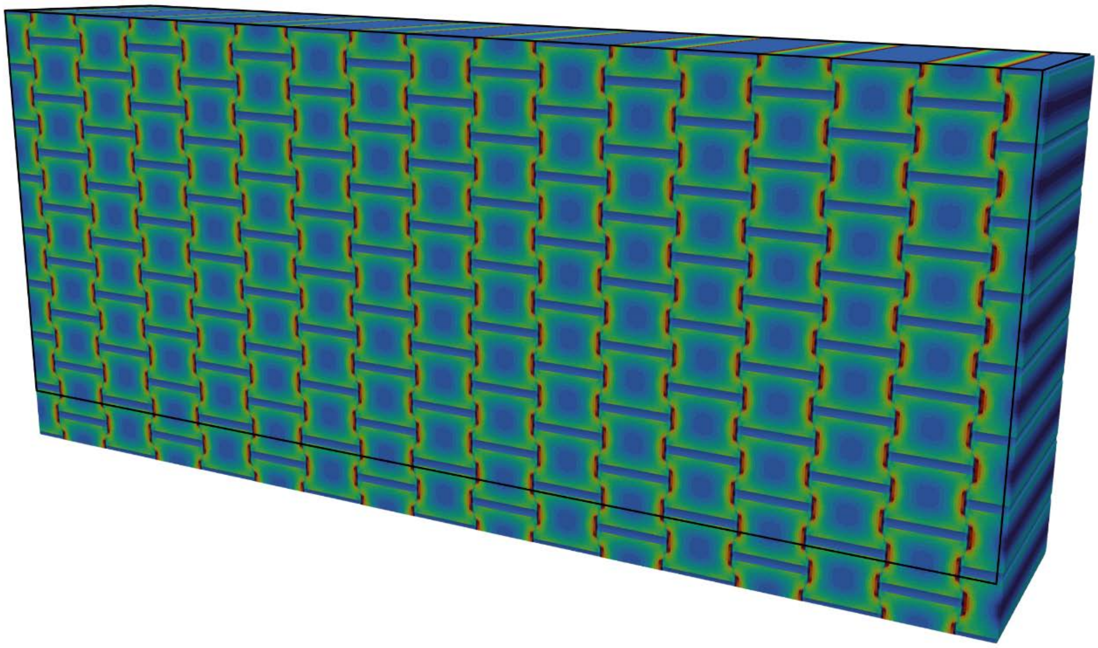
Mathematical Models in Solid Mechanics

Volume II: Further Models in Elasticity and Plasticity



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Mathematical Models in Solid Mechanics Volume II: Further Models in Elasticity and Plasticity

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This text is the continuation of the handbook **Modelos Matemáticos en Mecánica de Sólidos. Tomo I: Elasticidad Lineal** by Prof. P. Quintela. It contains the latest version of notes on the final topics of the *Solid Mechanics* subject of the *Master's degree in Industrial Mathematics*, which is taught jointly by the universities of Santiago de Compostela, A Coruña, Vigo, Carlos III and the Politechnique University of Madrid. First volume focuses on mathematical models relating to static and dynamic problems in solid mechanics associated with linear elastic and isotropic materials.

This second volume introduces more general behaviour laws. In *Chapter 11*, the behaviour law in anisotropic linear elasticity is deduced, with a focus on the orthotropic case. Thermal effects are then incorporated into the linear elasticity model studied in Volume I to obtain thermoelasticity models.

Chapter 12 studies permanent deformations and their formulation using the theory of plasticity. The concepts of isotropic and kinematic hardening and their mathematical formulation are analysed in one and three-dimensional cases.

Finally, *Chapter 13* provides an overview of the most common contact conditions in Solid Mechanics Modelling.

Patricia Barral
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Chapter 11

Anisotropic linear elasticity. Linear thermoelasticity

One of the main difficulties when modelling a real problem is the correct choice of the material behaviour law. The isotropic linear elasticity, which corresponds to Hooke's law, is suitable for metals, ceramics, crystals and polymers under "soft" forces. It has the advantage that to characterize the material we only need two constants, whose values, in general, can be found in the literature. In this chapter we will introduce two new models in elasticity:

Anisotropic linear elasticity: It considers that the behaviour of the material can be different in certain directions. It allows modelling some composites, wood, ... Between 3 and 21 constants are necessary.

Linear thermoelasticity: In many industrial processes it is necessary to take into account the thermal process and the mechanical deformations due to temperature changes. We will see how to include those deformations in a linear elastic law.

11.1 Anisotropic linear elasticity

The isotropic elastic model does not allow to describe the response of some materials that have a specific orientation, for example, wood, as it is assumed that the deformations are similar in all directions.

Let us consider the linear elastic law in its general form

$$\sigma_{ij}(x) = C_{ijkl}\varepsilon_{kl}(x),$$

where ($\mathbf{C}$) is the fourth order elasticity tensor (see, for instance [Gur90]). This tensor has 81 components, but, due to symmetry properties the number of constants is reduced to 21. Indeed, it verifies:

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{ijlk}.$$

Minor symmetry: $C_{ijkl} = C_{jikl}$**.** Since $\mathbf{C} \in \mathcal{L}(\text{Lin}, \text{Sym})$, given any $\mathbf{H} \in \text{Lin}$, then $\boldsymbol{\tau} = \mathbf{CH} \in \text{Sym}$, and, therefore

$$\tau_{ij} = C_{ijkl} H_{kl} = \tau_{ji} = C_{jikl} H_{kl}.$$

Major symmetry: $C_{ijkl} = C_{klij}$**.** Since by definition

$$C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}},$$

and using the folowing relation between the stress tensor and the strain energy density U:

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}},$$

we can deduce

$$C_{ijkl} = \frac{\partial^2 U}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = \frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = C_{klij}.$$

Finally, from both previous symmetries we can conclude the third one:

$$C_{ijkl} = C_{klij} = C_{lkij} = C_{ijlk}.$$

These symmetry properties allow us to write the behaviour law in matrix form:

$$\{\boldsymbol{\sigma}\} = [\mathbf{C}]\{\boldsymbol{\varepsilon}\},$$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix},$$

where $c_{11} = C_{1111}$, $c_{12} = C_{1122}$, $c_{13} = C_{1133}$, $c_{14} = C_{1123} = C_{1132}$, $c_{15} = C_{1113} = C_{1131}$, $c_{16} = C_{1112} = C_{1121}$,... Notice that, in the isotropic case, $[\mathbf{C}]$ matches the matrix of elasticity $[\mathbf{E}]$.

The above expression can be written in inverse form as

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{S}]\{\boldsymbol{\sigma}\}, \tag{11.1}$$

$$[\mathbf{S}] = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{pmatrix}$$

To understand what $s_{11}, s_{12}, \ldots$ represent, we are going to perform two tests.

Test 1: uniaxial stress test on an anisotropic body. Let us perform a uniaxial test in the direction $\mathbf{e}_1$. Unlike the isotropic case, this tension will induce an extension and a shear in the solid. In effect, if we consider

$$(\boldsymbol{\sigma}) = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from equation (11.1) we deduce that the deformation tensor verifies:

$$\varepsilon_{11} = s_{11}\sigma_{11}, \tag{11.2}$$

$$\varepsilon_{22} = s_{21}\sigma_{11}, \tag{11.3}$$

$$\varepsilon_{33} = s_{31}\sigma_{11} \tag{11.4}$$

$$2\varepsilon_{23} = s_{41}\sigma_{11}, \tag{11.5}$$

$$2\varepsilon_{13} = s_{51}\sigma_{11} \tag{11.6}$$

$$2\varepsilon_{12} = s_{61}\sigma_{11}, \tag{11.7}$$

where we have not considered the symmetry of matrix $[\mathbf{S}]$ in order to explain the orthotropic case.
Note that all constants have dimensions m^2/N.

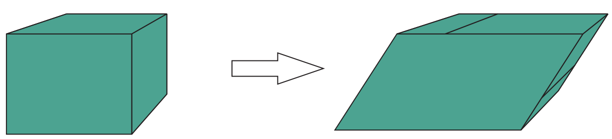

Figure 11.1: Uniaxial load on an anisotropic material

From (11.2), we can see that the constant

$$s_{11} = \frac{\varepsilon_{11}}{\sigma_{11}}$$

shows the proportionality between ε_{11} and σ_{11}, and therefore makes the paper made by $1/E$ in linear elasticity in the direction $\mathbf{e}_1$. So, we can denote

$$s_{11} = \frac{\varepsilon_{11}}{\sigma_{11}} = \frac{1}{E_1}, \tag{11.8}$$

where E_1 is the *elasticiy modulus in the direction* $\mathbf{e}_1$.
From (11.2)-(11.3) we can write

$$\frac{s_{21}}{s_{11}} = \frac{\varepsilon_{22}}{\varepsilon_{11}}.$$

By similarity with linear elasticity this quotient quantifies the *contraction in the direction* $\mathbf{e}_2$, *orthogonal to the stretch direction* $\mathbf{e}_1$. So, this value can be consider as *a generalized version of the Poisson's ratio*, and we denote it by ν_{12}:

$$\frac{s_{21}}{s_{11}} = \frac{\varepsilon_{22}}{\varepsilon_{11}} = -\nu_{12}. \tag{11.9}$$

Notice that from (11.8) and (11.9) it results

$$s_{21} = -\frac{\nu_{12}}{E_1}.$$

Analogously, from (11.2)-(11.4) the quotient

$$\frac{s_{31}}{s_{11}} = \frac{\varepsilon_{33}}{\varepsilon_{11}} = -\nu_{13},$$

quantifies the contraction in the $\mathbf{e}_3$ direction.

Finally, notice that shear terms are not null, whereas in an isotropic body a uniaxial tension does not induce shearing strains.

Test 2: shear stress test in directions $\mathbf{e}_1$, $\mathbf{e}_2$ on an anisotropic body. Let us consider

$$(\boldsymbol{\sigma}) = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From equation (11.1) it can be deduced that the deformation tensor verifies:

$$2\varepsilon_{12} = s_{66}\sigma_{12}.$$

Therefore,

$$\frac{1}{s_{66}} = \frac{\sigma_{12}}{2\varepsilon_{12}}.$$

If we compare this with the isotropic linear elastic case, $1/s_{66}$ represents the coefficient of proportionality between the shear stress in the directions $\mathbf{e}_1$, $\mathbf{e}_2$ and the deviation from the right angle in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$, which was called *shear modulus.* So, we can denote:

$$\mu_{12} = \frac{1}{s_{66}} = \frac{\sigma_{12}}{2\varepsilon_{12}},$$

the shear modulus in $\mathbf{e}_1$, $\mathbf{e}_2$ directions.

11.1.1 Orthotropic materials

An orthotropic material has three perpendicular symmetry planes (wood, laminates, reinforced polymers, ...). The intersection of these three planes of symmetry define the main axes of orthotropy.

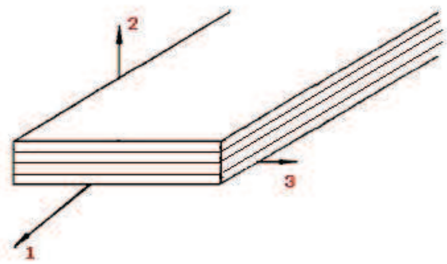

Figure 11.2: Laminated material. Axes of orthotropy

Let us assume that the planes of symmetry are parallel to the coordinate planes. It can be proved (see [Cia85; Cha12]) that the elasticity matrix has the form:

$$[\mathbf{C}] = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}.$$

Notice that the elasticity matrix of these materials has only 9 non-zero constants, as in the isotropic case.

This behavior law is usually expressed in its inverse form, using the matrix [**S**]. Following the reasoning used in the previous traction test we can obtain the expression of s_{11}, s_{21} and s_{31} in terms of the Young's modulus in direction $\mathbf{e}_1$ and the Poisson's ratios ν_{12} and ν_{13}. Repeating the traction test in directions $\mathbf{e}_2$ and $\mathbf{e_3}$ the terms of second and third column of [**S**] can be deduced. Shear traction tests analogous to the previous test 2 leads to the remaining columns of the matrix. Summarizing:

$$[\mathbf{S}] = \begin{pmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu_{12} \end{pmatrix}.$$

Notice that the Poisson's ratios must verify

$$\nu_{ij}/E_i = \nu_{ji}/E_j,$$

so the matrix is symmetric, as it was expected. The material is characterized by 9 independent coefficients, which can be obtained from 3 tests of traction in the three directions of orthotropy and three tests of shearing (see [LC94]):

- 3 elasticity moduli: E_1, E_2, E_3, in the orthotropy directions.
- 3 shear moduli: μ_{12}, μ_{23}, μ_{13}.
- 3 contraction coefficients: ν_{12}, ν_{23}, ν_{13}, for example.

Transversely isotropic materials A particular case of an orthotropic solid is when it has a plane of isotropy (the behavior is the same in that plane). That is the case in some masonry walls, reinforced concrete,...

Let us consider that the plane of isotropy is the x_1, x_2 one. We denote:

- E_p, ν_p: the elasticity constants in the plane $x_1 x_2$.
- E_t: the elasticity modulus in the transversal direction $\mathbf{e}_3$.
- ν_{pt}: the contraction in the direction $\mathbf{e}_3$ due to an stretch in an orthogonal direction.
- ν_{tp}: the contraction in the direction $\mathbf{e}_1$ or $\mathbf{e}_2$ due to an stretch in the direction $\mathbf{e}_3$.
- μ_t: the shear modulus between the direction $\mathbf{e}_3$ and the plane $x_1 x_2$.

Then the elasticity matrix has the form

$$[\mathbf{S}] = \begin{pmatrix} 1/E_p & -\nu_p/E_p & -\nu_{tp}/E_t & 0 & 0 & 0 \\ -\nu_p/E_p & 1/E_p & -\nu_{tp}/E_t & 0 & 0 & 0 \\ -\nu_{pt}/E_p & -\nu_{pt}/E_p & 1/E_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu_p \end{pmatrix},$$

where $\mu_p = \dfrac{E_p}{2(1+\nu_p)}$ and the elasticity constants verify:

$$\nu_{tp}/E_t = \nu_{pt}/E_p.$$

	E_p (GPa)	E_t (GPa)	ν_p	ν_{tp}	ν_{pt}	μ_t (GPa)	μ_p (GPa)
Ti	104.37	143.27	0.48	0.27	0.20	46.70	35.20
Co	211.30	313.15	0.49	0.22	0.15	78.30	71.00

Table 11.1: Some values of transversely isotropic materials obtained from [Bow10].

11.2 Linear thermoelasticity

Consider a pair of forces $(\mathbf{s}, \mathbf{b})$ and a thermal system (g, f) for the body $\mathcal{B}$ during the movement $\mathbf{X}$. Remember that the *surface heat*, $g(\mathbf{n}, x, t)$, is defined for each unit vector $\mathbf{n}$ and every (x, t) in the trajectory of the movement with the following property: given an oriented surface $\mathcal{S}$ in $\mathcal{B}_t$ with outwards normal unit vector $\mathbf{n}$ in x, $g(\mathbf{n}, x, t)$ represents the heat per unit area and time flowing from the negative side of $\mathcal{S}$ towards its positive side. On the other hand, the density *inner heat* $f(x, t)$, represents heat, per unit of volume and time, provided by the environment to point x at time t.

11.2.1 First principle of Thermodynamics. Energy conservation law

Definition The heat rate supplied into the part $\mathcal{P}$ of the body $\mathcal{B}$ at a time t is

$$\mathcal{Q}(\mathcal{P}, t) = -\int_{\partial\mathcal{P}_t} g(\mathbf{n})\mathrm{d}A_x + \int_{\mathcal{P}_t} f\mathrm{d}V_x.$$

The *first principle of Thermodynamics*, or *energy conservation law*, establishes the existence of a scalar field E, called **specific total energy** per unity of mass, such that the increment of that field is equal to the power of exterior forces applied to the body plus the increment of the supplied heat. This can be expressed on every part $\mathcal{P}$ and for each instant t as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}_t} \rho E \mathrm{d}V_x = \int_{\partial\mathcal{P}_t} \mathbf{s}(\mathbf{n})\cdot\mathbf{v}\mathrm{d}A_x + \int_{\mathcal{P}_t} \mathbf{b}\cdot\mathbf{v}\mathrm{d}V_x - \int_{\partial\mathcal{P}_t} g(\mathbf{n})\mathrm{d}A_x + \int_{\mathcal{P}_t} f\mathrm{d}V_x, \ \forall\mathcal{P}_t \subset \mathcal{B}_t,$$

where $\mathbf{v}$ denotes the velocity field. We have the following result (see [Ber05]):

Theorem (Cauchy). Suppose that the momentum balance laws hold. Then, a necessary and sufficient condition for the energy conservation law to be satisfied is the existence of a spatial vector field $\mathbf{q}$ (called **heat flux** vector) such that

1. For each unit vector $\mathbf{n}$, $g(\mathbf{n}, x, t) = \mathbf{q}(x, t) \cdot \mathbf{n}$,
2. $\rho\dot{E} = \mathrm{div}(\mathbf{T}\mathbf{v}) + \mathbf{b} \cdot \mathbf{v} - \mathrm{div}\mathbf{q} + f$.

We denote by **specific internal energy** the scalar field

$$e = E - \frac{|\mathbf{v}|^2}{2}.$$

From the first principle of Thermodynamics and the Theorem of power expanded the next expression can be deduced (see [Ber05]):

$$\rho\dot{e} = \mathbf{T} : \mathbf{D} - \mathrm{div}\mathbf{q} + f, \tag{11.10}$$

where $\mathbf{D}$ denotes the symmetric part of the velocity gradient, $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$, $\mathbf{L} = \mathrm{grad}(\mathbf{v})$.

11.2.2 The conservation equation in terms of temperature

Let's see how we can write the energy conservation law (11.10) in terms of temperature.

Let $\theta(x,t)$ be the *absolute temperature* at a point x at time instant t. We define an **elastic material with heat conduction** as a material body, the constitutive class of which consists of all thermodynamic processes satisfying

$$\mathbf{T}(x,t) = \hat{\mathbf{T}}(\mathbf{F}(p,t), \theta(x,t), p), \tag{11.11}$$
$$e(x,t) = \hat{e}(\mathbf{F}(p,t), \theta(x,t), p), \tag{11.12}$$
$$\mathbf{q}(x,t) = \hat{\mathbf{q}}(\mathbf{F}(p,t), \theta(x,t), \mathrm{grad}\theta(x,t), p) \tag{11.13}$$

with $x = \mathbf{X}(p,t)$, for some "smooth enough" mappings

$$\hat{\mathbf{T}} : Lin^+ \times \mathbb{R} \times \mathcal{B} \to Sym \tag{11.14}$$
$$\hat{e} : Lin^+ \times \mathbb{R} \times \mathcal{B} \to \mathbb{R} \tag{11.15}$$
$$\hat{\mathbf{q}} : Lin^+ \times \mathbb{R} \times \mathcal{V} \times \mathcal{B} \to \mathcal{V} \tag{11.16}$$

which are called **response mappings** of the body.

Let us derive (11.12) with respect to time:

$$\dot{e}(x,t) = \frac{\partial \hat{e}}{\partial \theta}(\mathbf{F}, \theta, p)\dot{\theta}(x,t) + \frac{\partial \hat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, p) : \dot{\mathbf{F}}(p,t). \tag{11.17}$$

Let us introduce the next definition:
Definition. The *specific heat at constant deformation* is the escalar field defined by

$$c_F(x,t) = \hat{c}_F(\mathbf{F}(p,t), \theta(x,t), p).$$

with

$$\hat{c}_F(\mathbf{F}, \theta, p) = \frac{\partial \hat{e}}{\partial \theta}(\mathbf{F}, \theta, p).$$

	c_F $(J/(KgK))$
Aluminum	900
Steel	480
Tantalum	140
Wood	1760

Table 11.2: Typical values of specific heat.

With this definition, equation (11.17) can be rewritten as

$$\dot{e}(x,t) = c_F(x,t)\dot{\theta}(x,t) + \frac{\partial \hat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, p) : \dot{\mathbf{F}}(p,t). \tag{11.18}$$

On the other hand, in [Ber05] it is proved that from the *second principle of Thermodynamics* it can be deduced that

$$\frac{\partial \hat{e}}{\partial \mathbf{F}}(\mathbf{F},\theta,p) = \frac{1}{\rho}\hat{\mathbf{T}}(\mathbf{F},\theta,p)\mathbf{F}^{-T} - \frac{\theta}{\rho}\frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F},\theta,p)\mathbf{F}^{-T}.$$

If we replace this expression in (11.18), and multiply by ρ, we obtain

$$\rho\dot{e} = \rho c_F\dot{\theta} + \hat{\mathbf{T}}(\mathbf{F},\theta,p)\mathbf{F}^{-T} : \dot{\mathbf{F}} - \theta\frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F},\theta,p)\mathbf{F}^{-T} : \dot{\mathbf{F}}. \tag{11.19}$$

Notice that, to simplify the notation, we have suppressed the dependence of the variables x, p and t.
Since $\hat{\mathbf{T}}(\mathbf{F},\theta,p) \in Sym$ we can deduce that

$$\begin{aligned}\hat{\mathbf{T}}(\mathbf{F},\theta,p)\mathbf{F}^{-T} : \dot{\mathbf{F}} &= \hat{\mathbf{T}}(\mathbf{F},\theta,p)\mathbf{F}^{-T} : \mathbf{L}\mathbf{F} = \hat{\mathbf{T}}(\mathbf{F},\theta,p)\mathbf{F}^{-T}\mathbf{F}^{T} : \mathbf{L} = \\ &= \hat{\mathbf{T}}(\mathbf{F},\theta,p) : \mathbf{L} = \hat{\mathbf{T}}(\mathbf{F},\theta,p) : \mathbf{D}\end{aligned}$$

where we have applied the tensor property $A : BC = AC^T : B$. Using the same reasoning for the last term of (11.19), since $\frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F},\theta,p) \in Sym$, we can conclude that

$$\rho\dot{e} = \rho c_F\dot{\theta} + \hat{\mathbf{T}}(\mathbf{F},\theta,p) : \mathbf{D} - \theta\frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F},\theta,p) : \mathbf{D}. \tag{11.20}$$

By replacing this equality in (11.10) we obtain the following expression for the *energy conservation law* in terms of temperature:

$$\rho c_F\dot{\theta} = \theta\frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F},\theta,p) : \mathbf{D} - \mathrm{div}(\mathbf{q}) + f. \tag{11.21}$$

Let us assume that the material is *isotropic* at p and the response function of the heat flux $\hat{\mathbf{q}}$ does not depend on $\mathbf{F}$, then in [Ber05] it is proved that there exists a map

$$\hat{k} : \mathbb{R} \times \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}$$

such that

$$\hat{\mathbf{q}}(\mathbf{F},y,\mathbf{w},p) = -\hat{k}(y,|\mathbf{w}|^2,p)\mathbf{w}.$$

Then, the heat flux is given by

$$\mathbf{q}(x,t) = -\hat{k}(\theta(x,t),|\mathrm{grad}\theta(x,t)|^2,p)\mathrm{grad}\theta(x,t).$$

The scalar field $k(x,t) = \hat{k}(\theta(x,t),|\mathrm{grad}\theta(x,t)|^2,p)$ is called *thermal conductivity* ($W/(mK)$) and is positive.

Obs: In the linear case we will consider $k = cte$, therefore

$$\mathbf{q}(x,t) = -k\,\mathrm{grad}\theta(x,t), \tag{11.22}$$

This is known as **Fourier law**.

Material	k $(W/(mK))$
Aluminum	209.3
Wood	0.13
Iron	80.2
Brick	0.80

Table 11.3: Thermal conductivity of some materials

11.2.3 Conservation equations in Lagrangian coordinates

Following the reasoning used to obtain the momentum equilibrium equations in Lagrangian coordinates, we will deduce the expression of the energy conservation law in such coordinates. There, it was deduced that the motion conservation equation in the reference configuration is

$$\rho_0 \ddot{\mathbf{u}} - \mathrm{Div}(\mathbf{S}) = \mathbf{b}_0 \text{ en } \mathcal{B}, \tag{11.23}$$

where $\mathbf{S}$ is the First Piola Kirchhoff stress tensor, which is now assumed to be dependent on temperature and

$$\mathbf{b}_0(p,t) = \mathbf{b}_m(x,t)\det(\mathbf{F}(p,t)).$$

Let us introduce a response function for the First Piola Kirchhoff stress tensor from the response function for the Cauchy stress tensor:

$$\mathbf{S}(p,t) = \hat{\mathbf{S}}(\mathbf{F},\theta,p) = \det(\mathbf{F})\hat{\mathbf{T}}_m(\mathbf{F},\theta,p)\mathbf{F}^{-T}.$$

Let $\mathcal{P} \subset \mathcal{B}$, to obtain the energy conservation law in the reference configuration we integrate the equation (11.21) into $\mathcal{P}_t$ and apply the Gauss theorem:

$$\int_{\mathcal{P}_t} \rho c_F \dot{\theta}\, \mathrm{d}V_x = \int_{\mathcal{P}_t} \theta \frac{\partial \hat{\mathbf{T}}}{\partial \theta} : \mathbf{D}\, \mathrm{d}V_x - \int_{\partial\mathcal{P}_t} \mathbf{q}\cdot\mathbf{n}\, \mathrm{d}A_x + \int_{\mathcal{P}_t} f \mathrm{d}V_x,$$

where $\mathbf{n}$ is the outward unit vector normal to the boundary $\partial\mathcal{P}_t$. Appling the theorem of change of variable with $x = \mathbf{X}(p,t)$, we obtain

$$\begin{aligned}
\int_{\mathcal{P}_t} \rho c_F \dot{\theta}\, \mathrm{d}V_x &= \int_{\mathcal{P}} \rho_0 c_{F\,m} \dot{\theta}_m\, \mathrm{d}V_p, \\
\int_{\mathcal{P}_t} \theta \frac{\partial \hat{\mathbf{T}}}{\partial \theta} : \mathbf{D}\, \mathrm{d}V_x &= \int_{\mathcal{P}_t} \theta \frac{\partial \hat{\mathbf{T}}}{\partial \theta} : \mathbf{L}\, \mathrm{d}V_x \\
&= \int_{\mathcal{P}} \theta \left(\frac{\partial \hat{\mathbf{T}}}{\partial \theta} \right)_m : \dot{\mathbf{F}}\mathbf{F}^{-1} \det(\mathbf{F})\, \mathrm{d}V_p = \int_{\mathcal{P}} \theta \det(\mathbf{F}) \left(\frac{\partial \hat{\mathbf{T}}}{\partial \theta} \right)_m \mathbf{F}^{-T} : \dot{\mathbf{F}}\, \mathrm{d}V_p \\
&= \int_{\mathcal{P}} \theta \frac{\partial \hat{\mathbf{S}}}{\partial \theta} : \nabla \dot{\mathbf{u}}\, \mathrm{d}V_p, \\
\int_{\partial \mathcal{P}_t} \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}A_x &= \int_{\partial \mathcal{P}} \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{q}_m \cdot \mathbf{m}\, \mathrm{d}A_p = \int_{\mathcal{P}} \mathrm{Div}(\det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{q}_m)\, \mathrm{d}V_p, \\
\int_{\mathcal{P}_t} f \mathrm{d}V_x &= \int_{\mathcal{P}} f_m \det(\mathbf{F}) \mathrm{d}V_p,
\end{aligned}$$

where $\mathbf{m}$ is the outward unit vector normal to the boundary $\partial \mathcal{P}$. Let us define

$$\begin{aligned}
\mathbf{q}_0(p,t) = \hat{\mathbf{q}}_0(\mathbf{F}, \theta, \nabla \theta, p) &= \det(\mathbf{F}) \mathbf{F}^{-1} \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathrm{grad}\theta, p), \\
f_0(p,t) &= f_m(p,t) \det(\mathbf{F}(p,t)).
\end{aligned}$$

From the Localization theorem we deduce

$$\rho_0 c_F \dot{\theta} = \theta \frac{\partial \hat{\mathbf{S}}}{\partial \theta}(\mathbf{F}, \theta, p) : \nabla \dot{\mathbf{u}} - \mathrm{Div}(\mathbf{q}_0) + f_0. \tag{11.24}$$

Summing up, we have the following equations in Lagrangian coordinates:

$$\begin{cases} \rho_0 \ddot{\mathbf{u}} - \mathrm{Div}(\mathbf{S}) = \mathbf{b}_0 \text{ in } \mathcal{B} \times (0, +\infty), \\ \rho_0 c_F \dot{\theta} = \theta \dfrac{\partial \hat{\mathbf{S}}}{\partial \theta}(\mathbf{F}, \theta, p) : \nabla \dot{\mathbf{u}} - \mathrm{Div}(\mathbf{q}_0) + f_0. \end{cases} \tag{11.25}$$

11.2.4 Linear thermoelasticity

In this section we will assume that the gradient of displacement from an initial equilibrium state $\nabla \mathbf{u}$ and the variation of temperature from the reference temperature θ_0 are small and the reference configuration is a natural estate.

Linear approximation of the behavior law

The linear thermoelastic model is obtained by using a Taylor development of the response function $\hat{\mathbf{T}}(\mathbf{F}, \theta, p)$ at $\mathbf{F} = \mathbf{I}$ and $\theta = \theta_0$:

$$\hat{\mathbf{T}}(\mathbf{F}, \theta, p) = \hat{\mathbf{T}}(\mathbf{I}, \theta_0, p) + \frac{\partial \hat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p)(\mathbf{F} - \mathbf{I}) + \frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{I}, \theta_0, p)(\theta - \theta_0) \tag{11.26}$$

$$+ \; o(\nabla \mathbf{u}) + o(\theta - \theta_0) + O[(\theta - \theta_0)(\nabla \mathbf{u})]. \tag{11.27}$$

So, if the reference configuration is a natural state, $\hat{\mathbf{T}}(\mathbf{I}, \theta_0, p) = \mathbf{0}$, under the assumption of small displacements and small temperature variations, we can approach

$$\hat{\mathbf{T}}(\mathbf{F}, \theta, p) \approx \frac{\partial \hat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p)(\nabla \mathbf{u}) + \frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{I}, \theta_0, p)(\theta - \theta_0). \tag{11.28}$$

Definition. The linear operator $\mathbf{C}(\theta_0, p) \in \mathcal{L}(Lin, Lin)$ defined by

$$\mathbf{C}(\theta_0, p) = \frac{\partial \hat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) \tag{11.29}$$

is called the (fourth order) **elasticity tensor** at point p and temperature θ_0.

Following the arguments used to deduce the linear elasticity model, the following result can be proved:

Proposition 11.2.1 *Let us suppose that the initial stress is null. Then, the elasticity tensor is symmetric and*

$$\mathbf{C}(\theta_0, p) = \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p).$$

Following [Ber05], let us assume that $\frac{\partial \hat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p)$ is invertible when considered as a linear mapping from *Sym* to *Sym*. Then, by the Implicit Function Theorem, the equation $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \theta, p)$ defines $\mathbf{F}$ as a function of $\mathbf{T}$ and θ in a neighborhood of $\mathbf{F} = \mathbf{I}$, $\theta = \theta_0$ and $\mathbf{T} = \mathbf{0}$, since $\hat{\mathbf{T}}(\mathbf{I}, \theta_0, p) = \mathbf{0}$. Furthermore, its derivative with respect to the temperature θ at constant $\mathbf{T}$ verifies

$$\frac{\partial \hat{\mathbf{F}}}{\partial \theta}(\mathbf{T}, \theta, p) = -\left(\frac{\partial \hat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, p)\right)^{-1} \frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F}, \theta, p),$$

with $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \theta, p)$. Let us evaluate this derivative at $\mathbf{F} = \mathbf{I}$, $\theta = \theta_0$ and $\mathbf{T} = \mathbf{0}$ and use the definition (11.29).

Definition. The **thermal expansion at constant elastic stress** is the tensor

$$\mathbf{A}(\theta_0, p) = -\mathbf{C}(\theta_0, p)^{-1} \left(\frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{I}, \theta_0, p)\right) \tag{11.30}$$

If we now consider a Taylor development for the first Piola-Kirchoff stress tensor analogous to (11.27), we obtain

$$\hat{\mathbf{S}}(\mathbf{F}, \theta, p) \approx \hat{\mathbf{S}}(\mathbf{I}, \theta_0, p) + \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p)(\nabla \mathbf{u}) + \frac{\partial \hat{\mathbf{S}}}{\partial \theta}(\mathbf{I}, \theta_0, p)(\theta - \theta_0) \tag{11.31}$$

Since $\hat{\mathbf{S}}(\mathbf{I}, \theta_0, p) = \hat{\mathbf{T}}(\mathbf{I}, \theta_0, p)$, if $\hat{\mathbf{T}}(\mathbf{I}, \theta_0, p) = \mathbf{0}$, then

$$\hat{\mathbf{S}}(\mathbf{F}, \theta, p) \approx \mathbf{C}(\theta_0, p)(\nabla \mathbf{u}) - \mathbf{C}(\theta_0, p)\mathbf{A}(\theta_0, p)(\theta - \theta_0). \tag{11.32}$$

As it was proved in linear elasticity,

$$\mathbf{C}(\theta_0, p)(\nabla \mathbf{u}) = \mathbf{C}(\theta_0, p)(\boldsymbol{\varepsilon}(\mathbf{u})),$$

where $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor. So, we can write the previous expresion as

$$\hat{\mathbf{S}}(\mathbf{F}, \theta, p) \approx \mathbf{C}(\theta_0, p)\left(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{A}(\theta_0, p)(\theta - \theta_0)\right).$$

Following the notation introduced in linear elasticity, from now on we denote by $\boldsymbol{\sigma}$ the linear approximation of the first Piola-Kirchhoff stress tensor, so we can write the *linear thermoelastic behavior law* as:

$$\boldsymbol{\sigma} = \mathbf{C}(\theta_0, p)\left(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{A}(\theta_0, p)(\theta - \theta_0)\right).$$

Observation 11.2.1 *If the temperature is constant $\theta = \theta_0$ we recover the linear elastic behavior law:*

$$\boldsymbol{\sigma} = \mathbf{C}(\theta_0, p)\left(\boldsymbol{\varepsilon}(\mathbf{u})\right).$$

Isotropic linear thermoelastic law

If the body is *isotropic*, tensors $\mathbf{C}$ and $\mathbf{A}$ are simpler, as it is proved in [Ber05] using the Representation theorem for linear isotropic functions.

Proposition 11.2.2 *Let us assume that the body is isotropic at p. Then, there exist functions $\hat{\lambda}(\theta_0, p)$, $\hat{\mu}(\theta_0, p)$ and $\hat{\alpha}(\theta_0, p)$ defined in $\mathbb{R}^+ \times \mathcal{B}$ such that*

$$\mathbf{C}(\theta_0, p)(\boldsymbol{E}) = 2\hat{\mu}(\theta_0, p)\boldsymbol{E} + \hat{\lambda}(\theta_0, p)\mathrm{tr}(\boldsymbol{E})\mathbf{I},\ \forall \boldsymbol{E} \in Sym \tag{11.33}$$

$$\mathbf{A}(\theta_0, p) = \hat{\alpha}(\theta_0, p)\mathbf{I}. \tag{11.34}$$

Functions $\hat{\lambda}(\theta_0, p)$, $\hat{\mu}(\theta_0, p)$ are the Lamé coefficients of the body at point p at temperature θ_0 and $\hat{\alpha}(\theta_0, p)$ is called the ***coefficient of linear thermal expansion at constant stress*** *of the body at point p at temperature θ_0.*

Therefore, the isotropic linear thermolastic behavior law in terms of the Lamé coefficients and the coefficient of linear thermal expansion becomes

$$\boldsymbol{\sigma} = 2\hat{\mu}(\theta_0, p)\boldsymbol{\varepsilon}(\mathbf{u}) + \hat{\lambda}(\theta_0, p)\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\hat{\mu}(\theta_0, p) + 3\hat{\lambda}(\theta_0, p))\hat{\alpha}(\theta_0, p)(\theta - \theta_0)\mathbf{I}. \tag{11.35}$$

Furthermore, if we assume that the initial state is spatially homogeneous, that is, initial fields are constant functions, we obtain

$$\boldsymbol{\sigma} = 2\mu_0\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda_0\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\mu_0 + 3\lambda_0)\alpha_0(\theta - \theta_0)\mathbf{I},$$

where $\mu_0 = \hat{\mu}(\theta_0, p)$, $\lambda_0 = \hat{\lambda}(\theta_0, p)$ and $\alpha_0 = \hat{\alpha}(\theta_0, p)$.

Thermal contribution in (11.35) can also be written in terms of Young's modulus and the Poisson's coefficient:

$$(2\mu_0 + 3\lambda_0)\alpha_0(\theta - \theta_0)\mathbf{I} = \frac{E_0}{1 - 2\nu_0}\alpha_0(\theta - \theta_0)\mathbf{I}. \tag{11.36}$$

It is also usual to write thermolastic behavior law in terms of the strain tensor. For that purpose, we assume that the strain tensor is the sum of an elastic part $\boldsymbol{\varepsilon}^e(\mathbf{u})$ and a thermal one $\boldsymbol{\varepsilon}^T$:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}^e(\mathbf{u}) + \boldsymbol{\varepsilon}^T,$$

where,

$$\boldsymbol{\varepsilon}^e(\mathbf{u}) = \mathbf{C}(\theta_0, p)^{-1}(\boldsymbol{\sigma}), \tag{11.37}$$

$$\boldsymbol{\varepsilon}^T = \mathbf{A}(\theta_0, p)(\theta - \theta_0). \tag{11.38}$$

In the isotropic case, if the initial state is spatially homogeneous, we have

$$\boldsymbol{\varepsilon}^e(\mathbf{u}) = \frac{1}{2\mu_0}\left(\boldsymbol{\sigma} - \frac{\lambda_0}{3\lambda_0 + 2\mu_0}\mathrm{tr}\boldsymbol{\sigma}\mathbf{I}\right), \tag{11.39}$$

$$\boldsymbol{\varepsilon}^T = \alpha_0(\theta - \theta_0)\mathbf{I}. \tag{11.40}$$

Notice that for isotropic materials no shear strains are originated by temperature changes. In the orthotropic case thermal expansion does not induce shear strains either, but expansion in the three orthotropic directions has not to be equal. So, it is necessary to determine three thermal expansion coefficients (see [Bow10]).

The coefficient of thermal expansion quantifies the volume change when a body is subjected to a variation of temperature in the absence of stresses. The units of the coefficient of thermal expansion are K^{-1}. Some values for different materials are given in Table 11.2.4.

	α ($10^{-6}\mathrm{K}^{-1}$)
Aluminum	24
Glass	4
Rubber	200

Table 11.4: Typical values of coefficients of thermal expansion.

If we denote $\alpha_0 = \alpha$, $E_0 = E$ and $\nu_0 = \nu$, linear thermoelastic isotropic behavior law can also be written in matrix mode as:

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} + \alpha(\theta-\theta_0) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Or:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} - \frac{E\alpha(\theta-\theta_0)}{1-2\nu} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear approximation of the energy conservation law

Let us consider the energy conservation law (11.24). In [Ber05] a linear approximation of the heat flux is obtained assuming that $\nabla \mathbf{u} = O(\varepsilon)$, $\theta - \theta_0 = O(\varepsilon)$ and $\nabla\theta - \nabla\theta_0 = O(\varepsilon)$. If the body is *isotropic* and the *Fourier law* is verified, in [Ber05] it is proved that, if the reference temperature is homogeneous, $\theta_0 = cte$, then we can approach

$$\hat{\mathbf{q}}_0(\mathbf{F}, \theta_0, \nabla\theta_0, p) \approx -\hat{k}(\theta_0, 0, p)\nabla\theta.$$

Furthermore, if we approximate in (11.24) $\theta\frac{\partial\hat{\mathbf{S}}}{\partial\theta}(\mathbf{F}, \theta, p)$ and $c_F(\mathbf{F}, \theta, p)$ by their values in the reference configuration, $\mathbf{F} = \mathbf{I}$ and $\theta = \theta_0$, we obtain the *linear energy equation for thermoelasticity*:

$$\rho_0 \hat{c}_F(\mathbf{I}, \theta_0, p)\dot{\theta} = \theta_0 \frac{\partial\hat{\mathbf{S}}}{\partial\theta}(\mathbf{I}, \theta_0, p) : \nabla\dot{\mathbf{u}} + \mathrm{Div}(\hat{k}(\theta_0, 0, p)\nabla\theta) + f_0.$$

Let us assume that *the initial state is spatially homogeneous* then, using (11.30) and (11.35) we obtain

$$\rho_0 c_{F0}\dot{\theta} = -\theta_0(2\mu_0 + 3\lambda_0)\alpha_0 \mathrm{Div}(\dot{\mathbf{u}}) + k_0\Delta\theta + f_0,$$

where $c_{F0} = \hat{c}_F(\mathbf{I}, \theta_0, p)$ and $k_0 = \hat{k}(\theta_0, 0, p)$.

Summing up, the *linear equations for thermoelasticity in the reference configuration, when the material is isotropic, the heat flux is given by de Fourier law and the initial state is spatially homogeneous* are:

$$\begin{cases} \rho_0\ddot{\mathbf{u}} - \mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b}_0 \text{ en } \mathcal{B}\times(0,+\infty), \\ \boldsymbol{\sigma} = 2\mu_0\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda_0 \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\mu_0 + 3\lambda_0)\alpha_0(\theta - \theta_0)\mathbf{I}, \\ \rho_0 c_{F0}\dot{\theta} = -\theta_0(2\mu_0 + 3\lambda_0)\alpha_0 \mathrm{Div}(\dot{\mathbf{u}}) + k_0\Delta\theta + f_0. \end{cases} \tag{11.41}$$

In the *quasistatic* case, terms $\rho_0\ddot{\mathbf{u}}$ and $\frac{\partial\hat{\mathbf{S}}}{\partial\theta}(\mathbf{I}, \theta_0, p) : \nabla\dot{\mathbf{u}}$, may be neglected. Then thermoelastic equations (11.41) can be reduced to

$$\begin{cases} -\mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b}_0 \text{ en } \mathcal{B}\times(0,+\infty), \\ \boldsymbol{\sigma} = 2\mu_0\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda_0 \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\mu_0 + 3\lambda_0)\alpha_0(\theta - \theta_0)\mathbf{I}, \\ \rho_0 c_{F0}\dot{\theta} = k_0\Delta\theta + f_0. \end{cases} \tag{11.42}$$

Boundary conditions

To complete the model, it is necessary to add the boundary conditions of both, the mechanical and thermal problem. The latter can be of three types: Dirichlet (temperature is known), convection or radiation.

When a fluid comes in contact with a solid whose contact surface is at a different temperature, heat transfer by *convection* occurs. Convection can be *free* or *natural* if the movement of the fluid is caused by differences in the density due to differences in temperature, or *forced* if it is due to external mechanisms, such as a fan.

Under the previous assumptions, Newton's law of cooling allows to express convection boundary condition as

$$-k\frac{\partial\theta}{\partial\mathbf{n}} = \alpha_c(\theta - \theta_c),$$

where α_c is the *heat transfer coefficient* $(W/(m^2K))$, θ_c the temperature of the fluid and $k = k_0$ is the thermal conductivity. The heat transfer coefficient is not a property of the material, it depends on the fluid and the geometry of the solid surface (see [Cha87]). The following table indicates the range of values for some fluids. The wide range indicates that one of the difficulties in the modelling of thermal processes is to determine the heat transfer coefficients.

	α_c $(W/(m^2K))$
Free convection with air	5-25
Free convection with water	500-1000
Forced convection with air	10-500
Forced convection with water	1000-15000

Table 11.5: Typical values of heat transfer coefficients

Whereas conduction and convection take place through a material medium, thermal *radiation* can transport heat through a fluid or vacuum. So, if a body is placed near another one in a vacuum, which is, for example, colder, it loses energy. This energy loss (thermal radiation) is an aspect of a more general phenomenon, known as electromagnetic emission, produced by the thermal excitation of the body's matter.

One of the main characteristics of radiation is that it is proportional to the fourth power of the absolute temperature, and therefore it is a non-linear condition. Note that the heat transfer by radiation is very important at high temperatures. Further, the amount of radiation emitted by a body depends on the size, the shape and relative orientation of the emitter and receptor.

The thermal radiation emitted by a black body (the perfect emitter) is given by the *Stefan-Boltzmann's law*

$$q_r = \sigma\theta^4,$$

where q_r is the heat flux by radiation, $\sigma = 5.67 \times 10^{-8}\, W/(m^2K^4)$ the Stefan-Boltzman constant and θ the absolute temperature of the body.

However, not all radiant bodies behave like ideal black bodies. A surface not so perfect, that will be called *gray surface*, emit radiation according with the expression

$$q_r = \varepsilon\sigma\theta^4,$$

where $\varepsilon \in [0,1]$ is the emissivity of the surface.

If a gray body at absolute temperature θ, with emissivity factor ε, is completely surrounded by another, for example the wall of a room, which is supposed to be a black body and very large in comparison with the gray body, the boundary condition that models the flow of radiation heat between the surface of the body and the wall is

$$-k\frac{\partial\theta}{\partial\mathbf{n}} = \varepsilon\sigma(\theta^4 - \theta_r^4),$$

where θ_r is the absolute temperature of the walls.

Indeed, the radiation at each point of the boundary can be decomposed as:

$$q_r(x) = R(x) - G(x),$$

where $R(x)$ is the radiosity, the outgoing energy of radiation, and $G(x)$ the irradiation, the incoming radiative energy. The radiosity is the sum of the emitted and the reflected radiation. So,

$$R(x) = \varepsilon\sigma\theta^4(x) + \gamma G(x),$$

where γ denotes the reflectivity. The Kirchoff's law for thermal radiation implies that $\gamma = 1-\varepsilon$. Therefore, if the wall is considered a black body, $G(x) = \sigma\theta_r^4(x)$, and :

$$R(x) = \varepsilon\sigma\theta^4(x)+\gamma G(x) = \varepsilon\sigma\theta^4(x)+(1-\varepsilon)\sigma\theta_r^4(x) = \varepsilon\sigma(\theta^4(x)-\theta_r^4(x))+\sigma\theta_r^4(x).$$

Then we conclude

$$q_r(x) = R(x) - G(x) = \varepsilon\sigma(\theta^4(x) - \theta_r^4(x)).$$

11.3 Exercises

1. The following relations were obtained in a laboratory test for a certain material:

$$\begin{aligned}\varepsilon_x &= \frac{1}{E_1}\sigma_x + \frac{-\nu_{21}}{E_2}\sigma_y + \frac{-\nu_{31}}{E_3}\sigma_z \\ \varepsilon_y &= \frac{-\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y + \frac{-\nu_{32}}{E_3}\sigma_z \\ \varepsilon_z &= \frac{-\nu_{13}}{E_1}\sigma_x + \frac{-\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z,\end{aligned}$$

   where $\nu_{21} = 0,2$, $\nu_{31} = 0,25$, $\nu_{32} = 0,25$, $E_1 = 1000MPa$, $E_2 = 2000MPa$, $E_3 = 1500MPa$. Knowing that it is an orthotropic material, calculate the values of ν_{12}, ν_{13} and ν_{23}.

2. We consider a circular-section cylinder with an axis $\mathbf{e}_3$, made of concrete reinforced with steel fibres along the beam's axis. This allows us to assume that it is a transversely isotropic, linear elastic material. We make the following assumptions:

   - The constants of elasticity in the isotropy plane are E_p, ν_p.
   - The modulus of elasticity in direction $\mathbf{e}_3$ is E_t.
   - The coefficient giving the contraction in the direction $\mathbf{e}_3$ due to a stretching in one of the directions of the isotropy plane is nu_{pt}.
   - The coefficient giving the contraction in one of the directions of the isotropy plane due to a stretching in direction $\mathbf{e}_3$ is ν_{tp}.
   - The shear modulus between the direction $\mathbf{e}_3$ and the isotropy plane is μ_t.

   Taking into account the axial symmetry and assuming that the applied forces are compatible with this symmetry, write the corresponding transversely isostatic behaviour law.

3. We consider a bar of 7.5 m length consisting of an elastic material with Young modulus $E = 2.0 \times 10^{11} Pa$. Initially the bar is at 15^oC and the temperature increases to 50^oC. If $\alpha = 20 \times 10^{-6} 1/^oC$, compute the elongation of the bar assuming that it can freely elongate. Consider the problem in one dimension.

4. Given a known temperature field, write the equations of the linear thermoelasticity in terms of the displacements in the quasi-static and isotropic case.

5. We consider a solid made of a linear homogeneous isotropic elastic material with the following parameters: $E = 10^6 Pa$, $\nu = 0.25$, $\alpha = 20 \times 10^{-6\,o}C^{-1}$.

Suppose that at a certain point of the solid the stress state is known.:

$$\boldsymbol{\sigma} = \begin{pmatrix} 12 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 6 \end{pmatrix} Pa$$

(a) Compute the strain tensor at that point.

(b) If the temperature in the solid increases $\Delta\theta = 50^o C$, compute the strain tensor at that point.

6. We consider a simply connected solid, with linear, homogeneous and isotropic elastic behaviour. In the absence of applied forces, the body is subjected to a constant temperature increase $\theta - \theta_0$ equal in all its points, being θ_0 the reference temperature.

   - Can the stress in the solid be null?
   - If the stress is null, is the strain also null?
   - Compute the displacement vector.

7. Look for the mechanical parameters for an aluminium alloy on the page www.matweb.com and write the thermoelastic isotropic linear law for a reference temperature of $20^o C$.

8. Explain what type of problem this system of equations models:

$$\begin{cases} -\mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b} \text{ in } \Omega, \\ \boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\mu + 3\lambda)\alpha(\theta - \theta_0)\mathbf{I} \text{ in } \Omega, \\ -k\Delta\theta = f \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D, \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_N, \\ \theta = \theta_D \text{ on } \Gamma_{DT}, \\ k\dfrac{\partial\theta}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_{NT}. \end{cases}$$

9. Explain what type of problem this system of equations models:

$$\begin{cases} -\mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b} \text{ in } \Omega, \\ \boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\mu + 3\lambda)\alpha(\theta - \theta_0)\mathbf{I} \text{ in } \Omega, \\ -k\Delta\theta = f \text{ in } \Omega, \\ \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D, \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_N, \\ \theta = \theta_D \text{ on } \Gamma_{DT}, \\ k\dfrac{\partial\theta}{\partial\mathbf{n}} = \alpha_c(\theta_c - \theta) \text{ on } \Gamma_{CT}, \\ k\dfrac{\partial\theta}{\partial\mathbf{n}} = \alpha_r(\theta_r^4 - \theta^4) \text{ on } \Gamma_{RT}. \end{cases}$$

10. We consider a cube of a linear isotropic elastic material whose centre is at the origin of coordinates. In the absence of applied forces, we subject the body to a constant temperature variation $\Delta\theta$. Assuming that the mechanical behaviour of the body is symmetric with respect to the coordinate planes, pose on an eighth part of the cube the mechanical problem which models the thermal expansion of the material due to the change of temperature. To do this, consider a linear thermoelastic behaviour law.

11. On the website www.matweb.com you can find the following data for an aluminium alloy:

    - Yield Strength 400MPa
    - Modulus of Elasticity 68.0GPa
    - Poisson's Ratio 0.420
    - Shear Modulus 2.39GPa
    - Density 2700 Kg/m^3
    - Hardening Modulus 15 GPa
    - Coefficient of Thermal Expansion, linear 24 μm/(mK)
    - Specific Heat Capacity 0.900 J/(gK)
    - Thermal Conductivity 210 W/(mK)

    Let

    $$\Omega = \{(x, y, z) : -1m < x,\ y,\ z < 1m\} \tag{11.43}$$

    and assume that:

    - Gravitational forces act on the solid in the direction of the z-axis.
    - The upper face, located on the plane $z = 1$, is free of forces.
    - The lower face, located on the plane $z = 0$, is clamped.
    - A pressure $p(t)$ is exerted on the lateral faces, which increases with time.

    (a) Write the set of equations that models the quasi-static deformation that the solid undergoes during 10 minutes assuming that the behaviour law is linear elastic.

    (b) The solid, which is initially at a temperature of 20^0C, is heated in a large forced convection oven with air at 250^0C, keeping the lower face of the solid controlled by means of an automaton which continuously raises its temperature to 250^0C in 10 minutes. Consider that the heat coefficient by forced convection with air inside the oven is 400 W/(m^{2}K). Also include the heat exchange by radiation with the oven walls, which are assumed to be at 250^0C and knowing that the emissivity is 0.25. Write the set of equations that allows modelling the evolution of the temperature in the solid.

    (c) Incorporate in the mechanical model of question (a) the deformations due to the temperature variation suffered by the solid.

Chapter 12

Plasticity

A material is said to have a *plastic* behaviour if, after applying a large enough load, it presents permanent deformations.

In this session we will study how to model this behaviour in metals. We will focus on isotropic plasticity and small strains. We will start with an analysis of the one-dimensional case, and after we will extend it to the three dimensional case.

12.1 1D plastic models

12.1.1 Tensile test. 1D stress-strain plastic curves

The simplest test to characterize a material is made in traction or traction-compression at constant temperature. For this, a tensile specimen of material is submitted to an axial force, which generates a uniform state of tension throughout test piece.

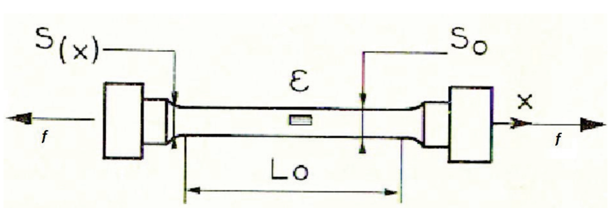

Figure 12.1: Tensile specimen. [LC94]

Let us denote by L_0 the initial length and by L the length of the deformed test piece. A point at distance x from the center of the specimen moves to $x + u$

position. Since deformation is uniform, the displacement at x is proportional to the specimen one, so:

$$\frac{x}{x+u} = \frac{L_0}{L},$$

and therefore

$$u = \frac{L - L_0}{L_0} x.$$

The longitudinal strain, named *engineering or nominal strain* is

$$\varepsilon(u) = \frac{L - L_0}{L_0},$$

which is constant on the specimen.

On the other hand, the *nominal or engineering stress* is defined as

$$\sigma = \frac{f}{S_0},$$

where f is the tensile force and S_0 the initial cross-sectional area, which is assumed to suffer small variations.

During the tensile test the specimen is elongated and the variation of engineering stress is measured as function of engineering strain to characterize the hardening of the material. Tensile tests with ductile metals produce stress-strain curves similar to that shown in Figure 12.2.

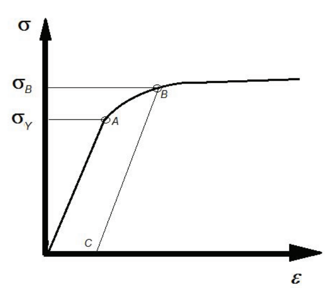

Figure 12.2: Stress-strain graphic during tensile test with a plastic material.

At the beginning, the specimen deforms almost nearly. If the bar is unloaded, the specimen returns to its original size and strains and stresses disappear. Thus, in segment OA the behaviour of the material is linear elastic. In this range the Young modulus can be easily calculated:

$$\sigma = E\varepsilon \Longrightarrow E = \frac{f}{S_0}\frac{L_0}{L - L_0}.$$

There is a limit for the stress, σ_Y, the so-called *elastic limit* or *yield stress*, from which the stress-strain curve changes dramatically. Beyond this point the behaviour of the material is irreversible: if the test piece is loaded from point A to B and then unloaded, the bar returns to an unstressed state via path BC and a permanent change in the length of the specimen is observed (see Figure 12.2). This permanent deformation, is called *plastic strain*, ε^p, and it verifies: $\varepsilon^p = \varepsilon$ when $\sigma = 0$. Note that the yield stress can be experimentally determined as the point at which the angle between the stress-strain curve and the axis is no longer constant. In practice, the curve becomes slightly curved but there is not any point with a sudden change of slope. So, it is adopted the international standard criterion of computing the elastic limit by taking elastic slope with 0.2% plastic strain ($\varepsilon^p = 0.002$) (see Figure 12.3).

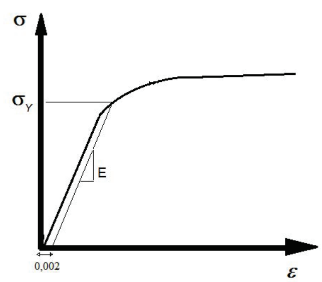

Figure 12.3: Yield stress calculus.

If the bar is reloaded again to a stress value σ_B, it follows the path CB and the behaviour is again elastic (see Figure 12.2). So, if it is unloaded before σ_B is reached, there is no plastic strain increment. When σ exceeds the new elastic limit σ_B, the curve representing the process without discharge is followed. Looking at Figure 12.2 we observe that the relationship between stress and strain is not linear and the new elastic limit is greater. This phenomenon is known as *hardening*.

We can summarize some important aspects on plastic behaviour:

- The existence of an *elastic domain*, where the behaviour of the material is purely elastic, without evolution of plastic strains.
- If the elastic limit or *yield stress* is exceeded, then the evolution of plastic strains (*plastic yielding* or *plastic flow*) takes place.
- An evolution of the yield stress with the plastic evolution is observed, which is known as *hardening*.

12.1.2 Elastic-plastic decomposition of strain

Looking at Figure 12.4, the total strain can be decomposed as the sum of the *elastic strain* ε^e and the *plastic strain* ε^p (see Figure 12.4):

$$\varepsilon = \varepsilon^e + \varepsilon^p,$$

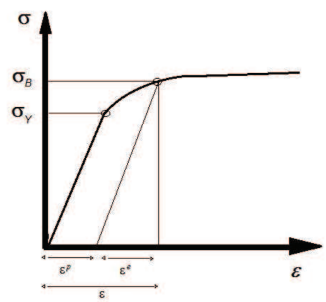

Figure 12.4: Strain decomposition.

The elastic component is given by

$$\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p).$$

Various formulas are given in the literature to fit the relation between stresses and plastic strains. The most common is the so-called *Ramberg-Osgood law*:

$$\varepsilon^p = \langle \frac{\sigma - \sigma_Y}{H} \rangle^M, \tag{12.1}$$

where H is called *hardening* or *plastic modulus*, M is the *hardening exponent* and $\langle x \rangle = \max\{x, 0\}$.

Material	T (^oC)	σ_Y (MPa)	H (Mpa)	M
Steel 35 NCD 16	20	1200	3340	3,1
Alloy IN 100	20	650	655	5,6

Table 12.1: Parameters of the Ramberg-Osgood law for some materials

Elastic-plastic solid with linear hardening Let us consider $M = 1$ in (12.1):

$$\varepsilon^p = \frac{\langle \sigma - \sigma_Y \rangle}{H}. \tag{12.2}$$

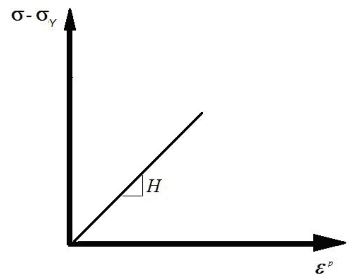

Figure 12.5: Linear hardening.

Then, the behaviour of the material during loading can be modelled as

$$\begin{cases} \varepsilon = \varepsilon^e = \dfrac{\sigma}{E}, & \text{if } \sigma < \sigma_Y, \\ \varepsilon = \varepsilon^e + \varepsilon^p = \dfrac{\sigma}{E} + \dfrac{(\sigma - \sigma_Y)}{H}, & \text{if } \sigma \geq \sigma_Y. \end{cases}$$

From this, we can obtain the expression for σ when $\sigma \geq \sigma_Y$ as

$$\sigma = \frac{EH}{E+H}\left(\varepsilon + \frac{\sigma_Y}{H}\right).$$

Notice that in this case, stress-strain graph corresponds to two straight lines (see Figure 12.6), the first with slope E and the second one with slope

$$H_{eq} = \frac{EH}{E+H}$$

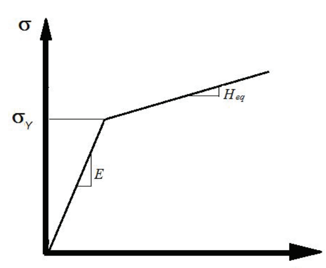

Figure 12.6: Elastic-plastic behaviour with linear hardening.

Furthermore, the actual yield stress can be computed from the plastic strain from (12.2) as follows:

$$\sigma = \sigma_Y + H\varepsilon^p.$$

12.1.3 Loading/unloading modelling

The main difficulty in plastic modelling is in the unloading processes, where there is not a univocal relation between stresses and strains, since the same stress can correspond to different strains in a loading/unloading cycle. So, it is necessary to take into account the history of the process. To overcome this difficulty we are going to introduce the strain rate in the plastic model.

Yield criterion Let us consider the tensile test during loading and let us assume that the behaviour in compression is identical to that in tension. We define the *yield function*:

$$F(\sigma) = |\sigma| - \sigma_Y. \tag{12.3}$$

So, the *set of plastically admissible stresses* can be defined as

$$K = \{\sigma; F(\sigma) = |\sigma| - \sigma_Y \leq 0\}.$$

In Figure 12.7 the corresponding stress-strain curve is represented. Materials with this behaviour are named *elastic-perfectly plastic* materials. Let us analyze when plastic strain increment may occur:

- The elastic domain coincides with the interior of K. There $F(\sigma) = |\sigma| - \sigma_Y < 0$ and there is not plastic strain increment, i.e. $\dot{\varepsilon}^p = 0$.
- On the boundary of the elastic domain, the *yield surface* (two points in the 1D case), $F(\sigma) = |\sigma| - \sigma_Y = 0$, either elastic unloading $\dot{\varepsilon}^p = 0$ or plastic loading $\dot{\varepsilon}^p \neq 0$ can take place.

Flow rule Notice that:

- $\dot{\varepsilon}^p \geq 0$ (stretching) when $\sigma > 0$ (tension).
- $\dot{\varepsilon}^p \leq 0$ (compressive) when $\sigma < 0$ (compression).

Let us consider a scalar $\dot{\lambda} \geq 0$, named the *plastic multiplier*. The *flow rule* can be established as:

$$\dot{\varepsilon}^p = \dot{\lambda} \geq 0, \text{ if } \sigma = \sigma_Y > 0, \tag{12.4}$$

$$\dot{\varepsilon}^p = -\dot{\lambda} \leq 0, \text{ if } \sigma = -\sigma_Y < 0. \tag{12.5}$$

(12.6)

This law can be summarized as

$$\dot{\varepsilon}^p = \dot{\lambda}\text{sign}(\sigma), \tag{12.7}$$

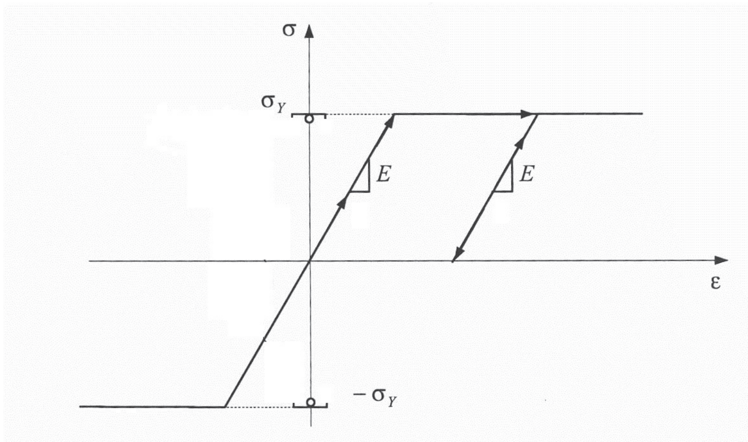

Figure 12.7: Stress-strain curve for an elastic-perfectly plastic material.

where sign(·) is the signum function. This law can be generalized as :

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial F}{\partial \sigma}.$$

We can summarize the yield criterion and the flow rule in this *Kuhn-Tucker condition*, also known as *loading/unloading condition*:

$$\dot{\lambda} \geq 0,\ F(\sigma) \leq 0,\ \dot{\lambda}F(\sigma) = 0.$$

Hardening law We have seen that in most plastic materials, the evolution of the plastic strain implies also the evolution of the yield stress. This phenomenon is known as *hardening*. It can be incorporated into the model by assuming in (12.3) that the yield stress is a function of the *accumulated plastic strain*, $\bar{\varepsilon}^p$:

$$\sigma_Y = Y(\bar{\varepsilon}^p).$$

To define the accumulated plastic strain, let us consider two instants t_i and t_f during traction. The increment of plastic deformation can be computed as:

$$|\varepsilon^p(t_f) - \varepsilon^p(t_i)| = \varepsilon^p(t_f) - \varepsilon^p(t_i) = \int_{t_i}^{t_f} \dot{\varepsilon}^p dt = \int_{t_i}^{t_f} |\dot{\varepsilon}^p|\, dt.$$

Analogously, during compression, the increment of plastic deformation is:

$$|\varepsilon^p(t_f) - \varepsilon^p(t_i)| = -(\varepsilon^p(t_f) - \varepsilon^p(t_i)) = \int_{t_i}^{t_f} -\dot{\varepsilon}^p dt = \int_{t_i}^{t_f} |\dot{\varepsilon}^p|\, dt.$$

So, we define the *accumulated plastic strain* as:

$$\bar{\varepsilon}^p(t) = \int_0^t |\dot{\varepsilon}^p(\tau)|\, d\tau, \tag{12.8}$$

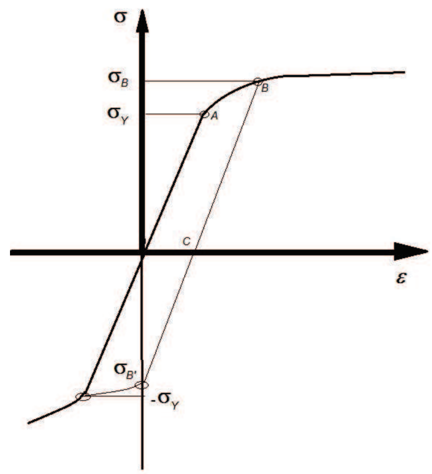

Figure 12.8: Traction-compression test for a plastic material with hardening.

and in this way both traction and compression contribute to $\bar{\varepsilon}^p$.

When considering hardening, the yield function (12.3) takes the form

$$F(\sigma, \bar{\varepsilon}^p) = |\sigma| - Y(\bar{\varepsilon}^p), \tag{12.9}$$

and the elastic condition is verified when

$$|\sigma| < Y(\bar{\varepsilon}^p).$$

In order to define the hardening function $Y(\bar{\varepsilon}^p)$ we can consider:

- A perfectly plastic material:

$$Y(\bar{\varepsilon}^p) = \sigma_Y, \tag{12.10}$$

- A plastic material with linear hardening:

$$Y(\bar{\varepsilon}^p) = \sigma_Y + H\bar{\varepsilon}^p, \tag{12.11}$$

- A plastic material with non-linear hardening:

$$Y(\bar{\varepsilon}^p) = \sigma_Y + H(\bar{\varepsilon}^p)^{1/M}. \tag{12.12}$$

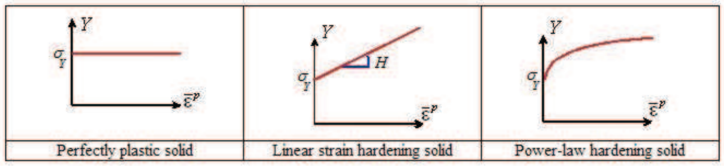

Figure 12.9: Some hardening functions [Bow10]

Notice that from definition (12.8) it follows that

$$\dot{\bar{\varepsilon}}^p(t) = |\dot{\varepsilon}^p(t)|.$$

Therefore, from the flow rule $\dot{\varepsilon}^p = \dot{\lambda}\mathrm{sign}(\sigma)$ we deduce

$$\dot{\lambda} = \dot{\bar{\varepsilon}}^p.$$

Summary of 1D elastic-plastic model with isotropic hardening

- Behaviour law: $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$.
- Elastic law: $\sigma = E\varepsilon^e$.
- Plastic flow rule: $\dot{\varepsilon}^p = \dot{\lambda}\mathrm{sign}(\sigma)$.
- Yield criterion: $F(\sigma, \bar{\varepsilon}^p) = |\sigma| - Y(\bar{\varepsilon}^p) \leq 0$.
- Isotropic hardening law: $\dot{\lambda} = \dot{\bar{\varepsilon}}^p$.
- Loading/unloading condition: $\dot{\lambda} \geq 0,\ F(\sigma, \bar{\varepsilon}^p) \leq 0,\ \dot{\lambda}F(\sigma, \bar{\varepsilon}^p) = 0$.

Determination of the plastic multiplier for an elastic-plastic model with isotropic linear hardening. Let us consider the plastic behaviour shown in Figure 12.10, corresponding with linear hardening.

In the previous model, the plastic multiplier is indeterminate during plastic yielding, since we only know that it vanishes during elastic behaviour and it takes a non-negative value during plastic flow. So, we need to add another equation to the model, which is known as *consistency condition*:

$$\dot{\lambda}\dot{F}(\sigma, \bar{\varepsilon}^p) = 0,$$

which implies that the rate of F is null when plastic flow occurs ($\dot{\lambda} > 0$), whereas during elastic strain $\dot{\lambda} = 0$ and $\dot{F}(\sigma, \bar{\varepsilon}^p)$ may assume any value.

Let us try to obtain the expression of $\dot{\lambda}$, which is non null during linear plastic yielding ($\dot{F}(\sigma, \bar{\varepsilon}^p) = 0$). By taking the time derivative of the yield function one obtains

$$\dot{F}(\sigma, \bar{\varepsilon}^p) = \frac{\partial F}{\partial \sigma}\dot{\sigma} + \frac{\partial F}{\partial \bar{\varepsilon}^p}\dot{\bar{\varepsilon}}^p = 0.$$

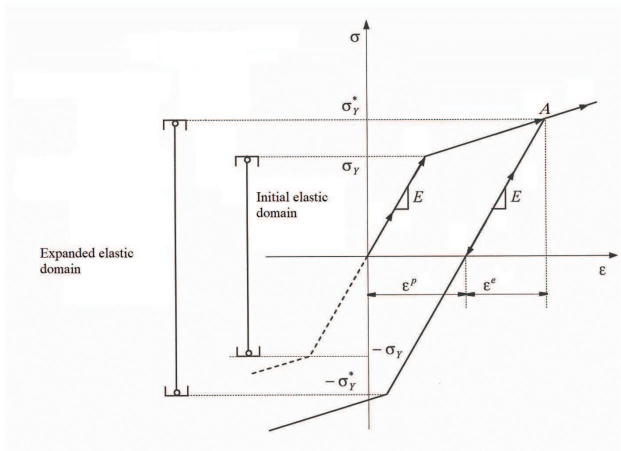

Figure 12.10: Stress-strain curve for an elastic-plastic material with isotropic linear hardening.

In the linear yielding case,

$$F(\sigma, \bar{\varepsilon}^p) = |\sigma| - (\sigma_Y + H\bar{\varepsilon}^p),$$

so

$$\dot{F}(\sigma, \bar{\varepsilon}^p) = \frac{\partial F}{\partial \sigma}\dot{\sigma} - H\dot{\bar{\varepsilon}}^p = 0.$$

Taking into account the flow rule and the behavior law it results

$$\dot{F}(\sigma, \bar{\varepsilon}^p) = \text{sign}(\sigma)E(\dot{\varepsilon} - \dot{\varepsilon}^p) - H\dot{\lambda} = \text{sign}(\sigma)E\dot{\varepsilon} - (E + H)\dot{\lambda} = 0.$$

Therefore,

$$\dot{\lambda} = \text{sign}(\sigma)\frac{E}{E + H}\dot{\varepsilon},$$

and using again the flow rule, we can obtain the following expression for the plastic strain rate:

$$\dot{\varepsilon}^p = \frac{E}{E + H}\dot{\varepsilon}$$

So, from the behaviour law, we deduce the following relation between stress and strain rate:

$$\dot{\sigma} = \frac{EH}{E + H}\dot{\varepsilon} = H_{eq}\dot{\varepsilon},$$

where H_{eq} is the *elastic-plastic tangent modulus.*

Bauschinger effect. Until now we have assumed that material behaviour in compression is the same as in tension, which is known as *isotropic hardening*. Nevertheless, the yield stress in tension and compression are different. If the yield stress in traction is σ_Y, in compression is greater than $-\sigma_Y$ (see Figure 12.8). This phenomenon is known as the *Bauschinger effect* or *kinematic hardening*.

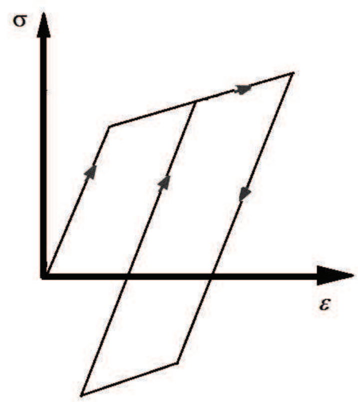

Figure 12.11: Kinematic hardening

To model the Bauschinger effect, we consider a function α, and we rewrite the flow condition as:

$$|\sigma - \alpha| = \sigma_Y.$$

To complete the model we must add another equation, showing that the variation of α depends linearly on the plastic strain rate:

$$\dot{\alpha} = c\dot{\varepsilon}^p$$

where c, the *kinematic hardening modulus*, is an experimental constant.

So, we can summarize the **elastic-plastic model with kinematic hardening** as follows:

- Behaviour law: $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$.
- Elastic law: $\sigma = E\varepsilon^e$.
- Plastic flow rule: $\dot{\varepsilon}^p = \dot{\lambda}\text{sign}(\sigma - \alpha)$.
- Yield criterion: $F(\sigma, \alpha) = |\sigma - \alpha| - \sigma_Y \leq 0$.
- Kinematic hardening law: $\dot{\alpha} = c\dot{\varepsilon}^p$.
- Loading/unloading condition: $\dot{\lambda} \geq 0,\ F(\sigma, \alpha) \leq 0,\ \dot{\lambda}F(\sigma, \alpha) = 0$.

Finally, we can summarize the **elastic-plastic model with isotropic and kinematic hardening** as follows:

- Behaviour law: $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$.
- Elastic law: $\sigma = E\varepsilon^e$.
- Plastic flow rule: $\dot{\varepsilon}^p = \dot{\lambda}\text{sign}(\sigma - \alpha)$.
- Yield criterion: $F(\sigma, \alpha, \bar{\varepsilon}^p) = |\sigma - \alpha| - Y(\bar{\varepsilon}^p) \leq 0$.
- Isotropic and kinematic hardening law: $\dot{\lambda} = \dot{\bar{\varepsilon}}^p$, $\dot{\alpha} = c\dot{\varepsilon}^p$.
- Loading/unloading condition: $\dot{\lambda} \geq 0,\ F(\sigma, \alpha, \bar{\varepsilon}^p) \leq 0,\ \dot{\lambda}F(\sigma, \alpha, \bar{\varepsilon}^p) = 0$.

12.1.4 Elastic-plastic rheological models

Rheological models allow to schematize mathematical models of certain physical phenomena. For this, we use different elements, which are combined in series or in parallel.
An elastic material is represented by a spring with constant E. The stress of the spring is related to its strain by the equation

$$\sigma = E\varepsilon.$$

The yield limit is represented by a sliding frictional element (see Figure 12.12). It represents a body resting on a surface, which can not move until the effort has exceeded a certain frictional limit.

Figure 12.12: Sliding frictional element.

By mechanical-electrical analogy, we can connect these devices in series or in parallel. If they are connected in series, their stresses are the same and the total strain is the sum of each component. For a system connected in parallel, its strain coincides and the total stress is the sum of stresses of each component (see Figure 12.13).

So, in Figure 12.13 we represent an elastic-perfectly plastic material. Indeed, as the stress increases all the strain is absorbed by the spring until the stress σ_Y is reached. After this threshold all the strain will be absorbed by the friction device, without an increase in stress. If the material is unloaded, the strain suffered by the friction device is no longer recovered, only the elastic strain due to the spring.

To represent an elastic-plastic material with linear isotropic hardening we use an spring and a frictional element in parallel, connected with another spring

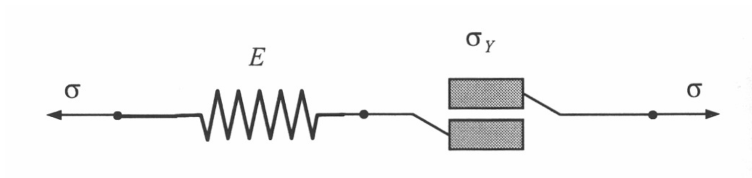

Figure 12.13: Rheological model for an elastic-perfectly plastic material [Cha09]

in series (Figure 12.14). Under σ_Y, there is only stress due to the spring with constant E. Once this value is exceeded, stress is absorbed by the spring with constant H. During plastic flow the spring is equivalent to that of constant H_{eq}:

$$\varepsilon = \frac{\sigma}{H_{eq}} = \frac{\sigma}{H} + \frac{\sigma}{E} = \frac{H+E}{HE}\sigma \Rightarrow H_{eq} = \frac{HE}{H+E}.$$

During unloading, the strain of frictional element remains, and therefore, also that of the spring with constant H. So, only the stress of the spring with constant E is recovered. Furthermore, since the other spring is deformed, its stress is the new yield stress: $\sigma_Y + H\varepsilon^p$.

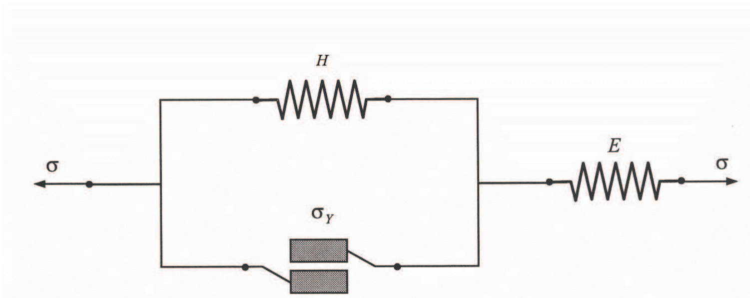

Figure 12.14: Rheological model for a elastic-plastic material with isotropic linear hardening.

12.2 3D plastic models

When modeling plastic behavior in metals, it is important to take into account two experimental observations:

- The hydrostatic part of the stress tensor does not have effects on plastic deformations.
- Plastic deformations do not induce a variation of volume.

12.2.1 Additive decomposition of the strain tensor

Following the one-dimensional case, we assume that the total strain is the sum of a small recoverable elastic part $\boldsymbol{\varepsilon}^e$ and a large, irreversible plastic component $\boldsymbol{\varepsilon}^p$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p,$$

where the relation between $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\sigma}$ is given by the Hooke's law:

$$\boldsymbol{\varepsilon}^e = \mathbf{C}^{-1}\boldsymbol{\sigma}.$$

To take into account unloaded processes this law can be formulated in rate form:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p = \mathbf{C}^{-1}\dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\varepsilon}}^p,$$

where we have assumed that $\mathbf{C}^{-1}$, and therefore also the elastic parameters of the material, are time independent.

12.2.2 Yield criterion

We recall that during the tensile test, there is an elastic limit or yield stress such that under this value the behaviour of the specimen is elastic, but, if the limit is exceeded, plastic flow takes place. To model this behaviour, in section 12.1.3 yield function and the corresponding yield criterion was introduced. In order to extend these concepts to the three-dimensional case, we introduce a scalar *yield function* $F(\boldsymbol{\sigma})$ such that:

- $\{\boldsymbol{\sigma}; F(\boldsymbol{\sigma}) < 0\}$ corresponds to the interior of the *elastic domain*, for which plastic yielding doesn't take place.

- $\{\boldsymbol{\sigma}; F(\boldsymbol{\sigma}) \leq 0\}$ is the set of *plastically admissible stresses.*

- $\{\boldsymbol{\sigma}; F(\boldsymbol{\sigma}) = 0\}$ is the boundary of the elastic domain, where plastic yielding may occur, and is named *yield surface.*

In metals, it has been experimentally justified that plastic strain does not depend on the hydrostatic component of the stress. Furthermore, if we assume that the material is *isotropic*, the yield criterion is defined in terms of an isotropic yield function. Since any isotropic scalar function of a symmetric tensor can be expressed as a function of the principal values of its argument, the yield function could be characterized with the principal values J_2 and J_3 of $\boldsymbol{\sigma}_d$, since $J_1 = \mathrm{tr}(\boldsymbol{\sigma}_d) = 0$. Furthermore, if we do not take into account the Bauschinger effect, the elastic limit should not change when changing the sign of the applied stresses. Since J_3 changes when the stresses are reversed, the yield function F should be an even function of J_3.

Some of the most common yield criteria used in engineering practice are the *Tresca* and *von Mises* criteria (see [Hil83]).

- **Von Mises criterium**: Von Mises suggested that yielding occurs when J_2 reaches a critical value, which is known as J_2 *theory of plasticity*. In this case the yield function is defined as

$$F(\boldsymbol{\sigma}) = \sigma_{vm} - \sigma_Y, \tag{12.13}$$

where $\sigma_{vm} = \sqrt{3J_2} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d}$, is the *von Mises effective* or *equivalent stress*. This name is due to the three-dimensional yield criterion

$$\sigma_{vm} = \sigma_Y$$

in the case of a uniaxial stress test is equivalent to the one-dimensional yield criterion $|\sigma| = \sigma_Y$. Analogously, we define the *von Mises equivalent strain* as $\varepsilon^p_{vm} = \sqrt{\frac{2}{3}\boldsymbol{\varepsilon}^p : \boldsymbol{\varepsilon}^p}$.

Since the yield function is isotropic, it can be graphically represented as a surface in the space of principal stresses. For that purpose, we can rewrite the yield criterion in function of principal stresses as

$$\frac{1}{\sqrt{2}}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right)^{1/2} = \sigma_Y. \tag{12.14}$$

This can be graphically represented in the space of principal stresses as an infinite circular cylinder with axis (1,1,1) and radius $\sqrt{2/3}\,\sigma_Y$ (see Figure 12.15). The elastic domain corresponds to the interior of the cylinder.

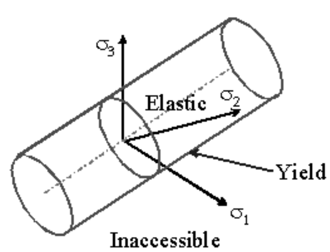

Figure 12.15: Von Mises yield surface [Bow10]

- **Tresca criterium**: This criterium assumes that plastic yielding occurs when the maximum shear stress reaches the critical value $\sigma_Y/2$. The yield function can be written in terms of the principal stresses as

$$F(\boldsymbol{\sigma}) = \max\{|\sigma_i - \sigma_j| - \sigma_Y,\ i,\ j = 1, 2, 3\}.$$

The graphical representation in the space of principal stresses is an infinite hexagonal prism with axis (1,1,1). Although the Tresca yield function

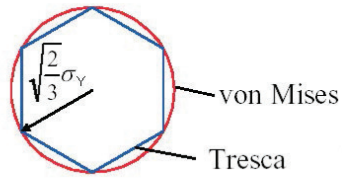

Figure 12.16: Transversal section of Tresca and von Mises yield surfaces.

seems to be very different from the von Mises one, this prism is inscribed in the von Mises cylinder (see Figure 12.16) and the difference between both criteria is small (see [SPO08]).

The Tresca yield function can also be expressed in terms of J_2 and J_3 using the Lode angle (see [Mor97; SPO08]).

To take into account the *anisotropy* of certain metals, Hill [Hil83] proposed a yield criterium for orthotropic materials which, in the isotropic case, coincides with the von Mises criterium. The proposed yield function is known as **Hill criterium**:

$$F(\boldsymbol{\sigma}) = J(\sigma_{11}-\sigma_{22})^2+G(\sigma_{22}-\sigma_{33})^2+H(\sigma_{33}-\sigma_{11})^2+2L\sigma_{12}^2+2M\sigma_{23}^2+2N\sigma_{13}^2-1,$$

where J, G, H, $L\,M\,N$ are scalar parameters. These constants can be written in terms of the traction and shear limits in the orthotropic directions (see [Hil83; LC94]).

12.2.3 Hardening law

Following the reasoning for the one-dimensional case, we model de isotropic hardening making the yield surface increasing in size, without modifying its shape (see Figure 12.17).

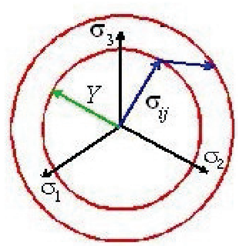

Figure 12.17: Isotropic hardening scheme.

The **von Mises yield function with isotropic hardening** can be written as:

$$F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sigma_{vm} - Y(\bar{\varepsilon}^p), \tag{12.15}$$

where

$$\bar{\varepsilon}^p = \int_0^t \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^p(\tau) : \dot{\boldsymbol{\varepsilon}}^p(\tau)}\mathrm{d}\tau,$$

is the *accumulated plastic strain* and the hardening function $Y(\bar{\varepsilon}^p)$ were defined in (12.10)-(12.12). Notice that the corresponding yield surface is a cylinder which radius increments with $\bar{\varepsilon}^p$, so the elastic region is increasing with plastic flow.

12.2.4 Kinematic hardening

This behaviour can be modelled assuming that the yield surface moves without modifying its shape (see Figure 12.18).

The **von Mises yield function with kinematic hardening**, is defined as follows:

$$F(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \sqrt{\frac{3}{2}\left(\boldsymbol{\sigma}_d - \boldsymbol{\alpha}\right) : \left(\boldsymbol{\sigma}_d - \boldsymbol{\alpha}\right)} - \sigma_Y.$$

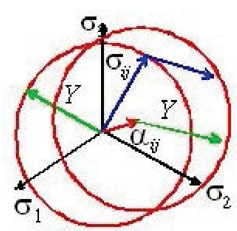

Figure 12.18: Kinematic hardening scheme.

Furthermore, we assume that the movement is in the direction of the plastic flow, and the relation between $\boldsymbol{\alpha}$ and the plastic strain is linear, so :

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}c\dot{\boldsymbol{\varepsilon}}^p,$$

where c is named *linear kinematic hardening modulus.*

12.2.5 Flow rule

As we have announced, the flow rule (12.7) can be generalized in terms of a plastic potential ϕ as

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial \phi}{\partial \boldsymbol{\sigma}}. \tag{12.16}$$

Many plasticity models assume that the potential coincides with the yield function. Such models are known as *associate plasticity* models and this is the case study here. Therefore, the flow rule reduces to

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\sigma}}. \tag{12.17}$$

Notice to this rule requires F to be differentiable. When this happens, the flow rule implies that plastic strain rate has the direction of the gradient of the yield surface. Nevertheless, this hypothesis is very restrictive and many yield functions, for example, the Tresca one, does not verify that. In those cases, the plastic flow rule is generalized using the concept of subdifferential (see [SPO08]).

12.2.6 Loading/unloading condition

It is the same as in the one-dimensional case. Then, if the hardening is isotropic, the condition is:

$$\dot{\lambda} \geq 0,\ F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) \leq 0,\ \dot{\lambda}F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = 0.$$

12.2.7 Example: Elastic-plastic behaviour law with von Mises yield criterium and isotropic hardening

We conclude by summarizing the complete elastic-plastic stress-strain relations for an isotropic solid with von-Mises yield surface and isotropic hardening. Then, the yield function is given by (12.15):

$$F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sigma_{vm} - Y(\bar{\varepsilon}^p).$$

- First, let us compute the derivative of the von Mises yield function. For that purpose, let us define $G(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}_d|^2 = \boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d$. We can rewrite (12.15) as

  $$F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sqrt{\frac{3}{2}\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d} - Y(\bar{\varepsilon}^p) = \sqrt{\frac{3}{2}}G(\boldsymbol{\sigma})^{1/2} - Y(\bar{\varepsilon}^p).$$

  It is verified that

  $$DG(\boldsymbol{\sigma}) = 2\boldsymbol{\sigma}_d,$$

  since $G(\boldsymbol{\sigma} + \boldsymbol{\tau}) - G(\boldsymbol{\sigma}) = 2(\boldsymbol{\sigma}_d : \boldsymbol{\tau}_d) + o(\boldsymbol{\tau}_d)$. Then,

  $$\frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sqrt{\frac{3}{2}}\frac{\boldsymbol{\sigma}_d}{|\boldsymbol{\sigma}_d|}. \tag{12.18}$$

  Using the definition of the von Mises effective stress, this derivative can be written as

  $$\frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \frac{3}{2}\frac{\boldsymbol{\sigma}_d}{\sigma_{vm}}. \tag{12.19}$$

- Notice that on the yield surface, $F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = 0$, and then, $\sigma_{vm} = Y(\bar{\varepsilon}^p)$, so

$$\frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sqrt{\frac{3}{2}} \frac{\boldsymbol{\sigma}_d}{|\boldsymbol{\sigma}_d|} = \frac{3}{2} \frac{\boldsymbol{\sigma}_d}{\sigma_{vm}} = \frac{3}{2} \frac{\boldsymbol{\sigma}_d}{Y(\bar{\varepsilon}^p)}. \qquad (12.20)$$

- Using (12.18) or (12.19), the flow rule (12.17) can be written as

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \sqrt{\frac{3}{2}} \frac{\boldsymbol{\sigma}_d}{|\boldsymbol{\sigma}_d|} = \dot{\lambda} \frac{3}{2} \frac{\boldsymbol{\sigma}_d}{\sigma_{vm}}.$$

- Then, by definition of the accumulated plastic strain,

$$\dot{\bar{\varepsilon}}^p = \sqrt{\frac{2}{3} \dot{\varepsilon}^p : \dot{\varepsilon}^p} = \dot{\lambda}.$$

- Notice that, from (12.20), on the yield surface we have

$$\dot{\varepsilon}^p = \dot{\bar{\varepsilon}}^p \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \dot{\bar{\varepsilon}}^p \frac{3}{2} \frac{\boldsymbol{\sigma}_d}{Y(\bar{\varepsilon}^p)}.$$

- Then, the behavior law can be written as

$$\dot{\boldsymbol{\varepsilon}} = \begin{cases} \dfrac{1+\nu}{E} \dot{\boldsymbol{\sigma}} - \dfrac{\nu}{E} \mathrm{tr} \dot{\boldsymbol{\sigma}}, & \text{if } \sigma_{vm} < Y(\bar{\varepsilon}^p), \\ \dfrac{1+\nu}{E} \dot{\boldsymbol{\sigma}} - \dfrac{\nu}{E} \mathrm{tr} \dot{\boldsymbol{\sigma}} + \dot{\bar{\varepsilon}}^p \dfrac{3}{2} \dfrac{\boldsymbol{\sigma}_d}{Y(\bar{\varepsilon}^p)}, & \text{if } \sigma_{vm} = Y(\bar{\varepsilon}^p). \end{cases}$$

- In order to obtain an expression for the plastic multiplier, we must use the *consistency condition* (see [LC94]): $\dot{\lambda} \dot{F} = 0$.

  Lets us assume that plastic flow occurs: $\dot{\lambda} > 0$ and $F(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = 0$. By the consistency condition, $\dot{F}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = 0$. Following a similar reason than in the 1D case, it can be proved that:

$$\dot{\bar{\varepsilon}}^p = \frac{\dot{\sigma}_{vm}}{Y'(\bar{\varepsilon}^p)}. \qquad (12.21)$$

  Indeed:

$$\dot{F}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial \bar{\varepsilon}^p} \dot{\bar{\varepsilon}}^p = 0.$$

  Using (12.19), and the definition of the yield function:

$$\dot{F}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \frac{3}{2} \frac{\boldsymbol{\sigma}_d}{\sigma_{vm}} : \dot{\boldsymbol{\sigma}} - Y'(\bar{\varepsilon}^p) \dot{\bar{\varepsilon}}^p = 0.$$

  Since

$$\frac{3}{2} \frac{\boldsymbol{\sigma}_d}{\sigma_{vm}} : \dot{\boldsymbol{\sigma}} = \dot{\sigma}_{vm},$$

we conclude

$$\dot{F}(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \dot{\sigma}_{vm} - Y'(\bar{\varepsilon}^p)\dot{\bar{\varepsilon}}^p = 0.$$

Therefore, it results (12.21).

Notice that, since $Y'(\bar{\varepsilon}^p) > 0$ and $\dot{\lambda} = \dot{\bar{\varepsilon}}^p > 0$, it must be $\dot{\sigma}_{vm} > 0$ and we can express

$$\dot{\bar{\varepsilon}}^p = \frac{< \dot{\sigma}_{vm} >}{Y'(\bar{\varepsilon}^p)}.$$

- Summing up, we obtain the following stress-strain rate relation:

$$\dot{\boldsymbol{\varepsilon}} = \begin{cases} \dfrac{1+\nu}{E}\dot{\boldsymbol{\sigma}} - \dfrac{\nu}{E}\mathrm{tr}\dot{\boldsymbol{\sigma}}, & \text{if } \sigma_{vm} < Y(\bar{\varepsilon}^p), \\ \dfrac{1+\nu}{E}\dot{\boldsymbol{\sigma}} - \dfrac{\nu}{E}\mathrm{tr}\dot{\boldsymbol{\sigma}} + \dfrac{3}{2}\dfrac{< \dot{\sigma}_{vm} >}{Y'(\bar{\varepsilon}^p)}\dfrac{\boldsymbol{\sigma}_d}{\sigma_{vm}}, & \text{if } \sigma_{vm} = Y(\bar{\varepsilon}^p). \end{cases}$$

12.3 Exercises

1. Prove that in the space of principal stresses the von Mises criterion can be expressed as

$$\frac{1}{\sqrt{2}}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)^{1/2}=\sigma_Y. \qquad (12.22)$$

2. Prove that the maximum tangential stress is given by

$$\boldsymbol{\sigma}_{tmax}=\frac{\sigma_I-\sigma_{III}}{2},$$

where $\sigma_I > \sigma_{II} > \sigma_{III}$ are the principal stresses ordered.
Hint: Given a stress $\boldsymbol{\sigma}$, from its principal stresses maximise the function $g(n_1,n_2,n_3)=|\boldsymbol{\sigma}_t|^2=|\boldsymbol{\sigma}\mathbf{n}|^2-|\sigma_n\mathbf{n}|^2=\sigma_1^2n_1^2+\sigma_2^2n_2^2+\sigma_3^2n_3^2-(\sigma_1 n_1^2+\sigma_2 n_2^2+\sigma_3 n_3^2)^2$ subject to the constraint $n_1^2+n_2^2+n_3^2=1$.

3. We consider a linear isotropic homogeneous linear material with the following parameters: $E=10^6Pa$, $\nu=0.25$. We assume that at certain point of the solid we know the stress tensor:

$$\boldsymbol{\sigma}=\begin{pmatrix}12 & 0 & 4\\ 0 & 0 & 0\\ 4 & 0 & 6\end{pmatrix}Pa$$

Compute the principal stresses and strains and the maximum tangential stress.

4. In a simple tensile test with a 100 mm specimen, the values of the following table were obtained: Estimate the Young modulus and yield stress.

σ (Pa)	$\varepsilon(\times 10^{-2})$
6,67	0,667
13,3	1,33
20	2
24	3

5. In a simple tensile test the following values were obtained:

   - Young modulus 200 GPa
   - Poisson's coefficient 0,29
   - Yield stress 200 MPa
   - Yield strain 10^{-3}
   - Tensile strength 600 MPa
   - Tensile stress 0.15

Obtain the 1D behaviour law assuming a linear isotropic hardening.

6. Using the consistency condition proof that in the unidimensional case a perfectly plastic material with Bauschinger effect verifies

$$\dot{\varepsilon}^p = \frac{Ec}{E+c}\dot{\varepsilon},$$

where E is the Young modulus and c the kinematic hardening modulus.

7. Using the consistency condition proof that in the unidimensional case an elastic-plastic material with linear hardening and Bauschinger effect verifies

$$\dot{\sigma} = \frac{E(c+H)}{E+c+H}\dot{\varepsilon},$$

where E is the Young modulus, c is the kinematic hardening modulus and H is the plastic modulus.

8. The mechanical parameters for a certain alloy are:

   - Young modulus 200 GPa
   - Poisson's coefficient 0,29
   - Mass density 2800 Kg/m^3
   - Yield stress 200 MPa
   - Plastic modulus 605 MPa

   Write the 3D elastic-plastic behaviour law assuming the von-Mises criterium and linear isotropic hardening.

9. Deduce the expression of the Tresca's criterium in the uniaxial tensile test. Compare the result with the one dimensional yield criterium.

10. A body is deformed under the condition of plane strains in the xy-plane. Assume that the behaviour of the body is governed by the von Mises criterion in the perfectly plastic case. Prove that, if in the initial state the stresses are zero, during the the process they verify that $\sigma_{13} = \sigma_{23} = 0$.

11. Write the set of equations to model the 3D quasi-static deformation of a solid assuming an elastic-plastic behaviour law, a von Mises criterion and isotropic linear hardening. Assume that the force of gravity acts on the body, the displacement $\mathbf{u_D}$ is known on a part of the boundary Γ_D and a pressure $p(x,t)$ acts on the rest of the boundary.

12. In the previous exercise assume that the body is subjected to a known temperature field $\theta(x,t)$ and introduce the deformations due to temperature variation into the model.

Chapter 13

Contact boundary conditions

Contact problems arise in a multitude of industrial problems: metal castings, cars design, bearing design, etc. The objective of this chapter is to study some contact conditions between solids, with and without friction.

Figure 13.1: Car impact simulation. [Wri06]

Figure 13.2: Butt curl deformation. Imaged provided by ALCOA-INESPAL S.A.

Let us consider $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with Γ split into three parts:

- Γ_N denotes the part of the boundary where we know the density of applied forces $\mathbf{g}$:

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_N,$$

  where $\mathbf{n}$ is the unit exterior vector normal to $\partial\Omega$.

- On Γ_D the displacement is known:

$$\mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D.$$

- On the remaining part of the boundary, denoted by Γ_C, we consider a contact condition with another solid. This contact can be *unilateral* –if the two bodies can separate– or *bilateral* –if the two bodies are in permanent contact. Furthermore, contact can be *without friction* –when tangential displacement is free– or *with friction*.

We denote by u_n and $\mathbf{u}_t$ the normal and tangential components of the displacement vector $\mathbf{u}$, respectively:

$$u_n = \mathbf{u} \cdot \mathbf{n}, \quad \mathbf{u}_t = \mathbf{u} - u_n \mathbf{n}. \tag{13.1}$$

Analogously, we define σ_n and $\boldsymbol{\sigma}_t$ as the normal and tangential components of the stress vector $\boldsymbol{\sigma}\mathbf{n}$:

$$\sigma_n = \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n}, \ \ \boldsymbol{\sigma}_t = \boldsymbol{\sigma}\mathbf{n} - \sigma_n \mathbf{n}. \tag{13.2}$$

13.1 Unilateral contact

13.1.1 Contact between a solid and a rigid base

Let us suppose that Ω rests on a rigid base. A priori, the effective contact zone is not known but it is part of Γ_C (see Figure 13.3). We denote by g the initial distance between the body and the rigid base, also known as gap.

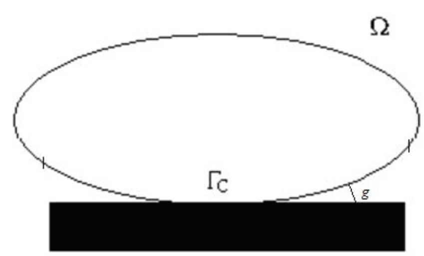

Figure 13.3: Contact with a rigid base.

Since the base is a rigid solid we have no penetration condition

$$u_n \leq g.$$

On the points of effective contact ($u_n = g$), by the principle of action-reaction, the base exerts a pressure on the body in the normal direction, so $\sigma_n \leq 0$. On the points where there is no contact, the movement is free: $\sigma_n = 0$. So, we can summarize the contact condition as

$$u_n \leq g, \ \sigma_n \leq 0, \ \sigma_n(u_n - g) = 0.$$

This condition is known as *Signorini contact condition* (see [KO98]).

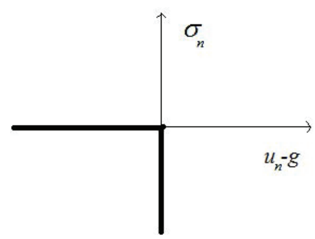

Figure 13.4: Normal stress versus gap in Signorini condition.

13.1.2 Contact between two deformable solids

Let Ω^j, $j = 1, 2$, be two solids and Γ^j their boundaries (see Figure 13.5). Let Γ^j_C denote the region of contact of Ω^j. We assume that both solids are initially in contact along $\Gamma_C = \Gamma^1_C = \Gamma^2_C$ and contact is without penetration and friction. Thanks to the action-reaction principle, normal stresses must be equal on the contact zone. So, the boundary condition on Γ_C can be formulated as

$$u_n^1 + u_n^2 \leq 0, \tag{13.3}$$

$$\sigma_n = \sigma_n^1 = \sigma_n^2 \leq 0,\ \sigma_n(u_n^1 + u_n^2) = 0, \tag{13.4}$$

where $\mathbf{u}^j$, $\boldsymbol{\sigma}^j$ denote $\mathbf{u}|_{\Omega^j}$, $\boldsymbol{\sigma}|_{\Omega^j}$, respectively. Notice that on the contact zone the normal vector has opposite sense on each solid: $\mathbf{n} = \mathbf{n}^1 = -\mathbf{n}^2$ and we denote $u_n^1 = \mathbf{u}^1 \cdot \mathbf{n}^1 = \mathbf{u}^1 \cdot \mathbf{n}$ and $u_n^2 = \mathbf{u}^2 \cdot \mathbf{n}^2 = -\mathbf{u}^2 \cdot \mathbf{n}$. Furthermore, by the action-reaction principle, $\boldsymbol{\sigma}^1\mathbf{n}^1 = -\boldsymbol{\sigma}^2\mathbf{n}^2$, so, multiplying by $\mathbf{n}$, it results $\boldsymbol{\sigma}^1\mathbf{n}^1 \cdot \mathbf{n} = -\boldsymbol{\sigma}^2\mathbf{n}^2 \cdot \mathbf{n}$. Therefore, $\sigma_n^1 = \boldsymbol{\sigma}^1\mathbf{n}^1 \cdot \mathbf{n}^1 = \boldsymbol{\sigma}^2\mathbf{n}^2 \cdot \mathbf{n}^2 = \sigma_n^2$. If we denote $\sigma_n = \sigma_n^1 = \sigma_n^2$ condition (13.4) is obtained.

13.2 Bilateral contact

We say that the contact between two solids is bilateral if they are in permanent contact: $u_n = 0$ on Γ_C, although tangential movements are allowed.

13.3 Frictionless contact

If we can assume that there is no friction between solids, for example to model perfectly lubricated contact, then $\boldsymbol{\sigma}_t = \mathbf{0}$ on Γ_C.

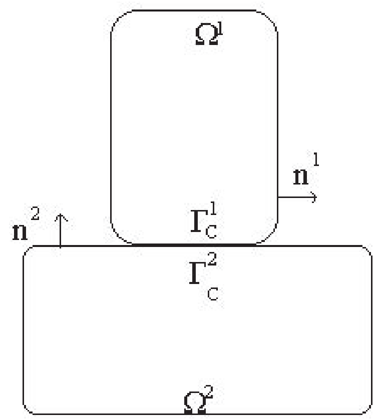

Figure 13.5: Contact between two deformable solids.

13.4 Contact with friction

Contact is a complex phenomenon. It is due to the force of attraction between the molecules of the two surfaces in contact and depends on many factors: type of materials, roughness of surfaces, etc. The simplest model to describe the friction is the *Coulomb's law* (see [Wri06]).

13.4.1 Coulomb's law

We distinguish two cases:

- Stick: The law establishes that, whereas the tangential stress is under a certain limit

$$|\boldsymbol{\sigma}_t| < \mu|\sigma_n|,$$

  there is not tangential displacement and, therefore, $\dot{\mathbf{u}}_t = \mathbf{0}$. The limit of the Coulomb's law is proportional to σ_n and μ is called *friction coefficient*, which depends on the material of the surfaces in contact (see table 13.1).

- Slip: When

$$|\boldsymbol{\sigma}_t| = \mu|\sigma_n|,$$

  then the surfaces in contact move relative to each other. The direction of the tangential velocity is opposite to the tangential stress:

$$\boldsymbol{\sigma}_t = -\mu|\sigma_n|\frac{\dot{\mathbf{u}}_t}{|\dot{\mathbf{u}}_t|}.$$

materials	μ
steel-steel	[0.2, 0.8]
steel-ice	[0.015, 0.03]
steel-concrete	[0.2, 0.4]

Table 13.1: Friction coefficients for some materials.

We can summarize the Coulomb's law in this way:

$$|\boldsymbol{\sigma}_t| \leq \mu|\sigma_n| \quad (13.5)$$

$$|\boldsymbol{\sigma}_t| < \mu|\sigma_n| \Rightarrow \dot{\mathbf{u}}_t = \mathbf{0}, \quad (13.6)$$

$$|\boldsymbol{\sigma}_t| = \mu|\sigma_n| \Rightarrow \boldsymbol{\sigma}_t = -\mu|\sigma_n|\frac{\dot{\mathbf{u}}_t}{|\dot{\mathbf{u}}_t|}, \quad (13.7)$$

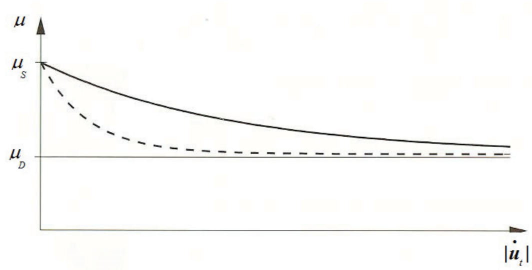

Figure 13.6: Coulomb's friction parameter [Wri06]

The friction coefficient depends on different physical and geometric characteristics: surface roughness, sliding speed, pressure of contact, temperature, ... If we introduce these effects we get a variant of Coulomb's law with variable friction coefficient

$$\mu(\dot{\mathbf{u}}_t) = \mu_D + (\mu_S - \mu_D)\mathrm{e}^{-c|\dot{\mathbf{u}}_t|}.$$

This coefficient depends on three parameters: μ_S, μ_C, c. When the sliding velocity is null, the friction coefficient coincides with the static one μ_S (see Figure 13.6). For high velocities this formula approaches the dynamic friction coefficient μ_D. The parameter c describes how this approach is done.

Example 1: Unilateral contact without friction between an elastic solid and a rigid base

$$\begin{cases} -\mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b} \text{ in } \Omega, \\ \boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}, \\ \boldsymbol{\sigma}_t = \mathbf{0},\ u_n \leq g,\ \sigma_n \leq 0,\ \sigma_n(u_n - g) = 0 \text{ on } \Gamma_C \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_N, \\ \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D. \end{cases}$$

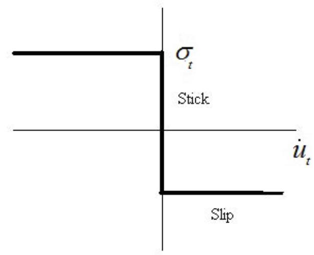

Figure 13.7: Tangential stress versus tangential velocity in Coulomb's friction.

Example 2: Unilateral contact without friction between two elastic solids

$$\begin{cases} -\mathrm{Div}(\boldsymbol{\sigma}^j) = \mathbf{b}^j \text{ in } \Omega^j,\ j = 1, 2, \\ \boldsymbol{\sigma}^j = 2\mu^j \varepsilon(\mathbf{u}^j) + \lambda^j \mathrm{tr}(\varepsilon(\mathbf{u}^j))\mathbf{I}, \\ \begin{cases} \boldsymbol{\sigma}_t = \boldsymbol{\sigma}_t^1 = \boldsymbol{\sigma}_t^2 = \mathbf{0},\ u_n^1 + u_n^2 \leq 0, \\ \sigma_n = \sigma_n^1 = \sigma_n^2 \leq 0,\ \sigma_n(u_n^1 + u_n^2) = 0,\ \text{on } \Gamma_C \end{cases} \\ \boldsymbol{\sigma}^j \mathbf{n}^j = \mathbf{g}^j \text{ on } \Gamma_N^j, \\ \mathbf{u}^j = \mathbf{u}_D^j \text{ on } \Gamma_D^j. \end{cases}$$

Example 3: Unilateral contact with friction between an elastic solid and a rigid base

$$\begin{cases} \rho\ddot{\mathbf{u}} - \mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b} \text{ in } \Omega \times [t_0, t_1], \\ \boldsymbol{\sigma} = 2\mu\varepsilon(\mathbf{u}) + \lambda\mathrm{tr}(\varepsilon(\mathbf{u}))\mathbf{I}, \\ u_n \leq g,\ \sigma_n \leq 0,\ \sigma_n(u_n - g) = 0 \\ |\boldsymbol{\sigma}_t| < \mu|\sigma_n| \Rightarrow \dot{\mathbf{u}}_t = \mathbf{0}, \\ |\boldsymbol{\sigma}_t| = \mu|\sigma_n| \Rightarrow \boldsymbol{\sigma}_t = -\mu|\sigma_n|\frac{\dot{\mathbf{u}}_t}{|\dot{\mathbf{u}}_t|},\ \text{on } \Gamma_C \times [t_0, t_1], \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_N \times [t_0, t_1], \\ \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D \times [t_0, t_1], \\ \mathbf{u}(t_0) = \mathbf{u}_0, \dot{\mathbf{u}}(t_0) = \mathbf{u}_1 \text{ in } \Omega. \end{cases}$$

Example 4: Thermomechanical contact Thermomechanical problems with contact arise in many industrial applications. There, the thermal variation can induce a dilatation or contraction that can in turn modify the contact area. This modification changes the heat flow between the body and the surface of contact. Let us suppose that an elastic body Ω is in contact with a rigid solid on its surface Γ_C. The set of equations modelling the static mechanical behaviour of the solid are:

$$\begin{cases} -\mathrm{Div}(\boldsymbol{\sigma}) = \mathbf{b} \text{ in } \Omega, \\ \boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - (2\mu + 3\lambda)\alpha(\theta - \theta_0)\mathbf{I}, \\ \boldsymbol{\sigma}_t = \mathbf{0},\ u_n \leq g,\ \sigma_n \leq 0,\ \sigma_n(u_n - g) = 0 \text{ on } \Gamma_C \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_N, \\ \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D, \end{cases}$$

where α is the thermal expansion coefficient and θ_0 the reference temperature. Furthermore, assuming that the thermal conductivity k is constant, the thermal field is a solution to the problem

$$\begin{cases} -k\Delta\theta = f \text{ in } \Omega, \\ \theta = \theta_D \text{ on } \Gamma_D \cup \Gamma_N, \\ k\dfrac{\partial\theta}{\partial\mathbf{n}} = \alpha_C(\theta_C, \theta, \sigma_n)(\theta_C - \theta) \text{ on } \Gamma_C, \end{cases}$$

where f is the density of inner heat. In the boundary condition on Γ_C, θ_C denotes the air temperature when there is no contact and the temperature of the rigid base in the other case. α_C is the heat transfer coefficient on Γ_C, which depends on the existence of effective contact (see [And+06]).

13.5 Exercises

1. Write the system of equations that allows to model the unilateral frictionless contact between an elastic solid and two rigid obstacles located at an initial distance g, considering the inertial term. Assume that:

   - The initial configuration is depicted in the figure below.
   - The forces of volume $\mathbf{b}$ are those due to gravity.
   - The surface forces $\mathbf{f}$ on the Neumann boundary are known.
   - The displacement is zero on the Dirichlet boundary.
   - The solid is free of forces on the rest of the boundary.

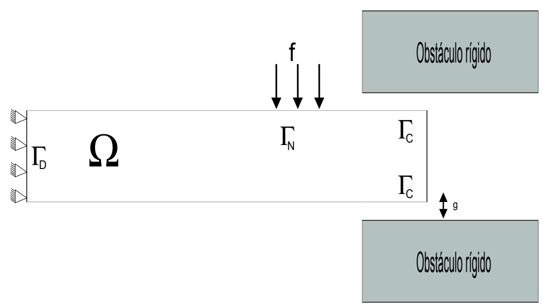

Figure 13.8: Inital configuration.

2. Let $D = (x_1, x_2, x_3); (x_2, x_3) \in B, |x_1| \leq l\}$, where l is a positive constant and B is the rectangle of vertices $\{(0,0),\ (0.5,0),\ (0.5,0.6),\ (0.5,0.6),\ (0,0.6)\}$. We denote by Γ_D the boundary of B given by $x_3 = 0.6$, Γ_N the boundaries $x_2 = 0$ and $x_2 = 0.5$ and Γ_C the face $x_3 = 0$. Let ρ be the density of the material and g be the acceleration of gravity. We consider the following problem of plane deformations:

$$\left\{\begin{array}{l} \partial_\beta \sigma_{2\beta} = 0,\ \partial_\beta \sigma_{3\beta} = \rho g \text{ in } B, \\ \sigma_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta} + \lambda(\varepsilon_{22} + \varepsilon_{33})\delta_{\alpha\beta},\ \sigma_{\alpha 1} = 0,\ \sigma_{11} = \nu(\sigma_{22} + \sigma_{33}) \text{ in } B, \\ \boldsymbol{\sigma}_t = \mathbf{0},\ u_n \leq 0,\ \sigma_n \leq 0,\ \sigma_n u_n = 0 \text{ on } \Gamma_C \\ \sigma_{\alpha\beta} n_\beta = g_\alpha \text{ on } \Gamma_N, \\ \mathbf{u}_\alpha = (\mathbf{u}_D)_\alpha,\ u_1 = 0, \text{ on } \Gamma_D, \end{array}\right.$$

for $\alpha, \beta = 2, 3$, where $\mathbf{u}_D = (0, -gh0.6^2)$, $h = \frac{(1+\nu)(1-2\nu)\rho}{2E(1-\nu)}$ and

$$\mathbf{g} = \left\{\begin{array}{ll} (\frac{-\nu}{(1-\nu)}\rho g(x_3 - 0.6), 0), & \text{on } x_2 = 0, \\ \\ (\frac{\nu}{(1-\nu)}\rho g(x_3 - 0.6), 0), & \text{on } x_2 = 0.5. \end{array}\right.$$

   - Explain what kind of problem this system of equations models.

- Let

$$\mathbf{u} = (0, 0, gh(x_3 - 0.6)^2 - gh0.6^2),$$
$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{\nu}{(1-\nu)}\rho g(x_3 - 0.6) & 0 & 0 \\ 0 & \frac{\nu}{(1-\nu)}\rho g(x_3 - 0.6) & 0 \\ 0 & 0 & \rho g(x_3 - 0.6) \end{pmatrix}.$$

  Are $\mathbf{u}$, $\boldsymbol{\sigma}$ solution of the previous problem?

3. On the website www.matweb.com you can find the following data for an aluminium alloy:

   - Yield Strength 400MPa
   - Modulus of Elasticity 68.0GPa
   - Poisson's Ratio 0.420
   - Shear Modulus 2.39GPa
   - Density 2700 Kg/m^3
   - Hardening Modulus 15 GPa
   - Coefficient of Thermal Expansion, linear 24 μm/(mK)
   - Specific Heat Capacity 0.900 J/(gK)
   - Thermal Conductivity 210 W/(mK)

   Let
$$\Omega = \{(x, y, z) : 0m < x,\ y,\ z < 1m\} \qquad (13.8)$$
   and assume that:

   - Gravitational forces act on the solid in the direction of the z-axis.
   - The upper face, located on the plane $z = 1$, is free of forces.
   - The lower face, located on the plane $z = 0$, is clamped.
   - A pressure $p(t)$ is exerted on the lateral faces, which increases with time.

   (a) Write the set of equations that models the quasi-static deformation that the solid undergoes during 10 minutes assuming that the behaviour law is elasto-plastic, considering the von Mises criterion and isotropic linear hardening.

   (b) Change the boundary condition on the bottom face if it rests on a rigid surface that is assumed to be lubricated, so that it can be assumed that there is no friction with the solid.

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