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Leibniz cohomology in low degrees. Some structure theory of Leibniz n-algebras

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Memoria para optar al grado de Doctor realizada en el Departamento de Álgebra de la Universidad de Santiago de Compostela bajo la dirección de los Profesores D. Bakhrom Omirov y D. Manuel Ladra González.

Santiago de Compostela, a 28 de octubre de 2015.

Fdo.: Bakhrom Omirov Fdo.: Manuel Ladra González

Leibniz cohomology in low degrees. Some structure theory of Leibniz n-algebras

Dr. Bakhrom Omirov y Dr. Manuel Ladra González,

AUTORIZAMOS la presentación de la Tesis Doctoral con título **Leibniz cohomology in low degrees. Some structure theory of Leibniz n-algebras**, realizada por D. Rustam Turdibaev bajo nuestra dirección en el Departamento de Álgebra de la Universidad de Santiago de Compostela, para optar al grado de Doctor por la Universidad de Santiago de Compostela.

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Resumen de la Tesis Doctoral:

Leibniz cohomology in low degrees. Some structure theory of Leibniz n-algebras

Resumen abreviado:

Cohomología de Leibniz en grados inferiores. Alguna estructura de la teoría de n-álgebras de Leibniz

En esta tesis se presentan algunas herramientas para estudiar grupos de cohomología de álgebras de Leibniz con valores en sí mismo. Usando la descomposición de Levi para álgebras de Leibniz semisimples establecemos una descomposición más precisa de sus grupos de cohomología. Una mirada cercana a las cohomologías en grados bajos da resultados sobre derivaciones exteriores de álgebras Leibniz semisimples. Además, se establece un análogo de la descomposición de Jordan-Chevalley para álgebras de Leibniz. Moviéndonos a un objeto más general, se introducen varias nociones de solubilidad y nilpotencia de n-álgebras de Leibniz y se establece su invariancia por derivaciones. Se estudian los subálgebras de Frattini y Cartan de n-álgebras de Leibniz. Algunos resultados clásicos sobre estas subálgebras se extienden a n-álgebras de Leibniz, mientras que otros no. En particular, se muestran ejemplos de que un enunciado sobre conjugación de subálgebras de Cartan de álgebras de Lie, que también se verifica en álgebras de Leibniz y n-álgebras de Lie, no se verifica para n-álgebras de Leibniz.

Las álgebras de Leibniz sobre un cuerpo $\mathbb{K}$ fueron introducidas por primera vez por A. Bloh [13] y más tarde redefinidas por J.-L. Loday en [35, 37] como un $\mathbb{K}$-módulo L con una aplicación bilineal llamada y denotada por un corchete $[-,-]\colon L \times L \to L$ que verifica la llamada identidad de Leibniz $[[x,y],z] = [[x,z],y] + [x,[y,z]]$. Si el corchete se factoriza a través de $L \wedge L$ entonces el álgebra de Leibniz se vuelve un álgebra de Lie, y así un álgebra de Leibniz es una versión no antisimétrica de un álgebra de Lie. Denotamos por I el ideal generado por los cuadrados de elementos de L. Existe una sucesión exacta corta $0 \to I \to L \xrightarrow{f} \mathfrak{g} \to 0$, donde $\mathfrak{g} := L/I$ es un álgebra de Lie llamada la liezación de L. La proyección f es universal en el sentido de que una aplicación de Leibniz de L a cualquier álgebra de Lie se factoriza a través de f. Se establecen también muchos resultados de la teoría de estructura de álgebras de Lie para álgebras de Leibniz.

Una *representación* M de un álgebra de Leibniz L, introducida en [37], es un $\mathbb{K}$-módulo dotado con dos acciones a la izquierda y derecha de L (denotado como corchetes) que verifica la identidad de Leibniz si un elemento es de M.

Una *cohomología* de un álgebra de Leibniz L con coeficientes en la representación M están definidas en [37] de la siguiente forma. Definimos el espacio de las *n-cocadenas* $CL^n(L,M) = \operatorname{Hom}_k(L^{\otimes n}, M)$ para $n \geq 0$ y un $\mathbb{K}$-homomorfismo $\partial^n \colon CL^n(L,M) \to CL^{n+1}(L,M)$ por

$$(\partial^n f)(x_1,\dots,x_{n+1}) = [x_1, f(x_2,\dots,x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1,\dots,\widehat{x}_i,\dots,x_{n+1}), x_i]$$
$$+ \sum_{1\leq i<j\leq n+1} (-1)^{j+1} f(x_1,\dots,x_{i-1},[x_i,x_j],x_{i+1},\dots,\widehat{x}_j,\dots,x_{n+1}).$$

Este $(CL^*(L,M),\partial)$ es un complejo de cocadenas. Su n-ésimo grupo de cohomología está bien definido por $HL^n(L,M) := ZL^n(L,M)/BL^n(L,M)$, donde los elementos $ZL^n(L,M) := \ker \partial^n$ y $BL^n(L,M) := \operatorname{im} \partial^{n-1}$ se llaman *n-cociclos* y *n-cofronteras*, respectivamente.

En el Capítulo 1 desarrollamos algunas técnicas para estudiar la cohomología de álgebras de Leibniz. Sea V un espacio vectorial sobre un $\mathbb{K}$ y $V = V_0 \oplus V_1$ una descomposición no-trivial. El objetivo de la Sección 1.2 es presentar la descomposición de $\operatorname{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$ que será usada en gran medida en las siguientes secciones.

Para un número natural n y un entero k definimos $\mathcal{I}(n,k)$ como el conjunto de las biyecciones de $\{1,2,\dots,n\}$ al conjunto de k unos $n-k$ ceros para $0 \leq k \leq n$ y $\mathcal{I}(n,k) = \emptyset$ en otro caso. Para una biyección $\pi \in \mathcal{I}(n,k)$ denotamos de una forma abreviada $H^j_{\pi(1)\dots\pi(n)} := \operatorname{Hom}_{\mathbb{K}}(V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}, V_j)$, donde j es 0 o 1.

Denotamos por $CL^n(V,V) = \operatorname{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$ y definimos para $-n \leq k \leq 1$ el subespacio $CL^n(V,V)_{(-k)} = \left(\underset{\pi\in\mathcal{I}(n,k)}{\oplus} H^0_{\pi(1)\dots\pi(n)}\right) \oplus \left(\underset{\pi\in\mathcal{I}(n,k+1)}{\oplus} H^1_{\pi(1)\dots\pi(n)}\right)$. Además, para cualquier $H^j_{\pi(1)\dots\pi(n)}$ en esta descomposición se tiene $k = j - (\pi(1) + \cdots + \pi(n))$.

Llamamos el subespacio $CL^n(V,V)_{(k)}$ de *nivel* k. El siguiente enunciado presenta una descomposición isomorfa a $CL^n(V,V)$ en subespacios de nivel $-n,\dots,0,1$.

Proposición 1.2.1. $CL^n(V,V) \cong \bigoplus_{k=-n}^{1} CL^n(V,V)_{(k)}$.

Se dice que un álgebra de Leibniz L es *semisimple* si su liezación $\mathfrak{g}$ es semisimple. L se dice *simple* si el único ideal no-trivial de L es $I \neq [L,L]$. Esto concuerda con la definición sugerida en [1].

Ahora consideramos L como una representación *adjunta* de L, i.e., L actuando sobre sí misma con el corchete de álgebra de Leibniz. Como fue señalado en [37] $HL^1(L,L)$ es el espacio de las derivaciones exteriores del álgebra de Leibniz L. Además, tenemos $L \cong I \oplus \mathfrak{g}$ como espacios vectoriales y por la Proposición 1.2.1 el espacio de n-cocadenas es isomorfo a una suma directa de subespacios de nivel.

Pirashvili [44] probó que dado un epimorfismo de Leibniz de un álgebra de Leibniz L a un álgebra de Lie semisimple, existe una sección. Por lo tanto, para un álgebra de Leibniz semisimple L tenemos una suma semidirecta $L = I \dotplus \mathfrak{g}$ del ideal I y su liezación $\mathfrak{g}$.

En la Proposición 1.3.1 se establece que ∂ conserva los espacios de nivel, i.e. $\partial(CL^n(L,L)_{(k)}) \subseteq CL^{n+1}(L,L)_{(k)}$ para todo $-n \leq k \leq 1$ y así tenemos

Teorema 1.3.3. *Sea L un álgebra de Leibniz semisimple de dimensión finita. Entonces*

$$HL^n(L,L) \cong HL^n(L,L)_{(-n+1)} \oplus \cdots \oplus HL^n(L,L)_{(-1)} \oplus HL^n(L,L)_{(0)}$$

para $n \geq 2$ y $HL^1(L,L) \cong HL^1(L,L)_{(0)} \oplus HL^1(\mathfrak{g},I)$.

El caso $n=2$ merece una atención especial a la cual se dedica la Sección 1.4. Existe una conjetura en [2] de que $HL^2(L,L)=0$ para cualquier álgebra de Leibniz semisimple L. Los autores en [2] validan la afirmación para un álgebra de Leibniz simple con $L/I \cong \mathfrak{sl}_2$. En general, establecemos lo siguiente

Teorema 1.4.4. *Para un álgebra de Leibniz semisimple de dimensión finita L sobre $\mathbb{C}$ tenemos un isomorfismo $HL^2(L,L) \cong HL^2(L,L)_{(-1)} \oplus HL^2(L,L)_{(0)}$, donde*

$$\begin{aligned} HL^2(L,L)_{(-1)} &= \ker \partial|_{H^0_{10}\oplus H^0_{01}\oplus H^1_{11}} / \partial(H^0_1) \qquad y \\ HL^2(L,L)_{(0)} &= \ker \partial|_{H^0_{00}\oplus H^1_{10}} / \partial(H^1_1 \oplus H^0_0). \end{aligned}$$

Con el fin de proceder con la conjetura estudiamos por separado cada uno de estas componentes y llegamos a los siguientes enunciados.

Proposición 1.4.5. *Cualquier 2-cociclo* $\varphi = \varphi^0_{10} + \varphi^0_{01} + \varphi^1_{11} \in \ker \partial|_{H^0_{10} \oplus H^0_{01} \oplus H^1_{11}}$ *está determinado de modo único por una aplicación* $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}) \to \mathfrak{g}$ *que verifica* $\phi(x_1, [y_0, z_0]) = \phi([x_1, y_0], z_0) - \phi([x_1, z_0], y_0)$*. Además,* φ *está determinado por las siguientes igualdades*

$$\begin{cases} \varphi^0_{01}(x_0, [z_1, y_0]) = & -[x_0, \phi(z_1, y_0)] \\ \varphi^1_{11}(x_1, [z_1, y_0]) = & -[x_1, \phi(z_1, y_0)] \\ \varphi^0_{10}([x_1, y_0], z_0) = & \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0]. \end{cases}$$

La liezación de L denotada por $\mathfrak{g}$, es un álgebra de Lie compleja semisimple de dimensión finita e I se puede considerar como un $\mathfrak{g}$-módulo de dimensión finita por la acción inducida del álgebra de Leibniz. Denotamos por $i.g$ la acción de $g \in \mathfrak{g}$ sobre $i \in I$. Gracias a la proposición anterior para comprobar si $HL^2(L, L) = 0$ llegamos a enunciados equivalentes que se expresan en términos de álgebras de Lie y sus módulos.

Proposición 1.4.6. *Sea* $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$*. Entonces* $HL^2(L, L)_{(-1)} = 0$ *si y solo si*

$$\phi(i, [g_1, g_2]) = \phi(i.g_1, g_2) - \phi(i.g_2, g_1),$$

para todo $g_1, g_2 \in \mathfrak{g}, i \in I$ *garantiza la existencia de un* $d \in \mathrm{Hom}(I, \mathfrak{g})$ *tal que* $\phi(i, g) = d(i.g)$*.*

Proposición 1.4.7. *Sea* L *un álgebra de Leibniz semisimple sobre* $\mathbb{C}$*. El enunciado* $HL^2(L, L)_{(0)} = 0$ *es válido si y solo si para cualquier aplicación* $\psi \in \mathrm{Hom}(I \otimes \mathfrak{g}, I)$ *que verifica*

$$[\psi(i, g_1), g_2] - [\psi(i, g_2), g_1] - \psi(i, [g_1, g_2]) + \psi(i.g_1, g_2) - \psi(i.g_2, g_1) = 0,$$

se deduce que existen $g_0 \in \mathfrak{g}$ *y* $d \in \mathrm{Hom}(I, I)$ *tal que*

$$\psi(i, g) = i.[g_0, g] + d(i).g - d(i.g).$$

En la última sección del Capítulo 1 comprobamos estas conjeturas para algunas álgebras de Leibniz semisimples.

Ya que el estudio de las propiedades de las derivaciones y automorfismos de álgebras de Lie juegan un papel esencial en la teoría de álgebras de Lie, la cuestión que surge de modo natural es si los resultados correspondientes se pueden extender al marco más general de las álgebras de Leibniz.

En el Capítulo 2 consideramos algunas propiedades de derivaciones y automorfismos de álgebras de Leibniz. Extendemos algunos resultados clásicos obtenidos para derivaciones y automorfismos de álgebras de Lie por Felix Gantmacher [25] y Nathan Jacobson [28] al caso de álgebras de Leibniz.

El siguiente resultado es un análogo de la descomposición de Jordan–Chevalley, que expresa una derivación de un álgebra de Leibniz como la suma de sus partes semisimples y nilpotentes que conmutan.

Teorema 2.2.2. *Sea D una derivación de un álgebra de Leibniz L. Entonces existe una única derivación diagonalizable D_0 y una única derivación nilpotente tal que $D = D_0 + T$ y $D_0 T = T D_0$.*

Gantmacher [25], en la teoría de álgebras de Lie, probó que cualquier automorfismo de álgebras de Lie se descompone en el producto de un automorfismo semisimple y el exponente de una derivación nilpotente que conmutan. La verificación de un resultado análoga en el caso de álgebras de Leibniz es exitosa.

Teorema 2.2.7. *Sea A un automorfismo de un álgebra de Leibniz. Entonces existe un único automorfismo diagonalizable A_0 y una única derivación nilpotente T tal que $A = A_0 \exp(T)$ y $A_0 T = T A_0$.*

Recordemos que para un álgebra de Leibniz las series derivada y central inferior se introducen como en álgebras de Lie

$$\text{(i)} \qquad L^{(1)} = L, \qquad L^{(n+1)} = [L^{(n)}, L^{(n)}], \qquad n > 1;$$

$$\text{(ii)} \qquad L^{1} = L, \qquad L^{n+1} = [L^{n}, L], \qquad n > 1.$$

Un álgebra L se dice soluble (nilpotente) si existe $s \in \mathbb{N}$ ($k \in \mathbb{N}$, respectivamente) tal que $L^{(s)} = 0$ ($L^k = 0$, respectivamente).

En 1955, Jacobson [28] probó que toda álgebra de Lie sobre un cuerpo de característica cero que admita una derivación no singular es nilpotente. El problema de si el recíproco de esta afirmación es correcto permaneció abierto hasta que fue construido en [23] un ejemplo de un álgebra de Lie nilpotente en la cual toda derivación era nilpotente (y por lo tanto, singular). Las álgebras de Lie nilpotentes con esta propiedad fueron denominadas álgebras de

Lie característicamente nilpotentes. En [32] se demostró que cada componente irreducible de la variedad de álgebras de Lie complejas filiformes de dimensión superior a 7 contiene un conjunto abierto de Zariski que consta de álgebras de Lie característicamente nilpotentes. Nótese que entre las álgebras de Lie nilpotentes de dimensión inferior a 7, no se producen álgebras de Lie característicamente nilpotentes debido a la clasificación dada en [26].

Establecemos en el Teorema 2.3.1 que un álgebra de Leibniz compleja de dimensión finita que admita una derivación no degenerada es nilpotente. Al igual que en el caso de Lie, la inversa de esta afirmación no se verifica debido a la aparición de familias de álgebras de Leibniz característicamente nilpotentes, i.e., álgebras en las que toda derivación es nilpotente. Se establece en [42] que las álgebras de Leibniz, no de Lie, filiformes característicamente nilpotentes aparecen a partir de dimensión 5.

Además, establecemos en el Teorema 2.3.5 que cualquier derivación deja invariante el ideal soluble maximal (radical) de un álgebra de Leibniz.

En la última sección del Capítulo 2 indagamos detenidamente en la primera cohomología de álgebras de Leibniz semisimples.

Teorema 2.4.1. *Para un álgebra de Leibniz semisimple de dimensión finita con liezación* $\mathfrak{g}$ *sobre* $\mathbb{C}$ *tenemos una descomposición*

$$HL^1(L, L) \cong \mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{g}, I) \oplus \mathrm{Hom}_{U\mathfrak{g}}(I, I),$$

donde $U\mathfrak{g}$ *es el álgebra envolvente universal de* $\mathfrak{g}$.

En términos de derivaciones se puede reformular el teorema anterior como sigue.

Corolario 2.4.2. *Sea* L *un álgebra de Leibniz semisimple compleja de dimensión finita con liezación* $\mathfrak{g}$. *Entonces cualquier derivación se descompone en una suma de una derivación interior y homomorfismos* $\mathfrak{g}$*-equivariantes* $f\colon \mathfrak{g} \to I$ *y* $\alpha\colon I \to I$.

En [46] fue obtenida una afirmación similar con métodos diferentes.

Si L es un álgebra de Leibniz simple, usando el famoso Lema de Schur, obtenemos

Corolario 2.4.3. *Sea* L *un álgebra de Leibniz simple compleja de dimensión finita con liezación* $\mathfrak{g}$. *Si* $\dim L = 2\dim \mathfrak{g}$, *entonces* $HL^1(L, L) \cong \mathbb{C} \oplus \mathbb{C}$. *En otro caso,* $HL^1(L, L) \cong \mathbb{C}$.

El último capítulo de esta tesis, está dedicado a la investigación de alguna estructura de la teoría de n-álgebras de Leibniz. En 1985, Filippov [24] introdujo una noción de n-álgebra de Lie con una multiplicación n-aria anti simétrica que verifica la identidad

$$[[x_1, x_2, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

teniendo en cuenta la noción general de Ω-álgebra considerada por Kurosh [33].

Anteriormente en 1973, Nambu [40] había construido un ejemplo de 3-álgebra Lie, donde la multiplicación de un triple de observables clásicos sobre el espacio de fases en dimensión tres $\mathbb{R}^3$ era dada por el jacobiano. Este corchete generaliza de manera natural, el corchete de Poisson habitual de un operador binario a uno ternario.

Como una generalización de las álgebras de Leibniz y n-álgebras de Lie, en 2002, Casas, Loday y Pirashvili [20] definieron n-álgebras de Leibniz como una versión no antisimétrica de las n-álgebras de Lie. También presentaron construcciones entre las variedades de álgebras de Leibniz y n-álgebras de Leibniz ($n \geq 3$) que no son inversibles. Los resultados necesarios sobre la teoría de n-álgebras de Leibniz y referencias se dan en la Sección 3.1.

Debido a la no antisimetría se pueden introducir algunas variaciones de nociones como la de ideal dependiendo de la posición de la multiplicación.

Definición 3.1.3. *Un subespacio I de una n-álgebra de Leibniz L se dice un s-ideal de L, si*

$$[\underbrace{L, \ldots, L}_{s-1}, I, \underbrace{L, \ldots, L}_{n-s}] \subseteq I.$$

Si I es un s-ideal para todo $1 \leq s \leq n$, entonces I es llamado un ideal.

En el estudio de n-álgebras de Leibniz usamos muchas propiedades de las derivaciones.

Definición 3.1.5. *Una aplicación lineal d definida sobre una n-álgebra de Leibniz L se llama una derivación si*

$$d([x_1, x_2, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots d(x_i), \ldots, x_n].$$

El espacio de todas las derivaciones de una n-álgebra de Leibniz L dada, se denota por $\mathrm{Der}(L)$.

El espacio $\mathrm{Der}(L)$ forma un álgebra de Lie con respecto al conmutador [4]. Sea $A^{\times k} = \underbrace{A \times A \times \cdots \times A}_{k-\text{times}}$. Dado un elemento arbitrario $x = (x_2, \ldots, x_n) \in L^{\times(n-1)}$ consideremos el operador $R[x]\colon L \to L$ de la multiplicación por la derecha definido para todo $z \in L$ por

$$R[x](z) = [z, x_2, \ldots, x_n].$$

Cualquier operador de la multiplicación por la derecha es una derivación y el espacio $R[L]$ de todos los operadores de la multiplicación por la derecha forma un ideal de Lie de $\mathrm{Der}(L)$ [4].

Filippov [24] señaló que los operadores de la multiplicación por la derecha juegan el mismo papel crucial en la teoría n-álgebra de Lie como en las álgebras de Lie ya que forman un álgebra de Lie con respecto al conmutador. El espacio de los operadores de la multiplicación por la derecha en n-álgebras de Leibniz también es un ideal del álgebra de Lie de derivaciones. Sin embargo, en el caso $n \geq 3$, algunas propiedades bien conocidas de los operadores de la multiplicación por la derecha no se verifican en general. Por ejemplo, en [4] se da un ejemplo de una n-álgebra de Leibniz que admite un operador de la multiplicación por la derecha no degenerado.

Nótese que el ideal generado por los cuadrados

$$I = \text{ideal}\,\langle [x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] \mid \exists i, j : x_i = x_j \rangle,$$

de manera similar a las álgebras de Leibniz tiene la propiedad que una n-álgebra de Leibniz cociente es una n-álgebra de Lie. Por otra parte, en este caso resulta que I puede coincidir con toda la n-álgebra de Leibniz.

Lema 3.2.2. *Sea L una n-álgebra de Leibniz. Si admite un operador de la multiplicación por la derecha no degenerado entonces $I = L$.*

Sin embargo, la afirmación recíproca no es verdad como señalamos en el Ejemplo 3.2.3.

En la Sección 3.3 estudiamos solubilidad y nilpotencia de n-álgebras de Leibniz y demostramos que los radicales solubles y nilpotentes son invariantes por todas las derivaciones. Ya que la multiplicación en n-álgebras de Leibniz no es antisimétrica en todas las variables, se pueden introducir nociones como nilpotencia y solubilidad de diferentes maneras dependiendo de la posición del multiplicando. El producto en la definición de la serie correspondientes no es necesariamente un ideal y esto hace que algunos argumentos sean difíciles de probar. Por lo tanto, introducimos nociones especiales, como k-solubilidad, s-nilpotencia, nilpotencia y K_1-nilpotencia de n-álgebras de Leibniz. Empezamos con la definición de sucesiones crecientes de subespacios para un ideal dado, I:

- $$I^{(m+1)_k} = \sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L,\ldots,L}_{i_1}, I^{(m)_k}, \underbrace{L,\ldots,L}_{i_2}, I^{(m)_k}, \ldots, \underbrace{L,\ldots,L}_{i_k}, I^{(m)_k}, \underbrace{L,\ldots,L}_{n-k-i_1-\cdots-i_k}],$$

  donde $I^{(1)_k} = I$, $1 \leq k \leq n$, $m \geq 1$.

  - Si existe un número $m \in \mathbb{N}$ tal que $I^{(m)_k} = 0$, entonces I se dice un ideal k-soluble. Tal número minimal m se dice índice de k-solubilidad.
  - La suma de ideales k-solubles es un ideal k-soluble.
  - El ideal k-soluble maximal es llamado el radical k-soluble de L.

- $I^{[k+1]} = [I^{[k]}, I, L, \ldots, L]$, donde $I^{[1]} = I$, $k \geq 1$.

  - Si existe un número $m \in \mathbb{N}$ tal que $I^{[m]} = 0$, entonces I es llamado un ideal K_1-nilpotente. Tal número minimal m se dice índice de K_1-nilpotencia.
  - La suma de ideales K_1-nilpotentes es un ideal K_1-nilpotente.
  - El ideal K_1-nilpotente maximal es llamado el K_1-nilradical de L.

- $I^{<k+1>_s} = [\underbrace{L,\ldots,L}_{(s-1)-\text{veces}}, I^{<k>_s}, \underbrace{L,\ldots,L}_{(n-s)-\text{veces}}]$, donde $I^{<1>_s} = I$, $1 \leq s \leq n$ y $k \geq 1$.

  - Si existe un número $m \in \mathbb{N}$ tal que $I^{<m>_s} = 0$, entonces I se dice un ideal s-nilpotente. Tal número minimal m se dice índice de s-nilpotencia.
  - La suma de ideales s-nilpotentes es un ideal s-nilpotente.

– El ideal s-nilpotente maximal es llamado el s-nilradical de L.

- $I^{k+1} = \sum_{i=1}^{n} [\underbrace{L, \ldots, L}_{(i-1)-\text{veces}}, I^k, \underbrace{L, \ldots, L}_{(n-i)-\text{veces}}]$, donde $I^1 = I$ y $k \geq 1$.

  – Si existe un número $m \in \mathbb{N}$ tal que $I^m = 0$, entonces I se dice un ideal nilpotente. Tal número minimal m se dice índice de nilpotencia.

  – La suma de ideales nilpotentes es un ideal nilpotente.

  – El ideal nilpotente maximal es llamado el nilradical de L.

En los Teoremas 3.3.8, 3.3.12, 3.3.16, 3.3.17 se demuestra que para una n-álgebra de Leibniz estos radicales y nilradicales introducidos son invariantes con respecto a una derivación de una n-álgebra. Estos son generalizaciones naturales de los resultados correspondientes al caso de un álgebra de Leibniz que es establecido en el Teorema 2.3.5.

La Sección 3.4 se dedica al estudio de subálgebras de Frattini de n-álgebras de Leibniz. La sección consta de 3 subsecciones. En la Subsección 3.4.1 introducimos la subálgebra de Frattini de una n-álgebra de Leibniz y establecemos algunas propiedades elementales que extienden resultados correspondientes de álgebras de Leibniz y de n-álgebras de Lie.

Definición 3.4.2. *La intersección de todas las subálgebras maximales de una n-álgebra de Leibniz L es una subálgebra denotada por $F(L)$ y llamada la subálgebra de Frattini.*

El ideal maximal de L que está contenido en $F(L)$ es llamado el ideal de Frattini y se denota por $\phi(L)$.

Definición 3.4.18. *En una n-álgebra de Leibniz L la intersección de todos los ideales maximales de L es llamado el radical de Jacobson radical y denotado por $J(L)$.*

La teoría de Frattini fue descubierta originalmente en la teoría de grupos [48] y más tarde se ha estudiado en álgebras de Lie en [39, 8, 49], en n-álgebras de Lie en [7, 51] y en álgebras de Leibniz en [9, 11]. Algunos resultados destacados relativos a subálgebras de Frattini e ideales de Frattini de la teoría de n-álgebras de Lie siguen siendo ciertos cuando omitimos la propiedad antisimétrica de la multiplicación n-aria. El siguiente resultado es una generalización del teorema correspondiente de la teoría de grupos probado en el libro de la teoría de grupos de W.R. Scott [48], de la teoría de álgebras de Lie [21] y de la teoría de n-álgebras de Lie [7], [8] y [51].

Teorema 3.4.20. *Sea L una n-álgebra de Leibniz nilpotente de dimensión finita. Entonces las siguientes afirmaciones se verifican:*

1. *Cualquier subálgebra maximal M de L es un ideal de L,*
2. $F(L) = \phi(L) = J(L) = [L, L, \dots, L]$.

En la Subsección 3.4.2, estudiamos los operadores de la multiplicación por la derecha en una n-álgebra de Leibniz. Debido a las propiedades curiosas de estos operadores para ser no degenerados en algunas n-álgebras, y para obtener algunos resultados sobre los operadores de la multiplicación a la derecha que sean válidos para álgebras de Leibniz y n-álgebras de Lie debemos considerarlos con condiciones adicionales. El estudio de su comportamiento lleva a las siguientes relaciones entre nilpotencia de una n-álgebra de Leibniz y el comportamiento de sus ideales maximales, así como las subálgebras de Frattini en la Subsección 3.4.3.

Corolario 3.4.23. *Supongamos que para cualesquiera $a_2, \dots, a_n$, elementos de una n-álgebra de Leibniz L tenemos que $a_2, \dots, a_n \in L_{0R[a_2,\dots,a_n]}$ y toda subálgebra maximal es un i- y j-ideal para algún $1 \leq i \neq j \leq n$ en L. Entonces L es una n-álgebra de Leibniz 1-nilpotente.*

Aquí por $L_{0R[a_2,\dots,a_n]}$ denotamos el espacio asociado al autovalor cero del operador lineal $R[a_2, \dots, a_n]$.

Proposición 3.4.24. *Sea L una n-álgebra de Leibniz de dimensión finita con la condición de que para un arbitrario $(a_2, \dots, a_n) \in L^{\times(n-1)}$ y para algún $2 \leq i \leq n$ tenemos $a_i \in L_{0R[a_2,\dots,a_n]}$. Supongamos que cualquier subálgebra maximal M de L es un ideal de L. Entonces L es 1-nilpotente.*

La última sección está dedicada a la investigación sobre subálgebras de Cartan de n-álgebras de Leibniz que fueron estudiadas en [20], [16] y [4].

Definición 3.5.1 ([4]). *Una subálgebra C de una n-álgebra de Leibniz L se dice subálgebra de Cartan si* *i) C es 1-nilpotente,*

ii) $C = N_1(C)$.

En [4], se demostró que el subespacio de las raíces nulas de los operadores de la multiplicación por la derecha con respecto a un elemento regular es una subálgebra nilpotente. Aquí obtenemos que esta subálgebra bajo algunas restricciones de la Subsección 3.4.2 es una subálgebra de Cartan.

Proposición 3.5.7. *Sea L una n-álgebra de Leibniz sobre un cuerpo infinito y $x = (x_2, \ldots, x_n) \in L^{\times(n-1)}$ un elemento regular de L tal que*

$$x_2, \ldots, x_n \in L_{0R[x]}.$$

Entonces $L_{0R[x]}$ es una subálgebra de Cartan de L.

Por otra parte, un resultado clásico acerca de conjugación de subálgebras de Cartan en álgebras de Lie que se extendió a los casos generales de álgebras de Leibniz [43] y n-álgebras de Lie [31], desgraciadamente no se verifica en el caso de n-álgebras de Leibniz ($n \geq 3$). En particular, construimos ejemplos que muestran la no conjugación de las subálgebras Cartan para n-álgebras de Leibniz.

Ejemplo 3.5.17. *Sea L_s $(1 \leq s \leq n+1)$ una n-álgebra con base*

$$\langle e_1, e_2, \ldots, e_{n+1}, x_1, \ldots, x_m \rangle$$

y la siguiente multiplicación:

$$\begin{aligned}
[e_1, \ldots, e_{p-1}, e_{p+1}, \ldots, e_{n+1}] &= e_p\,, & 1 \leq p \leq n+1,\\
[x_k, e_k, e_k, \ldots, e_k] &= x_k\,, & 1 \leq k \leq s,\\
[x_{s+i}, e_s, e_s, \ldots, e_s] &= x_{s+i}\,, & 1 \leq i \leq m-s\,,
\end{aligned}$$

donde la multiplicación es antisimétrica en todas las variables sobre el subespacio $\langle e_1, e_2, \ldots, e_{n+1} \rangle$. Entonces L_s es una n-álgebra de Leibniz.

Entonces, como mostramos en las Proposiciones 3.5.18–3.5.21 para esta n-álgebra de Leibniz los siguientes espacios son subálgebras de Cartan de dimensiones diferentes:

- de dimensión $n-1$

$$\begin{aligned} R_1 = & \quad \langle e_1, e_2, \ldots, e_s, e_{s+1}, \ldots, e_{n-1}\rangle, \\ R_2 = & \quad \langle e_1, e_2, \ldots, e_s, e_{s+2}, \ldots, e_{n-1}, e_n\rangle, \\ R_3 = & \quad \langle e_1, e_2, \ldots, e_s, e_{s+3}, \ldots, e_{n-1}, e_n, e_{n+1}\rangle. \end{aligned}$$

- de dimensión n

$$S_i = \langle e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n\rangle \quad \text{para todo} \quad 1 \leq i \leq s-1.$$

- de dimensión $m+n-s$

$$\begin{aligned} T_1 &= \langle e_1, e_2, \ldots, e_{s-1}, e_{s+1}, e_{s+2}, \ldots, e_n, x_s, x_{s+1}, \ldots, x_m\rangle, \\ T_2 &= \langle e_1, e_2, \ldots, e_{s-1}, e_{s+2}, e_{s+3}, \ldots, e_{n+1}, x_s, x_{s+1}, \ldots, x_m\rangle. \end{aligned}$$

- de dimensión $m+n-s+1$

$$U_i = \langle e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_{s-1}, e_{s+1}, \ldots, e_{n+1}, x_s, \ldots, x_m\rangle$$

para todo $1 \leq i \leq s-1$.

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Introduction

In 1993, Loday [35, 37] introduced a non skew-symmetric version of Lie algebras, the so-called Leibniz algebras. Leibniz algebras over $\mathbb{K}$ were first introduced by A. Bloh [13] and called D-algebras. However, it is due to J.-L. Loday's rediscovery in [35, 37] that they become an interesting topic of research. Considering a lifting of a classical Chevalley-Eilenberg chain complex $(C_*(\wedge^\bullet \mathfrak{g}, M), d)$ to $(C_*(\otimes^\bullet \mathfrak{g}, M), \partial)$ Loday noticed that the only property needed from algebra is a so called Leibniz rule. It leads to definition of Leibniz algebras and Leibniz homology. Leibniz algebra is a $\mathbb{K}$-module L with a bilinear map called and denoted as a bracket $[-,-]\colon L \times L \to L$ that satisfies a so-called Leibniz identity $[[x,y],z] = [[x,z],y] + [x,[y,z]]$. If the bracket factors through $L \wedge L$ then Leibniz algebra turns into a Lie algebra, thus Leibniz algebra is a non-antisymmetric version of Lie algebra. Denote by I an ideal generated by squares of elements of L. There is a short exact sequence $0 \to I \to L \xrightarrow{f} \mathfrak{g} \to 0$, where $\mathfrak{g} := L/I$ is a Lie algebra called liezation of L. The projection f is universal in the sense that a Leibniz map from L to any Lie algebra factors through f. Many results of the structure theory of Lie algebras are also established in Leibniz algebras.

A Leibniz algebra is semi-simple if its liezation is semi-simple Lie algebra. They can be naturally consider as a $\mathbb{Z}_2$-graded algebras with L_0 being isomorphic to their liezation and $L_1 = I$. Following the idea of work of [2] in Chapter 1 some technics to study cohomology of Leibniz algebras are developed and a decomposition of cochain space $CL^n(L,L)$ is presented. Due to structural properties of semi-simple Leibniz algebras, coboundary operator preserves the level spaces. It makes possible to establish in Theorem 1.3.3 a decomposition of $HL^n(L,L)$ into direct sum of n subspaces, that we refer to as level subspaces from $-n+1$ to 0.

Our goal is to check proposed conjecture in [2] that $HL^2(L,L) = 0$ for any finite-dimensional complex semisimple Leibniz algebra L. Authors in [2] validate the claim for simple Leibniz algebra with liezation $L/I \cong \mathfrak{sl}_2$ over $\mathbb{C}$.

In order to conjecture to hold, we elaborate on each of the subspaces in the decomposition of $HL^2(L, L)$ to be zero and present equivalent conditions to the conjecture to be true in Proposition 1.4.6 and Proposition 1.4.7.In the last section, we establish that space of level -1 is zero for all finite-dimensional Leibniz algebras with liezation $\mathfrak{sl}_2$ in Theorem 1.5.1 and we check analogous statement for another Leibniz algebra with liezation $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ in Theorem 1.5.2.

Since the study of the properties of derivations and automorphisms of Lie algebras play an essential role in the theory of Lie algebras, the question naturally arises whether the corresponding results can be extended to the more general framework of the Leibniz algebras.

In Chapter 2 we consider some general properties of derivations and automorphisms of Leibniz algebras. We extend some classical results obtained for derivations and automorphisms of Lie algebras by Felix Gantmacher [25] and Nathan Jacobson [28] to the case of Leibniz algebras in Theorems 2.2.7 and 2.2.2, respectively.

In 1955, Jacobson [28] proved that every Lie algebra over a field of characteristic zero admitting a nonsingular derivation is nilpotent. The problem whether the converse of this statement is correct remained open until an example of a nilpotent Lie algebra in which every derivation is nilpotent (and hence, singular) was constructed in [23]. Nilpotent Lie algebras with this property were named characteristically nilpotent Lie algebras. In [32] it was proved that every irreducible component of the variety of complex filiform Lie algebras of dimension greater than 7 contains a Zariski open set consisting of characteristically nilpotent Lie algebras. Note that among nilpotent Lie algebras of dimension less than 7, characteristically nilpotent Lie algebras do not occur due to the classification given in [26].

In Theorem 2.3.1 it is proved that a finite dimensional complex Leibniz algebra admitting a non-degenerate derivation is nilpotent. Similar to the Lie case, the inverse of this statement does not hold due to occurrence of families of characteristically nilpotent Leibniz algebras, i.e. algebras in which every derivation is nilpotent. It is established in [42] that characteristically nilpotent non-Lie filiform Leibniz algebra occurs starting with dimension 5.

Moreover, any derivation keeps maximal solvable ideal (radical) of Leibniz algebra invariant (Theorem 2.3.5). Section 2.4 is devoted to a more close look on first cohomology of semisimple Leibniz algebras which provides information on outer derivations.

The last chapter of this thesis is devoted to investigation of some structure theory of Leibniz n-algebras. The general notion of an algebra with a system Ω of polylinear operations was introduced by A.G. Kurosh in [33] under the term of Ω-algebra. In 1973, Nambu [40] constructed an example of 3-Lie algebra, where the multiplication for a triple of classical observables on the three-dimensional phase space $\mathbb{R}^3$ was given by the Jacobian. They are applied in the formalism of mechanics of Nambu, which generalizes the classical Hamiltonian formalism and the bracket naturally generalizes the usual Poisson bracket from a binary to a ternary operation. Later V.T. Filippov [24] defined an n-Lie algebra as an algebra with one n-ary polylinear operation ($n \geq 2$) which is anti-symmetric in all variables and satisfies a generalized Jacobi identity:

$$[[x_1, x_2, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

In n-Lie algebras, similarly to the case of Lie algebras, operators of right multiplication are derivations with respect to the given n-ary multiplication and generate the Lie algebra of inner derivations. In this manner a natural generalization of Lie algebras was suggested for the case where the given multiplication is an n-ary operation. For further examples and methods of construction of n-Lie algebras we refer to [22]-[24], [31]. The chapter is dedicated to investigation of some structure theory of a relatively new algebraic notion – so called Leibniz n-algebras which was introduced in [20] and has been further investigated in [18]-[19], [45]. These algebras are "non antisymmetric" generalization of n-Lie algebras and generalization of Leibniz algebras in terms of arity of multiplication.

Note that an ideal generated by squares

$$I = \mathrm{ideal}\langle [x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] \mid \exists i, j : x_i = x_j \rangle,$$

similarly as in Leibniz algebras has a property that a quotient Leibniz n-algebra is an n-Lie algebra. Moreover, in this case turns out that I can coincide with whole Leibniz n-algebra as shown in Lemma 3.2.2.

In Section 3.3 we study solvability and nilpotency in Leibniz n-algebras and show that the solvable and nilpotent radicals are invariant under all derivations. Since multiplication in Leibniz n-algebras is not anti-symmetric in all the variables, notions such as nilpotency and solvability may be introduced in different ways depending on the position of the multiplicand. The product in the definition of the corresponding series is not necessarily an ideal and this makes some arguments difficult to prove. Hence, we introduce special notions, as k-solvability, nilpotency and K_1-nilpotency of Leibniz n-algebras. Most of them agree with the corresponding notions in particular cases: n-Lie algebras [31] and Leibniz algebras. We establish some properties of k-solvable (nilpotent) ideals, as well.

Section 3.4 is devoted to the study of Frattini subalgebras of Leibniz n-algebras. The section consist of 3 subsections. In Subsection 3.4.1 we introduce the Frattini subalgebra of a Leibniz n-algebra and establish some elementary properties extending corresponding results of Leibniz algebras and of n-Lie algebras. Frattini theory was originally discovered in group theory [48] and further have been studied in Lie algebras in [39, 8, 49], in n-Lie algebras in [7, 51] and in Leibniz algebras in [9, 11]. Here we show that many results concerning Frattini subalgebras and Frattini ideals from the theory of n-Lie algebras remain true when we omit the skew-symmetrical property of the n-ary multiplication. In Subsection 3.4.2, we study the right multiplication operators in a Leibniz n-algebra. Filippov [24] noted that the so-called right multiplication operators play the same crucial role in the theory of n-Lie algebras as in Lie algebras since they form a Lie algebra with respect to the commutator. The space of the right multiplication operators in Leibniz n-algebras also forms an ideal in the Lie algebra of derivations.

However, in the case $n \geq 3$, some well-known properties of the right multiplication operators do not hold in general; for instance, in [4] it was given an example of a Leibniz n-algebra which admits a non-degenerate right multiplication operator. Because of that curious properties of these operators, to obtain some results on right multiplication operators which are valid for Leibniz and n-Lie algebras we must consider them with additional conditions. In Subsection 3.4.3 using the results on right multiplication operators presented in Subsection 3.4.2 we establish relation between nilpotency of Leibniz n-algebra and behavior of its maximal ideals, as well as Frattini subalgebras.

Finally, in Subsection 3.5.2, we continue the investigation on Cartan subalgebra of Leibniz n-algebras started in [20], [16] and [4]. In [4], it was proved that the null root subspace of the right multiplication operators with respect to a regular element is a nilpotent subalgebra. Here we obtain that this subalgebra under some restrictions from Subsection 3.4.2 is a Cartan subalgebra. Moreover, a classical result about conjugacy of Cartan subalgebras in Lie algebras that was extended to the general cases —Leibniz algebras [43] and n-Lie algebras [31]. However, it does not hold in the case of Leibniz n-algebras ($n \geq 3$). Notably, we construct examples that show the non-conjugacy of Cartan subalgebras for Leibniz n-algebras. Starting with a particular n-Lie algebra, we construct Leibniz n-algebras which quotient by the ideal I are isomorphic to the given n-Lie algebra under some conditions. These Leibniz n-algebras have Cartan subalgebras of different dimensions and therefore they are not conjugated.

Contents

Chapter 1

Leibniz algebras and Leibniz cohomology

In this chapter we present some known notions and results concerning Leibniz algebras in the first section. For a semisimple Leibniz algebra L in Theorem 1.3.3 an isomorphic decomposition of $HL^n(L, L)$ is established. Our goal is to check proposed conjecture in [2] that $HL^2(L, L) = 0$ for any finite-dimensional complex semisimple Leibniz algebra L. In order to conjecture to hold, we elaborate on each of the subspaces in the decomposition of $HL^2(L, L)$ to be zero and present equivalent conditions to the conjecture to be true in Proposition 1.4.6 and Proposition 1.4.7 in Section 1.4. In the last section, we establish that space of level -1 is zero for all finite-dimensional Leibniz algebras with liezation $\mathfrak{sl}_2$ in Theorem 1.5.1 and we check analogous statement for another Leibniz algebra with liezation $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ in Theorem 1.5.2 .

1.1 Introduction to Leibniz algebra and Leibniz cohomology

Before giving a definition of a Leibniz algebra let us remind a notion of Lie algebra.

Definition 1.1.1. *([15]) An algebra $\mathfrak{g}$ over K is called a Lie algebra over K if its multiplication (denoted by $(x, y) \mapsto [x, y]$) satisfies the identities:*
(1) $[x, x] = 0$
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

for all x, y, z *in* $\mathfrak{g}$.

The product $[x, y]$ is called the bracket of x and y. Identity (2) is called the Jacobi identity.

A representation of a Lie algebra $\mathfrak{g}$ is usually defined as a homomorphism of Lie algebra $\rho : \mathfrak{g} \to \mathrm{End}(M)$, where M is vector space and the Lie structure on $\mathrm{End}(M)$ is defined by commutator $[A, B] = AB - BA$. However, we will use an equivalent notion of a Lie algebra module defined below.

Definition 1.1.2. *A* $\mathfrak{g}$*-module is* K*-vector space* M *together with a map* $\mathfrak{g} \otimes M \to M,\ x \otimes m \mapsto x.m$ *such that*

$$[x, y].m = x.(y.m) - y.(x.m)$$

for all $x, y \in \mathfrak{g}$ *and all* $m \in M$.

Given a Lie algebra and its non-trivial module a different algebraic structure arises on the direct sum.

Example 1.1.3. *([34]) Let* $\mathfrak{g}$ *be a Lie algebra and* M *be a* $\mathfrak{g}$*-module. Consider* $L = \mathfrak{g} \oplus M$ *with a bracket* $[(g_1, m_1), (g_2, m_2)] := ([g_1, g_2], -g_2.m_1)$. *Then*

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

holds for any $x, y, z \in L$ *and this algebra is not a Lie algebra if the action of* $\mathfrak{g}$ *on* M *is not trivial.*

Looking to the above identity, one notices that the right multiplication operator $[-, z]$ by any element z is a derivation (see [35]) and satisfies a so called Leibniz rule. Although defined earlier by A. Bloh [13] these objects attracted more attention after series of J.–P. Loday and his collaborators' works.

Definition 1.1.4. *An algebra* L *over a field* $\mathbb{K}$ *is called a Leibniz algebra if for any* $x, y, z \in L$, *the Leibniz identity*

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

is satisfied, where $[-, -]$ *is the multiplication in* L.

Leibniz algebra of Example 1.1.3 is denoted as $\mathfrak{g} \dotplus M$ and is called hemisemidirect product Leibniz algebra in [34].

Any Lie algebra is a Leibniz algebra but not the converse. Given a Leibniz algebra L its two sided ideal generated by elements $[x,x]$ for all $x \in L$ (originally denoted by L^{ann} in [35], also called Leibniz kernel of L and denoted by $Leib(L)$ in [12]) is very important. In this thesis we denote this ideal by I. The quotient Leibniz algebra L/I is easily seen to be a Lie algebra, called the liezation of Leibniz algebra L.

Definition 1.1.5. *The set* $\mathrm{Ann}_R(L) = \{x \in L \,|\, [L,x] = 0\}$ *of a Leibniz algebra* L *is called the right annihilator of* L.

One can show that $\mathrm{Ann}_R(L)$ is an ideal of L. Note that due to Leibniz identity it follows that $[L,I] = 0$. Thus I is a subset of a right annihilator $\mathrm{Ann}_R(L) = \{x \in L \mid [L,x] = 0\}$. The center is defined as $Z(L) = \{x \in L \mid [x,L] = [L,x] = 0\}$. If $I = L$ then Leibniz algebra is a trivial algebra with all products being zero. If $I = 0$ then L is a Lie algebra. In this work we assume I to be non trivial, eliminating Lie and trivial Leibniz algebras from the study. One has a short exact sequence $0 \to I \to L \xrightarrow{f} \mathfrak{g} \to 0$. Note that projection f is universal in the sense that a Leibniz map from L to any Lie algebra factors through f.

A *representation* M of a Leibniz algebra L is introduced in [35].

Definition 1.1.6. *A vector space* M *is called a representation or bimodule over a Leibniz algebra* L *if there are two bilinear maps:*

$$[-,-]\colon L \times M \to M \quad \text{and} \quad [-,-]\colon M \times L \to M$$

satisfying the following three axioms

$$\begin{aligned}
[m,[x,y]] &= [[m,x],y] - [[m,y],x],\\
[x,[m,y]] &= [[x,m],y] - [[x,y],m],\\
[x,[y,m]] &= [[x,y],m] - [[x,m],y],
\end{aligned}$$

for any $m \in M$, $x, y \in L$.

A Leibniz bimodule is called *anti-symmetric* when $[x,m] = 0$ for $x \in L$, $m \in M$. Provided that, the last two required identities vanish. Since ideal I is in the right-annihilator, if $x \in I$ or $y \in I$ the first identity vanishes as well. Therefore, the action of L on anti-symmetric L-bimodule M is determined by the action of complement of I that is required to satisfy only the first axiom.

Now for $g \in \mathfrak{g}$ define $g \circ m := -[m, \phi(g)]$. This action is well-defined, since if $g = g'$ then $\phi(g - g') \in I$ and I is in the right annihilator. Moreover, the first identity turns into

$$\begin{aligned}&[g_1, g_2] \circ m = -[m, \phi([g_1, g_2])] = -[m, [\phi(g_1), \phi(g_2)]] = \\ &-[[m, \phi(g_1)], \phi(g_2)] + [[m, \phi(g_2)], \phi(g_1)] = g_2 \circ [m, \phi(g_1)] - g_1 \circ [m, \phi(g_2)] \\ &\qquad = g_1 \circ (g_2 \circ m) - g_2 \circ (g_1 \circ m),\end{aligned}$$

i.e. M becomes a left $\mathfrak{g}$-module (or a right $\mathfrak{g}$-module with action $m \circ g = [m, \phi(g)]$).

Conversely, if M is a left $\mathfrak{g}$-module with an action $\circ : \mathfrak{g} \times M \to M$, then by defining $[m, l] := -f(l) \circ m$ for $l \in L$ implies by the arguments above that M is an anti-symmetric L-bimodule.

Thus, anti-symmetric Leibniz L-bimodule is equivalent to a Lie algebra $\mathfrak{g}$-module of it liezation $\mathfrak{g}$. For instance, an ideal generated by squares I can be considered as a Lie algebra module over liezation $\mathfrak{g}$.

Call Leibniz algebra L *semisimple* if $\mathfrak{g}$ is semisimple. L is called *simple* if the only non-trivial ideal of L is $I \neq [L, L]$. These agrees with suggested definition in [1]. T. Pirashvili proved the following statement.

Proposition 1.1.7. *([44]) Let $f : L \to \mathfrak{g}$ be an epimorphism from an arbitrary finite dimensional Leibniz algebra L to semisimple Lie algebra $\mathfrak{g}$. Then f admits a section.*

Considering a semisimple Leibniz algebra L its liezation by definition is semisimple Lie algebra. The fundamental projection onto its liezation admits a section due to the above proposition. This leads to a key fact that is the base of the study on decomposition of cohomology of semisimple Leibniz algebras in the following sections.

Corollary 1.1.8. *Let L be a finite dimensional semisimple Leibniz algebra with liezation $\mathfrak{g}$. Then $L \cong \mathfrak{g} \dot{+} I$.*

A *cohomology* of a Leibniz algebra L with coefficients in the representation M are defined in [37] as follows.

Define the space of *n-cochains* $CL^n(L, M) = \mathrm{Hom}_{\mathbb{K}}(L^{\otimes n}, M)$ for $n \geq 0$

and a $\mathbb{K}$-homomorphism $\partial^n : CL^n(L, M) \to CL^{n+1}(L, M)$ by

$$(\partial^n f)(x_1, \dots, x_{n+1}) := [x_1, f(x_2, \dots, x_{n+1})] + \\ + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}), x_i] + \\ + \sum_{1 \le i < j \le n+1} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x}_j, \dots, x_{n+1}).$$

This $(CL^*(L, M), \partial)$ is a cochain complex. Its n-th cohomology group is well defined by $HL^n(L, M) := ZL^n(L, M)/BL^n(L, M)$, where the elements $ZL^n(L, M) := \ker \partial^n$ and $BL^n(L, M) := \operatorname{im} \partial^{n-1}$ are called *n-cocycles* and *n-coboundaries*, respectively.

Recall that Loday [37] notes that

$$HL^0(L, M) = \{m \in M \mid [l, m] = 0, \forall l \in L\} = \{m \in M \mid R_m = 0\} =: M^L$$

which is called *bisubmodule of left invariants* of M.

In Leibniz algebras a derivation is defined as usual.

Definition 1.1.9. *A linear map $d\colon L \to M$ is called a derivation of L in M if*

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for any } x, y \in L.$$

Space of all derivations from L to M is denoted by $\operatorname{Der}(L, M)$.

Moreover, for a given $m \in M$ a map $\operatorname{ad}_m : L \to M$ defined by $\operatorname{ad}_m(l) = [l, m]$ is a derivation. It is called an inner derivation and running through all of the elements of the bimodule we obtain space of inner derivation $\operatorname{Der}_{\mathrm{Inn}}(L, M)$.

It is known that $HL^1(L, M) = \operatorname{Der}(L, M)/\operatorname{Der}_{\mathrm{Inn}}(L, M)$.

If M is an anti-symmetric L-bimodule then inner derivations are zero and

$$HL^1(L, M) = \operatorname{Der}(L, M) = \{f : L \to M \mid f([l_1, l_2]) = [f(l_1), l_2]\}.$$

Since for Leibniz algebra L ideal I is an antisymmetric bimodule over L we have

$$HL^1(L, I) = \operatorname{Der}(L, I) = \{f : L \to I \mid f([l_1, l_2]) = [f(l_1), l_2]\}$$

and as Loday notes $id|_L \in HL^1(L, I)$ which implies that $HL^1(L, I) \neq 0$.

Note that when M is symmetric L-bimodule [37] establishes

$$HL^1(L,M) = HL^1(\mathfrak{g},M) = H^1(\mathfrak{g},M)$$

for liezation $\mathfrak{g}$ of Leibniz algebra L.

However, similar statement for anti-symmetric M does not hold.

One of the main tools in this thesis relies on T. Pirashvili's result [44] on Leibniz cohomology of semisimple Lie algebras.

Theorem 1.1.10. *[44] Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and M be its module. Then $HL^n(\mathfrak{g},M) = 0$ for all $n \geq 2$.*

It was proved by constructing some spectral sequences. The same result was proved by P. Ntolo [41] using Casimir element and constructing the explicit homotopy.

1.2 Decomposition of $CL^n(L,L)$ for Leibniz algebra

Let V be a vector space over a field $\mathbb{K}$ and $V = V_0 \oplus V_1$ be a non-trivial decomposition. The goal of this section is to present the decomposition of $\mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$ that will be heavily used in the next sections.

For a natural number n and an integer k define $\mathcal{I}(n,k)$ to be a set of bijections from $\{1,2,\dots,n\}$ to a set of k ones and $n-k$ zeros for $0 \leq k \leq n$ and $\mathcal{I}(n,k) = \emptyset$ otherwise.

For a bijection $\pi \in \mathcal{I}(n,k)$ and a number j equal to 0 or 1 let us denote for short $H^j_{\pi(1)\dots\pi(n)} := \mathrm{Hom}_{\mathbb{K}}(V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}, V_j)$.

Denote by $CL^n(V,V) = \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$ and define for $-n \leq k \leq 1$ a subspace $CL^n(V,V)_{(-k)} = \left(\bigoplus_{\pi\in\mathcal{I}(n,k)} H^0_{\pi(1)\dots\pi(n)}\right) \oplus \left(\bigoplus_{\pi\in\mathcal{I}(n,k+1)} H^1_{\pi(1)\dots\pi(n)}\right)$. Note that there are $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ summands in the above decomposition. Moreover, for any $H^j_{\pi(1)\dots\pi(n)}$ in this decomposition one has $k = j - (\pi(1) + \cdots + \pi(n))$.

Let us call subspace $CL^n(V,V)_{(k)}$ of *level* k. The following statement presents a decomposition isomorphic to $CL^n(V,V)$ into subspaces of all possible levels $-n,\dots,0,1$.

Proposition 1.2.1. $CL^n(V,V) \cong \bigoplus_{k=-n}^{1} CL^n(V,V)_{(-k)}$.

Proof. We have

$$\begin{aligned}
CL^n(V,V) = &\ \mathrm{Hom}(V^{\otimes n},V) = \mathrm{Hom}((V_0\oplus V_1)^{\otimes n}, V_0\oplus V_1)\\
\cong &\ \mathrm{Hom}((V_0\oplus V_1)^{\otimes n}, V_0) \oplus \mathrm{Hom}((V_0\oplus V_1)^{\otimes n}, V_1)\\
\cong &\ \mathrm{Hom}(V_1\otimes V_1\otimes\cdots\otimes V_1, V_0) \bigoplus\\
&\bigoplus \left(\underset{i\in\mathcal{I}(n,n-1)}{\oplus} H^0_{i(1)\dots i(n)} \oplus \underset{i\in\mathcal{I}(n,n)}{\oplus} H^1_{i(1)\dots i(n)} \right) \bigoplus\\
&\qquad\vdots\\
&\bigoplus \left(\underset{i\in\mathcal{I}(n,0)}{\oplus} H^0_{i(1)\dots i(n)} \oplus \underset{i\in\mathcal{I}(n,1)}{\oplus} H^1_{i(1)\dots i(n)} \right) \bigoplus\\
&\bigoplus \mathrm{Hom}(V_0\otimes V_0\otimes\cdots\otimes V_0, V_1),
\end{aligned}$$

or in more compact form

$$CL^n(V,V) \cong \bigoplus_{k=-1}^{n} \left(\underset{i\in\mathcal{I}(n,k)}{\oplus} H^0_{i(1)\dots i(n)} \right) \oplus \left(\underset{i\in\mathcal{I}(n,k+1)}{\oplus} H^1_{i(1)\dots i(n)} \right).$$

Hence, the claim is proved. □

From Proposition 1.2.1 we have the following decompositions vital for consideration in the next section.

$$\begin{aligned}
CL^1(V,V) &\cong H^0_1 \bigoplus (H^1_1\oplus H^0_0) \bigoplus H^1_0,\\
CL^2(V,V) &\cong H^0_{11} \bigoplus (H^0_{10}\oplus H^0_{01}\oplus H^1_{11}) \bigoplus (H^0_{00}\oplus H^1_{01}\oplus H^1_{10}) \bigoplus H^1_{00},\\
CL^3(V,V) &\cong CL^3(V,V)_{(-3)} \oplus CL^3(V,V)_{(-2)}\\
&\quad \oplus CL^3(V,V)_{(-1)} \oplus CL^3(V,V)_{(0)} \oplus CL^3(V,V)_{(1)},
\end{aligned}$$

where

$$\begin{aligned}
CL^3(V,V)_{(-3)} &= H^0_{111},\\
CL^3(V,V)_{(-2)} &= H^0_{110}\oplus H^0_{101}\oplus H^0_{110}\oplus H^1_{111},\\
CL^3(V,V)_{(-1)} &= H^0_{100}\oplus H^0_{010}\oplus H^0_{001}\oplus H^1_{110}\oplus H^1_{101}\oplus H^1_{011},\\
CL^3(V,V)_{(0)} &= H^0_{000}\oplus H^1_{100}\oplus H^1_{010}\oplus H^1_{001},\\
CL^3(V,V)_{(1)} &= H^1_{000}.
\end{aligned}$$

1.3 Decomposition of $HL^n(L,L)$ for semisimple Leibniz algebra

Now consider a semi-simple Leibniz algebra L as an *adjoint* representation of L, i.e. L acting on itself by Leibniz algebra bracket. Due to Corollary 1.1.8 we have $L \cong I \dotplus \mathfrak{g}$. Setting $L_0 = \mathfrak{g}$ and $L_1 = I$ cochain spaces $CL^n(L,L)$ admits the decomposition of Proposition 1.2.1. However, due to $[L, I] = 0$ one can obtain more properties of the coboundary operator.

Proposition 1.3.1. *Coboundary operator ∂ preserves the level spaces.*

Proof. For a non-zero $\varphi \in CL^n(L,L)_{(k)}$ consider $\partial\varphi(x_1, \ldots, x_n, x_{n+1})$, where exactly m of $x_1, \ldots, x_{n+1}$ belong to I. It is evident that if $m < k$ or $m > k+1$ then $\partial\varphi(x_1, \ldots, x_n, x_{n+1}) = 0$. We will consider the other possible cases.

Let $\varphi \in H^0_{i(1)\ldots i(n)}$ for some $i \in \mathcal{I}(n,k)$. If $m = k$ then due to $\operatorname{im}\varphi \in \mathfrak{g}$ it follows that $\partial\varphi(x_1, \ldots, x_n, x_{n+1}) \in \mathfrak{g}$. If $m = k+1$ then $\partial\varphi(x_1, \ldots, x_n, x_{n+1})$ belongs to I if $x_1 \in I$ and zero otherwise. Hence,

$$\partial(H^0_{i(1)\ldots i(n)}) \subseteq \left(\bigoplus_{j \in \mathcal{I}(n+1,k)} H^0_{j(1)\ldots j(n+1)} \right) \oplus H^1_{1i(1)\ldots i(n)}.$$

Now consider $\varphi \in H^1_{i(1)\ldots i(n)}$ for some $i \in \mathcal{I}(n, k+1)$. It is obvious that $m = k$ implies $\partial\varphi(x_1, \ldots, x_n, x_{n+1}) = 0$. One verifies the last possibility $m = k+1$ to yield $\partial(H^1_{i(1)\ldots i(n)}) \subseteq \bigoplus_{j \in \mathcal{I}(n+1,k+1)} H^1_{j(1)\ldots j(n+1)}$.

Thus $\partial(CL^n(L,L)_{(-k)}) \subseteq CL^{n+1}(L,L)_{(-k)}$ for all $-1 \leq k \leq n$. □

Let us denote by $ZL^n(L,L)_{(k)}$ the kernel of restriction of ∂ on $CL^n(L,L)_{(n)}$ and by $BL^n(L,L)_{(k)}$ the image of restriction of ∂ on $CL^{n-1}(L,L)_{(k)}$ for all $-n \leq k \leq 1$.

Proposition 1.3.2. *Let L be a finite-dimensional semisimple Leibniz algebra. Then $ZL^n(L,L)_{(-n)} = 0$ for all positive integers n.*

Proof. Let $\varphi \in ZL^n(L,L)_{(-n)}$. Then $\varphi : \otimes^n I \to \mathfrak{g}$ and $\partial\varphi(g, i_1, \ldots, i_n) = [g, \varphi(i_1, \ldots, i_n)] = 0$ for all $g \in \mathfrak{g}, i_1, \ldots, i_n \in I$. This yields $\operatorname{im}\varphi \in Z(\mathfrak{g})$ which is zero since $\mathfrak{g}$ is semi-simple. Therefore, $\varphi = 0$ and we are done. □

Provided with Proposition 1.3.1 one has $BL^n(L,L)_{(k)} \subseteq ZL^n(L,L)_{(k)}$ and consider subspace $ZL^n(L,L)_{(k)}/BL^n(L,L)_{(k)}$. Let us denote it by $HL^n(L,L)_{(k)}$, for all $-n \leq k \leq 1$. As a consequence on Propositions 1.2.1 and 1.3.1 we have the following

Theorem 1.3.3. *Let L be a finite-dimensional semisimple Leibniz algebra. Then*

$$HL^n(L,L) \cong HL^n(L,L)_{(-n+1)} \oplus \cdots \oplus HL^n(L,L)_{(-1)} \oplus HL^n(L,L)_{(0)}$$

for $n \geq 2$ and $HL^1(L,L) \cong HL^1(L,L)_{(0)} \oplus HL^1(\mathfrak{g}, I)$.

Proof. Clearly, we have

$$\begin{aligned} BL^n(L,L) &\cong \bigoplus_{k=-n}^{1} BL^n(L,L)_{(-k)}, \\ ZL^n(L,L) &\cong \bigoplus_{k=-n}^{1} ZL^n(L,L)_{(-k)}. \end{aligned}$$

Note that by definition $BL^n(L,L)_{(-n)} = 0$ and together with Proposition 1.3.2 we obtain $HL^n(L,L)_{(-n)} = 0$.

Observe that $HL^n(L,L)_{(1)} = HL^n(\mathfrak{g}, I)$. If $n > 1$ by Theorem 1.1.10 it is zero. Hence, the result follows. □

1.4 $HL^2(L,L)$ for semisimple Leibniz algebra

Let us describe $BL^2(L,L) = BL^2(L,L)_{(-1)} \oplus BL^2(L,L)_{(0)} \oplus BL^2(L,L)_{(1)}$ first.

Proposition 1.4.1. *A coboundary operator ∂ acts on $CL^1(L,L)$ as follows:*

$$H_1^0 \hookrightarrow H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1, \ \ H_1^1 \oplus H_0^0 \to H_{10}^0 \oplus H_{00}^0, \ \ H_0^1 \to H_{00}^1.$$

Proof. Consider ∂d for $d \in CL^1(L,L)$. We list the only non-zero actions of ∂d on an element $(x,y) \in (L_{i(1)}, L_{i(2)})$ depending on d.

1. Let $d \in CL^1(L,L)_{(-1)} = H_1^0$. Then $\partial(H_1^0) \subseteq H_{11}^1 \oplus H_{01}^0 \oplus H_{1,0}^0$. Indeed,
$\partial d(x_0, y_1) = [x_0, d(y_1)] \in L_0$,
$\partial d(x_1, y_0) = [d(x_1), y_0] - d([x_1, y_0]) \in L_0$,
$\partial d(x_1, y_1) = [x_1, d(y_1)] \in L_1$.

Moreover, if $\partial d = 0$ then from $[x_0, d(y_1)] = 0$ we obtain $d = 0$, i.e. ∂ sends H_1^0 to $H_{11}^1 \oplus H_{01}^0 \oplus H_{1,0}^0$ injectively. In particular, this implies $\dim BL_{(-1)}^2(L,L) = \dim H_1^0 = \dim \mathrm{Hom}(L_1, L_0) = \dim L_1 \cdot \dim L_0$.

2. Let $d \in CL^1(L,L)_{(0)} = H_1^1 \oplus H_0^0$. Then we analyse separate cases.
2.1 Let $d_1 \in H_1^1$. Then $\partial(H_1^1) \subseteq H_{10}^1$. Indeed,
$\partial d_1(x_1, y_0) = [d_1(x_1), y_0] - d_1([x_1, y_0]) \in L_1$.
2.2 Let $d_0 \in H_0^0$. Then $\partial(H_0^0) \subseteq H_{00}^0 \oplus H_{1,0}^1$. Indeed,
$\partial d_0(x_0, y_0) = [d_0(x_0), y_0] + [x_0, d_0(y_0)] - d_0([x_0, y_0]) \in L_0$,
$\partial d_0(x_1, y_0) = [x_1, d_0(y_0)] \in L_1$.

3. Let $d \in CL^1(L,L)_{(1)} = H_0^1$. Then $\partial(H_0^1) \subseteq H_{00}^1$. Indeed, $\partial d(x_0, y_0) = [d(x_0), y_0] - d([x_0, y_0]) \subseteq L_1$. □

Now we will concentrate on $ZL^2(L,L)$. Consider $\partial\varphi$ for $\varphi \in CL^2(L,L)$. Let us rewrite the equality

$$\partial\varphi(x,y,z) = [x,\varphi(y,z)] - [\varphi(x,y),z] + [\varphi(x,z),y] + \varphi(x,[y,z]) - \varphi([x,y],z) + \varphi([x,z],y)$$

for $(x_{i(1)}, y_{i(2)}, z_{i(3)}) \in (L_{i(1)}, L_{i(2)}, L_{i(3)})$ using structure of our algebra:

$$\begin{cases} \partial\varphi(x_1,y_1,z_1) &= [x_1,\varphi(y_1,z_1)] \\ \partial\varphi(x_0,y_1,z_1) &= [x_0,\varphi(y_1,z_1)] \\ \partial\varphi(x_1,y_0,z_1) &= [x_1,\varphi(y_0,z_1)] + [\varphi(x_1,z_1),y_0] - \varphi([x_1,y_0],z_1) \\ \partial\varphi(x_0,y_0,z_1) &= [x_0,\varphi(y_0,z_1)] + [\varphi(x_0,z_1),y_0] - \varphi([x_0,y_0],z_1) \\ \partial\varphi(x_0,y_1,z_0) &= [x_0,\varphi(y_1,z_0)] - [\varphi(x_0,y_1),z_0] + \varphi(x_0,[y_1,z_0]) \\ & \quad +\varphi([x_0,z_0],y_1) \\ \partial\varphi(x_1,y_0,z_0) &= [x_1,\varphi(y_0,z_0)] - [\varphi(x_1,y_0),z_0] + [\varphi(x_1,z_0),y_0] \\ & \quad +\varphi(x_1,[y_0,z_0]) - \varphi([x_1,y_0],z_0) + \varphi([x_1,z_0],y_0) \\ \partial\varphi(x_1,y_1,z_0) &= [x_1,\varphi(y_1,z_0)] - [\varphi(x_1,y_1),z_0] + \varphi(x_1,[y_1,z_0]) \\ & \quad +\varphi([x_1,z_0],y_1) \\ \partial\varphi(x_0,y_0,z_0) &= [x_0,\varphi(y_0,z_0)] - [\varphi(x_0,y_0),z_0] + [\varphi(x_0,z_0),y_0] \\ & \quad +\varphi(x_0,[y_0,z_0]) - \varphi([x_0,y_0],z_0) + \varphi([x_0,z_0],y_0) \end{cases} \tag{1.4.1}$$

By Proposition 1.3.2 there is a decomposition $ZL^2(L,L) = ZL^2(L,L)_{(-1)} \oplus ZL^2(L,L)_{(0)} \oplus ZL^2(L,L)_{(1)}$. We have $ZL^2(L,L)_{(-1)} = \ker \partial|_{H^0_{10}\oplus H^0_{01}\oplus H^1_{11}}$ and proposition below shows that coboundary operator takes every component of $CL^2(L,L)_{(-1)}$ injectively.

Proposition 1.4.2. *A coboundary operator ∂ acts on $CL^2(L,L)_{(-1)}$ as follows*

$$\begin{array}{rcl} H^0_{01} &\hookrightarrow& H^1_{101} \oplus H^0_{010} \oplus H^0_{001}, \\ H^0_{10} &\hookrightarrow& H^1_{110} \oplus H^0_{100} \oplus H^0_{010}, \\ H^1_{11} &\hookrightarrow& H^1_{101} \oplus H^1_{110}. \end{array}$$

Proof. We have $CL^2(L,L)_{(-1)} = H^0_{10} \oplus H^0_{01} \oplus H^1_{11}$ and

$$\partial(CL^2(L,L)_{(-1)}) \subseteq H^0_{100} \oplus H^0_{010} \oplus H^0_{001} \oplus H^1_{110} \oplus H^1_{101} \oplus H^1_{011}.$$

Now let us examine how ∂ acts on each of the subspaces.

1. Let $\varphi \in H^0_{01}$. Then in the system (1.4.1) the only non-zero equalities are

$$\begin{aligned} \partial\varphi(x_1,y_0,z_1) &= [x_1,\varphi(y_0,z_1)] \in L_1, \\ \partial\varphi(x_0,y_0,z_1) &= [x_0,\varphi(y_0,z_1)] + [\varphi(x_0,z_1),y_0] - \varphi([x_0,y_0],z_1) \in L_0, \\ \partial\varphi(x_0,y_1,z_0) &= -[\varphi(x_0,y_1),z_0] + \varphi(x_0,[y_1,z_0]) + \varphi([x_0,z_0],y_1) \in L_0. \end{aligned}$$

Hence, $H^0_{01} \to H^1_{101} \oplus H^0_{010} \oplus H^0_{001}$. Now if $\varphi \in \ker \partial|_{H^0_{01}}$ then $[x_1, \varphi(y_0, z_1)] = 0$ implies $\varphi(y_0, z_1) \subseteq I = L_1$ while $\operatorname{im} \varphi \subseteq L_0$.

Thus $\varphi = 0$ and $H^0_{01} \hookrightarrow H^1_{101} \oplus H^0_{010} \oplus H^0_{001}$.

2. Let $\varphi \in H^0_{10}$. Similarly, we have the following equalities from system (1.4.1):

$$\begin{aligned}
\partial\varphi(x_0, y_1, z_0) = \;& [x_0, \varphi(y_1, z_0)] \in L_0 \\
\partial\varphi(x_1, y_0, z_0) = \;& -[\varphi(x_1, y_0), z_0] + [\varphi(x_1, z_0), y_0] + \\
& + \varphi(x_1, [y_0, z_0]) - \varphi([x_1, y_0], z_0) + \varphi([x_1, z_0], y_0) \in L_0 \\
\partial\varphi(x_1, y_1, z_0) = \;& [x_1, \varphi(y_1, z_0)] \in L_1
\end{aligned}$$

Hence, $H^0_{10} \to H^1_{110} \oplus H^0_{100} \oplus H^0_{010}$. Analogously as above, if $\varphi \in \ker \partial|_{H^0_{10}}$ then equality $[x_1, \varphi(y_1, z_0)] = 0$ implies $\varphi = 0$. Therefore, $H^0_{10} \hookrightarrow H^1_{110} \oplus H^0_{100} \oplus H^0_{010}$.

3. Let $\varphi \in H^1_{11}$, then system (1.4.1) implies

$$\begin{aligned}
\partial\varphi(x_1, y_0, z_1) = \;& [\varphi(x_1, z_1), y_0] - \varphi([x_1, y_0], z_1) \in L_1, \\
\partial\varphi(x_1, y_1, z_0) = \;& -[\varphi(x_1, y_1), z_0] + \varphi(x_1, [y_1, z_0]) + \varphi([x_1, z_0], y_1) \in L_1.
\end{aligned}$$

Assuming $\varphi \in \ker \partial|_{H^1_{11}}$ from the equalities $\partial\varphi(x_1, z_0, y_1) = \partial\varphi(x_1, y_1, z_0) = 0$ we obtain $\varphi(x_1, [y_1, z_0]) = 0$. Due to $[I, \mathfrak{g}] = I$ it implies $\varphi = 0$. Hence, $H^1_{11} \hookrightarrow H^1_{101} \oplus H^1_{110}$.

□

We have $\partial(H^0_{00} \oplus H^1_{01} \oplus H^1_{10}) \subseteq H^0_{000} \oplus H^1_{100} \oplus H^1_{010} \oplus H^1_{001}$. Now let us examine how ∂ acts on each of the subspaces.

Proposition 1.4.3. *Coboundary operator acts on $CL^2(L,L)_{(0)}$ as follows*

$$\partial : H^0_{00} \hookrightarrow H^1_{100} \oplus H^0_{000}$$

$$\partial : H^1_{10} \to H^1_{100}$$

$$\partial : H^1_{01} \hookrightarrow H^1_{010} \oplus H^1_{001}$$

and $ZL^2(L,L)_{(0)} = \ker \partial|_{H^0_{00} \oplus H^1_{10}}$.

Proof. **1**. Assume $\varphi \in H^0_{00}$, by system (1.4.1) we have

$$\begin{aligned}
\partial\varphi(x_1, y_0, z_0) = \;& [x_1, \varphi(y_0, z_0)] \in L_1, \\
\partial\varphi(x_0, y_0, z_0) = \;& [x_0, \varphi(y_0, z_0)] - [\varphi(x_0, y_0), z_0] + [\varphi(x_0, z_0), y_0] + \\
& + \varphi(x_0, [y_0, z_0]) - \varphi([x_0, y_0], z_0) + \varphi([x_0, z_0], y_0) \in L_0.
\end{aligned}$$

Given that $\varphi \in \ker \partial|_{H^0_{00}} = \Phi^0_{GG}$ then $[x_1, \varphi(y_0, z_0)] = 0$ which implies $\varphi = 0$. Thus, $\partial : H^0_{00} \hookrightarrow H^1_{100} \oplus H^0_{000}$.

2. Assuming $\varphi \in H^1_{10}$ non-zero equalities of system (1.4.1) yields

$$\begin{aligned}\partial\varphi(x_1, y_0, z_0) = &-[\varphi(x_1, y_0), z_0] + [\varphi(x_1, z_0), y_0] \\ &+ \varphi(x_1, [y_0, z_0]) - \varphi([x_1, y_0], z_0) + \varphi([x_1, z_0], y_0) \subseteq L_1.\end{aligned}$$

Hence, $\partial : H^1_{10} \to H^1_{100}$.

3. Let $\varphi \in H^1_{01}$. System (1.4.1) provides

$$\begin{aligned}\partial\varphi(x_0, y_1, z_0) = &\quad -[\varphi(x_0, y_1), z_0] + \varphi(x_0, [y_1, z_0]) + \varphi([x_0, z_0], y_1) \subseteq L_1; \\ \partial\varphi(x_0, y_0, z_1) = &\quad [\varphi(x_0, z_1), y_0] - \varphi([x_0, y_0], z_1) \in L_1.\end{aligned}$$

Then $\partial : H^1_{01} \hookrightarrow H^1_{010} \oplus H^1_{001}$ since assuming $\varphi \in \ker \partial|_{H^1_{01}}$ the last two equalities yield $[x_1, \varphi(y_0, z_0)] = 0$ which implies $\varphi = 0$.

From above it follows that

$$ZL^2(L, L)_{(0)} = \ker \partial|_{H^0_{00} \oplus H^1_{01} \oplus H^1_{10}} = \ker \partial|_{H^0_{00} \oplus H^1_{10}} \oplus \ker \partial|_{H^1_{01}} = \ker \partial|_{H^0_{00} \oplus H^1_{10}},$$

which finishes the proof. □

As a consequence of the last two propositions we obtain

Theorem 1.4.4. *For a finite dimensional semisimple Leibniz algebra L over $\mathbb{C}$ we have an isomorphism*

$$HL^2(L, L) \cong HL^2(L, L)_{(-1)} \oplus HL^2(L, L)_{(0)},$$

where

$$HL^2(L, L)_{(-1)} = \ker \partial|_{H^0_{10} \oplus H^0_{01} \oplus H^1_{11}} / \partial(H^0_1) \text{ and}$$

$$HL^2(L, L)_{(0)} = \ker \partial|_{H^0_{00} \oplus H^1_{10}} / \partial(H^1_1 \oplus H^0_0).$$

From this point we will concentrate our study on each of the subspaces of level -1 and 0 separately.

Proposition 1.4.5. *Any 2-cocycle* $\varphi = \varphi_{10}^0 + \varphi_{01}^0 + \varphi_{11}^1 \in \ker \partial|_{H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1}$ *is uniquely determined by a map* $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}) \to \mathfrak{g}$ *that satisfies*

$$\phi(x_1, [y_0, z_0]) = \phi([x_1, y_0], z_0) - \phi([x_1, z_0], y_0).$$

Moreover, φ *is determined by the following equalities*

$$\begin{cases} \varphi_{01}^0(x_0, [z_1, y_0]) = & -[x_0, \phi(z_1, y_0)] \\ \varphi_{11}^1(x_1, [z_1, y_0]) = & -[x_1, \phi(z_1, y_0)] \\ \varphi_{10}^0([x_1, y_0], z_0) = & \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0] \end{cases} \tag{1.4.2}$$

Proof. Let $\varphi = \varphi_{11}^1 + \varphi_{01}^0 + \varphi_{10}^0 \in \ker \partial|_{H_{11}^1 \oplus H_{01}^1 \oplus H_{10}^1} = ZL_{-1}^2$. Then $\partial\varphi = 0$ implies the following:

$$\begin{aligned} \partial\varphi(x_1, y_0, z_1) = &\ [\varphi_{11}^1(x_1, z_1), y_0] - \varphi_{11}^1([x_1, y_0], z_1) + [x_1, \varphi_{01}^0(y_0, z_1)] \\ \partial\varphi(x_0, y_0, z_1) = &\ [x_0, \varphi_{01}^0(y_0, z_1)] + [\varphi_{01}^0(x_0, z_1), y_0] - \varphi_{01}^0([x_0, y_0], z_1) \\ \partial\varphi(x_0, y_1, z_0) = &\ -[\varphi_{01}^0(x_0, y_1), z_0] + \varphi_{01}^0(x_0, [y_1, z_0]) + \varphi_{01}^0([x_0, z_0], y_1) \\ &+ [x_0, \varphi_{10}^0(y_1, z_0)] \\ \partial\varphi(x_1, y_0, z_0) = &\ -[\varphi_{10}^0(x_1, y_0), z_0] + [\varphi_{10}^0(x_1, z_0), y_0] + \\ &+ \varphi_{10}^0(x_1, [y_0, z_0]) - \varphi_{10}^0([x_1, y_0], z_0) + \varphi_{10}^0([x_1, z_0], y_0) \\ \partial\varphi(x_1, y_1, z_0) = &\ -[\varphi_{11}^1(x_1, y_1), z_0] + \varphi_{11}^1(x_1, [y_1, z_0]) + \varphi_{11}^1([x_1, z_0], y_1) \\ &+ [x_1, \varphi_{10}^0(y_1, z_0)] \end{aligned}$$

Let us re-write the third and the last equation as

$$\begin{aligned} \partial\varphi(x_0, z_1, y_0) = &\ -[\varphi_{01}^0(x_0, z_1), y_0] + \varphi_{01}^0(x_0, [z_1, y_0]) + \varphi_{01}^0([x_0, y_0], z_1) \\ &+ [x_0, \varphi_{10}^0(z_1, y_0)] \\ \partial\varphi(x_1, z_1, y_0) = &\ -[\varphi_{11}^1(x_1, z_1), y_0] + \varphi_{11}^1(x_1, [z_1, y_0]) + \varphi_{11}^1([x_1, y_0], z_1) \\ &+ [x_1, \varphi_{10}^0(z_1, y_0)] \end{aligned}$$

So we have

$$0 = [\varphi^1_{11}(x_1, z_1), y_0] - \varphi^1_{11}([x_1, y_0], z_1) + [x_1, \varphi^0_{01}(y_0, z_1)]$$

$$0 = [x_0, \varphi^0_{01}(y_0, z_1)] + [\varphi^0_{01}(x_0, z_1), y_0] - \varphi^0_{01}([x_0, y_0], z_1)$$

$$0 = -[\varphi^0_{01}(x_0, z_1), y_0] + \varphi^0_{01}(x_0, [z_1, y_0]) + \varphi^0_{01}([x_0, y_0], z_1) + [x_0, \varphi^0_{10}(z_1, y_0)]$$

$$\begin{aligned} 0 = \; & -[\varphi^0_{10}(x_1, y_0), z_0] + [\varphi^0_{10}(x_1, z_0), y_0] + \\ & + \varphi^0_{10}(x_1, [y_0, z_0]) - \varphi^0_{10}([x_1, y_0], z_0) + \varphi^0_{10}([x_1, z_0], y_0) \end{aligned}$$

$$0 = -[\varphi^1_{11}(x_1, z_1), y_0] + \varphi^1_{11}(x_1, [z_1, y_0]) + \varphi^1_{11}([x_1, y_0], z_1) + [x_1, \varphi^0_{10}(z_1, y_0)]$$

Adding the first and the last equalities we obtain

$$\varphi^1_{11}(x_1, [z_1, y_0]) = -[x_1, \varphi^0_{01}(y_0, z_1) + \varphi^0_{10}(z_1, y_0)].$$

Adding the second and the third equalities we obtain

$$\varphi^0_{01}(x_0, [z_1, y_0]) = -[x_0, \varphi^0_{01}(y_0, z_1) + \varphi^0_{10}(z_1, y_0)].$$

Summarizing, we have a system of relations

$$\left\{ \begin{aligned} 0 &= [\varphi^1_{11}(x_1, z_1), y_0] - \varphi^1_{11}([x_1, y_0], z_1) + [x_1, \varphi^0_{01}(y_0, z_1)] \\ 0 &= [x_0, \varphi^0_{01}(y_0, z_1)] + [\varphi^0_{01}(x_0, z_1), y_0] - \varphi^0_{01}([x_0, y_0], z_1) \\ \varphi^0_{01}(x_0, [z_1, y_0]) &= -[x_0,\ \varphi^0_{01}(y_0, z_1) + \varphi^0_{10}(z_1, y_0)] \\ 0 &= -[\varphi^0_{10}(x_1, y_0), z_0] + [\varphi^0_{10}(x_1, z_0), y_0] + \\ & \quad + \varphi^0_{10}(x_1, [y_0, z_0]) - \varphi^0_{10}([x_1, y_0], z_0) + \varphi^0_{10}([x_1, z_0], y_0) \\ \varphi^1_{11}(x_1, [z_1, y_0]) &= -[x_1,\ \varphi^0_{01}(y_0, z_1) + \varphi^0_{10}(z_1, y_0)] \end{aligned} \right.$$

Taking third and last equations one can deduce the first equation using the fact that any element $z_1 \in L_1$ has a decomposition $z_1 = \sum_{l \le k \le m} [z^1_k, z^0_k]$ for some

$z_k^i \in L_i$ for $i = 0, 1$ and $1 \le k \le m$, $m \in \mathbb{N}$. Indeed,

$$\begin{aligned}
[\varphi_{11}^1(x_1, z_1), y_0] \quad & -\varphi_{11}^1([x_1, y_0], z_1) + [x_1, \varphi_{01}^0(y_0, z_1)] = \\
= \quad & -\Big[[x_1, \textstyle\sum_{l\le k\le m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)\,], y_0\Big] \\
& + \left[[x_1, y_0], \sum_{l\le k\le m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)\right] - \\
& - \left[x_1, [y_0, \sum_{l\le k\le m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)]\right] = 0.
\end{aligned}$$

Similarly, we can check that third equality implies the second:

$$\begin{aligned}
& [x_0, \varphi_{01}^0(y_0, z_1)] + [\varphi_{01}^0(x_0, z_1), y_0] - \varphi_{01}^0([x_0, y_0], z_1) \\
& \quad = -[x_0, [y_0, \sum_{l\le k\le m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)]\,] \\
& \quad - \left[\left[x_0, \sum_{l\le k\le m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)\right], y_0\right] \\
& \quad + \left[[x_0, y_0], \sum_{l\le k\le m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)\right] = 0.
\end{aligned}$$

Define $\phi \in \mathrm{Hom}(L_1 \otimes L_0, L_0)$ by $\phi(i, g) = \varphi_{01}^0(g, i) + \varphi_{10}^0(i, g)$. Then substituting $\varphi_{10}^0(i, g) = \phi(i, g) - \varphi_{01}^0(g, i)$ into the fourth equation we obtain the following:

$$\begin{aligned}
0 = & - [\phi(x_1, y_0) - \varphi_{01}^0(y_0, x_1), z_0] + \big[\phi(x_1, z_0) - \varphi_{01}^0(z_0, x_1), y_0\big] + \\
& + \big(\phi(x_1, [y_0, z_0]) - \varphi_{01}^0([y_0, z_0], x_1)\big) - \big(\phi([x_1, y_0], z_0) - \varphi_{01}^0(z_0, [x_1, y_0])\,\big) + \\
& + \big(\phi([x_1, z_0], y_0) - \varphi_{01}^0(y_0, [x_1, z_0])\,\big) =
\end{aligned}$$

$$\begin{aligned}
= & - [\phi(x_1, y_0), z_0] + [\phi(x_1, z_0), y_0] + \phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) \\
& + \phi([x_1, z_0], y_0) + [\varphi_{01}^0(y_0, x_1), z_0] - \big[\varphi_{01}^0(z_0, x_1), y_0\big] - \varphi_{01}^0([y_0, z_0], x_1) \\
& + \varphi_{01}^0(z_0, [x_1, y_0]) - \varphi_{01}^0(y_0, [x_1, z_0]) =
\end{aligned}$$

$$
\begin{aligned}
&= \big(-[\phi(x_1, y_0), z_0] + \varphi^0_{01}(z_0, [x_1, y_0])\big) + \big([\phi(x_1, z_0), y_0] - \varphi^0_{01}(y_0, [x_1, z_0])\big) + \\
&\quad + \big([\varphi^0_{01}(y_0, x_1), z_0] - \left[\varphi^0_{01}(z_0, x_1), y_0\right] - \varphi^0_{01}([y_0, z_0], x_1)\big) + \\
&\quad + (\phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) + \phi([x_1, z_0], y_0)) =
\end{aligned}
$$

(since the bilinear forms involved here map to Lie algebra, we can use antisymmetry)

$$
\begin{aligned}
&= \big([z_0, \phi(x_1, y_0)] + \varphi^0_{01}(z_0, [x_1, y_0])\big) - \big([y_0, \phi(x_1, z_0)] + \varphi^0_{01}(y_0, [x_1, z_0])\big) + \\
&\quad + \big(\left[y_0, \varphi^0_{01}(z_0, x_1)\right] + [\varphi^0_{01}(y_0, x_1), z_0] - \varphi^0_{01}([y_0, z_0], x_1)\big) + \\
&\quad + (\phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) + \phi([x_1, z_0], y_0))\,.
\end{aligned}
$$

Now the first two expressions in parentheses are zero due to third equality. The third expression in parentheses is also zero due to second equality. Hence, we obtain

$$\phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) + \phi([x_1, z_0], y_0) = 0.$$

Since any element $z_1 \in L_1$ admits a decomposition $z_1 = \sum\limits_{l \le k \le m} [z^1_k, z^0_k]$ for some $z^i_k \in L_i$ for $i = 0, 1$ and $1 \le k \le m$, $m \in \mathbb{N}$, we have

$$\varphi^0_{10}(z_1, x_0) = \phi(z_1, x_0) - \varphi^0_{01}(x_0, z_1) = \phi(z_1, x_0) - \sum_{l \le k \le m} \varphi^0_{01}(x_0, [z^1_k, z^0_k]) =$$

$$= \phi(z_1, x_0) + [x_0, \sum_{l \le k \le m} \phi(z^1_k, z^0_k)].$$

In particular,

$$\varphi^0_{10}([x_1, y_0], z_0) = \phi([x_1, y_0], z_0) + [z_0, \phi(x_1, y_0)] = \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0].$$

Summarizing, we can determine all components of $\varphi = \varphi^1_{11} + \varphi^0_{01} + \varphi^0_{10} \in ZL^2_{-1}$ in terms of $\phi \in \mathrm{Hom}(L_1 \otimes L_0) \to L_0$

$$
\begin{cases}
\varphi^0_{01}(x_0, [z_1, y_0]) = & -[x_0, \phi(z_1, y_0)] \\
\varphi^1_{11}(x_1, [z_1, y_0]) = & -[x_1, \phi(z_1, y_0)] \\
\varphi^0_{10}([x_1, y_0], z_0) = & \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0],
\end{cases}
$$

where ϕ satisfies

$$\phi(x_1, [y_0, z_0]) = \phi([x_1, y_0], z_0) - \phi([x_1, z_0], y_0).$$

For the sake of convenience, let us re-denote ϕ by $\phi(i, g) = -\varphi_{01}^0(g, i) - \varphi_{10}^0(i, g)$. Then

$$\begin{cases} \varphi_{01}^0(x_0, [z_1, y_0]) = & [x_0, \phi(z_1, y_0)] \\ \varphi_{11}^1(x_1, [z_1, y_0]) = & [x_1, \phi(z_1, y_0)] \\ \varphi_{10}^0([x_1, y_0], z_0) = & [\phi(x_1, y_0), z_0] - \phi([x_1, y_0], z_0) \end{cases}$$

which finishes the proof of the proposition. □

It is conjectured in [2] that $HL^2(L, L) = 0$ for any semisimple Leibniz algebra L. Authors in [2] validate the claim for simple Leibniz algebra with $L/I \cong \mathfrak{sl}_2$. Armed with the proposition above in order to check if $HL^2(L, L)_{(-1)} = 0$ we arrive into an equivalent statement.

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra and I be a finite dimensional $\mathfrak{g}$-module. Denote by $i.g$ the action of $g \in \mathfrak{g}$ on $i \in I$.

Proposition 1.4.6. *Let $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$. Then $HL^2(L, L)_{(-1)} = 0$ if and only if*

$$\phi(i, [g_1, g_2]) = \phi(i.g_1, g_2) - \phi(i.g_2, g_1) \qquad (*)$$

for all $g_1, g_2 \in \mathfrak{g}, i \in I$ yields existence of a $d \in \mathrm{Hom}(I, \mathfrak{g})$ such that $\phi(i, g) = d(i.g)$.

Proof. Recall that $BL^2(L, L)_{(-1)}$ consists of $\partial d =: \psi = \psi_{01}^0 + \psi_{11}^1 + \psi_{10}^0$, where $d \in \mathrm{Hom}(L_1, L_0)$ and $\psi_{ij}^k \in H_{ij}^k$ are given by

$$\begin{cases} \psi_{01}^0(x_0, y_1) = & [x_0, d(y_1)] \\ \psi_{11}^1(x_1, y_1) = & [x_1, d(y_1)] \\ \psi_{10}^0(x_1, y_0) = & [d(x_1), y_0] - d([x_1, y_0]) \end{cases}$$

Consider $\varphi \in ZL^2(L, L)_{(-1)}$. By Proposition 1.4.5 its components $\varphi = \varphi_{11}^1 + \varphi_{01}^0 + \varphi_{10}^0$ are completely determined by system of equations 1.4.2 and a map $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$ that satisfies $(*)$.

Our conjecture holds, if and only $\varphi = \psi$. Consider $\varphi_{01}^0 = \psi_{01}^0$. Then $\varphi_{01}^0(g_0, [i,g]) = [g_0, \phi(i,g)] = [g_0, d([i,g])]$ which yields $[g_0, \phi(i,g) - d([i,g])] = 0$ for any $g_0, g \in \mathfrak{g}, i \in I$. Due to semi-simplicity it follows that $\phi(i,g) = d([i,g])$. Note that by construction $[i,g] = i.g$ Conversely, if $\phi(i,g) = d([i,g])$ holds then one can check easily that $\varphi = \psi$. □

Let us consider $HL^2(L,L)_{(0)}$ space. Verification if the later one is zero is not known to authors. However, we present an equivalent conjecture in the next proposition.

Proposition 1.4.7. *Let L be a semisimple Leibniz algebra over $\mathbb{C}$. Statement $HL^2(L,L)_{(0)} = 0$ is valid if and only if for any map $\psi \in \mathrm{Hom}(I \otimes \mathfrak{g}, I)$ that satisfies*

$$[\psi(i,g_1), g_2] - [\psi(i,g_2), g_1] - \psi(i,[g_1,g_2]) + \psi(i.g_1, g_2) - \psi(i.g_2, g_1) = 0,$$

it follows that there exist $g_0 \in \mathfrak{g}$ and $d \in \mathrm{Hom}(I,I)$ such that

$$\psi(i,g) = i.[g_0,g] + d(i).g - d(i.g).$$

Proof. By Theorem 1.4.4 the space $ZL^2(L,L)_{(0)}$ consists of $\varphi = \varphi_{00}^0 + \varphi_{10}^1$, where $\varphi_{00}^0 \in H_{00}^0$ and $\varphi_{10}^1 \in H_{10}^1$. We have the following defining equalities for φ :

$\partial\varphi_{00}^0(x_1, y_0, z_0) = [x_1, \varphi_{00}^0(y_0, z_0)]$;

$\partial\varphi_{00}^0(x_0, y_0, z_0) = [x_0, \partial\varphi_{00}^0(y_0, z_0)] - [\partial\varphi_{00}^0(x_0, y_0), z_0] + [\partial\varphi_{00}^0(x_0, z_0), y_0] + \partial\varphi_{00}^0(x_0, [y_0, z_0]) - \partial\varphi_{00}^0([x_0, y_0], z_0) + \partial\varphi_{00}^0([x_0, z_0], y_0)$;

$\partial\varphi_{10}^1(x_1, y_0, z_0) = -[\varphi_{10}^1(x_1, y_0), z_0] + [\varphi_{10}^1(x_1, z_0), y_0] + \varphi_{10}^1(x_1, [y_0, z_0]) - \varphi_{10}^1([x_1, y_0], z_0) + \varphi_{10}^1([x_1, z_0], y_0)$, where elements with subscripts equal to 1 belong to $L_1 := I$, and $L_0 := \mathfrak{g}$ otherwise.

Therefore, for $\varphi = \varphi_{00}^0 + \varphi_{10}^1 \in ZL^2(L,L)_{(0)}$ it is necessary and sufficient that

(1) $[x_1, \varphi_{00}^0(y_0, z_0)] - [\varphi_{10}^1(x_1, y_0), z_0] + [\varphi_{10}^1(x_1, z_0), y_0] + \varphi_{10}^1(x_1, [y_0, z_0]) - \varphi_{10}^1([x_1, y_0], z_0) + \varphi_{10}^1([x_1, z_0], y_0) = 0$,

(2) $\varphi_{00}^0|_{H_{000}^0}$ is a Lie 2-cocycle.

Recall, that for $d = d_1 + d_0 \in H_1^1 \oplus H_0^0$ where $d_0 \in H_0^0$, $d_1 \in H_1^1$ we have $\partial(H_0^0) \subseteq H_{00}^0 \oplus H_{10}^1$ and $\partial(H_1^1) \subseteq H_{10}^1$ with

$$\partial d(x_1, y_0) = \partial d_0(x_1, y_0) + \partial d_1(x_1, y_0) = [x_1, d_0(y_0)] + [d_1(x_1), y_0] - d_1([x_1, y_0]),$$

$$\partial d(x_0, y_0) = [d_0(x_0), y_0] + [x_0, d_0(y_0)] - d_0([x_0, y_0]).$$

Since $H^2(L_0, L_0) = 0$ we have $\varphi_{00}^0 = \partial d_0|_{L_0 \otimes L_0}$ for some $d_0 \in H_0^0$.
Using Leibniz identity one can check the following equality

$$\begin{aligned}[x_1, \varphi_{00}^0(y_0, z_0)] = -\partial d(x_1, [y_0, z_0]) + \partial d([x_1, y_0], z_0) \\ - \partial d([x_1, z_0], y_0) + [\partial d([x_1, y_0]), z_0] - [\partial d([x_1, z_0]), y_0],\end{aligned}$$

where $d = d_0 + d_1$ and $d_1 \in \mathrm{Hom}(L_1, L_1)$ is arbitrary.

Let us denote by $\psi = \varphi_{10}^1 - \partial d|_{L_1 \otimes L_0}$. Then $\psi \in \mathrm{Hom}(L_1 \otimes L_0, L_1)$ and condition (1) is equivalent to

$$[\psi(x_1, y_0), z_0] - [\psi(x_1, z_0), y_0] - \psi(x_1, [y_0, z_0]) + \psi([x_1, y_0], z_0) - \psi([x_1, z_0], y_0) = 0.$$

Observe that $\varphi_{00}^0 + \varphi_{10}^1 = \partial d_0 + \partial d_1 + \psi$ and we are done if there exist $\bar{d}_0 \in H_0^0$ and $\bar{d}_1 \in H_1^1$ such that

$$\partial \bar{d}_0|_{L_0 \otimes L_0} + \partial(\bar{d}_0 + \bar{d}_1)|_{L_1 \otimes L_0} = \partial d_0|_{L_0 \otimes L_0} + \partial(d_0 + d_1)|_{L_1 \otimes L_0} + \psi.$$

Therefore, for $\varphi_{00}^0 + \varphi_{10}^1 \in ZL^2(L, L)_{(0)}$ to be a 2-coboundary it is necessary and sufficient to the following conditions to take place:
(i) $\partial \bar{d}_0|_{L_0 \otimes L_0} = \partial d_0|_{L_0 \otimes L_0}$.
(ii) $\partial(\bar{d}_0 + \bar{d}_1)|_{L_1 \otimes L_0} = \partial(d_0 + d_1)|_{L_1 \otimes L_0} + \psi$.

Now (i) implies that $\bar{d}_0 - d_0 \in Z^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$. Since $\mathfrak{g}$ is semisimple it is known that $\mathrm{Der}(\mathfrak{g}) = \mathrm{Innder}(\mathfrak{g})$. Therefore, there exists $g_0 \in \mathfrak{g}$ such that $(\bar{d}_0 - d_0)(g) = [g_0, g]$. Putting this into (ii) yields

$$\psi(i, g) = [i, [g_0, g]] + [(\bar{d}_1 - d_1)(i), g] - (\bar{d}_1 - d_1)([i, g]).$$

Re-denoting $d = \bar{d}_1 - d_1$ finishes the proof. □

Summarizing, results of this section we have the following conjectures for semi-simple finite-dimensional Lie algebra $\mathfrak{g}$ and its right module I that are equivalent to $HL^2(L,L)_{(-1)} = 0$ and $HL^2(L,L)_{(0)} = 0$, correspondingly.

Conjecture 1. Let $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$ satisfy

$$\phi(i, [g_1, g_2]) = \phi(i.g_1, g_2) - \phi(i.g_2, g_1).$$

Then there exists $d \in \mathrm{Hom}(I, \mathfrak{g})$ such that $\phi(i, g) = d(i.g)$.

Conjecture 2. Let $\psi \in \mathrm{Hom}(I \otimes \mathfrak{g}, I)$ satisfy

$$[\psi(i, g_1), g_2] - [\psi(i, g_2), g_1] - \psi(i, [g_1, g_2]) + \psi(i.g_1, g_2) - \psi(i.g_2, g_1) = 0.$$

Then there exist $g_0 \in \mathfrak{g}$ and $d \in \mathrm{Hom}(I, I)$ such that

$$\psi(i, g) = i.[g_0, g] + d(i).g - d(i.g).$$

1.5 Verification of $HL^2(L,L)_{(-1)} = 0$ for some algebras

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra and I be finite dimensional right $\mathfrak{g}$-module. Denote by $i.g$ the action of $g \in \mathfrak{g}$ on $i \in I$.

For $\phi \in \mathrm{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$ let us introduce a map $\Phi_\phi \in \mathrm{Hom}(I \otimes \mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ defined by

$$\Phi_\phi(i, g_1, g_2) = \phi(i, [g_1, g_2]) - \phi(i.g_1, g_2) + \phi(i.g_2, g_1).$$

Note that $\Phi_\phi(i, g_1, g_2) = -\Phi_\phi(i, g_2, g_1)$ and $\Phi_\phi(i, g, g) = 0$.

Propositions 1.4.6 and 1.4.7 claim:

- $HL^2(L,L)_{(-1)} = 0 \iff$ [$\Phi_\phi = 0$ implies $\phi(i, g) = d(i.g)$ for some $d \in \mathrm{Hom}(I, \mathfrak{g})$;]

- $HL^2(L,L)_{(0)} = 0 \iff$ [$\Phi_\phi = [\psi(i, g_1), g_2] - [\psi(i, g_2), g_1]$ implies that there exist $g_0 \in \mathfrak{g}$ and $d \in \mathrm{Hom}(I, I)$ such that $\psi(i, g) = i.[g_0, g] + d(i).g - d(i.g)$.]

In [2] it was verified that $HL^2(L,L)_{(-1)} = 0$ for simple Leibniz algebra with liezation $\mathfrak{sl}_2$. Below we present more general result when L is not necessarily simple Leibniz algebra with liezation $\mathfrak{sl}_2$.

Let $I = \{x_0, x_1, \ldots, x_m\}$ be an irreducible right $\mathfrak{sl}_2$-module. The action is very well-known to be as follows:

$$\begin{array}{ll} x_k.e = -k(m+1-k)x_{k-1}, & k = 1, \ldots, m. \\ x_k.f = x_{k+1}, & k = 0, \ldots, m-1, \\ x_k.h = (m-2k)x_k & k = 0, \ldots, m, \end{array}$$

and Lie algebra multiplication on $\mathfrak{sl}_2 = \{e, f, h\}$ to be

$$\begin{array}{lll} [e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\ [h,e] = -2e & [f,h] = -2f, & [f,e] = -h. \end{array}$$

Theorem 1.5.1. *Let L be a finite dimensional Leibniz algebra with liezation $\mathfrak{sl}_2$. Then $HL^2(L,L)_{(-1)} = 0$.*

Proof. First assume that I is irreducible $\mathfrak{sl}_2$-module. As mentioned above, we have the basis $\{x_0, x_1, \ldots, x_m\}$ of I .

Define a map $d : I \to \mathfrak{g}$ by $d(x_k) = \phi(x_{k-1}, f)$ for all $1 \le k \le m$ and $d(x_0) = \frac{1}{m}\phi(x_0, h)$.

From $\Phi_\phi(x_m, f, h) = 0$ one obtains $\phi(x_m, f) = 0$ which is in accordance with $d(x_m.f) = 0$. Hence, we have $\phi(x_k, f) = d(x_k.f)$ for all $0 \le k \le m$.

Condition $\Phi_\phi(x_k, f, h) = 0$ for $0 \le k \le m-1$ simplifies to

$$-2\phi(x_k, f) = \phi(x_k, [f,h]) = \phi(x_k.f, h) - \phi(x_k.h, f) = \phi(x_{k+1}, h) - (m-2k)\phi(x_k, f)$$

which yields

$$\phi(x_{k+1}, h) = (m - 2(k+1))\phi(x_k, f) = (m - 2(k+1))d(x_{k+1}) = d(x_{k+1}.h).$$

Together with $\phi(x_0, h) = md(x_0) = d(x_0.h)$ we obtain $\phi(x_k, h) = d(x_k.h)$ for all $0 \le k \le m$.

Using $\Phi_\phi(x_k, e, f) = 0$ for $1 \le k \le m-1$ we have the following chain of equalities

$$\begin{array}{rcl} (m-2k)d(x_k) = & & d(x_k.h) = \phi(x_k, h) = \phi(x_k, [e,f]) \\ & = & \phi(x_k.e, f) - \phi(x_k.f, e) = \\ & = & -k(m+1-k)\phi(x_{k-1}, f) - \phi(x_{k+1}, e) \\ & = & -k(m+1-k)d(x_k) - \phi(x_{k+1}, e). \end{array}$$

This results in $\phi(x_{k+1}, e) = -(k+1)(m-k)d(x_k) = d(x_{k+1}.e)$ for $1 \le k \le m-1$.

Now $\Phi_\phi(x_0, e, f) = 0$ yields $\phi(x_1, e) = -\phi(x_0, h) = -md(x_0) = d(x_1.e)$ and $\Phi_\phi(x_0, e, h) = 0$ results in $\phi(x_0, e) = 0$ which is the same as $d(x_0.e) = 0$. Thus we have $\phi(x_k, e) = d(x_k.e)$ for all $0 \leq k \leq m$.

Thus $\phi(i, g) = d(i.g)$ for all $i \in I, g \in \mathfrak{g}$ and by Theorem 1.4.6 in this case we obtain $HL^2(L, L)_{(-1)} = 0$.

Now assume that I is a finite dimensional $\mathfrak{sl}_2$-module. Then it is completely reducible. Let J be an irreducible submodule of I. Then for restriction of ϕ on $\mathrm{Hom}(J \otimes \mathfrak{g}, \mathfrak{g})$ by the previous construction we have a map $d_J \in \mathrm{Hom}(J, \mathfrak{g})$ such that $\phi(j, g) = d(j.g)$. Defining $d \in \mathrm{Hom}(I, \mathfrak{g})$ as a direct sum of d_J of all irreducible submodules we obtain the desired map. □

Consider Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and let $I = I_1 \oplus I_2$ be sum of two irreducible $\mathfrak{sl}_2$-modules of the same dimension (hence, isomorphic). As shown in work [17] the action of $\mathfrak{g}$ on I is as follows:

$$\begin{array}{ll} x_k.e_1 = -k(m+1-k)x_{k-1}, & k = 1, \dots, m. \\ x_k.f_1 = y_{k+1}, & k = 0, \dots, m-1, \\ x_k.h_1 = (m-2k)y_k, & k = 0, \dots, m, \\ y_k.e_1 = -k(m+1-k)y_{k-1}, & k = 1, \dots, m. \\ y_k.f_1 = y_{k+1}, & k = 0, \dots, m-1, \\ y_k.h_1 = (m-2k)y_k, & k = 0, \dots, m, \\ & \\ x_j.e_2 = y_j.h_2 = y_j & j = 0, \dots, m \\ x_j.h_2 = y_j.f_2 = -x_j & j = 0, \dots, m \end{array}$$

where $I_1 = \mathrm{Span}\{x_0, x_1, \dots, x_m\}$, $I_2 = \mathrm{Span}\{y_0, y_1, \dots, y_m\}$ and $\mathfrak{sl}_2^i = \langle e_i, f_i, h_i \rangle$ for $i = 1, 2$.

Let $L_1 = I_1 \oplus I_2 \oplus \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2$ be a Leibniz algebra with table of multiplication as above.

Proposition 1.5.2. *For Leibniz algebra L_1 we have $HL^2(L_1, L_1)_{(-1)} = 0$.*

Proof. Define a map $d : I \to \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2$ by $d(x_k) = \phi(x_{k-1}, f_1), d(y_k) = \phi(y_{k-1}.f_1)$ for all $1 \leq k \leq m$ and $d(x_0) = \frac{1}{m}\phi(x_0, h_1), d(y_0) = \frac{1}{m}\phi(y_0, h_1)$. Then by the proof of Theorem 1.5.1 we have $\phi(i, g) = d(i.g)$ for all $i \in I_1 \oplus I_2$ and $g \in \mathfrak{sl}_2^1$. Hence, we only need to show the desired equality when $g \in \mathfrak{sl}_2^2$.

Conditions $\Phi_\phi(x_k, f_2, h_2) = 0$ and $\Phi_\phi(y_k, e_2, h_2) = 0$ yield

$$\phi(x_k, f_2) = 0 = d(x_k.f_2), \ \phi(y_k, e_2) = 0 = d(y_k.e_2),$$

respectively, for $0 \leq k \leq m$.

Next conditions $\Phi_\phi(x_k, e_2, f_2) = 0$ and $\Phi_\phi(x_k, e_2, h_2) = 0$ gives us

$$\phi(x_k, h_2) = \phi(y_k, f_2),$$

$$\phi(x_k, e_2) = \phi(y_k, h_2)$$

for $0 \leq k \leq m$. Recall $x_k.h_2 = y_k.f_2$ and $x_k.e_2 = y_k.h_2$.

Using $\Phi_\phi(x_k, e_2, f_1) = 0$ one obtain

$$\phi(x_{k+1}, e_2) = \phi(y_k, f_1) = d(y_k.f_1) = d(y_{k+1}) = d(x_{k+1}.e_2)$$

for $0 \leq k \leq m-1$.

One derives equality $\phi(x_0, e_2) = \frac{1}{m}\phi(y_0, h_1) = d(y_0) = d(x_0.e_2)$ using $\Phi_\phi(x_0, e_2, h_1) = 0$.

Similarly, using $\Phi_\phi(x_k, f_1, h_2) = 0$ one obtain

$$\phi(x_{k+1}, h_2) = -\phi(x_k, f_1) = -d(x_k.f_1) = -d(x_{k+1}) = d(x_{k+1}.h_2)$$

for $0 \leq k \leq m-1$. As for the only undefined missing part one verifies $\phi(x_0, h_2) = -\frac{1}{m}\phi(x_0, h_1) = -d(x_0) = d(x_0.h_2)$ using $\Phi_\phi(x_0, h_1, h_2) = 0$.

Hence, we have proved that $\phi(i, g) = d(i.g)$ for all $i \in I_1 \oplus I_2, g \in \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2$ and by Theorem 1.4.6 in this case we obtain $HL^2(L_1, L_1)_{(-1)} = 0$. $\square$

Chapter 2

Derivations and automorphisms of Leibniz algebras

This chapter is devoted to the extension of known results for Lie algebras on automorphisms and derivations to Leibniz algebras. After reminding preliminary results in Section 2.1 we establish Jordan-Chevalley decomposition for derivations of Leibniz algebra in Section 2.2. Relation of derivations and automorphisms with nilpotency of Leibniz algebra are discussed in Section 2.3. Some results on describing outer derivations of semisimple Leibniz algebras in Section 2.4 using the results of Chapter 1.

2.1 Preliminary results from Lie algebra

As in Lie algebra theory for a Leibniz algebra L consider the following derived and lower central series:

(i) $\quad L^{(1)} = L, \qquad L^{(n+1)} = [L^{(n)}, L^{(n)}], \qquad n > 1;$

(ii) $\quad L^{1} = L, \qquad L^{n+1} = [L^{n}, L], \qquad n > 1.$

Definition 2.1.1. *An algebra L is called solvable (nilpotent) if there exists $s \in \mathbb{N}$ ($k \in \mathbb{N}$, respectively) such that $L^{(s)} = 0$ ($L^{k} = 0$, respectively).*

For an arbitrary element $x \in L$, we consider the right multiplication operator $R_x \colon L \to L$ defined by $R_x(z) = [z, x]$. Right multiplication operators are derivations of the algebra L. The set $R(L) = \{R_x \mid x \in L\}$ is a Lie algebra with respect to the commutator and the following identity holds:

$$R_xR_y - R_yR_x = R_{[y,x]}\,. \tag{2.1.1}$$

The classical Engel's theorem from Lie algebras has the following analogue in Leibniz algebras.

Theorem 2.1.2 ([5], Engel's theorem)**.** *A Leibniz algebra L is nilpotent if and only if R_x is nilpotent for any $x \in L$.*

For a Leibniz algebra L, let H be a maximal solvable ideal in the sense that H contains any solvable ideal of L. Since the sum of solvable ideals is again a solvable ideal (see [5]), this implies the existence of a unique maximal solvable ideal, which is said to be the *radical* of L.

Similarly, let K be a maximal nilpotent ideal of Leibniz algebra L. Since the sum of nilpotent ideals is a nilpotent ideal (see [5]), this implies the existence of a unique maximal nilpotent ideal, which is said to be the *nilradical* of L. Notice that, the nilradical does not possess the properties of the radical in the sense of Kurosh.

The following theorem from linear algebra characterizes the decomposition of a vector space into a direct sum of characteristic (eigen) subspaces.

Theorem 2.1.3 ([38])**.** *Let A be a linear transformation of vector space V. Then V decomposes into the direct sum of characteristic subspaces $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}$ with respect to A, where $V_{\lambda_i} = \{x \in V \mid (A - \lambda_i I)^k(x) = 0$ for some $k \in \mathbb{N}\}$ and $\lambda_i, 1 \leq i \leq k$, are eigenvalues of A.*

The following proposition gives the (additive) Jordan–Chevalley decomposition of an endomorphism.

Proposition 2.1.4 ([27])**.** *Let V be a finite dimensional vector space over $\mathbb{C}$, $x \in End(V)$.*

(i) There exist unique $x_d, x_n \in End(V)$ satisfying the conditions: $x = x_d + x_n$, x_d is diagonalizable, x_n is nilpotent, x_d and x_n commute.

(ii) There exist polynomials $p(t),\ q(t) \in \mathbb{C}[t]$, without constant term, such that $x_d = p(x),\ x_n = q(x)$. In particular, x_d and x_n commute with any endomorphism commuting with x.

(iii) If $A \subseteq B \subseteq V$ are subspaces and x maps B in A, then x_d and x_n also map B in A.

Definition 2.1.5 ([29]). *A subset S of an associative algebra A over a field $\mathbb{K}$ is called weakly closed if for every pair $(a,b) \in S \times S$ an element $\gamma(a,b) \in \mathbb{K}$ is defined such that $ab + \gamma(a,b)\, ba \in S$.*

Further, we need a powerful result concerning the weakly closed sets.

Theorem 2.1.6 ([29]). *Let S be a weakly closed subset of the associative algebra A of linear transformations of a finite-dimensional vector space V over F. Assume every $W \in S$ is nilpotent, that is, $W^k = 0$ for some positive integer k. Then the enveloping associative algebra S^* of S is nilpotent.*

2.2 Decomposition of a derivation and an automorphism of a Leibniz algebra

Lemma 2.2.1. *Let L be a finite dimensional Leibniz algebra with a derivation d defined on it and $L = L_{\rho_1} \oplus \cdots \oplus L_{\rho_s}$ be a decomposition of L into characteristic spaces with respect to d. Then for any $\alpha, \beta \in Spec(d)$ we have*

$$[L_\alpha, L_\beta] \subseteq \begin{cases} L_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is an eigenvalue of } d \\ 0 & \text{if } \alpha+\beta \text{ is not an eigenvalue of } d\ . \end{cases}$$

Proof. First observe that $(d-(\alpha+\beta)I)([x,y]) = [d(x),y]+[x,d(y)]-(\alpha+\beta)[x,y] = [(d-\alpha I)(x),y]+[x,(d-\beta I)(y)]$. Now assume that

$$(d-(\alpha+\beta)I)^k([x,y]) = \sum_{i=0}^{k} C_k^i[(d-\alpha I)^i(x),(d-\beta I)^{k-i}(y)] \qquad (2.2.1)$$

for some $k > 1$. Then

$$\begin{aligned}
&(d-(\alpha+\beta)I)^{k+1}([x,y]) \\
=&(d-(\alpha+\beta)I)\left(\sum_{i=0}^{k} C_k^i[(d-\alpha I)^i(x),(d-\beta I)^{k-i}(y)]\right) \\
=&\sum_{i=0}^{k} C_k^i[(d-\alpha I)^{i+1}(x),(d-\beta I)^{k-i}(y)] + \sum_{i=0}^{k} C_k^i[(d-\alpha I)^i(x),(d-\beta I)^{k-i+1}(y)] \\
&= [(d-\alpha I)^{k+1}(x),(y)] + \sum_{i=0}^{k-1} C_k^i[(d-\alpha I)^{i+1}(x),(d-\beta I)^{k+1-(i+1)}(y)]
\end{aligned}$$

$$+\sum_{i=1}^{k} C_k^i[(d-\alpha I)^i(x),(d-\beta I)^{k+1-i}(y)]+[x,(d-\beta I)^{k+1}(y)]$$

$$=[(d-\alpha I)^{k+1}(x),y]+\sum_{i=1}^{k}(C_k^{i-1}+C_k^i)[(d-\alpha I)^i(x),(d-\beta I)^{k+1-i}(y)] + [x,(d-\beta I)^{k+1}(y)]$$

$$=[(d-\alpha I)^{k+1}(x),y]+\sum_{i=1}^{k}C_{k+1}^i[(d-\alpha I)^i(x),(d-\beta I)^{k+1-i}(y)] + [x,(d-\beta I)^{k+1}(y)]$$

$$=\sum_{i=0}^{k+1}C_{k+1}^i[(d-\alpha I)^i(x),(d-\beta I)^{k+1-i}(y)]\,.$$

Hence (2.2.1) holds for any $k \in \mathbb{N}$.

Consider $x \in L_\alpha$, $y \in L_\beta$. Then there exist natural numbers p, q such that $(d-\alpha I)^p(x) = 0$ and $(d-\beta I)^q(y) = 0$. In (2.2.1) taking $k = p+q$ we have that $\big(d-(\alpha+\beta)I\big)^k([x,y]) = 0$ which completes the proof of the statement of the lemma. □

Let d be a derivation of a Leibniz algebra L. From the definition of derivation it is straightforward that $\ker d$ is a subalgebra. Moreover, by Lemma 3.3.1 we have $[L_0, L_0] \subseteq L_0$ and hence L_0 is also a subalgebra of L.

The following theorem is a generalization of the analogous result in the theory of Lie algebras established in [25].

Theorem 2.2.2. *Let D be a derivation of a Leibniz algebra L. Then there exists a unique diagonalizable derivation D_0 and a unique nilpotent derivation T such that $D = D_0 + T$ and $D_0T = TD_0$.*

Proof. Let $L = L_{\rho_1} \oplus \cdots \oplus L_{\rho_s}$ be a decomposition of L into characteristic spaces with respect to d. Let us define a linear operator $D_0 \colon L \to L$ as $D_0(x) = \rho_i x$ for $x \in L_{\rho_i}$. Then D_0 is obviously diagonalizable and $D_0D = DD_0$.

Now we show that D_0 is a derivation of L.

By Lemma 3.3.1 if $x \in L_{\rho_i}, y \in L_{\rho_j}$ we obtain $[x,y] \in L_{\rho_i+\rho_j}$ if $\rho_i + \rho_j$ is an eigenvalue and $[x,y] = 0$ otherwise. If $\rho_i + \rho_j$ is an eigenvalue of D, then we obtain

$$D_0([x,y]) = (\rho_i + \rho_j)[x,y]\,,$$
$$[D_0(x),y] + [x,D_0(y)] = [\rho_i x,y] + [x,\rho_j y] = (\rho_i + \rho_j)[x,y]\,.$$

So $D_0([x,y]) = [D_0(x),y] + [x,D_0(y)]$.

If $\rho_i + \rho_j$ is not an eigenvalue, then $[x,y] = 0$ and we obtain $D_0([x,y]) = 0$ and $[D_0(x),y] + [x,D_0(y)] = (\rho_i + \rho_j)[x,y] = 0$. Hence, D_0 is a derivation.

Now denote by $T = D - D_0$. Obviously, T is a derivation of L and T is nilpotent. Moreover, T commutes with D_0.

The uniqueness of such decomposition follows from Proposition 2.1.4. $\square$

In order to obtain a similar result for automorphisms of Leibniz algebras we need the following lemma.

Lemma 2.2.3. *Let P be a nilpotent transformation of a Leibniz algebra L such that $P + I$ is an automorphism. Then*

$$P^k([x,y]) = \sum_{i=0}^{k}\sum_{j=0}^{i} C_k^i C_i^j [P^{k-j}(x), P^{k-i+j}(y)] \tag{2.2.2}$$

for all $k \in \mathbb{N}$.

Proof. Let us denote $Q = P + I$. Since Q is an automorphism we obtain

$$\begin{aligned} P([x,y]) &= (Q-I)([x,y]) = [Q(x),Q(y)] - [x,y] \\ &= [Q(x)-x, Q(y)-y] + [Q(x)-x, y] + [x, Q(y)-y] \end{aligned}$$

$$= [P(x),P(y)] + [P(x),y] + [x,P(y)] = \sum_{i=0}^{1}\sum_{j=0}^{i} C_1^i C_i^j [P^{1-j}(x), P^{1-i+j}(y)]\,.$$

Now assume that (2.2.2) holds for some natural $k > 1$. Then

$$P^{k+1}([x,y]) = \sum_{i=0}^{k}\sum_{j=0}^{i} C_k^i C_i^j P([P^{k-j}(x), P^{k-i+j}(y)])$$

$$= \sum_{i=0}^{k} C_k^i \sum_{j=0}^{i} C_i^j ([P^{k-j+1}(x), P^{k-i+j+1}(y)] + [P^{k-j+1}(x), P^{k-i+j}(y)]$$
$$+ [P^{k-j}(x), P^{k-i+j+1}(y)]).$$

Consider

$$\sum_{j=0}^{i} C_i^j [P^{k-j+1}(x), P^{k-i+j}(y)] + \sum_{j=0}^{i} C_i^j [P^{k-j}(x), P^{k-i+j+1}(y)]$$
$$= C_i^0 [P^{k+1}(x), P^{k-i}(y)] + \sum_{j=1}^{i} \big(C_i^j [P^{k+1-j}(x), P^{k-i+j}(y)]$$
$$+ C_i^{j-1} [P^{k+1-j}(x), P^{k-i+j}(y)] \big) + C_i^i [P^{k-i}(x), P^{k+1}(y)]$$
$$= C_{i+1}^0 [P^{k+1}(x), P^{k+1-(i+1)}(y)]$$
$$+ \sum_{j=1}^{i} \left(C_i^j + C_i^{j-1} \right) [P^{k+1-j}(x), P^{k+1-(i+1)+j}(y)] + C_{i+1}^{i+1} [P^{k+1-(i+1)}(x), P^{k+1}(y)] .$$

Using the fact $C_i^j + C_i^{j-1} = C_{i+1}^j$ we obtain

$$\sum_{j=0}^{i} C_i^j [P^{k-j+1}(x), P^{k-i+j}(y)] + \sum_{j=0}^{i} C_i^j [P^{k-j}(x), P^{k-i+j+1}(y)]$$
$$= \sum_{j=0}^{i+1} C_{i+1}^j [P^{k+1-j}(x), P^{k+1-(i+1)+j}(y)].$$

Now

$$P^{k+1}([x,y]) = \sum_{i=0}^{k} \sum_{j=0}^{i} C_k^i C_i^j [P^{k-j+1}(x), P^{k-i+j+1}(y)]$$
$$+ \sum_{i=0}^{k} \sum_{j=0}^{i+1} C_k^i C_{i+1}^j [P^{k+1-j}(x), P^{k+1-(i+1)+j}(y)]$$

$$= [P^{k+1}(x), P^{k+1}(y)] + \sum_{i=0}^{k-1} \sum_{j=0}^{i+1} C_k^{i+1} C_{i+1}^j [P^{k-j+1}(x), P^{k+1-(i+1)+j}(y)]$$
$$+ \sum_{i=0}^{k-1} \sum_{j=0}^{i+1} C_k^i C_{i+1}^j [P^{k+1-j}(x), P^{k+1-(i+1)+j}(y)] + \sum_{j=0}^{k+1} C_{k+1}^j [P^{k+1-j}(x), P^j(y)]$$

$$= [P^{k+1}(x), P^{k+1}(y)] + \sum_{i=0}^{k-1}\sum_{j=0}^{i+1} \left(C_k^{i+1} + C_k^i\right) C_{i+1}^j [P^{k-j+1}(x), P^{k+1-(i+1)+j}(y)]$$
$$+ \sum_{j=0}^{k+1} C_{k+1}^j [P^{k+1-j}(x), P^j(y)]$$
$$= [P^{k+1}(x), P^{k+1}(y)] + \sum_{i=0}^{k-1}\sum_{j=0}^{i+1} C_{k+1}^{i+1} C_{i+1}^j [P^{k-j+1}(x), P^{k+1-(i+1)+j}(y)]$$
$$+ \sum_{j=0}^{k+1} C_{k+1}^j [P^{k+1-j}(x), P^j(y)]$$
$$=[P^{k+1}(x), P^{k+1}(y)] + \sum_{i=1}^{k}\sum_{j=0}^{i} C_{k+1}^i C_i^j [P^{k-j+1}(x), P^{k+1-i+j}(y)]$$
$$+ \sum_{j=0}^{k+1} C_{k+1}^j [P^{k+1-j}(x), P^j(y)] = \sum_{i=0}^{k+1}\sum_{j=0}^{i} C_{k+1}^i C_i^j [P^{k-j+1}(x), P^{k+1-i+j}(y)]\,.$$

Thus, (2.2.2) is proved. □

The next lemma presents the similar result for automorphisms of Leibniz algebras as Lemma 3.3.1 does for derivations. Notice that, it also generalizes the result for Lie algebras given in [25].

Lemma 2.2.4. *Let L be a finite dimensional Leibniz algebra and $L = L_{\rho_1} \oplus \cdots \oplus L_{\rho_s}$ be a decomposition of L into characteristic spaces with respect to an automorphism A. Then for any $\alpha, \beta \in Spec(A)$ we have*

$$[L_\alpha, L_\beta] \subseteq \begin{cases} L_{\alpha\beta} & \text{if } \alpha\beta \text{ is an eigenvalue of } A \\ 0 & \text{if } \alpha\beta \text{ is not an eigenvalue of } A\,. \end{cases}$$

Proof. First observe that

$$(A - \alpha\beta I)([x,y]) = [A(x), A(y)] - \alpha\beta[x,y]$$
$$= [(A-\alpha I)(x), (A-\beta I)(y)] + [(A-\alpha I)(x), \beta y] + [\alpha x, (A-\beta I)(y)]\,.$$

Similarly to the proof of Lemma 2.2.3 one can establish by induction

$$(A - \alpha\beta I)^k([x,y]) = \sum_{j=0}^{k}\sum_{i=0}^{j} \alpha^i \beta^{j-i} C_k^j C_j^i [(A-\alpha I)^{k-i}(x), (A-\beta I)^{k-j+i}(y)]\,. \tag{2.2.3}$$

Now let $x \in L_\alpha$ and $y \in L_\beta$. Then there exist natural numbers p, q such that $(A-\alpha I)^p(x)=0$ and $(A-\beta I)^q(y)=0$. In (2.2.3) taking $k=p+q$ we have that $(A-\alpha\beta I)^k([x,y])=0$ which completes the proof of the lemma. □

Below, we establish a technical lemma and a corollary in order to obtain a similar result to Theorem 2.2.2 for automorphisms of Leibniz algebra.

Lemma 2.2.5. *For any polynomial P of degree less than n, where $n \in \mathbb{N}$, the following equality holds:*

$$\sum_{i=0}^{n}(-1)^i C_n^i P(i) = 0\,.$$

Proof. Since $\deg P(x) < n$, applying Lagrange interpolation formula to the points $x_k = k$, $0 \le k \le n-1$ we obtain $P(x) = \sum\limits_{k=0}^{n-1} q_k(x)P(k)$, where $q_k(x) = \dfrac{x(x-1)\cdots\big(x-(k-1)\big)\cdot\big(x-(k+1)\big)\cdots\big(x-(n-1)\big)}{k(k-1)\cdots 1\cdot(-1)(-2)\cdots\big(-(n-1-k)\big)}$.

Now

$$q_k(n) = \frac{n(n-1)\cdots\big(n-(k-1)\big)\cdot\big(n-(k+1)\big)\cdots\big(n-(n-1)\big)}{k(k-1)\cdots 1\cdot(-1)(-2)\cdots\big(-(n-1-k)\big)} = \frac{n!}{(-1)^{n-1-k}k!(n-k)!} = \frac{1}{(-1)^{n-1}}\cdot(-1)^k C_n^k\,.$$

Thus $P(n) = \sum\limits_{k=0}^{n-1} q_k(n)P(k) = \dfrac{1}{(-1)^{n-1}}\sum\limits_{k=0}^{n-1}(-1)^k C_n^k \cdot P(k)$.

Hence, $0 = \sum\limits_{k=0}^{n-1}(-1)^k C_n^k\cdot P(k) + (-1)^n C_n^n P(n) = \sum\limits_{i=0}^{n}(-1)^i C_n^i P(i)$. □

Corollary 2.2.6. *Let n, m be non-negative integers such that $n < m$. Then*

$$\sum_{i=0}^{n}\frac{(-1)^i}{m-i}C_n^i C_{m-i}^n = \begin{cases} \frac{1}{m} & \text{if } n=0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Proof. Let $n > 1$ and consider the polynomial

$$P(x) = \frac{1}{n!}(m-1-x)(m-2-x)\cdots(m-(n-1)-x) = \frac{1}{m-x}\cdot C_{m-x}^n$$

of degree $n-1$.

By Lemma 2.2.5 we obtain

$$0=\sum_{i=0}^{n}(-1)^i C_n^i P(i)=\sum_{i=0}^{n}\frac{(-1)^i}{m-i}C_n^i C_{m-i}^n\,.$$

For $n=0,1$, simple calculations verify the statement of the corollary. □

The following result shows that the analogous one established for Lie algebras [25] is also valid for Leibniz algebras.

Theorem 2.2.7. *Let A be an automorphism of a Leibniz algebra. Then there exists a unique diagonalizable automorphism A_0 and a unique nilpotent derivation T such that $A=A_0\exp(T)$ and $A_0T=TA_0$.*

Proof. Let $L=L_{\rho_1}\oplus\cdots\oplus L_{\rho_s}$ be a decomposition of a Leibniz algebra L into characteristic spaces with respect to A.

Let us define a linear operator $A_0\colon L\to L$ as $A_0(x)=\rho_i x$ for $x\in L_{\rho_i}$. Then A_0 is obviously diagonalizable and $A_0A=AA_0$. Notice that if $x\in L_{\rho_i}, y\in L_{\rho_j}$ then $[A_0(x),A_0(y)]=\rho_i\rho_j[x,y]$ and by Lemma 2.2.4 we have $[x,y]\in L_{\rho_i\rho_j}$. Therefore, $A_0([x,y])=\rho_i\rho_j[x,y]$, which implies that A_0 is an automorphism.

Let us denote by $Q=A_0^{-1}A$. Then $A=A_0Q$ and $A_0Q=QA_0$. Also note that $Spec(Q)=\{1\}$.

Consider $P=Q-I$. Obviously, P is nilpotent and hence $\log Q=\log(I+P)=P-\frac{1}{2}P^2+\cdots+\frac{(-1)^{n-1}}{n}P^n+\cdots$ diverges.

Since P is nilpotent, $\log Q$ is also a nilpotent transformation. We will prove that $\log(I+P)$ is a derivation, i.e.,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}P^k([x,y])=\Big[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}P^k(x),y\Big]+\Big[x,\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}P^k(y)\Big]\,. \tag{2.2.4}$$

By Lemma 2.2.3, formula (2.2.4) is valid for P. Putting $C_k^i=C_k^{k-i}$ and substituting $r=k-i$, we obtain $P^k([x,y])=\sum\limits_{r=0}^{k}\sum\limits_{j=0}^{k-r}C_k^rC_{k-r}^j[P^{k-j}(x),P^{r+j}(y)]$.

Now denote by $B_{k,r}=\sum\limits_{j=0}^{k-r}C_k^rC_{k-r}^j[P^{k-j}(x),P^{j+r}(y)]$ for all $0\le r\le k$.

Then $P^k([x,y])=B_{k,0}+B_{k,1}+\cdots+B_{k,k}$.

Therefore,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}P^k([x,y]) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(B_{k,0}+B_{k,1}+\cdots+B_{k,k})$$

$$= \sum_{m=0}^{\infty}\left(\frac{1}{2m+1}B_{2m+1,0}-\frac{1}{2m}B_{2m,1}+\cdots+\frac{(-1)^m}{m+1}B_{m+1,m}\right)$$

$$-\sum_{m=1}^{\infty}\left(\frac{1}{2m}B_{2m,0}-\frac{1}{2m-1}B_{2m-1,1}+\cdots+\frac{(-1)^m}{m}B_{m,m}\right)$$

$$= \sum_{m=0}^{\infty}\left(\sum_{t=0}^{m}\frac{(-1)^t}{2m+1-t}B_{2m+1-t,t}\right)-\sum_{m=1}^{\infty}\left(\sum_{t=0}^{m}\frac{(-1)^t}{2m-t}B_{2m-t,t}\right)$$

$$= \sum_{m=0}^{\infty}\left(\sum_{t=0}^{m}\frac{(-1)^t}{2m+1-t}\sum_{j=0}^{2m+1-2t}C^t_{2m+1-t}C^j_{2m+1-2t}[P^{2m+1-t-j}(x),P^{j+t}(y)]\right)$$

$$-\sum_{m=1}^{\infty}\left(\sum_{t=0}^{m}\frac{(-1)^t}{2m-t}\sum_{j=0}^{2m-2t}C^t_{2m-t}C^j_{2m-2t}[P^{2m-t-j}(x),P^{j+t}(y)]\right)$$

$$= \sum_{m=0}^{\infty}\left(\frac{1}{2m+1}[P^{2m+1}(x),y]\right.$$

$$+\sum_{s=1}^{2m}\left(\sum_{t=0}^{s}\frac{(-1)^t}{2m+1-t}C^t_{2m+1-t}C^{s-t}_{2m+1-2t}\right)\cdot\ [P^{2m+1-s}(x),P^s(y)]$$

$$\left.+\frac{1}{2m+1}[x,P^{2m+1}(y)]\right)$$

$$-\sum_{m=1}^{\infty}\left(\frac{1}{2m}[P^{2m}(x),y]\right.$$

$$+\sum_{s=1}^{2m-1}\left(\sum_{t=0}^{s}\frac{(-1)^t}{2m-t}C^t_{2m-t}C^{s-t}_{2m-2t}\right)[P^{2m-s}(x),P^s(y)]$$

$$\left.+\frac{1}{2m}[x,P^{2m}(y)]\right).$$

Now since $C^t_{2m+1-t}C^{s-t}_{2m+1-2t} = C^t_sC^s_{2m+1-t}$ and $C^t_{2m-t}C^{s-t}_{2m-2t} = C^t_sC^s_{2m-t}$

we obtain

$$\sum_{t=0}^{s} \frac{(-1)^t}{2m+1-t} C_{2m+1-t}^{t} C_{2m+1-2t}^{s-t} = \sum_{t=0}^{s} \frac{(-1)^t}{2m+1-t} C_s^t C_{2m+1-t}^{s},$$
$$\sum_{t=0}^{s} \frac{(-1)^t}{2m-t} C_{2m-t}^{t} C_{2m-2t}^{s-t} = \sum_{t=0}^{s} \frac{(-1)^t}{2m-t} C_s^t C_{2m-t}^{s}.$$

However, by Corollary 2.2.6 the last sums are zero for all $1 \leq s \leq 2m$ and ($1 \leq s \leq 2m-1$, respectively). Hence,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} P^k([x,y]) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} ([P^n(x), y] + [x, P^n(y)])$$
$$= [\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} P^n(x), y] + [x, \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} P^n(y)]$$

and (2.2.4) is proved.

Thus, $T = \log Q$ is a nilpotent derivation of L and $A = A_0 \exp(T)$, $A_0 T = T A_0$. Now since $\exp(T) - I$ is nilpotent, we obtain the additive Jordan–Chevalley decomposition $A = A_0 + A_0(\exp(T) - I)$ of A. Therefore, by Proposition 2.1.4, A_0 and as consequence T, are determined uniquely. □

2.3 Sufficient conditions of nilpotency of a Leibniz algebra in terms of derivations and automorphisms

The following theorems generalize the results from the theory of Lie algebras [28] to Leibniz algebras.

Theorem 2.3.1. *Let L be a finite-dimensional complex Leibniz algebra which admits a non-degenerate derivation. Then L is a nilpotent algebra.*

Proof. Let d be a non-singular derivation of a Leibniz algebra L and $L = L_{\rho_1} \oplus L_{\rho_2} \oplus \cdots \oplus L_{\rho_k}$ be a decomposition of L into characteristic spaces with respect to d.

Let $\alpha, \beta \in Spec(d)$. Then by Lemma 3.3.1 we have

$$[\dots[[L_\alpha, \underbrace{L_\beta], L_\beta], \dots, L_\beta]}_{k-\text{times}} \subseteq L_{\alpha+k\beta}.$$

Since for sufficiently large $k \in \mathbb{N}$ we have $\alpha + k\beta \notin Spec(d)$, and by Lemma 3.3.1 we obtain $[\dots[[L_\alpha, L_\beta], L_\beta], \dots, L_\beta] = 0$.

Thus, for $x \in L_\beta$ any right multiplication operator R_x is nilpotent, and due to the fact that α, β were taken arbitrarily, it follows that every operator from $\bigcup_{i=1}^{k} R(L_{\rho_i})$ is nilpotent.

Now from identity (2.1.1) and Lemma 3.3.1 it follows that $\bigcup_{i=1}^{k} R(L_{\rho_i})$ is a weakly closed set of an associative algebra $R(L)$. Hence, by Theorem 2.1.6 it follows that every operator from $R(L)$ is nilpotent.

Now by Theorem 3.1.6 we obtain the result, i.e., L is nilpotent. □

Remark 2.3.2. *The following family $L(\beta) = \langle e_1, \dots, e_n\rangle$ of characteristically nilpotent Leibniz algebras, i.e. algebras in which every derivation is nilpotent, with the following multiplication*

$$\begin{aligned}
&[e_0, e_0] = e_2, \qquad [e_i, e_0] = e_{i+1} \qquad (1 \le i \le n-1)\,,\\
&[e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \dots + \alpha_{n-1} e_{n-1} + \theta e_n\,,\\
&[e_i, e_1] = \alpha_3 e_{i+2} + \alpha_4 e_{i+3} + \dots + \alpha_{n+1-i} e_n \quad (1 \le i \le n-2)\,,
\end{aligned}$$

where $(\alpha_3, \dots, \alpha_n,\ \theta \in \mathbb{C})$ and $\alpha_i\alpha_j \neq 0$ for some $3 \le i \neq j \le n$, was constructed in [42]. This implies that the statement of Theorem 2.3.1 in the opposite direction does not hold.

Theorem 2.3.3. *Let L be a finite dimensional complex Leibniz such that it admits an automorphism of prime order with no fixed-points. Then L is a nilpotent algebra.*

Proof. Let A be an automorphism of Leibniz algebra L with the properties given in the statement of the theorem. Since A has no fixed points then 1 is not an eigenvalue of A.

Let $L = L_{\rho_1} \oplus L_{\rho_2} \oplus \dots \oplus L_{\rho_k}$ be a decomposition of L into characteristic spaces with respect to A. From the condition that A is an automorphism of prime order we obtain that the spectrum of A consists of primitive $p-$th roots of unity. Therefore, for any $\alpha, \beta \in Spec(A)$ there exists $k \in \mathbb{N}$ such that $\alpha\beta^k = 1 \notin Spec(A)$. Hence, by Lemma 2.2.4 we obtain

$$[\dots[[L_\alpha, \underbrace{L_\beta], L_\beta], \dots, L_\beta]}_{k-\text{times}} \subseteq L_{\alpha\beta^k} = 0\,.$$

Thus, for $x \in L_\beta$ any right multiplication operator R_x is nilpotent, and similarly as in the proof of Theorem 2.3.1 we obtain that L is nilpotent. □

Let D be a derivation of a Leibniz algebra L such that D commutes with any inner derivation. Then $D(L) \subseteq \mathrm{Ann}_R(L)$. Indeed, since D commutes with any right multiplication operator we have $[D(x), y] = (R_y \circ D)(x) = (D \circ R_y)(x) = D([x, y]) = [D(x), y] + [x, D(y)]$ which implies $[x, D(y)] = 0$ for any $x, y \in L$. Thus, $[L, D(L)] = 0$ and $D(L) \subseteq \mathrm{Ann}_R(L)$.

Lemma 2.3.4. *Let J be an ideal of Leibniz algebra L and D be a derivation given on L. Then $J + D(J)$ is also an ideal of L.*

Proof. Since for any $x \in J, y \in L$ we have

$$[y, D(x)] = D([x, y]) - [D(x), y] \in D([J, L]) + [J, L] \subseteq D(J) + J\,,$$

and so $[L, D(J)] \subseteq D(J) + J$. Therefore, $[L, J + D(J)] \subseteq J + D(J)$.

Similarly, since for any $x \in J, y \in L$ we have

$$[D(x), y] = D([x, y]) - [x, D(y)] \in D([J, L]) + [J, L] \subseteq D(J) + J\,,$$

and so $[D(J), L] \subseteq D(J) + J$. Therefore, $[J + D(J), L] \subseteq J + D(J)$. This implies that $J + D(J)$ is an ideal of L. □

Theorem 2.3.5. *Let J be the solvable radical of a Leibniz algebra L and D be a derivation. Then $D(J) \subseteq J$.*

Proof. By Lemma 2.3.4 it follows that $J + D(J)$ is an ideal of Leibniz algebra L. We have

$$\big(J + D(J)\big)^{(2)} = [J + D(J), J + D(J)] \subseteq J + [D(J), D(J)] \subseteq J + D^2(J^{(2)})\,.$$

Now assume that

$$\big(J + D(J)\big)^{(k)} \subseteq J + D^{2^{k-1}}\big(J^{(k)}\big) \tag{2.3.1}$$

for some natural $k > 1$. Then

$$\begin{aligned}\big(J + D(J)\big)^{(k+1)} &= [\big(J + D(J)\big)^{(k)}, \big(J + D(J)\big)^{(k)}] \\ &\subseteq [J + D^{2^{k-1}}(J^{(k)}), J + D^{2^{k-1}}(J^{(k)})] \subseteq J + [D^{2^{k-1}}\big(J^{(k)}\big), D^{2^{k-1}}\big(J^{(k)}\big)] \\ &\subseteq J + D^{2^{k-1}+2^{k-1}}\big([J^{(k)}, J^{(k)}]\big) = J + D^{2^k}\big(J^{(k+1)}\big)\,.\end{aligned}$$

Hence, (2.3.1) is verified.

Let $J^{(m)} = 0$. Then $\big(J + D(J)\big)^{(m)} \subseteq J + D^{2^{m-1}}\big(J^{(m)}\big) = J$. Now $\big(J + D(J)\big)^{(2m-1)} = \big((J + D(J))^{(m)}\big)^{(m)} \subseteq J^{(m)} = 0$.

Hence, $J + D(J)$ is a solvable ideal of Leibniz algebra L. Since J is the solvable radical of L, it follows that $J + D(J) \subseteq J$ and therefore, $D(J) \subseteq J$. □

Remark 2.3.6. *In Theorem 2.3.5 if J is the nilradical, analogous arguments establish the invariance of J with respect to any derivation of L.*

It is not difficult to verify that a derivation in a Leibniz algebra induces a derivation in the corresponding Lie quotient algebra. However, the following example shows that the inverse is not necessarily true, i.e., not every derivation in the Lie quotient algebra can be extended to a derivation of the Leibniz algebra.

Example 2.3.7. *Consider a Leibniz algebra $L = \langle e_1, \dots, e_m, f_1, \dots, f_m \rangle$ with the following multiplication*

$$[e_i, e_i] = f_i, \quad 1 \le i \le m, \quad [e_1, e_i] = f_i, \quad 1 \le i \le m, \quad \text{and } 0 \text{ in other case.}$$

Then $L^{\text{ann}} = \langle f_1, \dots, f_m \rangle$ and L/L^{ann} is an abelian Lie algebra. Therefore, any linear operator in L/L^{ann} is a derivation.

Now consider an arbitrary derivation $d \colon L \to L$.

Since $[e_p, e_1] = 0$ for $p > 1$, we have that $0 = d([e_p, e_1]) = [d(e_p), e_1] + [e_p, d(e_1)]$.

If $d(e_p) = d_{1p}e_1 + \dots + d_{mp}e_m + c_{1p}f_1 + \dots + c_{mp}f_m$ then $[d(e_p), e_1] = d_{1p}[e_1, e_1] = d_{1p}f_1$.

Now if $d(e_1) = d_{11}e_1 + \dots + d_{m1}e_m + c_{11}f_1 + \dots + c_{m1}f_m$ then $[e_p, d(e_1)] = d_{p1}[e_p, e_p] = d_{p1}f_p$. Hence we obtain a condition $d_{1p}f_1 + d_{p1}f_p = 0$ which implies $d_{1p} = d_{p1} = 0$ for all $2 \le p \le m$. Therefore, not every derivation of L/L^{ann} can be extended to L.

2.4 Outer derivations of semisimple Leibniz algebras

In this section we establish some results concerning the outer derivations of semisimple and simple Leibniz algebras using information from Chapter 1.

Since I is an anti-symmetric Leibniz $\mathfrak{g}$-bimodule, as discussed in Section 1.1 we have

$$HL^1(\mathfrak{g}, I) = \mathrm{Der}(\mathfrak{g}, I) = \{f : L \to I \mid f([g_1, g_2]) = [f(g_1), g_2]\} \cong \mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{g}, I).$$

Theorem 2.4.1. *For a finite dimensional semisimple Leibniz algebra L with liezation $\mathfrak{g}$ over $\mathbb{C}$ we have a decomposition*

$$HL^1(L, L) \cong \mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{g}, I) \oplus \mathrm{Hom}_{U\mathfrak{g}}(I, I).$$

Proof. By Theorem 1.3.3 and the observation above the space of outer derivations is isomorphic to

$$HL^1(L,L) \cong HL^1(L,L)_{(0)} \oplus \mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{g}, I).$$

Let us concentrate on level zero. Straightforward calculations from the proof of Proposition 1.4.1 show that $ZL^1_{(0)} = \ker \partial|_{H^1_1 \oplus H^0_0}$ and therefore $HL^1(L,L)_{(0)}$ corresponds to the exactness of the following short sequence

$$\mathfrak{g} \longrightarrow H^1_1 \oplus H^0_0 \longrightarrow H^0_{00} \oplus H^1_{10}$$

or in old notations as

$$\mathfrak{g} \longrightarrow \mathrm{Hom}(I,I) \oplus \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \longrightarrow \mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}) \oplus \mathrm{Hom}(I\otimes\mathfrak{g}, I).$$

For $g \in \mathfrak{g}$ we have $\partial(g) = R_g = R_g|_{\mathfrak{g}} + R_g|_I$. So

$$BL^1_{(0)} = \mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}, L) = \mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}) \oplus \mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}, I).$$

Now

$$ZL^1_{(0)} = \ker \partial|_{H^1_1 \oplus H^0_0} = \{d_1 + d_0 \mid d_0 \in \mathrm{Der}(\mathfrak{g}),\ [i, d_0(g)] = -[d_1(i), g] + d_1([i,g])\}.$$

Since for semi-simple $\mathfrak{g}$ every derivation is inner we have that $d_0 = R_a|_I$ for some $a \in \mathfrak{g}$. Putting this into $[i, d_0(g)] = -[d_1(i), g] + d_1([i,g])$ yields

$$\begin{aligned} -[d_1(i), g] + d_1([i,g]) &= [i,[g,a]] \\ &= -[i,[a,g]] = -[[i,a],g] + [[i,g],a] \\ &= -[R_a(i), g] + R_a([i,g]). \end{aligned}$$

The last equality gives $(R_a|_I - d_1)([i,g]) = [(R_a|_I - d_1)(i), g]$.

Denoting by $\alpha := R_a|_I - d_1$ one obtains $\alpha([i,g]) = [\alpha(i), g]$ and $\alpha : I \to I$.

Hence, $d_0 + d_1 = R_a|_{\mathfrak{g}} + R_a|_I + \alpha = R_a + \alpha$.

Therefore,

$$\begin{aligned} HL^1(L,L)_0 &= ZL^2_{(0)}/BL^2_{(0)} \\ &\cong \{\alpha \in \mathrm{Hom}(I,I) \mid \alpha([i,g]) = [\alpha(i), g],\ \forall i \in I, g \in \mathfrak{g}\} = \mathrm{Hom}_{U\mathfrak{g}}(I,I). \end{aligned}$$

and the statement of the theorem is proved. □

In terms of derivations one can reformulate the above theorem as follows.

Corollary 2.4.2. *Let L be a finite dimensional complex semisimple Leibniz algebra with liezation $\mathfrak{g}$. Then any derivation decomposes into a sum of an inner derivation and $\mathfrak{g}$-equivariant homomorphisms $f : \mathfrak{g} \to I$ and $\alpha : I \to I$.*

Similar statement was obtained with different methods in [46].

In case L is a simple Leibniz then both $\mathfrak{g}$ and I are simple $\mathfrak{g}$-modules. By Schur's Lemma $\mathrm{Hom}_{U\mathfrak{g}}(I, I) \cong \mathbb{C}$ and any element in $\mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{g}, I)$ is either zero or is an isomorphism.

Hence, unless $\dim \mathfrak{g} = \dim I$ (which is equivalent to $\dim L = 2 \dim \mathfrak{g}$) for a simple Leibniz algebra L we have $\mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{g}, I) = 0$ and $HL^1(L, L) \cong \mathbb{C}$.

If L is simple and $\dim L = 2 \dim \mathfrak{g}$, then $HL^1(L, L) \cong \mathrm{Isom}_{\mathfrak{g}}(\mathfrak{g}, I) \oplus \mathbb{C}$.

In case of $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$ direct calculations shows that $\mathrm{Isom}_{\mathfrak{g}}(\mathfrak{g}, I) \cong \mathbb{C}$.

In the case when L is a simple Leibniz algebra, using famous Schur's Lemma the next statement follows.

Corollary 2.4.3. *Let L be a finite dimensional complex simple Leibniz algebra with liezation $\mathfrak{g}$. If $\dim L = 2 \dim \mathfrak{g}$, then $HL^1(L, L) \cong \mathbb{C} \oplus \mathbb{C}$. Otherwise, $HL^1(L, L) \cong \mathbb{C}$.*

Chapter 3

Structure theory of Leibniz n-algebras

Investigations of Leibniz algebras and n-Lie algebras show that many properties of solvable and nilpotent radicals, Frattini and Cartan subalgebras and regular elements of Lie algebras may be extended to these more general algebras. Therefore a natural question occurs whether the corresponding classical results on the structure theory are valid in Leibniz n-algebras. This problem is the main objective of this chapter. We present the necessary results concerning the theory of Leibniz n-algebras and give references in Section 3.1. All spaces in this chapter are assumed to be finite dimensional and if otherwise stated, over the field of complex numbers. Section 3.2 is devoted to certain properties of an ideal generated by squares that occurs only in case $n \geq 3$. In Section 3.3 notions as k-solvability, nilpotency and K_1-nilpotency of Leibniz n-algebras are introduced and their invariance with respect to any derivation is proved. Section 3.4 is devoted to the study of Frattini subalgebras of Leibniz n-algebras. A relation between nilpotency of Leibniz n-algebra and behavior of its maximal ideals, as well as Frattini subalgebras are established. Finally, in Section 3.5, Cartan subalgebras are investigated and examples that show the non-conjugacy of Cartan subalgebras for Leibniz n-algebras are constructed.

3.1 Preliminary definitions and results

Definition 3.1.1 ([20]). *A vector space L with a linear map called n-ary multiplication $[-,-,\ldots,-] : L^{\otimes n} \to L$ is called a Leibniz n-algebra if it satisfies* *for all $x_1, \dots, x_n, y_2, \dots, y_n \in L$ the following identity*

$$[[x_1, x_2, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n]. \quad (3.1.1)$$

An n-Lie algebra L is a Leibniz n-algebra, where the bracket multiplication $[-, -, \dots, -]$ factors through the exterior product to a morphism

$$\Lambda^n L = \underbrace{L \wedge \cdots \wedge L}_{n \text{ factors}} \to L.$$

Obviously, if the multiplication $[-, -, \dots, -]$ is skew-symmetric in each pair of variables, i.e.

$$[x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_n] = -[x_1, x_2, \dots, x_j, \dots, x_i, \dots, x_n]\,,$$

then we obtain an $n-$Lie algebra.

Example 3.1.2. [20] Let L be a Leibniz algebra with the product $[-, -]$. Then the vector space L can be equipped with the Leibniz n-algebra structure with the following product:

$$[x_1, x_2, \dots, x_n] := [x_1, [x_2, \dots, [x_{n-1}, x_n]]].$$

Since in Leibniz n-algebras the n-ary multiplication is not necessarily skew-symmetrical, it is possible to consider some variation of basic notions such as ideals with different conditions.

Definition 3.1.3. *A subspace I of a Leibniz n-algebra L is called an s-ideal of L, if*

$$[\underbrace{L, \dots, L}_{s-1}, I, \underbrace{L, \dots, L}_{n-s}] \subseteq I.$$

If I is an s-ideal for all $1 \leq s \leq n$, then I is called an ideal.

Given an arbitrary Leibniz n-algebra L consider the following sequences (s is a fixed natural number, $1 \leq s \leq n$):

$$L^{<1>_s} = L, \qquad L^{<k+1>_s} = [\underbrace{L, \dots, L}_{(s-1)-\text{times}}, L^{<k>_s}, \underbrace{L, \dots, L}_{(n-s)-\text{times}}],$$

$$L^1 = L, \qquad L^{k+1} = \sum_{i=1}^{n} [\underbrace{L, \dots, L}_{(i-1)-\text{times}}, L^k, \underbrace{L, \dots, L}_{(n-i)-\text{times}}].$$

Definition 3.1.4. *A Leibniz n-algebra L is said to be s-nilpotent (nilpotent) if there exists a natural number $k \in \mathbb{N}$ ($l \in \mathbb{N}$) such that $L^{<k>_s} = 0$ ($L^l = 0$, respectively).*

It should be noted that for n-Lie algebras the above notions of an s-nilpotency and a nilpotency coincide. Recall also that for Leibniz algebras (i.e. Leibniz 2-algebras) the notions of 1-nilpotency and nilpotency also coincide [5].

In [4, Example 2.2], it is shown that the s-nilpotency property for Leibniz n-algebra ($n \geq 3$) essentially depends on s.

Definition 3.1.5. *A linear map d defined on a Leibniz n-algebra L is called a derivation if*

$$d([x_1, x_2, \dots, x_n]) = \sum_{i=1}^{n} [x_1, \dots d(x_i), \dots, x_n].$$

The space of all derivations of a given Leibniz n-algebra L is denoted by $\mathrm{Der}(L)$.

The space $\mathrm{Der}(L)$ forms a Lie algebra with respect to the commutator [4]. Set $A^{\times k} = \underbrace{A \times A \times \cdots \times A}_{k-\text{times}}$.

Given an arbitrary element $x = (x_2, \dots, x_n) \in L^{\times(n-1)}$ consider the operator $R[x] : L \to L$ of right multiplication defined for all $z \in L$ by

$$R[x](z) = [z, x_2, \dots, x_n].$$

Any right multiplication operator is a derivation and the space $R[L]$ of all right multiplication operators forms a Lie ideal of $\mathrm{Der}(L)$ [4]. Furthermore, they possess the following property.

Theorem 3.1.6 ([4] Engel's Theorem)**.** *A Leibniz n-algebra L is 1-nilpotent if and only if $R[x]$ is nilpotent for all $x \in L^{\times(n-1)}$.*

The following lemma yields a decomposition of a given vector space into a direct sum of two subspaces which are invariant with respect to a given linear transformation. The proof is given in [29, Chapter II, §4].

Lemma 3.1.7 (Fitting Lemma)**.** *Let L be a vector space and $R : L \to L$ be a linear transformation. Then $L = L_{0R} \oplus L_{1R}$, where $R(L_{0R}) \subseteq L_{0R}$, $R(L_{1R}) \subseteq$*

L_{1R} and $L_{0R} = \{v \in L \mid R^i(v) = 0$ for some $i\}$ and $L_{1R} = \bigcap_{i=1}^{\infty} R^i(L)$. Moreover, $R_{|L_{0R}}$ is a nilpotent transformation and $R_{|L_{1R}}$ is an automorphism. L_{0R} is called the Fitting null-component and L_{1R} is called the Fitting one-component of L with respect to R.

It should be noted that in the case of n-Lie algebras the operator $R[h]$ of right multiplication is degenerate. In particular, if the dimension of an n-Lie algebra L is less than n then we have $V_{0R[h]} = L$. If $\dim L \geq n$ then $\dim V_{0R[h]} \geq n-1$.

Note that for Leibniz algebras (i.e. $n = 2$) the operator $R[h]$ is also degenerate [42]. Let us give an example of a Leibniz n-algebra $(n \geq 3)$ which admits a non degenerate operator of right multiplication.

Example 3.1.8. *Consider an m-dimensional Leibniz n-algebra L over a field $\mathbb{K}$ with the following multiplication:*

$$[e_i, e_1, \ldots, e_{n-1}] = \alpha_i e_i, \quad \alpha \in \mathbb{K}$$

where $\{e_1, \ldots, e_m\}$ is the basis of the algebra, $\alpha_i \neq 0$ for all $1 \leq i \leq m$, $\sum_{i=1}^{n-1} \alpha_i = 0$ and all other products are zero.

In this algebra the operator $R(e_1, \ldots, e_{n-1})$ is non-degenerate.

This is the significant difference between Leibniz n-algebras $(n \geq 3)$ on one hand and Leibniz algebras and n-Lie algebras on the other.

We also have the following generalization of Fitting's Lemma for Lie algebras of nilpotent transformations of a vector space that is proved in [29] (chapter II, §4).

Theorem 3.1.9. *Let G be a nilpotent Lie algebra of linear transformations of a vector space V and $V_0 = \bigcap_{A \in G} V_{0A}$, $V_1 = \bigcap_{i=1}^{\infty} G^i(V)$. Then the subspaces V_0 and V_1 are invariant with respect to G (i.e. they are invariant with respect to every transformation B of G) and $V = V_0 \oplus V_1$. Moreover, $V_1 = \sum_{A \in G} V_{1A}$.*

Remark 3.1.10. *From [29] (chapter III, p. 117) in the case of a vector space V over an infinite field and under the conditions of Theorem 3.1.9, we have the existence of an element $B \in G$ such that $V_0 = V_{0B}$ and $V_1 = V_{1B}$.*

From the theory of Linear Algebra we have the following well-known theorem [38] that presents a decomposition of a vector space L into a direct sum of its characteristics spaces with respect to a linear operator.

Theorem 3.1.11. *Let L be a finite dimensional vector space over a field $\mathbb{C}$ and $R : L \to L$ be a linear operator. Then for the eigenvalues $\alpha_1, \dots, \alpha_k$ of R we have the following decomposition*

$$L = L_{\alpha_1} \oplus L_{\alpha_2} \oplus \cdots \oplus L_{\alpha_k},$$

where $L_{\alpha_i} = \{x \in L \mid \exists\, m \in \mathbb{N} \text{ such that } (R - \alpha_i I)^m(x) = 0\}$.

Remark 3.1.12. *If α is an eigenvalue of R, then $\{0\} \neq \ker(R-\alpha I) \subseteq L_\alpha$ and therefore $\dim L_\alpha \geq 1$. Now if α is not an eigenvalue, then $R-\alpha I$ is non-singular operator, which in turns implies that $L_\alpha = \{0\}$. Therefore, in decomposition of Theorem 3.1.11 we can always add zero subspaces L_α for non-eigenvalues of operator R.*

We will be using frequently the following

Lemma 3.1.13 ([4])**.** *Let M be an invariant subspace of a vector space L with respect to a linear transformation $Q : L \to L$. Let $x = x_0 + x_\alpha + x_\beta + \cdots + x_\gamma$ be any decomposition of an element x into a sum of characteristic vectors from the corresponding characteristic spaces $L_\xi (\xi \in \{0, \alpha, \beta, \dots, \gamma\})$. Let $Q(x) \in M$. Then $x - x_0 \in M$.*

The following lemma is a generalization of the correspondings result from Lie, n-Lie and Leibniz algebra theories.

Lemma 3.1.14 ([16])**.** *Let L be a finite dimensional complex Leibniz n-algebra with given derivation d, and let $L = L_\alpha \oplus L_\beta \oplus \cdots \oplus L_\gamma$ be the decomposition from Theorem 3.1.11. Then*

$$[L_{\alpha_1}, L_{\alpha_2}, \dots, L_{\alpha_n}] \subseteq \begin{cases} 0 & \text{if } \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ is not a root of } d \\ L_{\alpha_1+\alpha_2+\cdots+\alpha_n} & \text{if } \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ is a root of } d. \end{cases}$$

Notice, from this lemma it immediately follows that Fitting null-component, i.e. null-characteristics space of the decomposition of a Leibniz n-algebra L with respect to a derivation is a subalgebra of L. In case of right multiplication operators, this subalgebras are called Engel's subalgebras in the theory of Leibniz [9] and n-Lie [10] algebras.

Definition 3.1.15 ([4])**.** *An element $h \in L^{\times(n-1)}$ is said to be regular for a Leibniz n-algebra L if the dimension of the Fitting null-component of the space L with respect to $R[h]$ is not greater than the dimension of the Fitting null-component of the space L with respect to $R[g]$ for any $g \in L^{\times(n-1)}$.*

Since multiplication in Leibniz n-algebras essentially depends on the position of multiplicand, the definition of normalizer is given as follows.

Definition 3.1.16 ([4])**.** *For a natural number $s \leq n$ and a given subset X in a Leibniz n-algebra, the s-normalizer of X is a set*

$$N_s(X) = \{a \in L \mid [x_1, \dots, x_{s-1}, a, x_{s+1}, \dots, x_n] \in X \text{ for all } x_i \in X\}.$$

The set $N(X) = \bigcap_{s=1}^{n} N_s(X)$ is called the normalizer of X.

Notice that, if X is a subalgebra of L, then $N_s(X)$ contains X for all $1 \leq s \leq n$ and whence, $N(X) \supseteq X$.

3.2 Ideal generated by squares

We need the following lemma which can easily be proved.

Lemma 3.2.1. *Let $[-, -, \dots, -] : V^{\otimes n} \to V$ be a polylinear operation on a vector space V. The following conditions are equivalent:*

1) $[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -[x_1, \dots, x_{i+1}, x_i, \dots, x_n]$ *for all $1 \leq i \leq n-1$,*

2) $[x_1, \dots, x_i, \dots, x_j, \dots, x_n] = -[x_1, \dots, x_j, \dots, x_i, \dots, x_n]$ *for all $1 \leq i \neq j \leq n$,*

3) $[x_1, \dots, x_i, \dots, x_j, \dots, x_n] = 0$ *if $x_i = x_j$ for some $1 \leq i \neq j \leq n$,*

4) $[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = 0$ *if $x_i = x_{i+1}$ for some $1 \leq i \leq n-1$.*

Consider the n-sided ideal

$$I = \text{ideal}\langle [x_1, \dots, x_i, \dots, x_j, \dots, x_n] \mid \exists i, j : x_i = x_j \rangle.$$

Lemma 3.2.2. *Let L be a Leibniz n-algebra. If it admits a non-degenerate operator of right multiplication then $I = L$.*

Proof. Let $x_2, \dots, x_n \in L$ be elements such that the operator $R(x_2, \dots, x_n)$ is non-degenerate.

Suppose first that $x_2, \dots, x_n$ are linearly dependent and

$$x_p = \sum_{i=2,\ i \neq p}^{n} \alpha_i x_i,\ \ \alpha_i \in \mathbb{C}.$$

For any $a \in L$ there exists $b \in L$ such that

$$a = [b, x_2, \dots, x_n] = \sum_{i=2, i \neq p}^{n} \alpha_i [b, x_2, \dots, x_{p-1}, x_i, x_{p+1}, \dots, x_n] \in I$$

and therefore $L = I$.

Now suppose that $x_2, \dots, x_n$ are linearly independent. Since the operator $R(x_2, \dots, x_n)$ is non-degenerate, there exists $y_k \in L$, $(2 \leq k \leq n)$ such that $[y_k, x_2, \dots, x_n] = x_k$.

It is clear that the elements $y_2, \dots, y_n$ are also linearly independent.

Note that if $x_k \in I$ for some k, then

$$L = [L, x_2, \dots, x_n] \subseteq I,$$

and therefore $L = I$.

Suppose that $x_k \notin I$ for all $2 \leq k \leq n$. Consider the equality

$$\begin{aligned} x_k = & [y_k, x_2, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n] = \\ & [y_k, x_2, \dots, x_{k-1}, [y_k, x_2, \dots, x_n], x_{k+1}, \dots, x_n]. \end{aligned}$$

The Leibniz n-algebra identity (3.1.1) implies

$$\begin{aligned} & [[y_k, x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n], x_2, \dots, x_n] \\ & = [[y_k, x_2, \dots, x_n], x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] \\ & \quad + [y_k, [x_2, x_2, \dots, x_n], x_3 \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] + \dots + \\ & \quad + [y_k, x_2, \dots, x_{k-2}, [x_{k-1}, x_2, \dots, x_n], y_k, \dots, x_n] \\ & \quad + [y_k, x_2, \dots, x_{k-1}, [y_k, x_2, \dots, x_n], x_{k+1}, \dots, x_n] \\ & \quad + [y_k, x_2, \dots, x_{k-1}, y_k, [x_{k+1}, x_2, \dots, x_n], \dots, x_n] + \dots \\ & \quad + [y_k, x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_{n-1}, [x_n, x_2, \dots, x_n]]. \end{aligned}$$

Since all summands on the right side except

$$[[y_k, x_2, \dots, x_n], x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n]\,,$$

$$[y_k, x_2, \dots, x_{k-1}, [y_k, x_2, \dots, x_n], x_{k+1}, \dots, x_n]$$

and the whole left side in the above equality belong to the ideal I, it follows that

$$[y_k, x_2, \dots, x_{k-1}, [y_k, x_2, \dots, x_n], x_{k+1}, \dots, x_n]$$
$$+ [[y_k, x_2, \dots, x_n], x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] \in I.$$

Therefore

$$x_k + [x_k, x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] \in I. \qquad (3.2.1)$$

If $a \in \langle x_2, \dots, x_n\rangle \cap I$ then $a = \sum_{i=2}^{n} \alpha_i x_i$.

We have that

$$[a, x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] \in I.$$

On the other hand

$$[a, x_2, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] = \sum_{i=2}^{n} \alpha_i [x_i, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n].$$

Therefore $\alpha_k [x_k, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n] \in I$. Thus (3.2.1) implies that $\alpha_k x_k \in I$ and hence $\alpha_k = 0$ $(2 \le k \le n)$, i.e. $a = 0$. This means that

$$\langle x_2, \dots, x_n\rangle \cap I = 0.$$

Note that $y_k \notin I$, because if $y_k \in I$ then (3.2.1) implies that $x_k \in I$ which contradicts our assumption that $x_k \notin I$ for all k.

If $a \in \langle y_2, \dots, y_n\rangle \cap \langle x_2, \dots, x_n\rangle$, then $a = \sum_{i=2}^{n} \beta_i y_i$ and $a = \sum_{i=2}^{n} \alpha_i x_i$ for some $\alpha_i, \beta_i \in \mathbb{C}$.

By applying the operator $R(x_2, \dots, x_n)$ to the element a we obtain

$$\sum_{i=2}^{n} \beta_i [y_i, x_2, \dots, x_n] = \sum_{i=2}^{n} \alpha_i [x_i, x_2, \dots, x_n] \in I,$$

i.e. $\sum_{i=2}^{n} \beta_i x_i \in I$. But $\langle x_2, \ldots, x_n \rangle \cap I = \{0\}$ and therefore $\beta_i = 0$ for all $2 \leq i \leq n$ and thus

$$\langle y_2, \ldots, y_n \rangle \cap \langle x_2, \ldots, x_n \rangle = 0.$$

If $a \in \langle y_2, \ldots, y_n \rangle \cap I$ then $a = \sum_{i=2}^{n} \alpha_i y_i$ and

$$\sum_{i=2}^{n} \alpha_i x_i = \sum_{i=2}^{n} \alpha_i [y_i, x_2, \ldots, x_n] = [a, x_2, \ldots, x_n] \in I.$$

Therefore $\alpha_i = 0$, i.e. $a = 0$ and

$$\langle y_2, \ldots, y_n \rangle \cap I = 0.$$

Since $R(x_2, \ldots, x_n)$ is non-degenerate, there exists $z_k \in L, (2 \leq k \leq n)$ such that $[z_k, x_2, \ldots, x_n] = y_k$.

In the same way one can prove that $z_2, \ldots, z_n$ are linearly independent and

$$\langle z_2, \ldots, z_n \rangle \cap \langle x_2, \ldots, x_n \rangle = 0,$$

$$\langle z_2, \ldots, z_n \rangle \cap \langle y_2, \ldots, y_n \rangle = 0,$$

$$\langle z_2, \ldots, z_n \rangle \cap I = 0.$$

Repeating the above process we obtain that

$$I \oplus \langle x_2, \ldots, x_n \rangle \oplus \langle y_2, \ldots, y_n \rangle \oplus \langle z_2, \ldots, z_n \rangle \oplus \cdots \subseteq L,$$

which contradicts the finiteness of $\dim L$. The proof is complete. □

The following example shows that the converse assertion to Lemma 3.2.2 is not true in general.

Example 3.2.3. *Consider a complex m-dimensional ($m \geq 4$) non Lie Leibniz algebra L with the basis $\{e, f, h, i_0, i_1, \ldots, i_{m-4}\}$ and the following table of multiplication:*

$$[i_k, h] = (m - 4 - 2k) i_k, \quad 0 \leq k \leq m - 4;$$

$$[i_k, f] = i_{k+1}, \quad 0 \leq k \leq m - 5;$$

$$[i_k, e] = k(k+3-n)i_{k-1}, \quad 1 \leq k \leq m-4;$$

$$[e, h] = 2e, \quad [h, e] = -2e, \quad [f, h] = -2f,$$

$$[h, f] = 2f, \quad [e, f] = h, \quad [f, e] = -h,$$

with other products are zero.

Following the construction of Leibniz n-algebras from Example 3.1.2 we obtain a Leibniz n-algebra. For $n > 4$ one has

$$I \ni [h, h, \dots, h, e] = [h, [h, \dots, [h, e] \dots]] = (-2)^{n-1}e,$$

$$I \ni [h, h, \dots, h, f] = [h, [h, \dots, [h, f] \dots]] = 2^{n-1}f,$$

$$I \ni [f, h, \dots, h, e] = [f, [h, \dots, [h, e] \dots]] = -(-2)^{n-2}h,$$

$$I \ni [i_k, h, \dots, h, f] = [i_k, [h, \dots, [h, f] \dots]] = 2^{n-2}i_{k+1}$$

for $0 \leq k \leq m-5$ and

$$I \ni [i_1, h, \dots, h, e] = [i_1, [h, \dots, [h, e] \dots]] = (-2)^{n-2}(4-n)i_0.$$

Therefore for this Leibniz n-algebra ($n > 4$) we have $I = L$.

Moreover, let us show that in this Leibniz n-algebra all operators of right multiplication are degenerate. Indeed, suppose that for some $a = (a_2, \dots, a_n) \in L^{\times(n-1)}$ the operator $R[a]$ is non-degenerate. Then for every $x \in L$ we have

$$0 \neq [x, a_2, \dots, a_n] = [x, [a_2, \dots, [a_{n-1}, a_n] \dots]],$$

and for the element $b = [a_2, \dots, [a_{n-1}, a_n] \dots]$ we obtain that $R[b]$ is a non-degenerate operator in the Leibniz algebra L, which contradicts the Lemma 2.6 from [42].

3.3 Invariance of Some Radicals under a Derivation

In the following section we introduce some versions of solvability and nilpotency of ideals of Leibniz n-algebras and extend some classical results from the theory of Lie algebras, that are also true in Leibniz algebras and n-Lie algebras to the case of Leibniz n-algebras.

3.3.1 Invariance of k-Solvable Radicals

Let H be an ideal of a Leibniz n-algebra L. Put $H^{(1)_k} = H$ and

$$H^{(m+1)_k} = \sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L,\ldots,L}_{i_1}, H^{(m)_k}, \underbrace{L,\ldots,L}_{i_2}, H^{(m)_k}, \ldots, \underbrace{L,\ldots,L}_{i_k}, H^{(m)_k}, \underbrace{L,\ldots,L}_{n-k-i_1-\cdots-i_k}]$$

for all $1 \le k \le n$ and $m \ge 1$.

Proposition 3.3.1. *For an ideal H of a Leibniz n-algebra L the equality $(H^{(m)_k})^{(r)_k} = H^{(m+r-1)_k}$ holds for all $m, r \in \mathbb{N}$ and $1 \le k \le n$.*

Proof. Let us fix a non-negative integer m. The assertion of the proposition is obvious if $r = 1$. Let it be true for some $r > 1$. Then

$$(H^{(m)_k})^{(r+1)_k} =$$
$$= \sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L,\ldots,L}_{i_1}, (H^{(m)_k})^{(r)_k}, \ldots, \underbrace{L,\ldots,L}_{i_k}, (H^{(m)_k})^{(r)_k}, \underbrace{L,\ldots,L}_{n-k-i_1-\cdots-i_k}] =$$
$$\Big(\text{and by induction hypothesis, i.e. } (H^{(m)_k})^{(r)_k} = H^{(m+r-1)_k} \text{ we have }\Big)$$
$$= \sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L,\ldots,L}_{i_1}, H^{(m+r-1)_k}, \ldots, \underbrace{L,\ldots,L}_{i_k}, H^{(m+r-1)_k}, \underbrace{L,\ldots,L}_{n-k-i_1-\cdots-i_k}] =$$
$$= H^{(m+r)_k}$$

which completes the proof by induction. □

Even though we can not state that $H^{(m)_k}$ is an ideal, we establish the following result.

Proposition 3.3.2. *For an ideal H of a Leibniz n-algebra L, $H^{(m)_k}$ is a 1-ideal of L for all $m \in \mathbb{N}$ and $1 \le k \le n$.*

Proof. Let k be an arbitrary fixed natural number not greater than n.

For $m = 1$ we have $[H^{(1)_k}, L, \ldots, L] \subseteq [H, L, \ldots, L] \subseteq H = H^{(1)_k}$ since H is an ideal.

Let $H^{(m)_k}$ be a 1-ideal, i.e. $[H^{(m)_k}, L, \ldots, L] \subseteq H^{(m)_k}$.

Then

$$[H^{(m+1)_k}, L, \dots, L] =$$
$$= \Big[\sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, H^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)_k}, \underbrace{L, \dots, L}_{n-k-i_1-\cdots-i_k}], L, \dots, L\Big] =$$
$$= \sum_{i_1+\cdots+i_k=0}^{n-k} \big[[\underbrace{L, \dots, L}_{i_1}, H^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)_k}, \underbrace{L, \dots, L}_{n-k-i_1-\cdots-i_k}], L, \dots, L\big] .$$

Since $H^{(m)_k}$ is a 1-ideal by induction hypothesis, using identity (3.1.1) we obtain that

$$\big[[\underbrace{L, \dots, L}_{i_1}, H^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)_k}, \underbrace{L, \dots, L}_{n-k-i_1-\cdots-i_k}], L, \dots, L\big] \subseteq$$
$$\subseteq [\underbrace{L, \dots, L}_{i_1}, H^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)_k}, \underbrace{L, \dots, L}_{n-k-i_1-\cdots-i_k}] .$$

Therefore

$$[H^{(m+1)_k}, L, \dots, L] \subseteq$$
$$\subseteq \sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, H^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)_k}, \underbrace{L, \dots, L}_{n-k-i_1-\cdots-i_k}] = H^{(m+1)_k}$$

and $H^{(m+1)_k}$ is a 1-ideal of L. □

Definition 3.3.3. *An ideal H of Leibniz n-algebra is said to be k-solvable with index of k-solvability equal to m if there exists $m \in \mathbb{N}$ such that $H^{(m)_k} = 0$ and $H^{(m-1)_k} \neq 0$.*

When $L = H$, L is called a k-solvable Leibniz n-algebra.

Notice that this definition agrees with the definition of k-solvability of n-Lie algebras given in [31].

The following proposition is a natural generalization of well-known result from the theory of Lie [29] and n-Lie algebras [30].

Proposition 3.3.4. *Let I be a k-solvable ideal of a Leibniz n-algebra L such that L/I is also k-solvable. Then L is k-solvable.*

Proof. Let $\phi\colon L \to L/I$ be the natural homomorphism. Since L/I is k-solvable, we have $0 = (L/I)^{(m)_k} = \big(\phi(L)\big)^{(m)_k} = \phi(L^{(m)_k})$ for some $m \in \mathbb{N}$. Thus $L^{(m)_k} \subseteq I$. Since I is k-solvable, there exists $p \in \mathbb{N}$ such that $I^{(p)_k} = 0$. Therefore by Proposition 3.3.1 we have $L^{(m+p-1)_k} = (L^{(m)_k})^{(p)_k} \subseteq I^{(p)_k} = 0$ and so L is k-solvable. □

Proposition 3.3.5. *Let I be a k-solvable ideal of a Leibniz n-algebra L. Then I is also $(k+p)$-solvable for all $1 \le p \le n-k$.*

Proof. We have $I^{(m+1)_{k+p}} =$

$$\sum_{i_1+\dots+i_{k+p}=0}^{n-(k+p)} [\underbrace{L,\dots,L}_{i_1}, I^{(m)_{k+p}},\dots,\underbrace{L,\dots,L}_{i_{k+p}}, I^{(m)_{k+p}}, \underbrace{L,L,\dots,L}_{n-(k+p)-i_1-\dots-i_{k+p}}] \subseteq$$

$$\subseteq \sum_{i_1+\dots+i_k=0}^{n-k} [\underbrace{L,\dots,L}_{i_1}, I^{(m)_{k+p}},\dots,\underbrace{L,\dots,L}_{i_k}, I^{(m)_{k+p}}, \underbrace{L,L,\dots,L}_{n-k-i_1-\dots-i_k}] = I^{(m+1)_k}$$

for all $m \in \mathbb{N}$. Therefore, for some $r \in \mathbb{N}$ we have $I^{(r)_{k+p}} \subseteq I^{(r)_k} = \{0\}$. □

Proposition 3.3.6. *Let I and J be k-solvable ideals of a Leibniz n-algebra L. Then $I+J$ is k-solvable.*

Proof. Since $I \cap J$ is an ideal in I, by the Second Isomorphism Theorem we have $(I+J)/J \cong I/(I\cap J)$.

Obviously, $I/(I\cap J)$ is k-solvable. Then by Proposition 3.3.4 we obtain that $I+J$ is also k-solvable. □

Let H be a maximal k-solvable ideal in a finite dimensional Leibniz n-algebra L and let K be an arbitrary k-solvable ideal of L. Then $H+K$ is also k-solvable and $H+K \supseteq H$. Since H is a maximal k-solvable ideal, we obtain that $H+K=H$. Therefore we can define the maximal k-solvable ideal as the sum of all the k-solvable ideals in L and call it the *k-solvable radical*.

The following identity for a derivation $d: L \to L$ of a Leibniz n-algebra L over a field $\mathbb{K}$ of characteristic zero, for any $k \in \mathbb{N}$, was given in [16]:

$$d^k([x_1,\dots,x_n]) = \sum_{i_1+i_2+\dots+i_n=k} \frac{k!}{i_1!i_2!\dots i_n!} [d^{i_1}(x_1), d^{i_2}(x_2),\dots,d^{i_n}(x_n)]\,. \tag{3.3.1}$$

Proposition 3.3.7. *Let I be an ideal of a Leibniz n-algebra L and $d \in \operatorname{Der}(L)$. Then*

$$\big(d(I)\big)^{(m)_k} \subseteq I + d^{k^{m-1}}\big(I^{(m)_k}\big)$$

for all $m \in \mathbb{N}$ and $1 \leq k \leq n$.

Proof. For $m = 1$ we have $d(I) \subseteq I + d(I)$ which obviously holds.

Assume that $\big(d(I)\big)^{(m)_k} \subseteq I + d^{k^{m-1}}\big(I^{(m)_k}\big)$ holds for some $m > 1$.

Using formula (3.3.1) we verify the inclusion for $m + 1$:

$$\big(d(I)\big)^{(m+1)_k} =$$

$$= \sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L, \ldots, L}_{i_1}, d(I)^{(m)_k}, \ldots, \underbrace{L, \ldots, L}_{i_k}, d(I)^{(m)_k}, \underbrace{L, \ldots, L}_{n-k-i_1-\cdots-i_k}] \subseteq$$

$$\sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L, \ldots, L}_{i_1}, I + d^{k^{m-1}}(I^{(m)_k}), \ldots, \underbrace{L, \ldots, L}_{i_k}, I + d^{k^{m-1}}(I^{(m)_k}), \underbrace{L, \ldots, L}_{n-k-i_1-\cdots-i_k}].$$

Now due to the fact that I is an ideal of L and identity (3.3.1), the sum of subspaces above is a subset of

$$I + d^{k^m}\Big(\sum_{i_1+\cdots+i_k=0}^{n-k} [\underbrace{L, \ldots, L}_{i_1}, I^{(m)_k}, \ldots, \underbrace{L, \ldots, L}_{i_k}, I^{(m)_k}, \underbrace{L, \ldots, L}_{n-k-i_1-\cdots-i_k}]\Big)$$

which in turn equals

$$I + d^{k^m}\big(I^{(m+1)_k}\big).$$

Therefore, the assertion of the proposition is true. □

Note that in [16], it was shown that for any ideal I of L and $d \in \operatorname{Der}(L)$ the $I + d(I)$ is also an ideal of L.

Theorem 3.3.8. *Let J be a k-solvable radical of a Leibniz n-algebra L. Then $d(J) \subseteq J$ for any $d \in \operatorname{Der}(L)$.*

Proof. Let $s \in \mathbb{N}$ be such $J^{(s)_k} = 0$. Then by Proposition 3.3.7 we have

$$\big(d(J)\big)^{(s)_k} \subseteq J + d^{k^{s-1}}\big(J^{(s)_k}\big) = J.$$

Using formula (3.3.1), we obtain that $\big(J+d(J)\big)^{(s)_k} \subseteq J + \big(d(J)\big)^{(s)_k} \subseteq J$. Now by Proposition 3.3.1 we have

$$\big(J+d(J)\big)^{(2s-1)_k} = \Big(\big(J+d(J)\big)^{(s)_k}\Big)^{(s)_k} \subseteq J^{(s)_k} = 0.$$

But this means that $J+d(J)$ is a k-solvable ideal. Since J is a k-solvable radical, we obtain that $J+d(J) \subseteq J$ and therefore $d(J) \subseteq J$. □

3.3.2 Invariance of K_1-Nilradicals

Similarly as in [6] we introduce the following series for a 1-ideal I of a Leibniz n-algebra L :

$$I^{[1]} = I, \quad I^{[k+1]} = [I^{[k]}, I, L, \ldots, L] \quad (k \geq 1).$$

By a simple induction using identity (3.1.1) it can be proved that for any 1-ideal I and for all $p \in \mathbb{N}$, $I^{[p]}$ is a 1-ideal.

Definition 3.3.9. *A* 1*-ideal I is called K_1-nilpotent, if there exists $k \in \mathbb{N}$ such that $I^{[k]} = 0$.*

The introduced type of nilpotency is also known as nilpotency in the sense of Kuzmin for n-Lie algebras. Identity (3.1.1) is organized in such way, that the elements of the above introduced series are 1-ideals. However, if we change the position of $I^{[k]}$ in the product defining $I^{[k+1]}$ from the first to any other, we are not able to state that the elements of the obtained series will be s-ideals of L for any $2 \leq s \leq n$.

Proposition 3.3.10. *Let I and J be K_1-nilpotent* 1*-ideals. Then $I+J$ is also a K_1-nilpotent* 1*-ideal.*

Proof. First, observe that

$$[I^{[p]} \cap J^{[q]}, I, L \ldots, L] \subseteq [I^{[p]}, I, L, \ldots, L] = I^{[p+1]},$$

and since $J^{[q]}$ is a 1-ideal we obtain

$$[I^{[p]} \cap J^{[q]}, I, L \ldots, L] \subseteq [J^{[q]}, I, L \ldots, L] \subseteq J^{[q]}.$$

Therefore,

$$[I^{[p]} \cap J^{[q]}, I, L \ldots, L] \subseteq I^{[p+1]} \cap J^{[q]}.$$

Analogously,

$$[I^{[p]} \cap J^{[q]}, J, L \ldots, L] \subseteq I^{[p]} \cap J^{[q+1]}.$$

We have $(I+J)^{[1]} = I + J = I^{[1]} + J^{[1]}$.

Now assume that

$$(I+J)^{[k]} \subseteq I^{[k]} + \left(I^{[k-1]} \cap J^{[1]}\right) + \cdots + \left(I^{[1]} \cap J^{[k-1]}\right) + J^{[k]}$$

holds for some $k > 1$ (induction hypothesis).

Then

$$\begin{aligned}(I+J)^{[k+1]} = [(I+J)^{[k]}, I+J, L, \ldots, L] \subseteq \\ \subseteq [(I+J)^{[k]}, I, L, \ldots, L] + [(I+J)^{[k]}, J, L, \ldots, L].\end{aligned}$$

Applying the induction hypothesis we obtain

$$\begin{aligned}&[(I+J)^{[k]}, I, L, \ldots, L] + [(I+J)^{[k]}, J, L, \ldots, L] \\ &\subseteq [I^{[k]}, I, L, \ldots, L] + \sum_{r=1}^{k-1} [I^{[k-r]} \cap J^{[r]}, I, L, \ldots, L] + [J^{[k]}, I, L, \ldots, L] \\ &+ [I^{[k]}, J, L, \ldots, L] + \sum_{r=1}^{k-1} [I^{[k-r]} \cap J^{[r]}, J, L, \ldots, L] + [J^{[k]}, J, L, \ldots, L] \\ &\subseteq I^{[k+1]} + \Big(\sum_{r=1}^{k-1} I^{[k-r+1]} \cap J^{[r]}\Big) + \left(I^{[1]} \cap J^{[k]}\right) \\ &+ \left(I^{[k]} \cap J^{[1]}\right) + \Big(\sum_{r=1}^{k-1} I^{[k-r]} \cap J^{[r+1]}\Big) + J^{[k+1]} \\ &\subseteq I^{[k+1]} + \left(I^{[k]} \cap J^{[1]}\right) + \cdots + \left(I^{[1]} \cap J^{[k]}\right) + J^{[k+1]}.\end{aligned}$$

Hence, for any $p \in \mathbb{N}$ we have

$$(I+J)^{[p]} \subseteq I^{[p]} + \left(I^{[p-1]} \cap J^{[1]}\right) + \cdots + \left(I^{[1]} \cap J^{[p-1]}\right) + J^{[p]}.$$

So if $I^{[p_1]} = 0$ and $J^{[p_2]} = 0$, then for $p = p_1 + p_2$ every summand in the above sum is zero. Therefore $(I+J)$ is also K_1-nilpotent. □

Let I be a maximal K_1-nilpotent ideal in a finite dimensional Leibniz n-algebra L and let J be an arbitrary K_1-nilpotent ideal of L. Then $I+J$ is also K_1-nilpotent and $I+J \supseteq I$. Since I is a maximal K_1-nilpotent ideal, we obtain that $I+J=I$. Therefore we can define the maximal K_1-nilpotent ideal as the sum of all the K_1-nilpotent ideals in L and call it the K_1*-nilradical*. Notice that, the K_1-nilradical does not possess the properties of the radical in the sense of Kurosh.

Using the same argumentation as in the proof of Proposition 3.3.7 and Theorem 3.3.8 the following statements can be established.

Proposition 3.3.11. *Let I be an ideal of a Leibniz n-algebra L. Then for any $d \in \mathrm{Der}(L)$ we have $\big(d(I)\big)^{[p]} \subseteq I + d^p(I^{[p]})$ for all $p \in \mathbb{N}$.*

Proof. For $p=1$ we have $d(I) \subseteq I + d(I)$.

For $p=2$ we have

$$(d(I))^{[2]} = [d(I), d(I), L, \dots, L] \subseteq I + d[I, I, L, \dots, L] = I + d(I^{[2]}).$$

Suppose for $n=k$ we have $(d(I))^{[k]} \subseteq J + d^k(I^{[k]})$. Then

$$(d(I))^{[k+1]} = [(d(I))^{[k]}, d(I), L, \dots, L] \subseteq [I + d^k(I^{[k]}), d(I), L, \dots, L]$$

$$\subseteq I + d^{k+1}([I^{[k]}, I, L, \dots, L]) = I + d^{k+1}(I^{[k+1]}).$$

Hence, $(d(I))^{[p]} \subseteq J + d^p(I^{[p]})$ for all $p \in \mathbb{N}$. □

Theorem 3.3.12. *Let J be a K_1-nilradical of a Leibniz n-algebra L. Then for any $d \in \mathrm{Der}(L)$ we have $d(J) \subseteq J$.*

Proof. Since J is a K_1-nilradical, for some $k \in \mathbb{N}$ we have $J^{[k]} = 0$.

By Proposition 3.3.11 for any derivation d we have

$$(d(J))^{[k]} \subseteq J + d^k(J^{[k]}) = J.$$

Then $(J + d(J))^{[k]} \subseteq J$ and therefore $(J + d(J))^{[2k]} \subseteq J^{[k]} = 0$. So, $J + d(J)$ is K_1-nilpotent ideal.

Since J is a K_1-nilradical, we have $J + d(J) = J$ and therefore $d(J) \subseteq J$. □

3.3.3 Invariance of Nilradicals

Using similar argumentations as in previous subsections, we define nilradicals and s-nilradicals of Leibniz n-algebras and prove that under a derivation they are invariant.

Proposition 3.3.13. *Let I and J be s-nilpotent (nilpotent) ideals of Leibniz n-algebra L. Then $I+J$ is also s-nilpotent (respectively, nilpotent) ideal of L.*

Proof. We have $(I+J)^{<1>_s} = I+J$ and

$$(I+J)^{<2>_s} = [\underbrace{L,\dots,L}_{s-1}, I+J, \underbrace{L,\dots,L}_{n-s}]$$
$$\subseteq [\underbrace{L,\dots,L}_{s-1}, I, \underbrace{L,\dots,L}_{n-s}] + [\underbrace{L,\dots,L}_{s-1}, J, \underbrace{L,\dots,L}_{n-s}] = I^{<2>_s} + J^{<2>_s}.$$

Now if for some $k>1$ we have $(I+J)^{<k>_s} \subseteq I^{<k>_s} + J^{<k>_s}$, then

$$(I+J)^{<k+1>_s} = [\underbrace{L,\dots,L}_{s-1}, (I+J)^{<k>_s}, \underbrace{L,\dots,L}_{n-s}] \subseteq$$
$$[\underbrace{L,\dots,L}_{s-1}, I^{<k>_s} + J^{<k>_s}, \underbrace{L,\dots,L}_{n-s}] \subseteq [\underbrace{L,\dots,L}_{s-1}, I^{<k>_s}, \underbrace{L,\dots,L}_{n-s}]$$
$$+ [\underbrace{L,\dots,L}_{s-1}, J^{<k>_s}, \underbrace{L,\dots,L}_{n-s}] = I^{<k+1>_s} + J^{<k+1>_s}.$$

Thus, $(I+J)^{<m>_s} = \{0\}$ for any $m \in \mathbb{N}$ greater than the maximum of the nilpotency indexes of I and J.

One can verify the statement of the proposition for the case of nilpotent ideals using the similar argumentation. □

Now let N be a maximal s-nilpotent (nilpotent) ideal in a finite dimensional Leibniz n-algebra L and let M be an arbitrary s-nilpotent ideal of L. Then $N+M$ is also s-nilpotent (nilpotent, respectively) and $N+M \supseteq N$. Since N is maximal s-nilpotent (nilpotent, respectively) ideal, we obtain $N+M=N$. Therefore we can define the maximal s-nilpotent (nilpotent, respectively) ideal as the sum of all the s-nilpotent (nilpotent, respectively) ideals in L and call it the *s-nilradical* (*nilradical*, respectively).

Proposition 3.3.14. *Let J be an s-nilradical of a Leibniz n-algebra L over a field $\mathbb{K}$ of characteristic zero. Then*

$$\big(d(J)\big)^{<m>_s} \subseteq J + d(J^{<m>_s}).$$

Proof. We will establish $(d(J))^{<m>_s} \subseteq J + d(J^{<m>_s})$ by induction on m.

Indeed, for $m = 1$ the inclusion holds.

Let $(d(J))^{<m>_s} \subseteq J + d(J^{<m>_s})$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned}
(d(J))^{<m+1>_s} &= [\underbrace{L, \dots, L}_{s-1}, (d(J))^{<m>_s}, \underbrace{L, \dots, L}_{n-s}] \\
&\subseteq [\underbrace{L, \dots, L}_{s-1}, J, \underbrace{L, \dots, L}_{n-s}] + [\underbrace{L, \dots, L}_{s-1}, d(J^{<m>_s}), \underbrace{L, \dots, L}_{n-s}] \\
&\subseteq J + d([\underbrace{L, \dots, L}_{s-1}, J^{<m>_s}, \underbrace{L, \dots, L}_{n-s}]) \subseteq J + d(J^{<m+1>_s}).
\end{aligned}$$

Hence, the proposition is proved. □

For nilradical we present the similar statement.

Proposition 3.3.15. *Let J be a nilradical of a Leibniz n-algebra L over a field $\mathbb{K}$ of characteristic zero. Then*

$$\big(J + d(J)\big)^m \subseteq J + \big(d(J)\big)^m.$$

Proof. The statement of proposition is evident if $m = 1$.

If for some $m \geq 1$ we have $(J + d(J))^m \subseteq J + d(J^m)$ then

$$\begin{aligned}
(J + d(J))^{m+1} &= \sum_{k=1}^{n} [\underbrace{L, \dots, L}_{k-1}, (J + d(J))^m, \underbrace{L, \dots, L}_{n-k}] \\
&\subseteq \sum_{k=1}^{n} [\underbrace{L, \dots, L}_{k-1}, J + d(J^m), \underbrace{L, \dots, L}_{n-k}] \\
&\subseteq \sum_{k=1}^{n} [\underbrace{L, \dots, L}_{k-1}, J, \underbrace{L, \dots, L}_{n-k}] + \sum_{k=1}^{n} [\underbrace{L, \dots, L}_{k-1}, d(J^m), \underbrace{L, \dots, L}_{n-k}].
\end{aligned}$$

Thus, we have

$$\begin{aligned}(J+d(J))^{m+1} &\subseteq I + d\Big(\sum_{k=1}^{n}[\underbrace{L,\ldots,L}_{k-1}, J^m, \underbrace{L,\ldots,L}_{n-k}]\Big)\\ &\subseteq J + d(J^{m+1}).\end{aligned}$$

Hence, $(J+d(J))^m \subseteq J + d(J^m)$ holds for all $m \in \mathbb{N}$. □

Similarly to Theorem 3.3.8 we establish that s-nilradical (nilradical) of Leibniz n-algebra is invariant under a derivation.

Theorem 3.3.16. *Let J be an s-nilradical of a Leibniz n-algebra L. Then $d(J) \subseteq J$ for any $d \in \mathrm{Der}(L)$.*

Proof. Since J is $s-$nilpotent, there exists $p \in \mathbb{N}$ such that $J^{<p>_s} = 0$. By formula (3.3.1) we have $(J + d(J))^{<p>_s} \subseteq J + (d(J))^{<p>_s}$. By Proposition 3.3.14 we obtain

$$(J + d(J))^{<p>_s} \subseteq J + (d(J))^{<p>_s} \subseteq J + d(J^{<p>_s}) = J$$

and therefore $(J + d(J))^{<2p>_s} \subseteq J^{<p>_s} = \{0\}$. Thus $J + d(J)$ is an s-nilpotent ideal.

Having in mind that J is the s-nilradical of L, it follows that $J + d(J) = J$. Thus $d(J) \subseteq J$. □

Once again, we have the analogous theorem for nilradical of Leibniz n-algebras.

Theorem 3.3.17. *Let J be a nilradical of a Leibniz n-algebra L. Then $d(J) \subseteq J$ for any $d \in \mathrm{Der}(L)$.*

Proof. Since J is nilpotent, there exists $p \in \mathbb{N}$ such that $J^p = 0$. Then by Proposition 3.3.15 we obtain

$$(J + d(J))^p \subseteq J + (d(J))^p \subseteq J + d(J^p) = J$$

and therefore $(J + d(J))^{2p} \subseteq J^p = \{0\}$ and $J + d(J)$ is a nilpotent ideal.

Since J is nilradical of L, it follows that $J + d(J) = J$. Thus $d(J) \subseteq J$. □

3.4 Frattini Subalgebras of Leibniz n-algebras

We introduce the Frattini subalgebra of a Leibniz n-algebra. We extend some properties and results known in the theory of Leibniz algebras and of n-Lie algebras to our case. Frattini theory have been studied in Lie algebras [39, 8, 49], in n-Lie algebras [7, 51] and in Leibniz algebras [9, 11].

3.4.1 Elementary Properties of Frattini Subalgebras

Here we present elementary properties of Frattini subalgebras and Frattini ideals that are known from the theory of n-Lie algebras [7, 51]. Interestingly, core of the proofs of many results remains the same even if we omit the skew-symmetrical property of the n-ary multiplication.

Definition 3.4.1. *A proper subalgebra M of a Leibniz n-algebra L is called maximal if the only subalgebra properly containing M is L.*

Definition 3.4.2. *The intersection of all maximal subalgebras of a Leibniz n-algebra L is a subalgebra denoted by $F(L)$ and it is called the Frattini subalgebra.*

The maximal ideal of L that is contained in $F(L)$ is called the Frattini ideal and it is denoted by $\phi(L)$.

Below we give the following statements which hold for n-Lie algebras [7] and extend in a similar way to the case of Leibniz n-algebras.

Proposition 3.4.3. *Let L be a Leibniz n-algebra. Then the following statements hold:*

1. *If B is a subalgebra of L such that $B + F(L) = L$, then $B = L$.*

2. *If B is a subalgebra of L such that $B + \phi(L) = L$, then $B = L$.*

Proof. 1. Let $B \neq L$ and $\overline{B}$ be a maximal subalgebra of l containing B. From the definition of Frattini subalgebras we have $F(L) \subseteq \overline{B}$. Then

$$L = B + F(L) \subseteq \overline{B} + F(L) = \overline{B}$$

and we obtain $\overline{B} = L$ which is a contradiction to the maximality of $\overline{B}$. Hence $B = L$.

2. The proof is similar to 1) using the inclusion $\phi(L) \subseteq F(L)$. □

Proposition 3.4.4. *Let L be a Leibniz n-algebra and B an ideal of L. Then there exists a proper subalgebra C of L such that $L = B + C$ if and only if $B \nsubseteq F(L)$.*

Proof. Let $B \nsubseteq F(L)$. Then there exists a maximal subalgebra C such that $B \nsubseteq C$, otherwise all maximal subalgebras contain B and therefore their intersection $F(L)$ contain B which is a contradiction. Since B is an ideal, $B + C$ is a subalgebra of L (generally might not be true for i-ideals). Since C is a maximal subalgebra and $C \subsetneq B + C$ we obtain $B + C = L$.

Conversely, let $B \subseteq F(L)$. Then for any proper subalgebra C, taking the maximal subalgebra $\overline{C}$ containing C we have $B + C \subseteq \overline{C}$. Thus $B + C \neq L$ since $\overline{C}$ is a maximal subalgebra. □

We have immediately the following

Corollary 3.4.5. *Let L be a Leibniz n-algebra and B an ideal of L. Then there exists a proper subalgebra C of L such that $L = B + C$ if and only if $B \nsubseteq \phi(L)$.*

Proof. Let $B \subsetneq \phi(L)$. Then $B \subsetneq F(L)$, or otherwise since B is an ideal it must be $B \subseteq \phi(L)$. Then by Proposition 3.4.4 it follows that there exists a proper subalgebra C of L such that $L = B + C$.

Conversely, let $B \subseteq \phi(L)$. Then $B \subseteq F(L)$ and again by Proposition 3.4.4 we obtain the desired result. □

Proposition 3.4.6. *Let L be a Leibniz n-algebra and C be a subalgebra of L, B an ideal of L such that $B \subseteq F(C)$ $\big(B \subseteq \phi(C)\big)$.*

Then $B \subseteq F(L)$ $\big(B \subseteq \phi(L)$, respectively$\big)$.

Proof. If $C = L$ then it is evident. Let $C \neq L$ and $B \nsubseteq F(L)$. Then by Proposition 3.4.4 there exists a subalgebra M such that $L = B + M$ and since $B \subseteq F(C) \subseteq C$, we have $L = B + M = C + M$. Then

$$C = B + C \cap M \subseteq F(C) + C \cap M \subseteq C$$

and therefore $C \subseteq M$. Consequently,

$$L = B + M \subseteq C + M \subseteq M$$

which is a contradiction with the maximality of M. Hence $B \subseteq F(L)$.

When $B \subseteq \phi(C)$, we have $B \subseteq F(C)$. Therefore, $B \subseteq F(L)$. Since B is an ideal, we finally obtain $B \subseteq \phi(L)$. □

From the above proposition taking $C = F(L)$ and $F(B)$ as B, we obtain the following

Corollary 3.4.7. *Let L be a Leibniz n-algebra and B a subalgebra of L such that $F(B)$ $\big(\phi(B)\big)$ is an ideal of L. Then $F(B) \subseteq F(L)$ $\big(\phi(B) \subseteq \phi(L)$, respectively$\big)$.*

Proposition 3.4.8. *Let L be a Leibniz n-algebra and B an ideal of L. Then the following statements hold:*

1. *$(F(L)+B)/B \subseteq F(L/B)$, $\big((\phi(L)+B)/B \subseteq \phi(L/B)\big)$;*
2. *If $B \subseteq F(L)$ then $F(L)/B = F(L/B)$, $\phi(L)/B = \phi(L/B)$;*
3. *If $F(L/B) = 0$ $(\phi(L/B) = 0)$, then $F(L) \subseteq B$ $(\phi(L) \subseteq B)$.*

Proof. 1. Let $\pi \colon L \to L/B$ be a natural homomorphism. Then $\pi^{-1}(F(L/B)) = T$ is a subalgebra of L and $F(L/B) = T/B$. So T is the intersection of maximal subalgebras containing B. Thus T contains $F(L)$. Then

$$(F(L)+B)/B \subseteq T/B = F(L/B).$$

Similarly one can prove the results for $\phi(L)$.

2. Since $B \subseteq L$ every maximal subalgebra contains B. But $F(L/B)$ is an image of intersection of maximal subalgebras containing B and in this case all maximal subalgebras. Thus, $F(L)/B = F(L/B)$. Since B is an ideal, $\phi(B)$ is also contained in B. Analogously, we obtain $\phi(L)/B = \phi(L/B)$.

3. Follows from 2. □

The following theorem establishes behavior of Frattini subalgebras and ideals of a decomposed Leibniz n-algebra.

Theorem 3.4.9. *If a Leibniz n-algebra L has a decomposition*

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_m,$$

where L_i $(1 \le i \le m)$ are ideals of L, then

1. *$F(L) \subseteq F(L_1) + \cdots + F(L_m)$;*
2. *$\phi(L) = \phi(L_1) + \cdots + \phi(L_m)$.*

Proof. 1. Let $\Omega(L_i)$ denote the set of maximal subalgebras of L_i. By definition we have $F(L_i) = \bigcap\limits_{M \subseteq \Omega(L_i)} M$ for all $i = 1, \dots, m$. Then for all $M_i \subseteq \Omega(L_i)$,

$$L_1 \oplus \cdots \oplus L_{i-1} \oplus M_i \oplus L_{i+1} \oplus \cdots \oplus L_m$$

is a maximal subalgebra of L. Thus

$$\begin{aligned} F(L) \subseteq \bigcap_{i=1}^{m} \bigcap_{M \subseteq \Omega(L_i)} & L_1 \oplus \cdots \oplus L_{i-1} \oplus M_i \oplus L_{i+1} \oplus \cdots \oplus L_m \\ &= \bigcap_{i=1}^{m} L_1 \oplus \cdots \oplus L_{i-1} \oplus F(L_i) \oplus L_{i+1} \oplus \cdots \oplus L_m \\ &= F(L_1) + F(L_2) + \cdots + F(L_m). \end{aligned}$$

2. First we show

$$\phi(L_i) = \phi(L) \cap L_i, \; i = 1, 2, \dots, m.$$

From 1) we have $\phi(L) \cap L_i \subseteq F(L) \cap L_i \subseteq F(L_i)$. Since $\phi(L) \cap L_i$ is an ideal of L_i, then it is a subset of $\phi(L_i)$.

On the other hand, $\phi(L_i)$ is an ideal in L and by Corollary 3.4.7 it follows that $\phi(L_i) \subseteq \phi(L)$. Then $\phi(L_i) \subseteq \phi(L) \cap L_i$.

Hence $\phi(L_i) = \phi(L) \cap L_i \subseteq \phi(L)$. Thus we have

$$\phi(L_1) + \phi(L_2) + \cdots + \phi(L_m) \subseteq \phi(L).$$

Now we prove the converse inclusion.

Let $x \in \phi(L)$, $x = x_1 + x_2 + \cdots + x_m$, where $x_i \in F(L_i)$, $1 \leq i \leq m$.

If $x \notin \phi(L_i)$, then using that

$$\begin{aligned} [\underbrace{L_i, \dots, L_i}_{s-1}, x, L_i, \dots, L_i] &= [\underbrace{L_i, \dots, L_i}_{s-1}, x_1 + x_2 + \cdots + x_m, L_i, \dots, L_i] \\ &= [\underbrace{L_i, \dots, L_i}_{s-1}, x_i, L_i, \dots, L_i] \subseteq \phi(L) \cap L_i = \phi(L_i) \end{aligned}$$

we obtain the following inclusion

$$[\underbrace{L_i, \dots, L_i}_{s-1}, \phi(L_i) + \mathbb{K}x_i, L_i, \dots, L_i] \subseteq \phi(L_i) \subseteq \phi(L_i) + \mathbb{K}x_i,$$

that holds for all $1 \leq s \leq n$. This means that $\phi(L_i) + \mathbb{K}x_i$ is an ideal of L_i that is contained in $F(L_i)$ but properly contains $\phi(L_i)$. This contradicts with the definition of $\phi(L_i)$.

Hence, $x_i \in \phi(L_i)$, $i = 1, 2, \ldots, m$ and we obtain

$$\phi(L) \subseteq \phi(L_1) + \phi(L_2) + \cdots + \phi(L_m).$$

Therefore, $\phi(L) = \phi(L_1) + \phi(L_2) + \cdots + \phi(L_m)$ which completes the proof. □

3.4.2 Degenerate Operators with Special Conditions

In order to obtain further results on Frattini subalgebras we need to establish some results concerning right multiplication operators which are either automatically valid for Leibniz and n-Lie algebras, or proved in corresponding works. However, not every property for right multiplication operators known in Lie and n-Lie case is expendable to Leibniz n-algebra case due to existence of non-degenerate right multiplication operators. The following is about degenerate ones.

Proposition 3.4.10. *In a Leibniz n-algebra L any degenerate right multiplication operator $R[a_2, \ldots, a_n]$ is a sum of right multiplication operators with zero weight space with respect to $R[a_2, \ldots, a_n]$.*

Proof. Let $0 = \alpha_0, \alpha_1, \ldots, \alpha_k$ be the eigenvalues of $R[a_2, \ldots, a_n]$. Then L is decomposed into a direct sum

$$L = L_0 \oplus L_{\alpha_1} \oplus \cdots \oplus L_{\alpha_k},$$

where $L_{\alpha_i} = \{x \mid (R[a_2, \ldots, a_n] - \alpha_i I)^m(x) = 0 \text{ for some } m \in \mathbb{N}\}$.

Consider $a_i = a_0^i + a_{\alpha_1}^i + \cdots + a_{\alpha_k}^i$, $a_{\alpha_m}^i \in L_m$, $2 \leq i \leq k$. Then for all $x \in L$, we have

$$\begin{aligned}
&R[a_2, \ldots, a_n](x) \\
&\quad = [x, a_2, \ldots, a_n] = [x, a_0^2 + a_{\alpha_1}^2 + \cdots + a_{\alpha_k}^2, \ldots, a_0^n + a_{\alpha_1}^n + \cdots + a_{\alpha_k}^n] \\
&\qquad = [x, a_0^2, \ldots, a_0^n] + [x, a_{\alpha_1}^2, a_0^3 \ldots, a_0^n] + \cdots + [x, a_{\alpha_k}^2, a_{\alpha_k}^3, \ldots, a_{\alpha_k}^n] \\
&\qquad\quad = R[a_0^2, a_0^3, \ldots, a_0^n](x) + R[a_{\alpha_1}^2, a_0^3 \ldots, a_0^n](x) + \cdots + \\
&\qquad\qquad + R[a_{\alpha_k}^2, a_{\alpha_k}^3, \ldots, a_{\alpha_k}^n](x)\,.
\end{aligned}$$

By Lemma 3.1.14, we obtain that $R[a_2, \ldots, a_n](x) = B(x) + C(x)$, where B is a sum of right multiplication operators with zero weight and C is a sum of right multiplication operators with nonzero weights. Then for any $x \in L_{\alpha_i}$, we have

$$C(x) = (R[a_2, \dots, a_n] - B)(x) = R[a_2, \dots, a_n](x) - B(x) \subseteq L_{\alpha_i} ,$$

which holds only if $C(x) = 0$ since C adds a weight. Therefore, C is a zero operator on L_{α_i}. Since $L = L_0 \oplus L_{\alpha_1} \oplus \cdots \oplus L_{\alpha_k}$, we obtain $C = 0$ on L.

So, $R[a_2, \dots, a_n] = B$, i.e. is a sum of right multiplication operators with zero weight with respect to $R[a_2, \dots, a_n]$. □

In [9] the following result was proved for left Leibniz algebras which is also valid for right Leibniz algebras, i.e. Leibniz 2-algebras.

Lemma 3.4.11 ([9]). *In a Leibniz algebra L for any $a \in L$ there exists $b \in L_{0R[a]}$ such that $L_{0R[b]} = L_{0R[a]}$.*

Concerning this lemma we establish the following result for the case $n \geq 3$.

Corollary 3.4.12. *Let the nonzero eigenvalues $\alpha_1, \dots, \alpha_k$ of the right multiplication operator $R[a_2, \dots, a_n]$ in a Leibniz n-algebra $(n \geq 3)$ satisfy*

$$\mu_1 \alpha_1 + \mu_2 \alpha_2 + \cdots + \mu_k \alpha_k \neq 0,$$

for all non-negative integers $\mu_1, \dots, \mu_k$ such that

$$0 < \mu_1 + \cdots + \mu_k \leq n - 1.$$

Then there exist $b_2, b_3, \dots, b_n \in L_{0R[a_2,\dots,a_n]}$ such that

$$L_{0R[b_2,\dots,b_n]} = L_{0R[a_2,\dots,a_n]}.$$

Proof. From Proposition 3.4.10 we obtain that $R[a_2, \dots, a_n] = B$. From the condition on the eigenvalues we conclude that B consists of just one right multiplication operator, namely $B = R[a_0^2, a_0^3, \dots, a_0^n]$. So, if we take $b_i = a_0^i$, we obtain $L_{0R[b_2,\dots,b_n]} = L_{0R[a_2,\dots,a_n]}$. □

A Leibniz n-algebra satisfying the conditions of Corollary 3.4.12 is given in the following

Example 3.4.13 ([4]). *Consider a Leibniz n-algebra $L = \langle e_1, e_2, \ldots, e_n \rangle$ with the following multiplication:*

$$[e_k, e_1, \ldots, e_1] = e_k \quad (2 \leq k \leq m).$$

The right multiplication operator $R[e_1, \ldots, e_1]$ has only two eigenvalues: 0 and 1. It is easy to see that the conditions of Corollary 3.4.12 are satisfied and $e_1 \in L_{0R[e_1,\ldots,e_1]}$.

Below, we present an example which shows that without the given condition in Corollary 3.4.12 the statement need not to be true.

Example 3.4.14. *Consider an m dimensional Leibniz n-algebra L with $m > n$ and the following multiplication:*

$$\begin{array}{rcl} [e_k, e_1, e_2 \ldots, e_{n-1}] & = & \alpha_k e_k \\ [e_{k+1}, e_1, e_2, \ldots, e_{n-1}] & = & \alpha_{k+1} e_{k+1} \\ & \vdots & \\ [e_m, e_1, e_2, \ldots, e_{n-1}] & = & \alpha_m e_m \end{array}$$

where $\{e_1, \ldots, e_m\}$ is a basis, $\alpha_k \cdot \ldots \cdot \alpha_m \neq 0$, $\sum\limits_{i=k}^{n-1} \alpha_i = 0$ and $1 < k < n-1$.

Then $L_{0R[e_1,\ldots,e_{n-1}]} = \langle e_1, \ldots, e_{k-1} \rangle$ and from the table of multiplication of n-algebra, it follows that any operator of right multiplication to $(n-1)$ elements of subspace $\langle e_1, \ldots, e_{k-1} \rangle$ is identically zero.

Hence, for any $b_2, \ldots, b_n \in L_{0R[e_1,\ldots,e_{n-1}]}$ we have

$$L_{0R[b_2,\ldots,b_n]} \neq L_{0R[e_1,\ldots,e_{n-1}]}.$$

3.4.3 Frattini Subalgebras and Nilpotent Leibniz n-algebras

Definition 3.4.15. *We say that a subalgebra U of a Leibniz n-algebra L is left subnormal if there exists a chain of subalgebras $U = U_k \subseteq \cdots \subseteq U_1 \subseteq U_0 = L$ with each U_{i+1} an r-ideal $(r \neq 1)$ in U_i.*

The following theorem is a generalization of [9, Theorem 3.6] from the case of Leibniz algebras to Leibniz n-algebras.

Theorem 3.4.16. *Let U be a left subnormal subalgebra of Leibniz n-algebra L and V an ideal in U such that $V \subseteq F(L)$. If U/V is 1-nilpotent, then U is 1-nilpotent.*

Proof. Let $u_2, \dots, u_n \in U$ and $U = U_k \subseteq \cdots \subseteq U_0 = L$ is a chain of subalgebras where U_i is an ideal of U_{i-1}. We obtain the following inclusions:

$$R[u_2, \dots, u_n](L) = [L, u_2, \dots, u_n,] \subseteq [L, U_1, \dots, U_1] \subseteq U_1$$

$$R[u_2, \dots, u_n](U_1) = [U_1, u_2, \dots, u_n,] \subseteq [U_1, U_2, \dots, U_2] \subseteq U_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$R[u_2, \dots, u_n](U_{k-1}) = [U_{k-1}, u_2, \dots, u_n,] \subseteq [U_{k-1}, U_k, \dots, U_k] \subseteq U_k.$$

Therefore, $R[u_2, \dots, u_n]^k(L) \subseteq U_k = U$.

Since U/V is 1-nilpotent, for some $m \in \mathbb{N}$ we have $R[u_2, \dots, u_n]^m(U) \subseteq V$. Thus, $R[u_2, \dots, u_n]^{k+m}(L) \subseteq V \subseteq F(L)$.

From Fitting Lemma 3.1.7 we have

$$L \subseteq R[u_2, \dots, u_n]^{k+m}(L) + L_{0R[u_2,\dots,u_n]} \subseteq F(L) + L_{0R[u_2,\dots,u_n]}.$$

Since $L_{0R[u_2,\dots,u_n]}$ is a subalgebra of L, by Proposition 3.4.3 it follows that $L = L_{0R[u_2,\dots,u_n]}$, i.e. $R[u_2, \dots, u_n]$ is nilpotent for any $(u_2, \dots, u_n) \in U^{\times(n-1)}$. Hence, U is 1-nilpotent. □

The following statements hold for n-Lie algebras [7] and are also true for Leibniz n-algebras.

Corollary 3.4.17. *If $I \subseteq F(L)$ is an r-ideal $(r \neq 1)$ of L, then I is 1-nilpotent. Particularly, $\phi(L)$ is a 1-nilpotent ideal of L.*

Proof. Since I is an r-ideal (with $r \neq 1$) in L, ideal I is subnormal. Now $I/I = 0$ is 1-nilpotent and taking $U = V = I$ from Theorem 3.4.16 we obtain that I is 1-nilpotent.

Taking $I = \phi(L)$, it follows the 1-nilpotency of $\phi(L)$. □

Definition 3.4.18. *In a Leibniz n-algebra L the intersection of all maximal ideals of L is called the Jacobson radical and it is denoted by $J(L)$.*

Proposition 3.4.19. *Let L be a finite dimensional Leibniz n-algebra. Then*

1. $F(L) \subseteq [L, L, \dots, L]$;
2. $J(L) \subseteq [L, L, \dots, L]$;
3. *If L is a k-solvable Leibniz n-algebra, then* $J(L) = [L, L, \dots, L]$.

Proof. 1. If $L = [L, L, \ldots, L]$ then the inclusion is clear.

Suppose that $L \neq [L, L, \ldots, L]$ and $F(L) \neq [L, L, \ldots, L]$. Then there exists $x \in F(L)$ such that $x \notin [L, L, \ldots, L]$.

Let $[L, L, \ldots, L] = \langle e_1, \ldots, e_k \rangle$. Complement this basis till the basis of n-algebra $L = \langle x, e_1, \ldots, e_k, e_{k+1}, \ldots, e_m \rangle$. Let $B = \langle e_1, \ldots, e_k, e_{k+1}, \ldots, e_m \rangle$. Then

$$[B, B, \ldots, B] \subseteq [L, L, \ldots, L] \subseteq B$$

and B is a subalgebra. Note that, the dimension of subalgebra B is m, which makes it maximal subalgebra of L. But $x \notin B$ and we obtain $F(L) \nsubseteq B$ which contradicts to the definition of Frattini subalgebra.

Therefore, $F(L) \subseteq [L, L, \ldots, L]$.

2. The proof uses the similar idea as the first part. For the sake of completeness, let us present it below.

If $L = [L, L, \ldots, L]$ then the inclusion is clear.

Suppose that $L \neq [L, L, \ldots, L]$ and $J(L) \neq [L, L, \ldots, L]$. Then there exists $x \in J(L)$ such that $x \notin [L, L, \ldots, L]$.

Let $[L, L, \ldots, L] = \langle e_1, \ldots, e_k \rangle$. Complement this basis till the basis of n-algebra $L = \langle x, e_1, \ldots, e_k, e_{k+1}, \ldots, e_m \rangle$. Let $B = \langle e_1, \ldots, e_k, e_{k+1}, \ldots, e_m \rangle$. Then

$$[\underbrace{L, \ldots, L}_{i-1}, B, L, \ldots, L] \subseteq [L, L, \ldots, L] \subseteq B$$

for all $1 \leq i \leq n$. Hence B is an ideal. Note that, the dimension of ideal B is m, which makes it maximal ideal of L. But $x \notin B$ and we obtain $J(L) \nsubseteq B$ which contradicts to the definition of Jacobson radical.

Therefore, $J(L) \subseteq [L, L, \ldots, L]$.

3. Due to part 2 all we need to show is the inclusion $[L, L, \ldots, L] \subseteq J(L)$.

Let I be a maximal ideal of L. Then L/I does not contain proper ideals. Thus $[L/I, L/I, \ldots, L/I]$ is either 0 or L/I. Since L is k-solvable, so is L/I which implies $[L/I, L/I, \ldots, L/I] \neq L/I$. Therefore, $[L/I, L/I, \ldots, L/I] = 0$ and $[L, L, \ldots, L] \subseteq I$.

Now since $J(L)$ is an intersection of all maximal ideals, we obtain

$$[L, L, \ldots, L] \subseteq J(L).$$

Thus, $J(L) = [L, L, \ldots, L]$. □

The following theorem is a generalization of corresponding theorem from the theory of groups proved in W.R. Scott's group theory book [48], from the theory of Lie algebras [21] and from the theory of n-Lie algebras [7], [8] and [51].

Theorem 3.4.20. *Let L be a finite dimensional nilpotent Leibniz n-algebra. Then the following statements hold:*

1. *Any maximal subalgebra M of L is an ideal of L;*
2. *$F(L) = \phi(L) = J(L) = [L, L, \dots, L]$.*

Proof. 1. Consider an arbitrary maximal subalgebra M of L. Since L is nilpotent there exists $k \in \mathbb{N}$ such that

$$L^k + M \supsetneq M = L^{k+1} + M.$$

Then

$$\begin{aligned}[][L^k + M, L^k + M, \dots, L^k + M] = [L^k, L^k, \dots, L^k] + [L^k, L^k, \dots, M] + \cdots + \\ + [M, L^k, \dots, L^k] + \cdots + [L^k, M, \dots, M] + [M, M, \dots, M] \subseteq \\ [L^k, L, \dots, L] + [L, L^k, L, \dots, L] + \cdots + [L, L, \dots, L^k] + [M, M, \dots, M] \\ \subseteq L^{k+1} + M = M \subseteq L^k + M.\end{aligned}$$

Therefore, $L^k + M$ is a subalgebra of L and $M \subsetneq L^k + M$. Since M is a maximal subalgebra of L it follows that $L^k + M = L$.

Then the chain of inclusions above implies that $[L, L, \dots, L] \subseteq M$ for any maximal subalgebra M. Therefore,

$$[\underbrace{L, \dots, L}_{i-1}, M, L, \dots, L] \subseteq [L, L, \dots, L] \subseteq M$$

for all $1 \leq i \leq n$. Hence, M is an ideal of L.

2. Since $[L, L, \dots, L] \subseteq M$ for any maximal subalgebra M, we have $[L, L, \dots, L] \subseteq F(L)$. From Proposition 3.4.19 we have $F(L) \subseteq [L, L, \dots, L]$. Therefore, $F(L) = [L, L, \dots, L]$. Since the last one is an ideal in L, we have $F(L) = \phi(L)$.

Also by the same proposition and the fact that nilpotency implies k-solvability, we obtain $J(L) = [L, L, \dots, L]$. Thus,

$$F(L) = J(L) = \phi(L) = [L, L, \dots, L]$$

and the assertion of the theorem is proved. □

The following lemma is an extension of [9, Lemma 3.2] under the condition $a_2, \dots, a_n \in L_{0R[a_2,\dots,a_n]}$.

Lemma 3.4.21. *Let L be a Leibniz n-algebra and $R[a_2, \dots, a_n] : L \to L$ a right multiplication operator such that $a_2, \dots, a_n \in L_{0R[a_2,\dots,a_n]}$. Then for any subalgebra U containing $L_{0R[a_2,\dots,a_n]}$ the equality $N(U) = U$ holds.*

Proof. Let $z \in N_1(U)$. Then $[z, U, \dots, U] \subseteq U$. Denote $L_0 = L_{0R[a_2,\dots,a_n]}$. Then

$$R[a_2, \dots, a_n](z) = [z, a_2, \dots, a_n] \in [z, L_0, \dots, L_0] \subseteq [z, U, \dots, U] \subseteq U.$$

Hence $R[a_2, \dots, a_n](z) \in U$.

Notice that

$$R[a_2, \dots, a_n](U) = [U, a_2, \dots, a_n] \subseteq [U, L_0, \dots, L_0] \subseteq [U, U, \dots, U] \subseteq U$$

since U is a subalgebra. Therefore, the conditions of Lemma 3.1.13 are satisfied. Thus $z - z_0 \in U$, which implies $z \in U$. So we have proved $N_1(U) = U$.

Since U is a subalgebra, $N_s(U) \supseteq U$ for all $2 \leq s \leq n$.

Then $N(U) = N_1 \bigcap \big(\cap_{s=2}^{n} N_s(U) \big) = U$. □

In [52, Theorem 2.2] it was proved that nilpotency of the finite dimensional n-Lie algebras is equivalent to the statement that every maximal subalgebra is an ideal, which in turns is equivalent to $F(L) = [L, L, \dots, L]$. For Leibniz n-algebras Theorem 3.4.20 verifies the statement in one direction. The other direction of the statement in our case is not true in general. However we establish similar results under some conditions to right multiplication operators.

Proposition 3.4.22. *Let $a_2, \dots, a_n$ be elements of a Leibniz n-algebra L such that $a_2, \dots, a_n \in L_{0R[a_2,\dots,a_n]}$. Let every maximal subalgebra be an i- and j-ideal for some $1 \leq i \neq j \leq n$ in L. Then $R[a_2, \dots, a_n]$ is nilpotent.*

Proof. Assume that $L_{0R[a_2,\dots,a_n]} \neq L$. Then there exists maximal algebra M such that $L_{0R[a_2,\dots,a_n]} \subseteq M$. Then by Lemma 3.4.21 we have $N(M) = M$.

Since M is an i- and a j-ideal ($i \neq j$), we have

$$[\underbrace{M, \dots M}_{s-1}, L, M, \dots, M] \subseteq M$$

for all $1 \leq s \leq n$. Thus, $L = N_s(M)$ for all $1 \leq s \leq n$ and $L = N(M)$ which is a contradiction.

Thus $L = L_{0R[a_2,\dots,a_n]}$ and $R[a_2, \dots, a_n]$ is a nilpotent operator. □

Corollary 3.4.23. *Let for any $a_2, \dots, a_n$ elements of a Leibniz n-algebra L we have $a_2, \dots, a_n \in L_{0R[a_2,\dots,a_n]}$ and let every maximal subalgebra be an i- and j-ideal for some $1 \leq i \neq j \leq n$ in L. Then L is 1-nilpotent Leibniz n-algebra.*

Proof. By Proposition 3.4.22 we have that $R[a_2, \dots, a_n]$ is nilpotent for any $a_2, \dots, a_n \in L$. Thus, by Engel's theorem 3.1.6 we obtain 1-nilpotency of Leibniz n-algebra L. □

The following proposition establishes similar result under some other conditions to maximal subalgebras and operators of right multiplication.

Proposition 3.4.24. *Let L be a finite dimensional Leibniz n-algebra with condition that for an arbitrary $(a_2, \dots, a_n) \in L^{\times(n-1)}$ and for some $2 \leq i \leq n$ we have $a_i \in L_{0R[a_2,\dots,a_n]}$. Let any maximal subalgebra M of L be an ideal of L. Then L is 1-nilpotent.*

Proof. Assume that L is not 1-nilpotent. Then there exists a non-nilpotent right multiplication operator $R[a_2, \dots, a_n]$. Since $R[a_2, \dots, a_n]$ is non-nilpotent, the Fitting null-component $L_{0R[a_2,\dots,a_n]} \neq L$.

Let M be a maximal subalgebra of L containing $L_{0R[a_2,\dots,a_n]}$. Then $a_i \in L_{0R[a_2,\dots,a_n]} \subseteq M$ for some $2 \leq i \leq n$ by assumption of the proposition. Since M is a maximal subalgebra, it is also an ideal of L. Then $R[a_2, \dots, a_n](L) \subseteq M$.

Since $R[a_2, \dots, a_n]$ is non-singular on $L_{1R[a_2,\dots,a_n]}$, we obtain that $L_1 = R[L_1] = L_1 \cap M$. Hence $L_1 \subseteq M$.

Then $L = L_0 \oplus L_1 \subseteq M \neq L$. This is a contradiction. Hence, all of the right multiplication operators are nilpotent. Therefore, by Engel's theorem 3.1.6 L is 1-nilpotent. □

3.5 Cartan Subalgebras of Leibniz n-algebras

In this section we consider Cartan subalgebras of Leibniz n-algebras. Most of the results concerning Cartan subalgebras of Leibniz n-algebras and their relation with Cartan subalgebras of the corresponding n-Lie algebras were studied in [4]. In this section we establish some new results on Cartan subalgebras of Leibniz n-algebras under additional conditions for the determining elements of right multiplication operators, as in the previous sections. Moreover, we give a detailed construction of Leibniz n-algebras in different dimensions with surprisingly non-conjugated Cartan subalgebras.

3.5.1 Basic properties and relation with n-Lie algebra

Definition 3.5.1 ([4]). *A subalgebra C of a Leibniz n-algebra L is said to be Cartan subalgebra if*

i) C is 1-nilpotent;

ii) $C = N_1(C)$.

The importance of considering 1-normalizer in the definition of Cartan subalgebras was shown in [42].

The following example shows the existence of such subalgebras.

Example 3.5.2. *Consider the algebra $L = \langle e_1, e_2, \dots, e_m \rangle$ with the following multiplication:*

$$[e_k, e_1, e_1, \dots, e_1] = e_k \quad (2 \leq k \leq m).$$

It easy to see that L is neither a nilpotent Leibniz n-algebra and nor an n-Lie algebra.

Consider the subspace $H = \langle e_1 \rangle$. It is clear that H is a nilpotent subalgebra. Put

$$a = \alpha e_1 + \sum_{k=2}^{m} \beta_k e_k \in N_1(H),$$

then $H \ni [a, e_1, \dots, e_1] = \sum\limits_{k=2}^{m} \beta_k e_k$ and therefore $\beta_k = 0$ for all $2 \leq k \leq m$. Thus $H = N_1(H)$ and H is a Cartan subalgebra in L.

Lemma 3.5.3. *Let L be the Leibniz n-algebra ($n \geq 3$) constructed from a Leibniz algebra as in Example 3.1.2 and let H be a Cartan subalgebra of the Leibniz algebra L. Then in the Leibniz n-algebra we have:*

a) H is a nilpotent subalgebra in Leibniz n-algebra L;

b) $N_1(H) = L$.

Proof. The nilpotency of the subalgebra H in the Leibniz n-algebra follows from its nilpotency in the Leibniz algebra L. From [43] it is known that under the natural homomorphism of a Leibniz algebra onto the corresponding factor algebra which is a Lie algebra, the image of the Cartan subalgebra H is a Cartan subalgebra of the Lie algebra. Further using abelianness of Cartan subalgebras in Lie algebras [29] we obtain that $[H, H]$ is contained in the ideal generated by the squares of elements from the Leibniz algebra L, and this ideal is contained in the right annihilator. Therefore for $n \geq 3$ we have $N_1(H) = \{x \in L | [x, H, \ldots, H] \subseteq H\} = L$. □

For Cartan subalgebras of n-Leibniz algebras similar to the case of n-Lie algebras and Leibniz algebras, there is a characterization in terms of the Fitting's null-component, namely, the following proposition is true.

Proposition 3.5.4. *Let H be a nilpotent subalgebra of a Leibniz n-algebra L. Then H is a Cartan subalgebra if and only if it coincides with L_0 in the Fitting decomposition of the algebra L with respect to $R[H]$.*

Proof. Let $x \in N_1(H)$, then $[x, h_2, \ldots, h_n] \in H$ for all $h_i \in H$ $(2 \leq i \leq n)$. Since H is nilpotent there exists $k \in \mathbb{N}$ such that $R^k(h_2, \ldots, h_n)(x) = 0$, i.e. $x \in L_0$.

Therefore we have $N_1(H) \subseteq L_0$. Since $H \subseteq N_1(H)$ we obtain that $H \subseteq L_0$.

Suppose that $H \subsetneqq L_0$.

Taking $R[H]$ instead of G in Theorem 3.1.9 we obtain that L_0 is invariant with respect to $R[H]$ and $R(h_2, \ldots, h_n)|_{L_0}$ is a nilpotent operator for all $h_i \in H$ $(2 \leq i \leq n)$.

Therefore we have

$$R[H] : L_0 \to L_0,$$

$$R[H] : H \to H,$$

where $R[H]$ is a Lie algebra.

Thus we obtain the induced Lie algebra $\overline{R[H]} : L_0/H \to L_0/H$, where L_0/H is a non-zero linear quotient space. If we consider $R[H] : L_0 \to L_0$, then as it was mentioned above the operator $R(h_2, \ldots, h_n)$ is nilpotent for all $h_i \in H$ $(2 \leq i \leq n)$. Then by Engel's theorem [29] it follows that there exists a non-zero element $\overline{x} = x + H$ $(x \notin H)$ such that $\overline{R[H]}(x + H) = \overline{0}$. This means that $[x, h_2, \ldots, h_n] \in H$ for every $h_i \in H$ $(2 \leq i \leq n)$. Therefore there exists an element $x \in N_1(H)$ such that $x \notin H$ – the contradiction shows that $H = L_0$. The proof is complete. □

Corollary 3.5.5. *Let H be a Cartan subalgebra of the Leibniz n-algebra L. Then H is a maximal nilpotent subalgebra of L.*

Proof. Let B be a nilpotent subalgebra of the L such that $H \subseteq B$ then by Proposition 3.5.4 we have $H \subseteq B \subseteq L_0(H) = H$. □

The following theorem establishes properties of the Fitting's null-component of the regular element of an n-Leibniz algebra.

Theorem 3.5.6. *Let L be a Leibniz n-algebra over an infinite field and let x be a regular element for L. Then the Fitting null-component $H = L_0$ with respect to the operator $R[x]$ is a 1-nilpotent subalgebra in L.*

Proof. Let us prove that both Fitting components with respect to $R[x]$ are invariant under $R[H]$. Indeed, let $a = (a_2, \dots, a_n) \in H^{\times(n-1)}$. Then from defining identity (3.1.1) it easily follows that

$$[[[R[a], \underbrace{R[x]], R[x]], \dots, R[x]]}_{m-\text{times}} = (-1)^m \sum_{i_2+\dots+i_n=m} R\left(R[x]^{i_2}(a_2), \dots, R[x]^{i_n}(a_n)\right).$$

For sufficiently large m we obtain that

$$[[[R[a], \underbrace{R[x]], R[x]], \dots, R[x]]}_{m-\text{times}} = 0.$$

From [29, Lemma 1, Chapter II, §4] we have that the Fitting components L_0 and L_1 with respect to $R[x]$ are invariant under $R[a]$.

Let us prove that the operator $R(h_2, \dots, h_n)|_{L_0}$ is nilpotent for $h_i \in H$ $(2 \leq i \leq n)$. Assume the opposite, i.e. there exists $h = (h_2, \dots, h_n) \in H^{\times(n-1)}$ such that $R(h_2, \dots, h_n)|_{L_0}$ is not nilpotent.

Consider $u^t = (u_2^t, \dots, u_n^t)$, where $u_i^t = tx_i + (1-t)h_i$ and t belongs to the underlying field. Then the elements of the matrices $R(u_2^t, \dots, u_n^t)|_{L_0}$ and $R(u_2^t, \dots, u_n^t)|_{L_1}$ are polynomials in t. Since for $t = 1$ we have $R[x]|_{L_1} = R(u^1)|_{L_1}$ and $R(u^1)|_{L_1}$ is non-degenerate, there exists t_0 such that $R(u^{t_0})|_{L_1}$ is non-degenerate and $R(u^{t_0})|_{L_0}$ is not nilpotent.

In this case the dimension of the Fitting null-component of the space L with respect to the operator $R(u^{t_0})$ is less than the dimension of the Fitting null-component with respect to the operator $R[a]$, which contradicts the regularity of the element x.

Therefore, $R[h]|_{L_0}$ is nilpotent for every $h \in H^{\times(n-1)}$ and by Theorem 3.1.6 H is 1-nilpotent. The proof is complete. □

Let us recall that the Fitting null-component with respect to the right multiplication operator by a regular element in n-Lie algebras [31] and Leibniz algebras [42] is a Cartan subalgebra. But in the case of Leibniz n-algebras the Example 3.1.8 shows that the Fitting null-component with respect to the operator of right multiplication by the regular element $e = (e_1, e_2, \dots, e_{n-1})$ is not a Cartan subalgebra, because $V_{0R[x]} = \{0\}$ and $N_1(\{0\}) = L$. Now we establish this result under same restrictions as in the previous sections: requiring the components of the right multiplication operator defined by a regular element to be in a Fitting's null-component of L defined by this operator.

Proposition 3.5.7. *Let L be a Leibniz n-algebra over an infinite field and let $x = (x_2, \dots, x_n) \in L^{\times(n-1)}$ be a regular element of L such that*

$$x_2, \dots, x_n \in L_{0R[x]}.$$

Then the Fitting null-component of L with respect to operator $R[x]$ is a Cartan subalgebra of L.

Proof. Due to the previous theorem, we need to prove $N_1(L_0) = L_0$.

Let $y \in N_1(L_0)$. Then $R[x](y) = [y, x_2, \dots, x_n] \in [y, L_0, \dots, L_0] \subseteq L_0$. Applying Lemma 3.1.13 we obtain $y - y_0 \in L_0$ and thus $y \in L_0$. Therefore, $N_1(L_0) \subseteq L_0$ and since L_0 is a subalgebra $N_1(L_0) \supseteq L_0$. Hence, $L_0 = N_1(L_0)$ and L_0 is a Cartan subalgebra of L. □

Proposition 3.5.8. *Let L be a Leibniz n-algebra over a field F and let Ω be an arbitrary extension of the field F. Put $L_\Omega = L_F \otimes \Omega$. Then H is a Cartan subalgebra in L if and only if $H_\Omega = H_F \otimes \Omega$ is a Cartan subalgebra in L_Ω.*

Proof. Let H be a Cartan subalgebra in L. Then H_Ω is a subalgebra in L_Ω.

Since $H^{<k>_1} = 0$, from the evident equality $H_\Omega^{<k>_1} = H_F^{<k>_1} \otimes \Omega$ it follows that H_Ω is a nilpotent subalgebra.

Consider $a_\Omega \in N_1(H_\Omega)$. Then $[a_\Omega, h_{2\Omega}, \dots, h_{n\Omega}] = [a, h_2, \dots, h_n] \otimes \alpha\gamma_2 \dots \gamma_n$, where $a_\Omega = a \otimes \alpha$, $h_{i\Omega} = h_i \otimes \gamma_i$ $(2 \le i \le n)$. Therefore, $N_1(H_\Omega) \subseteq N_1(H) \otimes \Omega$. But H is a Cartan subalgebra and thus $N_1(H) = H$ and $N_1(H_\Omega) \subseteq H_F \otimes \Omega = H_\Omega$.

Therefore, H_Ω is a Cartan subalgebra in L_Ω.

Conversely, suppose that H_Ω is a Cartan subalgebra in L_Ω. Then from $H_F^{<k>_1} \otimes \Omega = H_\Omega^{<k>_1} = 0$ it follows that $H_F^{<k>_1} = 0$.

Consider $a \in N_1(H_F)$. We have $[a, h_2, \dots, h_n] \in H_F$ for all $h_i \in H$ $(2 \le i \le n)$. Since $[a_\Omega, h_{2\Omega}, \dots, h_{n\Omega}] = [a, h_2, \dots, h_n] \otimes \alpha\gamma_2 \dots \gamma_n \in H_F \otimes \Omega$, where $a_\Omega = a \otimes \alpha$, $h_{i\Omega} = h_i \otimes \gamma_i$ $(2 \le i \le n)$, one has $a \otimes \alpha = a_\Omega \in N_1(H_\Omega) = H_\Omega = H_F \otimes \Omega$. Therefore, $a \in H_F$ and H_F is a Cartan subalgebra in L_F. The proof is complete. □

Theorem 3.5.9. *Let $\varphi : L \to L'$ be an epimorphism of Leibniz n-algebras and suppose that H is a Cartan subalgebra in L and $\varphi(H) = H'$. Then H' is a Cartan subalgebra in L'.*

Proof. In view of Proposition 3.5.8 we may assume that the field F is algebraically closed.

Consider the decomposition into the sum of characteristic subspaces:

$$L = L_\alpha \oplus L_\beta \oplus \cdots \oplus L_\gamma$$

with respect to the nilpotent Lie algebra $R[H]$ of linear transformations of the vector space L, where

$$L_\alpha = \left\{ x \in L \mid (R[h] - \alpha(h)id)^k\,(x) = 0 \text{ for some } k \text{ and for any } h \in H^{\times(n-1)} \right\}.$$

Then

$$\varphi(L) = \varphi(L_\alpha) + \varphi(L_\beta) + \cdots + \varphi(L_\gamma).$$

By using the properties of homomorphisms we obtain by induction that

$$\varphi \circ R(x_2, \ldots, x_n)^k = R(\varphi(x_2), \ldots, \varphi(x_n))^k \circ \varphi$$

for every $k \in \mathbb{N}$.

Further we have

$$\begin{gathered} \varphi \circ (R[h] - \alpha id)^k = \quad \varphi \circ \sum_{i=0}^{k} C_k^i \alpha^{n-k} R[h]^k = \\ \sum_{i=0}^{k} C_k^i \alpha^{n-k} \varphi \circ R[h]^k = \quad \sum_{i=0}^{k} C_k^i \alpha^{n-k} R(\varphi(h))^k \circ \varphi = \\ (R(\varphi(h)) - \alpha id)^k \circ \varphi, \end{gathered}$$

where C_k^i are binomial coefficients.

Therefore from

$$(R[h] - \alpha(h)id)^k(x) = 0$$

we obtain

$$(R(h') - \alpha(h')id)^k \varphi(x) = 0,$$

where $h' = \varphi(h)$.

Thus, if $x \in L_\alpha$, then $x' \in L'_\alpha$ (where $\varphi(L_\alpha) = L'_\alpha$). Since φ is epimorphic, we have the following decomposition of the space L' with respect to $R(H')$:

$$L' = L'_\alpha \oplus L'_\beta \oplus \cdots \oplus L'_\gamma,$$

where $\varphi(L_\sigma) = L'_\sigma$ and $\sigma \in \{\alpha, \beta, \ldots, \gamma\}$.

If $\alpha \neq 0$, the action of H' on L'_α is non-degenerate and therefore $L'_0 = \varphi(L_0) = \varphi(H) = H'$. Now Proposition 3.5.4 implies that H' is a Cartan subalgebra of L'. The proof is complete. □

For a Leibniz n-algebra L consider the natural homomorphism φ onto the factor algebra $\overline{L} = L/I$. It is clear that $\overline{L}$ is an n-Lie algebra.

Corollary 3.5.10. *Let $b \in L^{\times(n-1)}$. Consider the decomposition of the element $x = x_0 + x_\alpha + x_\beta + \cdots + x_\gamma$ with respect to $R[b]$, where $x_\sigma \in L_\sigma$, $\sigma \in \{0, \alpha, \beta, \ldots, \gamma\}$. If there exists $k \in \mathbb{N}$ such that $R[b]^k(x) \in I$, then $\overline{x} = \overline{x}_0$.*

Proof. Let $R[b]^k(x) \in I$ and $R[b]^{k-1}(x) \notin I$.

Setting $Q := R[b]^k$, we obtain $Q(x) \in I$. On the other hand $Q(I) \subseteq I$ since I is an ideal in L. Proposition 3.1.13 implies that $x - x_0 \in I$, i.e. $\overline{x} = \overline{x}_0$. The proof is complete. □

Remark 3.5.11. *For the Cartan subalgebra H of the Leibniz n-algebra L, we consider the Lie algebra $R[H]$ of linear transformations L (which evidently is nilpotent) and the decomposition of L with respect to $R[H]$. Remark 3.1.10 implies the existence of an element $R[b] \in R[H]$ such that the Fitting's null-component with respect to the nilpotent Lie algebra of linear transformations $R[H]$ coincides with the Fitting's null component with respect to the transformation $R[b]$, i.e. $L_0 = L_0(b)$. Using Proposition 3.5.4 we obtain $H = L_0(b)$.*

Lemma 3.5.12. *Let $b \in H^{\times(n-1)}$ and $H = L_0(b)$. Then $\overline{H} = L_{\overline{0}}(\overline{b})$.*

Proof. Let $\overline{H}$ be the image of the Cartan subalgebra H under the homomorphism $\varphi : L \to L/I$. From the theory of n-Lie algebras [31] we know that there exists a regular element $\overline{a} = (\overline{a}_2, \overline{a}_3, \ldots, \overline{a}_n) \in \overline{H}^{\times(n-1)}$ such that $\overline{H} = L_{\overline{0}}(\overline{a})$.

Without loss of generality we may assume that $a = (a_2, a_3, \ldots, a_n) \in H^{\times(n-1)}$. It is clear that $L_0(b) \subseteq L_0(a)$. Since $\overline{a}$ is a regular element we have that $\overline{H} \subseteq L_{\overline{0}}(\overline{b})$.

If there exists i such that $a_i \in I$, then $\overline{L} = L_{\overline{0}}(\overline{a}) \subseteq L_{\overline{0}}(\overline{b})$ and therefore $L_{\overline{0}}(\overline{a}) = L_{\overline{0}}(\overline{b})$ and $\overline{H} = L_{\overline{0}}(\overline{b})$.

Suppose that for any i we have $a_i \notin I$ and $\overline{H} \subsetneqq L_{\overline{0}}(\bar{b})$. Then there exists x such that $\overline{x} = x + I \in L_{\overline{0}}(\bar{b}) \setminus L_{\overline{0}}(\overline{a})$. Therefore for the element x we have $R[b]^k(x) \in I$ for some k and $R[a]^s(x) \notin I$ for any $s \in \mathbb{N}$.

Note that $R[b]^t(x) \neq 0$ for any $t \in \mathbb{N}$, because in the other case $x \in L_0(b) \subseteq L_0(a)$ which contradicts the condition $\overline{x} \notin L_{\overline{0}}(\overline{a})$. Therefore $x \notin H$.

Thus for the element x we have $R[b]^k(x) \in I$ and $x \neq x_0$. Corollary 3.5.10 implies that $\overline{x} = \overline{x}_0 \in \overline{H} = L_{\overline{0}}(\overline{a})$, which contradicts the choice of x. Therefore $\overline{H} = L_{\overline{0}}(\bar{b})$. The proof is complete. □

Theorem 3.5.13. *The image of a regular element for a Leibniz n-algebra L under the natural homomorphism $\varphi : L \to L/I$ is a regular element for the n-Lie algebra L/I.*

Proof. Suppose that $a = (a_2, a_3, \dots, a_n) \in L^{\times(n-1)}$ is a regular element for L and $\overline{a} = (a_2 + I, a_3 + I, \dots, a_n + I)$ is not regular in L/I. Let $\bar{b} = (b_2 + I, b_3 + I, \dots, b_n + I)$ be an arbitrary regular element for L/I, then $a_i - b_i \notin I$ for some i.

Since I is an ideal in L, for every $x \in L^{\times(n-1)}$ we have $R[x](I) \subseteq I$ and the matrix of the operator $R[x]$ in the basis $\{e_1, e_2, \dots, e_m, i_1, i_2, \dots, i_l\}$ of the algebra L (where $\{i_1, i_2, \dots, i_l\}$ the basis of I) has the following form:

$$R[x] = \begin{pmatrix} X, & 0 \\ Z_x, & I_x \end{pmatrix},$$

where X is the matrix of the operator $R[x]|_{\{e_1,\dots,e_m\}}$ and I_x is the matrix of the operator $R[x]|_I$.

Let

$$R[a] = \begin{pmatrix} A, & 0 \\ Z_a, & I_a \end{pmatrix}, \quad R[b] = \begin{pmatrix} B, & 0 \\ Z_b, & I_b \end{pmatrix}$$

be the matrices of the transformations $R[a]$ and $R[b]$ respectively.

Denote by k (respectively by k') the order of the 0 characteristic value of the matrix A (respectively B) and by s (respectively by s') the order of the 0 characteristic value of the matrix I_a (respectively I_b). Then we have $k' < k,\ s < s'$.

Put $U = \left\{ y \in L^{\times(n-1)} \,\middle|\, R[y] = \begin{pmatrix} Y, & 0 \\ Z_y, & I_y \end{pmatrix} \text{ and } Y \text{ has the 0 characteristic value of the order less than } k \right\}$ and $V = \left\{ y \in L^{\times(n-1)} \,\middle|\, R[y] = \begin{pmatrix} Y, & 0 \\ Z_y, & I_y \end{pmatrix} \text{ and } I_y \text{ has the 0 characteristic value of the order less than } s+1 \right\}$.

Since $b \in U$ and $a \in V$, the above sets are non empty. Similar to considerations in [42] one can prove that the sets U and V are open in the Zariski topology and therefore they have non-empty intersection. Let $y \in U \cap V$, i.e. $y \in L^{\times(n-1)}$ is such an element that Y has the order of the 0 characteristic value less than k and I_y has the order of the 0 characteristic value less than $s+1$. But in this case R_y has the order of the 0 characteristic value less than $k+s$, i.e. $dim L_0(y) \leq k+s-1$. Therefore we come to a contradiction with the regularity of the element a, if we suppose that $\overline{a}$ is not regular. The proof is complete. □

It should be noted that the preimage under the natural homomorphism of a regular element (Cartan subalgebra) is not necessarily regular element (respectively, Cartan subalgebra).

3.5.2 Construction of Leibniz n-Algebras with Non-Conjugated Cartan Subalgebras

In this subsection, we will construct examples of Leibniz n-algebras with non-conjugated Cartan subalgebras. For the sake of convenience, let us first present the following 5 dimensional Leibniz 3-algebra which admits 2 and 3 dimensional Cartan subalgebras.

Proposition 3.5.14. *Let L be a 5-dimensional Leibniz 3-algebra with the following multiplication table:*

$$
\begin{array}{llll}
[e_1,e_2,e_3]=e_4 & [e_1,e_2,e_4]=e_3 & [e_1,e_3,e_4]=e_2 & [e_2,e_3,e_4]=e_1 \\
[e_1,e_3,e_2]=-e_4 & [e_1,e_4,e_2]=-e_3 & [e_1,e_4,e_3]=-e_2 & [e_2,e_4,e_3]=-e_1 \\
[e_3,e_1,e_2]=e_4 & [e_4,e_1,e_2]=e_3 & [e_4,e_1,e_3]=e_2 & [e_4,e_2,e_3]=e_1 \\
[e_3,e_2,e_1]=-e_4 & [e_4,e_2,e_1]=-e_3 & [e_4,e_3,e_1]=-e_2 & [e_4,e_3,e_2]=-e_1 \\
[e_2,e_3,e_1]=e_4 & [e_2,e_4,e_1]=e_3 & [e_3,e_4,e_1]=e_2 & [e_3,e_4,e_2]=e_1 \\
[e_2,e_1,e_3]=-e_4 & [e_2,e_1,e_4]=-e_3 & [e_3,e_1,e_4]=-e_2 & [e_3,e_2,e_4]=-e_1
\end{array}
$$

$$[e_5,e_1,e_1]=e_5$$

Then $H=\langle e_1,e_2\rangle$ and $K=\langle e_3,e_4,e_5\rangle$ are Cartan subalgebras of L.

Proof. Indeed, checking the identity (3.1.1) it is easy to verify that L is a Leibniz 3-algebra.

Also we have $[H,H,H]=0$ which means that H is nilpotent and if $x=\sum_{i=1}^{5}\alpha_i e_i \in N_1(H)$, then

$$[\sum_{i=1}^{5}\alpha_i e_i, e_1, e_2] = \alpha_3[e_3,e_1,e_2]+\alpha_4[e_4,e_1,e_2] = \alpha_3 e_4+\alpha_4 e_3 \in H.$$

Hence, $\alpha_3=\alpha_4=0$. Also, if $x=\sum_{i=1}^{5}\alpha_i e_i \in N_1(H)$, then

$$[\sum_{i=1}^{5}\alpha_i e_i, e_1, e_1] = \alpha_5[e_5,e_1,e_1] = \alpha_5 e_5 \in H.$$

Thus, $\alpha_5=0$ and $N_1(H)\subseteq\{\alpha_1e_1+\alpha_2e_2\}=H$. Since $N_1(H)\supseteq H$, we obtain $H=N_1(H)$. Therefore H is a 2−dimensional Cartan subalgebra.

Obviously, K is nilpotent $[K,K,K]=0$. Now let $x=\sum_{i=1}^{5}\alpha_i e_i \in N_1(K)$, then

$$[\sum_{i=1}^{5}\alpha_i e_i, e_3, e_4] = \alpha_1[e_1,e_3,e_4]+\alpha_2[e_2,e_3,e_4] = \alpha_1 e_2+\alpha_2 e_1 \in K.$$

Hence, $\alpha_1=\alpha_2=0$ and $N_1(K)\subseteq\{\alpha_3e_3+\alpha_4e_4+\alpha_5e_5\}=K$. Since $N_1(K)\supseteq K$, we obtain $N_1(K)=K$. Therefore K is a 3−dimensional Cartan subalgebra. □

The Leibniz 3-algebra from this proposition is closely related to so-called simple n-Lie algebras. It is well-known that a free $n+1$ dimensional n-Lie algebras admit the basis $\{e_1,e_2\dots,e_{n+1}\}$ such that

$$[e_1,\dots,e_{i-1},e_{i+1},\dots,e_n]=e_i$$

for all $1\le i\le n+2$.

Now let us construct a Leibniz n-algebra L such that the quotient n-algebra L/I is a simple $n+1$ dimensional n-Lie algebra.

Example 3.5.15. *Let $\{e_1, \ldots, e_{n+1}, x_1, \ldots, x_m\}$ be a basis of L.*

Consider an n-algebra with the following multiplication:

$$\begin{aligned} [e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n+1}] &= e_i \\ [x_k, e_j, \ldots, e_j] &= \alpha_{kj} x_k \,, \end{aligned}$$

where $1 \leq i, j \leq n+1$, $1 \leq k \leq m$, $|\alpha_{k1}|^2 + \cdots + |\alpha_{k\,n+1}|^2 \neq 0$ for all k, and the multiplication is skew symmetric in all the variables on $\langle e_1, \ldots, e_{n+1} \rangle$.

Then this n-algebra is a Leibniz n-algebra and $I = \langle x_1, \ldots, x_m \rangle$.

To be sure, we need to check the identity (3.1.1) on basis vectors:

$$\begin{aligned} &[[a_1, a_2 \ldots, a_n], b_2, \ldots, b_n] = [[a_1, b_2, \ldots, b_n], a_2, \ldots, a_n] \\ &\quad + [a_1, [a_2, b_2, \ldots, b_n], a_3, \ldots, a_n] + \cdots + [a_1, a_2, \ldots, a_{n-1}, [a_n, b_2, \ldots, b_n]]. \end{aligned}$$

If among $a_1, \ldots, a_n, b_2, \ldots b_n$ there are no vectors from $\{x_1, \ldots, x_m\}$, then the identity holds since L/I is an n-Lie algebra.

If among $b_2, \ldots, b_n$ there are vectors from $\{x_1, \ldots, x_m\}$, then the identity holds again ($0 = 0$). Similarly, if among $a_2, \ldots, a_n$ there are vectors from $\{x_1, \ldots, x_m\}$, then the identity holds ($0 = 0$).

The only case left to consider is $a_1 = x_k$, for $1 \leq k \leq m$. Then all $a_2, \ldots, a_n$ must be equal, otherwise again we get identity $0 = 0$.

Let $a_2 = \cdots = a_n = e_p$. Then the fundamental identity transforms to

$$\begin{aligned} &[[x_k, e_p, \ldots, e_p], b_2, \ldots, b_n] = [[x_k, b_2, \ldots, b_n], e_p, \ldots, e_p] \\ &\qquad + [x_k, [e_p, b_2, \ldots, b_n], e_p, \ldots, e_p] + \cdots + [x_k, e_p, \ldots, e_p, [e_p, b_2, \ldots, b_n]]. \end{aligned}$$

From the multiplication table we conclude that $[e_p, b_2, \ldots, b_n]$ can not be collinear to e_p, this is why all the summands except the first one annihilate:

$$[[x_k, e_p, \ldots, e_p], b_2, \ldots, b_n] = [[x_k, b_2, \ldots, b_n], e_p, \ldots, e_p]$$

If $b_2 = \cdots = b_n = e_q$, then we obtain identity $\alpha_{kp}\alpha_{kq}x_k = \alpha_{kq}\alpha_{kp}x_k$. Otherwise we derive into identity $0 = 0$.

Hence, this n-algebra is indeed Leibniz n-algebra.

Note that since L/I is a simple n-Lie algebra and by [7, Theorem 2.2] we have that $F(L/I) = 0$. Hence $F(L) \subseteq I$.

Proposition 3.5.16. *In Example 3.5.15, $F(L) = 0$.*

Proof. Consider the subspaces

$$L_k = \langle e_1, \dots, e_{n+1}, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m \rangle,\ 1 \le k \le m.$$

From the multiplication table it follows that they are subalgebras. Since the dimension of these subalgebras is $n + m = \dim L - 1$, they are maximal subalgebras.

Hence, $F(L) \subseteq \bigcap_{k=1}^{m} L_k = \langle e_1, \dots, e_{n+1} \rangle$.

But $F(L) \subseteq I = \langle x_1, \dots, x_m \rangle$. Thus $F(L) = 0$. □

Below, we present a more general construction of Leibniz n-algebra from special type of n-Lie algebras.

Let us consider an arbitrary n-Lie algebra with the basis $e_1, \dots e_{n+1}$ and the conditions

$$[e_i, f_2, \dots, f_n] \in \mathrm{Span}\langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1} \rangle,$$

for all $f_2, \dots, f_n \in \{e_1, \dots, e_{n+1}\}$, $1 \le i \le n+1$.

One of the n-Lie algebras with these conditions is a simple n-Lie algebra.

Complement this algebra with independent vectors $x_1, \dots, x_m$, with the following multiplication

$$[x_k, e_p, \dots, e_p] = \alpha_{kp}^1 x_1 + \alpha_{kp}^2 x_2 + \dots + \alpha_{kp}^m x_m$$

for all $1 \le k \le m, 1 \le p \le n+1$. Checking identity (3.1.1) we will find restrictions on the coefficients α_{ij}^k :

$$[[a_1, a_2 \dots, a_n], b_2, \dots, b_n] = [[a_1, b_2, \dots, b_n], a_2, \dots, a_n]$$
$$+ [a_1, [a_2, b_2, \dots, b_n], a_3, \dots, a_n] + \dots + [a_1, a_2, \dots, a_{n-1}, [a_n, b_2, \dots, b_n]].$$

Similar to above arguments, we find that if among $a_1, \dots, a_n, b_2, \dots b_n$ there are no vectors from $\{x_1, \dots, x_m\}$, then the identity holds since L/I is an n-Lie algebra.

If among $b_2, \dots, b_n$ there are vectors from $\{x_1, \dots, x_m\}$, then identity $0 = 0$ holds. Analogously, if among $a_2, \dots, a_n$ there are vectors from $\{x_1, \dots, x_m\}$, then we obtain identity $(0 = 0)$.

The only case left to consider is $a_1 = x_k$ for $1 \le k \le m$. Then all $a_2, \dots, a_n$ must be equal, otherwise again we derive identity $0 = 0$. Let $a_2 = \dots = a_n =$

e_p. Then the fundamental identity transforms to

$$[[x_k, e_p, \dots, e_p], b_2, \dots, b_n] = [[x_k, b_2, \dots, b_n], e_p, \dots, e_p] \\ + [x_k, [e_p, b_2, \dots, b_n], e_p, \dots, e_p] + \dots + [x_k, e_p, \dots, e_p, [e_p, b_2, \dots, b_n]].$$

From the conditions on n-Lie algebra that we are considering we have

$$[e_p, b_2, \dots, b_n] \in \langle e_1, \dots, e_{p-1}, e_{p+1}, \dots, e_{n+1} \rangle.$$

Hence, all the summands on the right hand side of the above written equality, except the first one turns to zero and we obtain

$$[[x_k, e_p, \dots, e_p], b_2, \dots, b_n] = [[x_k, b_2, \dots, b_n], e_p, \dots, e_p].$$

If not all of $b_i, 2 \leq i \leq n$ are equal to each other, we obtain identity $0 = 0$.

If $b_2 = \dots = b_n = e_q$, then

$$[[x_k, e_p, \dots, e_p], e_q, \dots, e_q] = [[x_k, e_q, \dots, e_q], e_p, \dots, e_p]$$

which is equivalent to

$$\sum_{i=1}^{m} \alpha_{kp}^{i} [x_i, e_q, \dots, e_q] = \sum_{i=1}^{m} \alpha_{kq}^{i} [x_i, e_p, \dots, e_p]$$

or

$$\sum_{i=1}^{m} \alpha_{kp}^{i} (\alpha_{iq}^{1} x_1 + \alpha_{iq}^{2} x_2 + \dots + \alpha_{iq}^{m} x_m) = \sum_{i=1}^{m} \alpha_{kq}^{i} (\alpha_{ip}^{1} x_1 + \alpha_{ip}^{2} x_2 + \dots + \alpha_{ip}^{m} x_m).$$

Finding the coefficients of corresponding basic vectors from both sides we obtain the following conditions for α_{ij}^{k} :

$$\sum_{i=1}^{m} \alpha_{kp}^{i} \alpha_{iq}^{j} = \sum_{i=1}^{m} \alpha_{kq}^{i} \alpha_{ip}^{j} \tag{3.5.1}$$

for all $1 \leq k, j \leq m$, $1 \leq p, q \leq n + 1$.

Hence, the satisfaction of the conditions (3.5.1) guaranties that the supplemented algebra is a Leibniz n-algebra.

In particular, in this way, one can supplement simple n-Lie algebras till Leibniz n-algebras.

On the ground of Example 3.5.15 let us consider more convenient and relatively simple for further argumentations example of a Leibniz n-algebra.

Example 3.5.17. *Let L_s $(1 \leq s \leq n+1)$ be an n-algebra with the basis $\langle e_1, e_2, \dots, e_{n+1}, x_1, \dots, x_m \rangle$ and the following multiplication:*

$$\begin{aligned} [e_1, \dots, e_{p-1}, e_{p+1}, \dots, e_{n+1}] &= e_p\,, && 1 \leq p \leq n+1, \\ [x_k, e_k, e_k, \dots, e_k] &= x_k\,, && 1 \leq k \leq s, \\ [x_{s+i}, e_s, e_s, \dots, e_s] &= x_{s+i}\,, && 1 \leq i \leq m-s\,, \end{aligned}$$

where the multiplication is skew symmetric in all the variables on the subspace $\langle e_1, e_2, \dots, e_{n+1} \rangle$. Then L_s is a Leibniz n-algebra.

Since this example is a particular case of Example 3.5.15 and obviously, satisfies the condition (3.5.1), it is clear that L_s is indeed a Leibniz n-algebra.

The following proposition shows the existence of $n-1$ dimensional Cartan subalgebras in Leibniz n-algebra L_s.

Proposition 3.5.18. *In a Leibniz n-algebra L_s the following $n-1$ dimensional subalgebras are Cartan subalgebras:*

$$\begin{aligned} R_1 = &\quad \langle e_1, e_2, \dots, e_s, e_{s+1}, \dots, e_{n-1} \rangle, \\ R_2 = &\quad \langle e_1, e_2, \dots, e_s, e_{s+2}, \dots, e_{n-1}, e_n \rangle, \\ R_3 = &\quad \langle e_1, e_2, \dots, e_s, e_{s+3}, \dots, e_{n-1}, e_n, e_{n+1} \rangle. \end{aligned}$$

Proof. Let us prove the assertion of proposition for R_1. The proof for R_2 and R_3 uses the similar argumentations.

Since $R_1 \subseteq \langle e_1, e_2, \dots, e_{n+1} \rangle$ and $\dim R_1 = n-1$, from the table of multiplication of Leibniz n-algebra L_s we obtain that

$$[R_1, R_1, \dots, R_1] = \{0\}.$$

Hence, R_1 is 1-nilpotent (and also nilpotent too).

Now we need to prove that $N_1(R_1) = R_1$.

$N_1(R_1)$ can not contain a vector from $\langle x_1, \dots, x_m \rangle$, since by table of multiplication of Leibniz n-algebra L_s we have

$$x_i = [x_i, e_i, \dots, e_i] \in [x_i, R_1, \dots, R_1]$$

and

$$x_{s+j} = [x_{s+j}, e_s, \dots, e_s] \in [x_{s+j}, R_1, \dots, R_1]$$

for $1 \leq i \leq s$ and $1 \leq j \leq m-s$.

Thus $N_1(R_1) \subseteq \langle e_1, e_2, \ldots, e_{n+1}\rangle$. By table of multiplication of Leibniz n-algebra L_s we have

$$e_n = -[e_{n+1}, e_1, \ldots, e_{n-1}]$$

$$e_{n+1} = -[e_n, e_1, \ldots, e_{n-1}].$$

Hence, if $\alpha e_n + \beta e_{n+1} \in N_1(R_1)$, then

$$-\alpha e_{n+1} - \beta e_n = [\alpha e_n + \beta e_{n+1}, e_1, \ldots, e_{n-1}] \in [N_1(R_1), R_1, \ldots, R_1] \subseteq R_1$$

which hold if and only if $\alpha = \beta = 0$. This implies $N_1(R_1) \subseteq \langle e_1, e_2, \ldots, e_{n-1}\rangle$. Since $R_1 \subseteq N_1(R_1)$, we obtain that $N_1(R_1) = R_1$ and therefore R_1 is a Cartan subalgebra of L_s. □

Notice that in a Leibniz n-algebra L_s, ideal I is precisely $\langle x_1, \ldots, x_m\rangle$ and $L_s/I \cong \langle e_1, e_2, \ldots, e_{n+1}\rangle$ is n-Lie algebra. Whence, Cartan subalgebras of L_s contained in $\langle e_1, e_2, \ldots, e_{n+1}\rangle$ are Cartan subalgebra in n-Lie algebras, and therefore are conjugated [31]. In order to construct Cartan subalgebras of dimensions different from $n-1$, we need to consider subalgebras not properly contained in $\langle e_1, e_2, \ldots, e_{n+1}\rangle$, which we succeed to construct in the following series of propositions.

Proposition 3.5.19. *In a Leibniz n-algebra L_s, the following n dimensional subalgebras*

$$S_i = \langle e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n\rangle$$

for $1 \leq i \leq s-1$ are Cartan subalgebras.

Proof. Denote by $SE = S_i \cap \langle e_1, e_2, \ldots, e_{n+1}\rangle = \langle e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\rangle$. Then $\dim SE = n-1$ and therefore $[SE, SE, \ldots, SE] = 0$. Now by the table of multiplication of Leibniz n-algebra L_s we have $[x_i, SE, SE, \ldots, SE] = 0$ since $e_i \notin SE$. Therefore

$$[S_i, S_i, \ldots, S_i] = \{0\}$$

which implies that S_i is 1-nilpotent (generally speaking, S_i is nilpotent).

Let $\alpha e_i + \beta e_{n+1} + \sum_{i=1}^{m} \gamma_i x_i \in N_1(S_i)$. Then

$$(-1)^{i-1}\alpha e_{n+1}+(-1)^{n-1}\beta e_i+\sum_{i=1}^{m}\gamma_i x_i$$

$$= [\alpha e_i + \beta e_{n+1}, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n] + \sum_{i=1}^{s-1} \gamma_i [x_i, e_i, \ldots, e_i]$$

$$+ \sum_{i=s}^{m} \gamma_i [x_i, e_s, \ldots, e_s] = \text{(and by multiplication table of } L_s \text{ we have)}$$

$$= [\alpha e_i + \beta e_{n+1}, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n] + \Big[\sum_{i=1}^{m} \gamma_i x_i, \sum_{i=1}^{s} e_i, \ldots, \sum_{i=1}^{s} e_i\Big] =$$

(multiplication between vectors $e_1, \ldots, e_{n+1}$ is skew-symmetric, therefore)

$$= \Big[\alpha e_i + \beta e_{n+1} + \sum_{i=1}^{m} \gamma_i x_i, \underbrace{\sum_{i=1}^{s} e_i, \ldots, \sum_{i=1}^{s} e_i}_{s-\text{times}}, e_{s+1} + \sum_{i=1}^{s} e_i, \ldots, e_n + \sum_{i=1}^{s} e_i\Big].$$

The last product belongs to $[N_1(S_i), S_i, \ldots, S_i] \subseteq S_i$ and thus

$$(-1)^{i-1}\alpha e_{n+1} + (-1)^{n-1}\beta e_i + \sum_{i=1}^{m} \gamma_i x_i \in S_i$$

which implies $\alpha = \beta = \gamma_1 = \cdots = \gamma_{i-1} = \gamma_{i+1} = \cdots = \gamma_m = 0$.

Hence $N_1(S_i) \subseteq S_i$ and we have $N_1(S_i) = S_i$.

Thus S_i is a Cartan subalgebra of L_s. □

Proposition 3.5.20. *In a Leibniz n-algebra L_s, the following $m+n-s$ dimensional subalgebras*

$$T_1 = \langle e_1, e_2, \ldots, e_{s-1}, e_{s+1}, e_{s+2}, \ldots, e_n, x_s, x_{s+1}, \ldots, x_m \rangle$$
$$T_2 = \langle e_1, e_2, \ldots, e_{s-1}, e_{s+2}, e_{s+3}, \ldots, e_{n+1}, x_s, x_{s+1}, \ldots, x_m \rangle$$

are Cartan subalgebras.

Proof. We prove the statement of proposition for T_1. The proof for T_2 is analogous.

From the multiplication table of L_s it is not difficult to see that

$$[T_1, T_1, \ldots, T_1] = \{0\}$$

and therefore T_1 is nilpotent (1-nilpotent).

Let $\sum_{i=1}^{n+1} \alpha_i e_i + \sum_{j=1}^{m} \beta_j x_j \in N_1(T_1)$. Then by definition of 1-normaliser we have

$$\big[\sum_{i=1}^{n+1} \alpha_i e_i + \sum_{j=1}^{m} \beta_j x_j, T_1, \ldots, T_1\big] \subseteq T_1.$$

Thus for $1 \leq k \leq s-1$ we have

$$T_1 \ni \big[\sum_{i=1}^{n+1} \alpha_i e_i + \sum_{j=1}^{m} \beta_j x_j,\ e_k, e_k, \ldots, e_k\big] = \beta_k x_k,$$

which is valid only if $\beta_k = 0$.

Also from

$$T_1 \ni \big[\sum_{i=1}^{n+1} \alpha_i e_i + \sum_{j=1}^{m} \beta_j x_j,\ e_1, \ldots, e_{s-1}, e_{s+1}, \ldots, e_n\big]$$
$$= (-1)^{s-1} \alpha_s e_{n+1} + (-1)^{n-1} \alpha_{n+1} e_s$$

it follows that $\alpha_s = \alpha_{n+1} = 0$.

Therefore $N_1(T_1) \subseteq T_1$ and we obtain $N_1(T_1) = T_1$.

Hence, T_1 is a Cartan subalgebra of L_s. □

Proposition 3.5.21. *In a Leibniz n-algebra L_s, the following $m+n-s+1$ dimensional subalgebras*

$$U_i = \langle e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_{s-1}, e_{s+1}, \ldots, e_{n+1}, x_s, \ldots, x_m \rangle$$

for $1 \leq i \leq s-1$ are Cartan subalgebras.

Proof. From the multiplication table of L_s it is not difficult to see that

$$[U_i, U_i, \ldots, U_i] = \{0\}$$

and therefore U_i is nilpotent (1-nilpotent).

Since U_i is a subalgebra, we already have $U_i \subseteq N_1(U_i)$.

Let $y = \beta_1 x_1 + \cdots + \beta_{i-1} x_{i-1} + \alpha_i e_i + \beta_{i+1} x_{i+1} + \cdots + \beta_{s-1} x_{s-1} + \alpha_s e_s \in N_1(T_1)$.

Then for $1 \leq k \leq s-1$, $k \neq i$ we have

$$U_i \supseteq [N_1(U_i), U_i, \ldots, U_i] \ni [y, e_k, e_k, \ldots, e_k] = \beta_k e_k$$

which is true if only $\beta_k = 0$.

Also we have

$$\begin{aligned} U_i \supseteq [N_1(U_i), U_i, \ldots, U_i] \ni &[y, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{s-1}, e_{s+1}, \ldots, e_{n+1}] \\ &= [\alpha_i e_i, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{s-1}, e_{s+1}, \ldots, e_{n+1}] \\ &\quad + [\alpha_s e_s, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{s-1}, e_{s+1}, \ldots, e_{n+1}] \\ &\qquad = (-1)^{i-1} \alpha_i e_s + (-1)^{n-1} \alpha_s e_i \end{aligned}$$

which holds if only $\alpha_i = \alpha_s = 0$.

Therefore $N_1(U_i) \subseteq U_i$ and hence $N_1(U_i) = U_i$.

Thus U_i is a Cartan subalgebra of L_s. □

In the considered Leibniz n-algebra we found Cartan subalgebras of dimensions $n-1$, n, $n+m-s$ and $n+m-s+1$. Hence, in general, Cartan subalgebras of a given Leibniz n-algebra are not conjugated.

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