Phi_i(s, x_i^*(s))\,\mathrm{d}\,g_i(s)\right| \\ &= \left| {}^{(KS)}\!\!\int_u^v f_i(s, \xi^*(s) + (x_i^*(s) - \xi_i^*(s))e_i)\,\mathrm{d}\,g_i(s)\right| \\ &\le {}^{(KS)}\!\!\int_u^v M_i(s)\,\mathrm{d}\,g_i(s) = h_i(v) - h_i(u)\end{aligned}$$

for every $[u,v] \subset \overline{I}$. This shows that x^* satisfies conditions (a) and (b) in the definition of $\mathcal{A}$, so $x^* \in \mathcal{A}$. Now, the definition of ξ^* implies that $x^* \le \xi^*$. In summary, $\xi^* = x^*$ and for each $i \in \{1,\dots,n\}$,

$$\xi_i^*(t) = x_{0,i} + {}^{(KS)}\!\!\int_{t_0}^t \Phi_i(s, \xi_i^*(s))\,\mathrm{d}\,g_i(s) = x_{0,i} + {}^{(KS)}\!\!\int_{t_0}^t f_i(s, \xi^*(s))\,\mathrm{d}\,g_i(s), \quad t \in \overline{I}.$$

Therefore, ξ^* is a solution of (5.35), and by (5.47), it is the greatest one in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$.

Claim 2 – The greatest solution of (5.35) *in* $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, $y^* = \xi^*$, *satisfies* (5.45). The lower solution $L \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ was fixed arbitrarily, so y^* is greater than or equal to any lower solution in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$. On the other hand, y^* is a lower solution itself and so (5.45) holds.

The proof of the existence of the least solution y_* as well the validity of (5.46) is analogous, and we omit it. □

5.3.1 Vectorial measure differential equations with functional arguments

We will now consider a more general vectorial measure differential equation. In this case, we will allow the function f to consider functional arguments. Specifically, given $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \dots, g_n)$, such that each map $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, n$, is nondecreasing and left–continuous; we will consider integral equations of the form

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^t f(s, x(s), x)\,\mathrm{d}\,\boldsymbol{g}(s), \tag{5.49}$$

where $x_0 \in \mathbb{R}^n$, $f : \overline{I} \times \mathbb{R}^n \times G(\overline{I},\mathbb{R}^n) \to \mathbb{R}^n$. It is interesting to note that this kind of problems include the so-called measure functional differential equations in the sense introduced in [27]. To see this it is enough to consider $\boldsymbol{g} = (g,\dots,g)$ for some nondecreasing and left–continuous map $g : \mathbb{R} \to \mathbb{R}$ and

$$f(t, y(t), y) = F(t, y_t),$$

where $r > 0$, $F : I \times G([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$, and for each $t \in I$, $y_t : [-r,0] \to \mathbb{R}^n$ denotes the history or memory of y in $[t-r,t]$, that is, $y_t(\theta) = y(t+\theta)$, $\theta \in [-r,0]$.

Once again, we start by formalizing the definition of solution of (5.49), which is analogous to that of (5.35).

Definition 5.80. *A* solution *of* (5.49) *on* $\overline{I}$ *is a map* $x \in G(\overline{I}, \mathbb{R}^n)$ *such that*

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f(s, x(s), x)\,\mathrm{d}\boldsymbol{g}(s), \quad t \in \overline{I}.$$

Similarly, we present the definitons of lower and upper solution for a vectorial measure differential equation of the form of (5.49), as well as the definition of extremal solutions between a pair of well–ordered lower and upper solutions.

Definition 5.81. *A* lower solution *of* (5.49) *on* $\overline{I}$ *is a function* $\alpha \in G(\overline{I}, \mathbb{R}^n)$ *such that* $\alpha(t_0) \leq x_0$ *and*

$$\alpha(v) - \alpha(u) \leq {}^{(KS)}\!\!\int_{u}^{v} f(s, \alpha(s), \alpha)\,\mathrm{d}\boldsymbol{g}(s), \quad [u, v] \subset \overline{I}.$$

Symmetrically, an upper solution *of* (5.49) *on* $\overline{I}$ *is a map* $\beta \in G(\overline{I}, \mathbb{R}^n)$ *such that* $x_0 \leq \beta(t_0)$ *and*

$$\beta(v) - \beta(u) \geq {}^{(KS)}\!\!\int_{u}^{v} f(s, \beta(s), \beta)\,\mathrm{d}\boldsymbol{g}(s), \quad [u, v] \subset \overline{I}.$$

Let α, β *be a lower and an upper solution of* (5.49) *on* $\overline{I}$*, respectively, such that* $\alpha \leq \beta$*. A solution of* (5.49) *on* $\overline{I}$*,* x^**, is said to be the* greatest solution of (5.49) on $\overline{I}$ between α and β *on* $\overline{I}$ *if* $x^* \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$ *and for any other solution,* $y \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$*, we have that* $y \leq x^*$*. Similarly, a solution of* (5.49) *on* $\overline{I}$*,* x_**, is said to be the* least solution of (5.49) on $\overline{I}$ between α and β *if* $x_* \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$ *and for any other solution,* $y \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$*, we have that* $x_* \leq y$*.*

The aim of this section is to obtain a result for (5.49) ensuring the existence of extremal solutions between a lower and an upper solution in a similar fashion to Section 4.2 in Chapter 4. There, we used a fixed point method approach based on an existence result for the nonfunctional problem. Here we will proceed in a similar manner. With that idea in mind, we present the following fixed point result obtained from Theorem 3.32 in the particular setting of the set of regulated functions.

Proposition 5.82. *Let* $\alpha, \beta \in G([a, b], \mathbb{R}^n)$ *be two functions such that* $\alpha \leq \beta$*, and consider* $T : [\alpha, \beta]_{G([a,b],\mathbb{R}^n)} \to [\alpha, \beta]_{G([a,b],\mathbb{R}^n)}$*. Assume that* T *is nondecreasing and such that for each* $i \in \{1, \ldots, n\}$ *there exists a nondecreasing map* $h_i : [a, b] \to \mathbb{R}$ *satisfying*

$$|(T\gamma)_i(v) - (T\gamma)_i(u)| \leq h_i(v) - h_i(u), \quad [u, v] \subset [a, b], \quad \gamma \in [\alpha, \beta]_{G([a,b],\mathbb{R}^n)}. \tag{5.50}$$

Then T *has the least fixed point,* γ_**, and the greatest fixed point,* γ^**, in* $[\alpha, \beta]_{G([a,b],\mathbb{R}^n)}$*, and they satisfy the following equalities:*

$$\gamma_* = \min\{\gamma \in [\alpha, \beta]_{G([a,b],\mathbb{R}^n)} : T\gamma \leq \gamma\}, \quad \gamma^* = \max\{\gamma \in [\alpha, \beta]_{G([a,b],\mathbb{R}^n)} : \gamma \leq T\gamma\}.$$

Proof. We will apply Theorem 3.32 assuming $X = Y = G([a, b], \mathbb{R}^n)$ equipped with the supremum norm and the partial ordering for functions defined above. Given a monotone sequence $\{\gamma_k\}_{k=0}^{\infty}$ in $[\alpha, \beta]_{G([a,b],\mathbb{R}^n)}$, it suffices to show that for each $i \in \{1, \ldots, n\}$, the sequence $\{(T\gamma_k)_i\}_{k=0}^{\infty}$ converges in $G([a, b], \mathbb{R}^n)$.

By (5.50) and Remark 5.75, for each $i \in \{1,\dots,n\}$, $\{(T\gamma_k)_i\}_{k=0}^{\infty}$ is a relatively compact subset of $G([a,b],\mathbb{R}^n)$, hence $\{(T\gamma_k)_i\}_{k=0}^{\infty}$ has a subsequence converging in X to a limit, say y. Now, since the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is monotone, we have that $\{(T\gamma_k)_i\}_{k=0}^{\infty}$ is monotone as well. Thus, $\{(T\gamma_k)_i\}_{k=0}^{\infty}$ also converges to y, which finishes the proof. □

We can now present a result ensuring the existence of extremal solutions between a lower and an upper solution that are well–ordered. Observe that Theorem 5.83 is a direct generalization of Theorem 5.79 to the context of functional measure differential equations.

Theorem 5.83. *Let $f : \overline{I} \times \mathbb{R}^n \times G(\overline{I},\mathbb{R}^n) \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a map. Suppose that* (5.49) *has a lower solution, α, and an upper solution, β, such that $\alpha \le \beta$. For each $\gamma \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, denote by $f_\gamma : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$ the function defined as $f_\gamma(t,x) = f(t,x,\gamma)$, $(t,x) \in \overline{I} \times \mathbb{R}^n$. Assume that for each $\gamma \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, the function f_γ is quasimonotone nondecreasing in*

$$E = \{(t,x) \in I \times \mathbb{R}^n : \alpha(t) \le x \le \beta(t)\}.$$

Furthermore, assume that for each $i \in \{1,2,\dots,n\}$, the following conditions hold:

(i) *For each $\gamma, \eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, the integral ${}^{(KS)}\!\!\int_{t_0}^{t_0+T} (f_\gamma)_i(s,\eta(t))\,\mathrm{d}\,g_i(s)$ exists.*

(ii) *There exists $M_i : \overline{I} \to \mathbb{R}$, Kurzweil-Stieltjes integrable with respect to g_i, such that*

$$\left| {}^{(KS)}\!\!\int_u^v (f_\gamma)_i(t,\eta(t))\,\mathrm{d}\,g_i(t) \right| \le {}^{(KS)}\!\!\int_u^v M_i(t)\,\mathrm{d}\,g_i(t)$$

for every $\gamma, \eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and $[u,v] \subset \overline{I}$.

(iii) *For each $\gamma, \eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t),\beta_i(t)] \mapsto (f_\gamma)_i(t,\eta(t) + (u - \eta_i(t))e_i)$$

is continuous.

(iv) *For each $\gamma, \eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t),\beta_i(t)] \mapsto u + (f_\gamma)_i(t,\eta(t) + (u - \eta_i(t))e_i)\,\Delta^+ g_i(t)$$

is nondecreasing.

If, moreover, the following condition is satisfied:

(v) *For each $t \in \overline{I}$ and $x \in \mathbb{R}^n$, the mapping $f(t,x,\cdot)$ is nondecreasing on $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$,*

then equation (5.49) *has the extremal solutions in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$.*

Proof. First of all, note that for each $\gamma \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, condition (v) implies that α and β are, respectively, a lower and an upper solution of the vectorial equation

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f_\gamma(s,x(s))\,\mathrm{d}\boldsymbol{g}(s). \tag{5.51}$$

Consider $T : [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)} \to [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ defined as follows: for each $\gamma \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, $T\gamma$ is the greatest solution of (5.51) in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$. The function T is well-defined as hypotheses (i)–(iv) together with Theorem 5.79 guarantee the existence of the extremal solutions of (5.51) in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$. Let us show that T satisfies the hypotheses of Proposition 5.82.

First, note that T clearly satisfies (5.50) with

$$h_i(t) = {}^{(KS)}\!\!\int_{t_0}^{t} M_i(s)\,\mathrm{d}\,g_i(s), \quad t \in \overline{I}, \quad i \in \{1,\dots,n\}.$$

Thus, we only need to show that T is nondecreasing. Let $\gamma,\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ be such that $\gamma \leq \eta$. Condition (v) yields that $f(s,T\eta(s),\gamma) \leq f(s,T\eta(s),\eta)$ for $s \in \overline{I}$. Thus, for each $i \in \{1,\dots,n\}$ and $[u,v] \subset \overline{I}$,

$${}^{(KS)}\!\!\int_{u}^{v} (f_\gamma)_i(s,T\eta(s))\,\mathrm{d}\,g_i(s) \leq {}^{(KS)}\!\!\int_{u}^{v} (f_\eta)_i(s,T\eta(s))\,\mathrm{d}\,g_i(s) = (T\eta)_i(v) - (T\eta)_i(u),$$

that is, $T\eta$ is an upper solution of

$$z(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} f_\gamma(s,z(s))\,\mathrm{d}\,\boldsymbol{g}(s). \tag{5.52}$$

Theorem 5.79 guarantees that (5.52) has the greatest solution between α and $T\eta$. Since $T\gamma$ is the greatest solution of (5.52) in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ it follows that $T\gamma \leq T\eta$. Therefore, T is nondecreasing. Now, Proposition 5.82 ensures that T has the greatest fixed point, γ^*, and it satisfies the following equality:

$$\gamma^* = \max\{\gamma \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)} : \gamma \leq T\gamma\}.$$

Naturally, γ^* is a solution of (5.49) in $[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$. Moreover, it is not hard to see that if $\gamma \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ is any other solution of (5.49), then $\gamma \leq T\gamma$. Therefore, by the definition of γ^*, we conclude that γ^* is the greatest solution of (5.49).

To prove the existence of the least solution for equation (5.49), proceed in a similar way but redefining the function T so that $T\gamma$ corresponds to the least solution of equation (5.51). Since the procedure in that case is analogous, we omit it. □

5.3.2 Applications to Stieltjes differential equations

It is possible to establish a relation between the solutions of (5.9) and the solutions of (5.35). It follows from Remark 5.29 that a solution of problem (5.9) solves (5.35) as well, since the integrals that exist in the Lebesgue–Stieltjes sense, also exist in the Kurzweil–Stieltjes sense. However, in order to apply the results obtained previously, we require some conditions that guarantee the converse process. That is, we need to investigate some conditions that ensure that the solutions (5.35) generate a solution of (5.9).

Along those lines, in [63] the authors showed that, under very general assumptions, the integral equation (5.34) is equivalent to

$$x_g'(t) = f(t,x(t)) \quad m_g\text{–a.e. } t \in I, \quad x(t_0) = x_0, \tag{5.53}$$

where m_g stands for the Thomson's variational measure induced by a function $g:\mathbb{R}\to\mathbb{R}$, see $\mathcal{S}_0$- μ_g in [81]. In the case when g is nondecreasing, as proved in [26], the variational measure m_g corresponds to the Lebesgue-Stieltjes outer measure, μ_g^*. Hence, if $E\subset I$ and $m_g(E)=0$, then $\mu_g^*(E)=0$ and, consequently, E is Lebesgue-Stieltjes measurable with $\mu_g(E)=0$. Accordingly, a solution of (5.53) also satisfies equation

$$x_g'(t)=f(t,x(t))\quad g\text{–a.a. } t\in I,\quad x(t_0)=x_0,$$

where $x\in\mathcal{AC}_g(\overline{I},\mathbb{R})$ if and only if $f(\cdot,x(\cdot))$ is integrable on I with respect to g in the Lebesgue-Stieltjes sense. Therefore, along similar lines of the results in [63], we can draw a correspondence between the solutions of

$$x_i(t)=x_{0,i}+\int_{t_0}^t f_i(s,x(s))\,\mathrm{d}\,g_i(s),\quad t\in\overline{I},\quad i\in\{1,\dots,n\},$$

and the solutions of

$$(x_i)'_{g_i}(t)=f_i(t,x(t))\quad \text{for } g_i\text{–a.a. } t\in I,\quad x_i(t_0)=x_{0,i},\quad i\in\{1,\dots,n\}.$$

Thus, we can now translate the results on existence of extremal solutions for vectorial measure differential equations into the corresponding results for Stieltjes differential equations with several derivators. To that end, we introduce the following definitions generalizing those in Definition 4.54.

Definition 5.84. *A* lower solution of (5.9) *is a function $\alpha\in\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$ such that $\alpha(t_0)\le x_0$ and*

$$\alpha'_{g_i}(t)\le f_i(t,\alpha(t))\quad g_i\text{–a.a. } t\in I,\quad i=1,2,\dots,n. \tag{5.54}$$

Analogously, an upper solution of (5.9) *is a map $\beta\in\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)$ such that $x_0\le\beta(t_0)$ and*

$$\beta'_{g_i}(t)\ge f_i(t,\beta(t))\quad g_i\text{–a.a. } t\in I,\quad i=1,2,\dots,n.$$

Remark 5.85. As in Remark 5.29, we could consider the lower and upper solution to belong to $E\mathcal{AC}_g(\overline{I},\mathbb{R}^n)$ instead of $\mathcal{AC}_g(\overline{I},\mathbb{R}^n)$. The reasoning behind this choice is the same: simplicity. In this case, it is even more relevant as we aim to conect lower and upper solutions of (5.9) to those of (5.35), which are defined on the closed interval.

Remark 5.86. Every lower solution of (5.9) is also a lower solution of (5.34). Indeed, given a lower solution α of (5.9), since $\alpha_i\in\mathcal{AC}_{g_i}(\overline{I},\mathbb{R})$ for each $i\in\{1,\dots,n\}$, the Fundamental Theorem of Calculus yields that for every $[u,v]\subset\overline{I}$ we have

$$\alpha_i(v)=\alpha_i(u)+\int_{[u,v)}(\alpha_i)'_{g_i}(s)\,\mathrm{d}\,g_i.$$

Therefore, (5.61) implies that

$$\alpha_i(v)-\alpha_i(u)\le\int_{[u,v)}f_i(s,\alpha(s))\,\mathrm{d}\,g_i\quad [u,v]\subset I,\quad i\in\{1,\dots,n\}.$$

Now, recalling that Lebesgue-Stieltjes integrability implies Kurzweil-Stieltjes integrability, together with the fact that g_i is left-continuous, ensures that the Lebesgue-Stieltjes integral on right-hand side coincides with the Kurzweil-Stieltjes integral ${}^{(KS)}\!\!\int_u^v f_i(s,\alpha(s))\,\mathrm{d}\,g_i(s)$. Since functions in the space $\mathcal{AC}_{g_i}(\overline{I},\mathbb{R})$ are of bounded variation, we conclude that α is a lower solution of the integral equation (5.34). A similar argument holds to show that every upper solution of (5.9) is also an upper solution of (5.34).

The following result, obtained from Theorem 5.79, ensures the existence of solution for the vectorial problem (5.9) in the presence of lower and upper solutions. For this result, it is important to note that $\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}) \subset G(\overline{I},\mathbb{R}^n)$ and, as a consequence,

$$[\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)} \cap \mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}) = [\alpha,\beta]_{\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R})},$$

with $[\alpha,\beta]_{\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R})}$ as in Proposition 3.33.

Theorem 5.87. *Let $f : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a map. Suppose that* (5.9) *has a lower solution, α, and an upper solution, β, such that $\alpha \leq \beta$. Assume that f is quasimonotone nondecreasing on*

$$E = \{(t,x) \in \overline{I} \times \mathbb{R}^n : \alpha(t) \leq x \leq \beta(t)\}.$$

Moreover, assume that for each $i \in \{1,\dots,n\}$ the following conditions hold:

(i) *For each $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$, the function $f_i(\cdot,\eta(\cdot))$ is g_i-measurable.*

(ii) *For every $r > 0$, there exists a function $h_{i,r} \in \mathcal{L}^1_{g_i}(\overline{I},[0,+\infty))$ such that*

$$|f_i(t,x)| \leq h_{i,r}(t), \quad \textit{for } g_i\textit{–a.a. } t \in \overline{I}, \quad x \in \mathbb{R}^n, \|x\| \leq r.$$

(iii) *For each $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and for g_i–a.a. $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t),\beta_i(t)] \mapsto f_i(t,\eta(t) + (u - \eta_i(t))e_i)$$

is continuous.

(iv) *For each $\eta \in [\alpha,\beta]_{G(\overline{I},\mathbb{R}^n)}$ and $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t),\beta_i(t)] \mapsto u + f_i(t,\eta(t) + (u - \eta_i(t))e_i)\Delta^+ g_i(t)$$

is nondecreasing.

Then (5.9) *has extremal solutions in $[\alpha,\beta]_{\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R})}$. Moreover, for $t \in \overline{I}$, the greatest solution, $x^* = (x_1^*, \dots, x_n^*)$, is given by*

$$x_i^*(t) = \sup\{l_i(t) : (l_1,\dots,l_n) \textit{ lower solution of } (5.9) \textit{ in } [\alpha,\beta]_{\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)}\},$$

and the least solution, $x_* = (x_{*,1},\dots,x_{*,n})$, is given by

$$x_{*i}(t) = \inf\{u_i(t) : (u_1,\dots,u_n) \textit{ upper solution of } (5.9) \textit{ in } [\alpha,\beta]_{\mathcal{AC}_{\boldsymbol{g}}(\overline{I},\mathbb{R}^n)}\}.$$

Proof. For each $i \in \{1, \dots, n\}$, let $N_i \subset \overline{I}$ be a g_i–null set such that both conditions (ii) and (iii) hold for all $t \in \overline{I} \setminus N_i$. Write $U_i : \overline{I} \times \mathbb{R}^n \to \mathbb{R}$

$$U_i(t,x) = \begin{cases} f_i(t,x) & \text{if } t \in I \setminus N_i,\ x \in \mathbb{R}^n \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = (U_1, \dots, U_n) : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$ and consider the modified problem

$$x(t) = x_0 + {}^{(KS)}\!\!\int_{t_0}^{t} U(s, x(s))\, \mathrm{d}\boldsymbol{g}(s), \quad t \in \overline{I}. \tag{5.55}$$

It follows that a solution of (5.55) is also a solution of (5.35). Moreover, conditions (i)–(iii) guarantee that the integrals in (5.55) also exist as Lebesgue-Stieltjes integrals when x is between α and β. Hence, the existence of extremal solutions for (5.55) in $[\alpha, \beta]$ yields the existence of greatest and least solutions for (5.9) in $[\alpha, \beta]$. Altogether, it is enough to show that the function U fulfills conditions (i)–(iv) of Theorem 5.79.

Clearly, U satisfies (iii) and (iv) in Theorem 5.79. Note that (i)–(iii) imply that for each $i \in \{1, \dots, n\}$ and $\eta \in [\alpha, \beta]$, the Lebesgue-Stieltjes integral $\int_I U_i(s, \eta(s))\, \mathrm{d}\, g_i$ exists. Consequently, (i) in Theorem 5.79 holds due to the relation between Lebesgue-Stieltjes and Kurzweil-Stieltjes integrals.

To prove (ii) in Theorem 5.79, take $r = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$ and let $h_{i,r}$, $i \in \{1, \dots, n\}$, be the corresponding function in (ii). Since

$$|U_i(t,x)| \le h_{i,r}(t), \quad t \in I, \quad x \in \mathbb{R}^n,\ \|x\| \le r,$$

and both functions are integrable with respect to g_i in the sense of Kurzweil-Stieltjes, we get

$$\left| {}^{(KS)}\!\!\int_u^v U_i(t,x)\, \mathrm{d}\, g_i(t) \right| \le {}^{(KS)}\!\!\int_u^v h_{i,r}(t)\, \mathrm{d}\, g_i(t)$$

for every $[u, v] \subset I$ and $x \in \mathbb{R}^n$ with $\|x\| \le r$. Proceeding as in the proof of Lemma 3.1 in [64], we can show that the inequality above still holds if we consider regulated functions $x : I \to \mathbb{R}^n$ with $\|x\|_\infty \le r$. Thus, (ii) follows and this concludes the proof. □

Next, we illustrate the applicability of Theorem 5.87 by considering a model of a bacteria population. This model represents the population of bacteria as well as the medium they live in, in this case water. We shall assume that the population depends directly on the amount of water available which, at the same time, varies as a consequence of the sun.

Example 5.88. Consider an open tank which contains an initial amount of water reaching a level of L meters high and assume that the changes on the level of water are exclusively caused by evaporation as a result of the effect of sunlight. According to this, during the day the level of water will change, whereas it will remain constant during night hours. Now consider a bacteria population whose resources depend directly on the volume of water. This means that the carrying capacity –that is, the number of bacteria that can be supported indefinitely in the tank– will be dependent on the level of water: the higher the level of water is, the bigger the carrying capacity will be. Finally, we will also assume that every morning, the tank is refilled until a certain level depending on the population of bacterias at that time.

We want to design a mathematical model for $w(t)$, the water height at time $t > 0$, and $p(t)$, the bacteria population at time $t > 0$, under the previous assumptions. For the latter, we will consider a logistic model where the carrying capacity will be given by a nondecreasing function $N : \mathbb{R} \to \mathbb{R}$ depending on $w(t)$. Hence, the population p at time t is represented by the equation

$$p'(t) = rp(t)(N(w(t)) - p(t)),$$

where $r > 0$ is a constant representing the reproduction rate of the population.

In order to find an expression that suits $w(t)$, we want to differentiate with respect to a function $g : \mathbb{R} \to \mathbb{R}$ which is constant during night hours and such that it assigns greater measure to middays, when the effect of sunlight is stronger. We also want this derivator to present jump discontinuities at the beginning of each day in order to introduce instantaneous changes in $w(t)$ due to refillings. Thus, if we identify day hours with the intervals $[2k, 2k+1]$, $k = 0, 1, 2, \dots$, and nights with $[2k + 1, 2k + 2]$, $k = 0, 1, 2, \dots$, a possible choice is

$$g(t) = \max\{k = 0, 1, 2, ... : 2k \leq t\} + \int_0^t \max\{\sin(\pi s), 0\} \, \mathrm{d}\, s, \quad t \in \mathbb{R},$$

since it presents the characteristics introduced above. That is, g is constant on $[2k+1, 2k+2]$, $k = 0, 1, 2, \dots$, presents jump discontinuities at times $t \in 2\,\mathbb{N}$, and has maximum slopes at $t = 1/2 + 2k$, $k = 1, 2, \dots$, which represent middays. You can observe this information in Figure 5.1.

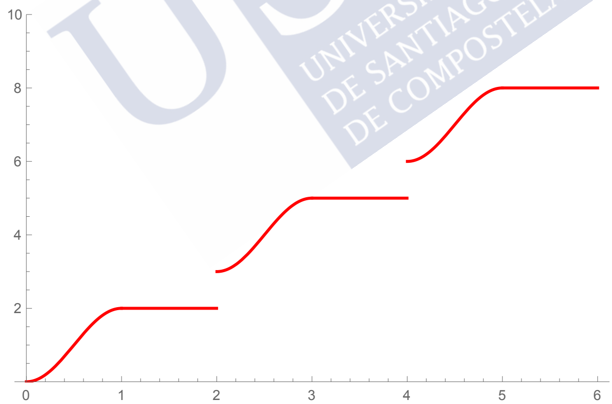


Figure 5.1: Graph of g.

Let $c > 0$ be the evaporation rate of the water and let $a \in \mathbb{R}$ be a proportionality parameter readjusting the mistakes arising from counting the number of bacteria. Fixed an arbitrary time $T > 0$, consider the described problem in the span interval $\overline{I} = [0, T]$. A simple first model for $w(t)$ is given by

$$w_g'(t) = F(t, p(t), w(t)),$$

where $F : \overline{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$F(t, p, w) = \begin{cases} \min\{\lfloor a\, p \rfloor w, 2L - w\}, & \text{if } t \in 2\mathbb{N} \cap \overline{I}, \\ -c, & \text{otherwise,} \end{cases}$$

and $\lfloor\cdot\rfloor$ denotes the floor function. Note that the definition of F at times $t \in 2\mathbb{N}\cap\overline{I}$ guarantees that $w(2k) \leq w(2k^+) \leq 2L$, $k \in \mathbb{N}$. Indeed, given that the jump discontinuities of g have length 1, it follows from the definition of g–derivative that

$$w_g'(2k) = w(2k^+) - w(2k) = \min\{\lfloor \alpha\, p(2k)\rfloor w(2k), 2L - w(2k)\}$$

so $w(2k^+) = \min\{(1 + \lfloor \alpha\, p(2k)\rfloor)w(2k), 2L\} \leq 2L$.

Therefore, we consider the following system of differential equations

$$\begin{cases} p'(t) = rp(t)(N(w(t)) - p(t)), & p(0) = p_0, \\ w_g'(t) = F(t, p(t), w(t)), & w(0) = L, \end{cases} \tag{5.56}$$

which can be regarded as a vectorial Stieltjes differential equation of the form (5.9) with unknown term $x = (p, w)$, initial condition $x_0 = (p_0, L)$, and the functions $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^2$, $f : \overline{I} \times \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$\boldsymbol{g}(t) = (t, g(t)), \quad f(t, (p, w)) = (rp(N(w) - p), F(t, p, w)). \tag{5.57}$$

We will use Theorem 5.87 to show that (5.56) has at least one solution. Clearly, the map $\alpha : \overline{I} \to \mathbb{R}^2$ defined as $\alpha(t) = (0, 0)$ is a lower solution of (5.56). On the other hand, consider the map $W : \overline{I} \to \mathbb{R}$ defined as

$$W(t) = \begin{cases} L - c\displaystyle\int_0^t \max\{\sin(\pi s), 0\}\, \mathrm{d}\, s, & \text{if } t \in [0, 2], \\ \min\{(1 + \lfloor a\, p(2k)\rfloor)w(2k)\,, 2L\} - c\displaystyle\int_{2k}^t \max\{\sin(\pi s), 0\}\, \mathrm{d}\, s, & \text{if } t \in I_k, \end{cases}$$

where $I_k = (2k, 2k + 2] \cap \overline{I}$, $k = 1, 2, \ldots$ In Figure 5.2 we have included a plot of this function for some values of L and c. It is easy to check that the map $\beta : \overline{I} \to \mathbb{R}$ defined as

$$\beta(t) = \left(p_0 \exp\left(\int_0^t rN(W(s))\, \mathrm{d}\, s\right), W(t)\right)$$

is a solution of the following simplification of (5.56):

$$\begin{cases} p'(t) = rp(t)N(w(t)), & p(0) = p_0, \\ w_g'(t) = F(t, p(t), w(t)), & w(0) = L. \end{cases} \tag{5.58}$$

Given the definition of (5.58), it follows that any solution of (5.58) is an upper solution of (5.56). In particular, we have that β is an upper solution of (5.56), so we are in condition to study the applicability of Theorem 5.87. and therefore, an upper solution of (5.56).

First, observe that the map f in (5.57) is quasimonotone nondecreasing in $I \times \mathbb{R}^2$, and in particular in $E = \{(t, x) : \alpha(t) \leq x \leq \beta(t)\}$. Moreover, given that the map N is nondecreasing, it is Borel measurable, so it follows that, $f_i(\cdot, \eta(\cdot))$ is g_i–measurable, $i = 1, 2$, for any regulated function on $\overline{I}$. Therefore, condition (i) in Theorem 5.87 is satisfied. The boundedness condition in (ii) in Theorem 5.87 is trivially satisfied for f_1 as it does not depend

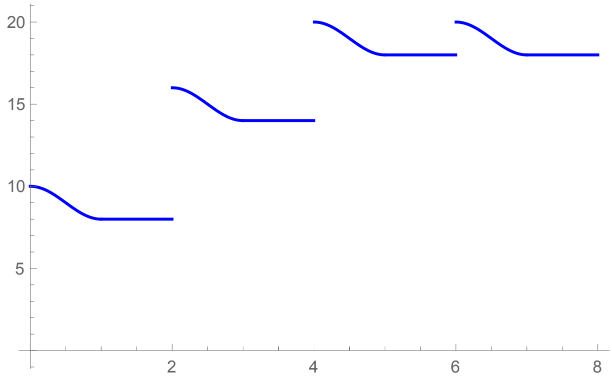


Figure 5.2: Graph of $W(t)$ for $L = 10$, $c = \pi$ and $a = 1/7$.

on the variable t and N is nondecreasing. For f_2, consider $R > 0$ and let $(p, w) \in \mathbb{R}^2$ be such that $\|(p, w)\| \leq R$. Then, for any $t \in \overline{I}$ we have that

$$\begin{aligned} |f_2(t,(p,w))| &\leq \max\{|\min\{\lfloor a\,p \rfloor w, 2L - w\}|, c\} \leq \max\{|\lfloor a\,p \rfloor w|, |2L - w|, c\} \\ &\leq |\lfloor a\,p \rfloor w| + |2L - w| + c \leq |\lfloor a\,p \rfloor||w| + |2L| + |w| + c \\ &\leq |\lfloor a\,R \rfloor| R + 2R + R + c. \end{aligned}$$

Hence, for each $R > 0$, it suffices to consider $h_{2,R}(t) = R(|\lfloor a\,R \rfloor| + 3) + c$, which is trivially integrable with respect to g on $\overline{I}$, to check that condition (ii) in Theorem 5.87 is satisfied. Finally, for conditions (iii) and (iv), for each regulated function $\eta : \overline{I} \to \mathbb{R}^2$, $\eta = (\eta_1, \eta_2)$, let us consider the maps $\varphi_\eta, \psi_\eta : \overline{I} \times \mathbb{R} \to \mathbb{R}$ defined as

$$\begin{aligned} \varphi_\eta(t,u) &= f_1(t, \eta(t) + (u - \eta_1(t))e_1) = ru(N(\eta_2(t)) - u), \quad && (t,u) \in \overline{I} \times \mathbb{R}, \\ \psi_\eta(t,u) &= f_2(t, \eta(t) + (u - \eta_2(t))e_2) = F(t, \eta_1(t), u), && (t,u) \in \overline{I} \times \mathbb{R}. \end{aligned}$$

Now, for any $t \in \overline{I}$, the maps $\varphi_\eta(t, \cdot), \psi_\eta(t, \cdot) : \mathbb{R} \to \mathbb{R}$ are continuous, so condition (iii) in Theorem 5.87 is satisfied. Furthermore, condition (iv) is trivially satisfied for $i = 1$ as the derivator is continuous. As for the case $i = 2$, it is enough to check that the map

$$\Psi_\eta(u) = u + \psi_\eta(t,u), \quad u \in \mathbb{R},$$

is nondecreasing for each $t \in D_g \cap \overline{I} = 2\mathbb{N} \cap \overline{I}$. For such points, we have that

$$\Psi_\eta(u) = u + \min\{\lfloor a\,\eta_1(t) \rfloor u, 2L - u\} = \min\{(\lfloor a\,\eta_1(t) \rfloor + 1)u, 2L\}, \quad u \in \mathbb{R},$$

which is clearly nondecreasing. Hence, the hypotheses of Theorem 5.87 are satisfied, which ensures that (5.56) has the extremal solution between α and β. Moreover, observe that the reasoning used to show that the conditions in Theorem 5.87 did not depend on the lower and upper solution selected. Hence, given a lower and an upper solution of (5.56) on I, Theorem 5.87 ensures the existence of extremal solutions between them.

So far, we have only used the fact that $N(w)$ is a nondecreasing function of w, so no matter if it is continuous or not, our theory applies. However, in some cases we may find it reasonable to allow the carrying capacity $N(w)$ to be piecewise constant because very small changes in the water level could have no influence on the carrying capacity. A simple example appears when we consider N to be the floor function, $N(t) = \lfloor t \rfloor$, for which we can obtain an explicit expression. As we mentioned before, W is a solution of

$$w_g'(t) = F(t, p(t), w(t)), \quad w(0) = L.$$

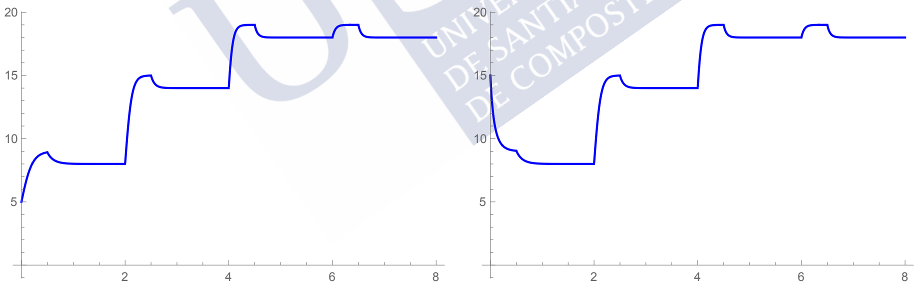
Hence, we obtain the following ODE

$$p'(t) = rp(t)(\lfloor W(t) \rfloor - p(t)), \quad p(0) = p_0. \tag{5.59}$$

It is easy to check that the map $\lfloor W(t) \rfloor$ has at most a countable number of discontinuities, which we will denote by $\{t_i\}_{i\in\mathbb{N}}$. Put $t_0 = 0$. Note that equation (5.59) is a linear equation in each interval $(t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots$, which can be solved exactly. Setting $p_i = p(t_i)$, $i \in \mathbb{N}$, then the solution of (5.59) is

$$p(t) = \frac{e^{\lfloor W(t) \rfloor rt}}{e^{\lfloor W(t_i) \rfloor rt_i}\left(\dfrac{1}{p_i} - \dfrac{1}{\lfloor W(t_i) \rfloor}\right) + \dfrac{e^{\lfloor W(t) \rfloor rt}}{\lfloor W(t) \rfloor}}, \qquad t \in (t_i, t_{i+1}].$$

In Figure 5.3 we have represented the solution for different values of the parameters.



(a) Solution of (5.59) for $L = 10$, $c = \pi$, $a = 1/7$, $r = 1$ and $p_0 = 5$.

(b) Solution of (5.59) for $L = 10$, $c = \pi$, $a = 1/7$, $r = 1$ and $p_0 = 15$.

Figure 5.3: Graphs of the solution of (5.59) for different values of the parameters.

Consider now the functional problem,

$$x_{\boldsymbol{g}}'(t) = f(t, x(t), x) \quad x(t_0) = x_0, \tag{5.60}$$

where $x_0 \in \mathbb{R}^n$, $f : \overline{I} \times \mathbb{R}^n \times G(\overline{I}, \mathbb{R}^n) \to \mathbb{R}^n$ and $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$ is such, for each $i \in \{1, 2, \ldots, n\}$ the map $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and left–continuous. As before, (5.60) denotes a system of Stieltjes differential equations subject to functional arguments; namely:

$$(x_i)_{g_i}'(t) = f_i(t, x(t), x) \qquad x_i(t_0) = x_{0,i}, \quad i \in \{1, \ldots, n\}.$$

A solution of this functional equation is defined analogously to those of problem (5.9), and so are the upper and lower solutions.

Definition 5.89. *A* solution *of* (5.60) *on I is a function $x \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$ such that $x(t_0) = x_0$ and*

$$x'_{g_i}(t) = f_i(t, x(t), x), \quad g_i\text{–a.a.}t \in I, \quad i \in \{1, 2, \ldots, n\}.$$

A lower solution of (5.60) *is a function $\alpha \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$ such that $\alpha(t_0) \leq x_0$ and*

$$\alpha'_{g_i}(t) \leq f_i(t, \alpha(t), \alpha) \quad g_i\text{–a.a. } t \in I, \quad i \in \{1, 2, \ldots, n\}. \tag{5.61}$$

Analogously, an upper solution of (5.60) *is a map $\beta \in \mathcal{AC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$ such that $x_0 \leq \beta(t_0)$ and*

$$\beta'_{g_i}(t) \geq f_i(t, \beta(t), \beta) \quad g_i\text{–a.a. } t \in I, \quad i \in \{1, 2, \ldots, n\}.$$

Remark 5.90. Similarly to Remarks 5.29 and 5.85, it is possible to consider the definitions replacing the space $\mathcal{AC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$ with $E\mathcal{AC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)$.

Applying Theorem 5.83, we present a result for (5.60) in its general formulation. Such a result relies on a correspondence between (5.60) and (5.49) which can be obtained from an extension of the argument used before for problems (5.9) and (5.35).

Theorem 5.91. *Let $f : \overline{I} \times \mathbb{R}^n \times G(\overline{I}, \mathbb{R}^n) \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, be a map. Suppose that* (5.60) *has a lower solution, α, and an upper solution, β, such that $\alpha \leq \beta$ and let*

$$E = \{(t, x) \in I \times \mathbb{R}^n : \alpha(t) \leq x \leq \beta(t)\}.$$

For each $\gamma \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$, denote by $f_\gamma : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$ the map defined as

$$f_\gamma(t, x) = f(t, x, \gamma), \quad (t, x) \in \overline{I} \times \mathbb{R}^n.$$

Assume that for each $\gamma \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$ and $i \in \{1, 2, \ldots, n\}$, the following conditions are satisfied:

(i) *For each $\eta \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$, the function $(f_\gamma)_i(\cdot, \eta(\cdot))$ is g_i-measurable.*

(ii) *For every $r > 0$, there exists a function $h_{i,r} \in \mathcal{L}^1_{g_i}(\overline{I}, [0, +\infty))$ such that*

$$|(f_\gamma)_i(t, x)| \leq h_{i,r}(t), \quad \text{for } g_i\text{–a.a. } t \in \overline{I}, \quad x \in \mathbb{R}^n, \|x\| \leq r.$$

(iii) *For each $\eta \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$ and for g_i–a.a. $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t), \beta_i(t)] \mapsto (f_\gamma)_i(t, \eta(t) + (u - \eta_i(t))e_i)$$

is continuous.

(iv) *For each $\eta \in [\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$ and $t \in \overline{I}$, the mapping*

$$u \in [\alpha_i(t), \beta_i(t)] \mapsto u + (f_\gamma)_i(t, \eta(t) + (u - \eta_i(t))e_i)\Delta^+ g_i(t)$$

is nondecreasing.

If for each $i \in \{1, 2, \dots, n\}$*, the following condition is satisfied:*

(v) *For* g_i*–a.a.* $t \in \overline{I}$ *and all* $x \in \mathbb{R}^n$*, the map* $f_i(t, x, \cdot)$ *is nondecreasing on* $[\alpha, \beta]_{G(\overline{I}, \mathbb{R}^n)}$*,*

then problem (5.60) *has the extremal solutions in* $[\alpha, \beta]_{\mathcal{AC}_{\boldsymbol{g}}(\overline{I}, \mathbb{R}^n)}$.

To finish this chapter, we include an application of Theorem 5.91. We consider a more complex model for the bacteria population than the one considered in Example 5.88.

Example 5.92. We shall show that we can still ensure the existence of extremal solutions for a more complex version of (5.56), where we allow functional arguments. Consider the following modified version of the problem in Example 5.88:

$$\begin{cases} p'(t) = rp(t)(N(w(t)) - p(t)), & p(0) = p_0, \\ w_g'(t) = \widetilde{F}(t, w(t), p), & w(0) = L, \end{cases} \tag{5.62}$$

where the map $\widetilde{F} : I \times \mathbb{R} \times L^1(I, \mathbb{R}) \to \mathbb{R}$ is defined as

$$\widetilde{F}(t, w, \varphi) = \begin{cases} \min\left\{ \left\lfloor a \displaystyle\int_{2(n-1)}^{2n} \varphi(s)\, \mathrm{d}\, s \right\rfloor w\,, 2L - w \right\}, & \text{if } t \in 2\mathbb{N} \cap \overline{I}, \\ -c, & \text{otherwise.} \end{cases}$$

In this case, $\alpha = (0, 0)$ is a lower solution of (5.62) and an upper solution can be obtained analogously to the one from problem (5.56). Either way, proceeding in a similar fashion to the one used in Example 5.88, we can show that the hypotheses of Theorem 5.91 are satisfied provided we can find a lower and an upper solution. To see that this is the case, it is enough to note that, for any $\varphi \in L^1(I, \mathbb{R})$ fixed, the functional arguments in (5.62) become constants, so the reasonings in the mentioned example still hold. The extra functional hypothesis in Theorem 5.91, condition (v), is satisfied by the monotonicity character of the integral. Hence, we can ensure the existence of extremal solutions on $\overline{I}$ between any given lower and upper solutions.

Results

Throughout this thesis, we have studied different mathematical objects in depth. In what follows, we revisit and highlight the most important results obtain in this work.

The first chapter of this manuscript consists, for the most part, of a bibliographical revision of integration theory. Nevertheless, we obtained some new results regarding the Lebesgue–Stieltjes integral, such as Theorem 1.50, in which we show that the Lebesgue–Stieltjes outer measure can be computed using families of pairwise disjoint intervals; Proposition 1.52, which confirms that the Lebesgue–Stieltjes measure associated to a sum of nondecreasing and left–continuous maps coincides with the sum of the corresponding Lebesgue–Stieltjes measures; or Proposition 1.51, a result that, essentially, shows that every simple Lebesgue–Stieltjes measurable function can be approximated by a sequence of step functions. This last result was fundamental for the contents of the next chapter.

The next chapter was devoted to the construction of a new type of derivative, called Δ–derivative, as well as the corresponding measure and their relations. In order to do so, we started by studying displacement spaces as topological objects and, from there, we moved on to the analytical structures. It is at this point that we find Proposition **??** and Definition 2.40 in which we can find the concept of integral with respect to a path of measures. However, the most important results in this chapter are, without a doubt, Theorems 2.62 and 2.71. These results are known as the Fundamental Theorems of the Displacement Calculus as they are the natural extension of the corresponding results for the usual derivative. Roughly speaking, the importance of these results comes from the fact that they show that the Δ–derivative and the Δ–integral are inverse processes.

In Chapter 3 we turned our attention to the particular case of the Stieltjes derivative. In this chapter, and through Proposition **??**, we established the relation between this derivative and the Δ–derivative showing that they are always equivalent. Furthermore, we also related the problems with Stieltjes derivatives to other known differential problems, see Theorems 3.39, 3.42, **??**, 3.49 and 3.51. It is in this chapter that we obtained some interesting properties for the Stieltjes derivative. Specifically, we want to highlight the importance of Proposition 3.13, in which we corrected the product and quotient formula for Stieltjes derivatives presented in [54].

The forth chapter was centered on the study of differential problems involving only one Stieltjes derivative. First, we focus on the results obtained for differential equations of the form

$$x'_g(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

with $f : [t_0, t_0+T) \times \mathbb{R}^n \to \mathbb{R}^n$. For this type of problems, we recreated some known results for the classical setting in the setting of Stieltjes differential equations. For example, we looked for explicit solutions for the linear equation and equations with separation of variables, see Proposition 4.21 and Theorem 4.23. We also proved some existence and uniqueness under the Lipschitz condition (Theorems 4.44 and 4.49) or the Montel–Osgood–Tonelli condition (Theorems 4.46 and 4.51). When we restricted ourselves to the context of the real line, we could use the method of lower and upper solutions to obtain some existence result. For example, in Theorem 4.88 we proved some conditions ensuring that the infimum of upper solutions and the supremum of lower solutions are the extremal solutions of the initial value problem. Similarly, in Theorem 4.61 we ensured the existence of extremal solutions between a lower and upper solutions. In fact, this result was fundamental for the study of the problem with functional arguments,

$$x'_g(t) = f(t, x(t), x), \quad B(x(t_0), x) = 0,$$

with $f : [t_0, t_0+T) \times \mathbb{R} \times \mathcal{AC}_g([t_0, t_0+T], \mathbb{R}) \to \mathbb{R}$ and $B : \mathbb{R} \times \mathcal{AC}_g([t_0, t_0+T], \mathbb{R}) \to \mathbb{R}$. For this problem, our main result was Theorem 4.94, which was obtained by reducing the functional problem to an initial value problem and applying the mentioned result. Finally, in Chapter 4, we also studied differential inclusions of the form

$$x'_g(t) \in F(t, x(t)), \quad x(t_0) = x_0,$$

with $F : [t_0, t_0+T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$. In this case, we found Theorem 4.110, a result ensuring the existence of solution for the differential inclusion which provides new information even in the context of the usual derivative.

Lastly, in Chapter 5 we focused on Stieltjes differential problems involving several different Stieltjes derivatives. Namely, given $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, n$, each of them nondecreasing and left–continuous, we considered

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad x_i(t_0) = x_{0,i}, \quad i = 1, 2, \dots, n,$$

with $f_i : [t_0, t_0+T) \times \mathbb{R}^n \to \mathbb{R}$. In this case, we were able to obtain analogous existence and uniqueness results to those for the case with just one Stieltjes derivative, see, for example, Theorems 5.53, 5.55, 5.58 or 5.62. Furthermore, by looking at the corresponding integral counterpart, we were able to obtain a new existence result in the context of lower and upper solutions. This is the case of Theorem 5.87, which can be regarded as generalization of the methods of lower and upper solutions in the previous chapter. Observe that in this case we did not restrict ourselves to the context of the real line.

Conclusions and future work

Throughout this Thesis we have studied several important concepts, including different derivatives and several differential problems. In what follows, we reflect on the information obtained for these aspects, while also commenting the limitations that we have encountered as well as the possibilities for future work on these topics.

The first notion that we consider is that of displacement. Displacement spaces are presented as a generalization of metric and pseudometric spaces. In particular, the hypothesis required for a map to be a displacement are shown to be the necessary conditions for open balls to be open. Other than that, very few topological aspects for this structure have been studied. It would be interesting to explore further topological properties in this context, especially those that play a major role in the context of mathematical analysis.

At the intersection between displacement spaces and mathematical analysis, we find the concept of derivative. In this work, we have considered two different type of derivatives: displacement derivatives and Stieltjes derivatives. The idea behind the displacement derivative was to come up with a generalization of many derivatives, including the Stieltjes derivative. However, under the hypotheses considered, both derivatives happen to be equivalent. Of course, in the light of this, the following question arises naturally: Is it possible to define a displacement derivative under a different set of hypotheses that still generalizes the Stieltjes derivative and cannot be reduced to such derivative? Recall that the connection between the displacement derivative and the Stieltjes derivative comes from the fact that the displacement measure is a Lebesgue–Stieltjes measure and, thus, has a nondecreasing and left–continuous map that represents it. We believe that a possible way to avoid the equivalence between the two derivatives might come from a new definition of integral based on Δ. It would be interesting to revisit the theory of displacement Calculus under the theory of the Kurzweil–Stieltjes integral. This is one of the possible ways that might lead to a stronger version of derivative. Another possible path to follow is to consider derivatives in a similar fashion to the absolute derivative in [15] in this context. That is, derivatives of functions where, not only the domain is “measured” by a map Δ satisfying the corresponding hypothesis, but the codomain is under these circumstances as well. In other words, the derivative of a function $f : X \to Y$ would be computed, roughly speaking, as

$$f'_{\Delta_X^Y}(t) = \lim_{s \to t} \frac{\Delta_Y(f(t), f(s))}{\Delta_X(t, s)},$$

where $\Delta_X : X \times X \to \mathbb{R}$ and $\Delta_Y : Y \times Y \to \mathbb{R}$ satisfy some set of hypotheses. Observe, however, that although this setting is more general than all the derivatives considered in this

Thesis, it is in detriment of some of the most basic properties of the derivatives. For example, the linearity of the derivative does not follow immediately from the definition, which could have a huge impact on the construction of a Theory of Calculus around this concept.

When it comes to Stieltjes derivatives, the topic of more relaxed hypothesis is interesting as well. In [34], the authors have established a theory where the hypothesis of monotonicity of the derivator was weakened to *functions of controlled variation*. Essentially, these functions are monotonic in each of the connected components determined by removing a negligible set. Exploring this type of derivative and further extensions of Stieltjes derivatives appears as an interesting task. In a similar fashion, and following Chapter 3, it would be interesting to explore more possible relations between Stieltjes differential equations and other problems. In particular, inspired by the ideas in [49], given a nondecreasing and left–continuous function, $g : \mathbb{R} \to \mathbb{R}$, we can consider one of the generalized inverses included in [29], $\gamma : \mathbb{R} \to \mathbb{R}$, defined as

$$\gamma(t) = \sup\{s \in \mathbb{R} : g(s) < t\} = \inf\{s \in \mathbb{R} : g(s) \geq t\}, \quad t \in \mathbb{R}.$$

As pointed out in [23, 44], this map is nondecreasing and left–continuous. Furthermore, as stated in such paper, this generalized inverse "swaps" the jumps and the points around which the map is constant. Exploring the relations between differential equations with Stieltjes derivatives for g and γ could provide new information on the problems as well as new resolution methods.

Finally, we need to reflect on the study of differential problems. Throughout this thesis we have extended some of the classical result for ODEs to the context of Stieltjes derivatives even in the setting where we consider several different derivators. Along those lines, there is still plenty of work to do. We need to consider different approaches and techniques to the study of this type of problems. For example, we could consider the qualitative study of systems of differential equations with Stieltjes derivatives. Similarly, it is necessary to continue to explore differential inclusions with Stieltjes derivatives and other differential problems that have not been studied in this thesis. Particularly, we could focus on problems that arise exclusively in the context of Stieltjes derivatives. For instance, we could look at problems where the derivator depends on the function we are looking for, such as differential equations of the form

$$\begin{aligned} x'_g(t) &= f(t, x(t)), \\ g(t) &= G(t, x), \end{aligned}$$

where $G : \mathbb{R} \times X \to \mathbb{R}$ for some functional space X. This could be used to describe processes where the magnitude represented by the derivator and the magnitude represented by the unknown function are closely related, such as the temperature and pressure of a gas contained in a tank.

With this work we pretend to show the applicability of differential problems with Stieltjes derivatives by means of their theoretical and real–world applications, while also motivating people to continue the study of the many possibilites that this framework offers.

Resumo

Esta Tese, titulada *Problemas diferenciais con derivadas de Stieltjes e aplicacións*, é unha recolección do traballo de investigación realizado polo autor durante a súa etapa predoutoral. Como indica o propio título, este traballo xira ao redor do concepto de derivada de Stieltjes. Dun xeito sinxelo, esta derivada é unha modificación da derivada usual utilizando unha función crecente e continua pola esquerda. Este pequeno cambio na definición permítenos estudar varios problemas diferenciais nun mesmo contexto.

No que segue, presentamos un resumo dos cinco capítulos que compón este manuscrito co fin de explicar en máis detalle os contidos do mesmo. Atendendo á súa temática, podemos dividir os capítulos en tres partes diferenciadas. A primeira delas corresponde ao Capítulo 1 e céntrase na teoría da medida e, en particular, nalgunhas integrais de Stieltjes, xa que son unha ferramenta fundamental para esta Tese. A segunda parte do manuscrito vén marcada polos Capítulos 2 e 3, onde nos centramos no estudo de dúas derivadas distintas, a derivada en desprazamentos e a derivada de Stieltjes, así como tamén na relación existente entre as mesmas. Por último, os Capítulos 4 e 5 conforman a terceira parte deste traballo. Aquí, estudamos algúns problemas diferenciais con derivadas de Stieltjes en profundidade, á vez que amosamos as súas aplicación en diversas situacións.

Capítulo 1: Teoría de integración

Coa intención de facer un traballo que sexa autosuficiente, o Capítulo 1 inclúe unha serie de conceptos e resultados fundamentais para os capítulos que seguen. En particular, este capítulo céntrase na teoría de integración.

Primeiro, presentamos os conceptos de espazo medible e espazo de medida seguindo, principalmente, [5, 73]. Desde aí, construímos a integral con respecto a unha medida e incluímos o Teorema de Radon–Nikodým. Estas dúas ferramentas son clave para a construción de integrais no Capítulo 2.

Na Sección 1.2 centrámonos no estudo de integrais de Stieltjes. En concreto, traballamos coas integrais de Lebesgue–Stieltjes e de Kurzweil–Stieltjes na recta real. Seguindo [5], definimos as integrais de Lebesgue–Stieltjes como a integral con respecto a unha medida de Lebesgue–Stieltjes, i.e. unha medida de Borel que asigna valor finito a conxuntos limitados. Despois, usando o Teorema de Extensión de Carathéodory, probamos que, dada unha función crecente e continua pola esquerda, podemos construír unha medida de Lebesgue–Stieltjes. Reciprocamente, vemos que dada unha medida de Lebesgue–Stieltjes, μ, podemos construír unha función satisfacendo esas hipóteses. É suficiente considerar a función $g : \mathbb{R} \to \mathbb{R}$

definida como

$$g(x) = \begin{cases} -\mu([x,0)), & \text{se } x < 0. \\ 0, & \text{se } x = 0, \\ \mu([0,x)), & \text{se } x > 0. \end{cases}$$

É dicir, existe unha bixección entre o conxunto das medidas de Lebesgue–Stieltjes e o conxunto de funcións crecentes e continuas pola esquerda. Isto é interesante tanto para a construción dunha medida no Capítulo 2, como para os contidos do Capítulo 3.

Doutra banda, a integral de Kurzweil–Stieltjes non é a integral con respecto a unha medida, senón que é construída dun xeito que recorda á integral de Riemann. Neste caso, a definición vén dada en termos dunha función chamada integrador. Isto, xunto coa bixección mencionada anteriormente, permítenos establecer certas conexións entre a integral de Kurzweil–Stieltjes e a de Lebesgue–Stieltjes, seguindo [65].

Capítulo 2: A derivada en desprazamentos

Neste capítulo centrámonos na construcción dunha nova derivada na recta real seguindo as ideas en [61]. Esta derivada xorde de notar que as definicións de derivada en [15, 54, 67] xiran ao redor da mesma idea: medir a loxitude entre dous obxectos, pero conservando o sentido de dirección. Con esta idea en mente, definimos o concepto de *desprazamento*. Un desprazamento nun conxunto X é unha aplicación $\Delta : X \times X \to \mathbb{R}$ satisfacendo as seguintes dúas propiedades:

(a) Para todo $x \in X$, $\Delta(x,x) = 0$.

(b) Para todo $x, y \in X$,

$$|\Delta(x,y)| = \sup\left\{\liminf_{n\to\infty} |\Delta(x,z_n)| : \{z_n\}_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} |\Delta(y,z_n)| = 0\right\}.$$

Os espazos en desprazamento son xeneralizacións dos espazos métricos e pseudométricos e, ao igual que para estes, é posible dotar ao espazo dunha estrutura topolóxica definida en termos do seu desprazamento. Na Sección 2.1 exploramos os aspectos topolóxicos destes obxectos.

Máis adiante, na Sección 2.2 buscamos construír certas estruturas analíticas na recta real que sexan compatibles co concepto en desprazamento. Especificamente, traballamos baixo a suposición de que existe unha aplicación $\Delta : [a,b] \times [a,b] \to \mathbb{R}$ satisfacendo as seguintes condicións:

(H1) Para todo $x \in [a,b]$, $\Delta(x,x) = 0$.

(H2) Para todo $x \in [a,b]$, a función $\Delta(x,\cdot)$ é crecente e continua pola esquerda.

(H3) Existe unha aplicación $\gamma : [a,b] \times [a,b] \to [1,+\infty)$ tal que

(i) Para todo $x, y, z, \overline{z} \in [a,b]$,

$$|\Delta(z,x) - \Delta(z,y)| \leq \gamma(z,\overline{z})|\Delta(\overline{z},x) - \Delta(\overline{z},y)|.$$

(ii) Para todo $z \in [a, b]$,

$$\lim_{\overline{z} \to z} \gamma(z, \overline{z}) = \lim_{\overline{z} \to z} \gamma(\overline{z}, z) = 1.$$

(iii) Para todo $z \in [a, b]$, as funcións $\gamma(z, \cdot), \gamma(\cdot, z) : [a, b] \to [1, +\infty)$ están limitadas.

É importante destacar que as condicións aquí consideradas non son as consideradas en [61]. Isto débese a que, aínda que non é evidente, as hipóteses (H1)–(H3) garanten que a aplicación Δ define un desprazamento en $[a, b]$, suprimindo algunha das condicións alí consideradas.

Na Sección 2.2.1, pasamos á construción dunha medida neste contexto, notando que as hipóteses garanten a existencia de medidas de Lebesgue–Stieltjes "locais" que son absolutamente continuas unhas con respecto das outras, i.e. estas medidas están nas hipóteses do Teorema de Radon–Nikodým. Despois, na Sección 2.2.2, introducimos e estudamos o concepto de derivada en desprazamentos, definida, esencialmente, como

$$f'_{\Delta}(x) = \lim_{y \to x} \frac{f(y) - f(x)}{\Delta(x, y)}.$$

Por último, na Sección 2.2.3, conectamos os conceptos de medida e derivada mediante o Teorema Fundamental do Cálculo utilizando razoamentos similares aos de [54] no contexto de derivadas de Stieltjes.

Capítulo 3: A derivada de Stieltjes

O terceito capítulo está adicado ao estudo da derivada de Stieltjes na recta real. Grosso modo, esta derivada calcúlase como

$$f'_g(x) = \lim_{y \to x} \frac{f(y) - f(x)}{g(y) - g(x)},$$

onde $g : \mathbb{R} \to \mathbb{R}$ é unha función crecente e continua pola esquerda que chamamos derivador. Naturalmente, esta derivada é un caso particular da derivada en desprazamentos. Non obstante, as conexións entre as dúas derivadas non rematan aí. Na Sección 3.1 exploramos outras relacións entre elas, probando que as dúas son equivalentes, a través da bixección existente entre as medidas de Lebesgue–Stieltjes e as funcións crecentes e continuas pola esquerda. Máis adiante, na Sección 3.2, recollemos toda a información existente para a derivada de Stieltjes, adaptando a este contexto os resultados no Capítulo 2, así coma incluíndo outros resultados dispoñibles en [33, 54].

O último que estudamos neste capítulo é a relación entre as ecuacións diferenciais con derivadas de Stieltjes, tamén chamadas *ecuacións diferenciais de Stieltjes*, da forma

$$x'_g(t) = f(t, x(t)), \quad t \in [a, b), \tag{1}$$

e outros problemas diferenciais, usando as ideas de [33,49,54]. É evidente que, por definición, as EDOs son un caso particular de (1). Non obstante, podemos dicir aínda máis. Ao longo da Sección 3.3.1 demostramos que, baixo certas hipóteses, os problemas diferenciais con derivadas de Stieltjes poden reducirse a unha EDO. En particular, isto aporta un método de resolución para ecuacións como (1) sempre que poidamos resolver a EDO asociada. Dun xeito similar, na Sección 3.3.2 estudamos problemas diferenciais con impulsos da forma

$$\begin{aligned} x'(t) &= f(t,x(t)), && t \in [a,b) \setminus J, \\ x(t^+) &= x(t) + I_t(x(t)), && t \in J, \end{aligned}$$

onde J é un conxunto numerable. Neste caso, podemos probar que este problema é equivalente a unha ecuación de Stieltjes para un derivador adecuado. Para rematar, na Sección 3.3.3 definimos a derivada de Hilger e os problemas diferenciais asociados a esta derivada, os cales mostramos, seguindo as ideas de [79], que poden ser estudados como unha ecuación da forma (1) para un determinado derivador.

Capítulo 4: Problemas diferenciais de Stieltjes cun único derivador

O Capítulo 4 céntrase, como o propio título indica, no estudo da existencia e unicidade de solución de varios problemas diferenciais con derivadas de Stieltjes. Nótese que ese mesmo título remarca o uso dun único derivador. Con isto referímonos a que, a pesar de poder considerar problemas en $\mathbb{R}^n$, imos derivar todas as compoñentes con respecto á mesma función. En particular, dada una función crecente e continua pola esquerda, $g : \mathbb{R} \to \mathbb{R}$, consideramos tres problemas diferenciais con derivadas de Stieltjes: problemas de valor inicial da forma

$$x'_g(t) = f(t,x(t)), \quad x(t_0) = x_0, \tag{2}$$

con $f : [t_0, t_0+T) \times \mathbb{R}^n \to \mathbb{R}^n$; ecuacións diferenciais de Stieltjes con argumentos funcionais como

$$x'_g(t) = f(t,x(t),x), \quad B(x(t_0),x) = 0, \tag{3}$$

onde $f : [t_0, t_0+T) \times \mathbb{R} \times X \to \mathbb{R}$ e $B : \mathbb{R} \times X \to \mathbb{R}$ para un certo espazo de Banach, X; e inclusións diferenciais de Stieltjes,

$$x'_g(t) \in F(t,x(t)), \quad x(t_0) = x_0, \tag{4}$$

con $F : [t_0, t_0+T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$.

Na Sección 4.1 traballamos con problemas de valor inicial coma (2), seguindo [48, 49, 51, 53, 58, 59]. Especificamente, na Sección 4.1.1, presentamos algúns métodos de resolución para a ecuación linear, tanto na súa formulación homoxénea coma non homoxénea, e para problemas en variables separables. Despois, na Sección 4.1.2, tratamos de adaptar algúns resultados coñecidos sobre existencia e unicidade de solución para EDOs ao novo contexto de ecuacións diferenciais de Stieltjes. En concreto, estudamos os problemas de valor inicial baixo a condición de Lipschitz,

$$\|f(t,x) - f(t,y)\| \le L(t)\|x-y\|, \quad g\text{–c.t.p. } t \in [t_0, t_0+T), \quad x,y \in \mathbb{R}^n;$$

a condición de Osgood,

$$\|f(t,x) - f(t,y)\| \le \omega(\|x-y\|) \quad g\text{–c.t.p. } t \in [t_0, t_0+T), \quad x,y \in \mathbb{R}^n;$$

a condición de Montel–Tonelli,

$$\|f(t,x) - f(t,y)\| \le \varphi(t)\omega(\|x-y\|), \quad g\text{–c.t.p. } t \in [t_0, t_0+T), \quad x, y \in \mathbb{R}^n;$$

e a condición de Perron,

$$\|f(t,x) - f(t,y)\| \le \omega(t, \|x-y\|), \quad g\text{–c.t.p. } t \in [t_0, t_0+T), \quad x, y \in \mathbb{R}^n.$$

A continuación, pasamos a estudar o problema (2) na recta real mediante o método de sub e sobresolucións. Isto faise na Sección 4.1.3, onde primeiro buscamos as solucións extremais entre unha subsolución e unha sobresolución ben ordeadas, e despois, procuramos condicións necesarias e suficientes para que o supremo puntual das subsolucións e o ínfimo puntual das sobresolucións sexan as solucións extremais de (2).

O estudo de (3) lévase a cabo na Sección 4.2. Alí, usando un resultado de existencia para (2) mediante o método de sub e sobresolucións, xunto coa técnica iterativa xeneralizada de Heikkilä, obtemos un resultado garantindo a existencia de solucións extremais entre unha subsolución e unha sobresolución. Utilizando este resultado, e seguindo as ideas de Biles e Binding en [9], conseguimos un novo resultado de existencia para (2) no contexto de sub e sobresolución que, de novo, é utilizado para obter outro resultado para (3).

Por último, a Sección 4.3 está adicada ao estudo de inclusións diferenciais con derivadas de Stieltjes. Aquí traballamos considerando a envoltura pechada e convexa da aplicación F no problema (4). Nestas circunstancias, obtemos un resultado de existencia de solución, o cal proporciona nova información incluso cando a derivada de Stieltjes resulta ser a derivada usual. Para rematar, voltamos ao problema (2) para o que obtemos un novo resultado de existencia de solución mediante á aplicación de Krasovskij asociada a aplicación f en (2), $\mathcal{K}f : [t_0, t_0+T) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, definida como

$$\mathcal{K}f(t,x) = \bigcap_{r>0} \overline{\text{co}} f\left(t, B(x,r)\right), \quad (t,x) \in [t_0, t_0+T) \times \mathbb{R}^n.$$

Ao longo deste capítulo, ilustramos os resultados que obtivemos para os diferentes problemas considerados con algúns exemplos analíticos, como tamén mediante algunhas aplicacións para situacións reais como cun modelo describindo o movemento dun vehículo impulsado por un motor eléctrico, ou un modelo representando unha poboación de vermes de seda.

Capítulo 5: Problemas diferenciais de Stieltjes con varios derivadores

Dun xeito similar ao Capítulo 4, o Capítulo 5 aborda tamén o tema de existencia e unicidade de solución para problemas diferenciais con derivadas de Stieltjes. Mentres que no Capítulo 4 consideramos un único derivador cando estudamos sistemas de ecuacións diferenciais, neste capítulo permitimos que cada compoñente sexa diferenciada con respecto a un derivador distinto. Para ser máis concretos, tomamos unha aplicación $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$, de xeito que cada g_i, $i = 1, 2, \ldots, n$, sexa un derivador, e consideramos o sistema de ecuacións

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad x_i(t_0) = x_{0,i}, \quad i = 1, 2, \ldots, n, \tag{5}$$

onde $f_i : [t_0, t_0 + T) \times \mathbb{R}^n \to \mathbb{R}$. Obsérvese que, dadas as relacións presentes no Capítulo 3, este novo contexto permítenos estudar fenómenos máis complexos nos que poidamos atopar procesos marcados por diferentes impulsos ou escalas de tempo. Podemos ver isto nalgúns dos exemplos incluídos neste capítulo.

Por suposto, os problemas de valor inicial da forma (5) son unha xeneralización do mesmo tipo de problemas cós que traballamos no Capítulo 4. Non obstante, e a pesar desta conexión, a maioría dos conceptos anteriormente utilizados precisan ser adaptados a este novo contexto. Así, a Sección 5.1 céntrase en estender os conceptos de continuidade no Capítulo 3 para que sexan adecuados para o estudo de (5), seguindo [50, 60]. Máis tarde, na Sección 5.2, xa coas definicións adecuadas e pensando en (5) como unha xeneralización de (2), tratamos de ver se é posible estender a este novo contexto os resultados de existencia e unicidade obtidos na Sección 4.1.2 do Capítulo 4. En particular, seguindo [50, 60], obtemos resultados análogos baixo as condicións de Lipschitz, Osgood, Montel–Tonelli e Perron que devolven os correspondentes resultados do Capítulo 4 cando (5) é da forma (2).

Finalmente, na Sección 5.3 traballamos de novo no contexto de sub e sobresolucións. Non obstante, esta vez non traballamos directamente con ecuacións da forma (5), senón que estudamos problemas da forma

$$x_i(t) = x_{0,i} + \int_{t_0}^{t} f_i(s, x(s))\, \mathrm{d}\, g_i(s), \quad i = 1, 2, \ldots, n, \tag{6}$$

onde consideramos a integral no sentido de Kurzweil–Stieltjes. Así, seguindo [52], obtemos un resultados garantindo a existencia de solucións maximais de (6) entre unha sub e unha sobresolución dadas. Utilizando este resultado e a conexión existente entre as integrais de Lebesgue–Stieltjes e as de Kurzweil–Stieltjes, conseguimos un novo resultado para (5). Tamén conseguimos un resultado de existencia para o caso no que consideramos argumentos funcionais que, de novo, é utilizado para obter o correspondente resultado no contexto de ecuacións diferenciais de Stieltjes con varios derivadores.

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