FOUR-DIMENSIONAL ALGEBRAIC SOLITONS ASSOCIATED TO GEOMETRIC FLOWS

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Abstract. We determine all the algebraic solitons for the geometric flows associated to the gradients of quadratic curvature functionals in dimension four.

1. Introduction

The search for optimal metrics is usually stated in terms of variational problems involving the invariants under consideration. Take for instance the Hilbert-Einstein functional, $g \mapsto \int_M \tau_g \, dvol_g$, which is defined in terms of the scalar curvature τ. Its gradient is given by the Einstein tensor $\frac{1}{2}\tau g - \rho$, where ρ denotes the Ricci tensor, so the critical metrics for variations with constant volume are the Einstein metrics.

Regarding Einstein metrics, the Ricci flow $\frac{\partial}{\partial t} g_t = -2\rho_{g_t}$ is the most widely investigated geometric evolution equation in an attempt to deform a given initial metric into an Einstein one. It was introduced by Hamilton in [14] and has been extensively used ever since. Self-similar solutions to the Ricci flow represent its geometric fixed points (up to diffeomorphism and scaling) and are known as Ricci solitons.

The classification of Ricci solitons is still an open problem, although partial results are available in some special situations. In the homogeneous situation any Ricci soliton is either rigid [23] or an algebraic Ricci soliton in low dimensions [2, 18]. As a consequence, the description of four-dimensional homogeneous Ricci solitons follows from the work in [19].

The scalar curvature is the only first-order scalar curvature invariant, but when it comes to quadratic scalar curvature invariants there are more of them. The space of such invariants has dimension four and is generated by $\{\tau^2, \|\rho\|^2, \|R\|^2, \Delta\tau\}$, where $\Delta\tau$ denotes the Laplacian of the scalar curvature and R is the curvature tensor, so it is also natural to consider the quadratic curvature functional associated to any quadratic curvature invariant. Note that quadratic curvature functionals are homothetically invariant in dimension four. Moreover, it follows from the Chern-Gauss-Bonnet Theorem that any such functional is equivalent either to the functional $\mathcal{S}\colon g \mapsto \int_M \tau^2 \, dvol_g$ or to some functional

$$\mathcal{F}[t]\colon g \mapsto \int_M \left\{ \|\rho\|^2 + t\tau^2 \right\} dvol_g, \quad t \in \mathbb{R}.$$

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The gradients of the functionals $\mathcal{F}[t]$ follow from the work by Berger [4], and one has that

$$(1)\quad \nabla\mathcal{F}[t] = -\Delta\rho + (2t+1)\,\mathrm{Hes}(\tau) - \tfrac{4t+1}{2}(\Delta\tau)g - 2R[\rho] + \tfrac{1}{2}\left(\|\rho\|^2 + t\tau^2\right)g - 2t\tau\rho,$$

where $\mathrm{Hes}(\tau)$ denotes the Hessian of the scalar curvature and $R[\rho]$ is the contraction of the curvature and the Ricci tensors $R[\rho]_{ij} = R_{ikj\ell}\rho^{k\ell}$. These gradients $\nabla\mathcal{F}[t]$ are trace-free and divergence-free in dimension four.

Some special cases of the functionals above have been widely investigated. For instance the functional $\mathcal{F}[-\frac{1}{4}]$ is equivalent to the one determined by the L^2-norm of the curvature tensor, $\mathcal{F}[-\frac{1}{3}]$ is equivalent to the functional determined by the L^2-norm of the Weyl tensor and $\mathcal{F}[0]$ to the one given by the L^2-norm of the Ricci tensor. To completely describe the $\mathcal{F}[t]$-critical metrics appears to be an unfeasible task, but the low-dimensional homogeneous situation is quite manageable and they have been completely determined in the three and four-dimensional homogeneous cases [6, 7].

Motivated by the work on the Ricci flow, a program has been recently developed to construct $\mathcal{F}[t]$-critical metrics through the flow associated to the negative gradient of the corresponding functional, $\mathfrak{F}[t] = -\nabla\mathcal{F}[t]$, in dimension four. The Streets flow, the Bach flow or the gradient flow of the norm of the Ricci tensor associated to the functional $\mathfrak{F}[t]$ for $t = -1/4$, $t = -1/3$ and $t = 0$, respectively, have received special attention (see, for example, [16, 20, 25, 26, 31]) as well as some more general situations [3, 5].

Self-similar solutions of the $\mathfrak{F}[t]$-flow are in correspondence with $\mathfrak{F}[t]$-solitons (see Section 2 below) and their classification significantly improves the understanding of the corresponding flows. Since $\mathfrak{F}[t]$-flows are specially relevant in dimension four (where the corresponding gradients are trace-free and divergence-free), our purpose in this work is to investigate self-similar solutions of quadratic curvature flows by focusing on the homogeneous case. We discuss Ricci solitons in Section 2, showing that any four-dimensional homogeneous Ricci soliton is also an $\mathfrak{F}[t]$-soliton for any quadratic curvature functional $\mathcal{F}[t]$. Hence we focus on the existence of *strict* algebraic $\mathfrak{F}[t]$-solitons, i.e., those which are not Ricci solitons nor $\mathcal{F}[t]$-critical, and determine all the algebraic solitons for any quadratic curvature flow in dimension four.

The Ricci operators of left-invariant metrics on four-dimensional Lie groups are not hard to manipulate. Consequently four-dimensional algebraic Ricci solitons are given in Remark 3.3, Remark 4.6 and Remark 6.5, making more explicit the description in [19]. In contrast, the gradients of quadratic curvature functionals are quite unmanageable expressions. This is why we approach the description of $\mathfrak{F}[t]$-solitons from a more general point of view. To do so we consider the algebraic solitons associated to an arbitrary isometrically invariant symmetric $(0,2)$-tensor field which is divergence-free, as it happens to the gradients $\mathfrak{F}[t]$. This general approach allows a simpler description of the underlying equations, which can be written in terms of the components of the generic tensor field and explicitly solved in every case.

Throughout this paper we will provide some graphic representations of the results obtained which must be read as follows. Each row indicates the range of t for the corresponding family of strict algebraic $\mathfrak{F}[t]$-solitons. Intervals indicating the range of t are represented with a segment for each homothetic class and are marked with a dotting above if the number of homothetic classes is infinite. The arrow on the left (resp. on the right) indicates that the interval extends to $-\infty$ (resp. to $+\infty$). An empty dot means that the point is not included in the interval, whereas a filled dot indicates that the point belongs to the range of t. Finally, colours are used to inform about the type of soliton. Thus, blue colour corresponds to expanding algebraic $\mathfrak{F}[t]$-solitons and red colour to shrinking ones. Moreover the numbering in the second column refers to the corresponding item in the classification result for the semi-direct extensions of the Abelian Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ (cf. Theorem 4.8) and the Heisenberg Lie group $\mathbb{R} \ltimes \mathcal{H}^3$ (cf. Theorem 6.6).

Non-rigid strict algebraic $\mathfrak{F}[t]$-solitons

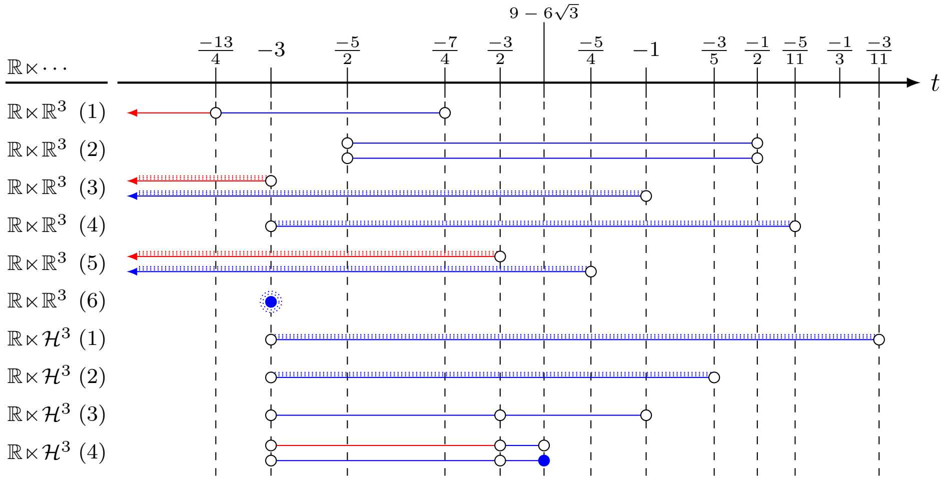

FIGURE 1. Range of the parameter t.

As we have already mentioned, the description of Bach and Street solitons follows from the study of $\mathfrak{F}[t]$-solitons for particular values of t. A direct application of the results in Theorem 3.4, Theorem 4.8, Theorem 5.2 and Theorem 6.6 provides a description of algebraic Bach solitons as follows.

Theorem 1.1. *Any algebraic Bach soliton is a Ricci soliton, a Bach-flat space, or otherwise it corresponds to one of the following:*

(i) *The product Lie group $SU(2)\times\mathbb{R}$ with the product left-invariant metric determined by*

$$[e_1,e_2]=4e_3,\quad [e_3,e_1]=4e_2,\quad [e_2,e_3]=e_1.$$

(ii) *The semi-direct extension $\mathbb{R}\ltimes\mathcal{H}^3$ with left-invariant metric determined by*

$$[e_1,e_2]=e_3,\;\; [e_1,e_4]=ae_1,\;\; [e_2,e_4]=\tfrac{1}{a}e_2,\;\; [e_3,e_4]=\tfrac{a^2+1}{a}e_3,\quad a\in(0,1),$$

where $\{e_1,\dots,e_4\}$ is an orthonormal basis.

The algebraic Bach soliton in Theorem 1.1-(i) corresponds to the rigid Bach soliton constructed by Helliwell in [15]. Furthermore, the algebraic Bach soliton on $\mathfrak{n}_4$ constructed by Thompson in [27] is a Ricci soliton, as well as the Bach solitons on $\mathbb{S}^2\times\mathbb{R}^2$ and $\mathbb{H}^2\times\mathbb{R}^2$ given by Ho in [16]. Hence they are omitted in Theorem 1.1.

On the other hand, the existence of algebraic Streets solitons (i.e., algebraic $\mathfrak{F}[t]$-solitons for $t=-1/4$) is more restrictive and one has:

Theorem 1.2. *Any algebraic soliton for the gradient of the L^2-norm of the curvature tensor is a Ricci soliton, a space homothetic to $\mathbb{S}^2 \times \mathbb{H}^2$, or otherwise it corresponds to the product Lie group $SU(2) \times \mathbb{R}$ with the product left-invariant metric determined by*

$$[e_1, e_2] = \tfrac{11}{4} e_3, \quad [e_3, e_1] = \tfrac{11}{4} e_2, \quad [e_2, e_3] = e_1,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

The left-invariant metrics on $SU(2)$ in Theorem 1.1 and Theorem 1.2 correspond to the Berger spheres determined by the only three-dimensional homogeneous $\mathcal{F}[t]$-critical metrics for the L^2-norm of the trace-free Ricci tensor for $t = -1/3$, and for the L^2-norm of the curvature tensor for $t = -1/4$, respectively (see [7, 17, 24]).

We establish some relations between Ricci solitons and $\mathfrak{F}[t]$-solitons in Section 2, showing that any Ricci soliton is an $\mathfrak{F}[t]$-soliton. Then the description of algebraic $\mathfrak{F}[t]$-solitons is considered separately for non-solvable and solvable Lie groups from Section 3. In every case, the description of algebraic $\mathfrak{F}[t]$-solitons is obtained by using the structural results previously obtained for general algebraic T-solitons.

2. Solitons and algebraic solitons associated to geometric flows

Given a geometric evolution equation $\frac{\partial}{\partial t} g_t = T_{g_t}$ associated to an isometrically invariant symmetric $(0,2)$-tensor field T on a manifold M, a solution that evolves by scaling and diffeomorphisms is said to be a *self-similar solution.* These solutions are of the form $g_t = \sigma(t)\psi_t^* g$, where ψ_t is a one-parameter family of diffeomorphisms of M and $\sigma(t)$ is a positive real-valued function.

A triple (M, g, X) is a *T-soliton* if (M, g) is a Riemannian manifold and there is a vector field X on M so that

$$\mathcal{L}_X g + T = \lambda g,$$

where $\mathcal{L}$ denotes the Lie derivative and $\lambda \in \mathbb{R}$. The soliton is *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively. Moreover, a T-soliton is *trivial* if T is a scalar multiple of the metric tensor.

Any self-similar solution gives rise to a T-soliton and the converse is true if the tensor field T is homogeneous of degree d, i.e., $\overline{T} = \kappa^d T$ for any homothetic transformation $\overline{g} = \kappa g$ (see [12, 30]). Since the Ricci tensor is homogeneous of degree $d = 0$ and the tensors $\mathfrak{F}[t] = -\nabla \mathcal{F}[t]$ are homogeneous of degree $d = -1$, it follows that self-similar solutions are equivalent to solitons for both the Ricci and $\mathfrak{F}[t]$ flows.

Remark 2.1. Let (M, g, X) be an $\mathfrak{F}[t]$-soliton (i.e., $\mathcal{L}_X g + \mathfrak{F}[t] = \lambda g$), and let $\overline{g} = \kappa g$ be a homothetic deformation of the metric. Then setting $\overline{X} = \kappa^{-2} X$, one has $\mathcal{L}_{\overline{X}} \overline{g} + \overline{\mathfrak{F}[t]} = \kappa^{-1} \mathcal{L}_X g + \kappa^{-1} \mathfrak{F}[t] = \kappa^{-1} \lambda g = \kappa^{-2} \lambda \overline{g}$, which shows that $(M, \kappa g, \kappa^{-2} X)$ remains an $\overline{\mathfrak{F}[t]}$-soliton. Hence one has that $\mathfrak{F}[t]$-solitons are invariant by homotheties, as well as $\mathcal{F}[t]$-critical metrics (i.e., $\nabla \mathcal{F}[t] = 0$).

If a Riemannian manifold (M, g) admits two distinct T-solitons, i.e., two vector fields X_i so that $\mathcal{L}_{X_i} g + T = \lambda_i g$ for $i = 1, 2$, then one has that $\xi = X_1 - X_2$ satisfies $\mathcal{L}_\xi g = (\lambda_1 - \lambda_2) g$ and so ξ is a homothetic vector field. What is more, if we assume that (M, g) is homogeneous, then either the manifold is flat or ξ is a Killing vector field. This shows that, if they exist, T-solitons are unique (up to Killing vector fields) in the homogeneous category.

It was shown in [12] that in the compact case, if the tensor field T is assumed to be trace-free and divergence-free, then T-solitons are trivial, since $T = 0$, which immediately applies to compact $\mathfrak{F}[t]$-solitons. Hence, even though some of the T-solitons constructed in this paper are realized on Lie groups admitting compact quotients, the T-soliton structure does not descend to the quotient.

2.1. Ricci solitons and $\mathfrak{F}[t]$-solitons.

Einstein metrics are $\mathcal{F}[t]$-critical for all functionals $\mathcal{F}[t]$ in dimensions three and four. In full analogy, Ricci solitons are also $\mathfrak{F}[t]$-solitons for all t in the four-dimensional homogeneous setting. Let (M, g) be a four-dimensional homogeneous Ricci soliton. Then the underlying metric g is critical for some functional $\mathcal{F}[t_0]$ with zero energy, i.e., for some $t_0 \in \mathbb{R}$ so that $\|\rho\|^2 + t_0\tau^2 = 0$ (see [6]). Homogeneity implies that the scalar curvature is constant, so the gradient of such functional given by Equation (1) reduces to

$$\nabla\mathcal{F}[t_0] = -\Delta\rho - 2R[\rho] - 2t_0\tau\rho.$$

Hence, the negative gradient of any quadratic curvature functional, $\mathfrak{F}[t] = -\nabla\mathcal{F}[t]$, in the homogeneous four-dimensional case becomes

$$\mathfrak{F}[t] = \Delta\rho + 2R[\rho] - \tfrac{1}{2}\left(\|\rho\|^2 + t\tau^2\right) g + 2t\tau\rho = -\tfrac{1}{2}\left(\|\rho\|^2 + t\tau^2\right) g + 2(t - t_0)\tau\rho.$$

Now, let X be a Ricci soliton vector field, i.e., $\mathcal{L}_X g + \rho = \lambda g$ for some $\lambda \in \mathbb{R}$. Then setting $\widetilde{X} = 2(t - t_0)\tau X$ one has

$$\begin{aligned}
\mathcal{L}_{\widetilde{X}} g + \mathfrak{F}[t] &= 2(t - t_0)\tau\mathcal{L}_X g + 2(t - t_0)\tau\rho - \tfrac{1}{2}\left(\|\rho\|^2 + t\tau^2\right) g \\
&= 2(t - t_0)\tau\left\{\mathcal{L}_X g + \rho\right\} - \tfrac{1}{2}\left(\|\rho\|^2 + t\tau^2\right) g \\
&= \left\{2(t - t_0)\tau\lambda - \tfrac{1}{2}\left(\|\rho\|^2 + t\tau^2\right)\right\} g,
\end{aligned}$$

which shows that any homogeneous four-dimensional Ricci soliton is an $\mathfrak{F}[t]$-soliton for any $t \in \mathbb{R}$.

As a possible converse of the above, note that the product $\mathbb{S}^2 \times \mathbb{H}^2$ is $\mathcal{F}[t]$-critical for all quadratic curvature functionals without being a Ricci soliton since all functionals $\mathcal{F}[t]$ have non-zero energy [6]. We therefore assume $\tau \neq 0$ in what follows. Indeed for any Ricci soliton (M, g, X) one has that $\frac{1}{2}\Delta_X\tau = \lambda\tau - \|\rho\|^2$ (see [10]), and thus the scalar curvature vanishes if and only if the manifold is flat. Now, if a homogeneous four-manifold (M, g) is an $\mathfrak{F}[t]$-soliton for two-distinct gradient functionals $\mathcal{F}[t_1]$ and $\mathcal{F}[t_2]$ with $t_1 \neq t_2$, then there are vector fields X_1, X_2 on M so that $\mathcal{L}_{X_\ell} g + \mathfrak{F}[t_\ell] = \lambda_\ell g$, $(\ell = 1, 2)$. Considering the difference vector field $\overline{\xi} = X_1 - X_2$, and subtracting the equations above one has

$$(\lambda_1 - \lambda_2)g = \mathcal{L}_{\overline{\xi}} g + \mathfrak{F}[t_1] - \mathfrak{F}[t_2] = \mathcal{L}_{\overline{\xi}} g + 2(t_1 - t_2)\tau\rho - \tfrac{1}{2}(t_1 - t_2)\tau^2 g.$$

Thus $\mathcal{L}_{\overline{\xi}} g + 2(t_1 - t_2)\tau\rho = \left\{(\lambda_1 - \lambda_2) + \frac{1}{2}(t_1 - t_2)\tau^2\right\} g$. Since the scalar curvature does not vanish, setting $\xi = \frac{1}{2(t_1 - t_2)\tau}\overline{\xi}$, one finally has

$$\mathcal{L}_\xi g + \rho = \tfrac{1}{2(t_1 - t_2)\tau}\left\{(\lambda_1 - \lambda_2) + \tfrac{1}{2}(t_1 - t_2)\tau^2\right\} g,$$

which shows that (M, g, ξ) is a Ricci soliton. Hence one has that

> *Homogeneous four-dimensional Ricci solitons are $\mathfrak{F}[t]$-solitons for any quadratic curvature functional $\mathcal{F}[t]$. Conversely, if (M, g) is a homogeneous $\mathfrak{F}[t]$-soliton for two-distinct quadratic curvature functionals then it is a Ricci soliton or locally homothetic to $\mathbb{S}^2 \times \mathbb{H}^2$.*

As a consequence, if a homogeneous $\mathfrak{F}[t]$-soliton is $\mathcal{F}[\nu]$-critical for some ν, then it is also a steady $\mathfrak{F}[\nu]$-soliton, and moreover if $\nu \neq t$, then it is a Ricci soliton or locally homothetic to $\mathbb{S}^2 \times \mathbb{H}^2$, as above.

2.1.1. *Ricci solitons and $\mathfrak{F}[t]$-solitons on symmetric spaces.* Four-dimensional symmetric spaces are either Einstein, or locally isometric to a product of the form $N^3(c) \times \mathbb{R}$ or $N_1^2(c_1) \times N_2^2(c_2)$ with $c_1 \neq c_2$. Any four-dimensional Einstein metric is $\mathcal{F}[t]$-critical for any quadratic curvature functional, as well as the products $N_1^2(c_1) \times N_2^2(c_2)$ with $c_1 = -c_2$. Hence they are trivial $\mathfrak{F}[t]$-solitons for any $t \in \mathbb{R}$. The products $N^3(c) \times \mathbb{R}$ are locally conformally flat and thus $\mathcal{F}[t]$-critical for $t = -1/3$ and moreover, they are rigid Ricci solitons and so they also are $\mathfrak{F}[t]$-solitons for any $t \in \mathbb{R}$. Products of two surfaces of constant Gauss curvature $N_1^2(c_1) \times N_2^2(c_2)$ with $c_1^2 \neq c_2^2$ are rigid Ricci solitons if $c_1 c_2 = 0$, in which case they are $\mathfrak{F}[t]$-solitons for any $t \in \mathbb{R}$ as well. If $c_1 c_2 \neq 0$, then products $N_1^2(c_1) \times N_2^2(c_2)$ are $\mathcal{F}[t]$-critical, and thus trivial $\mathfrak{F}[t]$-solitons, for $t = -1/2$. Since products $N_1^2(c_1) \times N_2^2(c_2)$ are not Ricci solitons, they cannot be $\mathfrak{F}[t]$-solitons for any $t \neq -1/2$. Hence one has that

> *Any four-dimensional Riemannian symmetric space is a trivial $\mathfrak{F}[t]$-soliton for some value of $t \in \mathbb{R}$. Moreover they are $\mathfrak{F}[t]$-solitons for any $t \in \mathbb{R}$ except the products $N_1^2(c_1) \times N_2^2(c_2)$ with $c_1 c_2 \neq 0$, $c_1^2 \neq c_2^2$, that are $\mathfrak{F}[t]$-solitons only for $t = -1/2$.*

2.1.2. *Homogeneous gradient $\mathfrak{F}[t]$-solitons.* Let (M, g, X) be a T-soliton. If the vector field X is the gradient of a real-valued function, $X = \frac{1}{2}\nabla f$, then the soliton equation becomes $\operatorname{Hes}(f) + T = \lambda g$. In such a case we say that (M, g, f) is a *gradient T-soliton* and refer to f as the potential function of the T-soliton. A gradient Ricci soliton (M, g) is *rigid* if the manifold M splits as $M = N \times \mathbb{R}^k$ so that (N, g_N) is Einstein and the potential function f is determined by the projection on the Euclidean factor, $f = \frac{\lambda}{2}\|\pi_{\mathbb{R}^k}\|^2$, which is the case of homogeneous gradient Ricci solitons [23].

Petersen and Wylie showed in [22] that if (M, g) is a homogeneous manifold and $\widetilde{T}$ is a divergence-free, symmetric, and isometrically invariant tensor field of type $(0, 2)$, then any non-constant solution of $\operatorname{Hes}(f) = \widetilde{T}$ induces a splitting of the manifold as a product $N \times \mathbb{R}^k$, and f is a function on the Euclidean factor. For a given T-flow, considering the tensor field $\widetilde{T} = \lambda g - T$, it is a direct consequence of the previous result that any homogeneous gradient T-soliton splits as a product if the tensor field T is divergence-free. The result above is no longer true if the T-soliton is not a gradient.

A three-dimensional manifold (N, g_N) is critical for a quadratic curvature functional $\mathcal{F}[t]$, restricted to variations with constant volume (i.e., $\nabla \mathcal{F}[t] = c[t] g$ for some constant $c[t]$), if and only if (see for example [9, 29])

$$-\Delta\rho_N + (2t+1)\operatorname{Hes}(\tau_N) - \tfrac{2t}{3}(\Delta\tau_N) g_N + \tfrac{2}{3}(\|\rho_N\|^2 + t\tau_N^2) g_N - 2R_N[\rho_N] - 2t\tau_N\rho_N = 0.$$

Equivalently, in the homogeneous three-dimensional case one has

$$-\Delta\rho_N + \tfrac{1}{2}(\|\rho_N\|^2 + t\tau_N^2) g_N - 2R_N[\rho_N] - 2t\tau_N\rho_N = -\tfrac{1}{6}(\|\rho_N\|^2 + t\tau_N^2) g_N.$$

For any such three-dimensional manifold (N, g_N), the product $M = N \times \mathbb{R}$ is a rigid $\mathfrak{F}[t]$-soliton, just considering the potential function $f = \frac{\lambda}{2}\|\pi_{\mathbb{R}}\|^2$, where the soliton constant $\lambda = \frac{1}{6}\left(\|\rho_N\|^2 + t\tau_N^2\right)$ is determined by the energy of the three-dimensional $\mathcal{F}[t]$-critical metric. Indeed, for such a potential function, using that $\tau = \tau_N$ and $\|\rho\|^2 = \|\rho_N\|^2$, it follows from the expression of the gradient $\nabla\mathcal{F}[t]$ in (1) that $\operatorname{Hes}(f) = \lambda dt \otimes dt$ and $\mathfrak{F}[t] = \frac{1}{6}(\|\rho_N\|^2 + t\tau_N^2)g_N$, since the tensor fields $\Delta\rho = \Delta\rho_N \oplus 0$ and $R[\rho] = R_N[\rho_N] \oplus 0$ split according to the decomposition $M = N \times \mathbb{R}$. Therefore $\operatorname{Hes}(f) + \mathfrak{F}[t] = \frac{1}{6}(\|\rho_N\|^2 + t\tau_N^2)g$, which shows that

> *Any three-dimensional homogeneous $\mathcal{F}[t]$-critical manifold (N, g_N) determines a rigid $\mathfrak{F}[t]$-soliton on $M = N \times \mathbb{R}$. Moreover the soliton is shrinking (resp., expanding) if and only if the energy of $\mathcal{F}[t]$ is positive (resp., negative).*

We refer to [7] for a complete description of homogeneous $\mathcal{F}[t]$-critical three-dimensional manifolds.

The structural result in [22] shows that any homogeneous gradient $\mathfrak{F}[t]$-soliton is a product $N \times \mathbb{R}^k$ with potential function depending only on the Euclidean factor. If $k = 2$, then N is a surface of constant Gauss curvature due to homogeneity and hence $N \times \mathbb{R}^2$ is a rigid Ricci soliton which is flat in the steady case. Moreover if $k = 1$, then the Hessian of the potential function $\operatorname{Hes}(f) = 0 \oplus f''$ and the Laplacian $\Delta f = f'' = 0$ just tracing on the steady soliton equation $\operatorname{Hes}(f) + \mathfrak{F}[t] = 0$. This shows that $\mathfrak{F}[t] = -\nabla\mathcal{F}[t] = 0$ and $M = N \times \mathbb{R}$ is $\mathcal{F}[t]$-critical. It is still an open question whether the analogous result remains true in the non-gradient case. As an application of our analysis, one has that *any steady algebraic $\mathfrak{F}[t]$-soliton is $\mathcal{F}[t]$-critical.* We refer to [13] for a proof that (strictly) extended steady T-solitons are T-flat, provided that T is divergence-free and trace-free.

2.2. **Algebraic T-solitons.**

Let T be a left-invariant symmetric $(0, 2)$-tensor field on a Lie group $(G, \langle\cdot,\cdot\rangle)$ that is endowed with a left-invariant Riemannian metric and denote by $\widehat{T}$ its associated $(1, 1)$-tensor field. Throughout this work, we will assume that T is divergence-free and isometrically invariant. $(G, \langle\cdot,\cdot\rangle)$ is said to be an *algebraic T-soliton* if

$$\widehat{T} = \mu \operatorname{Id} + \mathfrak{D}$$

for some derivation $\mathfrak{D}$ of the corresponding Lie algebra $\mathfrak{g}$ and some $\mu \in \mathbb{R}$, and it will be expanding, steady or shrinking if $\mu < 0$, $\mu = 0$, or $\mu > 0$, respectively. Moreover, if a simply connected Riemannian Lie group $(G, \langle\cdot,\cdot\rangle)$ is an algebraic T-soliton, then it is a T-soliton (cf. [18, 30]), where the soliton vector field is determined by the one parameter group of automorphisms associated to the derivation $\mathfrak{D}$. Hence the T-soliton is a self-similar solution of the T-flow whenever T is homogeneous under homotheties (see [30]). This immediately applies to algebraic Ricci solitons and algebraic $\mathfrak{F}[t]$-solitons.

2.3. **Notation.**

We say that an algebraic T-soliton is *trivial* if $T = \kappa\langle\cdot,\cdot\rangle$ for some real constant κ (or equivalently, $\widehat{T}$ and $\mathfrak{D}$ are multiples of the identity). Note that trivial algebraic Ricci solitons correspond to Einstein spaces, while trivial algebraic $\mathfrak{F}[t]$-solitons are precisely $\mathcal{F}[t]$-critical metrics. Moreover, we will say that an algebraic $\mathfrak{F}[t]$-soliton is *strict* if it is neither trivial (i.e., $\mathcal{F}[t]$-critical) nor a Ricci soliton. In particular, any symmetric four-manifold is a non-strict $\mathfrak{F}[t]$-soliton as shown in § 2.1.1.

3. ALGEBRAIC SOLITONS ON $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ AND $SU(2)\times\mathbb{R}$

Milnor proved in [21] that there exists an orthonormal basis $\{e_1,e_2,e_3\}$ of the Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ in terms of which their Lie brackets can be written as $[e_1,e_2]=\lambda_3 e_3$, $[e_2,e_3]=\lambda_1 e_1$, $[e_3,e_1]=\lambda_2 e_2$. We complement the above basis to an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ of $\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}$ or $\mathfrak{su}(2)\times\mathbb{R}$ so that

$$\begin{aligned}&[e_1,e_2]=\lambda_3 e_3, \qquad [e_1,e_3]=-\lambda_2 e_2, \qquad [e_2,e_3]=\lambda_1 e_1,\\ &[e_1,e_4]=k_3\lambda_2 e_2-k_2\lambda_3 e_3, \qquad [e_2,e_4]=k_1\lambda_3 e_3-k_3\lambda_1 e_1,\\ &[e_3,e_4]=k_2\lambda_1 e_1-k_1\lambda_2 e_2,\end{aligned}\tag{2}$$

where $\lambda_1\lambda_2\lambda_3\neq 0$. The associated Lie group corresponds to $SU(2)\times\mathbb{R}$ if λ_1, λ_2 and λ_3 have the same sign, and to $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ otherwise (see [6]).

Remark 3.1. $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ is never locally symmetric while $SU(2)\times\mathbb{R}$ is locally symmetric if and only if $\lambda_1=\lambda_2=\lambda_3$, in which case it is homothetic to $\mathbb{S}^3\times\mathbb{R}$.

3.1. Algebraic T-solitons on $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$.

The existence of algebraic T-solitons on $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$ is a very restrictive condition which is essentially given by the decomposition of $\widehat{T}$ according to the product such that its restriction to the semi-simple Lie algebra is a multiple of the identity.

Theorem 3.2. *$\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$ is a non-trivial algebraic T-soliton with soliton constant μ if and only if*

$$\widehat{T}=\operatorname{diag}[\mu,\mu,\mu,T_{44}], \quad T_{44}\neq\mu,$$

and the left-invariant metric is a product one, i.e., $k_1=k_2=k_3=0$.

Proof. Let $\{e_1,e_2,e_3,e_4\}$ be a basis of $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}$ or $\mathfrak{g}=\mathfrak{su}(2)\times\mathbb{R}$. The endomorphism $\mathfrak{D}=\widehat{T}-\mu\operatorname{Id}$ is a derivation of the corresponding Lie algebra $\mathfrak{g}$ if it satisfies the condition

$$\mathfrak{D}[e_i,e_j]-[\mathfrak{D}e_i,e_j]-[e_i,\mathfrak{D}e_j]=0, \quad i,j=1,\dots,4,$$

which, when expressed with respect to the basis $\{e_1,e_2,e_3,e_4\}$, is equivalent to

$$\mathfrak{P}_{ijk}=\mathfrak{D}_\ell{}^k c_{ij}{}^\ell-\mathfrak{D}_i{}^\ell c_{\ell j}{}^k-\mathfrak{D}_j{}^\ell c_{i\ell}{}^k=0,$$

where $\mathfrak{D}_s{}^r=\widehat{T}_s{}^r-\mu\delta_s{}^r$ and the structure constants $c_{ij}{}^\ell$ are given by $[e_i,e_j]=c_{ij}{}^\ell e_\ell$. Note that $T_{rs}=T(e_r,e_s)=\sum_{\alpha=1}^4\widehat{T}_\alpha{}^r\langle e_\alpha,e_s\rangle$ and therefore $\widehat{T}_s{}^r=T_{rs}$ since the basis $\{e_1,e_2,e_3,e_4\}$ is orthonormal.

First of all, note that the vanishing of the divergence of T leads to

$$\begin{aligned}&(\lambda_2-\lambda_3)T_{23}-k_3\lambda_2T_{24}+k_2\lambda_3T_{34}=0,\\ &(\lambda_1-\lambda_3)T_{13}-k_3\lambda_1T_{14}+k_1\lambda_3T_{34}=0,\\ &(\lambda_1-\lambda_2)T_{12}-k_2\lambda_1T_{14}+k_1\lambda_2T_{24}=0,\\ &k_3(\lambda_1-\lambda_2)T_{12}-k_2(\lambda_1-\lambda_3)T_{13}+k_1(\lambda_2-\lambda_3)T_{23}=0.\end{aligned}\tag{3}$$

The conditions for $\mathfrak{D}=\widehat{T}-\mu\operatorname{Id}$ to be a derivation amount to a system of twenty four – up to duplicity – polynomial equations on the soliton constant μ, the structure constants (2) and the components T_{ij}, corresponding to $\{\mathfrak{P}_{ijk}=0\}$, where the polynomials $\mathfrak{P}_{ijk}$ are given by

$$\mathfrak{P}_{211}=-(\lambda_1+\lambda_3)T_{13}+k_3\lambda_1T_{14},$$

$$\begin{aligned}
&\mathfrak{P}_{212} = -(\lambda_2+\lambda_3)T_{23} + k_3\lambda_2 T_{24},\\
&\mathfrak{P}_{213} = \lambda_3(T_{11}+T_{22}-T_{33}-k_1T_{14}-k_2T_{24}-\mu),\\
&\mathfrak{P}_{214} = -\lambda_3 T_{34},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{P}_{311} = (\lambda_1+\lambda_2)T_{12} - k_2\lambda_1 T_{14},\\
&\mathfrak{P}_{312} = -\lambda_2(T_{11}-T_{22}+T_{33}-k_1T_{14}-k_3T_{34}-\mu),\\
&\mathfrak{P}_{313} = (\lambda_2+\lambda_3)T_{23} - k_2\lambda_3 T_{34},\\
&\mathfrak{P}_{314} = \lambda_2 T_{24},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{P}_{321} = -\lambda_1(T_{11}-T_{22}-T_{33}+k_2T_{24}+k_3T_{34}+\mu),\\
&\mathfrak{P}_{322} = -(\lambda_1+\lambda_2)T_{12} + k_1\lambda_2 T_{24},\\
&\mathfrak{P}_{323} = -(\lambda_1+\lambda_3)T_{13} + k_1\lambda_3 T_{34},\\
&\mathfrak{P}_{324} = -\lambda_1 T_{14},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{P}_{411} = -k_3(\lambda_1+\lambda_2)T_{12} + k_2(\lambda_1+\lambda_3)T_{13},\\
&\mathfrak{P}_{412} = k_3\lambda_2(T_{11}-T_{22}+T_{44}-\mu) - k_1\lambda_2T_{13} + k_2\lambda_3T_{23} - \lambda_2T_{34},\\
&\mathfrak{P}_{413} = -k_2\lambda_3(T_{11}-T_{33}+T_{44}-\mu) + k_1\lambda_3T_{12} - k_3\lambda_2T_{23} + \lambda_3T_{24},\\
&\mathfrak{P}_{414} = -k_3\lambda_2T_{24} + k_2\lambda_3T_{34},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{P}_{421} = k_3\lambda_1(T_{11}-T_{22}-T_{44}+\mu) - k_1\lambda_3T_{13} + k_2\lambda_1T_{23} + \lambda_1T_{34},\\
&\mathfrak{P}_{422} = k_3(\lambda_1+\lambda_2)T_{12} - k_1(\lambda_2+\lambda_3)T_{23},\\
&\mathfrak{P}_{423} = k_1\lambda_3(T_{22}-T_{33}+T_{44}-\mu) - k_2\lambda_3T_{12} + k_3\lambda_1T_{13} - \lambda_3T_{14},\\
&\mathfrak{P}_{424} = k_3\lambda_1T_{14} - k_1\lambda_3T_{34},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{P}_{431} = -k_2\lambda_1(T_{11}-T_{33}-T_{44}+\mu) + k_1\lambda_2T_{12} - k_3\lambda_1T_{23} - \lambda_1T_{24},\\
&\mathfrak{P}_{432} = k_1\lambda_2(T_{22}-T_{33}-T_{44}+\mu) - k_2\lambda_1T_{12} + k_3\lambda_2T_{13} + \lambda_2T_{14},\\
&\mathfrak{P}_{433} = -k_2(\lambda_1+\lambda_3)T_{13} + k_1(\lambda_2+\lambda_3)T_{23},\\
&\mathfrak{P}_{434} = -k_2\lambda_1T_{14} + k_1\lambda_2T_{24}.
\end{aligned}$$

We start by considering the polynomials

$$\begin{aligned}
&\mathfrak{P}_{324} = -\lambda_1T_{14}, \quad \mathfrak{P}_{314} = \lambda_2T_{24}, \quad \mathfrak{P}_{214} = -\lambda_3T_{34},\\
&\mathfrak{P}_{321} = -\lambda_1(T_{11}-T_{22}-T_{33}+k_2T_{24}+k_3T_{34}+\mu),\\
&\mathfrak{P}_{312} = -\lambda_2(T_{11}-T_{22}+T_{33}-k_1T_{14}-k_3T_{34}-\mu),\\
&\mathfrak{P}_{213} = \lambda_3(T_{11}+T_{22}-T_{33}-k_1T_{14}-k_2T_{24}-\mu),
\end{aligned}$$

from where it follows that

$$T_{14}=T_{24}=T_{34}=0, \quad T_{11}=T_{22}=T_{33}=\mu. \tag{4}$$

Under the conditions above we obtain

$$\mathfrak{P}_{311} = (\lambda_1+\lambda_2)T_{12}, \quad \mathfrak{P}_{211} = -(\lambda_1+\lambda_3)T_{13}, \quad \mathfrak{P}_{212} = -(\lambda_2+\lambda_3)T_{23},$$

and now the necessary conditions for the vanishing of the divergence of T, given by Equation (3), lead to

$$(\lambda_1-\lambda_2)T_{12}=0, \quad (\lambda_1-\lambda_3)T_{13}=0, \quad (\lambda_2-\lambda_3)T_{23}=0.$$

Since $\lambda_1\lambda_2\lambda_3 \neq 0$, it follows that

$$T_{12} = T_{13} = T_{23} = 0. \tag{5}$$

Hence, Equations (4) and (5) imply that $\widehat{T} = \operatorname{diag}[\mu, \mu, \mu, T_{44}]$ and T is divergence-free. Moreover, the system $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$k_1(T_{44} - \mu) = 0, \quad k_2(T_{44} - \mu) = 0, \quad k_3(T_{44} - \mu) = 0,$$

from where the proof immediately follows. □

Remark 3.3. A direct calculation from Theorem 3.2, using the expression of the Ricci tensor of any left-invariant metric on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$, shows that any algebraic Ricci soliton is homothetic to the metric determined by the Lie algebra (2) with $k_1 = k_2 = k_3 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$, which corresponds to the rigid Ricci soliton $\mathbb{S}^3 \times \mathbb{R}$.

3.2. Algebraic $\mathfrak{F}[t]$-solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$.

The next result shows that the non-solvable Lie groups $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ provide strict algebraic $\mathfrak{F}[t]$-solitons for the gradient of any quadratic curvature functional $\mathcal{F}[t]$ with $t \neq -3, -\frac{1}{2}, \frac{1}{3}$ which are in one to one correspondence with the $\mathcal{F}[t]$-critical metrics on $\widetilde{SL}(2,\mathbb{R})$ and $SU(2)$ (see [7]) as pointed out in Section 2.1.2.

Theorem 3.4. *A left-invariant metric on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$ is a strict algebraic $\mathfrak{F}[t]$-soliton if and only if it is isomorphically homothetic to one of the following:*

1. $[e_1, e_2] = \lambda e_3$, $[e_3, e_1] = \lambda e_2$, $[e_2, e_3] = e_1$, *with* $\lambda \in \mathbb{R} \setminus \left\{0, 1, \frac{1}{4}\right\}$. *In this case,* $\mu = \frac{1}{4}\lambda$ *and* $t = -\frac{2\lambda-3}{4\lambda-1}$.
2. $[e_1, e_2] = \lambda_3 e_3$, $[e_3, e_1] = \lambda_2 e_2$, $[e_2, e_3] = e_1$, *with* $\lambda_2 \in (-1, -\frac{1}{2}\varphi) \cup (0, \frac{1}{2}\varphi^{-1})$, *where* $\varphi = \frac{1}{2}(1 + \sqrt{5})$ *is the Golden number. For a fixed* λ_2, *the parameter* λ_3 *corresponds to the only real solution of the equation*
$$\lambda_3^3 - (\lambda_2 + 1)\lambda_3^2 - (\lambda_2 + 1)^2\lambda_3 + (\lambda_2 - 1)^2(\lambda_2 + 1) = 0$$
satisfying $-1 < \lambda_2 < \lambda_3 < 1$. *In this case,* $\mu = \frac{1}{2}\lambda_2\lambda_3(\lambda_2 + \lambda_3 + 1)$ *and* $t = -\frac{2\left(\lambda_2^2+\lambda_3^2+1\right)}{\lambda_2^2+\lambda_3^2-2(\lambda_2\lambda_3+\lambda_2+\lambda_3)+1}$.

Remark 3.5. The strict algebraic $\mathfrak{F}[t]$-solitons in Theorem 3.4 are realized on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$ as indicated in Figure 2 below.

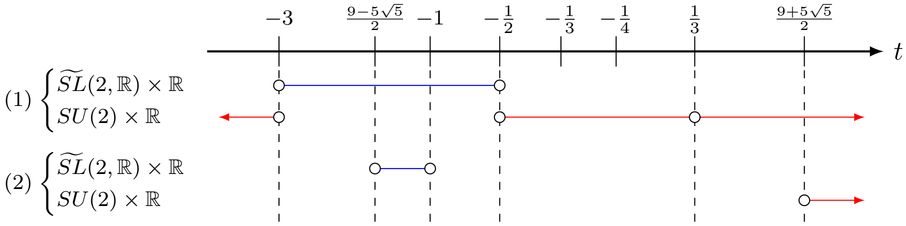

FIGURE 2. Range of the parameter t for strict algebraic $\mathfrak{F}[t]$-solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$.

A straightforward calculation shows that the homothetic invariants $\{t,\ \|\rho\|^2\,\tau^{-2},\ \|R\|^2\,\tau^{-2},\ \|\nabla\rho\|^2\,\tau^{-3},\ \|\nabla R\|^2\tau^{-3}\}$ suffice to prove that there are no homotheties between metrics in the two families in Theorem 3.4. Moreover, it follows from the restrictions on the parameters in each one of the two families that different values of the parameters correspond to different homothety classes. Furthermore, for each admissible value of $t\in\mathbb{R}\setminus\{-3,-\frac{1}{2},\frac{1}{3}\}$ there is a single strict algebraic $\mathfrak{F}[t]$-soliton (up to homotheties) for each family in Figure 2.

Note that none of the algebraic $\mathfrak{F}[t]$-solitons in Theorem 3.4 is trivial (i.e., $\nabla\mathcal{F}[t]\neq 0$) just comparing with the results in [6] where all homogeneous $\mathcal{F}[t]$-critical metrics on four-dimensional Riemannian manifolds were described.

Remark 3.6. It follows from the results in Section 2.1.2 that the rigid $\mathfrak{F}[t]$-soliton on $SU(2)\times\mathbb{R}$ for $t=-1/3$ corresponds to the rigid Bach soliton obtained by Helliwell in [15]. This soliton is determined by the only non-Einstein homogeneous $\mathcal{F}[-\frac{1}{3}]$-critical metric constructed by Sekigawa, where $\mathcal{F}[-\frac{1}{3}]$ is the functional associated to the L^2-norm of the traceless Ricci tensor in dimension three [7, 24].

In a completely analogous way, the $\mathfrak{F}[t]$-soliton corresponding to $t=-1/4$ is obtained as a rigid soliton from the only non-Einstein homogeneous $\mathcal{F}[-\frac{1}{4}]$-critical metric constructed by Lamontagne in [17].

Proof of Theorem 3.4. Since a strict algebraic $\mathfrak{F}[t]$-soliton must be non-symmetric, the structure eigenvalues λ_1, λ_2 and λ_3 are not all equal (see Remark 3.1). We make use of Theorem 3.2 to study the existence of non-trivial algebraic $\mathfrak{F}[t]$-solitons in this setting. Hence the only non-zero components of the symmetric $(0,2)$-tensor $\mathfrak{F}[t]$ must be $\mathfrak{F}[t]_{ii}$, where

$$\mathfrak{F}[t]_{11}=\mathfrak{F}[t]_{22}=\mathfrak{F}[t]_{33}=\mu,\quad \mathfrak{F}[t]_{44}\neq\mu.$$

Besides, $k_1=k_2=k_3=0$ so that the metric is the product one. In what follows we work modulo isomorphic homotheties by assuming $\lambda_1=1$. In this situation, a straightforward calculation shows that $\mathfrak{F}[t]_{ij}=0$ for all $i\neq j$, and

$$\begin{aligned}
8\mathfrak{F}[t]_{11}={}&\left(\lambda_2^2+(\lambda_3-1)^2-2\lambda_2(\lambda_3+1)\right)\left((3\lambda_2+2)\lambda_2+(3\lambda_3+2)\lambda_3-6\lambda_2\lambda_3-5\right)t\\
&+\left(9\lambda_2^3-4\lambda_2^2-2\lambda_2+12\right)\lambda_2+\left(9\lambda_3^3-4\lambda_3^2-2\lambda_3+12\right)\lambda_3\\
&-2\left(6\lambda_2^2+6\lambda_3^2-3\lambda_2\lambda_3-2\lambda_2-2\lambda_3+2\right)\lambda_2\lambda_3-15,\\
-8\mathfrak{F}[t]_{22}={}&\left(\lambda_2^2+(\lambda_3-1)^2-2\lambda_2(\lambda_3+1)\right)\left(5\lambda_2^2-3(\lambda_3-1)^2-2\lambda_2(\lambda_3+1)\right)t\\
&+\left(15\lambda_2^3-12\lambda_2^2+2\lambda_2+4\right)\lambda_2-3\left(3\lambda_3^3-4\lambda_3^2+2\lambda_3-4\right)\lambda_3\\
&-2\left(6\lambda_2^2-2\lambda_3^2-\lambda_2\lambda_3-2\lambda_2+2\lambda_3+2\right)\lambda_2\lambda_3-9,\\
8\mathfrak{F}[t]_{33}={}&\left(\lambda_2^2+(\lambda_3-1)^2-2\lambda_2(\lambda_3+1)\right)\left(3(\lambda_2-1)^2-5\lambda_3^2+2(\lambda_2+1)\lambda_3\right)t\\
&+3\left(3\lambda_2^3-4\lambda_2^2+2\lambda_2-4\right)\lambda_2-\left(15\lambda_3^3-12\lambda_3^2+2\lambda_3+4\right)\lambda_3\\
&-2\left(2\lambda_2^2-6\lambda_3^2+\lambda_2\lambda_3-2\lambda_2+2\lambda_3-2\right)\lambda_2\lambda_3+9,
\end{aligned}$$

while $\mathfrak{F}[t]_{44}=-\mathfrak{F}[t]_{11}-\mathfrak{F}[t]_{22}-\mathfrak{F}[t]_{33}$ since $\mathfrak{F}[t]$ is trace-free. Bearing in mind the relation between the components $\mathfrak{F}[t]_{ii}$, we consider the following linear combination

$$\begin{aligned}
0&=(\lambda_2-1)\left(\mathfrak{F}[t]_{22}-\mathfrak{F}[t]_{33}\right)+(\lambda_2-\lambda_3)\left(\mathfrak{F}[t]_{11}-\mathfrak{F}[t]_{22}\right)\\
&=-2(\lambda_2-\lambda_3)(\lambda_2-1)(\lambda_3-1)\\
&\quad\times\left(\left(\lambda_2^2+\lambda_3^2-2(\lambda_2\lambda_3+\lambda_2+\lambda_3)+1\right)t+2\left(\lambda_2^2+\lambda_3^2+1\right)\right).
\end{aligned}\tag{6}$$

This expression gives us four cases in terms of the vanishing of each factor.

3.2.1. **Case $\boldsymbol{\lambda_2 = \lambda_3}$.** We have

$$\begin{aligned}-8\,\mathfrak{F}[t]_{11} &= \left(16\lambda_2^2-24\lambda_2+5\right)t+8\lambda_2^2-24\lambda_2+15,\\ 8\,\mathfrak{F}[t]_{22}=8\,\mathfrak{F}[t]_{33} &= \left(16\lambda_2^2-16\lambda_2+3\right)t+8\lambda_2^2-16\lambda_2+9,\end{aligned}$$

so the condition $\mathfrak{F}[t]_{11}=\mathfrak{F}[t]_{22}$ gives

$$(\lambda_2-1)\left((4\lambda_2-1)\,t+2\lambda_2-3\right)=0.$$

Since the case $\lambda_2=1$ corresponds to the symmetric situation (see Remark 3.1) we only need to analyse when the second factor of the expression above vanishes. If $\lambda_2=\frac{1}{4}$, clearly there are no algebraic $\mathfrak{F}[t]$-solitons. Therefore, for $\lambda_2\notin\left\{0,1,\frac{1}{4}\right\}$ we can take $t=-\frac{2\lambda_2-3}{4\lambda_2-1}\in\mathbb{R}\setminus\left\{-3,-\frac{1}{2},\frac{1}{3}\right\}$, so that

$$\widehat{\mathfrak{F}[t]}=\tfrac{1}{4}\operatorname{diag}[\lambda_2,\lambda_2,\lambda_2,-3\lambda_2],\quad \mu=\tfrac{\lambda_2}{4},$$

and the corresponding left-invariant metric is given by

$$[e_1,e_2]=\lambda_2e_3,\quad [e_1,e_3]=-\lambda_2e_2,\quad [e_2,e_3]=e_1.$$

A direct calculation shows that these metrics are never critical [6], so we get strict shrinking algebraic $\mathfrak{F}[t]$-solitons on $SU(2)\times\mathbb{R}$ when $t\in(-\infty,-3)\cup\left(-\frac{1}{2},\frac{1}{3}\right)\cup\left(\frac{1}{3},+\infty\right)$ and strict expanding algebraic $\mathfrak{F}[t]$-solitons on $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ when the parameter $t\in\left(-3,-\frac{1}{2}\right)$. These metrics correspond to Family (1).

3.2.2. **Case $\boldsymbol{\lambda_2 = 1}$.** We proceed as in the previous case and obtain that the left-invariant metric

$$[e_1,e_2]=\lambda_3e_3,\quad [e_1,e_3]=-e_2,\quad [e_2,e_3]=e_1,\quad \lambda_3\notin\{0,1,4\},$$

determines a strict algebraic $\mathfrak{F}[t]$-soliton for $t=-\frac{3\lambda_3-2}{\lambda_3-4}$. Now, the transformation $(e_1,e_2,e_3,e_4)\mapsto\frac{1}{\lambda_3}(e_3,e_2,-e_1,e_4)$ is an isomorphic homothety between these metrics and the ones in the previous case.

3.2.3. **Case $\boldsymbol{\lambda_3 = 1}$.** Proceeding exactly as in the previous cases we get that the left-invariant metric

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=-\lambda_2e_2,\quad [e_2,e_3]=e_1,\quad \lambda_2\notin\{0,1,4\},$$

determines a strict algebraic $\mathfrak{F}[t]$-soliton for $t=-\frac{3\lambda_2-2}{\lambda_2-4}$. In this case, the transformation $(e_1,e_2,e_3,e_4)\mapsto(e_1,-e_3,e_2,e_4)$ is an isomorphic isometry between these metrics and the ones in the previous case.

3.2.4. **Case $\boldsymbol{\lambda_2 \neq 1}$, $\boldsymbol{\lambda_3 \neq 1}$ and $\boldsymbol{\lambda_2 \neq \lambda_3}$.** In this case, Equation (6) implies that

$$\left(\lambda_2^2+\lambda_3^2-2\left(\lambda_2\lambda_3+\lambda_2+\lambda_3\right)+1\right)t+2\left(\lambda_2^2+\lambda_3^2+1\right)=0.$$

If the coefficient of t vanishes, which occurs if $\lambda_2>0$ and $\lambda_3=(1\pm\sqrt{\lambda_2})^2$, then we have

$$\mathfrak{F}[t]_{22}-\mathfrak{F}[t]_{11}=4\sqrt{\lambda_2}\,\left(\lambda_2^2-1\right)\left(3\sqrt{\lambda_2}\pm2(\lambda_2+1)\right),$$

which does not vanish for $\lambda_2>0$, $\lambda_2\neq1$. Therefore there are no algebraic $\mathfrak{F}[t]$-solitons if the coefficient of t vanishes, so we assume $\lambda_3\neq\left(1\pm\sqrt{\lambda_2}\right)^2$ and obtain

$$t=-\frac{2\left(\lambda_2^2+\lambda_3^2+1\right)}{\lambda_2^2+\lambda_3^2-2(\lambda_2\lambda_3+\lambda_2+\lambda_3)+1}. \tag{7}$$

In this situation, since $\lambda_2\neq1$, the vanishing of

$$\mathfrak{F}[t]_{11}-\mathfrak{F}[t]_{22}=(\lambda_2-1)\left(\lambda_3^3-(\lambda_2+1)\lambda_3^2-(\lambda_2+1)^2\lambda_3+(\lambda_2-1)^2(\lambda_2+1)\right)$$

implies that

$$\lambda_3^3-(\lambda_2+1)\lambda_3^2-(\lambda_2+1)^2\lambda_3+(\lambda_2-1)^2(\lambda_2+1)=0. \tag{8}$$

A straightforward calculation using this condition shows that

$$\widehat{\mathfrak{F}[t]}=\tfrac{1}{2}\lambda_2\lambda_3(\lambda_2+\lambda_3+1)\,\mathrm{diag}[1,1,1,-3],\quad \mu=\tfrac{1}{2}\lambda_2\lambda_3(\lambda_2+\lambda_3+1),$$

so the corresponding left-invariant metric, which is given by

$$[e_1,e_2]=\lambda_3e_3,\quad [e_1,e_3]=-\lambda_2e_2,\quad [e_2,e_3]=e_1,$$

is an algebraic $\mathfrak{F}[t]$-soliton whenever Equation (8) is satisfied, where $\lambda_2\neq 0,1$, $\lambda_3\neq 0,1,\lambda_2$ and, if $\lambda_2>0$, $\lambda_3\neq\left(1\pm\sqrt{\lambda_2}\right)^2$. A straightforward calculation shows that these metrics are never critical (see [6]). Moreover, $(e_1,e_2,e_3,e_4)\mapsto(e_1,e_3,-e_2,e_4)$ defines an isomorphic isometry which interchanges (λ_2,λ_3) and (λ_3,λ_2), while the isomorphic homothety $(e_1,e_2,e_3,e_4)\mapsto\frac{1}{\lambda_3}(-e_3,e_2,e_1,e_4)$ interchanges (λ_2,λ_3) and $\left(\frac{\lambda_2}{\lambda_3},\frac{1}{\lambda_3}\right)$. Thus, we can restrict the parameters λ_2 and λ_3 to $-1<\lambda_2<\lambda_3<1$ satisfying Equation (8). Indeed, for a fixed $\lambda_2\in\left(-1,-\frac{1+\sqrt{5}}{4}\right)\cup\left(0,-\frac{1-\sqrt{5}}{4}\right)$, the parameter λ_3 corresponds to the only real solution of the Equation (8) satisfying $-1<\lambda_2<\lambda_3<1$. Besides, the parameter t given by Equation (7) belongs to $\left(\frac{9-5\sqrt{5}}{2},-1\right)\cup\left(\frac{9+5\sqrt{5}}{2},+\infty\right)$, and the strict algebraic $\mathfrak{F}[t]$-solitons are expanding and realized on $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ when $t\in\left(\frac{9-5\sqrt{5}}{2},-1\right)$, while they are shrinking on $SU(2)\times\mathbb{R}$ when $t\in\left(\frac{9+5\sqrt{5}}{2},+\infty\right)$. These metrics correspond to Family (2), which concludes the proof. □

4. Algebraic solitons on $\mathbb{R}\ltimes\mathbb{R}^3$

Proceeding as in Section 3, the expression of left-invariant metrics on semi-direct extensions of the Abelian Lie group follows from [6, 21]. They depend upon six parameters as follows:

$$\begin{aligned}&[e_1,e_4]=ae_1+be_2+ce_3, && [e_2,e_4]=-be_1+fe_2+he_3,\\ &[e_3,e_4]=-ce_1-he_2+pe_3,\end{aligned} \tag{9}$$

where $\{e_1,e_2,e_3,e_4\}$ is an orthonormal basis of the Lie algebra. Considering the endomorphism ψ associated to the semi-direct extension of the Abelian Lie group, $\mathbb{R}\ltimes_\psi\mathbb{R}^3$, the parameters a,f,p correspond to the eigenvalues of the self-adjoint part of ψ, while b,c,h denote the components of the skew-symmetric part on the basis of eigenvectors of ψ.

Remark 4.1. A left-invariant metric on $\mathbb{R}\ltimes\mathbb{R}^3$ is locally symmetric if and only if it is Einstein (which corresponds to the case when the self-adjoint part of the derivation determining $\mathbb{R}\ltimes_\psi\mathbb{R}^3$ is a multiple of the identity, i.e., $a=f=p$), in which case it is homothetic to $\mathbb{H}^4$, or otherwise it is isomorphically isometric to a metric described by (9) with $a=f=c=h=0$, $p\neq 0$ (thus being homothetic to $\mathbb{H}^2\times\mathbb{R}^2$), or with $a=f\neq 0$, $p=c=h=0$ (thus being homothetic to $\mathbb{H}^3\times\mathbb{R}$).

In the Abelian case any metric is flat and any tensor field T is an algebraic T-soliton. Hence, we will exclude this case in what follows and assume that at least one of the structure constants in (9) is non-zero.

Remark 4.2. Note that there are some specific isomorphic isometries that allow us to simplify our study by eliminating redundancies in the different cases we consider. The isomorphic isometry determined by $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, e_3, e_4)$ implies that $(a, f, p, b, c, h) \sim (f, a, p, -b, h, c)$, and $(e_1, e_2, e_3, e_4) \mapsto (e_3, e_2, e_1, e_4)$ gives $(a, f, p, b, c, h) \sim (p, f, a, -h, -c, -b)$. Analogously, the isomorphic isometry $(e_1, e_2, e_3, e_4) \mapsto (e_1, e_3, e_2, e_4)$ shows that $(a, f, p, b, c, h) \sim (a, p, f, c, b, -h)$.

4.1. **Algebraic T-solitons on $\mathbb{R} \ltimes \mathbb{R}^3$.** Recall that we are assuming that the left-invariant tensor field T is divergence-free, so the following conditions hold:

$$\begin{aligned} &(2a + f + p)T_{14} + bT_{24} + cT_{34} = 0, \\ &bT_{14} - (a + 2f + p)T_{24} - hT_{34} = 0, \\ &cT_{14} + hT_{24} - (a + f + 2p)T_{34} = 0, \\ &aT_{11} + fT_{22} + pT_{33} - (a + f + p)T_{44} = 0. \end{aligned} \tag{10}$$

The conditions for $\mathfrak{D} = \widehat{T} - \mu \operatorname{Id}$ to be a derivation are determined by a system of polynomial equations (proceeding as in the proof of Theorem 3.2) on the soliton constant μ, the structure constants in (9) and the components T_{ij}, given by $\{\mathfrak{P}_{ijk} = 0\}$, where

$$\begin{aligned} &\mathfrak{P}_{211} = bT_{14} + aT_{24}, \quad \mathfrak{P}_{212} = -fT_{14} + bT_{24}, \quad \mathfrak{P}_{213} = -hT_{14} + cT_{24}, \\ &\mathfrak{P}_{311} = cT_{14} + aT_{34}, \quad \mathfrak{P}_{312} = hT_{14} + bT_{34}, \quad \mathfrak{P}_{313} = -pT_{14} + cT_{34}, \\ &\mathfrak{P}_{321} = cT_{24} - bT_{34}, \quad \mathfrak{P}_{322} = hT_{24} + fT_{34}, \quad \mathfrak{P}_{323} = -pT_{24} + hT_{34}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{411} = a(T_{44} - \mu) - 2bT_{12} - 2cT_{13}, \\ &\mathfrak{P}_{412} = b(T_{11} - T_{22} + T_{44} - \mu) - (a - f)T_{12} - hT_{13} - cT_{23}, \\ &\mathfrak{P}_{413} = c(T_{11} - T_{33} + T_{44} - \mu) + hT_{12} - (a - p)T_{13} - bT_{23}, \\ &\mathfrak{P}_{414} = -aT_{14} - bT_{24} - cT_{34}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{421} = b(T_{11} - T_{22} - T_{44} + \mu) + (a - f)T_{12} - hT_{13} - cT_{23}, \\ &\mathfrak{P}_{422} = f(T_{44} - \mu) + 2bT_{12} - 2hT_{23}, \\ &\mathfrak{P}_{423} = h(T_{22} - T_{33} + T_{44} - \mu) + cT_{12} + bT_{13} - (f - p)T_{23}, \\ &\mathfrak{P}_{424} = bT_{14} - fT_{24} - hT_{34}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{431} = c(T_{11} - T_{33} - T_{44} + \mu) + hT_{12} + (a - p)T_{13} - bT_{23}, \\ &\mathfrak{P}_{432} = h(T_{22} - T_{33} - T_{44} + \mu) + cT_{12} + bT_{13} + (f - p)T_{23}, \\ &\mathfrak{P}_{433} = p(T_{44} - \mu) + 2cT_{13} + 2hT_{23}, \\ &\mathfrak{P}_{434} = cT_{14} + hT_{24} - pT_{34}. \end{aligned}$$

In order to determine all the algebraic T-solitons in this case the distinguished direction $\operatorname{span}\{e_4\}$ plays an essential role, since it must be an eigenspace of the endomorphism $\widehat{T}$ in most cases.

Lemma 4.3. *Let $\mathbb{R} \ltimes \mathbb{R}^3$ be an algebraic T-soliton with soliton constant μ. Then the following relations hold.*

(a) *With no further assumptions, either e_4 is an eigenvector of $\widehat{T}$, (i.e., $T_{14} = T_{24} = T_{34} = 0$), or a, f and p are all distinct with $afp = 0$ and $bch = 0$. In* *the latter case, the original metric is isomorphically isometric to a metric with* $a = 0$ *and* $b = c = 0$.

(b) *If* e_4 *is an eigenvector of* $\widehat{T}$ *with associated eigenvalue* $T_{44} \neq \mu$*, then* $a + f + p = 0$*. Moreover, if* a*,* f *and* p *are all distinct, then* $afp = 0$ *and the original metric is isomorphically isometric to a metric with* $a = 0$ *and* $f = -p$.

Proof. We prove each assertion separately in what follows.

4.1.1. **The proof of Assertion (a).** Let us assume that e_4 is not an eigenvector of $\widehat{T}$, which means that at least one of the components T_{14}, T_{24} and T_{34} does not vanish. Firstly, we show that a, f and p are necessarily different. If two of them are equal, by Remark 4.2 we may assume that $f = a$, and

$$\mathfrak{P}_{211} = bT_{14} + aT_{24}, \quad \mathfrak{P}_{212} = -aT_{14} + bT_{24}.$$

Hence, either $T_{14} = T_{24} = 0$ or $a = b = 0$. If $T_{14} = T_{24} = 0$ then

$$\mathfrak{P}_{311} = aT_{34}, \quad \mathfrak{P}_{434} = -pT_{34}, \quad \mathfrak{P}_{312} = bT_{34}, \quad \mathfrak{P}_{313} = cT_{34}, \quad \mathfrak{P}_{323} = hT_{34},$$

and the fact that $T_{34} \neq 0$ implies that the Lie group is Abelian. If either T_{14} or T_{24} does not vanish, then $a = b = 0$, and

$$\mathfrak{P}_{313} + \mathfrak{P}_{414} = -pT_{14}, \quad \mathfrak{P}_{323} + \mathfrak{P}_{424} = -pT_{24},$$

together with

$$\mathfrak{P}_{311} = cT_{14}, \quad \mathfrak{P}_{321} = cT_{24}, \quad \mathfrak{P}_{312} = hT_{14}, \quad \mathfrak{P}_{322} = hT_{24},$$

imply that $p = c = h = 0$, so that the Lie group is Abelian. This shows that a, f and p must be different.

Secondly, we show that $bch = 0$ and $afp = 0$. The linear combinations

$$\begin{aligned}
\mathfrak{P}_{213} - \mathfrak{P}_{312} - \mathfrak{P}_{321} &= -2hT_{14},\\
\mathfrak{P}_{213} + \mathfrak{P}_{312} + \mathfrak{P}_{321} &= 2cT_{24},\\
\mathfrak{P}_{213} + \mathfrak{P}_{312} - \mathfrak{P}_{321} &= 2bT_{34},
\end{aligned}$$

clearly lead to $bch = 0$. If $b = 0$, then

$$\mathfrak{P}_{212} = -fT_{14}, \quad \mathfrak{P}_{211} = aT_{24}, \quad f\,\mathfrak{P}_{311} + c\,\mathfrak{P}_{212} = afT_{34}.$$

Consequently, $afp = 0$. The same conclusion is obtained if $c = 0$, since

$$\mathfrak{P}_{313} = -pT_{14}, \quad p\mathfrak{P}_{211} + b\mathfrak{P}_{313} = apT_{24}, \quad \mathfrak{P}_{311} = aT_{34},$$

or if $h = 0$, using

$$p\,\mathfrak{P}_{212} + b\,\mathfrak{P}_{323} = -fpT_{14}, \quad \mathfrak{P}_{323} = -pT_{24}, \quad \mathfrak{P}_{322} = fT_{34}.$$

Thus, $afp = 0$ and $bch = 0$. Finally, by Remark 4.2 we can take $a = 0$ working, if necessary, with an isomorphically isometric metric. Now, if $b = 0$, we compute

$$\mathfrak{P}_{311} = cT_{14}, \quad \mathfrak{P}_{321} = cT_{24}, \quad \mathfrak{P}_{414} = -cT_{34},$$

so necessarily $c = 0$. If $c = 0$, then $b = 0$ since

$$\mathfrak{P}_{211} = bT_{14}, \quad \mathfrak{P}_{414} = -bT_{24}, \quad \mathfrak{P}_{321} = -bT_{34}.$$

Finally, if $h = 0$ we use

$$\mathfrak{P}_{211} = bT_{14}, \quad \mathfrak{P}_{212} = -fT_{14} + bT_{24}, \quad \mathfrak{P}_{312} = bT_{34},$$

to obtain that $b = 0$ and, therefore, necessarily $c = 0$. This proves Assertion (a).

4.1.2. **The proof of Assertion (b).** Given the linear combination

$$\mathfrak{P}_{411}+\mathfrak{P}_{422}+\mathfrak{P}_{433}=(a+f+p)(T_{44}-\mu),$$

and the assumption that $T_{44}\neq\mu$, it follows that $a+f+p=0$. Therefore, we set $p=-a-f$, and obtain

$$\begin{aligned}\mathfrak{P}_{421}-\mathfrak{P}_{412}&=2\left((a-f)T_{12}-b(T_{44}-\mu)\right),\\ \mathfrak{P}_{431}-\mathfrak{P}_{413}&=2\left((2a+f)T_{13}-c(T_{44}-\mu)\right),\\ \mathfrak{P}_{432}-\mathfrak{P}_{423}&=2\left((a+2f)T_{23}-h(T_{44}-\mu)\right).\end{aligned}$$

Since $p=-a-f$, assuming that a, f, p are different, we have $a-f\neq 0$, $2a+f\neq 0$ and $a+2f\neq 0$, so

$$T_{12}=\tfrac{b}{a-f}(T_{44}-\mu),\quad T_{13}=\tfrac{c}{2a+f}(T_{44}-\mu),\quad T_{23}=\tfrac{h}{a+2f}(T_{44}-\mu).$$

Since e_4 is an eigenvector of $\widehat{T}$, we set $T_{14}=T_{24}=T_{34}=0$. In order to show that $afp=0$, (or equivalently $af(a+f)=0$), we compute the polynomials

$$\begin{aligned}\mathfrak{P}_{411}&=\left(a-\tfrac{2b^2}{a-f}-\tfrac{2c^2}{2a+f}\right)(T_{44}-\mu),\\ \mathfrak{P}_{422}&=\left(f-\tfrac{2h^2}{a+2f}+\tfrac{2b^2}{a-f}\right)(T_{44}-\mu),\\ \mathfrak{P}_{433}&=\left((a+f)-\tfrac{2c^2}{2a+f}-\tfrac{2h^2}{a+2f}\right)(\mu-T_{44}),\\ \mathfrak{P}_{421}&=b(T_{11}-T_{22})-\tfrac{3ch(a+f)}{(2a+f)(a+2f)}(T_{44}-\mu),\\ \mathfrak{P}_{413}&=c(T_{11}-T_{33})+\tfrac{3bhf}{(a-f)(a+2f)}(T_{44}-\mu),\\ \mathfrak{P}_{432}&=h(T_{22}-T_{33})+\tfrac{3bca}{(a-f)(2a+f)}(T_{44}-\mu).\end{aligned}\tag{11}$$

We now split our analysis into two different cases depending on whether $bch=0$ or not. If $bch=0$, it immediately follows that $af(a+f)=0$. For instance, assume that $b=0$. Hence, $\mathfrak{P}_{411}=\mathfrak{P}_{422}=0$ in Equation (11) imply that either $ch\neq 0$ or $af=0$. Now, if $ch\neq 0$, $\mathfrak{P}_{421}=0$ in the same equation leads to $a+f=0$. Thus, in any case, $af(a+f)=0$. One proceeds similarly in the cases $c=0$ and $h=0$.

Next we analyse the case $bch\neq 0$. We assume that $af(a+f)\neq 0$ and argue for a contradiction. At this point, we consider Equation (10), which characterizes the vanishing of the divergence of T. A direct calculation shows that such equation reduces to

$$a(T_{11}-T_{33})+f(T_{22}-T_{33})=0.\tag{12}$$

Since $ch\neq 0$, we can isolate $T_{11}-T_{33}$ and $T_{22}-T_{33}$ in the equations given by $\mathfrak{P}_{413}=0$ and $\mathfrak{P}_{432}=0$ in (11), respectively. Replacing them in Equation (12) we get

$$abf(T_{44}-\mu)\left((a+2f)c^2+(2a+f)h^2\right)=0$$

and therefore $(a+2f)c^2+(2a+f)h^2=0$, which leads to

$$c^2=-\frac{(2a+f)h^2}{a+2f}.$$

Finally, using this last expression in $\mathfrak{P}_{433}$ in Equation (11) we obtain

$$\mathfrak{P}_{433}=-(a+f)(T_{44}-\mu),$$

which leads to a contradiction since the polynomial $\mathfrak{P}_{433}$ above does not vanish.

We have shown that $afp = 0$. Now, according to Remark 4.2, we can take $a = 0$ working, if necessary, with an isomorphically isometric metric. This proves Assertion (b). $\square$

The information provided by the previous lemma significantly simplifies the proof of the following classification result.

Theorem 4.4. *Let $\mathbb{R} \ltimes \mathbb{R}^3$ be not Einstein. Then it is a non-trivial algebraic T-soliton with soliton constant μ if and only if one of the following holds.*

(i) *The self-adjoint part of* $\operatorname{ad}(e_4)$ *has two equal eigenvalues. In this case, the space is isomorphically isometric to a Lie group given by $a = f \neq p$, and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ * & T_{22} & 0 & 0 \\ * & * & T_{33} & 0 \\ * & * & * & \mu \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle,$$

with

$$bT_{12} = 0, \quad b(T_{11} - T_{22}) = 0, \quad a(T_{11} + T_{22}) + pT_{33} - (2a + p)\mu = 0,$$
$$c(T_{11} - T_{33}) + hT_{12} = 0, \quad h(T_{22} - T_{33}) + cT_{12} = 0.$$

(ii) *The self-adjoint part of* $\operatorname{ad}(e_4)$ *has three different eigenvalues and the tensor field $\widehat{T}$ is diagonal,*

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{22}, T_{33}, \mu] \neq \mu \operatorname{Id},$$

with

$$b(T_{11} - T_{22}) = 0, \quad c(T_{11} - T_{33}) = 0, \quad h(T_{22} - T_{33}) = 0,$$
$$aT_{11} + fT_{22} + pT_{33} - (a + f + p)\mu = 0.$$

(iii) *The self-adjoint part of* $\operatorname{ad}(e_4)$ *has three different eigenvalues and the tensor field $\widehat{T}$ is not diagonal. In this case, the space is isomorphically isometric to a Lie group given by one of the following:*

(iii.a) *$a = 0$, $p = -f \neq 0$, $c = b$, $2b^2 - f^2 + h^2 = 0$, and*

$$T = \begin{pmatrix} T_{11} & -\frac{b}{f}(T_{44} - \mu) & \frac{b}{f}(T_{44} - \mu) & 0 \\ * & T_{22} & \frac{h}{2f}(T_{44} - \mu) & 0 \\ * & * & T_{22} & 0 \\ * & * & * & T_{44} \end{pmatrix},$$

with

$$T_{44} \neq \mu, \quad b\left(2f(T_{11} - T_{22}) - 3h(T_{44} - \mu)\right) = 0.$$

(iii.b) *$a = 0$, $p = -f \neq 0$, $b = c = 0$, $h = f$ and*

$$T = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ * & T_{22} & \frac{1}{2}(T_{44} - \mu) & T_{24} \\ * & * & T_{22} & -T_{24} \\ * & * & * & T_{44} \end{pmatrix},$$

with $T_{24} \neq 0$.

Remark 4.5. Following the notation in [1], the underlying Lie algebra in Theorem 4.4-(iii.a) is isomorphic to $\mathfrak{n}_4$, and that in Theorem 4.4-(iii.b) is isomorphic to $\mathbb{R}\times\mathfrak{h}_3$ and the metric is the product one. Hence both situations also appear as semi-direct extensions of the Heisenberg group in Theorem 6.3.

Proof. We consider the self-adjoint part of $-\operatorname{ad}(e_4)$, given by $\operatorname{diag}[a,f,p]$, and analyse separately the case of two-equal eigenvalues and that of three-distinct eigenvalues. Recall that the metric is Einstein if $a=f=p$ (see Remark 4.1), and if two of the structure constants a, f, p are equal, we may assume $a=f\neq p$ just working, if necessary, with an isomorphically isometric metric (see Remark 4.2).

4.1.3. **Case $\boldsymbol{a=f\neq p}$.** By Lemma 4.3–(a), we know that $T_{14}=T_{24}=T_{34}=0$. If we now consider the linear combination

$$\mathfrak{P}_{411}+\mathfrak{P}_{422}+\mathfrak{P}_{433}=(a+f+p)(T_{44}-\mu), \tag{13}$$

we see that there are two different possibilities depending on whether $T_{44}=\mu$ or $a+f+p=0$. In what follows we study these two cases separately.

4.1.3.1. *Case $T_{44}=\mu$*. Considering the linear combinations

$$\mathfrak{P}_{431}-\mathfrak{P}_{413}=2(a-p)T_{13},\quad \mathfrak{P}_{432}-\mathfrak{P}_{423}=2(a-p)T_{23},$$

one has that

$$T_{13}=T_{23}=0.$$

Now, a direct calculation shows that the system of polynomial equations $\{\mathfrak{P}_{ijk}=0\}$ is determined by

$$\begin{aligned}&\mathfrak{P}_{422}=2bT_{12}=0, && \mathfrak{P}_{421}=b(T_{11}-T_{22})=0,\\ &\mathfrak{P}_{413}=c(T_{11}-T_{33})+hT_{12}=0, && \mathfrak{P}_{423}=h(T_{22}-T_{33})+cT_{12}=0,\end{aligned}$$

while the vanishing of the divergence of the tensor T given by Equation (10) reduces to

$$a(T_{11}+T_{22})+pT_{33}-(2a+p)\mu=0.$$

This proves Assertion (i).

4.1.3.2. *Case $T_{44}\neq\mu$ and $a+f+p=0$*. In this case we set $f=a$ and $p=-2a$, with $a\neq0$. A direct calculation now shows that

$$\begin{aligned}\mathfrak{P}_{421}-\mathfrak{P}_{412}&=-2b(T_{44}-\mu),\\ \mathfrak{P}_{431}-\mathfrak{P}_{413}&=\ \ 6aT_{13}-2c(T_{44}-\mu),\\ \mathfrak{P}_{432}-\mathfrak{P}_{423}&=\ \ 6aT_{23}-2h(T_{44}-\mu),\end{aligned}$$

and the conditions $T_{44}\neq\mu$ and $a\neq0$ imply that

$$b=0,\quad T_{13}=\tfrac{c}{3a}(T_{44}-\mu),\quad T_{23}=\tfrac{h}{3a}(T_{44}-\mu).$$

Now, the polynomials $\mathfrak{P}_{421}$, $\mathfrak{P}_{411}$ and $\mathfrak{P}_{433}$ reduce to

$$\begin{aligned}\mathfrak{P}_{421}&=-\tfrac{2}{3a}ch(T_{44}-\mu),\\ \mathfrak{P}_{411}&=\ \ \tfrac{1}{3a}(3a^2-2c^2)(T_{44}-\mu),\\ \mathfrak{P}_{433}&=-\tfrac{2}{3a}(3a^2-c^2-h^2)(T_{44}-\mu),\end{aligned}$$

which imply

$$ch=0,\quad 3a^2-2c^2=0,\quad 3a^2-c^2-h^2=0.$$

Since $a\neq0$ these equations are incompatible, so we conclude that there are no algebraic T-solitons in this case.

4.1.4. **Case $\boldsymbol{a \neq f \neq p}$ and $\boldsymbol{a \neq p}$.** If a, f and p are all distinct we distinguish two possibilities depending on whether e_4 is an eigenvector of $\widehat{T}$, i.e., T_{14}, T_{24} and T_{34} are all zero or not. In the former case we analyse the two cases given by $T_{44} = \mu$ and $T_{44} \neq \mu$, separately.

4.1.4.1. *Case* $T_{14} = T_{24} = T_{34} = 0$ *and* $T_{44} = \mu$. Since $T_{44} = \mu$, one has that

$$\begin{aligned} \mathfrak{P}_{421} - \mathfrak{P}_{412} &= 2(a-f)T_{12}, \\ \mathfrak{P}_{431} - \mathfrak{P}_{413} &= 2(a-p)T_{13}, \\ \mathfrak{P}_{432} - \mathfrak{P}_{423} &= 2(f-p)T_{23}, \end{aligned}$$

and so

$$T_{12} = T_{13} = T_{23} = 0.$$

Applying these relations, a direct calculation shows that the tensor $\widehat{T}$ is diagonal,

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{22}, T_{33}, \mu],$$

and the vanishing of the divergence of T (see Equation (10)) is given by

$$aT_{11} + fT_{22} + pT_{33} - (a+f+p)\mu = 0.$$

The system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ is now determined by

$$\begin{aligned} \mathfrak{P}_{412} &= b(T_{11} - T_{22}) = 0, \\ \mathfrak{P}_{413} &= c(T_{11} - T_{33}) = 0, \\ \mathfrak{P}_{423} &= h(T_{22} - T_{33}) = 0. \end{aligned}$$

This proves Assertion (ii).

4.1.4.2. *Case* $T_{14} = T_{24} = T_{34} = 0$, $T_{44} \neq \mu$. Lemma 4.3–(b) allows us to consider $a = 0$ and $p = -f \neq 0$ and work, if necessary, with an isomorphically isometric metric. Now, we consider the following linear combinations

$$\begin{aligned} \mathfrak{P}_{421} - \mathfrak{P}_{412} &= -2\left(fT_{12} + b(T_{44} - \mu)\right), \\ \mathfrak{P}_{431} - \mathfrak{P}_{413} &= 2\left(fT_{13} - c(T_{44} - \mu)\right), \\ \mathfrak{P}_{432} - \mathfrak{P}_{423} &= 2\left(2fT_{23} - h(T_{44} - \mu)\right), \end{aligned}$$

to obtain

$$T_{12} = -\tfrac{b}{f}(T_{44} - \mu), \quad T_{13} = \tfrac{c}{f}(T_{44} - \mu), \quad T_{23} = \tfrac{h}{2f}(T_{44} - \mu).$$

Using these expressions, the condition $\operatorname{div} T = 0$ given by Equation (10) reduces to

$$f(T_{22} - T_{33}) = 0,$$

and

$$\mathfrak{P}_{411} = \tfrac{2}{f}\left(b^2 - c^2\right)(T_{44} - \mu).$$

As a consequence

$$T_{22} = T_{33}, \quad c = \varepsilon b, \quad \text{with } \varepsilon^2 = 1,$$

and a direct calculation shows that the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\begin{aligned} \mathfrak{P}_{412} = \varepsilon\,\mathfrak{P}_{413} = \mathfrak{P}_{421} = \varepsilon\,\mathfrak{P}_{431} &= \tfrac{1}{2f} b\left(2f(T_{11} - T_{22}) - 3\varepsilon h(T_{44} - \mu)\right) = 0, \\ \mathfrak{P}_{422} = -\mathfrak{P}_{433} &= -\tfrac{1}{f}\left(2b^2 - f^2 + h^2\right)(T_{44} - \mu) = 0. \end{aligned}$$

Now, the associated left-invariant metric is given by

$$\begin{aligned}
[e_1, e_4] &= be_2 + \varepsilon be_3,\\
[e_2, e_4] &= -be_1 + fe_2 + he_3,\\
[e_3, e_4] &= -\varepsilon be_1 - he_2 - fe_3,
\end{aligned}$$

and the isometry $e_3 \mapsto -e_3$ interchanges (ε, f, b, h) and $(-\varepsilon, f, b, -h)$, so we can take $\varepsilon = 1$ and work with an isomorphically isometric metric if necessary. Thus Assertion (iii.a) is obtained.

4.1.4.3. *Some of T_{14}, T_{24} and T_{34} does not vanish.* According to Lemma 4.3–(a) we can assume that $a = b = c = 0$ and work, if necessary, with an isomorphically isometric metric. In this situation

$$\mathfrak{P}_{212} = -fT_{14}, \quad \mathfrak{P}_{313} = -pT_{14}$$

and, since $f \neq p$, it follows that

$$T_{14} = 0. \tag{14}$$

Now, we compute

$$\begin{aligned}
\mathfrak{P}_{322} &= hT_{24} + fT_{34}, & \mathfrak{P}_{323} &= -pT_{24} + hT_{34},\\
\mathfrak{P}_{424} &= -fT_{24} - hT_{34}, & \mathfrak{P}_{434} &= hT_{24} - pT_{34}.
\end{aligned}$$

Since T_{24} and T_{34} cannot vanish simultaneously, we have $f^2 = p^2 = h^2$. Moreover, $f \neq p$ and $fp \neq 0$ lead to

$$p = -f \neq 0, \quad h = \varepsilon f, \quad T_{34} = -\varepsilon T_{24} \neq 0, \tag{15}$$

where $\varepsilon^2 = 1$.

Using Equations (14) and (15), the condition $\operatorname{div} T = 0$ reduces to

$$f(T_{22} - T_{33}) = 0,$$

and

$$\mathfrak{P}_{412} = f(T_{12} - \varepsilon T_{13}), \quad \mathfrak{P}_{421} = -f(T_{12} + \varepsilon T_{13}), \quad \mathfrak{P}_{433} = f(2\varepsilon T_{23} - T_{44} + \mu).$$

Therefore,

$$T_{22} = T_{33}, \quad T_{12} = T_{13} = 0, \quad T_{23} = \tfrac{\varepsilon}{2}(T_{44} - \mu), \tag{16}$$

and we obtain an algebraic T-soliton with associated left-invariant metric given by

$$[e_2, e_4] = fe_2 + \varepsilon fe_3, \quad [e_3, e_4] = -\varepsilon fe_2 - fe_3.$$

Note that the isometry $e_3 \mapsto -e_3$ interchanges ε and $-\varepsilon$. Thus, working with an isomorphically isometric metric if necessary, we can take $\varepsilon = 1$. Now Equations (14), (15) and (16) lead to the proof of Assertion (iii.b). □

Remark 4.6. Following Theorem 4.4, non-trivial algebraic Ricci solitons on $\mathbb{R} \ltimes \mathbb{R}^3$ occur in the following situations:

(i) If the self-adjoint part of the derivation determining $\mathbb{R} \ltimes \mathbb{R}^3$ has two equal eigenvalues, then the metric is determined by

$$[e_1, e_4] = ae_1 + be_2, \ [e_2, e_4] = -be_1 + ae_2, \ [e_3, e_4] = pe_3, \ \text{with } \mu = -2a^2 - p^2,$$

where $a \neq p$. If $a = 0$ or $p = 0$ then the space is locally symmetric and homothetic to $\mathbb{H}^2 \times \mathbb{R}^2$ or $\mathbb{H}^3 \times \mathbb{R}$, respectively.

(ii) If the self-adjoint part of the derivation has three-distinct eigenvalues as in Theorem 4.4-(ii), then its skew-symmetric part vanishes and the metric is determined by

$$[e_1, e_4] = e_1,\ [e_2, e_4] = fe_2,\ [e_3, e_4] = pe_3, \quad \text{with } \mu = -(f^2 + p^2 + 1),$$

where $\{(f, p) \in \mathbb{R}^2;\ -1 \le f < p < 1\} \setminus \{(-1, p);\ -1 < p < 0\}$.

(iii) If $\mathbb{R} \ltimes \mathbb{R}^3$ is two-step nilpotent or three-step nilpotent as in case (iii.a) of Theorem 4.4, then the metric is determined by

(iii.a.1) $[e_2, e_4] = e_2 + e_3,\ [e_3, e_4] = -e_2 - e_3$, with $\mu = -6$, or

(iii.a.2) $[e_1, e_4] = \frac{1}{\sqrt{2}}(e_2 + e_3)\quad [e_2, e_4] = -\frac{1}{\sqrt{2}}e_1 + e_2\quad [e_3, e_4] = -\frac{1}{\sqrt{2}}e_1 - e_3$ with $\mu = -3$.

We determined in Theorem 4.4 the algebraic T-solitons in the non-Einstein case. Even though the Einstein case can usually be handled directly when dealing with particular tensor fields as in the case of our analysis, we consider this situation in the following result for the sake of completeness.

Theorem 4.7. *Let $\mathbb{R} \ltimes \mathbb{R}^3$ be Einstein. Then it is a non-trivial algebraic T-soliton with soliton constant μ if and only if it is isomorphically isometric to a metric with $a = f = p$ and*

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & 0 \\ * & T_{22} & T_{23} & 0 \\ * & * & T_{33} & 0 \\ * & * & * & \mu \end{pmatrix} \ne \mu \langle \cdot, \cdot \rangle,$$

with

$$\begin{aligned} &a(T_{11} + T_{22} + T_{33} - 3\mu) = 0, && b(T_{11} - T_{22}) - hT_{13} - cT_{23} = 0, \\ &bT_{12} + cT_{13} = 0, && c(T_{11} - T_{33}) + hT_{12} - bT_{23} = 0, \\ &cT_{13} + hT_{23} = 0, && h(T_{22} - T_{33}) + cT_{12} + bT_{13} = 0. \end{aligned}$$

Proof. Recall that $\mathbb{R} \ltimes \mathbb{R}^3$ is Einstein if and only if $a = f = p$ (see Remark 4.1) so, according to Lemma 4.3–(a), $T_{14} = T_{24} = T_{34} = 0$. The combination

$$\mathfrak{P}_{411} + \mathfrak{P}_{422} + \mathfrak{P}_{433} = 3a(T_{44} - \mu)$$

shows that either $T_{44} = \mu$ or $a = 0$.

If $T_{44} = \mu$, then the condition $\operatorname{div} T = 0$ given by Equation (10) reduces to

$$a(T_{11} + T_{22} + T_{33} - 3\mu) = 0,$$

and a direct calculation shows that the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ is determined by

$$\begin{aligned} \mathfrak{P}_{411} &= -2(bT_{12} + cT_{13}) = 0, \\ \mathfrak{P}_{433} &= 2(cT_{13} + hT_{23}) = 0, \\ \mathfrak{P}_{412} &= b(T_{11} - T_{22}) - hT_{13} - cT_{23} = 0, \\ \mathfrak{P}_{413} &= c(T_{11} - T_{33}) + hT_{12} - bT_{23} = 0, \\ \mathfrak{P}_{423} &= h(T_{22} - T_{33}) + cT_{12} + bT_{13} = 0. \end{aligned}$$

Assume now that $T_{44} \ne \mu$ and $a = 0$. Since either b, c or h must be non-zero, Remark 4.2 implies that we can assume that $b \ne 0$ (working, if necessary, with an isomorphically isometric metric). Now, a direct calculation shows that

$$\mathfrak{P}_{412} - \mathfrak{P}_{421} = 2b(T_{44} - \mu),$$

which implies that this situation cannot occur, thus completing the proof. ☐

4.2. **Algebraic $\mathfrak{F}[t]$-solitons on $\mathbb{R}\ltimes\mathbb{R}^3$.** In addition to the Ricci solitons described in Remark 4.6, the algebraic $\mathfrak{F}[t]$-solitons on $\mathbb{R}\ltimes\mathbb{R}^3$ are given as follows.

Theorem 4.8. *A left-invariant metric on $\mathbb{R}\ltimes\mathbb{R}^3$ is a strict algebraic $\mathfrak{F}[t]$-soliton if and only if it is isomorphically homothetic to one of the following:*

(1) $[e_1,e_4]=e_1$, $[e_2,e_4]=-2e_2-\kappa e_3$, $[e_3,e_4]=\kappa e_2+e_3$, *with* $\kappa\in(0,+\infty)\setminus\{\frac{3}{\sqrt{2}}\}$. *In this case,* $\mu=\frac{9}{2}\left(2\kappa^2-9\right)$ *and* $t=-\frac{4\kappa^2+21}{12}$.

(2) $[e_1,e_4]=e_1$, $[e_2,e_4]=-\frac{4\kappa^2+3}{6}e_2-\kappa e_3$, $[e_3,e_4]=\kappa e_2+e_3$, *with* $\kappa\in(0,+\infty)\setminus\left\{\frac{3}{2}\right\}$. *In this case, the soliton constant is given by* $\mu=-\frac{1}{36}\kappa^2(4\kappa^2+9)^2$ *and* $t=-\frac{16\kappa^4+168\kappa^2+81}{2(16\kappa^4-24\kappa^2+81)}$.

(3) $[e_1,e_4]=ae_1$, $[e_2,e_4]=(1-\frac{a}{2})e_2+he_3$, $[e_3,e_4]=-he_2-(1+\frac{a}{2})e_3$, *with* $a\in[0,+\infty)\setminus\{\frac{2}{3}\}$ *and* $h\in(0,+\infty)$. *Moreover the soliton constant* $\mu=-\frac{3}{2}\left(3a^2(h^2+1)-4(h^2-1)\right)$ *is not null, and* $t=-\frac{3a^2+4h^2+8}{3a^2+4}$.

(4) $[e_1,e_4]=ae_1$, $[e_2,e_4]=fe_2+he_3$, $[e_3,e_4]=-he_2+(1-a-f)e_3$, *with* $a\in\mathbb{R}\setminus\{\frac{1}{3}\}$ *and* $h>0$ *satisfying* $9a^2-6a-4h^2\neq 0$. *For* a *and* h *fixed, the parameter* $f=\frac{1}{2}\left(1-a+\sqrt{4h^2+1}\right)$. *In this case,* $\mu=-(3a-1)^2h^2$ *and* $t=-\frac{3a^2-2a+12h^2+2}{3a^2-2a+4h^2+4}$.

(5) $[e_1,e_4]=e_1+ce_3$, $[e_2,e_4]=-(p+1)e_2+he_3$, $[e_3,e_4]=-ce_1-he_2+pe_3$, *with* $p\in(-2,0]\setminus\{-\frac{1}{2}\}$ *and* $c>0$ *satisfying that* $(2p+1)(2p^2+p-c^2)<0$. *The soliton constant*

$$\mu=-\tfrac{3}{2(2p+1)}\left((2p+1)\left(8p^3+15p^2+3p+1\right)+c^2(p+2)\left(5p^2-p-1\right)\right)$$

is different from zero and for any c *and* p *fixed, the parameter* h *is the only positive solution of the equation* $(2p+1)h^2-(p-1)\left(2p^2+p-c^2\right)=0$. *In this case, the parameter* $t=-\frac{(2p+1)(12p^2+4p+5)+(p+2)c^2}{4(2p+1)(p^2+p+1)}$.

(6) $[e_1,e_4]=e_2+e_3$, $[e_2,e_4]=-e_1+fe_2+he_3$, $[e_3,e_4]=-e_1-he_2-fe_3$, *with* $h\neq 0$. *For* h *fixed, the parameter* f *is the only positive solution of* $f^2=h^2+2$. *In this case,* $\mu=-30(h^2+2)$ *and* $t=-3$.

Remark 4.9. Metrics in Theorem 4.8-(4) with $a=0$, in which case $t\in(-3,-\frac{1}{2})$, given by

$$[e_2,e_4]=fe_2+he_3,\quad [e_3,e_4]=-he_2+(1-f)e_3,\quad \text{with } f=\frac{1}{2}\left(1+\sqrt{4h^2+1}\right),$$

correspond to product metrics on $\mathbb{R}\times(\mathbb{R}\ltimes_\varphi\mathbb{R}^2)$ where $\mathbb{R}\ltimes_\varphi\mathbb{R}^2$ is a non-unimodular Lie group determined by a derivation $\varphi=\operatorname{ad}(e_4)$ of the two-dimensional Abelian Lie algebra with $\det(\varphi)=0$. It follows from the work in [28] (see also [8]) that $\mathbb{R}\ltimes_\varphi\mathbb{R}^2$ is homothetic (although not isomorphically homothetic) to a left-invariant metric on $\widetilde{SL}(2,\mathbb{R})$. Hence, although the algebraic $\mathfrak{F}[t]$-soliton in Theorem 4.8-(4) for $a=0$ and the algebraic $\mathfrak{F}[t]$-soliton corresponding to Theorem 3.4-(1) for $4h^2\lambda=-1$ are not isomorphic, the induced $\mathfrak{F}[t]$-solitons are equivalent at the Riemannian level.

Remark 4.10. The Lie algebra structure in Theorem 4.8-(6) is determined by a three-step nilpotent derivation of $\mathbb{R}^3$. Hence it is isomorphic to the Lie algebra $\mathfrak{n}_4$ determined by $[e_4, e_1] = e_2$, $[e_4, e_2] = e_3$, which is also a semi-direct extension of the Heisenberg Lie algebra.

Remark 4.11. The range of the parameter t in each family of Theorem 4.8 is indicated in Figure 3 below.

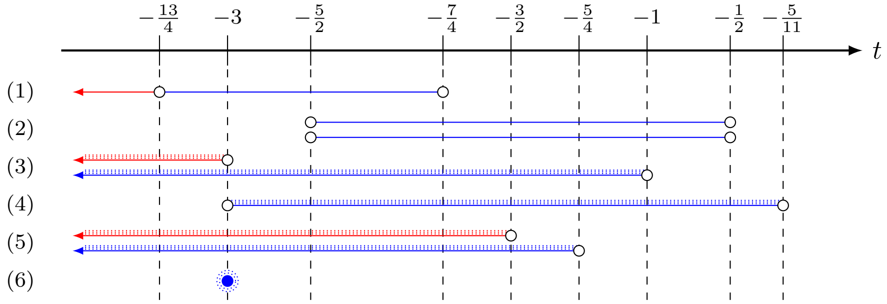

FIGURE 3. Range of the parameter t for strict algebraic $\mathfrak{F}[t]$-solitons on $\mathbb{R} \ltimes \mathbb{R}^3$.

Proceeding as in Remark 3.5, the homothetic invariants $\{t, \|\rho\|^2\,\tau^{-2}, \|R\|^2\,\tau^{-2}, \|\nabla\rho\|^2\,\tau^{-3}, \|\nabla R\|^2\tau^{-3}\}$ show that there are no homotheties between metrics in different families of Theorem 4.8 and, moreover, that different values of the parameters in any of the families give rise to different homothetic classes.

Besides, for each value of $t \in (-\infty, -\frac{7}{4}) \setminus \{-\frac{13}{4}\}$ there is a single strict algebraic $\mathfrak{F}[t]$-soliton (up to homotheties) for Family (1) in Figure 3. In Family (2) there are two-distinct homothety classes for any value of $t \in (-\frac{5}{2}, -\frac{1}{2})$. Finally, for each admissible value of $t \in (-\infty, -\frac{5}{11})$ there is an infinite number of non-homothetic strict algebraic $\mathfrak{F}[t]$-solitons in each Family (3)-(6).

Proof of Theorem 4.8. First of all note that a strict algebraic $\mathfrak{F}[t]$-soliton on $\mathbb{R} \ltimes \mathbb{R}^3$ must be non-Einstein. Hence, in what follows we will analyse the four cases obtained in Theorem 4.4 separately. Moreover, a direct calculation shows that, with no further assumptions,

$$\mathfrak{F}[t]_{14} = \mathfrak{F}[t]_{24} = \mathfrak{F}[t]_{34} = 0,$$

so Case (iii.b) in Theorem 4.4 is not possible. Next we examine the remaining possibilities.

4.2.1. **Case (i) in Theorem 4.4.** Take $a = f \neq p$. The conditions that give an algebraic $\mathfrak{F}[t]$-soliton are determined by the vanishing of the polynomials

$$\begin{aligned} &\mathfrak{Q}_1 = \mathfrak{F}[t]_{13}, \quad \mathfrak{Q}_2 = \mathfrak{F}[t]_{23}, \quad \mathfrak{Q}_3 = b\,\mathfrak{F}[t]_{12}, \quad \mathfrak{Q}_4 = b\,(\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{22})\,, \\ &\mathfrak{Q}_5 = c\,(\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{33}) + h\,\mathfrak{F}[t]_{12}, \quad \mathfrak{Q}_6 = h\,(\mathfrak{F}[t]_{22} - \mathfrak{F}[t]_{33}) + c\,\mathfrak{F}[t]_{12}, \\ &\mathfrak{Q}_7 = \mathfrak{F}[t]_{44} - \mu, \quad \mathfrak{Q}_8 = a\,(\mathfrak{F}[t]_{11} + \mathfrak{F}[t]_{22}) + p\,\mathfrak{F}[t]_{33} - (2a+p)\mu, \end{aligned} \tag{17}$$

and a straightforward calculation shows that

$$\mathfrak{Q}_1 = (p-a)\big(4c(3a^2+p^2+2ap)t + c(a^2+2p^2-6ap+b^2+4(c^2+h^2)) - 2bh(2a+p)\big)\,,$$
$$\mathfrak{Q}_2 = (p-a)\big(4h(3a^2+p^2+2ap)t + h(a^2+2p^2-6ap+b^2+4(c^2+h^2)) + 2bc(2a+p)\big)\,,$$

$$\begin{aligned}
\mathfrak{Q}_3 &= b(p-a)\left(3b(c^2-h^2)-4ch(2a+p)\right),\\
\mathfrak{Q}_4 &= 4b(a-p)\left((c^2-h^2)(2a+p)+3bch\right),\\
\mathfrak{Q}_5 &= (a-p)\left(4c(2a+p)(3a^2+p^2+2ap)t\right.\\
&\qquad\left.+2c(2a+p)(2a^2+p^2+4(c^2+h^2))+3bh(c^2+h^2)\right),\\
\mathfrak{Q}_6 &= (a-p)\left(4h(2a+p)(3a^2+2ap+p^2)t\right.\\
&\qquad\left.+2h(2a+p)(2a^2+p^2+4(c^2+h^2))-3bc(c^2+h^2)\right),\\
\mathfrak{Q}_7 &= (a-p)^2\left(2(3a^2+p^2+2ap)t+2a^2+p^2+3(c^2+h^2)\right)-\mu,\\
\mathfrak{Q}_8 &= (2a+p)\mathfrak{Q}_7.
\end{aligned}$$

Considering now the combination

$$h\mathfrak{Q}_1 - c\mathfrak{Q}_2 = 2b\left(c^2+h^2\right)(2a+p)(a-p),$$

it gives rise to different possibilities depending on the vanishing of each factor (note that $a-p\neq 0$).

4.2.1.1. Case $c=h=0$. In this case the left-invariant metric, given by

$$[e_1,e_4]=ae_1+be_2,\quad [e_2,e_4]=-be_1+ae_2,\quad [e_3,e_4]=pe_3,$$

is an algebraic Ricci soliton with soliton constant $-2a^2-p^2$ (see Remark 4.6).

4.2.1.2. Case $p=-2a$ and $c^2+h^2\neq 0$. Since $p=-2a$ and $a\neq p$, then necessarily $a\neq 0$. Computing the polynomials $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ in Equation (17) we get

$$\begin{aligned}
\mathfrak{Q}_1 &= -3ac\left(12a^2t+21a^2+b^2+4(c^2+h^2)\right),\\
\mathfrak{Q}_2 &= -3ah\left(12a^2t+21a^2+b^2+4(c^2+h^2)\right),
\end{aligned}$$

so it follows that $12a^2t+21a^2+b^2+4(c^2+h^2)=0$ and therefore

$$t=-\frac{21a^2+b^2+4(c^2+h^2)}{12a^2}. \tag{18}$$

At this point we have

$$\mathfrak{Q}_5 = 9abh(c^2+h^2),\qquad \mathfrak{Q}_6=-9abc(c^2+h^2),$$

so $b=0$ and a direct calculation shows that the polynomial system $\{\mathfrak{Q}_i=0\}$ determined by Equation (17) reduces to

$$2\mathfrak{Q}_7 = -9a^2\left(9a^2-2(c^2+h^2)\right)-2\mu=0.$$

Hence the left-invariant metric, determined by

$$[e_1,e_4]=ae_1+ce_3,\quad [e_2,e_4]=ae_2+he_3,\quad [e_3,e_4]=-ce_1-he_2-2ae_3,$$

is an algebraic $\mathfrak{F}[t]$-soliton with soliton constant $\mu=-\frac{9}{2}a^2\left(9a^2-2(c^2+h^2)\right)$ for the parameter $t=-\frac{21a^2+4(c^2+h^2)}{12a^2}$ given by (18). Moreover these metrics are not algebraic Ricci solitons, and they are critical for the value of t determined by $\mu=0$, which corresponds to $t=-\frac{13}{4}$ (see [6]). Since $a\neq 0$, we may take $a=1$ and work with an isomorphically isometric metric. Besides, we set $\kappa^2=c^2+h^2$, so that $\kappa\notin\{0,\pm\frac{3}{\sqrt{2}}\}$, and consider the orthonormal basis

$$\bar e_1=-\tfrac{1}{\kappa}(he_1-ce_2),\quad \bar e_2=e_3,\quad \bar e_3=\tfrac{1}{\kappa}(ce_1+he_2),\quad \bar e_4=e_4,$$

whose non-zero brackets are given by

$$[\bar e_1,\bar e_4]=\bar e_1,\quad [\bar e_2,\bar e_4]=-2\bar e_2-\kappa\bar e_3,\quad [\bar e_3,\bar e_4]=\kappa\bar e_2+\bar e_3.$$

Now, the isomorphic isometry $\bar{e}_2 \mapsto -\bar{e}_2$ interchanges the sign of κ, so we may assume $\kappa > 0$, $\kappa \neq \frac{3}{\sqrt{2}}$. In this setting, the soliton constant $\mu = \frac{9}{2}(2\kappa^2 - 9)$ and the parameter $t = -\frac{4\kappa^2+21}{12} \in (-\infty, -\frac{7}{4})\backslash\{-\frac{13}{4}\}$ determine strict algebraic $\mathfrak{F}[t]$-solitons, which are shrinking when $t \in (-\infty, -\frac{13}{4})$ and expanding when $t \in (-\frac{13}{4}, -\frac{7}{4})$. These metrics correspond to Family (1).

4.2.1.3. *Case $b = 0$, $p \neq -2a$ and $c^2 + h^2 \neq 0$.* In this case, the polynomials $\mathfrak{Q}_5$ and $\mathfrak{Q}_6$ in Equation (17) transform into

$$\begin{aligned}
\mathfrak{Q}_5 &= 2c(a-p)(2a+p)\left(2(3a^2+p^2+2ap)t + 2a^2 + p^2 + 4(c^2+h^2)\right), \\
\mathfrak{Q}_6 &= 2h(a-p)(2a+p)\left(2(3a^2+p^2+2ap)t + 2a^2 + p^2 + 4(c^2+h^2)\right),
\end{aligned}$$

and since $p \neq a$ it follows that $2(3a^2+p^2+2ap)t + 2a^2 + p^2 + 4(c^2+h^2) = 0$. Note that the coefficient of t vanishes if and only if $a = p = 0$ which is not possible. Thus, we get

$$t = -\frac{2a^2 + p^2 + 4(c^2+h^2)}{2(3a^2+p^2+2ap)}.$$

Now, the two conditions

$$\begin{aligned}
\mathfrak{Q}_1 &= c(a-p)\left(3a^2 + 6ap + 4(c^2+h^2)\right), \\
\mathfrak{Q}_2 &= h(a-p)\left(3a^2 + 6ap + 4(c^2+h^2)\right),
\end{aligned}$$

imply that $3a^2 + 6ap + 4(c^2+h^2) = 0$. In this situation, a cannot vanish – this would lead to $c = h = 0$, which is not possible in this case – and so we obtain

$$p = -\frac{3a^2 + 4(c^2+h^2)}{6a}.$$

Thus, $p \neq a$ is satisfied while the condition $p \neq -2a$ means $9a^2 - 4(c^2+h^2) \neq 0$. At this point, the polynomial system $\{\mathfrak{Q}_i = 0\}$ given by Equation (17) is determined by

$$-36a^2\mathfrak{Q}_7 = \left(9a^2 + 4(c^2+h^2)\right)^2\left(c^2+h^2\right) + 36a^2\mu,$$

from where we determine the soliton constant μ.

Therefore, the left-invariant metric

$$[e_1, e_4] = ae_1 + ce_3, \quad [e_2, e_4] = ae_2 + he_3, \quad [e_3, e_4] = -ce_1 - he_2 - \tfrac{3a^2+4(c^2+h^2)}{6a}e_3,$$

is an algebraic $\mathfrak{F}[t]$-soliton for the soliton constant $\mu = -\frac{\left(9a^2+4(c^2+h^2)\right)^2\left(c^2+h^2\right)}{36a^2}$ and the parameter $t = -\frac{16(c^2+h^2)^2+168a^2(c^2+h^2)+81a^4}{32(c^2+h^2)^2-48a^2(c^2+h^2)+162a^4}$. Moreover, a direct calculation shows that it is strict. As in the previous case, we work at the homothetical level to take $a = 1$ and set $\kappa^2 = c^2 + h^2$. Since c or h is non-zero and $9a^2 - 4(c^2+h^2) \neq 0$, we have $\kappa \notin \{0, \pm\frac{3}{2}\}$. Taking the same orthonormal basis as in the previous case, the non-zero brackets are given by

$$[\bar{e}_1, \bar{e}_4] = \bar{e}_1, \quad [\bar{e}_2, \bar{e}_4] = -\tfrac{4\kappa^2+3}{6}\bar{e}_2 - \kappa\bar{e}_3, \quad [\bar{e}_3, \bar{e}_4] = \kappa\bar{e}_2 + \bar{e}_3.$$

The isomorphic isometry $\bar{e}_2 \mapsto -\bar{e}_2$ interchanges the sign of κ, so we may assume $\kappa > 0$, $\kappa \neq \frac{3}{2}$. In this setting, we have $\mu = -\frac{1}{36}\kappa^2(4\kappa^2+9)^2$ and the parameter $t = -\frac{16\kappa^4+168\kappa^2+81}{2(16\kappa^4-24\kappa^2+81)} \in (-\frac{5}{2}, -\frac{1}{2})$. Clearly the strict algebraic $\mathfrak{F}[t]$-solitons are always expanding and they correspond to Family (2).

4.2.2. **Case (ii) in Theorem 4.4.** Take $a \neq f \neq p$, $a \neq p$ and assume that the operator $\widehat{\mathfrak{F}[t]}$ is diagonal. In this case, the algebraic $\mathfrak{F}[t]$-solitons are determined by the vanishing of the polynomials

$$\begin{aligned}
&\mathfrak{Q}_1 = \mathfrak{F}[t]_{12}, \quad \mathfrak{Q}_2 = \mathfrak{F}[t]_{13}, \quad \mathfrak{Q}_3 = \mathfrak{F}[t]_{23},\\
&\mathfrak{Q}_4 = b(\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{22}), \quad \mathfrak{Q}_5 = c(\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{33}),\\
&\mathfrak{Q}_6 = h(\mathfrak{F}[t]_{22} - \mathfrak{F}[t]_{33}), \quad \mathfrak{Q}_7 = \mathfrak{F}[t]_{44} - \mu,\\
&\mathfrak{Q}_8 = a\mathfrak{F}[t]_{11} + f\mathfrak{F}[t]_{22} + p\mathfrak{F}[t]_{33} - (a+f+p)\mu.
\end{aligned} \tag{19}$$

Next we split the analysis in several cases, depending on whether bch vanishes or not. According to Remark 4.2, it suffices to consider the case $b = c = h = 0$, the case $b = c = 0$, $h \neq 0$ and the case $ch \neq 0$.

4.2.2.1. *Case $b = c = h = 0$*. In this case the left-invariant metric, which is given by

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = fe_2, \quad [e_3, e_4] = pe_3,$$

is an algebraic Ricci soliton with soliton constant $\mu = -a^2 - f^2 - p^2$ (see Remark 4.6).

4.2.2.2. *Case $b = c = 0$ and $h \neq 0$*. We start by considering the polynomials $\mathfrak{Q}_3$ and $\mathfrak{Q}_6$ in Equation (19). We have

$$\mathfrak{Q}_3 = h(p-f)\left(4(a^2+f^2+p^2+af+ap+fp)t + a^2 + 2(f-p)^2 - 2a(f+p) + 4h^2\right).$$

Since $h(p-f) \neq 0$ and, moreover, the coefficient of t in the last factor vanishes only if $a = f = p = 0$, which cannot occur, we obtain

$$t = -\frac{a^2 + 2(f-p)^2 - 2a(f+p) + 4h^2}{4(a^2+f^2+p^2+af+ap+fp)}. \tag{20}$$

Now, using this expression, we compute

$$\mathfrak{Q}_6 = h(f-p)(a+f+p)\left((a+2f)(a+2p) + 4h^2\right), \tag{21}$$

which leads to two different cases depending on whether $a + f + p$ vanishes or not.

(i) If $a + f + p = 0$, we set $p = -a - f$ and Equation (19) reduces to

$$-2\mathfrak{Q}_7 = 3(a+2f)^2(a^2+f^2+af) + 6(a^2 - 2f^2 - 2af)h^2 + 2\mu.$$

At this point we introduce a new variable, $\kappa = \frac{a}{2} + f$. Since a, f and p are all different, we have $\kappa \notin \{0, \pm\frac{3}{2}a\}$. Now, $\mathfrak{Q}_7 = 0$ determines the soliton constant

$$\mu = -\tfrac{3}{2}\left(3a^2(h^2+\kappa^2) - 4\kappa^2(h^2-\kappa^2)\right)$$

for the corresponding algebraic $\mathfrak{F}[t]$-soliton, with associated left-invariant metric

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = (\kappa - \tfrac{a}{2})e_2 + he_3, \quad [e_3, e_4] = -he_2 - (\kappa + \tfrac{a}{2})e_3.$$

These metrics are critical for $t = -\frac{3a^2+4h^2+8\kappa^2}{3a^2+4\kappa^2}$ given by (20) whenever $\mu = 0$ (see also [6]). Furthermore, in this situation they are algebraic Ricci solitons for $a = 0$ and $\kappa = \pm h$, in which case they correspond to Remark 4.6-(iii.a.1).

Note that, since $\kappa \neq 0$, we work with a representative of the homothetic class which has $\kappa = 1$. Moreover, the isomorphic isometry $e_3 \mapsto -e_3$ interchanges (a, h) and $(a, -h)$, while $(e_1, e_2, e_3, e_4) \mapsto (-e_1, -e_3, e_2, -e_4)$ interchanges (a, h) and $(-a, -h)$. Hence we can restrict the parameters to

$a \geq 0$, $a \neq \frac{2}{3}$, and $h > 0$, satisfying $\mu = -\frac{3}{2}\left(3a^2(h^2+1) - 4(h^2-1)\right) \neq 0$. In this setting, the parameter $t = -\frac{3a^2+4h^2+8}{3a^2+4} \in (-\infty, -1)$, and we get strict expanding algebraic $\mathfrak{F}[t]$-solitons for any value $t \in (-\infty, -1)$ and strict shrinking algebraic $\mathfrak{F}[t]$-solitons for $t \in (-\infty, -3)$. They correspond to Family (3).

(ii) Assuming that $a + f + p \neq 0$ we work with a homothetic metric satisfying $a+f+p = 1$, so we take $p = 1-a-f$. Since the last factor in Equation (21) must vanish it follows that

$$4f^2 + 4(a-1)f + (a-2)a - 4h^2 = 0,$$

which leads to

$$f = \tfrac{1}{2}\left(1 - a + \varepsilon\sqrt{4h^2+1}\right), \quad \varepsilon^2 = 1.$$

We add the condition $9a^2 - 6a - 4h^2 \neq 0$ to ensure that a, f and p are all different. Hence Equation (19) reduces to

$$\mathfrak{Q}_7 = \mathfrak{Q}_8 = -(3a-1)^2h^2 - \mu,$$

which determines an algebraic $\mathfrak{F}[t]$-soliton with soliton constant given by $\mu = -(3a-1)^2h^2$ and associated left-invariant metric

$$\begin{aligned}
&[e_1, e_4] = ae_1,\\
&[e_2, e_4] = -\tfrac{1}{2}\left(a - 1 - \varepsilon\sqrt{4h^2+1}\right)e_2 + he_3,\\
&[e_3, e_4] = -he_2 - \tfrac{1}{2}\left(a - 1 + \varepsilon\sqrt{4h^2+1}\right)e_3.
\end{aligned}$$

A straightforward calculation shows that these metrics are critical only for the functional determined by $t = -\frac{36h^2+5}{12h^2+11}$, which corresponds to (20) when $\mu = 0$, i.e., $a = \frac{1}{3}$, but they are never algebraic Ricci solitons (see [6]). Now, the isomorphic isometry $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_3, e_2, e_4)$ interchanges the sign of ε, while $e_2 \mapsto -e_2$ interchanges the sign of h. Hence we can restrict the parameters to $a \in \mathbb{R} \setminus \{\frac{1}{3}\}$, $h > 0$ and $\varepsilon = 1$, satisfying $9a^2 - 6a - 4h^2 \neq 0$. In this setting, taking $\mu = -(3a-1)^2h^2$ and $t = -\frac{3a^2-2a+12h^2+2}{3a^2-2a+4h^2+4} \in (-3, -\frac{5}{11})$ we get the strict expanding algebraic $\mathfrak{F}[t]$-solitons corresponding to Family (4).

4.2.2.3. _Case $ch \neq 0$._ Recall that we are assuming that a, f and p are all different. Hence, by Remark 4.2, we may assume $a = 1$ working, if necessary, with an isomorphically isometric metric. Considering the polynomials

$$\begin{aligned}
&\mathfrak{Q}_5 = c(\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{33}), \quad \mathfrak{Q}_6 = h(\mathfrak{F}[t]_{22} - \mathfrak{F}[t]_{33}),\\
&\mathfrak{Q}_7 = \mathfrak{F}[t]_{44} - \mu = -\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{22} - \mathfrak{F}[t]_{33} - \mu,\\
&\mathfrak{Q}_8 = \mathfrak{F}[t]_{11} + f\mathfrak{F}[t]_{22} + p\mathfrak{F}[t]_{33} - (f+p+1)\mu,
\end{aligned}$$

given in Equation (19), it follows that

$$\mathfrak{F}[t]_{11} = \mathfrak{F}[t]_{22} = \mathfrak{F}[t]_{33}, \quad \mu = -3\mathfrak{F}[t]_{11}, \quad (f+p+1)(\mathfrak{F}[t]_{11} - \mu) = 0.$$

Therefore, $(f+p+1)\mathfrak{F}[t]_{11} = 0$. Since the operator $\widehat{\mathfrak{F}[t]}$ must be diagonal, the vanishing of $\mathfrak{F}[t]_{11}$ would lead to trivial algebraic $\mathfrak{F}[t]$-solitons. Thus, we may take

$f=-p-1$, with $p\notin\left\{-2,-\frac{1}{2},1\right\}$ to guarantee that $a=1$, f and p are all distinct. A direct calculation shows that

$$\mathfrak{Q}_6=-18bch^2.$$

Since $ch\neq 0$, necessarily $b=0$. Now, we compute the polynomial $\mathfrak{Q}_2$ in (19), obtaining

$$\mathfrak{Q}_2=c\left(4\left(p^3-1\right)t+(p-1)\left(5p^2+2p+5+4c^2\right)+(7p+2)h^2\right).$$

Note that the coefficient of t in the second factor vanishes if and only if $p=1$, which is not possible. Therefore, we get

$$t=-\frac{(p-1)\left(5p^2+2p+5+4c^2\right)+(7p+2)h^2}{4\left(p^3-1\right)} \tag{22}$$

and $\mathfrak{Q}_3$ in (19) transforms into

$$(p-1)\mathfrak{Q}_3=-3h(p+2)\left((2p+1)h^2-(p-1)\left(2p^2+p-c^2\right)\right).$$

Since $(p-1)(p+2)h\neq 0$ in the case at hand, $\mathfrak{Q}_3=0$ implies that

$$h=\varepsilon\left(\frac{(p-1)\left(2p^2+p-c^2\right)}{2p+1}\right)^{\frac{1}{2}},\quad \varepsilon^2=1, \tag{23}$$

and finally the system of polynomial equations $\{\mathfrak{Q}_i=0\}$ given by (19) is determined by the vanishing of

$$-2(2p+1)\mathfrak{Q}_7=3\left((2p+1)\left(8p^3+15p^2+3p+1\right)+c^2(p+2)\left(5p^2-p-1\right)\right)+2(2p+1)\mu.$$

Hence one has an algebraic $\mathfrak{F}[t]$-soliton, with associated left-invariant metric

$$[e_1,e_4]=e_1+ce_3,\quad [e_2,e_4]=-(p+1)e_2+he_3,\quad [e_3,e_4]=-ce_1-he_2+pe_3,$$

and soliton constant

$$\mu=-\tfrac{3}{2(2p+1)}\left((2p+1)\left(8p^3+15p^2+3p+1\right)+c^2(p+2)\left(5p^2-p-1\right)\right),$$

where $p\notin\left\{-2,-\frac{1}{2},1\right\}$, $c\neq 0$ and the parameter $h\neq 0$ is given by (23). The $\mathfrak{F}[t]$-solitons above are critical for $t=-\frac{(2p+1)(12p^2+4p+5)+(p+2)c^2}{4(2p+1)(p^2+p+1)}$ given by (22) whenever $\mu=0$, in which case the energy vanishes. However they are not Ricci solitons unless $p=0$ and $c=h=\frac{1}{\sqrt{2}}$, which correspond to those in Remark 4.6-(iii.a.2). Moreover, the isomorphic isometry $e_2\mapsto -e_2$ interchanges the sign of h, and $e_1\mapsto -e_1$ interchanges the sign of c. Hence, we may restrict to $h>0$ and $c>0$.

At this point, a key observation to delimit the homothety classes is the existence of the following isomorphic homothety whenever $p\neq -1$. Denoting by $\varepsilon_{(p+1)}=\pm 1$ the sign of $p+1$, $(e_1,e_2,e_3,e_4)\mapsto\frac{1}{|p+1|}(e_2,e_1,-\varepsilon_{(p+1)}e_3,-\varepsilon_{(p+1)}e_4)$ transforms the parameter p into $-\frac{p}{p+1}$, interchanges $c\mapsto\frac{\varepsilon_{(p+1)}h}{p+1}$ and $h\mapsto\frac{\varepsilon_{(p+1)}c}{p+1}$, in accordance with the expression of h in (23). Hence, the transformation $p\mapsto -\frac{p}{p+1}$ maps the interval $(-\infty,-2)$ into $(-2,-1)$. Moreover, it maps $[0,+\infty)$ into $(-1,0]$, from where it follows that p may be restricted to $(-2,0]\setminus\{-\frac{1}{2}\}$. This restriction implies that the parameter h given by (23) is real whenever $(2p+1)\left(2p^2+p-c^2\right)<0$.

In this setting, the parameter t belongs to the interval $(-\infty,-\frac{5}{4})$ and we get strict expanding algebraic $\mathfrak{F}[t]$-solitons for any value $t\in(-\infty,-\frac{5}{4})$, and strict shrinking algebraic $\mathfrak{F}[t]$-solitons for $t\in(-\infty,-\frac{3}{2})$. These metrics correspond to Family (5).

4.2.3. **Case (iii.a) in Theorem 4.4.** In this case the structure constants satisfy $a=0$, $p=-f\neq 0$, $c=b$ and $2b^2-f^2+h^2=0$, and the algebraic $\mathfrak{F}[t]$-solitons are determined by the system of polynomial equations $\{\mathfrak{Q}_i=0\}$, where

$$\begin{aligned}&\mathfrak{Q}_1=\mathfrak{F}[t]_{22}-\mathfrak{F}[t]_{33}, \quad \mathfrak{Q}_2=\mathfrak{F}[t]_{12}+\tfrac{b}{f}(\mathfrak{F}[t]_{44}-\mu),\\ &\mathfrak{Q}_3=\mathfrak{F}[t]_{12}+\mathfrak{F}[t]_{13}, \quad \mathfrak{Q}_4=\mathfrak{F}[t]_{23}-\tfrac{h}{2f}(\mathfrak{F}[t]_{44}-\mu),\\ &\mathfrak{Q}_5=b\left(2f(\mathfrak{F}[t]_{11}-\mathfrak{F}[t]_{22})-3h(\mathfrak{F}[t]_{44}-\mu)\right),\end{aligned}$$

with $\mu\neq\mathfrak{F}[t]_{44}$ and $h^2=f^2-2b^2$. A straightforward calculation shows that $\mathfrak{Q}_1=\mathfrak{Q}_3=0$ and

$$\begin{aligned}\mathfrak{Q}_2&=\ \tfrac{b}{f}\left(10f^4t+30f^4-30b^2f^2-\mu\right),\\ \mathfrak{Q}_4&=-\tfrac{h}{2f}\left(22f^4t+66f^4-30b^2f^2-\mu\right),\\ \mathfrak{Q}_5&=-3bh\left(6f^4t+18f^4-30b^2f^2-\mu\right),\end{aligned}\tag{24}$$

where $\mu\neq 6f^2(f^2t+3f^2-3b^2)$ and $h^2=f^2-2b^2$. Note that

$$3fh\mathfrak{Q}_2+\mathfrak{Q}_5=12(t+3)f^4bh.$$

In what follows we analyse the vanishing of the different factors in the expression above separately. Recall that we are assuming $f\neq 0$ and, moreover, b and h cannot vanish simultaneously since $h^2=f^2-2b^2$.

4.2.3.1. *Case $t=-3$*. Equation (24) reduces to

$$\mathfrak{Q}_2=-\tfrac{b}{f}\left(30b^2f^2+\mu\right), \quad \mathfrak{Q}_4=\tfrac{h}{2f}\left(30b^2f^2+\mu\right), \quad \mathfrak{Q}_5=3bh\left(30b^2f^2+\mu\right).$$

Hence, if $h^2=f^2-2b^2$ and $f\neq 0$, then we get an algebraic $\mathfrak{F}[-3]$-soliton with soliton constant $\mu=-30b^2f^2$, and associated left-invariant metric given by

$$[e_1,e_4]=b(e_2+e_3), \quad [e_2,e_4]=-be_1+fe_2+he_3, \quad [e_3,e_4]=-be_1-he_2-fe_3.$$

A direct calculation shows that the above $\mathfrak{F}[t]$-solitons are $\mathcal{F}[\nu]$-critical if and only if $b=0$ and $\nu=-3$ or $h=0$ and $\nu=-\frac{3}{2}$. In both cases they are Ricci solitons (cf. Remark 4.6-(iii.a)). Hence, we take $bh\neq 0$ and therefore we fix $b=1$ working at the homothetic level. Moreover, the isomorphic isometry $(e_1,e_2,e_3,e_4)\mapsto(e_1,e_3,e_2,e_4)$ interchanges (f,h) and $(-f,-h)$, so we can restrict the parameter f to $f>0$. In this setting, we get the strict expanding algebraic $\mathfrak{F}[t]$-solitons which correspond to Family (6).

4.2.3.2. *Case $b=0$, $h\neq 0$ and $t\neq -3$*. Since $h^2=f^2$ we set $h=\varepsilon f$, with $\varepsilon^2=1$, and the corresponding left-invariant metric, given by

$$[e_2,e_4]=f(e_2+\varepsilon e_3), \quad [e_3,e_4]=-f(\varepsilon e_2+e_3)$$

is an algebraic Ricci soliton with soliton constant $-6f^2$ (see Remark 4.6-(iii.a.1)).

4.2.3.3. *Case $h=0$, $b\neq 0$ and $t\neq -3$*. Since $2b^2=f^2$ we set $b=\frac{\varepsilon}{\sqrt{2}}f$, with $\varepsilon^2=1$. In this case, the left-invariant metric corresponds to

$$[e_1,e_4]=\tfrac{\varepsilon}{\sqrt{2}}f(e_2+e_3), \quad [e_2,e_4]=-f\left(\tfrac{\varepsilon}{\sqrt{2}}e_1-e_2\right), \quad [e_3,e_4]=-f\left(\tfrac{\varepsilon}{\sqrt{2}}e_1+e_3\right),$$

which is an algebraic Ricci soliton with soliton constant $-3f^2$ (cf. Remark 4.6-(iii.a.2)). □

5. Algebraic solitons on $\mathbb{R} \ltimes E(1,1)$ and $\mathbb{R} \ltimes \widetilde{E}(2)$

It follows from the work of Milnor [21] that any left-invariant metric on the Poincaré or the Euclidean group is determined by the corresponding Lie algebras $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ with $[e_3, e_1] = \lambda_2 e_2$ and $[e_2, e_3] = \lambda_1 e_1$, where $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}_3$, and $\mathfrak{g}_3 = \mathfrak{e}(2)$ if $\lambda_1\lambda_2 > 0$, or $\mathfrak{g}_3 = \mathfrak{e}(1,1)$ if $\lambda_1\lambda_2 < 0$. Now, proceeding as in [6] one has that left-invariant metrics on semi-direct extensions $\mathbb{R} \ltimes E(1,1)$ or $\mathbb{R} \ltimes \widetilde{E}(2)$ are determined by

$$\begin{aligned} &[e_1, e_3] = -\lambda_2 e_2, && [e_2, e_3] = \lambda_1 e_1, && \\ &[e_1, e_4] = b e_1 - A\lambda_2 e_2, && [e_2, e_4] = A\lambda_1 e_1 + b e_2, && [e_3, e_4] = C e_1 + D e_2, \end{aligned} \tag{25}$$

where $\lambda_1\lambda_2 \neq 0$.

Remark 5.1. The parameters $(\lambda_1, \lambda_2, A, b, C, D)$ in Equation (25) can be transformed into $(\lambda_2,\ \lambda_1,\ A,\ b,\ -D,\ C)$ through the isometry

$$(e_1, e_2, e_3, e_4) \mapsto (-e_2, e_1, e_3, e_4).$$

Therefore, any left-invariant metric (25) with $D = 0$ is isomorphically isometric to a left-invariant metric with $C = 0$.

5.1. Algebraic T-solitons on $\mathbb{R} \ltimes E(1,1)$ and $\mathbb{R} \ltimes \widetilde{E}(2)$.

Next we show that if a semi-direct extension $\mathbb{R} \ltimes E(1,1)$ or $\mathbb{R} \ltimes \widetilde{E}(2)$ is an algebraic T-soliton, then the underlying group is unimodular and hence a product $E(1,1) \times \mathbb{R}$ or $\widetilde{E}(2) \times \mathbb{R}$ (cf. [1]), which can also be viewed as a semi-direct extension of $\mathbb{R}^3$. Hence the existence of algebraic T-solitons on $\mathbb{R} \ltimes E(1,1)$ and $\mathbb{R} \ltimes \widetilde{E}(2)$ reduces to those already considered in Section 4.

Theorem 5.2. *$\mathbb{R} \ltimes E(1,1)$ or $\mathbb{R} \ltimes \widetilde{E}(2)$ is a non-trivial algebraic T-soliton with soliton constant μ if and only if the group is unimodular (i.e., $b = 0$) and*

$$T = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ * & T_{11} & 0 & 0 \\ * & * & A^2(T_{44} - \mu) + \mu & -A(T_{44} - \mu) \\ * & * & * & T_{44} \end{pmatrix} \neq \mu\langle\cdot,\cdot\rangle,$$

with either $C = D = 0$ or $T_{11} = A^2(T_{44} - \mu) + T_{44}$.

Remark 5.3. Assume $b = 0$ as in the previous theorem. Setting $\zeta = \sqrt{A^2 + 1}$ and considering the orthonormal basis given by

$$\bar{e}_1 = e_1, \quad \bar{e}_2 = e_2, \quad \bar{e}_3 = -\tfrac{1}{\zeta}(A e_3 - e_4), \quad \bar{e}_4 = -\tfrac{1}{\zeta}(e_3 + A e_4),$$

the Lie bracket transforms into

$$[\bar{e}_1, \bar{e}_4] = \zeta\lambda_2 \bar{e}_2, \quad [\bar{e}_2, \bar{e}_4] = -\zeta\lambda_1 \bar{e}_1, \quad [\bar{e}_3, \bar{e}_4] = C\bar{e}_1 + D\bar{e}_2.$$

Thus, these metrics are isomorphically isometric to metrics on $\mathbb{R} \ltimes \mathbb{R}^3$ and therefore the corresponding non-trivial algebraic T-solitons have already been described in Theorems 4.4 and 4.7.

Proof. The vanishing of the divergence of T is equivalent to

$$\begin{aligned} &3bT_{14} - \lambda_2 T_{23} - A\lambda_2 T_{24} = 0,\\ &\lambda_1 T_{13} + A\lambda_1 T_{14} + 3bT_{24} = 0,\\ &(\lambda_1 - \lambda_2)T_{12} - CT_{14} - DT_{24} - 2bT_{34} = 0,\\ &bT_{11} + bT_{22} - 2bT_{44} + A(\lambda_1 - \lambda_2)T_{12} + CT_{13} + DT_{23} = 0, \end{aligned} \tag{26}$$

while the conditions for $\mathfrak{D} = \widehat{T} - \mu \operatorname{Id}$ to be a derivation are determined by a system of twenty two – up to duplicity – polynomial equations on the soliton constant μ, the structure constants (25) and the components T_{ij}, given by $\{\mathfrak{P}_{ijk} = 0\}$, where

$$\mathfrak{P}_{211} = -\lambda_1(T_{13} + AT_{14}) + bT_{24}, \quad \mathfrak{P}_{212} = -bT_{14} - \lambda_2(T_{23} + AT_{24}),$$

$$\begin{aligned} &\mathfrak{P}_{311} = (\lambda_1 + \lambda_2)T_{12} - CT_{14} + bT_{34}, && \mathfrak{P}_{313} = \lambda_2 T_{23},\\ &\mathfrak{P}_{312} = -\lambda_2(T_{11} - T_{22} + T_{33} - \mu) - DT_{14} - A\lambda_2 T_{34}, && \mathfrak{P}_{314} = \lambda_2 T_{24}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{321} = -\lambda_1(T_{11} - T_{22} - T_{33} + \mu) - CT_{24} + A\lambda_1 T_{34}, && \mathfrak{P}_{323} = -\lambda_1 T_{13},\\ &\mathfrak{P}_{322} = -(\lambda_1 + \lambda_2)T_{12} - DT_{24} + bT_{34}, && \mathfrak{P}_{324} = -\lambda_1 T_{14}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{411} = b(T_{44} - \mu) + A(\lambda_1 + \lambda_2)T_{12} + CT_{13}, && \mathfrak{P}_{413} = -bT_{13} + A\lambda_2 T_{23},\\ &\mathfrak{P}_{412} = -A\lambda_2(T_{11} - T_{22} + T_{44} - \mu) + DT_{13} - \lambda_2 T_{34}, && \mathfrak{P}_{414} = -bT_{14} + A\lambda_2 T_{24}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{421} = -A\lambda_1(T_{11} - T_{22} - T_{44} + \mu) + CT_{23} + \lambda_1 T_{34}, && \mathfrak{P}_{423} = -A\lambda_1 T_{13} - bT_{23},\\ &\mathfrak{P}_{422} = b(T_{44} - \mu) - A(\lambda_1 + \lambda_2)T_{12} + DT_{23}, && \mathfrak{P}_{424} = -A\lambda_1 T_{14} - bT_{24}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{P}_{431} = -C(T_{11} - T_{33} - T_{44} + \mu) - DT_{12} + bT_{13} + \lambda_1(AT_{23} - T_{24}),\\ &\mathfrak{P}_{432} = -D(T_{22} - T_{33} - T_{44} + \mu) - CT_{12} - \lambda_2(AT_{13} - T_{14}) + bT_{23},\\ &\mathfrak{P}_{433} = -CT_{13} - DT_{23}, \quad \mathfrak{P}_{434} = -CT_{14} - DT_{24}. \end{aligned}$$

Since $\lambda_1\lambda_2 \neq 0$ and

$$\begin{aligned} &\mathfrak{P}_{323} = -\lambda_1 T_{13}, \quad \mathfrak{P}_{324} = -\lambda_1 T_{14}, \quad \mathfrak{P}_{313} = \lambda_2 T_{23}, \quad \mathfrak{P}_{314} = \lambda_2 T_{24},\\ &\mathfrak{P}_{312} = -\lambda_2(T_{11} - T_{22} + T_{33} - \mu) - DT_{14} - A\lambda_2 T_{34},\\ &\mathfrak{P}_{321} = -\lambda_1(T_{11} - T_{22} - T_{33} + \mu) - CT_{24} + A\lambda_1 T_{34}, \end{aligned}$$

it immediately follows that

$$T_{13} = T_{14} = T_{23} = T_{24} = 0, \quad T_{22} = T_{11}, \quad T_{33} = \mu - AT_{34}. \tag{27}$$

Using these conditions, we have

$$\mathfrak{P}_{412} = -\lambda_2\left(T_{34} + A(T_{44} - \mu)\right),$$

which leads to

$$T_{34} = -A(T_{44} - \mu), \quad T_{33} = A^2(T_{44} - \mu) + \mu. \tag{28}$$

Now, a direct computation leads to

$$\begin{aligned} \mathfrak{P}_{311} &= (\lambda_1 + \lambda_2)T_{12} - Ab(T_{44} - \mu),\\ \mathfrak{P}_{322} &= -(\lambda_1 + \lambda_2)T_{12} - Ab(T_{44} - \mu), \end{aligned}$$

while the third equation in (26) now becomes

$$(\lambda_1 - \lambda_2)T_{12} + 2Ab(T_{44} - \mu) = 0.$$

Hence one has that

$$T_{12} = 0, \tag{29}$$

and Equations (27), (28) and (29) show that

$$T = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ * & T_{11} & 0 & 0 \\ * & * & A^2(T_{44}-\mu)+\mu & -A(T_{44}-\mu) \\ * & * & * & T_{44} \end{pmatrix}.$$

The system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\begin{aligned} &\mathfrak{P}_{311} = \mathfrak{P}_{322} = -Ab(T_{44}-\mu) = 0, \\ &\mathfrak{P}_{411} = \mathfrak{P}_{422} = b(T_{44}-\mu) = 0, \\ &\mathfrak{P}_{431} = -C\left(T_{11} - A^2(T_{44}-\mu) - T_{44}\right) = 0, \\ &\mathfrak{P}_{432} = -D\left(T_{11} - A^2(T_{44}-\mu) - T_{44}\right) = 0, \end{aligned}$$

and the conditions for $\operatorname{div} T = 0$ in Equation (26) are

$$Ab(T_{44}-\mu) = 0, \quad b(T_{11} - T_{44}) = 0.$$

Note that if $b \neq 0$ then the solution is determined by $T_{11} = T_{44} = \mu$ and the algebraic T-soliton is trivial. Finally, for $b = 0$ the solution is given by $\mathfrak{P}_{431} = \mathfrak{P}_{432} = 0$ and therefore $C = D = 0$ or, otherwise, $T_{11} = A^2(T_{44}-\mu) + T_{44}$, which completes the proof. □

6. Algebraic solitons on $\mathbb{R} \ltimes \mathcal{H}^3$

It follows from the work of Milnor [21] that any left-invariant metric on the Heisenberg group $\mathcal{H}^3$ is determined by the Lie algebra $[e_1, e_2] = \gamma e_3$, where $\{e_i\}$ is an orthonormal basis and $\gamma \neq 0$. Now, proceeding as in [6], one has that left-invariant metrics on semi-direct extensions $\mathbb{R} \ltimes \mathcal{H}^3$ are determined by

$$\begin{aligned} &[e_1, e_2] = \gamma e_3, && [e_1, e_4] = a e_1 - c e_2 + H e_3, \\ &[e_3, e_4] = (a+d) e_3, && [e_2, e_4] = c e_1 + d e_2 + F e_3, \end{aligned} \tag{30}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of the Lie algebra of $\mathbb{R} \ltimes \mathcal{H}^3$.

Remark 6.1. $\mathbb{R} \ltimes \mathcal{H}^3$ is locally symmetric if and only if the structure constants satisfy $a = d = \frac{\varepsilon}{2}\gamma$ and $F = H = 0$, where $\varepsilon^2 = 1$. The isomorphism determined by $e_4 \mapsto -e_4$ is an orientation reversing isometry so we can set $\varepsilon = 1$, up to a change of orientation. A straightforward calculation shows that the anti-self-dual Weyl curvature operator W^- vanishes and so the underlying manifold is homothetic to the complex hyperbolic plane $\mathbb{CH}^2$ (see [11]). This is the only Einstein metric on $\mathbb{R} \ltimes \mathcal{H}^3$.

Remark 6.2. The isometry $(e_1, e_2, e_3, e_4) \mapsto (-e_2, e_1, e_3, e_4)$ transforms the parameters (γ, a, c, d, H, F) in Equation (30) into $(\gamma, d, c, a, -F, H)$. Thus, any left-invariant metric (30) with $H = 0$ is isomorphically isometric to a left-invariant metric with $F = 0$.

6.1. **Algebraic T-solitons on $\mathbb{R} \ltimes \mathcal{H}^3$.** The algebraic T-solitons on semi-direct extensions of the Heisenberg group are given as follows.

Theorem 6.3. *$\mathbb{R} \ltimes \mathcal{H}^3$ is a non-trivial algebraic T-soliton with soliton constant μ if and only if one of the following conditions holds.*

(i) *In the non-unimodular case, $a + d \neq 0$, it is isomorphically isometric to a Riemannian Lie group given by one of the following possibilities.*

(i.a) *$a \neq \pm d$, $c = 0$, $F = 0$, and the tensor field $\widehat{T}$ is diagonal,*

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{22}, T_{11} + T_{22} - \mu, \mu] \neq \mu \operatorname{Id},$$

with

$$H(T_{22} - \mu) = 0, \quad (2a + d)T_{11} + (a + 2d)T_{22} - 3(a + d)\mu = 0.$$

(i.b) *$a = d \neq 0$, $c = 0$, $F = H = 0$ and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ * & 2\mu - T_{11} & 0 & 0 \\ * & * & \mu & 0 \\ * & * & * & \mu \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle.$$

(ii) *In the unimodular case, $a + d = 0$, it is isomorphically isometric to a Riemannian Lie group given by one of the following possibilities.*

(ii.a) *$a = -d$, $a^2 \neq c^2$, $F = H = 0$ and the tensor field $\widehat{T}$ is diagonal,*

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{11}, 2T_{11} - \mu, \mu], \quad T_{11} \neq \mu.$$

(ii.b) *$a = d = c = 0$ and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & \frac{1}{\gamma}\left(F(T_{22} - T_{33} + T_{44} - \mu) + HT_{12}\right) \\ * & T_{22} & 0 & \frac{-1}{\gamma}\left(H(T_{11} - T_{33} + T_{44} - \mu) + FT_{12}\right) \\ * & * & T_{33} & 0 \\ * & * & * & T_{44} \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle,$$

with

$$\begin{aligned} &(H^2 - \gamma^2)T_{11} + (F^2 - \gamma^2)T_{22} - (F^2 + H^2 - \gamma^2)T_{33} \\ &\qquad + (F^2 + H^2)T_{44} + 2FHT_{12} - (F^2 + H^2 - \gamma^2)\mu = 0. \end{aligned}$$

(ii.c) *$a = -d$, $c = a \neq 0$ and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & T_{14} \\ * & T_{11} & 0 & T_{14} \\ * & * & 2T_{11} - \frac{1}{\gamma}(F - H)T_{14} - \mu & 0 \\ * & * & * & \mu - 2T_{12} \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle$$

where

$$\begin{aligned} HT_{11} - (F - 2H)T_{12} - \tfrac{1}{\gamma}\left(H(F - H) + \gamma^2\right)T_{14} - H\mu = 0, \\ FT_{11} + (2F - H)T_{12} - \tfrac{1}{\gamma}\left(F(F - H) - \gamma^2\right)T_{14} - F\mu = 0. \end{aligned}$$

Remark 6.4. If $a = c = d = 0$ as in Assertion (ii.b) in the previous theorem, then the left-invariant metrics associated to the corresponding algebraic T-solitons are given by

$$[e_1, e_2] = \gamma e_3, \quad [e_1, e_4] = He_3, \quad [e_2, e_4] = Fe_3.$$

Taking $\zeta=\sqrt{F^2+H^2+\gamma^2}$ and considering the orthonormal basis

$$\begin{aligned}
\bar{e}_1 &= \tfrac{1}{\zeta}\left(Fe_1-He_2+\gamma e_4\right),\\
\bar{e}_2 &= \tfrac{1}{\sqrt{2(F^2+H^2)}}\left(He_1+Fe_2-\sqrt{F^2+H^2}e_3\right),\\
\bar{e}_3 &= \tfrac{-1}{\sqrt{2(F^2+H^2)}}\left(He_1+Fe_2+\sqrt{F^2+H^2}e_3\right),\\
\bar{e}_4 &= \tfrac{1}{\zeta\sqrt{F^2+H^2}}\left(\gamma Fe_1-\gamma He_2-(F^2+H^2)e_4\right),
\end{aligned}$$

the only non-zero brackets now correspond to

$$[\bar{e}_2,\bar{e}_4]=\tfrac{\zeta}{2}(\bar{e}_2+\bar{e}_3),\quad [\bar{e}_3,\bar{e}_4]=-\tfrac{\zeta}{2}(\bar{e}_2+\bar{e}_3).$$

These metrics are isomorphically isometric to metrics on $\mathbb{R}\ltimes\mathbb{R}^3$ and the corresponding algebraic T-solitons have already been covered by Theorem 4.4-(iii.b) (see also Remark 4.5).

An analogous conclusion holds true when the structure constants $a=-d$ and $a=c\neq 0$ as in Theorem 6.3–(ii.c) if, in addition, $H=-F$. In this case the left-invariant metrics associated to the algebraic T-solitons are given by

$$[e_1,e_2]=\gamma e_3,\quad [e_1,e_4]=ae_1-ae_2-Fe_3,\quad [e_2,e_4]=ae_1-ae_2+Fe_3,$$

and taking $\zeta=\sqrt{2F^2+\gamma^2}$ and the orthonormal basis determined by

$$\begin{aligned}
\bar{e}_1 &= \tfrac{1}{\zeta}\left(Fe_1+Fe_2+\gamma e_4\right), & \bar{e}_2 &= \tfrac{-1}{\sqrt{6}}\left(e_1-e_2+2e_3\right),\\
\bar{e}_3 &= \tfrac{1}{\sqrt{3}}\left(e_1-e_2-e_3\right), & \bar{e}_4 &= \tfrac{1}{\sqrt{2}\,\zeta}\left(\gamma e_1+\gamma e_2-2Fe_4\right),
\end{aligned}$$

the non-zero brackets correspond to

$$[\bar{e}_1,\bar{e}_4]=\tfrac{2\sqrt{3}}{3}a\bar{e}_2-\tfrac{2\sqrt{6}}{3}a\bar{e}_3,\quad [\bar{e}_2,\bar{e}_4]=\tfrac{\sqrt{2}\,\zeta}{3}\bar{e}_2+\tfrac{\zeta}{3}\bar{e}_3,\quad [\bar{e}_3,\bar{e}_4]=-\tfrac{2\zeta}{3}\bar{e}_2-\tfrac{\sqrt{2}\,\zeta}{3}\bar{e}_3.$$

Thus, the above metrics are isomorphically isometric to metrics on $\mathbb{R}\ltimes\mathbb{R}^3$ and the corresponding algebraic T-solitons have already been covered by Theorem 4.4-(iii.a) (see also Remark 4.5).

Proof of Theorem 6.3. A direct calculation shows that T is divergence-free if and only if

$$\begin{aligned}
&(3a+2d)T_{14}+\gamma T_{23}-cT_{24}+HT_{34}=0,\\
&\gamma T_{13}-cT_{14}-(2a+3d)T_{24}-FT_{34}=0,\\
&3(a+d)T_{34}=0,\\
&aT_{11}+dT_{22}+(a+d)(T_{33}-2T_{44})+HT_{13}+FT_{23}=0.
\end{aligned}\tag{31}$$

The conditions for $\mathfrak{D}=\widehat{T}-\mu\,\mathrm{Id}$ to be a derivation can be expressed in terms of a system of twenty two – up to duplicity – polynomial equations on the soliton constant μ, the structure constants given in (30) and the components T_{ij}, given by $\{\mathfrak{P}_{ijk}=0\}$, where

$$\mathfrak{P}_{211} = -\gamma T_{13} - cT_{14} + aT_{24}, \qquad \mathfrak{P}_{212} = -dT_{14} - \gamma T_{23} - cT_{24},$$
$$\mathfrak{P}_{213} = \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24}, \quad \mathfrak{P}_{214} = -\gamma T_{34},$$

$$\mathfrak{P}_{311} = aT_{34}, \quad \mathfrak{P}_{312} = -cT_{34}, \quad \mathfrak{P}_{313} = -(a+d)T_{14} + \gamma T_{23} + HT_{34},$$

$$\mathfrak{P}_{321} = cT_{34}, \quad \mathfrak{P}_{322} = dT_{34}, \quad \mathfrak{P}_{323} = -\gamma T_{13} - (a+d)T_{24} + FT_{34},$$

$$\mathfrak{P}_{411} = a(T_{44} - \mu) + 2cT_{12} - HT_{13},$$
$$\mathfrak{P}_{412} = -c(T_{11} - T_{22} + T_{44} - \mu) - (a-d)T_{12} - HT_{23},$$
$$\mathfrak{P}_{413} = H(T_{11} - T_{33} + T_{44} - \mu) + FT_{12} + dT_{13} + cT_{23} + \gamma T_{24},$$
$$\mathfrak{P}_{414} = -aT_{14} + cT_{24} - HT_{34},$$

$$\mathfrak{P}_{421} = -c(T_{11} - T_{22} - T_{44} + \mu) + (a-d)T_{12} - FT_{13},$$
$$\mathfrak{P}_{422} = d(T_{44} - \mu) - 2cT_{12} - FT_{23},$$
$$\mathfrak{P}_{423} = F(T_{22} - T_{33} + T_{44} - \mu) + HT_{12} - cT_{13} - \gamma T_{14} + aT_{23},$$
$$\mathfrak{P}_{424} = -cT_{14} - dT_{24} - FT_{34},$$

$$\mathfrak{P}_{431} = -dT_{13} + cT_{23}, \qquad \mathfrak{P}_{432} = -cT_{13} - aT_{23},$$
$$\mathfrak{P}_{433} = (a+d)(T_{44} - \mu) + HT_{13} + FT_{23}, \quad \mathfrak{P}_{434} = -(a+d)T_{34}.$$

We start by considering

$$\begin{aligned} \mathfrak{P}_{214} &= -\gamma T_{34}, \\ \mathfrak{P}_{211} + \mathfrak{P}_{323} - \mathfrak{P}_{424} &= -2(\gamma T_{13} - FT_{34}), \\ \mathfrak{P}_{212} - \mathfrak{P}_{313} + \mathfrak{P}_{414} &= -2(\gamma T_{23} + HT_{34}), \end{aligned}$$

which imply that

$$T_{13} = T_{23} = T_{34} = 0, \tag{32}$$

and split our analysis differentiating the cases $a + d \neq 0$ and $a + d = 0$.

6.1.1. Case $a + d \neq 0$.

Using Equation (32), since

$$\begin{aligned} \mathfrak{P}_{313} &= -(a+d)T_{14}, \\ \mathfrak{P}_{323} &= -(a+d)T_{24}, \\ \mathfrak{P}_{433} &= (a+d)(T_{44} - \mu), \\ \mathfrak{P}_{213} &= \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24}, \end{aligned}$$

and $a + d \neq 0$ we obtain

$$T_{14} = T_{24} = 0, \quad T_{44} = \mu, \quad T_{33} = T_{11} + T_{22} - \mu. \tag{33}$$

Next we consider separately the cases depending on whether c vanishes or not.

6.1.1.1. *Case $c \neq 0$*. In this case Equations (32) and (33) imply that

$$\mathfrak{P}_{411} = 2cT_{12}, \quad \mathfrak{P}_{421} = -c(T_{11} - T_{22}) + (a-d)T_{12},$$

so the condition $c \neq 0$ leads to

$$T_{12} = 0, \quad T_{22} = T_{11}. \tag{34}$$

Now Equations (32), (33) and (34) show that $\widehat{T}$ is diagonal,

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{11}, 2T_{11} - \mu, \mu]$$

and, moreover, the vanishing conditions of the divergence of T given in Equation (31) reduce to

$$(a+d)(T_{11}-\mu)=0.$$

Hence, we conclude that there are no non-trivial algebraic T-soliton in this case.

6.1.1.2. <u>*Case $c=0$*</u>. Using Equation (33) together with Equation (32) and $c=0$, it is straightforward to see that

$$T=\begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ * & T_{22} & 0 & 0 \\ * & * & T_{11}+T_{22}-\mu & 0 \\ * & * & * & \mu \end{pmatrix}.$$

The system $\{\mathfrak{P}_{ijk}=0\}$ now corresponds to

$$\begin{aligned} \mathfrak{P}_{412} &= -\mathfrak{P}_{421} = -(a-d)T_{12}=0, \\ \mathfrak{P}_{413} &= FT_{12}-H(T_{22}-\mu)=0, \\ \mathfrak{P}_{423} &= -F(T_{11}-\mu)+HT_{12}=0, \end{aligned}$$

while the vanishing of the divergence of T given by Equation (31) reduces to

$$(2a+d)T_{11}+(a+2d)T_{22}-3(a+d)\mu=0.$$

If $a-d\neq 0$, then $T_{12}=0$. Besides, if $FH\neq 0$, then $T_{11}=T_{22}=\mu$ and the tensor field T is trivial. Thus, according to Remark 6.2, we may assume $F=0$ (working with an isomorphically isometric metric if necessary) and Assertion (i.a) immediately follows.

If $a=d$, they are both different from zero, since $a+d\neq 0$. Therefore, the last equation above implies that

$$T_{22}=2\mu-T_{11}$$

while the other equations become

$$FT_{12}+H(T_{11}-\mu)=0, \quad HT_{12}-F(T_{11}-\mu)=0.$$

Note that if either F or H is non-zero, then the corresponding algebraic T-soliton is trivial. Therefore, $F=H=0$ and Assertion (i.b) follows.

6.1.2. **Case $\boldsymbol{a+d=0}$.** In this situation, we distinguish two cases depending on whether a^2 and c^2 are equal or not.

6.1.2.1. <u>*Case $a^2\neq c^2$*</u>. A direct calculation involving Equation (32) and the condition $d=-a$ shows that

$$\begin{aligned} \mathfrak{P}_{211} &= -cT_{14}+aT_{24}, \\ \mathfrak{P}_{212} &= aT_{14}-cT_{24}, \\ 2a\mathfrak{P}_{411}+c(\mathfrak{P}_{412}-\mathfrak{P}_{421}) &= 2(a^2-c^2)(T_{44}-\mu), \\ \mathfrak{P}_{213} &= \gamma(T_{11}+T_{22}-T_{33}-\mu)-FT_{14}+HT_{24}, \end{aligned}$$

so $a^2\neq c^2$ implies that

$$T_{14}=T_{24}=0, \quad T_{44}=\mu, \quad T_{33}=T_{11}+T_{22}-\mu. \tag{35}$$

This last equation together with (32) leads to

$$\mathfrak{P}_{411}=2cT_{12}, \quad \mathfrak{P}_{421}=2aT_{12}-c(T_{11}-T_{22}),$$

while the condition $\operatorname{div} T = 0$ given in Equation (31) reduces to

$$a(T_{11} - T_{22}) = 0.$$

Hence, since $a^2 \neq c^2$, it follows that

$$T_{12} = 0, \quad T_{11} = T_{22}. \tag{36}$$

Putting together Equations (32), (35) and (36) we obtain that $\widehat{T}$ is diagonal,

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{11}, 2T_{11} - \mu, \mu],$$

and the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\mathfrak{P}_{413} = H(\mu - T_{11}) = 0, \quad \mathfrak{P}_{423} = F(\mu - T_{11}) = 0.$$

Thus, there are non-trivial algebraic T-solitons when $T_{11} \neq \mu$ and $F = H = 0$, so Assertion (ii.a) is obtained.

6.1.2.2. Case $a^2 = c^2$. We set $c = \varepsilon a$, with $\varepsilon^2 = 1$, and assume that the conditions in Equation (32) hold. In this case Equation (31), which corresponds to $\operatorname{div} T = 0$, reduces to

$$a(T_{11} - T_{22}) = 0, \quad a(\varepsilon T_{14} - T_{24}) = 0.$$

If $a = 0$, then $c = d = 0$ and the tensor T is divergence-free, while the system $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\begin{aligned}
\mathfrak{P}_{423} &= F(T_{22} - T_{33} + T_{44} - \mu) + HT_{12} - \gamma T_{14} = 0,\\
\mathfrak{P}_{413} &= H(T_{11} - T_{33} + T_{44} - \mu) + FT_{12} + \gamma T_{24} = 0,\\
\mathfrak{P}_{213} &= \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24} = 0.
\end{aligned}$$

Clearing T_{14} and T_{24} in the first and second equations above, respectively, Assertion (ii.b) is immediately obtained.

If $a \neq 0$, then we compute

$$\mathfrak{P}_{411} = a(2\varepsilon T_{12} + T_{44} - \mu),$$

which together with the conditions for $\operatorname{div} T = 0$ gives

$$T_{22} = T_{11}, \quad T_{24} = \varepsilon T_{14}, \quad T_{44} = \mu - 2\varepsilon T_{12}.$$

Thus, the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\begin{aligned}
\mathfrak{P}_{213} &= \gamma(2T_{11} - T_{33} - \mu) - (F - \varepsilon H)T_{14} = 0,\\
\mathfrak{P}_{413} &= H(T_{11} - T_{33}) + (F - 2\varepsilon H)T_{12} + \varepsilon\gamma T_{14} = 0,\\
\mathfrak{P}_{423} &= F(T_{11} - T_{33}) + (H - 2\varepsilon F)T_{12} - \gamma T_{14} = 0,
\end{aligned}$$

and from the first equation above we obtain the expression for T_{33}. Note that the corresponding left-invariant metrics are given by

$$[e_1, e_2] = \gamma e_3, \quad [e_1, e_4] = ae_1 - \varepsilon ae_2 + He_3, \quad [e_2, e_4] = \varepsilon ae_1 - ae_2 + Fe_3,$$

and the isometry $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_2, -e_3, e_4)$ interchanges (ε, a, F, H) and $(-\varepsilon, a, F, -H)$. Thus we can take $\varepsilon = 1$ working, if necessary, with an isomorphically isometric metric and Assertion (ii.c) is obtained. □

Remark 6.5. Following Theorem 6.3, non-trivial algebraic Ricci solitons on $\mathbb{R}\ltimes\mathcal{H}^3$ correspond to the nilpotent cases (iii.a.1) and (iii.a.2) in Remark 4.6 and the left-invariant metrics on $\mathbb{R}\ltimes\mathcal{H}^3$ determined by

$$[e_1,e_2]=e_3, \quad [e_1,e_4]=ae_1, \quad [e_2,e_4]=de_2, \quad [e_3,e_4]=(a+d)e_3,$$

with $a\in[-\frac{\sqrt{3}}{2},\frac{1}{2})$. For a fixed a, the parameter d is given by the only positive solution of $4(a^2+d^2+ad)-3=0$.

The semi-direct extension $\mathbb{R}\ltimes\mathcal{H}^3$ is unimodular for $a=-\frac{\sqrt{3}}{2}$ (which corresponds to case (ii.a)) and non-unimodular otherwise (corresponding to case (i.a) in Theorem 6.3).

6.2. **Algebraic $\mathfrak{F}[t]$-solitons on $\mathbb{R}\ltimes\mathcal{H}^3$.** In addition to the Ricci solitons described in Remark 4.6 and Remark 6.5, the algebraic $\mathfrak{F}[t]$-solitons on $\mathbb{R}\ltimes\mathcal{H}^3$ which do not correspond to any equivalent algebraic $\mathfrak{F}[t]$-soliton on $\mathbb{R}\ltimes\mathbb{R}^3$ are given as follows.

Theorem 6.6. *A left-invariant metric on $\mathbb{R}\ltimes\mathcal{H}^3$ is a strict algebraic $\mathfrak{F}[t]$-soliton if and only if it is isomorphically homothetic to one of the following:*

1. $[e_1,e_2]=e_3$, $[e_1,e_4]=ae_1$, $[e_2,e_4]=de_2$, $[e_3,e_4]=(a+d)e_3$, *with* $|a|<d$ *and* $4(a^2+d^2+ad)-3\neq 0$. *In this case,* $\mu=-(a-d)^2$ *and* $t=-\frac{4(a^2+d^2+ad)+3}{4(3a^2+3d^2+5ad)+1}$.
2. $[e_1,e_2]=e_3$, $[e_1,e_4]=ae_1+He_3$, $[e_2,e_4]=-2ae_2$, $[e_3,e_4]=-ae_3$, *with* $a>0$ *and* $H>0$. *In this case,* $\mu=-a^2(4H^2+9)$ *and the parameter* $t=-\frac{3(4a^2+H^2+1)}{20a^2+H^2+1}$.
3. $[e_1,e_2]=e_3$, $[e_1,e_4]=ae_1$, $[e_2,e_4]=-ae_2$, *with* $a\in(0,+\infty)\setminus\{\frac{\sqrt{3}}{2}\}$. *In this case,* $\mu=-4a^2$ *and* $t=-\frac{4a^2+3}{4a^2+1}$.
4. $[e_1,e_2]=e_3$, $[e_1,e_4]=ae_1-ce_2$, $[e_2,e_4]=ce_1-ae_2$, *with* $a\in(0,+\infty)\setminus\{\frac{1}{2}\}$. *For a fixed, c is the only positive solution of the equation* $4(4a^2+3)c^2-(4a^2-3)^2=0$. *In this setting,* $\mu=-\frac{12a^2(4a^2-1)}{4a^2+3}$ *and* $t=-\frac{3(16a^4+3)}{16a^4+16a^2+3}$.

Remark 6.7. The range of the parameter t in each family of Theorem 6.6 is indicated in Figure 4 below.

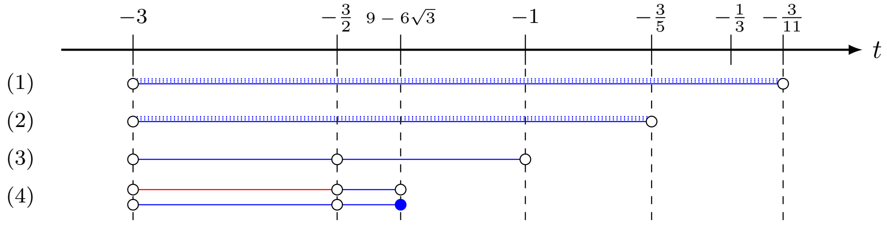

FIGURE 4. Range of the parameter t for strict algebraic $\mathfrak{F}[t]$-solitons on $\mathbb{R}\ltimes\mathcal{H}^3$.

Proceeding as in the previous sections, one has that the homothetical invariants $\{t, \|\rho\|^2\tau^{-2}, \|R\|^2\tau^{-2}, \|\nabla\rho\|^2\tau^{-3}, \|\nabla R\|^2\tau^{-3}\}$ distinguish homothety classes among metrics given in Theorem 4.8 and Theorem 6.6. Furthermore, in Theorem 6.6, it follows from the restrictions on the parameters in each case, that there are no homotheties neither between metrics in different families nor inside each family for different values of the parameters.

Besides, for each value of $t \in (-3,-1) \setminus \{-\frac{3}{2}\}$ there is a single strict algebraic $\mathfrak{F}[t]$-soliton (up to homotheties) for Family (3) in Figure 4, and the same occurs for $t = 9 - 6\sqrt{3}$ in Family (4). This last family provides two non-homothetic strict algebraic $\mathfrak{F}[t]$-solitons for the rest of values of the parameter $t \in (-3, 9-6\sqrt{3}) \setminus \{-\frac{3}{2}\}$. Finally, for each admissible value of $t \in (-3, -\frac{3}{11})$ there is an infinite number of non-homothetic strict algebraic $\mathfrak{F}[t]$-solitons in each Family (1) and (2).

Remark 6.8. It follows from Remark 3.6 and the results in Theorem 4.8 and Theorem 6.6 (see also the corresponding Figure 3 and Figure 4), that the only non-rigid strict algebraic Bach soliton corresponds to case (1) in Theorem 6.6 for the parameters $d = \frac{1}{a}$, $a \in (0,1)$, i.e., the one-parameter family of left-invariant metrics on $\mathbb{R} \ltimes \mathcal{H}^3$ determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = a e_1, \quad [e_2, e_4] = \tfrac{1}{a} e_2, \quad [e_3, e_4] = \tfrac{a^2+1}{a} e_3,$$

for any $a \in (0,1)$. The underlying Lie algebra is $\mathfrak{d}_{4,\lambda}$ with $\lambda = \frac{a^2}{a^2+1}$ (cf. [1]), so that the corresponding Lie group is diffeomorphic to $\mathbb{R}^4$, and the left-invariant metric becomes

$$g = e^{-\frac{2a^2}{a^2+1}t} dx \circ dx + e^{-2t}\left(e^{\frac{2a^2}{a^2+1}t} + x^2\right) dy \circ dy + e^{-2t}(dz \circ dz - 2x\, dy \circ dz) + \tfrac{a^2}{(a^2+1)^2} dt \circ dt$$

where (x, y, z, t) are coordinates on $\mathbb{R}^4$.

Proof of Theorem 6.6. The proof is based on the analysis of the five cases obtained in Theorem 6.3. Note that, in view of Remark 6.4, we can omit the case Theorem 6.3-(ii.b) since the possible algebraic $\mathfrak{F}[t]$-solitons have already been covered in §4.2. Moreover note that, without loss of generality, we may set $\gamma = 1$ remaining in the same homothety class.

6.2.1. **Case (i.a) in Theorem 6.3.** In this first case, $c = 0$, $F = 0$ and $d \neq \pm a$. A direct calculation shows that

$$\mathfrak{F}[t]_{12} = \mathfrak{F}[t]_{14} = \mathfrak{F}[t]_{23} = \mathfrak{F}[t]_{34} = 0.$$

Moreover, $\mathfrak{F}[t]_{44} = -\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{22} - \mathfrak{F}[t]_{33}$ since the tensor $\mathfrak{F}[t]$ is trace-free. Now, imposing the diagonal form and the conditions in Theorem 6.3–(i.a) to the operator $\widehat{\mathfrak{F}[t]}$, we have that an algebraic $\mathfrak{F}[t]$-soliton is determined by the vanishing of the polynomials

$$\begin{aligned} &\mathfrak{Q}_1 = \mathfrak{F}[t]_{13}, \quad \mathfrak{Q}_2 = \mathfrak{F}[t]_{24}, \quad \mathfrak{Q}_3 = -2(\mathfrak{F}[t]_{11} + \mathfrak{F}[t]_{22}), \\ &\mathfrak{Q}_4 = H(\mathfrak{F}[t]_{11} + 2\mathfrak{F}[t]_{22} + \mathfrak{F}[t]_{33}), \\ &\mathfrak{Q}_5 = (5a+4d)\mathfrak{F}[t]_{11} + (4a+5d)\mathfrak{F}[t]_{22} + 3(a+d)\mathfrak{F}[t]_{33}, \\ &\mathfrak{Q}_6 = \mathfrak{F}[t]_{11} + \mathfrak{F}[t]_{22} + \mathfrak{F}[t]_{33} + \mu. \end{aligned} \tag{37}$$

Computing

$$2\mathfrak{Q}_2 = H\left((12a^2 + 12d^2 + 20ad + H^2 + 1)t + 4a^2 + 3d^2 + 2ad + 3H^2 + 3\right) \tag{38}$$

we analyse the vanishing of each one of the two factors above separately.

6.2.1.1. *Case $H=0$*. A direct calculation shows that Equation (37) reduces to

$$\begin{aligned} 2\mathfrak{Q}_3 &= \left(4(a^2+d^2+ad)-3\right)\left((4(3a^2+3d^2+5ad)+1)t+4(a^2+d^2+ad)+3\right), \\ -\mathfrak{Q}_5 &= (a+d)\mathfrak{Q}_3, \\ -8\mathfrak{Q}_6 &= \left(4(a^2+d^2-ad)-1\right)\left((4(3a^2+3d^2+5ad)+1)t+4(a^2+d^2+ad)+3\right) \\ &\quad -8(a-d)^2-8\mu. \end{aligned}$$

The corresponding left-invariant metric, given by

$$[e_1,e_2]=e_3, \quad [e_1,e_4]=ae_1, \quad [e_2,e_4]=de_2, \quad [e_3,e_4]=(a+d)e_3,$$

is an algebraic Ricci soliton with soliton constant $-\frac{3}{2}$ if $4(a^2+d^2+ad)-3=0$ (see Remark 6.5). Otherwise, we use $\mathfrak{Q}_3=0$ and $\mathfrak{Q}_6=0$ to obtain the values of t and μ, respectively, given by

$$t=-\frac{4(a^2+d^2+ad)+3}{4(3a^2+3d^2+5ad)+1} \quad \text{and} \quad \mu=-(a-d)^2.$$

Thus, for $d\neq\pm a$ and $4(a^2+d^2+ad)-3\neq 0$, we obtain an expanding algebraic $\mathfrak{F}[t]$-soliton for $t\in(-3,-\frac{3}{11})$ and a straightforward calculation shows that it is strict since it is never critical. The isomorphic isometry $(e_1,e_2,e_3,e_4)\mapsto(e_2,-e_1,e_3,e_4)$ interchanges (a,d) and (d,a), while $e_4\mapsto -e_4$ transforms (a,d) into $(-a,-d)$. Thus, we may assume $|a|<d$ and Family (1) is obtained.

6.2.1.2. *Case $H\neq 0$*. In this case from Equation (38) we get

$$t=-\frac{4a^2+3d^2+2ad+3H^2+3}{4(3a^2+5ad+3d^2)+H^2+1}$$

and (37) reduces to

$$\begin{aligned} 2\mathfrak{Q}_3 &= d(2a+d)\left(4(a^2+d^2+ad)+7H^2-3\right), \\ -2\mathfrak{Q}_4 &= H(2a+d)\left(4a^2d-(4a-3d)(H^2+1)\right), \\ -\mathfrak{Q}_5 &= (a+d)\mathfrak{Q}_3, \\ -8\mathfrak{Q}_6 &= 2d\left(2(2a+d)(a^2+d^2-ad)-5d\right)-(8a^2-d^2-14ad)(H^2+1)-8\mu. \end{aligned}$$

If $d=-2a$, then $\mu=-a^2(4H^2+9)$ determines an algebraic $\mathfrak{F}[t]$-soliton with associated left-invariant metric given by

$$[e_1,e_2]=e_3, \quad [e_1,e_4]=ae_1+He_3, \quad [e_2,e_4]=-2ae_2, \quad [e_3,e_4]=-ae_3,$$

with $aH\neq 0$. A straightforward calculation shows that these metrics are never critical. Moreover, the isomorphic isometries $(e_1,e_2,e_3,e_4)\mapsto(-e_1,-e_2,e_3,-e_4)$ and $(e_1,e_2,e_3,e_4)\mapsto(-e_1,-e_2,e_3,e_4)$ transform the parameters (a,H) into $(-a,H)$ and $(a,-H)$, respectively. Hence, we may restrict the parameters to $a>0$ and $H>0$. In this setting, $\mu=-a^2(4H^2+9)$ and $t=-\frac{3(4a^2+H^2+1)}{20a^2+H^2+1}\in(-3,-\frac{3}{5})$ determine the strict expanding algebraic $\mathfrak{F}[t]$-solitons given in Family (2).

If $d\neq -2a$, we note that d cannot vanish since, otherwise, the vanishing of $\mathfrak{Q}_4=4a^2H(H^2+1)$ would lead to $a=d=0$, which is not possible. Hence, $\mathfrak{Q}_3=\mathfrak{Q}_4=0$ yields

$$4(a^2+d^2+ad)+7H^2-3=0, \quad 4a^2d-(4a-3d)(H^2+1)=0,$$

and a direct calculation shows that, under these conditions, the algebraic $\mathfrak{F}[t]$-soliton is trivial since it corresponds to a critical metric (see [6]).

6.2.2. **Case (i.b) in Theorem 6.3.** We have $a = d \neq 0$ and $c = F = H = 0$. A direct calculation shows that the operator $\widehat{\mathfrak{F}[t]}$ is diagonal,

$$\widehat{\mathfrak{F}[t]} = \tfrac{(4a^2-1)\left((44a^2+1)t+12a^2+3\right)}{8}\operatorname{diag}[-3,-3,5,1].$$

Since $\mathfrak{F}[t]_{33}$ and $\mathfrak{F}[t]_{44}$ must be equal in this setting, any algebraic $\mathfrak{F}[t]$-soliton is trivial in this case.

6.2.3. **Case (ii.a) in Theorem 6.3.** The structure constants must satisfy $d = -a$, $a^2 \neq c^2$ and $F = H = 0$. A direct calculation shows that $\mathfrak{F}[t]_{13}$, $\mathfrak{F}[t]_{14}$, $\mathfrak{F}[t]_{23}$, $\mathfrak{F}[t]_{24}$ and $\mathfrak{F}[t]_{34}$ vanish, so that the diagonalizability of the operator $\widehat{\mathfrak{F}[t]}$ depends only on $\mathfrak{F}[t]_{12}$. Moreover, $\mathfrak{F}[t]_{11} = \mathfrak{F}[t]_{22}$. Also, $\mathfrak{F}[t]_{44} = -\mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{22} - \mathfrak{F}[t]_{33}$ since $\widehat{\mathfrak{F}[t]}$ is trace-free. Thus, the existence of algebraic $\mathfrak{F}[t]$-solitons is determined by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \mathfrak{F}[t]_{12}, \quad \mathfrak{Q}_2 = \mathfrak{F}[t]_{11}, \quad \mathfrak{Q}_3 = 2\mathfrak{F}[t]_{11} + \mathfrak{F}[t]_{33} + \mu,$$

which transform into

$$\begin{aligned} \mathfrak{Q}_1 &= 2ac\left((4a^2+1)t + 4(2a^2+c^2)\right), \\ -8\mathfrak{Q}_2 &= (4a^2+1)(4a^2-3)t + 16a^4 + 32a^2c^2 - 9, \\ -8\mathfrak{Q}_3 &= (4a^2+1)(12a^2-1)t + 48a^4 + 96a^2c^2 - 3 - 8\mu. \end{aligned} \tag{39}$$

We analyse the vanishing of the three factors in $\mathfrak{Q}_1$ separately.

6.2.3.1. <u>*Case $a = 0$*</u>. The expressions above lead to $t = -3$ and $\mu = 0$, and the corresponding left-invariant metric, given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -ce_2, \quad [e_2, e_4] = ce_1,$$

is $\mathcal{F}[-3]$-critical, and thus a Ricci soliton [6].

6.2.3.2. <u>*Case $c = 0$ and $a \neq 0$*</u>. The corresponding left-invariant metric, given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1, \quad [e_2, e_4] = -ae_2,$$

is an algebraic Ricci soliton with soliton constant $-\frac{3}{2}$ whenever $4a^2 - 3 = 0$. Otherwise, we use (39) to obtain $t = -\frac{4a^2+3}{4a^2+1}$ and $\mu = -4a^2$, so that the metric above is and algebraic $\mathfrak{F}[t]$-soliton. A straightforward calculation shows that these metrics are never critical. Moreover, since the isomorphic isometry $e_4 \mapsto -e_4$ interchanges a and $-a$, we may restrict the parameter to $a > 0$. Hence, taking $a > 0$ with $a \neq \frac{\sqrt{3}}{2}$, the soliton constant $\mu = -4a^2$ and the parameter $t = -\frac{4a^2+3}{4a^2+1} \in (-3, -1) \setminus \{-\frac{3}{2}\}$ determine the strict expanding algebraic $\mathfrak{F}[t]$-solitons corresponding to Family (3).

6.2.3.3. <u>*Case $ac \neq 0$*</u>. In this case, $\mathfrak{Q}_1 = 0$ in Equation (39) yields

$$t = -\frac{4(2a^2+c^2)}{4a^2+1}$$

and (39) reduces to

$$-8\mathfrak{Q}_2 = 4(4a^2+3)c^2 - (4a^2-3)^2, \quad \mathfrak{Q}_3 = 3\mathfrak{Q}_2 + 8a^2 + 4c^2 - 3 + \mu.$$

Hence, the conditions $4(4a^2+3)c^2 - (4a^2-3)^2 = 0$ and $\mu = -8a^2 - 4c^2 + 3$ make the left-invariant metric

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1 - ce_2, \quad [e_2, e_4] = ce_1 - ae_2,$$

with $ac \neq 0$ and $a^2 \neq c^2$, an algebraic $\mathfrak{F}[t]$-soliton. Moreover, a straightforward calculation shows that these metrics are never critical. Furthermore, the isomorphic isometry $(e_1, e_2, e_3, e_4) \mapsto (e_2, -e_1, e_3, -e_4)$ interchanges c and $-c$, while $(e_1, e_2, e_3, e_4) \mapsto (-e_1, e_2, -e_3, -e_4)$ interchanges a and $-a$, in accordance with the relation $4(4a^2+3)c^2 - (4a^2-3)^2 = 0$ in both cases. Hence, we may restrict the parameters to $a > 0$, $c > 0$, with $(a, c) \neq (\frac{1}{2}, \frac{1}{2})$ satisfying the previous relation. In this setting, $\mu = -\frac{12a^2(4a^2-1)}{4a^2+3}$ and $t = -\frac{3(16a^4+3)}{16a^4+16a^2+3} \in (-3, 9-6\sqrt{3}] \setminus \{-\frac{3}{2}\}$ determine strict algebraic $\mathfrak{F}[t]$-solitons which are shrinking when $t \in (-3, -\frac{3}{2})$ and expanding for any value of $t \in (-3, 9-6\sqrt{3}] \setminus \{-\frac{3}{2}\}$. These metrics correspond to Family (4).

6.2.4. **Case (ii.c) in Theorem 6.3.** We take $d = -a$ and $c = a \neq 0$. The tensor $\mathfrak{F}[t]$ must have the matrix form given in Theorem 6.3–(ii.c) and a direct calculation shows that $\mathfrak{F}[t]_{34} = 0$ and $\mathfrak{F}[t]_{13} = -\mathfrak{F}[t]_{23}$. Hence, an algebraic $\mathfrak{F}[t]$-soliton is characterized by the vanishing of the polynomials

$$\begin{aligned}
&\mathfrak{Q}_1 = \mathfrak{F}[t]_{13}, \qquad \mathfrak{Q}_2 = \mathfrak{F}[t]_{11} - \mathfrak{F}[t]_{22},\\
&\mathfrak{Q}_3 = \mathfrak{F}[t]_{33} - 2\mathfrak{F}[t]_{11} + (F-H)\mathfrak{F}[t]_{14} + \mu,\\
&\mathfrak{Q}_4 = \mathfrak{F}[t]_{44} + 2\mathfrak{F}[t]_{12} - \mu, \qquad \mathfrak{Q}_5 = \mathfrak{F}[t]_{24} - \mathfrak{F}[t]_{14},\\
&\mathfrak{Q}_6 = H\mathfrak{F}[t]_{11} - (F-2H)\mathfrak{F}[t]_{12} - (H(F-H)+1)\,\mathfrak{F}[t]_{14} - H\mu,\\
&\mathfrak{Q}_7 = F\mathfrak{F}[t]_{11} + (2F-H)\mathfrak{F}[t]_{12} - (F(F-H)-1)\,\mathfrak{F}[t]_{14} - F\mu.
\end{aligned} \tag{40}$$

Computing

$$-2\mathfrak{Q}_1 = a(F+H)(4a^2+F^2+H^2+1)(t+3)$$

it follows that either $t = -3$ or $H = -F$. If $t = -3$, then a straightforward calculation shows that Equation (40) reduces to

$$\begin{aligned}
2\mathfrak{Q}_3 = -2\mathfrak{Q}_4 &= 15a^2\left((F-H)^2+2\right) + 2\mu,\\
2\mathfrak{Q}_6 &= 3a^2(F-4H)\left((F-H)^2+2\right) - 2H\mu,\\
-2\mathfrak{Q}_7 &= 3a^2((F-H)^2+2)(4F-H) + 2F\mu,
\end{aligned}$$

which leads to

$$H\mathfrak{Q}_3 + \mathfrak{Q}_6 = \tfrac{3}{2}a^2\left((F-H)^2+2\right)(F+H).$$

Hence, in any case, $H = -F$ and therefore these metrics are isomorphically isometric to metrics on $\mathbb{R} \ltimes \mathbb{R}^3$ (see Remark 6.4) and thus the possible algebraic $\mathfrak{F}[t]$-solitons have already been covered by Theorem 4.8. □

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