

## PIECEWISE SMOOTH STATIONARY EULER FLOWS WITH COMPACT SUPPORT VIA OVERDETERMINED BOUNDARY PROBLEMS

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ABSTRACT. We construct new stationary weak solutions of the 3D Euler equation with compact support. The solutions, which are piecewise smooth and discontinuous across a surface, are axisymmetric with swirl. The range of solutions we find is different from, and larger than, the family of smooth stationary solutions recently obtained by Gavrilov and Constantin–La–Vicol; in particular, these solutions are not localizable. A key step in the proof is the construction of solutions to an overdetermined elliptic boundary value problem where one prescribes both Dirichlet and (nonconstant) Neumann data.

### 1. INTRODUCTION

In two dimensions, it is easy to construct stationary solutions with compact support to the incompressible Euler equation

$$(1.1) \quad \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0.$$

For instance, an explicit  $C^\infty$  stationary Euler flow supported in the unit disk  $\mathbb{D}$  is given by the stream function  $\psi := e^{1/(|x|^2-1)} \mathbf{1}_{\mathbb{D}}(x)$ , which determines the velocity field as  $u = \nabla^\perp \psi$ .

The question of whether there are stationary Euler flows of compact support in three dimensions is much harder, and has attracted much recent attention. In the important particular case that the stationary Euler flow is a generalized Beltrami field, Nadirashvili [9] and Chae–Constantin [1] have shown that no compactly supported solutions exist, and that in fact there are not even any generalized Beltrami fields with finite energy. It is also known that axisymmetric stationary Euler flows of compact support without swirl do not exist [8]. In contrast, it has long been known that there exist  $C^{1,\alpha}$  stationary Euler flows whose vorticity is compactly supported [6]. Weak stationary Euler flows in  $L^\infty$  of compact support can also be constructed with the convex integration technique developed in [2].

A major recent breakthrough was Gavrilov’s construction of compactly supported stationary Euler flows in three dimensions [7], which are axisymmetric and of class  $C^\infty$ . More concretely, these solutions are of the form

$$(1.2) \quad u = G(p_R) u_R, \quad p = \int_0^{p_R} G(q)^2 dq,$$

where  $(u_R, p_R)$  are certain concrete functions depending on a positive parameter  $R$  that solve the stationary Euler equation in a toroidal domain, and  $G$  is an essentially arbitrary function of one variable (compactly supported). Slightly more general solutions can be constructed using the same method.

These solutions have been revisited and put in a broader context using the perspective of the Grad–Shafranov equation by Constantin, La and Vicol [3], which allowed them to develop nontrivial applications to other equations of fluid mechanics as well. As stressed by these authors, the key property of these solutions is that they are *localizable*, meaning that the pressure is constant along the stream lines of the flow:  $u_R \cdot \nabla p_R = 0$  (which in turn implies that  $u \cdot \nabla p = 0$ ).

Our objective in this paper is to derive a different approach to the construction of stationary Euler flows with compact support. The solutions we construct are very different from those obtained by Gavrilov and Constantin–La–Vicol: they are piecewise smooth stationary weak solutions with axial symmetry, and they are not localizable. Each stationary solution we construct is bounded (with bounded vorticity) and supported on a toroidal domain with a smooth boundary; the flow is smooth in the interior of this domain (up to the boundary) and possibly discontinuous across the boundary. As we will make precise below, this approach yields a wide range of axisymmetric Euler flows of compact support.

Our construction of stationary Euler flows with compact support is based on showing the existence of nontrivial solutions to a boundary value problem for an elliptic equation where both Dirichlet and Neumann data are prescribed. This kind of boundary problems are usually called *overdetermined*.

To see how overdetermined boundary problems appear in this context, let us start by recalling the Grad–Shafranov formulation of the axisymmetric Euler equation in three dimensions. This consists in writing an axisymmetric solution to the Euler equation in cylindrical coordinates (in terms of the orthonormal basis  $\{e_z, e_r, e_\varphi\}$ ) as

$$(1.3) \quad u = \frac{1}{r} [\partial_r \psi e_z - \partial_z \psi e_r + F(\psi) e_\varphi],$$

where the function  $\psi(r, z)$  satisfies the equation

$$(1.4) \quad L\psi = r^2 H'(\psi) - \frac{1}{2} (F^2)'(\psi)$$

for some function  $H$ . The pressure is then given by

$$(1.5) \quad p = H(\psi) - \frac{1}{2r^2} [|\nabla \psi|^2 + F(\psi)^2]$$

and we have set

$$(1.6) \quad L\psi := \partial_{rr}\psi + \partial_{zz}\psi - \frac{1}{r}\partial_r\psi.$$

The functions  $F$  and  $H$  can be picked freely.

The first observation of this paper, which explains why we are interested in overdetermined boundary problems in this context, is the following:

**Lemma 1.1.** *Let  $\Omega$  be a  $C^2$  bounded domain whose closure is contained in the half-space  $\{(r, z) \in \mathbb{R}^2 : r > 0\}$ . Suppose that there is a function  $\psi \in C^2(\Omega)$  satisfying Equation (1.4) in  $\Omega$  and the boundary conditions*

$$(1.7) \quad \psi|_{\partial\Omega} = 0,$$

$$(1.8) \quad \left. \frac{(\partial_\nu \psi)^2 + F(0)^2}{r^2} \right|_{\partial\Omega} = c,$$

where  $c$  is a constant, and  $\nu$  is a unit normal field on  $\partial\Omega$ . Then the vector field  $u$  defined by (1.3) inside  $\Omega$  and as  $u := 0$  outside  $\Omega$  is a weak solution of the Euler equation, the pressure being given by (1.5) inside  $\Omega$  and by  $p := H(0) - c/2$  outside  $\Omega$ .

Therefore, the key step of our construction of stationary Euler flows with compact support is to show the existence of nontrivial solutions to a (non-standard) overdetermined boundary problem for a certain semilinear elliptic equation in two variables. While the investigation of overdetermined boundary value problems traces back to Serrin's seminal paper [11] in 1971, until very recently the literature on overdetermined boundary problems was essentially limited to proofs that solutions need to be radial in cases that could be handled using the method of moving planes.

However, in two surprising papers, Pacard and Sicbaldi [10] and Delay and Sicbaldi [4] proved the existence of extremal domains with small volume for the first eigenvalue  $\lambda_1$  of the Laplacian in any compact Riemannian manifold, which guarantees the existence of solutions to a certain overdetermined problem for the linear elliptic operator  $\Delta + \lambda_1$  in a domain with both zero Dirichlet data and constant Neumann data. Very recently we managed to show the existence of nontrivial solutions, still with the same Dirichlet and Neumann data, for fairly general semilinear elliptic equations of second order with possibly nonconstant coefficients [5]. Our strategy here is to start by tweaking the proof of this result to accommodate to the non-standard boundary condition we must impose.

The main difficulty to solve Equation (1.4) with the overdetermined boundary conditions (1.7) and (1.8) is that the Neumann data depend on the point of the boundary  $\partial\Omega$ , a situation that was not considered in our paper [5]. The technique used there is based on a variational technique that relates the existence of overdetermined solutions with the critical points of certain functional, a strategy that is successful only for constant boundary data. Roughly speaking, the gist of the argument is that the overdetermined condition with constant data is connected with the local extrema for a natural energy functional, restricted to a specific class of functions labeled by points in the physical space. This fact ultimately permits to derive the existence of solutions from the fact that a continuous function attains its maximum on a compact manifold. We do not know of an analog of this fact for the nonconstant boundary conditions considered in this paper. Instead, we rely on a new, hands-on approach to the problem that results in a flexible, very explicit existence theorem:

**Theorem 1.2.** *Consider any functions  $\tilde{F}, H \in C^s((-1, 0])$  satisfying*

$$\tilde{F}(0) = 0, \quad \tilde{F}'(0) < 0, \quad H'(0) > 0,$$

where  $s > 2$  is not an integer. Then the following statements hold:

- (i) *For each small enough  $\varepsilon > 0$  and any large enough  $R > 0$ , there exists a nontrivial, piecewise  $C^s$ , axisymmetric stationary Euler flow of compact support  $u$  of the form described in Lemma 1.1 for a suitable  $C^{s+1}$  planar domain  $\Omega_{R,\varepsilon}$ .*
- (ii) *The domain  $\Omega_{R,\varepsilon}$  is a small deformation of a disk of radius  $\varepsilon$  centered at the point  $(R, 0)$  of the  $(r, z)$ -plane.*

(iii) *The functions that define the solution are*

$$(1.9) \quad F(\psi) := [\varepsilon^2 F_R + \tilde{F}(\psi)]^{1/2}$$

and  $H(\psi)$ , where  $F_R$  is a positive constant depending on  $R$ .

(iv) *The function  $\psi$ , which is of class  $C^{s+1}$  in  $\Omega_{R,\varepsilon}$  up to the boundary, is approximately radial. Moreover,  $\varepsilon^2 F_R + \tilde{F} \circ \psi > 0$ ,  $F \circ \psi > 0$  and  $H \circ \psi$  are of class  $C^s$  in  $\Omega_{R,\varepsilon}$ .*

*Remark 1.3.* In fact, the value of the constants and the structure of the solutions is completely explicit:

- (i)  $R$  must be larger than  $[-\frac{3}{2}\tilde{F}'(0)/H'(0)]^{1/2}$ .
- (ii) The boundary of  $\partial\Omega_{R,\varepsilon}$  is the curve defined by an equation of the form  $z^2 + (r - R)^2 - \varepsilon^2 = O(\varepsilon^3)$ .
- (iii) The constant  $F_R$  is

$$F_R := \frac{1}{16} \left[ H'(0)R^2 - \frac{1}{2}\tilde{F}'(0) \right] \left[ H'(0)R^2 + \frac{3}{2}\tilde{F}'(0) \right] > 0,$$

and the constant  $c$  in the Neumann condition is of the form

$$c = \frac{\varepsilon^2}{16R^2} \left[ H'(0)R^2 - \frac{1}{2}\tilde{F}'(0) \right] \left[ 5H'(0)R^2 - \frac{1}{2}\tilde{F}'(0) \right] + O(\varepsilon^3).$$

(iv) The function  $\psi$  is of the form

$$\psi = \frac{1}{4} \left[ H'(0)R^2 - \frac{1}{2}\tilde{F}'(0) \right] [(r - R)^2 + z^2 - \varepsilon^2] + O(\varepsilon^3).$$

(v) The vorticity,

$$\omega = \frac{F'(\psi)}{r} (\partial_r \psi e_z - \partial_z \psi e_r) + \left[ -rH'(\psi) + \frac{(F^2)'(\psi)}{2r} \right] e_\varphi,$$

is also of class  $C^{s-1}$  up to the boundary.

Several comments are in order. First, let us emphasize that the solutions we construct are piecewise smooth but discontinuous across a smooth surface; hence, from the point of view of their regularity, they stand between the smooth solutions of [7, 3] and the rough weak solutions of [2]. Concerning the flexibility of the construction, it is apparent that the range of solutions we obtain is much larger than that of [7, 3]. Indeed, our solutions are not localizable in general and are of class  $C^s$  in the toroidal domain of  $\mathbb{R}^3$  defined by the planar domain  $\Omega$ . In contrast, the basic vector field  $u_R$  that appears in (1.2) is not defined at an inside point (“the origin”), so the function  $G$  in that equation is chosen so that it vanishes to infinite order there; in fact, the supports of the solutions constructed so far are toroidal shells instead of solid tori. Actually, the functions  $H$  and  $F$  in [7, 3] defining the vector field  $u_R$  are precisely  $H(\psi) = a\psi$  and a certain function of the form  $F(\psi) = Rb\sqrt{\psi}[1 + O(\psi)]$ , where again the positive constants  $a$  and  $b$  are fixed.

It should be noted that the philosophy that underlies the proof of Theorem 1.2 has, in fact, a wider range of applicability. To illustrate this fact, we include in Section 7 a brief discussion on the existence of compactly supported solutions for a class of functions  $F$  and  $H$  different from that considered above.

The paper is organized as follows. In Section 2 we will prove Lemma 1.1 as a corollary of a more general result about piecewise smooth weak solutions to the stationary Euler equation. The rest of the paper is devoted to the proof of Theorem 1.2. In Section 3 we construct solutions to the Dirichlet problem for the Equation (1.4), and subsequently compute their asymptotic expansion in the parameter  $\varepsilon$  (Section 4). The way these solutions change when the domain is perturbed a little is discussed in Section 5. This is key for the proof that there exists a domain which is  $\varepsilon^2$ -close to a disk of radius  $\varepsilon$  where the Dirichlet solution also satisfies the additional Neumann condition (1.8). This result, which we prove in Section 6, allows us to complete the proof of Theorem 1.2. To conclude, in Section 7 we briefly discuss the existence of compactly supported solutions defined by functions  $F$  and  $H$  different from those considered in Theorem 1.2.

## 2. STATIONARY EULER FLOWS VIA AN OVERDETERMINED BOUNDARY PROBLEM

In this short section we prove that if one has a stationary Euler flow on a bounded domain  $\Omega$  which is tangent to the boundary and whose pressure is constant on  $\partial\Omega$ , then one can trivially extend it to a stationary Euler flow on the whole space with compact support. In the particular case when the initial flow is axisymmetric, and hence described by the Grad–Shafranov formulation (Equation (1.3)), this reduces to Lemma 1.1 stated in the Introduction.

Let us start by recalling the definition of a weak stationary Euler flow. We say that a pair  $(u, p)$  of class, say,  $L^2_{\text{loc}}(\mathbb{R}^3)$  is a *weak solution to the stationary Euler equation* if

$$\int_{\mathbb{R}^3} [(u \otimes u) \cdot \nabla w + p \operatorname{div} w] dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} u \cdot \nabla \phi dx = 0$$

for any vector field  $w \in C_c^\infty(\mathbb{R}^3)$  and any scalar function  $\phi \in C_c^\infty(\mathbb{R}^3)$ . Of course, if  $u$  and  $p$  are smooth enough, this is equivalent to saying that they satisfy Equation (1.1) in  $\mathbb{R}^3$ .

**Lemma 2.1.** *Given a bounded domain  $\Omega$  in  $\mathbb{R}^3$  with a  $C^2$  boundary, suppose that  $v \in C^1(\Omega) \cap L^2(\Omega)$  is a solution to the stationary Euler equation in  $\Omega$  with pressure  $\tilde{p} \in C^1(\Omega) \cap L^1(\Omega)$ . Assume that  $\nu \cdot v|_{\partial\Omega} = 0$  and  $\tilde{p}|_{\partial\Omega} = c$ , where  $c$  is a constant. Then*

$$u(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

is a weak solution of the stationary Euler equation on  $\mathbb{R}^3$  with pressure

$$p(x) := \begin{cases} \tilde{p}(x) & \text{if } x \in \Omega, \\ c & \text{if } x \notin \Omega. \end{cases}$$

*Proof.* We start by noticing that, for all  $\phi \in C_c^\infty(\mathbb{R}^3)$ ,

$$\int_{\mathbb{R}^3} u \cdot \nabla \phi dx = \int_{\Omega} v \cdot \nabla \phi dx = - \int_{\Omega} \phi \operatorname{div} v dx + \int_{\partial\Omega} \phi \nu \cdot v dS = 0,$$

where we have used that  $\operatorname{div} v = 0$  in  $\Omega$  and  $\nu \cdot v = 0$  on  $\partial\Omega$ . Hence  $\operatorname{div} u = 0$  in the sense of distributions.

Let us now take an arbitrary vector field  $w \in C_c^\infty(\mathbb{R}^3)$ . As  $\int_{\mathbb{R}^3} \operatorname{div} w \, dx = 0$ , we can write

$$\begin{aligned} \int_{\mathbb{R}^3} [(u \otimes u) \cdot \nabla w + p \operatorname{div} w] \, dx &= \int_{\Omega} (u \otimes u) \cdot \nabla w \, dx + \int_{\mathbb{R}^3} (p - c) \operatorname{div} w \, dx \\ &= \int_{\Omega} [(v \otimes v) \cdot \nabla w - w \cdot \nabla \tilde{p}] \, dx + \int_{\partial\Omega} (\tilde{p} - c) w \cdot \nu \, dS \\ &= - \int_{\Omega} [\operatorname{div}(v \otimes v) + \nabla \tilde{p}] \cdot w \, dx + \int_{\partial\Omega} [(v \cdot w)(v \cdot \nu) + (\tilde{p} - c) w \cdot \nu] \, dS. \end{aligned}$$

The volume integral is zero because  $v$  satisfies the stationary Euler equation in  $\Omega$ . Since  $v \cdot \nu = \tilde{p} - c = 0$  on  $\partial\Omega$ , the surface integral vanishes too. It then follows that  $u$  is a weak solution of the Euler equation in  $\mathbb{R}^3$ , as claimed.  $\square$

*Remark 2.2.* With  $u$  given by (1.3), and using the notation in Lemma 1.1, the conditions that  $u$  be tangent to the axisymmetric domain defined by  $\Omega$  and that the pressure (1.5) be constant on the boundary amount to saying that  $\psi$  takes a constant value  $c_0$  on  $\partial\Omega$  and that  $[(\partial_\nu \psi)^2 + F(c_0)^2]/r^2$  is also constant on  $\partial\Omega$ . Setting  $c_0 := 0$  without loss of generality, Lemma 2.1 results in the statement of Lemma 1.1.

### 3. SOLUTIONS TO THE DIRICHLET BOUNDARY PROBLEM

Let us take any  $R > 0$  that will be fixed during the whole construction and introduce suitably translated and rescaled variables  $(x, y) \in \mathbb{R}^2$  as

$$r =: R + \varepsilon x, \quad z =: \varepsilon y.$$

Throughout,  $\varepsilon > 0$  denotes a small parameter. We will also consider the polar coordinates  $(\rho, \theta) \in (0, \infty) \times \mathbb{T}$ , where  $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ , that are defined in terms of  $(x, y)$  through the formulas

$$x = \rho \cos \theta, \quad y = \rho \sin \theta.$$

In these variables, the Grad–Shafranov equation (1.4) reads as

$$(3.1) \quad \Delta \psi - \frac{\varepsilon}{R + \varepsilon x} \partial_x \psi = \varepsilon^2 (R + \varepsilon x)^2 H'(\psi) - \frac{1}{2} \varepsilon^2 (F^2)'(\psi),$$

where  $F$  and  $H$  are of the form described in Theorem 1.2.

We look for solutions to Equation (3.1) in domains of the form

$$\Omega_{\varepsilon B} := \{\rho < 1 + \varepsilon B(\theta)\}$$

for some function  $B \in C^{s+1}(\mathbb{T})$ ; notice that  $\Omega_{\varepsilon B}$  only depends on the product  $\varepsilon B$  and not on  $\varepsilon$  and  $B$  separately. To keep track of the size of  $\psi$ , we will set

$$\psi =: \varepsilon^2 \phi.$$

For  $\varepsilon \neq 0$ , Equation (3.1) can be written in terms of  $\phi$  as

$$(3.2) \quad \Delta \phi - \frac{\varepsilon}{R + \varepsilon x} \partial_x \phi = aR^2 + b + 2aR\varepsilon x + \mathcal{R}(x, \phi),$$

where we have defined the positive constants

$$a := H'(0) \quad b := -\frac{1}{2} \tilde{F}'(0),$$

and where

$$\mathcal{R}(x, \phi) := \varepsilon^2 a x^2 + (R + \varepsilon x)^2 H_1'(\varepsilon^2 \phi) - \frac{1}{2} \tilde{F}_1'(\varepsilon^2 \phi).$$

Here

$$H_1(\psi) := H(\psi) - a\psi - H(0), \quad \tilde{F}_1(\psi) := \tilde{F}(\psi) + 2b\psi$$

are functions that vanish to second order at  $\psi = 0$ , so we have the obvious bound

$$\sup_{\|\phi\|_{C^{s-1}(\Omega)} < C} \|\mathcal{R}\|_{C^{s-1}(\Omega)} \leq C' \varepsilon^2.$$

The Dirichlet boundary condition  $\psi = 0$  on  $\partial\Omega_{\varepsilon B}$  can then be rewritten in terms of  $\phi(\rho, \theta)$  as

$$(3.3) \quad \phi(1 + \varepsilon B(\theta), \theta) = 0.$$

Henceforth we will say that a function  $f(\rho, \theta)$  is *even* if

$$f(\rho, \theta) = f(\rho, -\theta),$$

and similarly for a function  $g(\theta)$ . Equivalently, a function is even if it is invariant under the reflection  $y \mapsto -y$ .

Since the function

$$\phi_0 := \frac{(aR^2 + b)(\rho^2 - 1)}{4}$$

satisfies Equation (3.2) and the Dirichlet condition (3.3) when  $\varepsilon = 0$ , it is straightforward to show that there are solutions to the problem for small  $\varepsilon$  using the implicit function theorem in Banach spaces:

**Proposition 3.1.** *For any small enough  $\varepsilon$  and any function  $B$  with  $\|B\|_{C^{s+1}} < 1$  there is a unique function  $\phi = \phi_{\varepsilon, B}$  in a small neighborhood of  $\phi_0$  in  $C^{s+1}(\Omega_{\varepsilon B})$  that satisfies Equation (3.2) and the Dirichlet boundary condition (3.3). Furthermore,  $\phi_{\varepsilon, B} < 0$  in  $\Omega_{\varepsilon B}$  and  $\phi_{\varepsilon, B}$  is even if  $B$  is.*

*Proof.* For small enough  $\varepsilon \neq 0$  and  $\|B\|_{C^{s+1}} < 1$ , let  $\chi_{\varepsilon B} : \mathbb{D} \rightarrow \Omega_{\varepsilon B}$  be the diffeomorphism defined in polar coordinates as

$$(\rho, \theta) \mapsto ([1 + \varepsilon \chi(\rho) B(\theta)]\rho, \theta),$$

where  $\chi(\rho)$  is a smooth cutoff function that is zero for  $\rho < 1/4$  and 1 for  $\rho > 1/2$ , and where  $\mathbb{D} := \{\rho < 1\}$  is the unit disk. Then one can define a map

$$\mathcal{H} : (-\varepsilon_0, \varepsilon_0) \times C_{\text{Dir}}^{s+1}(\mathbb{D}) \rightarrow C^{s-1}(\mathbb{D})$$

as

$$\begin{aligned} \mathcal{H}(\varepsilon, \bar{\phi}) := & \left[ \Delta(\bar{\phi} \circ \chi_{\varepsilon B}^{-1}) - \frac{\varepsilon}{R + \varepsilon x} \partial_x(\bar{\phi} \circ \chi_{\varepsilon B}^{-1}) \right. \\ & \left. - [aR^2 + b + 2aR\varepsilon x + \mathcal{R}(x, \bar{\phi} \circ \chi_{\varepsilon B}^{-1})] \right] \circ \chi_{\varepsilon B}. \end{aligned}$$

Here

$$C_{\text{Dir}}^{s+1}(\mathbb{D}) := \{\bar{\phi} \in C^{s+1}(\mathbb{D}) : \bar{\phi}|_{\partial\mathbb{D}} = 0\}.$$

Notice that  $\phi := \bar{\phi} \circ \chi_{\varepsilon B}^{-1} \in C^{s+1}(\Omega_{\varepsilon B})$  and  $\phi = 0$  on  $\partial\Omega_{\varepsilon B}$ . It is obvious that if  $\bar{\phi}$  solves  $\mathcal{H}(\varepsilon, \bar{\phi}) = 0$ , then  $\phi$  is a solution to the Dirichlet problem (3.2)–(3.3).

Since  $\mathcal{H}(0, \phi_0) = 0$  and the Fréchet derivative

$$D_{\bar{\phi}} \mathcal{H}(0, \phi_0) = \Delta$$

is an invertible mapping  $C_{\text{Dir}}^{s+1}(\mathbb{D}) \rightarrow C^{s-1}(\mathbb{D})$ , it follows from the implicit function theorem that for any small enough  $\varepsilon$  and  $B$  there is a unique solution  $\bar{\phi}_{\varepsilon,B}$  to the equation  $\mathcal{H}(\varepsilon, \bar{\phi}_{\varepsilon,B}) = 0$  in a neighborhood of  $\phi_0$ . As  $B$  only appears in the problem through the product  $\varepsilon B$ , this is equivalent to the first part of the statement. Also, this uniqueness property immediately implies that  $\phi_{\varepsilon,B} := \bar{\phi}_{\varepsilon,B} \circ \chi_{\varepsilon B}^{-1}$  is even when  $B$  is. The property that  $\phi_{\varepsilon,B} < 0$  in  $\Omega_{\varepsilon B}$  follows from Equation (3.2), i.e.,

$$\Delta \phi_{\varepsilon,B} = aR^2 + b + O(\varepsilon) > 0,$$

via the maximum principle.  $\square$

#### 4. ANALYSIS OF THE SOLUTION

In the following proposition we compute an asymptotic expansion for the function  $\phi_{\varepsilon,B}$  for small  $\varepsilon$ . The constants  $A_0$  and  $A_1$  appearing in this expansion, which depend on  $R$  but not on  $\varepsilon$ , will play a major role in the rest of the paper.

Throughout, we will denote by  $\mathbb{P}_{\varepsilon B}$  the Poisson integral operator of the domain  $\Omega_{\varepsilon B}$  in the coordinates  $(\rho, \theta)$ . That is,  $v(\rho, \theta) := \mathbb{P}_{\varepsilon B} f$  denotes the only harmonic function in  $\Omega_{\varepsilon B}$  satisfying the boundary condition

$$v(1 + \varepsilon B(\theta), \theta) = f(\theta).$$

Note that  $\mathbb{P}_{\varepsilon B}$  only depends on the product  $\varepsilon B$ , not on  $\varepsilon$  and  $B$  separately. It is standard that  $\mathbb{P}_{\varepsilon B}$  defines a map  $C^{s+1}(\mathbb{T}) \rightarrow C^{s+1}(\Omega_{\varepsilon B})$ . A convenient explicit formula for the Poisson operator in the case of the disk is

$$\mathbb{P}_0 f(\rho, \theta) := \sum_{n \in \mathbb{Z}} f_n \rho^{|n|} e^{in\theta} \quad \text{if} \quad f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}.$$

**Proposition 4.1.** *For small enough  $\varepsilon$ , the function  $\phi_{\varepsilon,B}$  has the asymptotic form*

$$\phi_{\varepsilon,B} = A_0(\rho^2 - 1) + \varepsilon[A_1(\rho^3 - \rho) \cos \theta - 2A_0 \mathbb{P}_{\varepsilon B} B] + O(\varepsilon^2),$$

with the constants

$$A_0 := \frac{aR^2 + b}{4}, \quad A_1 := \frac{5aR^2 + b}{16R}.$$

*Proof.* As it is clear that  $\phi_0 := A_0(\rho^2 - 1)$  is the solution to the boundary value problem under consideration when  $\varepsilon = 0$ , let us assume  $\varepsilon$  is nonzero. Note that the equation for  $\phi = \phi_{\varepsilon,B}$  is of the form

$$\Delta \phi - \frac{\varepsilon}{R} \partial_x \phi - (aR^2 + b + 2aR\varepsilon x) = O(\varepsilon^2), \quad \phi(1 + \varepsilon B(\theta), \theta) = 0.$$

One can then set  $\phi_1 := (\phi - \phi_0)/\varepsilon$  and arrive at the equation

$$\Delta \phi_1 = 8A_1 x + O(\varepsilon), \quad \phi_1(1 + \varepsilon B(\theta), \theta) = -2A_0 B(\theta) + O(\varepsilon).$$

A short computation then shows that  $h := \phi_1 - \frac{4}{3}A_1 x^3$  satisfies

$$\Delta h = O(\varepsilon), \quad h(1 + \varepsilon B(\theta), \theta) = -2A_0 B(\theta) - \frac{4}{3}A_1 \cos^3 \theta + O(\varepsilon).$$

Hence

$$\begin{aligned} h &= -2A_0 \mathbb{P}_{\varepsilon B} B - \frac{4}{3}A_1 \mathbb{P}_{\varepsilon B}(\cos^3 \theta) + O(\varepsilon) \\ &= -2A_0 \mathbb{P}_{\varepsilon B} B - \frac{4}{3}A_1 \mathbb{P}_0(\cos^3 \theta) + O(\varepsilon). \end{aligned}$$

As  $\cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$ , we then have

$$\begin{aligned} \phi_1 &= -2A_0 \mathbb{P}_{\varepsilon B} B + \frac{4A_1}{3} [\rho^3 \cos^3 \theta - \mathbb{P}_0(\cos^3 \theta)] + O(\varepsilon) \\ &= -2A_0 \mathbb{P}_{\varepsilon B} B + \frac{A_1}{3} [\rho^3 (\cos 3\theta + 3 \cos \theta) - (\rho^3 \cos 3\theta + 3\rho \cos \theta)] + O(\varepsilon) \\ &= -2A_0 \mathbb{P}_{\varepsilon B} B + A_1 (\rho^3 - \rho) \cos \theta + O(\varepsilon). \end{aligned}$$

This is the desired expression for  $\phi$ .  $\square$

*Remark 4.2.* Note that we cannot replace  $\mathbb{P}_{\varepsilon B}$  by  $\mathbb{P}_0$  in the formula presented in Proposition 4.1 because, generally,  $\mathbb{P}_0 B$  would only be defined on the unit disk, not on the possibly larger domain  $\{\rho < 1 + \varepsilon B(\theta)\}$ .

For future reference, we record some formulas that stem from Proposition 4.1 and will be useful later on:

$$\begin{aligned} \nabla \phi_{\varepsilon, B} &= [2A_0 \rho + \varepsilon A_1 (3\rho^2 - 1) \cos \theta] e_\rho - 2A_0 \varepsilon \nabla \mathbb{P}_{\varepsilon B} B \\ &\quad + \varepsilon A_1 (\rho^3 - \rho) \nabla \cos \theta + O(\varepsilon^2), \\ |\nabla \phi_{\varepsilon, B}|^2 &= 4A_0^2 \rho^2 + 4\varepsilon A_0 [A_1 (3\rho^3 - \rho) \cos \theta - 2A_0 \rho \partial_\rho \mathbb{P}_{\varepsilon B} B] + O(\varepsilon^2). \end{aligned} \tag{4.1a}$$

Here  $e_\rho := (x/\rho, y/\rho)$  is the unit vector field in the radial direction and we have used that  $e_\rho \cdot \nabla \cos \theta = 0$ .

Eventually we will need to evaluate the above formulas on the boundary of the domain, that is, at  $\rho = 1 + \varepsilon B(\theta)$ . In this direction, recall that the Dirichlet–Neumann map of the disk, defined as

$$\Lambda_0 f(\theta) = \partial_\rho \mathbb{P}_0 f(1, \theta),$$

is the operator  $C^{s+1}(\mathbb{T}) \rightarrow C^s(\mathbb{T})$  given by

$$\Lambda_0 f(\theta) := \sum_{n \in \mathbb{Z}} f_n |n| e^{in\theta} \quad \text{if} \quad f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}.$$

Also, note that the Dirichlet–Neumann map of the domain  $\Omega_{\varepsilon B}$  is an elliptic pseudodifferential operator of first order of the form

$$\Lambda_{\varepsilon B} f := \partial_\rho \mathbb{P}_{\varepsilon B} f|_{\rho=1+\varepsilon B} = \Lambda_0 f + O(\varepsilon), \tag{4.1c}$$

where the above notation can be taken to mean that the  $C^s$  norm of the error is bounded by  $C\varepsilon \|f\|_{C^{s+1}}$ .

## 5. COMPUTING THE VARIATIONS WITH RESPECT TO THE DOMAIN

Our next objective is to compute how  $\phi$  changes as we change the domain by perturbing the function  $B$ . More precisely, we aim to compute the derivative of  $\phi$  with respect to  $B$ , which we will denote as

$$\Phi_{\varepsilon, B, \mathbf{B}} := \left. \frac{\partial}{\partial t} \right|_{t=0} \phi_{\varepsilon, B+t\mathbf{B}},$$

where  $\mathbf{B}(\theta)$  is a function defined on  $\mathbb{T}$ . In this section, we are mainly interested in the derivative at  $B = 0$ ,  $\Phi_{\varepsilon, 0, \mathbf{B}}$ .

In the statement of the next proposition, we will need the operator  $T : C^{s+1}(\mathbb{T}) \rightarrow C^{s+1}(\mathbb{D})$  defined as

$$Tf(\rho, \theta) := \sum_{n \in \mathbb{Z} \setminus \{0\}} f_n (\rho^{|n|-1} - \rho^{|n|+1}) e^{i[n - \text{sign}(n)]\theta}$$

for  $f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}$ . Here and in what follows,  $\text{sign}(n) := n/|n|$  is the sign of the nonzero integer  $n$ .

**Proposition 5.1.** *For  $\mathbf{B} \in C^{s+1}(\mathbb{T})$  and any small enough  $\varepsilon$ ,*

$$\Phi_{\varepsilon, 0, \mathbf{B}} = -2\varepsilon A_0 \mathbb{P}_0 \mathbf{B} + \varepsilon^2 \left[ \frac{A_0}{2R} T\mathbf{B} - 2A_1 \mathbb{P}_0(\cos \theta \mathbf{B}) \right] + O(\varepsilon^3).$$

*Proof.* Differentiating Equation (3.2) with respect to  $B$  at  $B = 0$ , we obtain that  $\Phi \equiv \Phi_{\varepsilon, 0, \mathbf{B}}$  satisfies the equation

$$(5.1) \quad \Delta \Phi - \frac{\varepsilon}{R + \varepsilon x} \partial_x \Phi = \varepsilon^2 \left[ (R + \varepsilon x)^2 H_1''(\varepsilon^2 \phi_{\varepsilon, 0}) - \frac{\varepsilon^2}{2} \tilde{F}_1''(\varepsilon^2 \phi_{\varepsilon, 0}) \right] \Phi$$

in  $\mathbb{D}$ . Likewise, differentiating the boundary condition (3.3) we obtain that

$$\Phi(1, \theta) = -\varepsilon \partial_\rho \phi_{\varepsilon, 0}(1, \theta) \mathbf{B}(\theta).$$

In view of the asymptotics for  $\phi_{\varepsilon, 0}$  computed in Proposition 4.1, this boundary condition can be rewritten as

$$\Phi(1, \theta) = -2\varepsilon A_0 \mathbf{B}(\theta) - 2\varepsilon^2 A_1 \mathbf{B}(\theta) \cos \theta + O(\varepsilon^3),$$

and  $\Phi$  has the expression

$$\Phi = -2\varepsilon A_0 \mathbb{P}_0 \mathbf{B} + O(\varepsilon^2).$$

Equation (5.1) is then of the form

$$\Delta \Phi - \frac{\varepsilon}{R} \partial_x \Phi = O(\varepsilon^2) \Phi + O(\varepsilon^2) \partial_x \Phi = O(\varepsilon^3).$$

Assuming that  $\varepsilon \neq 0$  (since otherwise  $\Phi = 0$ ), let us set

$$\Phi_1 := (\Phi - \varepsilon \Phi_0) / \varepsilon^2,$$

with  $\Phi_0 := -2A_0 \mathbb{P}_0 \mathbf{B}$ . A short calculation shows that  $\Phi_1$  must solve the equation

$$\Delta \Phi_1 = \frac{1}{R} \partial_x \Phi_0 + O(\varepsilon)$$

with the boundary condition

$$\Phi_1(1, \theta) = -2A_1 \mathbf{B}(\theta) \cos \theta + O(\varepsilon).$$

Since

$$\partial_x = \cos \theta \partial_\rho - \frac{1}{\rho} \sin \theta \partial_\theta = \frac{1}{2} e^{i\theta} \left( \partial_\rho + \frac{i}{\rho} \partial_\theta \right) + \frac{1}{2} e^{-i\theta} \left( \partial_\rho - \frac{i}{\rho} \partial_\theta \right),$$

one readily finds that

$$\partial_x \Phi_0 = -2A_0 \sum_{n \in \mathbb{Z} \setminus \{0\}} |n| \mathbf{B}_n \rho^{|n|-1} e^{i[n - \text{sign}(n)]\theta},$$

if  $\mathbf{B} = \sum_{n \in \mathbb{Z}} \mathbf{B}_n e^{in\theta}$ . Let us now note that if we set

$$\Phi_2 := -\frac{A_0}{2R} \sum_{n \in \mathbb{Z} \setminus \{0\}} \mathbf{B}_n \rho^{|n|+1} e^{i[n - \text{sign}(n)]\theta},$$

then  $\Delta\Phi_2 = \frac{\partial_x \Phi_0}{R}$ . Consequently, the function  $\Phi_3 := \Phi_1 - \Phi_2$  satisfies the equation

$$\Delta\Phi_3 = O(\varepsilon)$$

and the boundary condition

$$\begin{aligned} \Phi_3(1, \theta) &= -2A_1\mathbf{B}(\theta) \cos \theta - \Phi_2(1, \theta) + O(\varepsilon) \\ &= -2A_1\mathbf{B}(\theta) \cos \theta + \frac{A_0}{2R} \sum_{n \in \mathbb{Z} \setminus \{0\}} \mathbf{B}_n e^{i[n - \text{sign}(n)]\theta} + O(\varepsilon). \end{aligned}$$

This shows that

$$\begin{aligned} \Phi_3 &= -2A_1\mathbb{P}_0(\mathbf{B} \cos \theta) + \frac{A_0}{2R} \sum_{n \in \mathbb{Z} \setminus \{0\}} \mathbf{B}_n \mathbb{P}_0(e^{i[n - \text{sign}(n)]\theta}) + O(\varepsilon) \\ &= -2A_1\mathbb{P}_0(\mathbf{B} \cos \theta) + \frac{A_0}{2R} \sum_{n \in \mathbb{Z} \setminus \{0\}} \mathbf{B}_n \rho^{|n|-1} e^{i[n - \text{sign}(n)]\theta} + O(\varepsilon), \end{aligned}$$

which results in

$$\Phi_1 = \Phi_2 + \Phi_3 = \frac{A_0}{2R} T\mathbf{B} - 2A_1\mathbb{P}_0(\cos \theta \mathbf{B}) + O(\varepsilon),$$

as claimed.  $\square$

As a consequence of Proposition 5.1, we record that

$$(5.2) \quad \partial_\rho \Phi_{\varepsilon, 0, \mathbf{B}}|_{\rho=1} = -2\varepsilon A_0 \Lambda_0 \mathbf{B} - 2\varepsilon^2 \left[ \frac{A_0}{R} T' \mathbf{B} + A_1 \Lambda_0(\cos \theta \mathbf{B}) \right] + O(\varepsilon^3),$$

where  $T' : C^{s+1}(\mathbb{T}) \rightarrow C^{s+1}(\mathbb{T})$  is the operator defined as

$$(5.3) \quad T' f(\theta) := \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} f_n e^{i[n - \text{sign}(n)]\theta}$$

for  $f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}$ .

## 6. ANALYSIS OF THE NEUMANN CONDITION AND CONCLUSION OF THE PROOF

Let us now set

$$(6.1) \quad \mathcal{F}(\varepsilon, B)(\theta) := |\nabla \phi_{\varepsilon, B}(1 + \varepsilon B(\theta), \theta)|^2 - c_{\varepsilon, B} [R + \varepsilon(1 + \varepsilon B(\theta)) \cos \theta]^2,$$

where the constant  $c_{\varepsilon, B}$  will be defined later. In view of the definition of the function  $F$  (Equation (1.9)), one should notice that the Neumann condition (1.8) holds with a constant  $c = \varepsilon^2 c_{\varepsilon, B}$  if and only if  $\mathcal{F}(\varepsilon, B) + F_R$  is the zero function.

Next we pick the constant  $c_{\varepsilon, B}$  so that  $\mathcal{F}(\varepsilon, B)$  is  $L^2$ -orthogonal to  $\cos \theta$ . The reason for which we do so will be clear later. This amounts to setting

$$(6.2) \quad c_{\varepsilon, B} := \frac{\int_0^{2\pi} |\nabla \phi_{\varepsilon, B}(1 + \varepsilon B(\theta), \theta)|^2 \cos \theta \, d\theta}{\int_0^{2\pi} [R + \varepsilon(1 + \varepsilon B(\theta)) \cos \theta]^2 \cos \theta \, d\theta}.$$

The following result guarantees that this choice of  $c_{\varepsilon, B}$  makes sense for all small enough  $\varepsilon$ , including  $\varepsilon = 0$ , and shows that  $\mathcal{F}(0, B)$  is in fact the constant

$$\kappa := 4A_0(A_0 - A_1R),$$

which depends on  $R$  but not on  $B$ . In what follows, we employ the notation

$$\langle f, g \rangle := \int_0^{2\pi} f(\theta) g(\theta) d\theta$$

for the  $L^2$  product on  $\mathbb{T}$ .

**Proposition 6.1.** *For small enough  $\varepsilon$  and any  $B$ ,*

$$\begin{aligned} c_{\varepsilon, B} &= \frac{4A_0A_1}{R} + O(\varepsilon), \\ \mathcal{F}(\varepsilon, B) &= \kappa + O(\varepsilon), \\ \mathcal{F}(\varepsilon, 0) &= \kappa + O(\varepsilon^2). \end{aligned}$$

*Proof.* Let us assume that  $\varepsilon \neq 0$ . In view of Equation (6.2), let us write  $c_{\varepsilon, B} = c_1/c_2$ , with

$$(6.3a) \quad c_1 := \int_0^{2\pi} |\nabla \phi_{\varepsilon, B}(1 + \varepsilon B(\theta), \theta)|^2 \cos \theta d\theta,$$

$$(6.3b) \quad c_2 := \int_0^{2\pi} [R + \varepsilon(1 + \varepsilon B(\theta)) \cos \theta]^2 \cos \theta d\theta.$$

It follows from the formula for  $|\nabla \phi_{\varepsilon, B}|^2$  derived in (4.1) that

$$\begin{aligned} (6.4) \quad c_1 &= \int_0^{2\pi} [4A_0^2 + 8A_0\varepsilon(A_0(B - \Lambda_0 B) + A_1 \cos \theta)] \cos \theta d\theta + O(\varepsilon^2) \\ &= 8\varepsilon A_0^2 \langle B - \Lambda_0 B, \cos \theta \rangle + 8\varepsilon A_0 A_1 \int_0^{2\pi} \cos^2 \theta d\theta + O(\varepsilon^2) \\ &= 8\pi \varepsilon A_0 A_1 + O(\varepsilon^2), \end{aligned}$$

where we have used that

$$\langle B - \Lambda_0 B, \cos \theta \rangle = \langle B, (1 - \Lambda_0) \cos \theta \rangle = 0$$

for any  $B$  because  $\Lambda_0$  is self-adjoint and  $\Lambda_0(\cos \theta) = \cos \theta$ .

The computation of  $c_2$  is straightforward:

$$(6.5) \quad c_2 = \int_0^{2\pi} [R^2 + 2\varepsilon R \cos \theta] \cos \theta d\theta + O(\varepsilon^2) = 2\pi \varepsilon R + O(\varepsilon^2).$$

This readily implies that  $c_{\varepsilon, B}$  can be defined at  $\varepsilon = 0$  by continuity and yields the formula for  $c_{\varepsilon, B}$  presented in the statement. Also, the above formulas immediately imply that

$$\begin{aligned} \mathcal{F}(\varepsilon, B) &= |\nabla \phi_{\varepsilon, B}(1 + \varepsilon B(\theta), \theta)|^2 - c_{\varepsilon, B} [R + \varepsilon(1 + \varepsilon B(\theta)) \cos \theta]^2 \\ &= 4A_0(A_0 - A_1 R) + O(\varepsilon), \end{aligned}$$

as claimed.

To prove that  $\mathcal{F}(\varepsilon, 0) = \kappa + O(\varepsilon^2)$ , it is convenient to define the ( $R$ -dependent) constant

$$c_3 := \lim_{\varepsilon \rightarrow 0} \frac{c_{\varepsilon, 0} - \frac{4A_0A_1}{R}}{\varepsilon}.$$

A straightforward computation using Equations (4.1) and (6.1) shows that

$$\mathcal{F}(\varepsilon, 0) = \kappa - R^2 c_3 \varepsilon + O(\varepsilon^2).$$

We claim that  $c_3 = 0$ . Indeed, noticing that the previous results imply that

$$(6.6) \quad 2\pi\varepsilon R c_{\varepsilon,0} = \int_0^{2\pi} |\nabla\phi_{\varepsilon,0}(1,\theta)|^2 \cos\theta \, d\theta = 8\pi\varepsilon A_0 A_1 + 2\pi\varepsilon^2 R c_3 + O(\varepsilon^3),$$

to compute  $c_3$  it is enough to obtain the  $\varepsilon^2$ -term of  $|\nabla\phi_{\varepsilon,0}(1,\theta)|^2$ . Recall that the function  $\phi_{\varepsilon,0}$  is the solution to the Equation (3.2) with Dirichlet boundary condition  $\phi_{\varepsilon,0}(1,\theta) = 0$ . According to Proposition 4.1, it is easy to check that the ( $\varepsilon$ -dependent) function  $\phi_2$  defined as

$$\phi_{\varepsilon,0} =: A_0(\rho^2 - 1) + \varepsilon A_1(\rho^2 - 1)\rho \cos\theta + \varepsilon^2 \phi_2,$$

satisfies the boundary value problem

$$\Delta\phi_2 = A_2 + A_3x^2 + A_4y^2 + O(\varepsilon), \quad \phi_2(1,\theta) = 0,$$

for some explicit constants  $A_2, A_3, A_4$  (depending on  $R$  but not on  $\varepsilon$ ) that are not relevant for our purposes. The solution to this problem is therefore of the form

$$\phi_2 = \frac{A_2}{4}(\rho^2 - 1) + \frac{A_3 + A_4}{32}(\rho^4 - 1) + \frac{A_3 - A_4}{24}\rho^2(\rho^2 - 1)\cos 2\theta + O(\varepsilon).$$

Using again Equation (4.1) we obtain that

$$|\nabla\phi_{\varepsilon,0}(1,\theta)|^2 = 4A_0^2 + 8\varepsilon A_0 A_1 \cos\theta + 4\varepsilon^2[A_1^2 \cos^2\theta + A_0 \nabla\phi_2(1,\theta) \cdot e_\rho] + O(\varepsilon^3),$$

where the scalar product  $\nabla\phi_2(1,\theta) \cdot e_\rho$  is given by

$$\nabla\phi_2(1,\theta) \cdot e_\rho = \frac{4A_2 + A_3 + A_4}{8} + \frac{A_3 - A_4}{12} \cos 2\theta + O(\varepsilon).$$

It is then immediate that the  $\varepsilon^2$ -term of  $|\nabla\phi_{\varepsilon,0}(1,\theta)|^2$  does not contribute to the integral in Equation (6.6), thus proving that  $c_3 = 0$  as claimed.  $\square$

It stems from Proposition 6.1 that the function

$$(6.7) \quad \mathcal{G}(\varepsilon, B) := \frac{1}{\varepsilon} [\mathcal{F}(\varepsilon, B) - \kappa]$$

can be defined at  $\varepsilon = 0$  by continuity, so that  $\mathcal{G}(0,0) = 0$ , resulting in a map defined for all  $|\varepsilon| < \varepsilon_0$ , where  $\varepsilon_0$  is some positive constant. A more convenient way of looking at this map, however, is by restricting our attention to those variations of the domain that are even and orthogonal to  $\cos\theta$ . Hence, let us now define, for each non-integer  $s > 2$ , the space

$$X_s := \{f \in C^s(\mathbb{T}) : f(\theta) = f(-\theta), \langle f, \cos\theta \rangle = 0\},$$

and its ball of radius 1,

$$X_s^1 := \{f \in C^s(\mathbb{T}) : \|f\|_{C^s} < 1, f(\theta) = f(-\theta), \langle f, \cos\theta \rangle = 0\}.$$

As  $\langle \mathcal{F}(\varepsilon, B), \cos\theta \rangle = 0$  by the definition of  $c_{\varepsilon,B}$ , and  $\phi_{\varepsilon,B}$  is an even function if  $B$  is (cf. Proposition 3.1), our previous results then immediately imply the following:

**Proposition 6.2.** *Given any  $R > 0$ , there is some  $\varepsilon_0 > 0$  such that the formula (6.7) defines a map*

$$\mathcal{G} : (-\varepsilon_0, \varepsilon_0) \times X_{s+1}^1 \rightarrow X_s.$$

In the following theorem we derive the key property of the map  $\mathcal{G}$ : as its domain consists of the even functions orthogonal to  $\cos\theta$ , we can show that its derivative with respect to  $B$  at certain points is an invertible map:

**Theorem 6.3.** *For any  $R > 0$  such that  $aR^2 - 3b \neq 0$ , the Fréchet derivative*

$$D_B \mathcal{G}(0, 0) : X_{s+1} \rightarrow X_s$$

*is one-to-one.*

*Proof.* It follows from the definition of  $\mathcal{F}$  (Equation (6.1)) and of  $\Phi_{\varepsilon, B, \mathbf{B}}$  that

$$\begin{aligned} D_B \mathcal{F}(\varepsilon, 0) \mathbf{B} &= (2\nabla \phi_{\varepsilon, 0} \cdot \nabla \Phi_{\varepsilon, 0, \mathbf{B}} + \varepsilon \mathbf{B} \partial_\rho |\nabla \phi_{\varepsilon, 0}|^2)|_{\rho=1} \\ &\quad - \mathcal{C}_{\varepsilon, \mathbf{B}} (R + \varepsilon \cos \theta)^2 - 2c_{\varepsilon, 0} \varepsilon^2 (R + \varepsilon \cos \theta) \mathbf{B} \cos \theta, \end{aligned}$$

where the constant  $\mathcal{C}_{\varepsilon, \mathbf{B}}$  is given by the derivative

$$\mathcal{C}_{\varepsilon, \mathbf{B}} := \left. \frac{\partial}{\partial t} \right|_{t=0} c_{\varepsilon, t\mathbf{B}}.$$

We readily obtain from formulas (4.1) and (5.2) that

$$\begin{aligned} (6.8) \quad &(2\nabla \phi_{\varepsilon, 0} \cdot \nabla \Phi_{\varepsilon, 0, \mathbf{B}} + \varepsilon \mathbf{B} \partial_\rho |\nabla \phi_{\varepsilon, 0}|^2)|_{\rho=1} = 8\varepsilon A_0^2 (\mathbf{B} - \Lambda_0 \mathbf{B}) \\ &+ 8\varepsilon^2 A_0 \left[ 4A_1 \mathbf{B} \cos \theta - A_1 \cos \theta \Lambda_0 \mathbf{B} - A_1 \Lambda_0 (\mathbf{B} \cos \theta) - \frac{A_0}{R} T' \mathbf{B} \right] + O(\varepsilon^3). \end{aligned}$$

Since  $\mathcal{C}_{\varepsilon, \mathbf{B}}$  is obviously of order  $O(\varepsilon)$ , cf. Proposition 6.1, it suffices to employ the leading order terms of this expression to arrive at

$$D_B \mathcal{F}(\varepsilon, 0) \mathbf{B} = 8\varepsilon A_0^2 (\mathbf{B} - \Lambda_0 \mathbf{B}) - \mathcal{C}_{\varepsilon, \mathbf{B}} R^2 + O(\varepsilon^2).$$

Hence, in order to compute this derivative modulo an error of order  $\varepsilon^2$  we only need to derive asymptotics for  $\mathcal{C}_{\varepsilon, \mathbf{B}}$ . To do so, we write

$$c_{\varepsilon, t\mathbf{B}} = c_1 / c_2$$

as in (6.3) (where now  $B := t\mathbf{B}$ ) and compute

$$\mathcal{C}_j := \left. \frac{\partial}{\partial t} \right|_{t=0} c_j.$$

Notice that, as we showed in the proof of Proposition 6.1 that  $c_j = O(\varepsilon)$ , we will need to compute  $\mathcal{C}_j$  to order  $O(\varepsilon^2)$ .

Let us start with  $\mathcal{C}_2$ . Since  $c_2 := \langle [R + \varepsilon(1 + t\varepsilon \mathbf{B}) \cos \theta]^2, \cos \theta \rangle$ , it is immediate that

$$\mathcal{C}_2 = 2\varepsilon^2 R \langle \mathbf{B}, \cos^2 \theta \rangle + O(\varepsilon^3).$$

To compute  $\mathcal{C}_1$ , we again employ the formula (6.8), now to second order:

$$\begin{aligned} \mathcal{C}_1 &= \int_0^{2\pi} (2\nabla \phi_{\varepsilon, 0} \cdot \nabla \Phi_{\varepsilon, 0, \mathbf{B}} + \varepsilon \mathbf{B} \partial_\rho |\nabla \phi_{\varepsilon, 0}|^2)|_{\rho=1} \cos \theta \, d\theta \\ &= 8\varepsilon A_0^2 \langle \mathbf{B} - \Lambda_0 \mathbf{B}, \cos \theta \rangle + 8\varepsilon^2 A_0 \left[ 4A_1 \langle \mathbf{B}, \cos^2 \theta \rangle \right. \\ &\quad \left. - A_1 \langle \Lambda_0 \mathbf{B}, \cos^2 \theta \rangle - A_1 \langle \Lambda_0 (\mathbf{B} \cos \theta), \cos \theta \rangle - \frac{A_0}{R} \langle T' \mathbf{B}, \cos \theta \rangle \right] + O(\varepsilon^3) \\ &= 8\varepsilon^2 A_0 \left( 3A_1 \langle \mathbf{B}, \cos^2 \theta \rangle - A_1 \langle \Lambda_0 \mathbf{B}, \cos^2 \theta \rangle - \frac{A_0}{R} \langle T' \mathbf{B}, \cos \theta \rangle \right) + O(\varepsilon^3). \end{aligned}$$

Here we have used that  $\Lambda_0$  is self-adjoint and that  $\Lambda_0(\cos \theta) = \cos \theta$ .

Now we need to compute the scalar products appearing in the two previous formulas in terms of the Fourier coefficients  $\mathbf{B}_n$  (note that  $\mathbf{B}_{-n} = \mathbf{B}_n$  because  $\mathbf{B}$  is even):

$$\begin{aligned}\langle \mathbf{B}, \cos^2 \theta \rangle &= \frac{1}{4} \langle \mathbf{B}, e^{2i\theta} + e^{-2i\theta} + 2 \rangle = \pi(\mathbf{B}_2 + \mathbf{B}_0), \\ \langle \Lambda_0 \mathbf{B}, \cos^2 \theta \rangle &= \frac{1}{4} \langle \mathbf{B}, \Lambda_0(e^{2i\theta} + e^{-2i\theta} + 2) \rangle = 2\pi \mathbf{B}_2, \\ \langle T' \mathbf{B}, \cos \theta \rangle &= \frac{1}{4} \sum_{n \in \mathbb{Z} \setminus \{0\}} \left\langle \mathbf{B}_n e^{i[n - \text{sign}(n)]\theta}, e^{i\theta} + e^{-i\theta} \right\rangle = \pi \mathbf{B}_2.\end{aligned}$$

Using the formulas for  $c_1$  and  $c_2$  derived in the proof of Proposition 6.1, this immediately yields

$$\begin{aligned}\mathcal{C}_{\varepsilon, \mathbf{B}} &= \frac{\mathcal{C}_1}{c_2} - c_{\varepsilon, B} \frac{\mathcal{C}_2}{c_2} \\ &= -\varepsilon \frac{8A_0^2}{R^2} \left( \frac{1}{2} \mathbf{B}_2 - \frac{A_1 R}{A_0} \mathbf{B}_0 \right) + O(\varepsilon^2),\end{aligned}$$

which results in

$$D_B \mathcal{F}(\varepsilon, 0) \mathbf{B} = 8\varepsilon A_0^2 \left( \mathbf{B} - \Lambda_0 \mathbf{B} + \frac{1}{2} \mathbf{B}_2 - \frac{A_1 R}{A_0} \mathbf{B}_0 \right) + O(\varepsilon^2).$$

We are now ready to analyze the differential of  $\mathcal{G}$ , which we have shown to be given by the formula

$$D_B \mathcal{G}(0, 0) \mathbf{B} = 8A_0^2 \left( \mathbf{B} - \Lambda_0 \mathbf{B} + \frac{1}{2} \mathbf{B}_2 - \frac{A_1 R}{A_0} \mathbf{B}_0 \right),$$

understood as a map  $X_{s+1} \rightarrow X_s$ . We recall that, as  $\mathbf{B}$  is an even function orthogonal to  $\cos \theta$ , the Fourier series  $\mathbf{B} = \sum_{n \in \mathbb{Z}} \mathbf{B}_n e^{in\theta}$  can be equivalently written as

$$\mathbf{B}(\theta) = \mathbf{B}_0 + 2 \sum_{n=2}^{\infty} \mathbf{B}_n \cos n\theta.$$

Therefore, the action of the linear elliptic operator  $D_B \mathcal{G}(0, 0)$  is given by

$$D_B \mathcal{G}(0, 0) \mathbf{B} = 8A_0^2 \left[ \left( 1 - \frac{A_1 R}{A_0} \right) \mathbf{B}_0 + \frac{1}{2} \mathbf{B}_2 - 2 \sum_{n=2}^{\infty} (n-1) \mathbf{B}_n \cos n\theta \right].$$

Note that  $A_1 R \neq A_0$  for all  $a, b, R > 0$  such that  $aR^2 - 3b \neq 0$  because

$$A_0 - A_1 R = \frac{3b - aR^2}{16}.$$

This implies that the kernel of the map  $D_B \mathcal{G}(0, 0) : X_{s+1} \rightarrow X_s$  is trivial, and that its range is the whole space  $X_s$ , as claimed.  $\square$

In the following corollary we show that, by the implicit function theorem for Banach spaces, Theorem 6.3 yields the existence of solutions to the overdetermined boundary value problem (1.4)–(1.8) for all small enough  $\varepsilon$  and all  $R$  such that  $aR^2 - 3b > 0$ . In turn, these define piecewise smooth stationary Euler flows of compact support via Lemma 1.1, thereby completing the proof of the main result of the paper (Theorem 1.2). Recall that the constant  $F_R$  appears in the definition of the function  $F$ , cf. Equation (1.9).

**Corollary 6.4.** *Fix any  $R > 0$  such that  $aR^2 - 3b > 0$ . Then, for any small enough  $\varepsilon$  there is a unique  $B \in X_{s+1}$  in a  $C^{s+1}$  neighborhood of 0 such that  $\psi := \varepsilon^2 \phi_{\varepsilon, B}$  satisfies Equation (1.4) in  $\Omega_{R, \varepsilon} := \Omega_{\varepsilon B}$  and the overdetermined boundary conditions (1.7)-(1.8) with  $F_R := -\kappa > 0$  and  $c = \varepsilon^2 c_{\varepsilon, B}$ .*

*Proof.* Since  $\mathcal{G}(0, 0) = 0$ , in view of Theorem 6.3, the implicit function theorem guarantees that if  $|\varepsilon|$  is small enough, there is a unique function  $B$  in a small neighborhood of 0 in  $X_{s+1}^1$  such that

$$\mathcal{G}(\varepsilon, B) = 0.$$

This is equivalent to saying that

$$|\nabla \psi|^2 - \varepsilon^2 c_{\varepsilon, B} r^2 - \varepsilon^2 \kappa = 0$$

on  $\partial\Omega_{\varepsilon B}$ , with  $\psi := \varepsilon^2 \phi_{\varepsilon, B}$ . The assumption that  $F^2(0) = \varepsilon^2 F_R = -\varepsilon^2 \kappa$  then ensures that we have a solution to the overdetermined boundary problem (1.4)-(1.8), as claimed. Observe that the condition  $aR^2 - 3b > 0$  implies that

$$\kappa = \frac{(aR^2 + b)(3b - aR^2)}{16} < 0$$

and hence  $F_R > 0$ . Accordingly, the function  $F(\psi)$  is well defined:

$$F(\psi) = \left( \varepsilon^2 F_R - 2b\psi + O(\psi^2) \right)^{1/2}$$

because  $\psi = O(\varepsilon^2)$  and  $\psi < 0$  in  $\Omega_{\varepsilon B}$  (cf. Proposition 3.1).  $\square$

## 7. DIFFERENT CHOICES FOR THE FUNCTIONS $F$ AND $H$

As we mentioned in the Introduction, for the sake of concreteness we have chosen the functions  $H$  and  $F$  as described in Theorem 1.2. However, the method introduced in this paper is flexible enough to construct compactly supported stationary Euler flows with other choices for the functions  $H$  and  $F$ . To illustrate this additional flexibility, in this section we show how a straightforward modification of the previous computations allows us to prove the following:

**Theorem 7.1.** *Take any non-integer  $s > 2$  and any functions  $\tilde{F}, H \in C^s((-1, 0])$  with*

$$\tilde{F}(0) = \tilde{F}'(0) = 0, \quad H'(0) > 0.$$

*Then the following statements hold:*

- (i) *For each small enough  $\varepsilon > 0$  and any  $R > 0$ , there exists a nontrivial, piecewise  $C^s$ , axisymmetric stationary Euler flow of compact support  $u$  of the form described in Lemma 1.1 for a suitable  $C^{s+1}$  planar domain  $\Omega_{R, \varepsilon}$ .*
- (ii) *The boundary of  $\Omega_{R, \varepsilon}$  is a small deformation of a disk of radius  $\varepsilon$ , given by an equation of the form  $z^2 + (r - R)^2 - \varepsilon^2 = O(\varepsilon^3)$ .*
- (iii) *The functions that define the solution are*

$$F(\psi) := \varepsilon F_R + \tilde{F}(\psi)$$

*and  $H(\psi)$ , where  $F_R$  is the positive constant*

$$(7.1) \quad F_R := \frac{R^2 H'(0)}{4}.$$

(iv) *The function  $\psi$  is of class  $C^{s+1}$  in  $\Omega_{R,\varepsilon}$  up to the boundary, and has the form*

$$\psi = \frac{1}{4}H'(0)R^2\left[(r-R)^2 + z^2 - \varepsilon^2\right] + O(\varepsilon^3).$$

*Moreover,  $F \circ \psi > 0$  and  $H \circ \psi$  are of class  $C^s$  in  $\Omega_{R,\varepsilon}$ . In particular, the vorticity is of class  $C^{s-1}$  up to the boundary.*

*Proof.* Indeed, using the same notation as in Section 3, and noticing that

$$(F^2)'(\psi) = \varepsilon O(\psi),$$

Equation (3.2) takes the form

$$\Delta\phi - \frac{\varepsilon}{R + \varepsilon x}\partial_x\phi = aR^2 + 2aR\varepsilon x + O(\varepsilon^2),$$

where we have defined the constant  $a := H'(0)$ . Notice that this is exactly the same as Equation (3.2) with  $b = 0$ . Repeating all the arguments in Sections 3–6, we obtain the same equations and results as in these sections with  $b = 0$ . In particular, the constant  $\kappa$  in Proposition 6.1 is given by

$$\kappa = -\frac{a^2R^4}{16} < 0,$$

$\mathcal{F}(\varepsilon, 0) = \kappa + O(\varepsilon^2)$ , and the invertibility condition in Theorem 6.3 is simply  $a \neq 0$ . The Neumann boundary condition is then satisfied taking  $F(0) = \varepsilon F_R$ , with  $F_R$  as in Equation (7.1). Notice that  $F(\psi) = \varepsilon F_R + O(\psi^2) = \varepsilon F_R + O(\varepsilon^4) > 0$  in  $\Omega_{R,\varepsilon}$ .  $\square$

#### ACKNOWLEDGEMENTS

M.D.-V. is supported by the grants MTM2016-75897-P, PID2019-105138GB-C21 (AEI/FEDER, Spain) and ED431C 2019/10, ED431F 2020/04 (Xunta de Galicia, Spain), and by the Ramón y Cajal program of the Spanish Ministry of Science. A.E. is supported by the ERC Starting Grant 633152. D.P.-S. is supported by the grants MTM2016-76702-P (MINECO/FEDER) and Europa Excelencia EUR2019-103821 (MCIU). This work is supported in part by the ICMAT–Severo Ochoa grant SEV-2015-0554 and the CSIC grant 20205CEX001.

#### REFERENCES

1. D. Chae, P. Constantin, Remarks on a Liouville-type theorem for Beltrami flows, *Int. Math. Res. Not.* 2015, 10012–10016.
2. A. Choffrut, L. Székelyhidi, Weak solutions to the stationary incompressible Euler equations, *SIAM J. Math. Anal.* 46 (2014) 4060–4074.
3. P. Constantin, J. La, V. Vicol, Remarks on a paper by Gavrillov: Grad–Shafranov equations, steady solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications, *Geom. Funct. Anal.* 29 (2019) 1773–1793.
4. E. Delay, P. Sicbaldi, Extremal domains for the first eigenvalue in a general compact Riemannian manifold, *Discrete Contin. Dyn. Syst.* 35 (2015) 5799–5825.
5. M. Domínguez-Vázquez, A. Enciso, D. Peralta-Salas, Solutions to the overdetermined boundary problem for semilinear equations with position-dependent nonlinearities, *Adv. Math.* 351 (2019) 718–760.
6. L.E. Fraenkel, M.S. Berger, A global theory of steady vortex rings in an ideal fluid, *Acta Math.* 132 (1974) 14–51.
7. A.V. Gavrillov, A steady Euler flow with compact support, *Geom. Funct. Anal.* 29 (2019) 190–197.

8. Q. Jiu, Z. Xin, Smooth approximations and exact solutions of the 3D steady axisymmetric Euler equations, *Comm. Math. Phys.* 287 (2009) 323–350.
9. N. Nadirashvili, Liouville theorem for Beltrami flow, *Geom. Funct. Anal.* 24 (2014) 916–921.
10. F. Pacard, P. Sicbaldi, Extremal domains for the first eigenvalue of the Laplace-Beltrami operator, *Ann. Inst. Fourier* 59 (2009) 515–542.
11. J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.* 43 (1971) 304–318.

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