INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

WILEY

INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

Intl. Trans. in Op. Res. 32 (2025) 1620–1640
DOI: 10.1111/itor.13518

Sequencing situations with position-dependent effects under cooperation

A. Saavedra-Nieves[a,d,*], M. A. Mosquera[b,d] and M. G. Fiestras-Janeiro[c,d]

[a] *Universidade de Santiago de Compostela, Departamento de Estatística, Análise Matemática e Optimización, Santiago de Compostela 15705, Spain*
[b] *Universidade de Vigo, Departamento de Estatística e Investigación Operativa, Ourense 32004, Spain*
[c] *Universidade de Vigo, Departamento de Estatística e Investigación Operativa, Vigo 36310, Spain*
[d] *CITMAga, Santiago de Compostela 15782, Spain*
E-mail: alejandro.saavedra.nieves@usc.es[Saavedra-Nieves]; mamrguez@uvigo.es[Mosquera]; fiestras@uvigo.es[Fiestras-Janeiro]

Received 14 February 2024; received in revised form 25 June 2024; accepted 12 July 2024

Abstract

This paper innovatively addresses the effect of cooperation on sequencing situations with position-dependent effects. Specifically, we ensure the convexity of the associated sequencing games under the fulfillment of certain conditions related to the neighbor switching gains. Additionally, we propose two families of allocation rules based on sharing the neighbor switching gains under two distinct procedures, each providing a path from the initial order to an optimal order. From a theoretical point of view, an axiomatization of both families of allocations is provided, and their stability is also ensured under the conditions related to convexity.

Keywords: game theory; position-dependent effects; neighbor switching gains; convexity; stable allocations

1. Introduction

The most basic sequencing situations naturally assume the processing of a collection of jobs on a single machine. In these situations, each job is associated with an agent (a player). An initial order of the jobs is assumed before processing begins. A common task in sequencing is to find an optimal order that minimizes the total costs of processing for all players. Interactive (time or cost) saving situations arise from the cooperation among players as they rearrange jobs from the initial order to an optimal order.

In standard sequencing situations (cf. Smith, 1956), specified by a linear cost function of the completion time of jobs, the nonincreasing order of the ratio of the linear cost coefficient to the

*Corresponding author.

processing time (the urgency index) minimizes the total costs. Curiel et al. (1989) defined a transferable utility cooperative game for each standard sequencing situation and analyzed its theoretical properties. Specific allocations of the total cost savings, based on the equal division of gains from switching two misplaced neighbors, are easily obtained using the so-called Equal Gain Splitting rule (EGS-rule). Various studies have extended the analysis of standard sequencing situations by relaxing or modifying some assumptions of the standard model. Borm et al. (2001) studied several classes of these games. Borm et al. (2002) studied due date criteria, while van Velzen (2006) assumed nonfixed processing times. Klijn and Sánchez (2006) introduced uncertainty in some parameters, and Estévez-Fernández et al. (2008) incorporated repeated players. Research by Gerichhausen and Hamers (2009), Grundel et al. (2013), or Çiftçi et al. (2013) focused on the effects of grouping players into families or batches. Musegaas et al. (2015, 2018) analyzed step out-step in sequencing games and their convexity. Curiel (2015) introduced multistage sequencing situations, and Liu et al. (2018) considered effects on machine availability. Yang et al. (2019) addressed cooperation in sequencing situations with externalities. More recently, Saavedra-Nieves et al. (2020) and Schouten et al. (2021) considered exponential and logarithmic cost functions in sequencing, respectively. Instead of imposing constraints on the jobs, Calleja et al. (2002) and Slikker (2005) analyzed cooperation in sequencing by extending the number of available machines. Slikker (2023) studied relaxed sequencing games arising from the profit by rearranging coalitions, and van Beek et al. (2023) analyzed standard sequencing problems from a noncooperative perspective.

Sequencing problems typically assume fixed job processing times. However, in practice, modeling sequencing situations often requires additional assumptions to more closely approximate reality, particularly when the processing position is critical. In many real-world problems, factors such as machine deterioration or learning can cause the processing time of a job to be nonfixed. This is common in industrial environments related to production or maintenance processes, or in firefighting, where a delay in action increases the time and effort required to complete the task. Broadly, two subclasses of sequencing problems arise based on these effects. Deterioration effects imply that the later a job starts, the longer it takes to process. Conversely, learning effects mean that the actual processing time of a job gets shorter the later the job is processed. Cheng et al. (2004) provide a survey on sequencing with time-dependent processing times, and Biskup (2008) gives an overview of sequencing situations with learning effects. Gordon et al. (2008) analyzed the problems of minimizing the makespan or the total flow time under various positional effects on processing times. Rustogi and Strusevich (2012a, 2012b) reviewed sequencing problems with position-dependent effects. Briskorn et al. (2021) studied sequencing problems involving an external resource for a noninterrupted period. Yang et al. (2020) analyzed cooperative sequencing problems considering potential position effects.

Given the variety of real-world applications, this paper examines a common model for allocating cost savings from cooperation in sequencing situations with position-dependent effects on the machine. To the authors' knowledge, a cooperative perspective on this class of problems has not yet been fully explored in the literature. Yang et al. (2020) provide a preliminary approach that serves as a basis for generalization to a broader context. This generalization encompasses a wider family of positional effects on processing times and allows for a broader range of objective functions, beyond just minimizing total processing costs typically considered in game-theoretic contexts. In contrast to standard sequencing situations, neighbor switching gains are now position dependent. The problem of allocating joint savings under position-dependent effects is approached from a game-theoretic perspective, requiring additional conditions on neighbor switching gains to analyze them as sequencing games (cf. Curiel et al., 1989). First, we provide a sufficient condition under which the games arising from these new sequencing situations are σ_0-component additive (cf. Curiel et al., 1989), ensuring stable savings allocations. Additionally, following the ideas of Saavedra-Nieves et al. (2020) and Schouten et al. (2021), we study the convexity of this new class of transferable utility (TU) games. We define two allocation rules for general sequencing situations with position-dependent effects, based on specific paths from the initial order to an optimal order. These allocations use the ideas of the Gain Splitting rules for standard sequencing situations in Hamers et al. (1996), which divide the neighbor switching gains at each step between the involved players. This allows their characterization, consistent with Yang et al. (2020).

The structure of the paper is as follows. Section 2 introduces preliminaries on sequencing situations with position-dependent effects and TU games. Section 3 introduces sequencing games with position-dependent effects and provides conditions under which convexity and σ_0-component additivity are satisfied. In Section 4, we study the stability of the proposed allocation rules for the resulting savings. Section 5 provides characterizations for each of these allocation rules. Concluding remarks are presented in Section 6.

2. Preliminaries

In this section, we formally introduce some theoretical terminology on sequencing problems and cooperative game theory to provide a better understanding of the rest of the article.

A *sequencing situation with position-dependent effects* is given by (N, σ_0, p, f, c), where $N = \{1, 2, \ldots, n\}$ is the *set of jobs* that need to be processed on a single machine. The order σ_0 is the *initial order* in which the jobs are arranged in front of the machine. Formally, an order is a bijection $\sigma : N \to \{1, 2, \ldots, n\}$ such that $\sigma(i) = k$ means that job i is at position k in the queue. $\Pi(N)$ is the *set of all orders* of N. The vector $p = (p_i)_{i\in N} \in \mathbb{R}^N_{++}$ indicates the original *processing times* of the jobs; $f : \{1, 2, \ldots, n\} \longrightarrow \mathbb{R}^{++}$ is a map that indicates the positional effect of processing a job i at position k, and the function $c : \Pi(N) \longrightarrow \mathbb{R}$ represents the global cost objective function, which evaluates the cost of processing jobs in N according to any order $\sigma \in \Pi(N)$ and is assumed to be additive and regular. Following Curiel et al. (1994), the global cost of an order is the sum of the individual costs of processing jobs, prescribed by a weakly monotonic function of the time each job spends in the system. Let $\sigma \in \Pi(N)$ be an order. The *set of predecessors* of $i \in N$ with respect to σ is denoted by $P(\sigma, i)$ and given by $P(\sigma, i) = \{k \in N : \sigma(k) < \sigma(i)\}$. The *set of followers* with respect to σ is denoted by $F(\sigma, i)$ and given by $F(\sigma, i) = \{k \in N : \sigma(k) > \sigma(i)\}$.

We consider positional effects on the processing times given by $f(k)$, for every $k \in \{1, \ldots, n\}$, satisfying that $f(1) = 1$. Thus, the *effective processing time* of job i at position k in a given order is exactly given, for all $k \in \{1, \ldots, n\}$, by

$$p_{ik} = p_i f(k), \text{ with } f(k) > 0.$$

Some models considering position-dependent effects under this scheme include those studied by Gordon et al. (2008) and Rustogi and Strusevich (2012b). Hereafter, (N, σ_0, p, f, c) refers to a sequencing situation with position-dependent effects and the set of all these situations is denoted by $PSEQ^N$.

Take $(N, \sigma_0, p, f, c) \in PSEQ^N$. In this sequencing context, the main purpose is to minimize the considered objective function c. The *completion times* of the jobs in N in an order $\sigma \in \Pi(N)$ are denoted by $C_i(\sigma)$ and given by

$$C_i(\sigma) = p_{i\sigma(i)} + \sum_{k \in P(\sigma,i)} p_{k\sigma(k)}, \text{ for all } i \in N.$$

Given a schedule for the jobs specified by an order $\sigma \in \Pi(N)$, we denote the value of the cost objective function by $c(\sigma)$. A useful example is the total flow time, which corresponds to the sum of the completion times for N according to σ. Thus, an order $\hat{\sigma}$ is *optimal for c* if it minimizes the cost criterion c considered over all orders of N. Note that the optimal order may not be unique. Accordingly, the set of *misplacements* with respect to an optimal order $\hat{\sigma}$ is given by the set of pairs of players that are ordered inversely in σ_0 and $\hat{\sigma}$. Formally, for a sequencing situation with positional effects $(N, \sigma_0, p, f, c) \in PSEQ^N$ and for an optimal order $\hat{\sigma}$, we have that

$$MP(\sigma_0, \hat{\sigma}) = \{(i, j) \in N \times N \ : \ \sigma_0(i) < \sigma_0(j) \text{ and } \hat{\sigma}(i) > \hat{\sigma}(j)\}. \tag{1}$$

In practice, any optimal order is obtained from the initial order by recursively interchanging pairs of consecutive jobs, because every permutation is the product of transpositions of pairs of consecutive misplaced neighbors. Thus, a sequence of permutations from σ_0 to an optimal order $\hat{\sigma}$ is a *path* $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m = \hat{\sigma})$ if, for every $k \in \{1, \ldots, m\}$, $\sigma_k \in \Pi(N)$ and σ_{k-1} and σ_k only differ in one pair of consecutive jobs. From now on, a path from σ_0 to $\hat{\sigma}$ will be denoted by $\mathcal{P}(\hat{\sigma})$. That is, there exist i_k and j_k in N such that $\sigma_{k-1}(j_k) = \sigma_{k-1}(i_k) + 1$ and $\sigma_k(i_k) = \sigma_{k-1}(j_k)$, $\sigma_k(j_k) = \sigma_{k-1}(i_k)$ and $\sigma_k(\ell) = \sigma_{k-1}(\ell)$ for every $\ell \in N \setminus \{i_k, j_k\}$. We only consider those paths $\mathcal{P}(\hat{\sigma}) = (\sigma_0, \sigma_1, \ldots, \sigma_m = \hat{\sigma})$ repairing all misplacements step by step.

In our context, the cost criterion considered varies in each step of a path from the initial order to the optimal order. In this sense, the *gain* from switching two consecutive jobs corresponds to the variation in the cost criterion of processing jobs. Let $\sigma \in \Pi(N)$ and $i, j \in N$ a pair of jobs such that $\sigma(i) = k$ and $\sigma(j) = k + 1$ for some $k \in \{1, \ldots, n - 1\}$. The *gain* from switching these jobs in σ is formally given by $c(\sigma) - c(\tau)$, where $\tau \in \Pi(N)$ is an order of N such that $\tau(i) = \sigma(j)$, $\tau(j) = \sigma(i)$ and $\tau(\ell) = \sigma(\ell)$, for all $\ell \in N \setminus \{i, j\}$. From now on, we denote $g_{ij}(\sigma)$ as the neighbor switching gain from switching the pair of consecutive jobs i, j of N in σ. Since the effective processing times of i and j change after switching, the neighbor switching gains become position-dependent. Clearly, the completion times of the followers of j are also modified under these circumstances even though they maintain their original positions relative to the machine. Therefore, the gain from switching i and j also includes the corresponding variation in the global cost caused by these changes in the completion times of the followers of job j in σ.

For some specific sequencing problems, an optimal order $\hat{\sigma}$ can be obtained if jobs are processed according to the nondecreasing order of certain priorities assigned to them. If this condition holds, we refer to the existence of an *index policy*. Two of the most popular index policies in sequencing problems are based on the processing times of jobs and are introduced below. Consider $(N, \sigma_0, p, f, c) \in PSEQ^N$ and $\hat{\sigma} \in \Pi(N)$.

- The *Shortest Processing Time rule* prescribes an optimal order $\hat{\sigma}$ for c if and only jobs with shorter processing times are processed earlier. Formally, it satisfies that

$$p_{\hat{\sigma}^{-1}(1)} \leq \cdots \leq p_{\hat{\sigma}^{-1}(k-1)} \leq p_{\hat{\sigma}^{-1}(k)} \leq p_{\hat{\sigma}^{-1}(k+1)} \leq \cdots \leq p_{\hat{\sigma}^{-1}(n)}. \quad (2)$$

- The *Largest Processing Time rule* prescribes an optimal order $\hat{\sigma}$ for c if and only if jobs with longer processing times are processed earlier. Formally, it satisfies that

$$p_{\hat{\sigma}^{-1}(1)} \geq \cdots \geq p_{\hat{\sigma}^{-1}(k-1)} \geq p_{\hat{\sigma}^{-1}(k)} \geq p_{\hat{\sigma}^{-1}(k+1)} \geq \cdots \geq p_{\hat{\sigma}^{-1}(n)}. \quad (3)$$

In sequencing situations, assuming that jobs belong to players, cooperation among the players is key to reaching the optimal order from an initial order. Hence, TU games are useful. A *TU game* is a pair (N, v), where N is the *set of players* and v is the *characteristic function*, that is, a map from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$. In general, $v(S)$ represents the worth of cooperation by players in S, for all $S \subseteq N$.

A TU game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for every pair of coalitions $S,\ T \subseteq N$ with $S \cap T = \emptyset$. A TU game (N, v) is *convex* (cf. Shapley, 1971) if, for all $i \in N$ and for all $S \subset T \subseteq N \setminus \{i\}$, it satisfies $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

TU games deal with coalitions and allocations, considering groups of agents willing to allocate the joint benefits derived from their cooperation. Let (N, v) be a TU game, an allocation of (N, v) is an element $x \in \mathbb{R}^N$. A solution concept (or a value) for TU games is a map ψ, which assigns to every TU game (N, v) a subset of allocations $\psi(v) \subset \mathbb{R}^N$. Two well-known solution concepts are the Shapley value (Shapley, 1953) and the τ-value (Tijs, 1981), among others. Additionally, the *core* of a TU game (N, v) (Gillies, 1953) is given by

$$Core(v) = \left\{ x \in \mathbb{R}^N \ : \ \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N \text{ with } S \neq \emptyset \right\}. \quad (4)$$

The core elements are considered efficient and stable, meaning that no coalition has an incentive to split off from the grand coalition N. The fulfillment of specific properties of the cooperative game is crucial. For instance, both the Shapley value and the τ-value of any TU game belong to the core if the game is convex.

3. Sequencing games with position-dependent effects

This section analyzes sequencing situations with position-dependent effects from a game-theoretic perspective. Let $(N, \sigma_0, p, f, c) \in PSEQ^N$ be a sequencing situation with position-dependent effects. A *sequencing game with position-dependent effects* is represented by the TU game (N, v), where players are identified with jobs (or each job is owned by only one player) and $v(S)$ denotes the maximal savings that coalition S achieves through admissible rearrangements of their jobs relative to σ_0. An order $\sigma \in \Pi(N)$ is *admissible* with respect to σ_0 for a nonempty coalition $S \subseteq N$, if $P(\sigma, i) = P(\sigma_0, i)$ for all $i \in N \setminus S$; that is, if the outsiders of S remain in exactly the same positions as in σ_0. The set of admissible orders for S with respect to $\sigma_0 \in \Pi(N)$ is denoted by $\mathcal{A}(\sigma_0, S)$.

Thus, the sequencing game with position-dependent effects associated with (N, σ_0, p, f, c) is formally given by

$$v(S) = \max_{\sigma\in\mathcal{A}(\sigma_0,S)} \{c(\sigma_0) - c(\sigma)\}, \text{ for all } S \subseteq N \text{ with } S \neq \emptyset. \tag{5}$$

A coalition $S \subseteq N$ is *connected* with respect to an order $\sigma \in \Pi(N)$ if for all $i, j \in S$ and $k \in N$ such that $\sigma(i) < \sigma(k) < \sigma(j)$, it holds that $k \in S$. Moreover, for any $S \subseteq N$ and $\sigma \in \Pi(N)$, a component of S with respect to σ is a maximally connected subset of S with respect to σ. The set of all components of a coalition $S \subseteq N$, with $S \neq \emptyset$, and an order $\sigma \in \Pi(N)$ is denoted by $S|_\sigma$.

Example 1 illustrates the determination of sequencing games with position-dependent effects.

Example 1. Let $(N, \sigma_0, p, f, c) \in PSEQ^N$ such that $N = \{1, 2, 3, 4\}, \sigma_0 = (1, 2, 3, 4)$, and $p_1 = 0.93$, $p_2 = 0.94$, $p_3 = 0.86$, and $p_4 = 0.9$. Consider $f(k) = b^{k-1}$ for all $k \in \{1, \dots, n-1\}$, with $b = 1.6$, and $c(\sigma) = \sum_{\ell\in N} e^{C_\ell(\sigma)}$. Below, we depict the global costs for all orders of N.

σ	$c(\sigma)$	σ	$c(\sigma)$	σ	$c(\sigma)$	σ	$c(\sigma)$
(1,2,3,4)	4230.407	(2,3,4,1)	4693.391	(3,4,1,2)	5189.909	(4,1,2,3)	4226.882
(2,1,3,4)	4205.141	(1,3,4,2)	4836.244	(3,4,2,1)	5115.430	(4,2,1,3)	4186.802
(2,3,1,4)	4495.271	(3,1,4,2)	5043.412	(1,4,2,3)	4151.598	(4,2,3,1)	4627.304
(1,3,2,4)	4565.621	(3,2,4,1)	4923.845	(2,4,1,3)	4087.673	(4,1,3,2)	4739.621
(3,1,2,4)	4761.181	(1,2,4,3)	3996.404	(2,4,3,1)	4517.729	(4,3,1,2)	5066.987
(3,2,1,4)	4715.984	(2,1,4,3)	3972.538	(1,4,3,2)	4655.190	(4,3,2,1)	4994.274

Let $S = \{1, 2\}$ and $T = \{3, 4\}$, two disjoint connected coalitions with respect to σ_0. By using (5), we have $v(S) = 25.266$ and $v(T) = 234.003$. If we take $S \cup T$, we obtain $v(S \cup T) = 257.869$. We directly check that the players in N have no incentives to cooperate because

$$25.266 + 234.003 = v(S) + v(T) > v(S \cup T) = 257.869.$$

That is, (N, v) is not superadditive.

The lack of superadditivity illustrated in the previous example indicates that sequencing games derived from sequencing situations with position-dependent effects generally do not belong to the class of σ_0-component additive games (cf. Curiel et al., 1993). Given $\sigma_0 \in \Pi(N)$, a TU game is called a σ_0-component additive game if the following conditions hold:

- $v(\{i\}) = 0$, for all $i \in N$;
- (N, v) is superadditive; and
- $v(S) = \sum_{T\in S|_{\sigma_0}} v(T)$ for all $S \subseteq N$, with $S \neq \emptyset$.

Let (N, v) be the sequencing game with position-dependent effects associated with the sequencing situation with position-dependent effects (N, σ_0, p, f, c). Consider $S \subseteq N$ as a nonempty and connected coalition with respect to σ_0. Formally, $\hat{\sigma}_S$ denotes an admissible optimal order for S resulting from only exchanging those pairs of players $i, j \in S$ that are misplaced. Thus, if σ^S_{ij} is the order in the path $\mathcal{P}(\hat{\sigma}_S)$ from σ_0 to $\hat{\sigma}_S$ in which i and j of S are consecutive and they switch their positions,

$$v(S) = \sum_{i,j \in S:\ (i,j) \in MP(\sigma_0, \hat{\sigma}_S)} g_{ij}(\sigma_{ij}^S). \tag{6}$$

If we further consider a condition on the position-dependent neighbor switching gains of misplacements, we can ensure that the associated sequencing game is a σ_0-component additive game. Let $(N, \sigma_0, p, f, c) \in PSEQ^N$ be a sequencing situation with position-dependent effects and consider i, j as a pair of players of N that are consecutive in a given order σ, that is, if $\sigma(j) = \sigma(i) + 1$. We say that the neighbor switching gains satisfy the property of *strict position dependence* if

$$g_{ij}(\sigma) = g_{ij}(\tau), \tag{7}$$

for any pair of orders of jobs $\sigma, \tau \in \Pi(N)$ such that $\sigma(i) = \tau(i)$ and $\sigma(j) = \tau(j)$ for any pair of consecutive players i and j of N. This property ensures that the gain from switching the consecutive jobs i and j is not influenced by the processing times of the remaining jobs but depends solely on their own parameters and position. Clearly, Example 1 does not satisfy this condition. It is enough to consider the orders (1,2,3,4) and (1,2,4,3), determine the gains from the exchange of players 1 and 2, and observe that the amounts are different.

Theorem 1. *Let $(N, \sigma_0, p, f, c) \in PSEQ^N$ be a sequencing situation with position-dependent effects, and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If the strict position dependence in (7) is satisfied for all pairs of players $i, j \in N$, then the corresponding sequencing game with position-dependent effects (N, v) is a σ_0-component additive game.*

Proof. Let (N, v) be the sequencing game with position-dependent effects associated with (N, σ_0, p, f, c) and take $\hat{\sigma}$ as an optimal order for the players in N.

Let S be a nonempty and connected coalition of N with respect to the initial order σ_0. We denote $\hat{\sigma}_S$ as an optimal order for S resulting, respectively, from exchanging only those pairs of players $i, j \in S$ that are misplaced in σ_0. That is, we can specify a path $\mathcal{P}(\hat{\sigma}_S) = (\sigma_0, \ldots, \hat{\sigma}_S)$ by only repairing pairs $(i, j) \in MP(\sigma_0, \hat{\sigma}_S)$, respectively.

Let T, without loss of generality, be a connected coalition with respect to σ_0 such that $S \cap T = \emptyset$ and $\sigma_0(i) < \sigma_0(j)$ for every $i \in S$ and $j \in T$. We denote $\hat{\sigma}_T$ as an optimal order for T and denote $\hat{\sigma}_{S\cup T}$ as an admissible and optimal order for $S \cup T$, when considering σ_0 as the initial order. Thus, we can specify two different paths from σ_0 to $\hat{\sigma}_{S\cup T}$, prescribed by $(\sigma_0, \ldots, \hat{\sigma}_S, \ldots, \bar{\sigma}_T, \ldots, \hat{\sigma}_{S\cup T})$ and $(\sigma_0, \ldots, \hat{\sigma}_T, \ldots, \hat{\sigma}_{S\cup T})$, where $\bar{\sigma}_T$ is an optimal order for T when considering $\hat{\sigma}_S$ as the initial order. Moreover, $\bar{\sigma}_T$ is an admissible order for $S \cup T$ obtained by correcting those pairs of misplaced players of T with respect to $\hat{\sigma}_S$ (and σ_0), following the steps of the sub-path from σ_0 to $\hat{\sigma}_T$.

By the property of strict position dependence, the values of the neighbor switching gains $g_{ij}(\sigma_{ij}^T)$ are identical regardless of whether we take the path from σ_0 or $\hat{\sigma}_S$ to optimally order the players in

T. Specifically, by construction, if σ_{ij}^T and $\bar{\sigma}_{ij}^T$ denote the steps in their respective paths where i and j are consecutive, it holds that

$$\begin{aligned} v(T) &= c(\sigma_0) - c(\hat{\sigma}_T) \\ &= \sum_{i,j \in T:\ (i,j) \in MP(\sigma_0, \hat{\sigma}_T)} g_{ij}(\sigma_{ij}^T) \\ &= \sum_{i,j \in T:\ (i,j) \in MP(\hat{\sigma}_S, \bar{\sigma}_T)} g_{ij}(\bar{\sigma}_{ij}^T) \\ &= c(\hat{\sigma}_S) - c(\bar{\sigma}_T). \end{aligned} \tag{8}$$

Below we prove that (N, v) belongs to the class of σ_0-component additive games as defined in Curiel et al. (1993). First, from its definition in (6), it directly follows that $v(\{i\}) = 0$ for all $i \in N$. To prove the additivity of connected components of (N, v), consider $S \cup T$, a nonconnected coalition with respect to σ_0. It also holds that

$$\begin{aligned} v(S) + v(T) &= (c(\sigma_0) - c(\hat{\sigma}_S)) + (c(\sigma_0) - c(\hat{\sigma}_T)) \\ &= (c(\sigma_0) - c(\hat{\sigma}_S)) + (c(\hat{\sigma}_S) - c(\bar{\sigma}_T)) \\ &= c(\sigma_0) - c(\bar{\sigma}_T) \\ &= c(\sigma_0) - c(\hat{\sigma}_{S\cup T}) \\ &= v(S \cup T). \end{aligned} \tag{9}$$

These equalities in (9) are justified by the equality in (8), the fact that players in $N \setminus (S \cup T)$ occupy the same positions in σ_0 and $\hat{\sigma}_S$ and that $\bar{\sigma}_T$ is admissible for $S \cup T$ by construction. However, this order is also optimal for $S \cup T$ since $S \cup T$ is a nonconnected coalition. Therefore, the exchanges of players between the two components are excluded. The case of more than two nonconnected components readily follows.

Now, to prove the superadditivity of (N, v), assume that $S \cup T$ is a connected coalition with respect to σ_0. Then, it holds that

$$\begin{aligned} v(S) + v(T) &= c(\sigma_0) - c(\bar{\sigma}_T) \\ &\le c(\sigma_0) - c(\hat{\sigma}_{S\cup T}) \\ &= v(S \cup T). \end{aligned} \tag{10}$$

Thus, the superadditivity of (N, v) is ensured by the arguments used to justify (9), and by the fact that, since $S \cup T$ is assumed to be connected, $\bar{\sigma}_T$ is an admissible but not necessarily optimal order for $S \cup T$.

Hence, we establish that (N, v) belongs to the class of σ_0-component additive games. □

Note that the previous condition is sufficient but not necessary to ensure that sequencing games belong to the class of σ_0-component additive games. For instance, Example 2 illustrates that the sequencing game associated with the considered sequencing situation is a σ_0-component additive game, despite the fact that the condition in (7) is not satisfied.

Example 2. Take $(N, \sigma_0, p, f, c) \in PSEQ^N$ such that $N = \{1, 2, 3, 4\}$, $\sigma_0 = (1, 2, 3, 4)$ and, $p_1 = 0.62$, $p_2 = 0.69$, $p_3 = 0.91$ and $p_4 = 0.95$. Consider again $f(k) = b^{k-1}$ for all $k \in \{1, \dots, n-1\}$, with $b = 1.51$, and $c(\sigma) = \sum_{\ell \in N} \mathrm{e}^{C_\ell(\sigma)}$. The next table displays the global costs for all orders of N.

σ	$c(\sigma)$	σ	$c(\sigma)$	σ	$c(\sigma)$	σ	$c(\sigma)$
(1,2,3,4)	1154.085	(2,3,4,1)	659.646	(3,4,1,2)	516.966	(4,1,2,3)	770.622
(2,1,3,4)	1113.811	(1,3,4,2)	762.695	(3,4,2,1)	488.332	(4,2,1,3)	731.397
(2,3,1,4)	895.094	(3,1,4,2)	658.719	(1,4,2,3)	910.670	(4,2,3,1)	561.734
(1,3,2,4)	977.411	(3,2,4,1)	590.337	(2,4,1,3)	834.146	(4,1,3,2)	626.639
(3,1,2,4)	843.915	(1,2,4,3)	1107.871	(2,4,3,1)	640.426	(4,3,1,2)	506.679
(3,2,1,4)	800.795	(2,1,4,3)	1069.218	(1,4,3,2)	740.296	(4,3,2,1)	478.623

The sequencing game with position-dependent effects (N, v) associated with (N, σ_0, p, f, c) is shown below.

S	∅	{1}	{2}	{3}	{4}	{1, 2}	{1, 3}	{1, 4}
$v(S)$	0	0	0	0	0	40.274	0	0

S	{2, 3}	{2, 4}	{3, 4}	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}	N
$v(S)$	176.674	0	46.214	353.289	40.274	46.214	413.789	675.462

It is easy to check that, in this case, (N, v) is superadditive. Hence, (N, v) belongs to the class of σ_0-component additive games since it also satisfies the other two required conditions.

In the following, we will focus exclusively on the analysis of sequencing situations with position-dependent effects that satisfy the strict position dependence property. This condition implies that the gains from switching a pair of consecutive jobs are independent of the parameters of the remaining jobs. To simplify the notation in the rest of the paper, we will denote these neighbor switching gains by $g_{ij}(k)$, where k is the position of job i in front of the machine. From now on, $\overline{PSEQ}^N$ refers to the subset of all sequencing situations with position-dependent effects such that the associated neighbor switching gains satisfy the property in (7). Although this property may seem restrictive, common objective functions in sequencing such as the makespan or the total flow time result in sequencing situations that fulfill this property.

Next, we study the convexity of sequencing games with position-dependent effects associated with any $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$. This property ensures that the larger the coalition of jobs a certain job joins, the larger the savings resulting from their optimal rearrangement. In general, Borm et al. (2002) characterized the property of the convexity for the class of σ_0-component additive games.

Proposition 1 establishes a condition that ensures the convexity of sequencing games with position-dependent effects. Following the approaches in Saavedra-Nieves et al. (2020) and Schouten et al. (2021), such condition is also based on the study of the increasing (or not) character of the neighbor switching gains, the fact that considerably simplifies the analysis of convexity with respect to the condition of Borm et al. (2002).

Proposition 1. *Let* $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ *and let* $\hat{\sigma}$ *be an optimal order such that*

$$g_{ij}(k) \geq 0 \text{ for all } (i, j) \in MP(\sigma_0, \hat{\sigma}) \text{ and } g_{ij}(k) \leq 0 \text{ for all } (i, j) \notin MP(\sigma_0, \hat{\sigma})$$

for all $k \in \{1, \ldots, n-1\}$. *Take* (N, v) *as the associated sequencing game with position-dependent effects. Then,* (N, v) *is convex if*

i) $g_{ij}(k) \geq g_{ij}(k')$ *for all* $(i, j) \in MP(\sigma_0, \hat{\sigma})$, *for all* $k \in \{1, \ldots, n-1\}$, *and for all* $k' \in \{k+1, \ldots, n-1\}$, *or*
ii) $g_{ij}(k) \leq g_{ij}(k')$ *for all* $(i, j) \in MP(\sigma_0, \hat{\sigma})$, *for all* $k \in \{1, \ldots, n-1\}$, *and for all* $k' \in \{k+1, \ldots, n-1\}$.

The previous result provides a sufficient condition for the convexity of the corresponding sequencing game. Ensuring convexity for the associated TU games is beneficial, as it guarantees that some well-known allocation procedures belong to the core. However, computing some of these procedures can be a challenging computational task (e.g., the Shapley value). For this reason, we aim to study simpler procedures for allocating the cost savings in general sequencing frameworks, as an alternative to those mentioned above.

4. Cost savings allocation rules under positional effects

In this section, we focus on allocating the savings obtained from optimally processing the jobs. For this purpose, we introduce two different families of allocation rules that can be computed directly for every sequencing situation with position-dependent effects $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$.

An allocation rule in this setting is formally defined as a mapping ψ, which assigns to each sequencing situation with position-dependent effects $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ an allocation $\psi(N, \sigma_0, p, f, c) \in \mathbb{R}^N$. Below, we outline the steps to derive two new allocation rules within our framework. Both are based on concepts similar to the Gain Splitting rules for standard sequencing situations (Smith, 1956). Analyzing the game-theoretic properties that these allocations satisfy is of particular interest. For instance, Curiel et al. (1989) proved the nonemptiness of the core for the general class of σ_0-component additive games, while Curiel et al. (1994) introduced and verified that the β-rule specifies a core allocation in such class of TU games. In our context, we also investigate whether our two new proposed allocations are at the core of the corresponding sequencing games with position-dependent effects.

First, we describe two general procedures that form the basis for both definitions, specifying paths from the initial order to an optimal one. The first procedure, named the *latest misplacement procedure*, begins by swapping the latest pair of consecutive misplaced players in the initial order relative to an optimal order. Subsequently, it identifies the latest pair of consecutive misplaced players in the resulting order from the previous step. This iterative process continues until all players are positioned according to the optimal order. Formally, the latest misplacement procedure is described as follows:

Procedure 1 (*Latest misplacement*). Take $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Fix $k = 1$.

Step 1: For the first step, set

$$j_1 = \arg\max \{\sigma_0(t) \ : \ (s, t) \in MP(\sigma_0, \hat{\sigma}), \ \text{with } \sigma_0(s) = \sigma_0(t) - 1\},$$

that is, (i_1, j_1) is the latest pair of misplaced players according to $\hat{\sigma}$ such that $\sigma_0(j_1) = \sigma_0(i_1) + 1$. Take σ_1 as the corresponding order of N obtained from switching the latest neighbors (i_1, j_1) and go to Step 2.

For $k > 1$ until $k = |MP(\sigma_0, \hat{\sigma})|$, do:

Step k: Take σ_{k-1} and do

$$j_k = \arg\max \{\sigma_{k-1}(t) \ : \ (s, t) \in MP(\sigma_0, \hat{\sigma}), \ \text{with } \sigma_{k-1}(s) = \sigma_{k-1}(t) - 1\},$$

that is, (i_k, j_k) is the latest pair of misplaced players according to $\hat{\sigma}$ such that $\sigma_{k-1}(j_k) = \sigma_{k-1}(i_k) + 1$. We obtain σ_k as the corresponding order of N resulting from switching the latest neighbors (i_k, j_k). Go to Step $k + 1$.

The second procedure, named the *earliest misplacement procedure*, naturally reverses the idea of the procedure above. Now, we start with the earliest pair of consecutive misplaced players. We sequentially switch the earliest pair of consecutive misplaced players in each step with respect to the optimal order. This procedure was initially introduced in Yang et al. (2020). Formally, the earliest misplacement procedure is defined as follows:

Procedure 2 (*Earliest misplacement*). Take $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Fix $k = 1$.

Step 1: For the first step, set

$$i_1 = \arg\min \{\sigma_0(s) \ : \ (s, t) \in MP(\sigma_0, \hat{\sigma}), \ \text{with } \sigma_0(t) = \sigma_0(s) + 1\},$$

that is, (i_1, j_1) is the earliest pair of misplaced players according to $\hat{\sigma}$ such that $\sigma_0(j_1) = \sigma_0(i_1) + 1$. Take σ_1 as the corresponding order of N obtained by switching (i_1, j_1) and go to Step 2.

For $k > 1$ until $k = |MP(\sigma_0, \hat{\sigma})|$, do:

Step k: Take σ_{k-1} and do

$$i_k = \arg\min \{\sigma_{k-1}(s) \ : \ (s, t) \in MP(\sigma_0, \hat{\sigma}), \ \text{with } \sigma_{k-1}(t) = \sigma_{k-1}(s) + 1\},$$

that is, (i_k, j_k) is the earliest pair of misplaced players according to $\hat{\sigma}$ such that $\sigma_{k-1}(j_k) = \sigma_{k-1}(i_k) + 1$. We obtain σ_k as the corresponding order of N resulting from switching the earliest neighbors (i_k, j_k). Go to Step $k + 1$.

Each procedure unequivocally determines a unique path from the initial order to an optimal one. From the set of misplaced agents, it is possible to identify which agents need to be switched at each step of the path from σ_0. Thus, after selecting a path from the initial order to an optimal order, we apply the concept behind the GS-rules from Hamers et al. (1996) to divide the corresponding neighbor switching gains at each step along this path.

Definition 1. *Take* $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$. *Let* $\hat{\sigma}$ *be an optimal order such that* $\mathcal{P}(\hat{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \ldots, \hat{\sigma})$ *is a path from* σ_0 *to* $\hat{\sigma}$, *and let* λ *be a collection* $\{\lambda_{ij}\}_{i,j\in N:\, i\neq j}$ *such that* $\lambda_{ij} \in (0,1)$ *for all* $i, j \in N, i \neq j$. *Thus, the* λ*-Gain Splitting rule for* $\hat{\sigma}$ *(*$GS^{\lambda,\mathcal{P}(\hat{\sigma})}$-rule*) specifies for every* (N, σ_0, p, f, c) *the following allocation:*

$$GS^{\lambda,\mathcal{P}(\hat{\sigma})}(N, \sigma_0, p, f, c) = \sum_{\ell=1}^{|MP(\sigma_0,\hat{\sigma})|} \left(\lambda_{i_\ell j_\ell} \cdot g_{i_\ell j_\ell}(\sigma_{\ell-1}(i_\ell))\mathbb{1}_{\{i_\ell\}} + (1-\lambda_{i_\ell j_\ell}) \cdot g_{i_\ell j_\ell}(\sigma_{\ell-1}(i_\ell))\mathbb{1}_{\{j_\ell\}}\right),$$

being (i_ℓ, j_ℓ) *the pair of misplaced players switched in step* ℓ *of the path* $\mathcal{P}(\hat{\sigma})$, *with* $\ell = 1, \ldots, |MP(\sigma_0, \hat{\sigma})|$, *and* $\mathbb{1}_{\{i\}} \in \mathbb{R}^N$, *for every* $i \in N$, *is such that, for all* $k \in N$,

$$(\mathbb{1}_{\{i\}})_k = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{otherwise.} \end{cases}$$

From here onward, for technical convenience, λ will denote the collection of weights given by $\{\lambda_{ij}\}_{i,j\in N:\, i\neq j}$, with $\lambda_{ij} \in (0,1)$ for all $i, j \in N$, $i \neq j$. The idea behind the family of the GS^{λ}-rules is straightforward: the neighbor switching gain for every misplaced pair of players (i, j) is divided between i and j with weights equal to λ_{ij} and $1 - \lambda_{ij}$, respectively. Specifically, the *Equal Gain Splitting rule for* $\hat{\sigma}$ *(*$EGS^{\mathcal{P}(\hat{\sigma})}$*-rule)* is defined by taking $\lambda_{ij} = \frac{1}{2}$ for all $i, j \in N, i \neq j$. This rule generalizes the EGS-rule for standard sequencing situations (Curiel et al., 1989) to accommodate general cost objective functions. From a game-theoretic perspective, the EGS-rule coincides with the β-rule for the sequencing game associated with standard sequencing situations (Curiel et al., 1994).

In contrast to Hamers et al. (1996), the position dependence of gains from switching players implies that the path chosen from the initial order to a fixed optimal one influences the definition of allocation rules. Therefore, we propose two types of allocation rules based on the two procedures mentioned. For an optimal order $\hat{\sigma}$ and each fixed choice of λ, the allocation given by the $\mathrm{GS}^{\lambda,\mathcal{P}(\hat{\sigma})}$-rule, when using the latest misplacement procedure, is named the λ*-Gain Splitting Latest rule (*$GS^{\lambda,\mathcal{L}(\hat{\sigma})}$*-rule)* and denoted by $GS^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c)$, where $\mathcal{L}(\hat{\sigma})$ denotes the path from σ_0 to $\hat{\sigma}$ specified by Procedure 1. Alternatively, for an optimal order $\hat{\sigma}$ and each fixed choice of λ, if the path is specified by the earliest misplacement procedure, the allocation is known as the λ-*Gain Splitting Earliest rule (*$GS^{\lambda,\mathcal{E}(\hat{\sigma})}$*-rule)* and denoted by $GS^{\lambda,\mathcal{E}(\hat{\sigma})}(N, \sigma_0, p, f, c)$, with $\mathcal{E}(\hat{\sigma})$ being the path from σ_0 to $\hat{\sigma}$ specified by Procedure 2. Now, due to the position dependence of neighbor switching gain, the allocation specified by the β-rule does not necessarily coincide with either of the two proposed allocation rules.

In general, it would be desirable for allocation rules to guarantee core elements. The following result ensures that for every fixed optimal order $\hat{\sigma}$ and every fixed choice of λ, the $\mathrm{GS}^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule, and $\mathrm{GS}^{\lambda,\mathcal{E}(\hat{\sigma})}$ -rule lead to allocations in the core, provided that the conditions on the position-dependent neighbor switching gains ensuring convexity for the associated sequencing game are met.

Theorem 2. *Let* $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$*, and let* $\hat{\sigma}$ *be an optimal order such that*

$$g_{ij}(k) \geq 0 \text{ for all } (i, j) \in MP(\sigma_0, \hat{\sigma}) \text{ and } g_{ij}(k) \leq 0 \text{ for all } (i, j) \notin MP(\sigma_0, \hat{\sigma})$$

for all $k \in \{1, \ldots, n-1\}$*. Take* (N, v) *as the associated sequencing game with position-dependent effects and take a fixed choice of* λ*.*

i) If $g_{ij}(k) \leq g_{ij}(k')$ *for all* $(i, j) \in MP(\sigma_0, \hat{\sigma})$*, for all* $k \in \{1, \ldots, n-1\}$*, and for all* $k' \in \{k+1, \ldots, n-1\}$*, then* $GS^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c)$ *provides a core element.*

ii) If $g_{ij}(k) \geq g_{ij}(k')$ *for all* $(i, j) \in MP(\sigma_0, \hat{\sigma})$*, for all* $k \in \{1, \ldots, n-1\}$*, and for all* $k' \in \{k+1, \ldots, n-1\}$*, then* $GS^{\lambda,\mathcal{E}(\hat{\sigma})}(N, \sigma_0, p, f, c)$ *provides a core element.*

Proof. Let (N, v) be the sequencing game with position-dependent effects associated with (N, σ_0, p, f, c) and take $\hat{\sigma}$ as an optimal order for the players in N. Thus, $\mathcal{L}(\hat{\sigma})$ and $\mathcal{E}(\hat{\sigma})$ denote the paths specified by the latest and earliest misplacement procedures, respectively. Take $S \subseteq N$ a nonempty and connected coalition with respect to the initial order σ_0, and take a fixed choice of λ.

By definition, the allocations prescribed by the $\mathrm{GS}^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule and the $\mathrm{GS}^{\lambda,\mathcal{E}(\hat{\sigma})}$-rule for $\hat{\sigma}$, denoted by $GS^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c)$ and $GS^{\lambda,\mathcal{E}(\hat{\sigma})}(N, \sigma_0, p, f, c)$, respectively, are efficient. Then, it only remains to check coalitional stability for those coalitions connected by σ_0. Take $\hat{\sigma}_S$ as the optimal order for S resulting from only exchanging of those pairs of players $i, j \in S$ such that $(i, j) \in MP(\sigma_0, \hat{\sigma})$. We denote k_{ij}^S as the position in which i and j of S switch in a given path from σ_0 to $\hat{\sigma}_S$.

Take $(i, j) \in MP(\sigma_0, \hat{\sigma})$ such that $i, j \in S$. In both cases, we obtain a direct expression for k_{ij}^S when considering the latest and the earliest procedures.

a) Using the latest misplacement procedure (Procedure 1), to determine the position where i and j are interchanged, we start from the initial position of i in σ_0 and move backward. At each step, we check if there is a player between i and j in σ_0 who follows j in $\hat{\sigma}$, or if a follower of j in σ_0 precedes j in $\hat{\sigma}$. Then,

$$k_{ij}^S = |P(\sigma_0, i)| + 1 + |P(\sigma_0, j) \cap F(\sigma_0, i) \cap P(\hat{\sigma}, j) \cap S| + |F(\sigma_0, j) \cap P(\hat{\sigma}, j) \cap S|.$$

Here, we establish that $k_{ij}^S \leq k_{ij}^N$, since S is connected. This follows because $P(\sigma_0, j) \cap F(\sigma_0, i) \cap P(\hat{\sigma}, j) \cap S = P(\sigma_0, j) \cap F(\sigma_0, i) \cap P(\hat{\sigma}, j)$ and $F(\sigma_0, j) \cap P(\hat{\sigma}, j) \cap S \subseteq F(\sigma_0, j) \cap P(\hat{\sigma}, j)$. Therefore, the nondecreasing character of $g_{ij}(k)$ under condition (i) ensures that $g_{ij}(k_{ij}^S) \leq g_{ij}(k_{ij}^N)$.

b) Using the earliest misplacement procedure (Procedure 2), to determine the position where i and j are interchanged, we start from the position of i in σ_0. We move backward each time a player between i and j in σ_0 precedes i in $\hat{\sigma}$, and move forwards each time a predecessor of i in σ_0 follows i in $\hat{\sigma}$. Then,

$$k_{ij}^S = |P(\sigma_0, i)| + 1 + |P(\sigma_0, j) \cap F(\sigma_0, i) \cap P(\hat{\sigma}, i) \cap S| - |P(\sigma_0, i) \cap F(\hat{\sigma}, i) \cap S|.$$

Furthermore, $k_{ij}^N \leq k_{ij}^S$ because S is connected. This is evident from $P(\sigma_0, j) \cap F(\sigma_0, i) \cap P(\hat{\sigma}, i) \cap S = P(\sigma_0, j) \cap F(\sigma_0, i) \cap P(\hat{\sigma}, i)$ and $P(\sigma_0, i) \cap F(\hat{\sigma}, i) \cap S \subseteq P(\sigma_0, i) \cap F(\hat{\sigma}, i)$. Therefore, the nonincreasing nature of $g_{ij}(k)$ under condition (ii) ensures that $g_{ij}(k_{ij}^S) \leq g_{ij}(k_{ij}^N)$.

As a consequence, for both statements, we have that

$$\sum_{i,j \in S:\ (i,j) \in MP(\sigma_0, \hat{\sigma})} g_{ij}(k_{ij}^N) \geq \sum_{i,j \in S:\ (i,j) \in MP(\sigma_0, \hat{\sigma})} g_{ij}(k_{ij}^S) = v(S).$$

Finally, note that for every $(i, j) \in MP(\sigma_0, \hat{\sigma})$ with $i, j \in S$, the corresponding neighbor switching gain $g_{ij}(k_{ij}^N)$ is divided among players i and j. Thus, by summing the allocations prescribed by the $\mathrm{GS}^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule or the $\mathrm{GS}^{\lambda,\mathcal{E}(\hat{\sigma})}$-rule, we find that

$$\sum_{i \in S} GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c) \geq \sum_{i,j \in S:\ (i,j) \in MP(\sigma_0, \hat{\sigma})} g_{ij}(k_{ij}^N)$$

and

$$\sum_{i \in S} GS_i^{\lambda,\mathcal{E}(\hat{\sigma})}(N, \sigma_0, p, f, c) \geq \sum_{i,j \in S:\ (i,j) \in MP(\sigma_0, \hat{\sigma})} g_{ij}(k_{ij}^N),$$

regardless of the fixed choice of λ considered. Thus, it immediately follows that

$$\sum_{i \in S} GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c) \geq v(S)$$

and

$$\sum_{i \in S} GS_i^{\lambda,\mathcal{E}(\hat{\sigma})}(N, \sigma_0, p, f, c) \geq v(S),$$

for statements (i) and (ii), respectively, concluding the proof. □

5. A characterization of the cost savings allocation rules

In this section, we aim to characterize the allocations prescribed by the $\mathrm{GS}^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule and the $\mathrm{GS}^{\lambda,\mathcal{E}(\hat{\sigma})}$-rule for every optimal order $\hat{\sigma}$ and every fixed choice of λ, for any sequencing situation with position-dependent effects in $\overline{PSEQ}^N$. To achieve this, we utilize a set of natural and comprehensive properties imposed on the rule prescribed by each procedure, for every fixed $\hat{\sigma}$ and every fixed λ, extending those originally introduced in Yang et al. (2020) in various aspects. These properties provide additional insights into the characteristics of the proposed allocations and enable comparisons with alternative allocations within the typically wide core. Specifically, we provide characterizations within a framework encompassing general positional effects on processing times, addressing cases not previously considered, such as allocations specified by the latest misplacement procedure and scenarios involving nonequalitarian and general weight distribution λ.

Consider $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ and, from now on and for technical reasons, take an optimal order $\hat{\sigma}$ and a fixed choice of λ specified by a collection $\{\lambda_{ij}\}_{i,j\in N:\, i\neq j}$ such that $\lambda_{ij} \in (0, 1)$ for all $i, j \in N,\ i \neq j$. We begin by examining the *λ-Gain Splitting Latest rule ($GS^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule)*. Now, we introduce some necessary properties. $U(\sigma_0, i)$ denotes the set of predecessors of player i that are ordered inversely in σ_0 and in $\hat{\sigma}$, defined as $U(\sigma_0, i) = \{k \in N : (k, i) \in MP(\sigma_0, \hat{\sigma})\}$. Player i is termed a *U-equivalence player* with respect to sequencing situations with position-dependent effects (N, σ_0, p, f, c) and (N, σ_1, p, f, c) if $P(\sigma_0, i) = P(\sigma_1, i)$ and for any pair of players $l, l' \in U(\sigma_0, i)$, $\sigma_0(l) < \sigma_0(l')$ implies $\sigma_1(l) < \sigma_1(l')$. A pair (j, i) is termed a *tail pair* with respect to (N, σ_0, p, f, c) if $(j, i) \in MP(\sigma_0, \hat{\sigma})$, $\sigma_0(i) = \sigma_0(j) + 1$ and for any $k \in F(\sigma_0, i)$, $(i, k) \notin MP(\sigma_0, \hat{\sigma})$.

Using these definitions, we can immediately formalize the following two properties, which are considered as variations of equivalence and switch properties for the EGS-rule (Curiel et al., 1993). Let (N, σ_0, p, f, c) and (N, σ_1, p, f, c) be two sequencing situations with position-dependent effects in $\overline{PSEQ}^N$, and let ϕ be any allocation rule defined on this class of problems. Using the previous notions, the following two properties can be immediately formalized.

- *U-equivalence*. If some player $i \in N$ is a U-equivalence player with respect to (N, σ_0, p, f, c) and (N, σ_1, p, f, c), then $\phi_i(N, \sigma_0, p, f, c) = \phi_i(N, \sigma_1, p, f, c)$.
- *λ-Latest switch property*. Take $i, j \in N$ such that $\sigma_0(i) = \sigma_0(j) + 1$, $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$, and $\sigma_1(k) = \sigma_0(k)$ for any $k \in N \setminus \{i, j\}$. If (j, i) is a tail pair with respect to (N, σ_0, p, f, c),

$$\lambda_{ji}(\phi_i(N, \sigma_1, p, f, c) - \phi_i(N, \sigma_0, p, f, c)) = (1 - \lambda_{ji})\big(\phi_j(N, \sigma_1, p, f, c) - \phi_j(N, \sigma_0, p, f, c)\big).$$

In addition, allocation rules for sequencing situations with position-dependent effects inherit some fundamental properties from cooperative game theory. A player $i \in N$ is said to be a *dummy player* in a sequencing situation with position-dependent effects $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ if $\sigma_0(j) > \sigma_0(i)$ implies that $(i, j) \notin MP(\sigma_0, \hat{\sigma})$ and $\sigma_0(k) < \sigma_0(i)$ implies that $(k, i) \notin MP(\sigma_0, \hat{\sigma})$. In other words, player i does not contribute to the cost savings.

- *Dummy property*. A division rule ϕ defined for sequencing situations with position-dependent effects satisfies *dummy property* if $\phi_i(N, \sigma_0, p, f, c) = 0$ for any dummy player $i \in N$.
- *Efficiency property*. A division rule ϕ in this setting satisfies *efficiency* if $\sum_{i\in N} \phi_i(N, \sigma_0, p, f, c) = v(N)$.

The following result ensures that, for every optimal order $\hat{\sigma}$ and a fixed choice of λ, the λ-Gain Splitting Latest rule for $\hat{\sigma}$ ($GS^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule) satisfies efficiency, dummy player property, U-equivalence, and λ-latest switch property. It is important to note that the result holds similarly for any λ considered, with adjustments made for the specific λ-latest switch property accordingly.

Theorem 3. *Let $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$, and let $\hat{\sigma}$ be an optimal order. Take a fixed choice of λ. Then, the $GS^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule is the unique allocation satisfying efficiency, dummy player, U-equivalence, and λ-latest switch property.*

Proof. Take $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$, $\hat{\sigma}$ as an optimal order for N, a fixed choice of λ, and the allocation prescribed by the $GS^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule for $\hat{\sigma}$, that is denoted by $GS^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c)$. The fulfillment of efficiency and dummy player for the $GS^{\lambda,\mathcal{L}(\hat{\sigma})}$-rule is straightforward. Therefore, we only need to prove the properties of U-equivalence and λ-latest switch property.

Fix any $i \in N$. Take $(N, \sigma_1, p, f, c) \in \overline{PSEQ}^N$ such that $P(\sigma_0, i) = P(\sigma_1, i)$ and for each pair of players $l, l' \in U(\sigma_0, i)$, $\sigma_0(l) < \sigma_0(l')$ implies $\sigma_1(l) < \sigma_1(l')$. Take $U(\sigma_0, i) = \{s_1, \ldots, s_u\}$ such that $\sigma_0(s_1) > \ldots > \sigma_0(s_u)$ and, thus, we have that $\sigma_1(s_1) > \ldots > \sigma_1(s_u)$. Thus, $GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c) = GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_1, p, f, c)$ by construction.

Below, we prove the λ-latest switch property. Take $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$ and (j, i) a tail pair with respect to (N, σ_0, p, f, c). Now, take $(N, \sigma_1, p, f, c) \in \overline{PSEQ}^N$ such that $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for any $k \in N \setminus \{i, j\}$. Take $U(\sigma_0, i) = \{s_1, \ldots, s_u\}$ satisfying that $\sigma_0(s_1) > \cdots > \sigma_0(s_u)$, $s_1 = j$ and, thus, that $U(\sigma_1, i) = U(\sigma_0, i) \setminus \{j\}$. For player i, as (j, i) is a tail pair, it holds that

$$GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c) = \sum_{\ell=1}^{|U(\sigma_0,i)|} (1 - \lambda_{s_\ell i}) \cdot g_{s_\ell i}(k^N_{s_\ell i}) = (1 - \lambda_{ji}) g_{ji}(k^N_{ji}) + \sum_{\ell=2}^{|U(\sigma_0,i)|} (1 - \lambda_{s_\ell i}) \cdot g_{s_\ell i}(k^N_{s_\ell i}) \tag{11}$$

for λ, and being

$$k^N_{s_\ell i} = |P(\sigma_0, s_\ell)| + 1 + |P(\sigma_0, i) \cap F(\sigma_0, s_\ell) \cap P(\hat{\sigma}, i)| + |F(\sigma_0, i) \cap P(\hat{\sigma}, i)|$$

the position where s_ℓ and i, with $\ell = 1, \ldots, |U(\sigma_0, i)|$, interchange their positions when using Procedure 1. Additionally, it also holds that

$$GS_i^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N, \sigma_1, p, f, c) = \sum_{\ell=2}^{|U(\sigma_0,i)|} (1 - \lambda_{s_\ell i}) \cdot g_{s_\ell i}(k^N_{s_\ell i}) \tag{12}$$

for (N, σ_0, p, f, c), with $\tilde{\mathcal{L}}(\hat{\sigma})$ being the path from σ_1 to $\hat{\sigma}$ that is obtained by repairing, step by step, the misplacements as in $\mathcal{L}(\hat{\sigma})$, with the exception of (j, i). Then, it holds that $GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c) - GS_i^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N, \sigma_1, p, f, c) = (1 - \lambda_{ji}) g_{ji}(k^N_{ji})$.

For player j, we must consider those followers of j in σ_0 that are misplaced with respect to j. That is, $D(\sigma_0, j) = \{k \in N : (j, k) \in MP(\sigma_0, \hat{\sigma})\} = \{t_1, \ldots, t_d\}$, with $t_1 = i$. We now assume that for any pair of players $s, t \in D(\sigma_0, j)$, it holds that $(s, t) \in MP(\sigma_0, \hat{\sigma})$, and for each pair of players $s, t \in D(\sigma_0, i)$, $(s, t) \notin MP(\sigma_0, \hat{\sigma})$. Thus, it holds that

$$GS_j^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c) = \lambda_{ji} \cdot g_{ji}(k^N_{ji}) + \sum_{\ell=2}^{|D(\sigma_0,j)|} \lambda_{jt_\ell} \cdot g_{jt_\ell}(k^N_{jt_\ell}) + \sum_{\ell=1}^{|U(\sigma_0,i)|} (1 - \lambda_{s_\ell j}) \cdot g_{s_\ell j}(k^N_{s_\ell j})$$

and

$$GS_j^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N, \sigma_1, p, f, c) = \sum_{\ell=2}^{|D(\sigma_0,j)|} \lambda_{jt_\ell} \cdot g_{jt_\ell}(k^N_{jt_\ell}) + \sum_{\ell=1}^{|U(\sigma_0,i)|} (1 - \lambda_{s_\ell j}) \cdot g_{s_\ell j}(k^N_{s_\ell j}).$$

In addition, we also have that $GS_j^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c) - GS_j^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c) = \lambda_{ji}\cdot g_{ji}(k_{ji}^N)$. Using these previous equalities, it directly satisfies that

$$\lambda_{ji}\Big(GS_i^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c) - GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c)\Big) = \\ = \big(1-\lambda_{ji}\big)\Big(GS_j^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c) - GS_j^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c)\Big)$$

and that

$$GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c) - GS_i^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c) \\ + GS_j^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c) - GS_j^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c) = g_{ji}(k_{ji}^N). \quad (13)$$

It only remains to be checked that this is the unique allocation rule fulfilling these properties. Take ϕ as an allocation rule for sequencing problems with position-dependent effects in $\overline{PSEQ}^N$ satisfying the four considered properties. For any $(N,\sigma_0,p,f,c)\in\overline{PSEQ}^N$ and $\hat{\sigma}$ an optimal order, we define $M_{\sigma_0}=\{(s,t)\in MP(\sigma_0,\hat{\sigma}):\ \sigma_0(s)=\sigma_0(t)-1\}$. We use a proof by induction on the cardinal of M_{σ_0}. If $|M_{\sigma_0}|=0$, $\phi_i(N,\sigma_0,p,f,c)=GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c)=0$ for all $i\in N$ by the dummy property. Now, suppose that for any $|M_{\sigma_0}|\le m$, it satisfies that $\phi(N,\sigma_0,p,f,c)=GS^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c)$.

Take $(N,\sigma_0,p,f,c)\in\overline{PSEQ}^N$ such that $|M_{\sigma_0}|=m+1$. Take $i\in N$ such that (j,i) is a tail pair with respect to (N,σ_0,p,f,c). We denote $\hat{\sigma}$ as an optimal order for (N,σ_0,p,f,c). Take $(N,\sigma_1,p,f,c)\in\overline{PSEQ}^N$ such that $\sigma_0(j)=\sigma_1(i)$, $\sigma_0(i)=\sigma_1(j)$, and $\sigma_0(k)=\sigma_1(k)$ for any $k\in N\setminus\{i,j\}$. Immediately, we have that $|M_{\sigma_1}|=m$. From the U-equivalence property and the induction assumption, it holds that

$$\phi_k(N,\sigma_0,p,f,c)=\phi_k(N,\sigma_1,p,f,c)=GS_k^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c)=GS_k^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c)$$

for all $k\in N\setminus\{i,j\}$. Besides, by efficiency and the induction hypothesis, it holds that $\phi_i(N,\sigma_0,p,f,c)+\phi_j(N,\sigma_0,p,f,c)-g_{ji}(k_{ji}^N)=\phi_i(N,\sigma_1,p,f,c)+\phi_j(N,\sigma_1,p,f,c)$. The λ-latest switch property implies that there is a fixed choice of $\lambda=\{\lambda_{ij}\}_{i,j\in N:\ i\neq j}$, with $\lambda_{ij}\in(0,1)$ for all $i,j\in N, i\neq j$, such that

$$\lambda_{ji}(\phi_i(N,\sigma_1,p,f,c)-\phi_i(N,\sigma_0,p,f,c)) = \\ = \big(1-\lambda_{ji}\big)\Big(\phi_i(N,\sigma_0,p,f,c)-\phi_i(N,\sigma_1,p,f,c)-g_{ji}(k_{ji}^N)\Big).$$

Now, using the previous equation, the induction assumption, and Expression (13), we have that

$$\begin{aligned}\phi_i(N,\sigma_0,p,f,c) &= \phi_i(N,\sigma_1,p,f,c)+\big(1-\lambda_{ji}\big)g_{ji}(k_{ji}^N)\\ &= GS_i^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N,\sigma_1,p,f,c)+\big(1-\lambda_{ji}\big)g_{ji}(k_{ji}^N)\\ &= GS_i^{\lambda,\mathcal{L}(\hat{\sigma})}(N,\sigma_0,p,f,c)\end{aligned}$$

for the fixed choice of λ considered. Similarly, we have that

$$\phi_j(N, \sigma_0, p, f, c) = GS_j^{\lambda,\tilde{\mathcal{L}}(\hat{\sigma})}(N, \sigma_1, p, f, c) + \lambda_{ji} g_{ji}(\sigma_0(j)).$$

Hence, $\phi(N, \sigma_0, p, f, c) = GS^{\lambda,\mathcal{L}(\hat{\sigma})}(N, \sigma_0, p, f, c)$. □

Similarly, we now characterize the allocations prescribed by the *λ-Gain Splitting Earliest rule for $\hat{\sigma}$ ($GS^{\lambda,\mathcal{E}(\hat{\sigma})}$ -rule)*, denoted by $GS^{\lambda,\mathcal{E}(\hat{\sigma})}(N, \sigma_0, p, f, c)$ for every optimal order $\hat{\sigma}$ and every fixed choice of λ, specified by the earliest misplacement procedure (Procedure 2). As mentioned previously, Yang et al. (2020) introduced this procedure for a specific sequencing problem and characterized the corresponding allocation. Now, we generalize the properties considered therein to a much broader context as follows.

Take (N, σ_0, p, f, c) as a sequencing situation with position-dependent effects in $\overline{PSEQ}^N$, and let $\hat{\sigma}$ be an optimal order. We denote $D(\sigma_0, i)$ as the set of followers of player i that are ordered inversely in σ_0 and in $\hat{\sigma}$, that is, $D(\sigma_0, i) = \{k \in N : (i, k) \in MP(\sigma_0, \hat{\sigma})\}$. Player i is said to be a *D-equivalence player* with respect to sequencing situations with position-dependent effects (N, σ_0, p, f, c) and (N, σ_1, p, f, c) if $F(\sigma_0, i) = F(\sigma_1, i)$ and for any pair of players $l, l' \in D(\sigma_0, i)$, $\sigma_0(l) < \sigma_0(l')$ implies that $\sigma_1(l) < \sigma_1(l')$. We refer to (i, j) as a *head pair* with respect to (N, σ_0, p, f, c) if $(i, j) \in MP(\sigma_0, \hat{\sigma})$, $\sigma_0(i) = \sigma_0(j) - 1$ and for any $k \in P(\sigma_0, i)$, $(k, i) \notin MP(\sigma_0, \hat{\sigma})$.

Take (N, σ_0, p, f, c) and (N, σ_1, p, f, c) in $\overline{PSEQ}^N$, and take ϕ as any allocation rule defined for sequencing situations with position-dependent effects.

- *D-equivalence*. If player $i \in N$ is a D-equivalence player with respect to (N, σ_0, p, f, c) and (N, σ_1, p, f, c), then $\phi_i(N, \sigma_0, p, f, c) = \phi_i(N, \sigma_1, p, f, c)$.
- *λ-Earliest switch property*. Take $i, j \in N$ such that $\sigma_0(i) = \sigma_0(j) - 1$, $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$, and $\sigma_1(k) = \sigma_0(k)$ for any $k \in N \setminus \{i, j\}$. If (i, j) is a head pair with respect to (N, σ_0, p, f, c),

$$(1 - \lambda_{ij})(\phi_i(N, \sigma_1, p, f, c) - \phi_i(N, \sigma_0, p, f, c)) = \lambda_{ij}\big(\phi_j(N, \sigma_1, p, f, c) - \phi_j(N, \sigma_0, p, f, c)\big).$$

Note that for the case of the EGS-rule (cf. Curiel et al., 1989), prescribed by setting $\lambda_{ij} = \frac{1}{2}$ for all $i, j \in N, i \neq j$, this property generalizes the *head switch property* introduced in Yang et al. (2020).

The next result ensures that the allocations specified by λ-Gain Splitting Earliest rule for an optimal order $\hat{\sigma}$ and for a fixed λ ($GS^{\lambda,\mathcal{E}(\hat{\sigma})}$-rule) satisfy efficiency, dummy player, and the properties above. This theorem generalizes the one already presented by Yang et al. (2020) to a wider family of sequencing problems and a more general family of allocations of the cost savings. The structure of the proof is quite similar to that of the previous theorem and, for the sake of readability, we omit it.

Theorem 4. *Let $(N, \sigma_0, p, f, c) \in \overline{PSEQ}^N$, and let $\hat{\sigma}$ be an optimal order. For a fixed choice of λ, the $GS^{\lambda,\mathcal{E}(\hat{\sigma})}$ -rule is the unique allocation satisfying efficiency, dummy player, D-equivalence, and λ-earliest switch property.*

6. Concluding remarks

In this paper, we analyze the cooperation among jobs in sequencing situations with position-dependent effects. First, we establish a sufficient condition to ensure that the associated sequencing games belong to the class of σ_0-component additive games, a property that is not always guaranteed in this context, and we study its convexity. Second, we specify a broad family of savings allocation rules for this class of sequencing situations, based on the concept of distributing gains among agents who have exchanged their positions due to misplacement. Specifically, our focus is on allocations defined by paths from the initial order to the optimal one, as prescribed by the earliest and the latest procedures. From a theoretical standpoint, the stability of these allocations is guaranteed under conditions that also ensure convexity. Additionally, we provide an axiomatization of these savings allocation rules. Within this family, a notable case occurs when the splitting coefficients are equal to 1/2.

It is noteworthy that the earliest and the latest procedures align with the principles underlying the growing tail procedure and the growing head procedure discussed in Schouten et al. (2021). The positions where players i and j in N switch during each step of the paths induced by the growing head and the growing tail procedures correspond to those associated with the latest and the earliest procedures, respectively. Consequently, allocations prescribed by the GS^{λ}-rules along paths specified by the latest and the earliest misplacement procedures coincide with those induced by the growing head and the growing latest procedures when applied to sequencing situations with positional effects, for every optimal order and every fixed choice of λ.

Furthermore, the results presented here on convexity and the stability of savings allocations are applicable to a wide range of sequencing situations with position-dependent effects, including those discussed in Yang et al. (2020), as well as scenarios involving positional effects and the cost objective functions, such as the total flow time or makespan, as discussed in Wang and Xia (2005) or Gordon et al. (2008).

Acknowledgments

This work has been supported under grants PID2021-124030NB-C32 and PID2021-124030NB-C33, funded by MCIN/AEI/10.13039/501100011033/ and by "ERDF A way of making Europe"/EU, and under grants *Grupos de Referencia Competitiva* ED431C-2021/24 and ED431C 2020/03, funded by *Consellería de Cultura, Educación e Universidades, Xunta de Galicia*. the authors would like to thank the associate editor and the anonymous referee for their valuable comments.

References

van Beek, A., Malmberg, B., Borm, P., Quant, M., Schouten, J., 2023. Competition and cooperation in linear production and sequencing processes. *Games and Economic Behavior* 139, 117–132.

Biskup, D., 2008. A state-of-the-art review on scheduling with learning effects. *European Journal of Operational Research* 188, 315–329.

Borm, P., Fiestras-Janeiro, G., Hamers, H., Sánchez, E., Voorneveld, M., 2002. On the convexity of games corresponding to sequencing situations with due dates. *European Journal of Operational Research* 136, 616–634.

Borm, P., Hamers, H., Hendrickx, R., 2001. Operations research games: a survey. *Top* 9, 139–199.

Briskorn, D., Davari, M., Matuschke, J., 2021. Single-machine scheduling with an external resource. *European Journal of Operational Research* 293, 457–468.
Calleja, P., Borm, P., Hamers, H., Klijn, F., Slikker, M., 2002. On a new class of parallel sequencing situations and related games. *Annals of Operations Research* 109, 265–277.
Cheng, T.E., Ding, Q., Lin, B.M., 2004. A concise survey of scheduling with time-dependent processing times. *European Journal of Operational Research* 152, 1–13.
Çiftçi, B., Borm, P., Hamers, H., Slikker, M., 2013. Batch sequencing and cooperation. *Journal of Scheduling* 16, 405–415.
Curiel, I., 2015. Compensation rules for multi-stage sequencing games. *Annals of Operations Research* 225, 65–82.
Curiel, I., Pederzoli, G., Tijs, S., 1989. Sequencing games. *European Journal of Operational Research* 40, 344–351.
Curiel, I., Potters, J., Prasad, R., Tijs, S., Veltman, B., 1993. Cooperation in one machine scheduling. *Zeitschrift fur Operations Research* 38, 113–129.
Curiel, I., Potters, J., Prasad, R., Tijs, S., Veltman, B., 1994. Sequencing and cooperation. *Operations Research* 42, 3, 566–568.
Estévez-Fernández, A., Borm, P., Calleja, P., Hamers, H., 2008. Sequencing games with repeated players. *Annals of Operations Research* 158, 189–203.
Gerichhausen, M., Hamers, H., 2009. Partitioning sequencing situations and games. *European Journal of Operations Research* 196, 207–216.
Gillies, D.B., 1953. Some theorems on n-person games. Ph.D. thesis, Princeton University.
Gordon, V.S., Potts, C.N., Strusevich, V.A., Whitehead, J.D., 2008. Single machine scheduling models with deterioration and learning: handling precedence constraints via priority generation. *Journal of Scheduling* 11, 357–370.
Grundel, S., Çiftçi, B., Borm, P., Hamers, H., 2013. Family sequencing and cooperation. *European Journal of Operations Research* 226, 414–424.
Hamers, H., Suijs, J., Tijs, S., Borm, P., 1996. The split core for sequencing games. *Games and Economic Behavior* 15, 2, 165–176.
Klijn, F., Sánchez, E., 2006. Sequencing games without initial order. *Mathematical Methods of Operations Research* 63, 53–62.
Liu, Z., Lu, L., Qi, X., 2018. Cost allocation in rescheduling with machine unavailable period. *European Journal of Operational Research* 266, 16–28.
Musegaas, M., Borm, P., Quant, M., 2015. Step out–step in sequencing games. *European Journal of Operational Research* 246, 894–906.
Musegaas, M., Borm, P., Quant, M., 2018. On the convexity of step out–step in sequencing games. *Top* 26, 68–109.
Rustogi, K., Strusevich, V., 2012a. Simple matching vs linear assignment in scheduling models with positional effects: a critical review. *European Journal of Operational Research* 222, 393–407.
Rustogi, K., Strusevich, V., 2012b. Single machine scheduling with general positional deterioration and rate-modifying maintenance. *Omega* 40, 791–804.
Saavedra-Nieves, A., Schouten, J., Borm, P., 2020. On interactive sequencing situations with exponential cost functions. *European Journal of Operational Research* 280, 78–89.
Schouten, J., Saavedra-Nieves, A., Fiestras-Janeiro, M.G., 2021. Sequencing situations and games with non-linear cost functions under optimal order consistency. *European Journal of Operational Research* 294, 734–745.
Shapley, L.S., 1953. A value for n-person games. In Kuhn, H.W. and Tucker, A.W. (eds), *Contributions to the Theory of Games, Volume II*, Princeton University Press, Princeton, NJ, pp. 307–318.
Shapley, L.S., 1971. Cores of convex games. *International Journal of Game Theory* 1, 11–26.
Slikker, M., 2005. Balancedness of sequencing games with multiple parallel machines. *Annals of Operations Research* 137, 177–189.
Slikker, M., 2023. The stable gain splitting rule for sequencing situations. *European Journal of Operational Research* 310, 2, 902–913.
Smith, W., 1956. Various optimizers for single-stage production. *Naval Research Logistics* 3, 59–66.
Tijs, S., 1981. Bounds for the core of a game and the τ-value. In *Game Theory and Mathematical Economics*. North-Holland Publishing, Amsterdam, pp. 123–132.
van Velzen, B., 2006. Sequencing games with controllable processing times. *European Journal of Operational Research* 172, 64–85.

Wang, J., Xia, Z., 2005. Flow-shop scheduling with a learning effect. *Journal of the Operational Research Society* 56, 1325–1330.

Yang, G., Sun, H., Hou, D., Xu, G., 2019. Games in sequencing situations with externalities. *European Journal of Operational Research* 278, 699–708.

Yang, G., Sun, H., Uetz, M., 2020. Cooperative sequencing games with position-dependent learning effect. *Operations Research Letters* 48, 428–434.