Dirichlet systems with discrete relativistic operator

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Abstract

We are concerned with Dirichlet systems involving the relativistic discrete operator

$$u \mapsto \Delta\left[\frac{\Delta u(n-1)}{\sqrt{1-|\Delta u(n-1)|^2}}\right] \qquad (n \in \{1,\dots,T\}).$$

Here, for $u : \{0,\dots,T+1\} \to \mathbb{R}^N$, we denote $\Delta u(n-1) := u(n)-u(n-1)$. Besides an "universal" existence result for a system with a general nonlinearity, we obtain multiplicity of solutions for systems with parameterized nonlinearities. Our approaches mainly rely on Brouwer degree arguments and critical point theory for convex, lower semicontinuous perturbations of C^1-functionals.

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1 Introduction

By $\mathbb{Z}[p,q]$ we mean the discrete interval $\{p, p+1, \dots, q\}$ for $p,q \in \mathbb{Z}$ $(p<q)$. The usual Euclidean norm $|\cdot|$ is considered on $\mathbb{R}^N$. We denote by B_σ the open ball

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in $\mathbb{R}^N$ centered in $0_{\mathbb{R}^N}$ of radius σ and let $\phi : B_1 \to \mathbb{R}^N$ be the homeomorphism

$$\phi(y) = \frac{y}{\sqrt{1-|y|^2}} \quad (y \in B_1). \tag{1.1}$$

In this paper, we firstly deal with Dirichlet systems of difference equations having the general form

$$\begin{cases} \Delta\left[\phi(\Delta u(n-1))\right] = f(n, u(n-1), u(n), u(n+1)) \qquad (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{1.2}$$

where $T \in \mathbb{Z}$ is positive and fixed, $\Delta u(n-1) := u(n) - u(n-1)$ is the usual forward difference operator and $f : \mathbb{Z}[1,T] \times (\mathbb{R}^N)^3 \to \mathbb{R}^N$ is continuous; notice, the unknown function u is $\mathbb{R}^N$-valued. As emphasized in [13], the operator $\Delta\left[\phi(\Delta u(n-1))\right]$ may be seen as a discretization of the acceleration in special relativity, hence the name *discrete relativistic operator*.

In recent years, much attention has been paid to equations with singular difference operators of type $\Delta\left[\varphi(\Delta u(n-1))\right]$, with $\varphi : B_\sigma \to \mathbb{R}^N$ a homeomorphism, associated with classical boundary conditions such as Dirichlet, Neumann, mixed or periodic. Results on this topic were achieved by using degree theory, lower and upper solutions and variational methods and focus on the cases when the unknown function is with either scalar values (most of them) [2, 3, 4, 10, 11, 16, 19] or, as in our case, is $\mathbb{R}^N$-valued [1, 13, 14, 15]. It is worth to point out that scalar unknown function ($N = 1$) correspond to the discrete version of a scalar differential equation, while the more general case of vector valued unknown function fits with discrete form of a differential system.

We prove that (1.2) is always solvable (Theorem 2.2), thus extending to systems with vector valued unknown function a known result when the unknown function is with scalar values [3, Theorem 1.2] (also, see [2, Theorem 1]). As the arguments in [3] (or [2]) do not work when $N > 1$, the technique we employ is different and essentially relies on Brouwer topological degree.

One of the main motivations of the present study originates from recent multiplicity results obtained in [12], concerning a Dirichlet autonomous differential system of type

$$\begin{cases} -\left[\phi(u')\right]' = \lambda \nabla H(u) \quad \text{in } [0,b], \\ u(0) = 0_{\mathbb{R}^N} = u(b), \end{cases} \tag{1.3}$$

where $\lambda > 0$ is parameter and the C^1-mapping $H : \mathbb{R}^N \to \mathbb{R}$ is even. Thus, if H has a certain at most quadratic asymptotic behavior near $0_{\mathbb{R}^N}$, then the existence of multiple distinct pairs of solutions for (1.3) is established in terms of the eigenvalues of the scalar operator $-u''$ with Dirichlet boundary conditions (see [12, Theorem 4.1]).

In this paper, using critical point theory for convex, lower semicontinuous perturbations of C^1-functionals developed by Szulkin [20], we obtain multiplicity of solutions for potential systems having the form

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda \nabla G(n, u(n)) \qquad (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a real parameter, the mapping $G(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is of class C^1 for all $n \in \mathbb{Z}[1, T]$ and ∇G stands for the gradient of G with respect to the second variable. When $G(n, \cdot)$ is asymptotically superquadratic near $0_{\mathbb{R}^N}$, we prove the existence of at least two nontrivial solutions for all sufficiently large values of the parameter λ (Theorem 4.1). Here, a key ingredient is the Mountain Pass Theorem. Then, when $G(n, \cdot)$ is even and has a certain at most quadratic asymptotic behavior near $0_{\mathbb{R}^N}$ (see (4.9)), we obtain multiple pairs of nontrivial solutions, in terms of eigenvalues of the scalar operator $-\Delta^2$ with Dirichlet boundary conditions (Theorem 4.2); this is the discrete analogue of [12, Theorem 4.1]. The proof make use of an extension of a result of Clark [5] to Szulkin type functionals. As applications, we derive the existence/non-existence and multiplicity of solutions for a Dirichlet problem involving parameterized Fisher-Kolmogorov nonlinearities (Corollary 4.1) and also address the case of a non-parametric system (Corollary 4.2).

The rest of the paper is organized as follows. In Section 2 we prove the solvability of (1.2). Section 3 is devoted to variational solutions for general potential systems–which actually means that in (1.2) one has $f(n, u(n-1), u(n), u(n+1)) = \nabla F(n, u(n))$, with $F(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ a C^1-mapping for all $n \in \mathbb{Z}[1, T]$. Existence of multiple solutions as well as of multiple pairs of solutions for (1.4) is discussed in Section 4; some illustrative examples of applications are also provided in this section. For the reader's convenience, in the final Section 5 we gather some notions and results in the frame of Szulkin's critical point theory which are needed throughout this paper.

2 Solvability of problem (1.2)

In this section we prove the existence of solutions to problem (1.2). The approach is a topological one and involves Brouwer degree arguments and Brouwer's fixed point theorem.

Denote by X_T the space of all functions $u : \mathbb{Z}[1, T] \to \mathbb{R}^N$ and let

$$X^0_{T+2} := \{u : \mathbb{Z}[0, T+1] \to \mathbb{R}^N \mid u(0) = 0_{\mathbb{R}^N} = u(T+1)\}.$$

Both of the spaces X_T and X^0_{T+2} will be endowed with the inner product and the corresponding norm

$$(u|v)_T = \sum_{j=1}^{T} \langle u(j), v(j) \rangle, \quad \|u\|_T = \left(\sum_{j=1}^{T} |u(j)|^2 \right)^{1/2},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^N$.

For $\ell \in X_T$, we define the operator $E_\ell : \mathbb{R}^N \to \mathbb{R}^N$ by

$$E_\ell(x) = \sum_{n=1}^{T+1} \phi^{-1} \left(\sum_{j=1}^{n-1} \ell(j) + x \right) \quad \left(x \in \mathbb{R}^N \right).$$

Notice, the inverse mapping $\phi^{-1} : \mathbb{R}^N \to B_1$ is given by

$$\phi^{-1}(y) = \frac{y}{\sqrt{1+|y|^2}} \quad \left(y \in \mathbb{R}^N\right). \tag{2.1}$$

Since ϕ satisfies [9, Lemma 2.1]:

$$\langle \phi(x) - \phi(y), x - y \rangle \geq |x-y|^2 \quad (x, y \in B_1),$$

it follows

$$\langle \xi - \eta, \phi^{-1}(\xi) - \phi^{-1}(\eta) \rangle \geq |\phi^{-1}(\xi) - \phi^{-1}(\eta)|^2 \quad \left(\xi, \eta \in \mathbb{R}^N\right),$$

showing that ϕ^{-1} is strictly monotone.

Lemma 2.1 *For any $\ell \in X_T$, the equation $E_\ell(x) = 0_{\mathbb{R}^N}$ has an unique solution $Q(\ell) \in \mathbb{R}^N$. Moreover, the mapping $Q : X_T \to \mathbb{R}^N$ is continuous and satisfies*

$$|Q(\ell)| < (1+\sqrt{2}) \sum_{j=1}^{T} |\ell(j)| + 1, \tag{2.2}$$

for all $\ell \in X_T$.

Proof. The uniqueness of the solution $Q(\ell)$ follows from the strictly monotonicity of the operator E_ℓ. To prove the existence, we use a topological degree argument. With this aim, set

$$\gamma_\ell := \sum_{j=1}^{T} |\ell(j)|, \qquad R_\ell := (1+\sqrt{2})\gamma_\ell + 1$$

and, for any $n \in \mathbb{Z}[1, T+1]$, let $c^n := \sum_{j=1}^{n-1} \ell(j)$; clearly, $|c^n| \leq \gamma_\ell$. Using that the function $[0, \infty) \ni t \mapsto t^2/\sqrt{1+t^2}$ is strictly increasing and that $|c^n + x| \geq ||x| - |c^n||$ for all $x \in \mathbb{R}^N$, one has

$$\frac{|c^n + x|^2}{\sqrt{1+|c^n+x|^2}} \geq \frac{(|x| - |c^n|)^2}{\sqrt{1+(|x|-|c^n|)^2}} \quad \left(x \in \mathbb{R}^N\right). \tag{2.3}$$

Let $x \in \mathbb{R}^N$ with $|x| = R_\ell$. From (2.3), we get

$$\frac{|c^n + x|^2}{\sqrt{1+|c^n+x|^2}} \geq \frac{(\sqrt{2}\gamma_\ell + 1)^2}{\sqrt{1+(\sqrt{2}\gamma_\ell+1)^2}} \geq \frac{(\sqrt{2}\gamma_\ell + 1)^2}{\sqrt{2(\sqrt{2}\gamma_\ell+1)^2}} \geq \gamma_\ell + \frac{\sqrt{2}}{2}. \tag{2.4}$$

On the other hand, it is easy to see that

$$\frac{|\langle c^n + x, c^n \rangle|}{\sqrt{1+|c^n+x|^2}} \leq \gamma_\ell. \tag{2.5}$$

By virtue of (2.4), (2.5) and (2.1), we obtain that, for all $x \in \mathbb{R}^N$ with $|x| = R_\ell$, it holds

$$\begin{aligned}\langle E_\ell(x), x\rangle &= \sum_{n=1}^{T+1} \frac{\langle c^n + x, x\rangle}{\sqrt{1+|c^n+x|^2}} \\ &= \sum_{n=1}^{T+1} \left(\frac{|c^n+x|^2}{\sqrt{1+|c^n+x|^2}} - \frac{\langle c^n + x, c^n\rangle}{\sqrt{1+|c^n+x|^2}} \right) \geq \frac{\sqrt{2}}{2}(T+1).\end{aligned}$$

Next, we introduce the homotopy $\mathcal{H} : [0,1] \times \overline{B}_{R_\ell} \to \mathbb{R}^N$ defined by

$$\mathcal{H}(t,x) = tx + (1-t)E_\ell(x), \quad \forall\, t \in [0,1],\ \forall\, x \in \overline{B}_{R_\ell}.$$

For any $t \in [0,1]$ and $x \in \partial B_{R_\ell}$, we have

$$\langle \mathcal{H}(t,x), x\rangle = t|x|^2 + (1-t)\langle E_\ell(x), x\rangle \geq tR_\ell^2 + (1-t)\frac{\sqrt{2}}{2}(T+1) > 0,$$

showing that $0_{\mathbb{R}^N} \notin \mathcal{H}([0,1] \times \partial B_{R_\ell})$. Hence, by the invariance under a homotopy of the Brouwer degree [6, Theorem 1.2.2], it follows

$$\begin{aligned}\deg[E_\ell, B_{R_\ell}, 0_{\mathbb{R}^N}] &= \deg[\mathcal{H}(0,\cdot), B_{R_\ell}, 0_{\mathbb{R}^N}] = \deg[\mathcal{H}(1,\cdot), B_{R_\ell}, 0_{\mathbb{R}^N}] \\ &= \deg[I_{\mathbb{R}^N}, B_{R_\ell}, 0_{\mathbb{R}^N}] = 1\end{aligned}$$

and consequently, the equation $E_\ell(x) = 0_{\mathbb{R}^N}$ has a (unique) solution $Q(\ell)$ and (2.2) holds true.

It remains to prove the continuity of the mapping $Q : X_T \to \mathbb{R}^N$. For this, let $\{\ell^m\}$ be a sequence in X_T such that $\ell^m \xrightarrow{\|\cdot\|_T} \ell$ in X_T, as $m \to \infty$. We show that for any subsequence of $\{Q(\ell^m)\}$, still denoted by $\{Q(\ell^m)\}$, there exists a subsequence $\{Q(\ell^{m_k})\}$ of $\{Q(\ell^m)\}$ such that $Q(\ell^{m_k}) \to Q(\ell)$, as $k \to \infty$. This will conclude the proof. Since $\{\ell^m\}$ is bounded in $(X_T, \|\cdot\|_T)$ and using (2.2) and the equivalence of the norms on the finite dimensional space X_T, we get that $\{Q(\ell^m)\}$ is bounded in $\mathbb{R}^N$; hence, there is a subsequence $\{Q(\ell^{m_k})\}$ of $\{Q(\ell^m)\}$ and some $\alpha \in \mathbb{R}^N$ such that $Q(\ell^{m_k}) \to \alpha$, as $k \to \infty$. Now, for all $n \in \mathbb{Z}[1, T+1]$, one has

$$\sum_{j=1}^{n-1} \ell^{m_k}(j) + Q(\ell^{m_k}) \to \sum_{j=1}^{n-1} \ell(j) + \alpha,$$

which implies that

$$\sum_{n=1}^{T+1} \phi^{-1}\left(\sum_{j=1}^{n-1} \ell^{m_k}(j) + Q(\ell^{m_k})\right) \to \sum_{n=1}^{T+1} \phi^{-1}\left(\sum_{j=1}^{n-1} \ell(j) + \alpha)\right),$$

i.e., $0_{\mathbb{R}^N} = E_{\ell^{m_k}}(Q(\ell^{m_k})) \to E_\ell(\alpha)$, as $k \to \infty$. Therefore, $E_\ell(\alpha) = 0_{\mathbb{R}^N}$ and by the uniqueness of solution $Q(\ell)$, we get $\alpha = Q(\ell)$. The proof is complete. ■

Remark 2.1 When $N = 1$, Lemma 2.1 was proved by specific to this case arguments in [2, Lemma 2] (also see the proof of Lemma 1.1 in [3]).

In the sequel $Q(\ell)$ will be understood as in Lemma 2.1.

Theorem 2.1 *For any $\ell \in X_T$, problem*

$$\begin{cases} \Delta\left[\phi(\Delta u(n-1))\right] = \ell(n) & (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) \end{cases} \tag{2.6}$$

has an unique solution $u_\ell \in X^0_{T+2}$ and this is given by

$$u_\ell(n) = \sum_{i=0}^{n-1} \phi^{-1}\left(\sum_{j=1}^{i} \ell(j) + Q(\ell)\right) \quad (n \in \mathbb{Z}[0,T+1]). \tag{2.7}$$

Proof. We first prove the uniqueness. To do this, let $u \in X^0_{T+2}$ be a solution of (2.6). Using that

$$\phi(\Delta u(i)) - \phi(\Delta u(i-1)) = \ell(i), \quad (i \in \mathbb{Z}[1,T]),$$

we get

$$\phi(\Delta u(k)) - a = \sum_{j=1}^{k} \ell(j) \quad (k \in \mathbb{Z}[1,T]),$$

where $a := \phi(\Delta u(0)) = \phi(u(1))$. So, we can write

$$\Delta u(k) = \phi^{-1}\left(\sum_{j=1}^{k} \ell(j) + a\right), \tag{2.8}$$

for all $k \in \mathbb{Z}[0,T]$. Summing (2.8) between 0 and $n \in \mathbb{Z}[0,T]$ we obtain

$$u(n+1) = \sum_{i=0}^{n} \phi^{-1}\left(\sum_{j=1}^{i} \ell(j) + a\right) \quad (n \in \mathbb{Z}[0,T]). \tag{2.9}$$

This implies

$$0_{\mathbb{R}^N} = u(T+1) = \sum_{i=1}^{T+1} \phi^{-1}\left(\sum_{j=1}^{i-1} \ell(j) + a\right) = E_\ell(a)$$

and on account of Lemma 2.1, we necessarily must have that $a = Q(\ell)$. Then, the uniqueness of the solution $u \equiv u_\ell$ given by (2.7) easily follows from (2.9).

Finally, checking the fact that u_ℓ in (2.7) satisfies the difference equation in (2.6) is straightforward. ∎

Now we can prove the following "universal" existence result.

Theorem 2.2 *For any continuous function $f : \mathbb{Z}[1,T] \times (\mathbb{R}^N)^3 \to \mathbb{R}^N$, problem* (1.2) *has at least one solution.*

Proof. We consider the Nemytskii type operator $N_f : X^0_{T+2} \to X_T$ defined by

$$N_f(u)(n) = f(n, u(n-1), u(n), u(n+1)) \quad (n \in \mathbb{Z}[1,T])$$

and let $\mathcal{D} : X^0_{T+2} \to X^0_{T+2}$ be given by

$$\mathcal{D}(u)(n) := \sum_{i=0}^{n-1} \phi^{-1}\left(\sum_{j=1}^{i} N_f(u)(j) + Q(N_f(u))\right) \quad (n \in \mathbb{Z}[0, T+1]).$$

It is a simple matter to check that the operator $\mathcal{D}$ is well defined and continuous (as N_f and Q are continuous). Also, since $|\mathcal{D}(u)(n)| \le T$, we deduce that $\|\mathcal{D}(u)\|_T \le T\sqrt{T}$. Thus, Brouwer's fixed point theorem implies that $\mathcal{D}$ has a fixed point $u \in X^0_{T+2}$. Then, from

$$u(n) = \sum_{i=0}^{n-1} \phi^{-1}\left(\sum_{j=1}^{i} N_f(u)(j) + Q(N_f(u))\right)$$

and Theorem 2.1 with $\ell = N_f(u)$ we get that u satisfies

$$\Delta\left[\phi(\Delta u(n-1))\right] = N_f(u)(n) \quad (n \in \mathbb{Z}[1,T]),$$

which ends the proof. ∎

Remark 2.2 In the case $N = 1$, Theorem 2.1 was proved in [3, Lemma 1.1]. The arguments therein do not work in the case $N > 1$ because they make use of the order relation in $\mathbb{R}$. In this respect, Theorem 2.1 is new, both as a result and as a proof – based on Brouwer degree (Lemma 2.1). Also, when $N = 1$, Theorem 2.2 is known to hold true from [3, Theorem 1.2] (also, see [2, Theorem 1]), where its proof, among others, again invokes the order in $\mathbb{R}$. Thus, besides the fact that Theorem 2.2 is more general, the operational approach in its proof is different. We note, however, that both proofs rely on Brouwer's fixed point theorem.

3 Variational solutions

In order to treat problem (1.4), we introduce in this section a variational formulation for a potential Dirichlet problem having the form

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \nabla F(n, u(n)), & (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{3.1}$$

where $F(n,\cdot):\mathbb{R}^N\to\mathbb{R}$ is a mapping of class C^1 for all $n\in\mathbb{Z}[1,T]$. We shall see that one of the solutions of (3.1) always appears as a minimizer of the associated energy functional. For this, denoting $|\Delta u|_\infty := \max_{n\in\mathbb{Z}[0,T]}|\Delta u(n)|$, we introduce the set

$$K := \{u\in X^0_{T+2}:\ |\Delta u|_\infty \le 1\}.$$

Obviously K is convex and, as for $u\in K$ and any $n\in\mathbb{Z}[1,T]$, one has

$$|u(n)|^2 = \left|\sum_{j=0}^{n-1}\Delta u(j)\right|^2 \le T\sum_{j=0}^{T-1}|\Delta u(j)|^2_\infty \le T^2,$$

which yields

$$\|u\|_T \le T\sqrt{T}, \tag{3.2}$$

it follows that the closed set K is compact in X^0_{T+2}. Then, setting $\Phi(y) := 1-\sqrt{1-|y|^2}$ $(y\in\overline{B}_1)$, we consider the functionals

$$\Psi: X^0_{T+2}\to(-\infty,+\infty], \qquad \Psi(u) = \begin{cases}\displaystyle\sum_{j=0}^{T}\Phi[\Delta u(j)], & \text{if } u\in K,\\ +\infty, & \text{if } u\in X^0_{T+2}\setminus K,\end{cases} \tag{3.3}$$

and

$$\mathbf{F}: X^0_{T+2}\to\mathbb{R}, \qquad \mathbf{F}(u) = -\sum_{j=1}^{T}F(j,u(j)).$$

It is not difficult to check that Ψ is proper, convex and lower semicontinuos, while $\mathbf{F}$ is of class C^1, its derivative being given by

$$\mathbf{F}'(u)(v) = -\sum_{j=1}^{T}\langle\nabla F(j,(u(j)),v(j)\rangle \qquad (u,v\in X^0_{T+2}). \tag{3.4}$$

Now, the action functional $I: X^0_{T+2}\to(-\infty,+\infty]$ associated to (3.1) will be

$$I = \Psi + \mathbf{F},$$

which fits the structure required by Szulkin's critical point theory [20] (also, see Appendix section).

Lemma 3.1 *If $\ell\in X_T$, then u_ℓ from Theorem 2.1 is the unique solution in K of the variational inequality*

$$\sum_{j=0}^{T}\{\Phi[\Delta v(j)]-\Phi[\Delta u(j)]\} + \sum_{j=1}^{T}\langle\ell(j),v(j)-u(j)\rangle \ge 0 \quad (v\in K). \tag{3.5}$$

Proof. It will be useful hereafter the summation by parts formula:

$$\sum_{k=m}^{m+n} \langle \Delta a(k-1), b(k)\rangle = \langle a(m+n), b(m+n)\rangle - \langle a(m-1), b(m)\rangle - \sum_{k=m+1}^{m+n} \langle a(k-1), \Delta b(k-1)\rangle, \tag{3.6}$$

for any $a,\ b : \mathbb{Z} \to \mathbb{R}^N$ and $m, n \in \mathbb{N}$.

Using that Ψ is convex, (3.6) and that u_ℓ is solution for problem (2.6), we get that, for all $v \in K$ it holds

$$\begin{aligned}\sum_{j=0}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta u_\ell(j)]\} &\geq \sum_{j=0}^{T} \langle \phi(\Delta u_\ell(j)), \Delta v(j) - \Delta u_\ell(j)\rangle \\ &= \sum_{j=1}^{T+1} \langle \phi(\Delta u_\ell(j-1)), \Delta(v(j-1) - u_\ell(j-1))\rangle \\ &= -\sum_{j=1}^{T} \langle \Delta[\phi(\Delta u_\ell(j-1))], v(j) - u_\ell(j)\rangle = -\sum_{j=1}^{T} \langle \ell(j), v(j) - u_\ell(j)\rangle,\end{aligned}$$

i.e., u_ℓ verifies (3.5). Next, defining $J : K \to \mathbb{R}$ by

$$J(u) = \sum_{j=0}^{T} \Phi[\Delta u(j)] + \sum_{j=1}^{T} \langle \ell(j), u(j)\rangle \quad (u \in K),$$

it is clear that $u \in K$ is a solution of the variational inequality (3.5) if and only if it is a minimum of J on K. Then, using that J is strictly convex, we deduce the uniqueness of the solution of (3.5). ■

The main result of this section is the following

Theorem 3.1 *Any critical point of I is a solution of problem* (3.1)*. Moreover, I satisfies the (PS) condition, is bounded from below and attains its infimum at some $u_0 \in K$, which solves problem* (3.1)*.*

Proof. Let $w \in K$ be a critical point of I. On account of (3.4), for any $v \in K$, one has

$$\sum_{j=0}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)]\} - \sum_{j=1}^{T} \langle \nabla F(j, w(j)), v(j) - w(j)\rangle \geq 0,$$

i.e., w is a solution of the variational inequality

$$\sum_{j=0}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)]\} + \sum_{j=1}^{T} \langle \ell_w(j), v(j) - w(j)\rangle \geq 0, \quad \forall\, v \in K, \tag{3.7}$$

with $\ell_w \in X_T$ given by $\ell_w(n) = -\nabla F(n,(w(n))$ $(n \in \mathbb{Z}[1,T])$. Then, by Lemma 3.1 we infer that actually w solves problem (3.1).

By the compactness of K it is easy to see that I is bounded from below and satisfies the (PS) condition. Finally, from [20, Theorem 1.7] (see Theorem 5.1 in Appendix section) it follows that I attains its infimum at some $u_0 \in K$ and u_0 is a critical point of I. The proof is now completed. ■

4 Multiplicity results for system (1.4)

To establish multiplicity of solutions for the parameterized potential system (1.4), we make use of some spectral properties of the operator $-\Delta^2$ on the space $\mathcal{X}^0_{T+2}$ of all scalar functions $u : \mathbb{Z}[0,T+1] \to \mathbb{R}$ with $u(0) = 0 = u(T+1)$. A number $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of $-\Delta^2$ on $\mathcal{X}^0_{T+2}$, if there is some $u \in \mathcal{X}^0_{T+2} \setminus \{0_{\mathcal{X}^0_{T+2}}\}$ such that

$$-\Delta^2 u(n-1) = \lambda u(n) \qquad (n \in \mathbb{Z}[1,T]) \tag{4.1}$$

and, in this case, u will be called an *eigenfunction* corresponding to the eigenvalue λ. It is straightforward to see that the eigenvalues of $-\Delta^2$ on $\mathcal{X}^0_{T+2}$ are precisely the characteristic roots of the $T \times T$ tridiagonal matrix

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and from [7, p. 45] we know that these are

$$\lambda_m = 4\sin^2 \frac{m\pi}{2(T+1)} \qquad (m \in \mathbb{Z}[1,T]).$$

Notice that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_T < 4$. Also, observe that multiplying equality (4.1) by an arbitrary $v \in \mathcal{X}^0_{T+2}$ and using summation by parts formula (3.6) (with $N = 1$), one has that if λ is an eigenvalue and u is an eigenfunction corresponding to λ, then it holds

$$\sum_{j=0}^{T} \Delta u(j)\Delta v(j) = \lambda \sum_{j=1}^{T} u(j)v(j) \qquad (v \in \mathcal{X}^0_{T+2}). \tag{4.2}$$

We shall need to consider an orthonormal basis $e^1, \ldots, e^T$ in $\mathcal{X}^0_{T+2}$, such that e^k is an eigenfunction corresponding to λ_k $(k \in \mathbb{Z}[1,T])$. From (4.2), we get

$$\sum_{j=0}^{T} \Delta e^i(j)\Delta e^k(j) = \lambda_k \delta_{ik} \qquad (i,k \in \mathbb{Z}[1,T]), \tag{4.3}$$

where δ_{ik} stands for the Kronecker delta function. Also, from [8, Theorem 9] we know that the following discrete Poincaré inequality holds true:

$$\sum_{j=1}^{T}(v(j))^2 \leq \frac{1}{\lambda_1}\sum_{j=0}^{T}(\Delta v(j))^2 \quad (v \in \mathcal{X}^0_{T+2});$$

this yields

$$\|u\|_T^2 \leq \frac{1}{\lambda_1}\sum_{j=0}^{T}|\Delta u(j)|^2 \quad (u \in X^0_{T+2}). \tag{4.4}$$

Next, with Ψ defined in (3.3), the energy functional associated with (1.4) will be now $I_\lambda : X^0_{T+2} \to (-\infty, +\infty]$ given by

$$I_\lambda = \Psi + \mathbf{G}_\lambda,$$

with

$$\mathbf{G}_\lambda : X^0_{T+2} \to \mathbb{R}, \qquad \mathbf{G}_\lambda(u) = -\lambda\sum_{j=1}^{T}G(j, u(j)).$$

Here and below, $G(n,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is of class C^1, for all $n \in \mathbb{Z}[1,T]$ - as assumed in the introductory part. Notice that, by virtue of Theorem 3.1, system (1.4) is solvable, I_λ satisfies the (PS) condition, is bounded from below and any critical point of I_λ is a solution of (1.4), for all $\lambda > 0$.

First, we provide a result concerning the existence of at least two nontrivial solutions (a ground-state and a mountain pass type solution) for system (1.4), when the null solution is know to exist.

Theorem 4.1 *Assume that* $G(\cdot, 0_{\mathbb{R}^N}) = 0$ *and that there is some* $\eta \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$ *with* $|\eta| \leq 1$ *and* $\sum_{j=1}^{T} G(j,\eta) > 0$. *If*

$$\lim_{|x|\to 0}\frac{G(n,x)}{|x|^2} = 0 \qquad (n \in \mathbb{Z}[1,T]), \tag{4.5}$$

then there exists $\Lambda > 0$ *such that* (1.4) *has at least two nontrivial solutions for all* $\lambda > \Lambda$. *If, in addition,* $G(n,\cdot) : \mathbb{R}^N \to \mathbb{R}$ *is even for all* $n \in \mathbb{Z}[1,T]$, *then* (1.4) *has at least two distinct pairs of nontrivial solutions, for all* $\lambda > \Lambda$.

Proof. First, we prove the existence of a constant $\Lambda > 0$ such that, for all $\lambda > \Lambda$, one has

$$\inf_{X^0_{T+2}} I_\lambda < 0. \tag{4.6}$$

Then, as $I_\lambda(0_{X^0_{T+2}}) = 0$, Theorem 3.1 and (4.6) will ensure that (1.4) has a solution which is a nontrivial minimizer of I_λ. Let $u^\eta \in X^0_{T+2}$ be with $u^\eta(n) = \eta$, for all $n \in \mathbb{Z}[1,T]$. Since $|\eta| \leq 1$, we see that $u^\eta \in K$ and $\Psi(u^\eta) = 2-2\sqrt{1-|\eta|^2}$. Choosing

$$\Lambda > \frac{2-2\sqrt{1-|\eta|^2}}{\sum_{j=1}^{T} G(j,\eta)} \geq 0,$$

we have

$$I_\lambda(u^\eta) = 2-2\sqrt{1-|\eta|^2} - \lambda \sum_{j=1}^{T} G(j,\eta) < (\Lambda - \lambda) \sum_{j=1}^{T} G(j,\eta) < 0$$

for all $\lambda > \Lambda$, and (4.6) follows.

Next, let $\lambda > \Lambda$ be fixed and $u_{\lambda,1} \in K$ be the corresponding (nontrivial) minimizer of I_λ with

$$I_\lambda(u_{\lambda,1}) < 0. \tag{4.7}$$

To produce a second nontrivial critical point of I_λ, we will use [20, Theorem 3.2] (see Theorem 5.2 in the Appendix section). Accordingly, on account of (4.7) and the fact that I_λ satisfies the (PS) condition, it suffices to show there exist $\alpha > 0$ and $\rho \in (0, \|u_{\lambda,1}\|_T)$, with

$$I_\lambda(u) \geq \alpha, \quad \text{for all } u \in K, \text{ with } \|u\|_T = \rho.$$

Let $\sigma \in (0, \lambda_1)$. From (4.5) there exists $\delta > 0$ such that

$$|G(j,x)| \leq \frac{\lambda_1 - \sigma}{2\lambda} |x|^2, \quad \text{for all } j \in \mathbb{Z}[1,T] \text{ and } x \in \mathbb{R}^N, \text{ with } |x| \leq \delta. \tag{4.8}$$

This, together with the elementary inequality

$$1 - \sqrt{1-s^2} \geq \frac{s^2}{2} \quad (s \in [0,1])$$

and (4.4) give

$$\begin{aligned} I_\lambda(u) &\geq \sum_{j=0}^{T} \left[1 - \sqrt{1-|\Delta u(j)|^2}\right] - \frac{\lambda_1 - \sigma}{2} \sum_{j=1}^{T} |u(j)|^2 \\ &\geq \sum_{j=0}^{T} \frac{|\Delta u(j)|^2}{2} - \frac{\lambda_1 - \sigma}{2} \|u\|_T^2 \geq \frac{\lambda_1}{2} \|u\|_T^2 - \frac{\lambda_1 - \sigma}{2} \|u\|_T^2 = \frac{\sigma}{2} \|u\|_T^2, \end{aligned}$$

for all $u \in K$ satisfying $\|u\|_T \leq \delta$. Then, taking $\rho \in (0, \min\{\delta, \|u_{\lambda,1}\|_T\})$, one has that $I_\lambda(u) \geq \alpha := \sigma\rho^2/2 > 0$, for all $u \in K$ with $\|u\|_T = \rho$. Therefore, Mountain Pass Theorem ensures the existence of a nontrivial critical point $u_{\lambda,2}$ of I_λ, with $I_\lambda(u_{\lambda,2}) > 0$. Obviously, $u_{\lambda,1} \neq u_{\lambda,2}$.

If $G(n,\cdot)$ is even ($n \in \mathbb{Z}[1,T]$), then so is $\mathbf{G}_\lambda$. It follows that $-u_{\lambda,1}$, $-u_{\lambda,2}$ are critical points of I_λ and, thus, $(u_{\lambda,1}, -u_{\lambda,1})$, $(u_{\lambda,2}, -u_{\lambda,2})$ will be distinct pairs of nontrivial solutions. The proof is complete. ■

Remark 4.1 The reader will emphasize that if $G(\cdot, 0_{\mathbb{R}^N}) = 0$ and (4.5) holds true - as in Theorem 4.1, then (1.4) always has the null solution. This easily follows from (4.8).

Example 4.1 If $w : \mathbb{Z}[1,T] \to \mathbb{R}$ is such that $0 \not\equiv w \geq 0$ and $a \in \mathbb{R}$ is a constant, then there exists $\Lambda > 0$ so that the system

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda \begin{pmatrix} w(n)\left(|u(n)|u(n)_1 + au(n)_1^2\right) \\ \vdots \\ w(n)\left(|u(n)|u(n)_N + au(n)_N^2\right) \end{pmatrix} \qquad (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) \end{cases}$$

has at least two nontrivial solutions for all $\lambda > \Lambda$. Theorem 4.1 applies with

$$G(n,x) = \frac{w(n)}{3}\left(|x|^3 + a\sum_{i=1}^{N} x_i^3\right) \quad (x = (x_1, \cdots, x_N) \in \mathbb{R}^N,\ n \in \mathbb{Z}[1,T]).$$

If $a = 0$ then at least two distinct pairs of nontrivial solutions exist, for all $\lambda > \Lambda$.

More about the existence and multiplicity of pairs of nontrivial solutions will be obtained in the sequel.

Theorem 4.2 *Assume that the C^1 mapping $G(n,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is even, $G(n, 0_{\mathbb{R}^N}) = 0$ and that*

$$\liminf_{|x|\to 0} \frac{2G(n,x)}{|x|^2} \geq 1 \qquad (n \in \mathbb{Z}[1,T]). \tag{4.9}$$

Then, the following hold true:

(i) if $\lambda > 2\lambda_m$ for some $m \in \mathbb{Z}[1,T]$, then (1.4) has at least mN distinct pairs of nontrivial solutions;

(ii) if $\lambda > \lambda_1$, then (1.4) has at least one pair of nontrivial solutions.

Proof. (i) On account of Theorem 3.1 and as Ψ and $\mathbf{G}_\lambda$ are even, we can apply [20, Theorem 4.3] (also, see Theorem 5.3 in the Appendix section). Accordingly, it suffices to prove that

$$\inf_{A\in\Gamma_{mN}} \sup_{v\in A} I_\lambda(v) < 0. \tag{4.10}$$

Since $\lambda > 2\lambda_m$, we can choose $\varepsilon \in (0,1)$ so that $\lambda > 2\lambda_m/(1-\varepsilon)$ and by (4.9), there exists $\delta \in (0, 1/2]$ such that

$$2G(j,x) \geq (1-\varepsilon)|x|^2, \quad \text{for all } j \in \mathbb{Z}[1,T] \text{ and } x \in \mathbb{R}^N, \text{ with } |x| \leq \delta. \tag{4.11}$$

Next, in X^0_{T+2}, we consider the subspace

$$E_m := \left\{\alpha^1 e^1 + ... + \alpha^m e^m \ : \ \alpha^1, ..., \alpha^m \in \mathbb{R}^N\right\},$$

equipped with the norm

$$\left\|\alpha^1 e^1 + ... + \alpha^m e^m\right\|_{E_m} = \left(\sum_{k=1}^{m} |\alpha^k|^2\right)^{\frac{1}{2}}.$$

Notice that $E_m = e^1\mathbb{R}^N \oplus e^2\mathbb{R}^N \oplus \cdots \oplus e^m\mathbb{R}^N$ and hence, dim $E_m = mN$. We set

$$A_m(\rho) := \{v \in E_m : \ \|v\|_{E_m} = \rho\},$$

with $\rho \in (0, \delta]$. Then, it is easily seen that the odd mapping $H : A_m(\rho) \to S^{mN-1}$ given by

$$H\left(\sum_{k=1}^{m} \alpha^k e^k\right) = \left(\frac{\alpha_1^1}{\rho}, \ldots, \frac{\alpha_N^1}{\rho}, \cdots, \frac{\alpha_1^m}{\rho}, \cdots, \frac{\alpha_N^m}{\rho}\right)$$

is a homeomorphism between $A_m(\rho)$ and S^{mN-1} (= the $mN-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^{mN}$). Hence, we estimate the genus $\gamma(A_m(\rho)) = mN$ and so, $A_m(\rho) \in \Gamma_{mN}$.

Let $v = \sum_{k=1}^{m} \alpha^k e^k \in A_m(\rho)$. Using (4.3), one has

$$\begin{aligned}
\sum_{j=0}^{T} |\Delta v(j)|^2 &= \sum_{j=0}^{T} \left|\Delta\left(\sum_{k=1}^{m} \alpha^k e^k(j)\right)\right|^2 = \sum_{j=0}^{T}\left[\sum_{i=1}^{N}\left(\sum_{k=1}^{m} \alpha_i^k \Delta e^k(j)\right)^2\right] \\
&= \sum_{j=0}^{T}\left[\sum_{i=1}^{N}\left(\sum_{k=1}^{m} (\alpha_i^k)^2 (\Delta e^k(j))^2 + \sum_{\substack{l,k=1 \\ l\neq k}}^{m} \alpha_i^k \alpha_i^l \Delta e^k(j) \Delta e^l(j)\right)\right] \\
&= \sum_{k=1}^{m}\left[\sum_{i=1}^{N} (\alpha_i^k)^2 \sum_{j=0}^{T} (\Delta e^k(j))^2\right] \\
&\quad + \sum_{\substack{l,k=1 \\ l\neq k}}^{m}\left[\sum_{i=1}^{N} \alpha_i^k \alpha_i^l \sum_{j=0}^{T} \Delta e^k(j) \Delta e^l(j)\right] \\
&= \sum_{k=1}^{m} \lambda_k |\alpha^k|^2 \leq \lambda_m \|v\|_{E_m}^2 = \lambda_m \rho^2. \qquad (4.12)
\end{aligned}$$

This gives $|\Delta v|_\infty \leq \sqrt{\lambda_m}\rho \leq 2\delta \leq 1$, meaning that $A_m(\rho) \subset K$. Also, it is clear that

$$\begin{aligned}\sum_{j=1}^{T}|v(j)|^2 &= \sum_{j=1}^{T}\left|\sum_{k=1}^{m}\alpha^k e^k(j)\right|^2 = \sum_{j=1}^{T}\left[\sum_{i=1}^{N}\left(\sum_{k=1}^{m}\alpha_i^k e^k(j)\right)^2\right]\\ &= \sum_{j=1}^{T}\left[\sum_{i=1}^{N}\left(\sum_{k=1}^{m}(\alpha_i^k)^2(e^k(j))^2 + \sum_{\substack{l,k=1\\ l\neq k}}^{m}\alpha_i^k\alpha_i^l e^k(j)e^l(j)\right)\right]\\ &= \sum_{k=1}^{m}\sum_{i=1}^{N}(\alpha_i^k)^2 = \sum_{k=1}^{m}|\alpha^k|^2 = \rho^2. \qquad (4.13)\end{aligned}$$

Then, from $|v(j)| \leq \rho \leq \delta$ ($j \in \mathbb{Z}[1,T]$), together with (4.11), (4.12) and (4.13), we estimate I_λ on $A_m(\rho)$ as follows

$$\begin{aligned}I_\lambda(v) &= \Psi(v) + \mathbf{G}_\lambda(v) \leq \sum_{j=0}^{T}|\Delta v(j)|^2 - \frac{\lambda}{2}(1-\varepsilon)\sum_{j=1}^{T}|v(j)|^2\\ &\leq \rho^2\lambda_m - \frac{\lambda}{2}(1-\varepsilon)\rho^2 = \rho^2\frac{2\lambda_m - \lambda(1-\varepsilon)}{2} < 0.\end{aligned}$$

This yields

$$\inf_{A\in\Gamma_{mN}}\sup_{v\in A} I_\lambda(v) \leq \sup_{v\in A_m(\rho)} I_\lambda(v) < 0,$$

showing that (4.10) holds true and the proof of (i) is accomplished.

(ii) Let $d \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$ be such that $\varphi_1 := de^1$ satisfies $|\Delta\varphi_1|_\infty \leq 1$. Then $\varphi_1 \in K \setminus \{0_{\mathbb{R}^N}\}$ and

$$\sum_{j=0}^{T}|\Delta\varphi_1(j)|^2 = \lambda_1\sum_{j=1}^{T}|\varphi_1(j)|^2.$$

We have

$$\lim_{s\to 0^+}\frac{\displaystyle\sum_{j=0}^{T}\left[1-\sqrt{1-|s\Delta\varphi_1(j)|^2}\right]}{\displaystyle\frac{1}{2}\sum_{j=1}^{T}|s\varphi_1(j)|^2} = \lim_{s\to 0^+}\frac{\displaystyle\sum_{j=0}^{T}\frac{s|\Delta\varphi_1(j)|^2}{\sqrt{1-|s\Delta\varphi_1(j)|^2}}}{\displaystyle s\sum_{j=1}^{T}|\varphi_1(j)|^2} = \lambda_1. \qquad (4.14)$$

Now, let $\lambda > \lambda_1$ and let us fix some $\varepsilon > 0$ with $\lambda_1 < \lambda - \varepsilon$. From (4.14) there exists $s_1 = s_1(\lambda,\varepsilon) \in (0,1)$ such that

$$\sum_{j=0}^{T}\left[1-\sqrt{1-|s\Delta\varphi_1(j)|^2}\right] < \frac{\lambda-\varepsilon}{2}\sum_{j=1}^{T}|s\varphi_1(j)|^2 \qquad (s \in (0,s_1)). \qquad (4.15)$$

On the other hand, from (4.9), there is some $\delta = \delta(\lambda, \varepsilon) > 0$ so that

$$2G(j,x) \geq \left(1 - \frac{\varepsilon}{\lambda}\right)|x|^2, \text{ for all } j \in \mathbb{Z}[1,T] \text{ and } x \in \mathbb{R}^N, \text{ with } |x| \leq \delta. \quad (4.16)$$

Then, choosing $s_2 = s_2(\lambda, \varepsilon) \in (0, s_1)$ with

$$\max_{n \in \mathbb{Z}[1,T]} |s_2 \varphi_1(n)| \leq \delta,$$

from (4.15) and (4.16), we infer

$$I_\lambda(s_2\varphi_1) \leq \sum_{j=0}^{T} \left[1 - \sqrt{1 - |s_2 \Delta\varphi_1(j)|^2}\right] - \frac{\lambda - \varepsilon}{2} \sum_{j=1}^{T} |s_2 \varphi_1(j)|^2 < 0 = I_\lambda(0_{\mathbb{R}^N})$$

and hence, if $\lambda > \lambda_1$, the even functional I_λ attains its infimum at some $u_\lambda \in K \setminus \{0_{\mathbb{R}^N}\}$. Therefore, by Theorem 3.1 system (1.4) has a pair of nontrivial solutions $(u_\lambda, -u_\lambda)$ and the proof is now complete. ■

As an application of Theorem 4.2, we obtain existence/non-existence and multiplicity of nontrivial solutions for the Dirichlet system involving Fisher-Kolmogorov type nonlinearities

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda(1 - w(n)|u(n)|^q)u(n) & (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \quad (4.17)$$

where $q > 0$ is fixed and and $w : \mathbb{Z}[1,T] \to \mathbb{R}$. Note that, in this case, one has

$$G(n,x) = \frac{|x|^2}{2} - w(n)\frac{|x|^{q+2}}{q+2} \qquad (x \in \mathbb{R}^N, \ n \in \mathbb{Z}[1,T]). \quad (4.18)$$

Corollary 4.1 *(i) If $\lambda > 2\lambda_m$ for some $m \in \mathbb{Z}[1,T]$, then* (4.17) *has at least mN distinct pairs of nontrivial solutions.*

(ii) If $\lambda > \lambda_1$, then (4.17) *has at least one pair of nontrivial solutions.*

(iii) If $\lambda \in (0, \lambda_1]$ and $w \geq 0$, then the only solution of (4.17) *is the trivial one.*

Proof. Statements (i) and (ii) are immediate by Theorem 4.2 and (4.18). To prove (iii) we proceed by contradiction. Assume that, for some $\lambda \in (0, \lambda_1]$, a function $u \in X^0_{T+2}$ is a nontrivial solution of (4.17). Then, using the summation by parts formula (3.6) and Poincaré inequality (4.4), one gets

$$\begin{aligned} \lambda \sum_{j=1}^{T} |u(j)|^2 (1 - w(j)|u(j)|^q) &= -\sum_{j=1}^{T} \langle \Delta\left[\phi(\Delta u(j-1))\right], u(j)\rangle \\ &= \sum_{j=0}^{T} \langle \phi(\Delta u(j)), \Delta u(j)\rangle = \sum_{j=0}^{T} \frac{|\Delta u(j)|^2}{\sqrt{1 - |\Delta u(j)|^2}} \\ &\geq \sum_{j=0}^{T} |\Delta u(j)|^2 \geq \lambda_1 \sum_{j=1}^{T} |u(j)|^2. \end{aligned} \quad (4.19)$$

This implies

$$0 \geq -\lambda \sum_{j=1}^{T} w(j)|u(j)|^{q+2} \geq (\lambda_1 - \lambda) \sum_{j=1}^{T} |u(j)|^2$$

and, in the case $0 < \lambda < \lambda_1$, we obtain the contradiction

$$0 \geq (\lambda_1 - \lambda) \sum_{j=1}^{T} |u(j)|^2 > 0.$$

If $\lambda = \lambda_1$, from (4.19) it follows

$$\lambda_1 \sum_{j=1}^{T} |u(j)|^2 \geq \lambda_1 \sum_{j=1}^{T} |u(j)|^2 (1 - w(j)|u(j)|^q)$$

$$= \sum_{j=0}^{T} \frac{|\Delta u(j)|^2}{\sqrt{1 - |\Delta u(j)|^2}} \geq \sum_{j=0}^{T} |\Delta u(j)|^2 \geq \lambda_1 \sum_{j=1}^{T} |u(j)|^2$$

and so, we have

$$\sum_{j=0}^{T} |\Delta u(j)|^2 \left(\frac{1}{\sqrt{1 - |\Delta u(j)|^2}} - 1 \right) = 0,$$

which yields $\Delta u(0) = \Delta u(1) = \ldots = \Delta u(T) = 0_{\mathbb{R}^N}$. This means that u is a constant vector and then, as $u \in K$, we infer that $u \equiv 0_{\mathbb{R}^N}$ – a contradiction, again. Thus, (4.17) has only the trivial solution and the proof is complete. ■

We end this section by a result of existence and multiplicity of nontrivial solutions for the non-parametric system (3.1).

Corollary 4.2 *Assume that, for all* $n \in \mathbb{Z}[1, T]$, *the* C^1*-mapping* $F(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ *is even and* $F(n, 0_{\mathbb{R}^N}) = 0$.
(i) If, for some $m \in \mathbb{Z}[1, T]$, F *satisfies*

$$\liminf_{|x| \to 0} \frac{F(n, x)}{|x|^2} > \lambda_m \qquad (n \in \mathbb{Z}[1, T]), \tag{4.20}$$

then (3.1) *has at least* mN *distinct pairs of nontrivial solutions.*

(ii) If

$$\liminf_{|x| \to 0} \frac{F(n, x)}{|x|^2} > \frac{\lambda_1}{2} \qquad (n \in \mathbb{Z}[1, T]), \tag{4.21}$$

holds true, then (3.1) *has at least one pair of nontrivial solutions.*

Proof. By (4.20), there exists $\overline{\lambda} > 0$ such that

$$\liminf_{|x|\to 0} \frac{2F(n,x)}{|x|^2} \geq \overline{\lambda} > 2\lambda_m \qquad (n \in \mathbb{Z}[1,T]).$$

From Theorem 4.2 (i) we have that problem

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \overline{\lambda}\,\nabla\left(\dfrac{F(n,u(n))}{\overline{\lambda}}\right) & (n \in \mathbb{Z}[1,T]),\\ u(0) = 0_{\mathbb{R}^N} = u(T+1) \end{cases}$$

has at least mN distinct pairs of nontrivial solutions. A similar argument works when (4.21) is fulfilled. ∎

Remark 4.2 Theorem 4.2, Corollaries 4.1 and 4.2 are nonautonomous discrete variants of Theorem 4.1, respectively, Corollaries 4.2 and 4.4 (i), (ii) in [12]. For analogous results regarding the multiplicity of periodic solutions of a system with discrete relativistic operator in the case of $N = 1$, we refer the reader to [11].

5 Appendix

We briefly recall here some notions and results in the frame of critical point theory for convex, lower semicontinuous perturbations of C^1-functionals, developed by A. Szulkin [20].

Let $(Y, \|\cdot\|)$ be a real Banach space and $\mathcal{I} : Y \to (-\infty, +\infty]$ be a functional having the structure

$$\mathcal{I} = \mathcal{F} + \psi, \tag{5.1}$$

with $\mathcal{F} \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ proper, convex and lower semicontinuous. An element $u \in D(\psi)$ is said to be *a critical point* of $\mathcal{I}$ if it satisfies the variational inequality

$$\langle \mathcal{F}'(u), v-u\rangle + \psi(v) - \psi(u) \geq 0 \quad (v \in D(\psi)).$$

A number $c \in \mathbb{R}$ such that $\mathcal{I}^{-1}(c)$ contains a critical point is called *a critical value* of the functional $\mathcal{I}$. A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-*sequence* if $\mathcal{I}(u_n) \to r \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v-u_n\rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v-u_n\| \quad (v \in D(\psi)),$$

where $\varepsilon_n \to 0$. The functional $\mathcal{I}$ is said *to satisfy the* (PS) *condition* if every (PS)-sequence possesses a convergent subsequence in Y.

Theorem 5.1 ([20, Theorem 1.7]) *If $\mathcal{I}$ is of type* (5.1)*, satisfies the (PS) condition and is bounded from below, then*

$$c := \inf_{Y} \mathcal{I}$$

is a critical value of $\mathcal{I}$.

The following theorem extends for functionals of type (5.1) the classical Mountain Pass Theorem.

Theorem 5.2 ([20, Theorem 3.2]) *Assume that $\mathcal{I}$ has the structure* (5.1)*, satisfies (PS) condition and that*

(i) $\mathcal{I}(0) = 0$ and there exist $\alpha, \rho > 0$ such that $\mathcal{I}(u) \geq \alpha$ if $\|u\| = \rho$;

(ii) $\mathcal{I}(e) \leq 0$ for some $e \in Y$, with $\|e\| > \rho$.

Then, $\mathcal{I}$ has a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{\theta \in \Theta} \sup_{t \in [0,1]} \mathcal{I}(\theta(t)),$$

where $\Theta = \{\theta \in C([0,1]; Y) \ : \ \theta(0) = 0, \ \theta(1) = e\}$.

Let Σ be the collection of all symmetric and closed subsets of $Y \setminus \{0\}$. The Krasnoselskii *genus* of a nonempty set $A \in \Sigma$, denoted $\gamma(A)$, is defined as being the smallest integer k with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$. If such an integer does not exist, then $\gamma(A) := +\infty$. It is known that if $A \in \Sigma$ is homeomorphic to S^{k-1} (= the $k-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^k$) by an odd homeomorphism, then $\gamma(A) = k$ (see e.g. [18, Corollary 5.5]). For other properties and more details of the notion of genus we refer the reader to [17, 18]. Denoting by Γ the collection of all nonempty compact, symmetric subsets of Y, endowed with the Hausdorff-Pompeiu distance and setting

$$\Gamma_k := cl\{A \in \Gamma : \ 0 \notin A, \ \gamma(A) \geq k\}$$

(cl is the closure in Γ), the following theorem is an immediate consequence of [20, Theorem 4.3].

Theorem 5.3 *Let $\mathcal{I}$ be of type* (5.1) *with $\mathcal{F}$ and ψ even. Also, suppose that $\mathcal{I}$ is bounded from below, satisfies the (PS) condition and $\mathcal{I}(0) = 0$. If*

$$\inf_{A \in \Gamma_k} \sup_{v \in A} \mathcal{I}(v) < 0,$$

then the functional $\mathcal{I}$ has at least k distinct pairs of nontrivial critical points.

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