Received: 2 February 2021 | Revised: 5 July 2021 | Accepted: 13 July 2021

DOI: 10.1002/mma.7698

RESEARCH ARTICLE

WILEY

Homogeneous generalized Einstein manifolds in dimension four

Eduardo García-Río[1] | **Rodrigo Mariño-Villar**[1] | **Maria Elena Vázquez-Abal**[1] | **Ramon Vázquez-Lorenzo**[2]

[1]Faculty of Mathematics, University of Santiago de Compostela, Santiago de Compostela, Spain

[2]Department of Mathematics, I.E.S. de Ribadeo Dionisio Gamallo, Ribadeo, Spain

Correspondence
Eduardo García-Río, Faculty of Mathematics, University of Santiago de Compostela, Santiago de Compostela, Spain.
Email: eduardo.garcia.rio@usc.es

Communicated by: G. Franssens

Funding information
Agencia Estatal de Investigación, Grant/Award Number: PID2019-105138GB-C21; Xunta de Galicia, Grant/Award Number: ED431C 2019/10 and ED431F 2020/04

We classify homogeneous four-dimensional manifolds satisfying a curvature condition which naturally generalizes the Einstein one. Our work leads to new examples of self-similar solutions of the two-loop renormalization group flow.

KEYWORDS

critical metric, Einstein metric, homogeneous space, Ricci soliton

MSC CLASSIFICATION

53C44; 53C30; 58B20

1 | INTRODUCTION

Einstein metrics, being central in Geometry and many Physical applications, have been broadly investigated. They appear naturally as critical points of the total scalar curvature functional $g \mapsto \int_M \tau dvol_g$ when restricting to metrics of constant volume. The classical Gauss–Bonnet theorem asserts that $\int_M \tau dvol_g$ is a topological invariant in dimension two, and hence all metrics are critical in that dimension. An immediate application is that any Riemannian surface (M, g) satisfies the universal curvature identity $\rho = \frac{1}{2}\tau g$, where ρ denotes the Ricci tensor.

Generalizations of the Gauss–Bonnet theorem to higher dimensions provide new universal curvature identities. Indeed, the functional defined by the Gauss–Bonnet integrand $g \mapsto \int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\} dvol_g$ is constant in dimension four, where R denotes the curvature tensor. Hence, any compact four-dimensional Riemannian manifold is critical, and Berger[1] (see also Besse[2]) showed that the curvature identity

$$\check{R} - \frac{\|R\|^2}{4} g = \frac{1}{3}\tau\rho_0 + 2W[\rho_0] \tag{1}$$

holds true for any four-dimensional metric, where $\rho_0 = \rho - \frac{1}{4}\tau g$ is the traceless Ricci tensor, W denotes the conformal Weyl curvature tensor, and the symmetric $(0,2)$-tensor fields $\check{R}$ and $W[\rho_0]$ are given by the contractions $\check{R}_{ij} = R_{i\alpha\beta\gamma} R_j{}^{\alpha\beta\gamma}$

and $W[\rho_0]_{ij} = W_{\alpha ij\beta}\rho_0{}^{\alpha\beta}$, respectively. If (M, g) is Einstein, then $\rho_0 = 0$, and thus Equation (1) shows that $\check{R}$ is a multiple of the metric (although $\|R\|^2$ is not necessarily constant) in dimension four.

The condition $W[\rho_0] = 0$, which is equivalent to $W[\rho] = 0$ since the Weyl tensor is traceless, clearly generalizes Einstein metrics (where $\rho_0 = 0$) and locally conformally flat metrics (where the Weyl tensor vanishes). Further, it follows from (1) that the weakly Einstein $\left(\check{R} = \frac{1}{4}\|R\|^2 g\right)$ and the generalized Einstein ($W[\rho]$=0) conditions are equivalent in dimension four if and only if the scalar curvature vanishes. Besides its own significance, the generalized Einstein condition $W[\rho] = 0$ is also interesting from the point of view of the two-loop renormalization group flow, which is mathematically described by $\frac{\partial}{\partial t} g_t = -2\rho - \frac{\alpha}{2}\check{R}$, where $\alpha \geq 0$ is a constant parameter. It follows from (1) that any four-dimensional generalized Einstein manifold with $\tau \neq 0$ satisfies $\rho - \frac{3}{\tau}\check{R} = \frac{1}{4}\left(1 - 3\frac{\|R\|^2}{\tau}\right) g$ and thus provides self-similar solutions of the RG-2-flow corresponding to the parameter $\alpha = -\frac{12}{\tau}$ whenever the scalar curvature is a negative constant. We refer to Carfora and Guenther and Gimre et al[3,4] for more information on the RG-2 flow.

It was shown by Jensen[5] that any four-dimensional homogeneous Einstein metric is locally symmetric and thus isometric to a (real or complex) space form or to the product of two real space forms with the same sectional curvature. Homogeneous four manifolds satisfying the weakly Einstein condition $\check{R} = \frac{1}{4}\|R\|^2 g$ have been classified in Arias-Marco and Kowalski,[6] showing that in the non-Einstein case, they are homothetic to $\mathbb{S}^2 \times \mathbb{H}^2$ or to the left-invariant metric on $\mathbb{R} \ltimes \mathbb{R}^3$ determined by the Lie algebra

$$[e_4, e_1] = e_1,\ [e_4, e_2] = -e_2,\ [e_4, e_3] = -e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. Locally conformally flat homogeneous manifolds are symmetric due to Takagi[7] and thus isometric to a real space form or a product $\mathbb{R} \times N(c)$, where $N(c)$ is a three-dimensional space form, or a product of two space forms of constant opposite sectional curvature.

The main purpose of this work is to generalize the above results of Jensen and Takagi, classifying four-dimensional homogeneous generalized Einstein manifolds. We show that they are either symmetric (and thus Einstein or locally conformally flat by Lemma 2.1) or they correspond to a left-invariant metric as follows.

Theorem 1.1. *Let (M, g) be a nonsymmetric four-dimensional homogeneous manifold. Then, the tensor field $W[\rho]$ vanishes if and only if (M, g) is homothetic to a semi-direct product $\mathbb{R} \ltimes H^3$ of the Heisenberg group with left-invariant metric determined by the Lie algebra*

$$[e_1, e_2] = e_3,\ [e_4, e_1] = \mu e_1,\ [e_4, e_2] = -\frac{1}{2\mu} e_2,\ [e_4, e_3] = \frac{2\mu^2 - 1}{2\mu} e_3,\ 0 < \mu \leq \frac{1}{\sqrt{2}},$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

Remark 1.2. The metrics in Theorem 1.1 have strictly negative scalar curvature and are self-similar solutions of the RG-2 flow corresponding to $\alpha = -\frac{12}{\tau} = \frac{8\mu^2}{4\mu^4 - 3\mu^2 + 1}$ with $\alpha \in (0, 8]$, and thus physically meaningful.

Remark 1.3. Ricci solitons are not only generalizations of Einstein metrics but also self-similar solutions of the Ricci flow. A left-invariant metric on a Lie group is an algebraic Ricci soliton if $\mathfrak{D} = Ric - \lambda id$ is a derivation of the corresponding Lie algebra. It was shown by Lauret[8] that algebraic Ricci solitons are Ricci solitons and the converse is true in the four-dimensional homogeneous case.[9] Now, a straightforward calculation shows that a left-invariant metric in Theorem 1.1 is an algebraic Ricci soliton if and only if $\mu = \frac{1}{2}$ and $\lambda = -\frac{3}{2}$. In this case, the underlying structure is Kähler, and it corresponds to the only nonsymmetric homogeneous Kähler–Ricci soliton in dimension four.[10]

Remark 1.4. The space of scalar quadratic curvature invariants of a Riemannian manifold is generated by $\{\tau^2, \Delta\tau, \|\rho\|^2, \|R\|^2\}$, where $\Delta\tau$ denotes the Laplacian of the scalar curvature. Due to the Gauss–Bonnet theorem, every functional given by a quadratic curvature invariant in dimension four is equivalent to

$$\mathcal{F}_t : g \mapsto \mathcal{F}_t(g) = \int_M \{\|\rho\|^2 + t\tau^2\} \mathrm{dvol}_g,\ t \in \mathbb{R},$$

with the exception of the functional given by the L^2-norm of the scalar curvature. A four-dimensional metric g is $\mathcal{F}_t$-critical if and only if

$$\Delta\rho - (1+2t)\nabla^2\tau + \frac{1+4t}{2}\Delta\tau g + 2\left(t+\frac{2}{3}\right)\tau\rho_0 - 2\check{\rho}_0 + 2W[\rho] = 0, \tag{2}$$

where $\check{\rho}_{ij} = \rho_{ia}\rho_j{}^a$ and $\check{\rho}_0$ is the traceless part of $\check{\rho}$. A long but straightforward calculation shows that a left-invariant metric in Theorem 1.1 is $\mathcal{F}_t$-critical if and only if $t = -\frac{1}{2}$ and $\mu = \frac{1}{2}$, which corresponds to the special case discussed in Remark 1.3

Remark 1.5. Let $\{e^1, \ldots, e^4\}$ be the dual basis of 1-forms of that in Theorem 1.1 and let

$$E_1^{\pm} = e^1 \wedge e^2 \pm e^3 \wedge e^4,\ E_2^{\pm} = e^1 \wedge e^3 \pm e^4 \wedge e^2,\ E_3^{\pm} = e^1 \wedge e^4 \pm e^2 \wedge e^3$$

be the associated self-dual and anti-self-dual 2-forms. A straightforward calculation shows that $dE_1^{\pm} = \theta_1^{\pm} E_1^{\pm}$, $dE_2^{\pm} = \theta_2 E_2^{\pm}$, and $dE_3^{\pm} = \theta_3^{\pm} E_3^{\pm}$, where the Lee 1-forms $\theta_1^{\pm}$, θ_2, and $\theta_3^{\pm}$ are given by

$$\theta_1^{\pm} = \left(\frac{1}{2\mu} - \mu \mp 1\right) e^4, \qquad \theta_2 = \left(\frac{1}{2\mu} - 2\mu\right) e^4, \qquad \theta_3^{\pm} = \left(\pm\frac{1}{\mu} \mp \mu\right) e^4.$$

Therefore, all of $\{E_1^{\pm}, E_2^{\pm}, E_3^{\pm}\}$ define locally conformally symplectic structures on M. A straightforward calculation shows that a left-invariant 2-form Ω on M is closed if and only if $\Omega = E_2^{\pm}$ with $\mu = \frac{1}{2}$ or $\Omega = E_1^{+}$ with $\mu = \frac{1}{2}\left(\sqrt{3}-1\right)$. The first case corresponds to the Kähler and opposite almost Kähler situation already discussed in Remarks 1.3 and 1.4. In the second case $dE_1^{+} = 0$ and $Ric = diag\left[\sqrt{3}-\frac{3}{2}, -\sqrt{3}-\frac{3}{2}, -\frac{3}{2}, -3\right]$ and $W^{\pm} = diag\left[\mp\frac{1}{2}, \pm\frac{1}{4}\left(1+\sqrt{3}\right), \pm\frac{1}{4}\left(1-\sqrt{3}\right)\right]$.

2 | FOUR-DIMENSIONAL HOMOGENEOUS METRICS

Bérard-Bergery[11] proved that a simply connected four-dimensional homogeneous Riemannian manifold is either symmetric or isometric to a Lie group with a left-invariant metric. Hence, any non-symmetric homogeneous metric is realized on the product Lie groups $SU(2)\times\mathbb{R}$ or $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$, or on the semi-direct products $\mathbb{R}\ltimes E(1,1)$, $\mathbb{R}\ltimes E(2)$, $\mathbb{R}\ltimes H^3$, and $\mathbb{R}\ltimes\mathbb{R}^3$, where $E(1,1)$, $E(2)$, H^3 and $\mathbb{R}^3$ are the Poincaré group, the Euclidean group, the Heisenberg group, and the Abelian group, respectively.

Let (M, g) be a four-dimensional symmetric space and let Ric be the Ricci operator $g(RicX, Y) = \rho(X, Y)$. If Ric has a single eigenvalue, then (M, g) is Einstein and trivially satisfies $W[\rho] = 0$. Assume Ric has two distinct eigenvalues. Then, (M, g) splits isometrically due to the parallelizability of the corresponding eigenspaces. If one of the eigenvalues has multiplicity one, then (M, g) is isometric to a product $\mathbb{R}\times N(c)$, where $N(c)$ is a space of constant curvature. Hence, (M, g) is locally conformally flat. On the other hand, if both eigenvalues have multiplicity two, then (M, g) splits as a product of two surfaces $N_1(c_1)\times N_2(c_2)$. A straightforward calculation now shows that $W[\rho] = 0$ if and only if $c_1^2 = c_2^2$ and thus (M, g) is Einstein or locally conformally flat.

Lemma 2.1. *Let (M, g) be a four-dimensional symmetric space. Then, $W[\rho] = 0$ if and only if either (M, g) is Einstein or locally conformally flat.*

The nonsymmetric situation allows nontrivial examples as shown in Theorem 1.1. The proof follows by a case-by-case analysis of the possible left-invariant metrics on four-dimensional Lie groups. In each case, the condition $W[\rho] = 0$ reduces to a polynomial system on the structure constants. Since these polynomials are rather involved, we make use of the theory of Gröbner basis to obtain "better" polynomials belonging to the ideal generated by the original polynomial system. We refer to Cox et al[12] for an introduction to Gröbner basis and to Decker et al[13] for a computer algebra system supporting our calculations. Calculations and explicit Gröbner basis for the corresponding ideals are available upon request.

3 | THE DIRECT PRODUCTS $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ AND $SU(2)\times\mathbb{R}$

Let $\mathfrak{g} = \mathfrak{g}_3 \times \mathbb{R}$ be a direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$, and let $\langle\cdot,\cdot\rangle_3$ denote its restriction to $\mathfrak{g}_3$. Following the work of Milnor,[14] there exists an orthonormal basis

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{3}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_1\lambda_2\lambda_3 \neq 0$. Moreover, the associated Lie group corresponds to $SU(2)$ (resp., $\widetilde{SL}(2,\mathbb{R})$) if $\lambda_1, \lambda_2, \lambda_3$ are all positive (resp., if any of $\lambda_1, \lambda_2, \lambda_3$ is negative).

Now, take $\mathbf{v}_4$ (not necessarily orthogonal to $\mathfrak{g}_3$) so that $[\mathbf{v}_4, \mathbf{v}_i] = 0$, for all $i = 1, 2, 3$. Finally, set $e_i = \mathbf{v}_i$ and $k_i = \langle \mathbf{v}_4, \mathbf{v}_i \rangle$ ($i = 1, 2, 3$), and normalize the vector $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ so that $\{e_1, \dots, e_4\}$ is an orthonormal basis with brackets given by

$$\begin{aligned}
&[e_1, e_2] = \lambda_3 e_3, \ [e_2, e_3] = \lambda_1 e_1, \ [e_3, e_1] = \lambda_2 e_2, \\
&[e_1, e_4] = \tfrac{1}{R}\{k_3\lambda_2 e_2 - k_2\lambda_3 e_3\}, \ [e_2, e_4] = \tfrac{1}{R}\{k_1\lambda_3 e_3 - k_3\lambda_1 e_1\}, \\
&[e_3, e_4] = \tfrac{1}{R}\{k_2\lambda_1 e_1 - k_1\lambda_2 e_2\}, \ R > 0.
\end{aligned} \tag{4}$$

Lemma 3.1. *Let G be a product $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$. Then, G does not admit any nonsymmetric left-invariant metric with $W[\rho] = 0$.*

Proof. A long but straightforward calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned}
&12R^4 W[\rho]_{11} = \mathfrak{W}_{11}, \ 24R^4 W[\rho]_{12} = \mathfrak{W}_{12}, \ 24R^4 W[\rho]_{13} = \mathfrak{W}_{13}, \\
&24R^3 W[\rho]_{14} = \mathfrak{W}_{14}, \ 12R^4 W[\rho]_{22} = \mathfrak{W}_{22}, \ 24R^4 W[\rho]_{23} = \mathfrak{W}_{23}, \\
&24R^3 W[\rho]_{24} = \mathfrak{W}_{24}, \ 12R^4 W[\rho]_{33} = \mathfrak{W}_{33}, \ 24R^3 W[\rho]_{34} = \mathfrak{W}_{34}, \\
&12R^4 W[\rho]_{44} = \mathfrak{W}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{W}_{11} =& -4(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3)k_1^4 - (\lambda_1-\lambda_3)^3(2\lambda_1+3\lambda_3)k_2^4 \\
&- (\lambda_1-\lambda_2)^3(2\lambda_1+3\lambda_2)k_3^4 - (\lambda_1-\lambda_3)(3\lambda_1+\lambda_3)(2\lambda_2^2-\lambda_3^2-\lambda_2\lambda_3)k_1^2k_2^2 \\
&+ (\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)(\lambda_2-\lambda_3)(\lambda_2+2\lambda_3)k_1^2k_3^2 \\
&- (4\lambda_1^4 - 3\lambda_1^3(\lambda_2+\lambda_3) + 2\lambda_1^2(\lambda_2^2+\lambda_3^2-5\lambda_2\lambda_3) - 6\lambda_2^2\lambda_3^2 + 7\lambda_1\lambda_2(\lambda_2+\lambda_3)\lambda_3)k_2^2k_3^2 \\
&+ (\lambda_2-\lambda_3)^2(3\lambda_1^2 - (\lambda_2-\lambda_3)^2 - 2\lambda_1(\lambda_2+\lambda_3))R^2k_1^2 \\
&- (\lambda_1-\lambda_3)(4\lambda_1^3 + 6\lambda_3^3 - \lambda_1^2(3\lambda_2+2\lambda_3) + \lambda_1(2\lambda_2^2 - 8\lambda_3^2 + 7\lambda_2\lambda_3) - 3\lambda_2^2\lambda_3)R^2k_2^2 \\
&- (\lambda_1-\lambda_2)((2\lambda_1-3\lambda_2)\lambda_3^2 + \lambda_1(7\lambda_2-3\lambda_1)\lambda_3 + 2(\lambda_1-\lambda_2)^2(2\lambda_1+3\lambda_2))R^2k_3^2 \\
&- \{2\lambda_1^4 - 3(\lambda_2^2-\lambda_3^2)^2 - 3\lambda_1^3(\lambda_2+\lambda_3) - \lambda_1^2(\lambda_2-3\lambda_3)(3\lambda_2-\lambda_3) \\
&+ 7\lambda_1(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)\}R^4,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{12} =& \, 2(\lambda_2-\lambda_3)(7\lambda_3^3 - 2\lambda_1\lambda_2^2 - (3\lambda_1+\lambda_2)\lambda_3^2 - \lambda_1\lambda_2\lambda_3)k_1^3k_2 \\
&- 2(\lambda_1-\lambda_3)(2\lambda_1^2\lambda_2 + (3\lambda_2 - 7\lambda_3)\lambda_3^2 + \lambda_1(\lambda_2+\lambda_3)\lambda_3)k_1k_2^3 \\
&- \{(5\lambda_1^2 + 5\lambda_2^2 - 16\lambda_1\lambda_2)\lambda_3^2 + 3(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2)\lambda_3 \\
&+ \lambda_1\lambda_2(\lambda_1^2+\lambda_2^2 - 8\lambda_1\lambda_2)\}k_1k_2k_3^2 \\
&- \{14\lambda_3^4 - 16(\lambda_1+\lambda_2)\lambda_3^3 + 2(\lambda_1^2+\lambda_2^2+9\lambda_1\lambda_2)\lambda_3^2 \\
&+ \lambda_1\lambda_2(\lambda_1^2+\lambda_2^2-8\lambda_1\lambda_2) + \lambda_1\lambda_2(\lambda_1+\lambda_2)\lambda_3\}R^2k_1k_2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{13} =& -2(\lambda_2-\lambda_3)(\lambda_2^2(7\lambda_2-\lambda_3) - \lambda_1(3\lambda_2^2 + 2\lambda_3^2 + \lambda_2\lambda_3))k_1^3k_3 \\
&- \{\lambda_1^3(3\lambda_2+\lambda_3) + \lambda_1^2(\lambda_2-\lambda_3)(5\lambda_2+8\lambda_3) \\
&- \lambda_1(16\lambda_2^2 - \lambda_3^2 - 3\lambda_2\lambda_3)\lambda_3 + \lambda_2(5\lambda_2+3\lambda_3)\lambda_3^2\}k_1k_2^2k_3 \\
&- 2(\lambda_1-\lambda_2)(2\lambda_1^2\lambda_3 - \lambda_2^2(7\lambda_2-3\lambda_3) + \lambda_1\lambda_2(\lambda_2+\lambda_3))k_1k_3^3 \\
&- \{\lambda_1^3\lambda_3 - \lambda_1(16\lambda_2^3 - \lambda_3^3 - 18\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) \\
&+ \lambda_1^2(2\lambda_2^2 - 8\lambda_3^2 + \lambda_2\lambda_3) + 2\lambda_2^2(7\lambda_2^2 + \lambda_3^2 - 8\lambda_2\lambda_3)\}R^2k_1k_3,
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{W}_{14} = & -2(\lambda_2-\lambda_3)^2(7\lambda_2^2+7\lambda_3^2-3\lambda_1(\lambda_2+\lambda_3)+10\lambda_2\lambda_3)k_1^3 \\
& +(\lambda_1-\lambda_3)(14\lambda_3^3-2(\lambda_1-\lambda_2)\lambda_3^2-\lambda_1(3\lambda_1+5\lambda_2)\lambda_2+2(2\lambda_1-5\lambda_2)\lambda_2\lambda_3)k_1k_2^2 \\
& -(\lambda_1-\lambda_2)((3\lambda_1^2-2\lambda_2^2-4\lambda_1\lambda_2)\lambda_3+2(\lambda_1-7\lambda_2)\lambda_2^2+5(\lambda_1+2\lambda_2)\lambda_3^2)k_1k_3^2 \\
& -2((\lambda_2-\lambda_3)^2)(\lambda_1^2+7\lambda_2^2+7\lambda_3^2-8\lambda_1(\lambda_2+\lambda_3)+10\lambda_2\lambda_3)R^2k_1,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{22} = & -(\lambda_2-\lambda_3)^3(2\lambda_2+3\lambda_3)k_1^4-4(\lambda_1-\lambda_3)^2(\lambda_1^2+\lambda_3^2+\lambda_1\lambda_3)k_2^4 \\
& +(\lambda_1-\lambda_2)^3(3\lambda_1+2\lambda_2)k_3^4-(\lambda_1-\lambda_3)(2\lambda_1+\lambda_3)(\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)k_1^2k_2^2 \\
& -\{\lambda_1^2(2\lambda_2^2-6\lambda_3^2+7\lambda_2\lambda_3)+\lambda_2^2(4\lambda_2^2+2\lambda_3^2-3\lambda_2\lambda_3) \\
& -\lambda_1\lambda_2(3\lambda_2^2-7\lambda_3^2+10\lambda_2\lambda_3)\}k_1^2k_3^2 \\
& -(\lambda_1-\lambda_2)(\lambda_1+3\lambda_2)(\lambda_1-\lambda_3)(\lambda_1+2\lambda_3)k_2^2k_3^2 \\
& -(\lambda_2-\lambda_3)(\lambda_1^2(2\lambda_2-3\lambda_3)+2(\lambda_2-\lambda_3)^2(2\lambda_2+3\lambda_3)-\lambda_1\lambda_2(3\lambda_2-7\lambda_3))R^2k_1^2 \\
& -(\lambda_1-\lambda_3)^2(\lambda_1^2-3\lambda_2^2+\lambda_3^2+2\lambda_1\lambda_2-2\lambda_1\lambda_3+2\lambda_2\lambda_3)R^2k_2^2 \\
& +(\lambda_1-\lambda_2)(2(\lambda_1-\lambda_2)^2(3\lambda_1+2\lambda_2)-(3\lambda_1-2\lambda_2)\lambda_3^2+(7\lambda_1-3\lambda_2)\lambda_2\lambda_3)R^2k_3^2 \\
& +\{3\lambda_1^3-\lambda_1^2(4\lambda_2+3\lambda_3)+(\lambda_2-\lambda_3)^2(2\lambda_2+3\lambda_3) \\
& -\lambda_1(\lambda_2^2+3\lambda_3^2-8\lambda_2\lambda_3)\}(\lambda_1-\lambda_2+\lambda_3)R^4,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{23} = & -2(\lambda_1-\lambda_3)(\lambda_1^2(7\lambda_1-3\lambda_2)-2\lambda_2\lambda_3^2-\lambda_1(\lambda_1+\lambda_2)\lambda_3)k_2^3k_3 \\
& -2(\lambda_1-\lambda_2)(7\lambda_1^3-\lambda_1^2(\lambda_2+3\lambda_3)-2\lambda_2^2\lambda_3-\lambda_1\lambda_2\lambda_3)k_2k_3^3 \\
& -(\lambda_1^2(5\lambda_2^2+5\lambda_3^2-16\lambda_2\lambda_3)+3\lambda_1(\lambda_2+\lambda_3)(\lambda_2^2+\lambda_3^2)+\lambda_2(\lambda_2^2+\lambda_3^2-8\lambda_2\lambda_3)\lambda_3)k_1^2k_2k_3 \\
& -\{14\lambda_1^4-16\lambda_1^3(\lambda_2+\lambda_3)+2\lambda_1^2(\lambda_2^2+\lambda_3^2+9\lambda_2\lambda_3) \\
& +\lambda_2(\lambda_2^2+\lambda_3^2-8\lambda_2\lambda_3)\lambda_3+\lambda_1\lambda_2(\lambda_2+\lambda_3)\lambda_3\}R^2k_2k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{24} = & -2(\lambda_1-\lambda_3)^2(7\lambda_1^2+7\lambda_3^2-3\lambda_1\lambda_2+10\lambda_1\lambda_3-3\lambda_2\lambda_3)k_2^3 \\
& -(\lambda_2-\lambda_3)(5\lambda_1^2(\lambda_2+2\lambda_3)+\lambda_1(3\lambda_2^2-2\lambda_3^2-4\lambda_2\lambda_3)+2(\lambda_2-7\lambda_3)\lambda_3^2)k_1^2k_2 \\
& -(\lambda_1-\lambda_2)(14\lambda_1^3-2\lambda_1^2(\lambda_2-\lambda_3)+2\lambda_1(2\lambda_2-5\lambda_3)\lambda_3-\lambda_2(3\lambda_2+5\lambda_3)\lambda_3)k_2k_3^2 \\
& -2(\lambda_1-\lambda_3)^2(7\lambda_1^2+\lambda_2^2+7\lambda_3^2-8\lambda_1\lambda_2+10\lambda_1\lambda_3-8\lambda_2\lambda_3)R^2k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{33} = & (\lambda_2-\lambda_3)^3(3\lambda_2+2\lambda_3)k_1^4+(\lambda_1-\lambda_3)^3(3\lambda_1+2\lambda_3)k_2^4 \\
& -4(\lambda_1-\lambda_2)^2(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)k_3^4+(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2)(\lambda_2-\lambda_3)(\lambda_2+3\lambda_3)k_1^2k_3^2 \\
& +(-4\lambda_3^4+3(\lambda_1+\lambda_2)\lambda_3^3+6\lambda_1^2\lambda_2^2-2(\lambda_1^2+\lambda_2^2-5\lambda_1\lambda_2)\lambda_3^2-7\lambda_1(\lambda_1+\lambda_2)\lambda_2\lambda_3)k_1^2k_2^2 \\
& -(\lambda_1-\lambda_2)(\lambda_1+2\lambda_2)(\lambda_1-\lambda_3)(\lambda_1+3\lambda_3)k_2^2k_3^2 \\
& +(\lambda_2-\lambda_3)(-\lambda_1^2(3\lambda_2-2\lambda_3)+2(3\lambda_2+2\lambda_3)(\lambda_2-\lambda_3)^2+\lambda_1(7\lambda_2-3\lambda_3)\lambda_3)R^2k_1^2 \\
& +(\lambda_1-\lambda_3)(6\lambda_1^3-8\lambda_1^2\lambda_3+(2\lambda_2^2+4\lambda_3^2-3\lambda_2\lambda_3)\lambda_3-\lambda_1(\lambda_2-2\lambda_3)(3\lambda_2-\lambda_3))R^2k_2^2 \\
& -(\lambda_1-\lambda_2)^2((\lambda_1-\lambda_2)^2-3\lambda_3^2+2(\lambda_1+\lambda_2)\lambda_3)R^2k_3^2 \\
& -\{2\lambda_3^4-3(\lambda_1^2-\lambda_2^2)^2-3(\lambda_1+\lambda_2)\lambda_3^3+7(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)\lambda_3 \\
& -(\lambda_1-3\lambda_2)(3\lambda_1-\lambda_2)\lambda_3^2\}R^4,
\end{aligned}
$$

$$\begin{aligned}\mathfrak{W}_{34} =& -2(\lambda_1-\lambda_2)^2(7\lambda_1^2+7\lambda_2^2+10\lambda_1\lambda_2-3(\lambda_1+\lambda_2)\lambda_3)k_3^3\\ &+(\lambda_2-\lambda_3)(5\lambda_1^2(2\lambda_2+\lambda_3)-2\lambda_2^2(7\lambda_2-\lambda_3)-\lambda_1(2\lambda_2^2-3\lambda_3^2+4\lambda_2\lambda_3))k_1^2k_3\\ &-(\lambda_1-\lambda_3)(14\lambda_1^3+2\lambda_1^2(\lambda_2-\lambda_3)-2\lambda_1\lambda_2(5\lambda_2-2\lambda_3)-\lambda_2(5\lambda_2+3\lambda_3)\lambda_3)k_2^2k_3\\ &-2(\lambda_1-\lambda_2)^2(7\lambda_1^2+7\lambda_2^2+\lambda_3^2+10\lambda_1\lambda_2-8(\lambda_1+\lambda_2)\lambda_3)R^2k_3,\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{44} =& \,3(\lambda_2^2-\lambda_3^2)^2k_1^4+3(\lambda_1^2-\lambda_3^2)^2k_2^4+3(\lambda_1^2-\lambda_2^2)^2k_3^4\\ &+(6\lambda_3^4+6\lambda_1^2\lambda_2^2-3(\lambda_1+\lambda_2)^2\lambda_3^2)k_1^2k_2^2+(6\lambda_1^4-3\lambda_1^2(\lambda_2+\lambda_3)^2+6\lambda_2^2\lambda_3^2)k_2^2k_3^2\\ &+3(2\lambda_2^4-\lambda_1^2(\lambda_2^2-2\lambda_3^2)-\lambda_2^2\lambda_3^2-2\lambda_1\lambda_2^2\lambda_3)k_1^2k_3^2\\ &+(\lambda_2-\lambda_3)^2(2\lambda_1^2-(\lambda_2-\lambda_3)^2-\lambda_1(\lambda_2+\lambda_3))R^2k_1^2\\ &-(\lambda_1-\lambda_3)^2(\lambda_1^2-2\lambda_2^2+\lambda_3^2+\lambda_1(\lambda_2-2\lambda_3)+\lambda_2\lambda_3)R^2k_2^2\\ &-(\lambda_1-\lambda_2)^2((\lambda_1-\lambda_2)^2-2\lambda_3^2+(\lambda_1+\lambda_2)\lambda_3)R^2k_3^2\\ &-4\{\lambda_1^4-\lambda_1^3(\lambda_2+\lambda_3)+\lambda_1^2\lambda_2\lambda_3-\lambda_1(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)\\ &+(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3)\}R^4.\end{aligned}$$

Since $\lambda_1\lambda_2\lambda_3\neq 0$, assume $\lambda_1 = 1$ just working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_1}e_i$. Now, $W[\rho_0]$ vanishes if and only if the structure constants in Equation (4) satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$. Let $\mathcal{I}_1 \subset \mathbb{R}[k_1,k_2,k_3,R,\lambda_2,\lambda_3]$ be the ideal generated by the polynomials $\mathfrak{W}_{ij}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and a detailed analysis of that basis shows that the polynomials

$$\begin{aligned}\mathbf{g}_{11} &= R^6\lambda_2^3\lambda_3^4(\lambda_3-1)^3(\lambda_3+1) \qquad \text{and}\\ \mathbf{g}_{12} &= -R^6\lambda_2^3\lambda_3^3(\lambda_3-1)^2(3\lambda_3^2+\lambda_3-2\lambda_2-2)\end{aligned}$$

belong to $\mathcal{G}_1$. Since the zero sets of $\{\mathfrak{W}_{ij}=0\}$ and $\mathcal{I}_1=\langle\mathfrak{W}_{ij}\rangle=\langle\mathcal{G}_1\rangle$ coincide, then necessarily $\lambda_3 = 1$.

Next, we compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1\cup\{\lambda_3-1\}\subset\mathbb{R}[k_1,k_2,k_3,R,\lambda_2,\lambda_3]$ with respect to the lexicographical order, obtaining that the polynomial $\mathbf{g}_{21} = R^6\lambda_2^2(\lambda_2-1)$ belongs to $\mathcal{G}_2$. Hence, $\lambda_1=\lambda_2=\lambda_3=1$ and a straightforward calculation shows that the manifold is locally symmetric, which finishes the proof. □

4 | THE SEMI-DIRECT PRODUCTS $\mathbb{R}\ltimes E(1,1)$ AND $\mathbb{R}\ltimes E(2)$

Let $\mathfrak{g}_3$ be either the Poincaré algebra $\mathfrak{e}(1,1)$ or the Euclidean algebra $\mathfrak{e}(2)$, and let $\mathfrak{g}=\mathbb{R}\ltimes\mathfrak{g}_3$ be a semi-direct extension. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ its restriction to $\mathfrak{g}_3$. Following the work of Milnor,[14] there exists an orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2,\mathbf{v}_3]=\lambda_1\mathbf{v}_1, \qquad [\mathbf{v}_3,\mathbf{v}_1]=\lambda_2\mathbf{v}_2, \qquad [\mathbf{v}_1,\mathbf{v}_2]=0, \tag{5}$$

where $\lambda_1,\lambda_2\in\mathbb{R}$ and $\lambda_1\lambda_2\neq 0$. Moreover, $\mathfrak{g}_3=\mathfrak{e}(2)$ (resp., $\mathfrak{g}_3=\mathfrak{e}(1,1)$) if $\lambda_1\lambda_2>0$ (resp., $\lambda_1\lambda_2<0$). The algebra of derivations of $\mathfrak{g}_3$ is given by

$$der(\mathfrak{g}_3)=\left\{\begin{pmatrix} b & a & c\\ -\frac{\lambda_2}{\lambda_1}a & b & d\\ 0 & 0 & 0\end{pmatrix}; a,b,c,d\in\mathbb{R}\right\}.$$

Let $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ are given by Equation (5) and $\mathfrak{g}=\mathbb{R}\mathbf{v}_4\oplus\mathfrak{g}_3$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, set $k_i=\langle\mathbf{v}_i,\mathbf{v}_4\rangle$, for $i=1,2,3$. Let $\hat{e}_4=\mathbf{v}_4-\sum_i k_i\mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1,\dots,e_4\}$ of $\mathfrak{g}=\mathbb{R}\oplus\mathfrak{g}_3$ so that

$$\begin{aligned}&[e_2,e_3]=\lambda_1e_1, && [e_3,e_1]=\lambda_2e_2,\\ &[e_4,e_1]=\tfrac{1}{R}\left\{be_1-\lambda_2\left(\tfrac{a}{\lambda_1}+k_3\right)e_2\right\}, && [e_4,e_2]=\tfrac{1}{R}\{(a+k_3\lambda_1)e_1+be_2\},\\ &[e_4,e_3]=\tfrac{1}{R}\{(c-k_2\lambda_1)e_1+(d+k_1\lambda_2)e_2\},\ R>0.\end{aligned} \tag{6}$$

Lemma 4.1. *Let G be a semi-direct product $\mathbb{R} \ltimes E(1,1)$ or $\mathbb{R} \ltimes E(2)$. Then G does not admit any non-symmetric left-invariant metric with $W[\rho] = 0$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be a left-invariant metric as described in (6). In order to simplify the notation, we set $\lambda_1 A = a + \lambda_1 k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$. A long but standard calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned}
&12R^4 W[\rho]_{11} = \mathfrak{W}_{11},\ 24R^4 W[\rho]_{12} = \mathfrak{W}_{12},\ 24R^4 W[\rho]_{13} = \mathfrak{W}_{13},\\
&24R^3 W[\rho]_{14} = \mathfrak{W}_{14},\ 12R^4 W[\rho]_{22} = \mathfrak{W}_{22},\ 24R^4 W[\rho]_{23} = \mathfrak{W}_{23},\\
&24R^3 W[\rho]_{24} = \mathfrak{W}_{24},\ 12R^4 W[\rho]_{33} = \mathfrak{W}_{33},\ 12R^3 W[\rho]_{34} = \mathfrak{W}_{34},\\
&12R^4 W[\rho]_{44} = \mathfrak{W}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{W}_{11} =& -(A^2+R^2)^2(2\lambda_1^4 - 3\lambda_2^4 - 3\lambda_1^3\lambda_2 + 7\lambda_1\lambda_2^3 - 3\lambda_1^2\lambda_2^2)\\
&- (A^2(8b^2+4C^2-3D^2) - (4b^2-4C^2+3D^2)R^2)\lambda_1^2\\
&+ (A^2(4b^2-2C^2-D^2) - (8b^2+2C^2+D^2)R^2)\lambda_2^2\\
&+ (A^2+R^2)(4b^2+3C^2-2D^2)\lambda_1\lambda_2 - AbCD(21\lambda_1 - 9\lambda_2)\\
&+ b^2(4C^2-5D^2) - 2(C^2+D^2)(C^2+2D^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{12} =& 10Ab(A^2+R^2)(\lambda_1^3 - \lambda_2^3 - 3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2) + CD(A^2+R^2)(\lambda_1^2+\lambda_2^2-8\lambda_1\lambda_2)\\
&+ Ab(24b^2+13C^2+D^2)\lambda_1 - Ab(24b^2+C^2+13D^2)\lambda_2 + 2CD(9b^2+2(C^2+D^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{13} =& -2AD(A^2+R^2)(7\lambda_2^3 + \lambda_1^2\lambda_2 - 8\lambda_1\lambda_2^2)\\
&+ 3bC(2A^2+3R^2)\lambda_1^2 + 9bC(2A^2-R^2)\lambda_2^2 - 24A^2bC\lambda_1\lambda_2\\
&+ 3AD(8b^2-C^2+2D^2)\lambda_1 - AD(24b^2+5C^2+14D^2)\lambda_2 + 9bC(C^2+D^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{14} =& -2D(A^2+R^2)(7\lambda_2^3 + \lambda_1^2\lambda_2 - 8\lambda_1\lambda_2^2) - 3AbC(\lambda_1^2 - 9\lambda_2^2 + 8\lambda_1\lambda_2)\\
&+ 3(2D^3 + 2b^2D - C^2D)\lambda_1 + D(6b^2 - 5C^2 - 14D^2)\lambda_2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{22} =& (A^2+R^2)^2(3\lambda_1^4 - 2\lambda_2^4 - 7\lambda_1^3\lambda_2 + 3\lambda_1\lambda_2^3 + 3\lambda_1^2\lambda_2^2)\\
&+ (A^2(4b^2-C^2-2D^2) - (8b^2+C^2+2D^2)R^2)\lambda_1^2\\
&- (A^2(8b^2-3C^2+4D^2) - (4b^2+3C^2-4D^2)R^2)\lambda_2^2\\
&+ (A^2+R^2)(4b^2-2C^2+3D^2)\lambda_1\lambda_2 - 3AbCD(3\lambda_1 - 7\lambda_2)\\
&- b^2(5C^2-4D^2) - 2(C^2+D^2)(2C^2+D^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{23} =& 2AC(A^2+R^2)(7\lambda_1^3 - 8\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2)\\
&+ 9bD(2A^2-R^2)\lambda_1^2 + 3bD(2A^2+3R^2)\lambda_2^2 - 24A^2bD\lambda_1\lambda_2\\
&+ AC(24b^2+14C^2+5D^2)\lambda_1 - 3AC(8b^2+2C^2-D^2)\lambda_2 + 9bD(C^2+D^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{24} =& 2C(A^2+R^2)(7\lambda_1^3 - 8\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2) + 3AbD(9\lambda_1^2 - \lambda_2^2 - 8\lambda_1\lambda_2)\\
&+ (14C^3 - 6b^2C + 5CD^2)\lambda_1 - 3C(2(b^2+C^2) - D^2)\lambda_2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{33} =& -(4A^4 + A^2R^2 - 3R^4)(\lambda_1^4 + \lambda_2^4) + 2(A^2+R^2)(2A^2\lambda_1^3\lambda_2 + 2A^2\lambda_1\lambda_2^3 - 3R^2\lambda_1^2\lambda_2^2)\\
&- (A^2(12b^2+C^2-2D^2) - 3(2C^2-D^2)R^2)\lambda_1^2\\
&- (A^2(12b^2-2C^2+D^2) + 3(C^2-2D^2)R^2)\lambda_2^2\\
&+ A^2(24b^2-C^2-D^2)\lambda_1\lambda_2 + 6AbCD(\lambda_1-\lambda_2) + 3(C^2+D^2)^2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{34} =& -A(A^2+R^2)(7\lambda_1^4 + 7\lambda_2^4 - 4\lambda_1^3\lambda_2 - 4\lambda_1\lambda_2^3 - 6\lambda_1^2\lambda_2^2)\\
&- A(12b^2+7C^2-5D^2)\lambda_1^2 - A(12b^2-5C^2+7D^2)\lambda_2^2\\
&+ A(24b^2-C^2-D^2)\lambda_1\lambda_2 - 9bCD(\lambda_1-\lambda_2),
\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{44} =& (3A^2-4R^2)(A^2+R^2)(\lambda_1^4+\lambda_2^4)+2(A^2+R^2)(2R^2\lambda_1^3\lambda_2+2R^2\lambda_1\lambda_2^3-3A^2\lambda_1^2\lambda_2^2)\\ &+(A^2(16b^2+6C^2-3D^2)+(4b^2-C^2+2D^2)R^2)\lambda_1^2\\ &+(A^2(16b^2-3C^2+6D^2)+(4b^2+2C^2-D^2)R^2)\lambda_2^2\\ &-(32A^2b^2+(8b^2+C^2+D^2)R^2)\lambda_1\lambda_2+24AbCD(\lambda_1-\lambda_2)\\ &+(b^2+3(C^2+D^2))(C^2+D^2).\end{aligned}$$

Since $\lambda_1\lambda_2\neq 0$, we work with a homothetic basis $\hat{e}_i = \frac{1}{\lambda_1}e_i$ so that we may assume $\lambda_1 = 1$. The $W[\rho_0]$-tensor field vanishes if and only if the structure constants in Equation (6) satisfy the system of polynomial equations $\{\mathfrak{W}_{ij}=0\}$, where $\mathfrak{W}_{ij}\in\mathbb{R}[A,b,C,D,R,\lambda_2]$. We compute a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{I}_1=\langle\mathfrak{W}_{ij}\rangle$ with respect to the graded lexicographical order and a detailed analysis of that basis shows that the polynomial

$$\mathbf{g}_{11} = D(32b^2+5C^2+5D^2)(9D^4+16b^2D^2+128b^2R^2+9C^2D^2)$$

belongs to $\mathcal{G}_1$. Thus, necessarily $D=0$. Now, we compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1\cup\{D\}\subset\mathbb{R}[A,b,C,D,R,\lambda_2]$ with respect to the graded lexicographical order obtaining that the polynomials

$$\mathbf{g}_{21} = C^2(A^2+C^2+R^2)^2 \text{ and } \mathbf{g}_{22} = Ab^4(\lambda_2-1)$$

belong to $\mathcal{G}_2$. Thus, $C=0$, and we are led to the cases $\lambda_2=1$, $b=0$, or $A=0$. If $\lambda_2=1$ then a straightforward calculation shows the manifold is locally symmetric. If $b=0$ then

$$\begin{aligned}\mathfrak{W}_{11} &= (A^2+R^2)^2(\lambda_2-1)^3(3\lambda_2+2) \qquad \text{and}\\ \mathfrak{W}_{22} &= -(A^2+R^2)^2(\lambda_2-1)^3(2\lambda_2+3).\end{aligned}$$

Since $\lambda_2=1$ was discussed previously, we conclude that $W[\rho_0]$ does not vanish in this case. Finally, if $A=0$, then we have $\mathfrak{W}_{33}=3R^4(\lambda_2^2-1)^2$. Since $\lambda_2=1$ was considered previously, it follows that $\lambda_2=-1$. This leads to $\mathfrak{W}_{11} = -8(b^2-R^2)R^2$, which implies $b=\pm R$ and a standard calculation shows the manifold is Einstein and locally symmetric. This finishes the proof. □

5 | THE SEMI-DIRECT PRODUCT $\mathbb{R}\ltimes H^3$

Let $\mathfrak{g}=\mathbb{R}\ltimes\mathfrak{h}^3$ be a semi-direct product of $\mathbb{R}$ with the Heisenberg algebra $\mathfrak{h}^3$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and let $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ be an orthonormal basis of $\mathfrak{h}^3$ so that

$$[\mathbf{v}_1,\mathbf{v}_2]=\gamma\mathbf{v}_3,\ [\mathbf{v}_2,\mathbf{v}_3]=0,\ [\mathbf{v}_1,\mathbf{v}_3]=0,\qquad \gamma\neq 0.$$

The algebra of derivations of $\mathfrak{h}^3$ with respect to a rotated basis that we also denote by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is given by (see Calviño-Louzao et al.[15])

$$\operatorname{der}(\mathfrak{h}^3)=\left\{\begin{pmatrix} a & c & 0\\ -c & d & 0\\ h & f & a+d\end{pmatrix}; a,c,d,h,f\in\mathbb{R}\right\}.$$

Let $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ where $\operatorname{ad}(e_4)$ is determined by a derivation as above. After normalization, as in the previous sections, there is an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ of $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ where the non-zero Lie brackets are given as follows:

$$\begin{aligned}&[e_1,e_2]=\gamma e_3, && [e_4,e_1]=\tfrac{1}{R}\{ae_1-ce_2+(h+k_2\gamma)e_3\},\\ &[e_4,e_3]=\tfrac{1}{R}(a+d)e_3, && [e_4,e_2]=\tfrac{1}{R}\{ce_1+de_2+(f-k_1\gamma)e_3\},\ R>0.\end{aligned} \tag{7}$$

Lemma 5.1. *Let G be a semi-direct product $\mathbb{R} \ltimes H^3$. Then, G admits a non-Einstein left-invariant metric with $W[\rho] = 0$ if and only if it is homothetic to the left-invariant metric determined by*

$$[e_1, e_2] = e_3,\ [e_1, e_4] = -\mu e_1,\ [e_2, e_4] = \frac{1}{2\mu} e_2,\ [e_3, e_4] = \left(\frac{1}{2\mu} - \mu\right) e_3,$$

with $\mu \in \left(0, \frac{1}{\sqrt{2}}\right]$ and where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

Remark 5.2. Let $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ be two Lie groups with negative scalar curvature τ_1 and τ_2, respectively. For $i = 1, 2$, let $\langle\cdot,\cdot\rangle_i^* = -\tau_i \langle\cdot,\cdot\rangle_i$ so that the scalar curvature of the normalized metric $\langle\cdot,\cdot\rangle_i^*$ is $\tau_i^* = -1$. Now, one has that $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ are homothetic if and only if the normalized metrics $\langle\cdot,\cdot\rangle_i^*$ are isometric. In this case, one has that $\|\rho_1^*\| = \|\rho_2^*\|$ and $\|R_1^*\| = \|R_2^*\|$, or equivalently, $\tau_1^{-2}\|\rho_1\|^2 = \tau_2^{-2}\|\rho_2\|^2$ and $\tau_1^{-2}\|R_1\|^2 = \tau_2^{-2}\|R_2\|^2$. The failure of any of these relations therefore implies that the left-invariant metrics $\langle\cdot,\cdot\rangle_i$ correspond to different homothetical classes.

Now, a standard calculation shows that left-invariant metrics in Lemma 5.1 corresponding to different values of the parameter μ are never homothetical since $\tau = -\frac{3(4\mu^4-3\mu^2+1)}{2\mu^2}$ and $\|R\|^2 = \frac{48\mu^8-40\mu^6+35\mu^4-10\mu^2+3}{4\mu^4}$.

Proof. Let $\langle\cdot,\cdot\rangle$ be a left-invariant metric on $\mathbb{R} \ltimes H^3$ determined by the Lie algebra inner product (7). We use the notation $F = f - k_1\gamma$ and $H = h + k_2\gamma$. A straightforward calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned}
&12R^4 W[\rho]_{11} = \mathfrak{W}_{11},\ 12R^4 W[\rho]_{12} = \mathfrak{W}_{12},\ 24R^4 W[\rho]_{13} = \mathfrak{W}_{13},\\
&12R^3 W[\rho]_{14} = \mathfrak{W}_{14},\ 12R^4 W[\rho]_{22} = \mathfrak{W}_{22},\ 24R^4 W[\rho]_{23} = \mathfrak{W}_{23},\\
&12R^3 W[\rho]_{24} = \mathfrak{W}_{24},\ 6R^4 W[\rho]_{33} = \mathfrak{W}_{33},\quad 8R^3 W[\rho]_{34} = \mathfrak{W}_{34},\\
&12R^4 W[\rho]_{44} = \mathfrak{W}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{W}_{11} =& - 16a^3d - 8a^2d^2 - (5F^2 - 4H^2 + 8\gamma^2R^2)a^2 - 3H^2c^2 - 12F^2d^2\\
&- 12FH(ac + cd) - (4(3F^2 + H^2) - 2\gamma^2R^2)ad\\
&- (F^2 + H^2 + \gamma^2R^2)(4F^2 - 3(H^2 + \gamma^2R^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{12} =& - 20a^2cd + 20acd^2 + 3FH(4a^2 - c^2 + 4d^2) + (2F^2 + 14H^2 - \gamma^2R^2)ac\\
&+ 5FHad - (2(7F^2 + H^2) - \gamma^2R^2)cd + 7FH(F^2 + H^2 + \gamma^2R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{13} =& 24Ha^3 + 12F(a^2c - 2cd^2) + 4H(5a^2d + 3ac^2 - 3c^2d + 3ad^2) - 4Facd\\
&+ H(13F^2 + 10(H^2 + \gamma^2R^2))a + F(4(F^2 + H^2) + \gamma^2R^2)c\\
&+ 3H(2F^2 + 3(H^2 + \gamma^2R^2))d,
\end{aligned}$$

$$\mathfrak{W}_{14} = -3\gamma F(a^2 - 4d^2) + \gamma(12Hac + 14Fad - 3Hcd) + 7\gamma F(F^2 + H^2 + \gamma^2R^2),$$

$$\begin{aligned}
\mathfrak{W}_{22} =& - 16ad^3 - 8a^2d^2 - 12H^2a^2 - 3F^2c^2 + (4F^2 - 5H^2 - 8\gamma^2R^2)d^2\\
&+ 12FH(ac + cd) - (4(F^2 + 3H^2) - 2\gamma^2R^2)ad\\
&+ (F^2 + H^2 + \gamma^2R^2)(3F^2 - 4H^2 + 3\gamma^2R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{23} =& 24Fd^3 + 12H(2a^2c - cd^2) + 4F(3a^2d - 3ac^2 + 3c^2d + 5ad^2) + 4Hacd\\
&+ 3F(3F^2 + 2H^2 + 3\gamma^2R^2)a - H(4(F^2 + H^2) + \gamma^2R^2)c\\
&+ F(10F^2 + 13H^2 + 10\gamma^2R^2)d,
\end{aligned}$$

$$\mathfrak{W}_{24} = -3\gamma H(4a^2 - d^2) - \gamma(3Fac + 14Had - 12Fcd) - 7\gamma H(F^2 + H^2 + \gamma^2R^2),$$

$$\mathfrak{W}_{33} = 4a^3d + 4ad^3 - 6a^2c^2 - 6c^2d^2 + 12ac^2d + 2(F^2 - 2H^2 + \gamma^2R^2)a^2 - 2(2F^2 - H^2 - \gamma^2R^2)d^2 - 9FH(ac - cd) + (F^2 + H^2 - 2\gamma^2R^2)ad - (F^2 + H^2 + \gamma^2R^2)^2,$$

$$\mathfrak{W}_{34} = 4\gamma(a^2c + cd^2) - 8\gamma acd - \gamma FH(a - d) - \gamma(F^2 + H^2)c,$$

$$\mathfrak{W}_{44} = 8a^3d + 8ad^3 + 12a^2c^2 + 16a^2d^2 + 12c^2d^2 - 24ac^2d + (F^2 + 16H^2 + 4\gamma^2R^2)a^2 + 3(F^2 + H^2)c^2 + (16F^2 + H^2 + 4\gamma^2R^2)d^2 + 18FH(ac - cd) + 14(F^2 + H^2)ad + (F^2 + H^2 + \gamma^2R^2)(3(F^2 + H^2) - 4\gamma^2R^2).$$

Note that since $\gamma \neq 0$, one may work with a homothetic basis $\hat{e}_i = \frac{1}{\gamma}e_i$ so that we may assume $\gamma = 1$. Now, $W[\rho_0]$ vanishes if and only if the structure constants in Equation (7) satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, F, H, R]$ be the ideal generated by the polynomials $\mathfrak{W}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded lexicographical order, and we get that the polynomials

$$\begin{aligned} \mathbf{g}_1 &= H^3(F^2 + H^2 + R^2)(d^2 + F^2 + H^2 + R^2), \\ \mathbf{g}_2 &= F^2R^2(F^2 + R^2)(4d^2 + F^2 + R^2) \\ &\quad - H^2R^2(2F^4 + F^2(7H^2 + 6R^2) + 4(H^2 + R^2)(d^2 + H^2 + R^2)) \qquad \text{and} \\ \mathbf{g}_3 &= 4c(a - d)^2 - FH(a - d) - (F^2 + H^2)c \end{aligned}$$

belong to $\mathcal{G}$. From $\mathbf{g}_1$, we get $H = 0$ and hence $\mathbf{g}_2$ leads to $F = 0$. Now, $\mathbf{g}_3$ implies that either $d = a$ or $c = 0$. If $d = a$, then $\mathfrak{W}_{11} = -3(8a^4 + 2a^2R^2 - R^4)$, from where we obtain $a = \pm\frac{R}{2}$ and a standard calculation shows the manifold is Einstein and locally symmetric. Now, if $c = 0$, then $\{\mathfrak{W}_{ij} = 0\}$ reduces to

$$\begin{aligned} \mathfrak{W}_{11} &= 3R^4 - 2a(4ad(2a + d) + (4a - d)R^2), \\ \mathfrak{W}_{22} &= -(4d(a + 2d) - 3R^2)(2ad + R^2), \\ \mathfrak{W}_{33} &= (2(a^2 + d^2) - R^2)(2ad + R^2), \\ \mathfrak{W}_{44} &= 4((a + d)^2 - R^2)(2ad + R^2). \end{aligned}$$

$\mathfrak{W}_{11}$ implies that a must be non-null. Moreover, since $d = a$ was discussed previously, the expressions of $\mathfrak{W}_{33}$ and $\mathfrak{W}_{44}$ easily leads to $d = -\frac{R^2}{2a}$. Thus, we get a non-Einstein manifold with $W[\rho_0] = 0$ and setting $\mu = \frac{a}{R} \neq 0$ the left-invariant metric is given by

$$[e_1, e_2] = e_3, \; [e_1, e_4] = -\mu e_1, \; [e_2, e_4] = \frac{1}{2\mu}e_2, \; [e_3, e_4] = \left(\frac{1}{2\mu} - \mu\right)e_3.$$

Note that the replacement $e_4 \mapsto -e_4$ defines an isometry which interchanges μ and $-\mu$. Hence, one may assume $\mu > 0$ without loss of generality. Moreover, $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, -e_4)$ defines an isometry interchanging μ and $\frac{1}{2\mu}$ which shows that one may restrict the parameter to $\mu \in \left(0, \frac{1}{\sqrt{2}}\right]$, finishing the proof. □

6 | THE SEMI-DIRECT PRODUCT $\mathbb{R} \ltimes \mathbb{R}^3$

Let $\mathfrak{r}^3$ be the Abelian algebra. The corresponding algebra of derivations is $\mathfrak{gl}(3, \mathbb{R})$. For any $D \in \mathfrak{gl}(3, \mathbb{R})$, decomposing it into its symmetric and skew-symmetric part, one has (see Calviño-Louzao et al.[15])

$$\operatorname{der}(\mathfrak{r}^3) = \left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix}; a, b, c, f, h, p \in \mathbb{R} \right\}.$$

The corresponding semi-direct product $\mathbb{R} \oplus \mathfrak{r}^3$ expresses in an orthonormal basis $\{e_1, \ldots, e_4\}$ as

$$\begin{aligned} &[e_4, e_1] = \tfrac{1}{R}(ae_1 + be_2 + ce_3), \quad [e_4, e_2] = \tfrac{1}{R}(-be_1 + fe_2 + he_3), \\ &[e_4, e_3] = \tfrac{1}{R}(-ce_1 - he_2 + pe_3), \; R > 0. \end{aligned} \tag{8}$$

Lemma 6.1. *Let G be a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$. Then, G does not admit any nonsymmetric left-invariant metric with $W[\rho] = 0$.*

Proof. A long but straightforward calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned}&3R^4W[\rho]_{11} = \mathfrak{W}_{11},\ 3R^4W[\rho]_{12} = \mathfrak{W}_{12},\ 3R^4W[\rho]_{13} = \mathfrak{W}_{13},\\ &3R^4W[\rho]_{22} = \mathfrak{W}_{22},\ 3R^4W[\rho]_{23} = \mathfrak{W}_{23},\ 3R^4W[\rho]_{33} = \mathfrak{W}_{33},\\ &3R^4W[\rho]_{44} = \mathfrak{W}_{44},\end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\mathfrak{W}_{11} = a^4 - (f+p)a^3 + (f^2+p^2-fp)a^2 - (f-p)^2(f^2+3h^2+p^2+fp),$$

$$\begin{aligned}\mathfrak{W}_{12} =& -2a^3b + (3f+p)a^2b + (p^2-3f^2)ab + 3h(f-p)ac\\ &+ f(f-p)(2f+p)b - 3hp(f-p)c,\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{13} =& -2a^3c + (f+3p)a^2c + 3h(f-p)ab + (f^2-3p^2)ac\\ &+ 3fh(p-f)b - p(f-p)(f+2p)c,\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{22} =& -a^4 - 3a^2c^2 + pa^3 + 6pac^2 + f^2a^2 - 3p^2c^2\\ &- (f^3-p^3+f^2p)a + (f-p)(f^3+p^3+fp^2),\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{23} =& 3a^2bc - 3(f+p)abc + h(f-p)a^2 + 3fpbc\\ &+ h(f^2-p^2)a - h(f-p)(2f^2+2p^2-fp),\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{33} =& -a^4 - 3a^2b^2 + fa^3 + 6fab^2 + p^2a^2 - 3f^2b^2\\ &+ (f^3-p^3-fp^2)a - (f-p)(f^3+p^3+f^2p),\end{aligned}$$

$$\begin{aligned}\mathfrak{W}_{44} =& a^4 + 3a^2b^2 + 3a^2c^2 - 6fab^2 - 6pac^2 - (2f^2+2p^2-fp)a^2 + 3f^2b^2 + 3p^2c^2\\ &+ fp(f+p)a + (f-p)^2((f+p)^2+3h^2).\end{aligned}$$

Let $\langle\cdot,\cdot\rangle$ be a left-invariant metric on $\mathbb{R} \ltimes \mathbb{R}^3$ determined by the Lie algebra inner product (8). $W[\rho_0]$ vanishes if and only if the structure constants satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$, where $\mathfrak{W}_{ij} \in \mathbb{R}[a,b,c,f,h,p]$. We compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} = \langle \mathfrak{W}_{ij}\rangle$ with respect to the lexicographical order and a detailed analysis of that basis shows that the polynomials

$$\begin{aligned}&\mathbf{g}_1 = h^4(f-p)^3(c^2+h^2+p^2) \qquad \text{and}\\ &\mathbf{g}_2 = -fp(f-p)^3(3h^4 - f^3p - fp^3 - f^2p^2)\end{aligned}$$

belong to $\mathcal{G}$. Hence, we are led to the cases $p=f$, $h = p = 0$, or $h = f = 0$.

If $p=f$ then $\mathfrak{W}_{11} = a^2(a-f)^2$ and $\mathfrak{W}_{44} = (a-f)^2(a^2+3(b^2+c^2)+2af)$. Thus, either $f=a$ or $a = b = c = 0$. In the first case the manifold is Einstein and in both cases the manifold is locally symmetric. If $p \neq f$ and $h = p = 0$ then $\mathfrak{W}_{33} = -(a-f)^2(a^2+3b^2+f^2+af)$ implies $f=a$ and, in that case, $\mathfrak{W}_{22} = -3a^2c^2$. Hence, $f=a$ and $c = 0$, which implies that the manifold is locally symmetric. Finally, if $0 \neq p \neq f$ and $h = f = 0$ then $\mathfrak{W}_{22} = -(a-p)^2(a^2+3c^2+p^2+ap)$ implies $p=a$ and, in that case, $\mathfrak{W}_{33} = -3a^2b^2$. Thus, $p=a$ and $b = 0$, from where it follows that the manifold is locally symmetric, finishing the proof. □

ACKNOWLEDGEMENT

Supported by projects PID2019-105138GB-C21(AEI/FEDER, Spain), and ED431C 2019/10, ED431F 2020/04 (Xunta de Galicia, Spain).

CONFLICT OF INTEREST

This work does not have any conflicts of interest

ORCID

Eduardo García-Río https://orcid.org/0000-0003-1195-1664
Rodrigo Mariño-Villar https://orcid.org/0000-0001-8554-8312
Maria Elena Vázquez-Abal https://orcid.org/0000-0002-2770-5380

REFERENCES

1. Berger M. Quelques formules de variation pour une structure riemannienne. *Ann Sci Éc Norm Super*. 1970;3(4):285-294.
2. Besse AL. *Manifolds all of whose geodesics are closed, Ergebnisse der Mathematik und ihrer Grenzgebiete **93***. Berlin-New York: Springer-Verlag; 1978.
3. Carfora M, Guenther C. Scaling and entropy for the RG-2 flow. *Comm Math Phys*. 2020;378(1):369-399.
4. Gimre K, Guenther C, Isenberg J. A geometric introduction to the two-loop renormalization group flow. *J Fixed Point Theory Appl*. 2013;14(1):3-20.
5. Jensen GR. Homogeneous Einstein spaces of dimension four. *J Differential Geometry*. 1969;3:309-349.
6. Arias-Marco T, Kowalski O. Classification of 4-dimensional homogeneous weakly Einstein manifolds. *Czech Math J*. 2015;65(1):21-59.
7. Takagi H. Conformally flat Riemannian manifolds admitting a transitive group of isometries. *Tohoku Math J*. 1975;27(2):103-110.
8. Lauret J. Ricci soliton homogeneous nilmanifolds. *Math Ann*. 2001;319(4):715-733.
9. Arroyo R, Lafuente R. Homogeneous Ricci solitons in low dimensions. *Int Math Res Not IMRN*. 2015;13:4901-4932.
10. Calviño-Louzao E, García-Río E, Gutiérrez-Rodríguez I, Vázquez-Lorenzo R. Four-dimensional homogeneous Kähler Ricci solitons, *Contemp. Math.*, Amer. Math. Soc., to appear.
11. Bérard-Bergery L. Les spaces homogènes Riemanniens de dimension 4. *Riemannian Geometry Dimension 4 (Paris 1978/1979)*. 1981;3:40-60.
12. Cox D, Little D, O'Shea D. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra, Undergraduate Texts in Mathematics*. Cham: Springer; 2015.
13. Decker W, Greuel GM, Pfister G, Schönemann H. Singular 4–1-0 — A computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2016.
14. Milnor J. Curvatures of left invariant metrics on Lie groups. *Adv Math*. 1976;21(3):293-329.
15. Calviño-Louzao E, Garcıa-Martınez X, Garcıa-Rıo E, Gutiérrez-Rodrıguez I, Vázquez-Lorenzo R. Conformally Einstein and Bach-flat four-dimensional homogeneous manifolds. *J Math Pures Appl*. 2019;130(9):347-374.

How to cite this article: García-Río E, Mariño-Villar R, Vázquez-Abal ME, Vázquez-Lorenzo R. Homogeneous generalized Einstein manifolds in dimension four. *Math Meth Appl Sci*. 2021;44: 14316-14327. https://doi.org/10.1002/mma.7698