UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

DEPARTAMENTO DE MATEMÁTICAS

Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebra

IKBOLJON KARIMJANOV

2017

Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebra

by

IKBOLJON KARIMJANOV

DOCTORAL DISSERTATION

Submitted for the degree of

DOCTOR EN MATEMÁTICAS

en la

UNIVERSIDAD DE SANTIAGO DE COMPOSTELA

Santiago de Compostela, 2017

Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebra

Fdo.: Ikboljon Karimjanov

Memoria para optar al grado de Doctor realizada en el Departamento de Matemáticas en el Programa de Doctorado de Matemáticas, de la Facultad de Matemáticas de la Universidad de Santiago de Compostela, bajo la dirección de los Profesores Dr. Bakhrom Omirov y Dr. Manuel Ladra González.

Santiago de Compostela, a 27 de abril de 2017.

Fdo.: Bakhrom Omirov   Fdo.: Manuel Ladra González

Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebra

AUTORIZACIÓN DEL DIRECTOR/TUTOR DE LA TESIS

Dr. Bakhrom Omirov, Profesor del Institute of Mathematics, National University of Uzbekistan (Uzbekistán) y Dr. Manuel Ladra González, Profesor del Departamento de Matemáticas de la Universidad de Santiago de Compostela, como Directores de la Tesis Doctoral titulada **Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebra**, presentada por D. **Ikboljon Karimjanov**, alumno del programa de Doctorado *Matemáticas*,

AUTORIZAMOS la presentación de la Tesis Doctoral indicada para optar al grado de Doctor por la Universidad de Santiago de Compostela, considerando que reúne los requisitos exigidos en el artículo 34 del reglamento de Estudos de Doutoramento, y que como Directores de la misma no incurre en causas de abstención establecidas en la ley 40/2015.

Santiago de Compostela, a 27 de abril de 2017.

Fdo.: Bakhrom Omirov

Fdo.: Manuel Ladra González

A maioría dos resultados presentados nesta memoria foron obtidos grazas ao financiamento da Consellería de Cultura, Educación e Ordenación Universitaria da Xunta de Galicia, na modalidade de Grupo de Referencia Competitiva, referencia GRC2013-045, incluído cofinanciamento do Fondo Europeo de Desenvolvemento Rexional (FEDER), (DOG 25/10/2013). Tamén queremos agradecer as axudas dos proxectos MTM2013-43687-P e MTM2016-79661-P (incluído cofinanciamento do FEDER) do Ministerio de Economía y Competitividad (España) e Agencia Estatal de Investigación.

Unión Europea
Fondo Europeo de Desarrollo Regional
GRC2013-045 (Xunta de Galicia) con fondos FEDER
Unha maneira de facer EUROPA

Acknowledgements

I must give my high, respectful gratitude to my mentor and advisor Professor Bakhrom Omirov for his complete support, motivation and patience. I wish to express my appreciative thanks to my supervisor Professor Manuel Ladra. Without they guidance and persistent help this thesis would not have been possible.

I am deeply grateful to my parents for upbringing and attention to my education. They have created all conditions for my education and research. A special thanks to my family for support and understanding.

I would like to acknowledge my teachers Acad. Ayupov Shavkat Abdullayevich, Professors Vladimir Ivanovich Chilin, Rasul Nabievich Ganikhodjaev, Ahmadjon Qodirov, Abdug'ulom Tohirov, Rasul Nazarov, Sohibjon Akbarovich Ahmedov and all of Mathematics faculty of the National University of Uzbekistan and Andijan State University of Uzbekistan that gave me knowledge.

In addition I am thankful to Professor Utkir Rozikov, Kamilyam Masutova, Jobir Adashev, Abror Khudoyberdiyev, Sherzod Murodov for their professional advice and all the team of Department of Algebra and functional analysis in Institute of Mathematics in Uzbekistan.

Finally, I thank Xabi García-Martínez and Rafael Fernández Casado for their valuable help and all the team of Department of Algebra, University of Santiago de Compostela.

Contents

Resumen de la Tesis Doctoral (in Spanish)

Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebra

Resumen abreviado:
Clasificación de álgebras de Leibniz con un nilradical dado y con alguna álgebra de Lie correspondiente

En esta tesis se estudia la clasificación de las álgebras de Leibniz con un nilradical dado y con alguna álgebra de Lie correspondiente (el álgebra de Leibniz módulo el ideal generado por los cuadrados de los elementos del álgebra). Para ello aplicamos en álgebras de Leibniz el método de Mubarakzjanov usado para álgebras de Lie. Utilizando dicho método clasificamos las álgebras de Leibniz solubles con nilradical nulo-filiforme, y extendemos dicha clasificación al caso en que el nilradical sea una suma directa de ideales nulo-filiformes y el espacio vectorial complementario del nilradical tenga dimensión uno. También estudiamos las álgebras de Leibniz solubles cuyo nilradical es el álgebra de Lie de las matrices triangulares superiores.

Por otra parte, también estudiamos las álgebras de Leibniz solubles con nilradical filiforme naturalmente graduado. Existen dos clases de álgebras de Leibniz filiformes naturalmente graduadas, que no son de Lie, F_n^1 y F_n^2. En particular, clasificamos las álgebras de Leibniz solubles con nilradical F_n^1 y F_n^2.

La última parte de la tesis está dedicada a la investigación de las álgebras de Leibniz correspondientes a las álgebras de Lie de tipo diamante. En concreto, describimos las álgebras de Leibniz cuyas álgebras de Lie correspondientes son las álgebras de Lie de tipo diamante y con cuatro tipos específicos de módulos indescomponibles. Finalmente encontramos una representación fiel del álgebra de Lie de tipo general de diamante, la cual es isomorfa a una subálgebra del álgebra de Lie simpléctica.

Las álgebras de Leibniz fueron introducidas e investigadas a principios de los años 90 del siglo pasado por J.-L. Loday [37–40]. Anteriormente, tales estructuras algebraicas fueron consideradas por Bloh quien las llamó D-álgebras [13]. Al estudiar propiedades de la homología de álgebras de Lie, Loday observó que la antisimetría del producto no era necesaria para probar la propiedad de derivación definida sobre las cadenas. Esto le motivó a introducir la noción de álgebra de Leibniz por la derecha (equivalentemente, por la izquierda), la cual es un álgebra no asociativa en la que el operador de la multiplicación por la derecha (equivalentemente, por la izquierda) es una derivación. Las álgebras de Leibniz generalizan álgebras de Lie de manera natural.

Es bien sabido que existen tres tipos diferentes de álgebras de Lie: semisimples, solubles, y las que no son ni semisimples ni solubles. Por lo tanto, determinar la clasificación de las álgebras de Lie, en general, equivale a revelar la clasificación de cada uno de estos tres tipos. Sin embargo, se pueden condensar en dos por el Teorema de Levi-Malcev, que es una combinación de los resultados formulados en primer lugar por Levi [34] en 1905, y más tarde por Malcev [41] en 1945: cualquier álgebra de Lie de dimensión finita sobre un cuerpo de característica cero se puede expresar como una suma semidirecta (la descomposición de Levi-Malcev) de una subálgebra semisimple (llamada el factor de Levi) y su radical (su ideal soluble maximal). Esto reduce la tarea de clasificar todas las álgebras de Lie restringiendo la clasificación a las álgebras de Lie semisimples y solubles.

La clasificación de las álgebras de Lie semisimples fue resuelta completamente por el bien conocido Teorema de Cartan: cualquier álgebra de Lie compleja o real semisimple se puede descomponer en una suma directa de ideales que son subálgebras simples las cuales son mutuamente ortogonales con respecto a la forma de Cartan-Killing. Así, el problema de clasificar las álgebras de Lie semisimples es equivalente a clasificar todas las álgebras de Lie simples no isomorfas; y la clasificación de las álgebras de Lie simples ya fue obtenida por Killing, Cartan y otros en la última década del siglo XIX (véase [25]). Por lo tanto, puede admitirse que el problema de la clasificación de las álgebras de Lie semisimples está totalmente resuelto en la actualidad. De hecho, principalmente Killing y Cartan, aunque otros autores también trabajaron en este tema, clasificaron las álgebras de Lie simples en cinco clases diferentes (las denominadas álgebras de Lie clásicas simples): las álgebras pertenecientes al grupo especial lineal, las álgebras ortogonales impares, las álgebras ortogonales pares, las álgebras simplécticas, más cinco álgebras de Lie que no tienen relación entre ellas y que no pertenecen a ninguna de las clases anteriores.

Con respecto a la clasificación de las álgebras de Lie solubles, a pesar de los primeros intentos hecho por Lie [35, 36] y por Bianchi [12], puede decirse que fue Dozias, en 1963, una de los primeros autores que se enfrentaron a este problema en serio: ella clasificó en su tesis doctoral las álgebras de Lie solubles de dimensiones inferiores a 6 sobre el cuerpo de los números reales [28]. En ese mismo año Mubarakzjanov (véase [42–44] y [48]) también clasificó estas álgebras hasta dimensión 6 sobre el cuerpo de los números reales.

Debido parcialmente a las dificultades para obtener una clasificación completa de las álgebras de Lie solubles, algunos autores consideraron la idea de una clasificación de extensiones solubles de ciertas clases de álgebras de Lie. En particular, las más relevantes fueron el análisis de todas las álgebras solubles, no nilpotentes, con un nilradical dado. Así, Rubin y Winternitz iniciaron una línea de investigación ([4, 5, 31, 45, 50–55, 57]) concerniente con la clasificación de las álgebras de Lie solubles con un nilradical dado, como las álgebras de Lie filiformes, las álgebras abelianas (también llamadas de 1-paso), las álgebras de Heisenberg, las álgebras de matrices triangulares estrictamente superiores, y así sucesivamente (para dimensiones arbitrariamente finitas). La investigación de las álgebras de Lie solubles con algunos tipos especiales de nilradical proviene de diferentes problemas de la física y fueron objeto de varios artículos [3, 5, 16, 24, 27], y muchas otras referencias dadas en esos trabajos.

Malcev [41] ya había reducido en 1945 la clasificación de las álgebras de Lie complejas solubles a la clasificación de un subconjunto, las álgebras de Lie nilpotentes. Para ello, Malcev definió un tipo particular de álgebra, que llamó el álgebra escindida, cuya estructura está completamente determinada a partir de su ideal nilpotente maximal (llamado nilradical) y demostró que un álgebra de Lie soluble arbitraria está contenida en una única álgebra escindida minimal. La relación entre un álgebra y sus escisiones le llevó a la construcción de todas las álgebras de Lie solubles con una escisión dada. Mubarakzjanov demostró que la dimensión del álgebra escindida no excede del número de derivaciones nil-independientes del nilradical [42]. Así, de esta manera, la clasificación de todas las álgebras de Lie solubles se había reducido a la clasificación de las álgebras de Lie nilpotentes. Se han realizado muchos progresos en las clasificaciones de las álgebras de Lie nilpotentes. Con respecto a la clasificación de las álgebras de Lie nilpotentes, se han hecho muchos intentos en este tema, y se han publicado muchas listas de álgebras con mayor o menor fortuna.

Durante los últimos 25 años se ha investigado activamente en la teoría de álgebras de Leibniz y se han dedicado numerosos trabajos al estudio de estas álgebras. Ayupov y Omirov clasificaron las álgebras de Leibniz complejas de dimensión 3 en 1999 [8]. Luego, comenzaron a investigar las álgebras de Leibniz nilpotentes. En la actualidad están clasificadas las álgebras de Leibniz nilpotentes cuya dimensión es menor que cinco. La clasificación de las álgebras de Leibniz complejas nilpotentes de dimensión finita ya es un problema complicado. Debido a la falta de antisimetría, el problema de clasificar las álgebras de Leibniz complejas nilpotentes es más difícil.

Se han realizado muchos progresos en el estudio de otras clasificaciones relativas a algunas propiedades particulares de las álgebras de Leibniz nilpotentes. Omirov consideró las álgebras de Leibniz nilpotentes graduadas. En [9], los autores clasificaron las álgebras de Leibniz nulo-filiformes y filiformes naturalmente graduadas. Después, muchos trabajos estuvieron dedicados al estudio de la sucesión característica de las álgebras de Leibniz nilpotentes [19–23].

Las álgebras de Leibniz simples y semisimples fueron definidas por Dzhumadil'daev [1]. En el trabajo [30], los autores investigaron las álgebras de Leibniz semisimples y demostraron que el teorema de escisión (Teorema de Cartan) para álgebras de Leibniz semisimples no es cierto en el caso general.

De hecho, muchos resultados de la teoría de álgebras de Lie se han extendido al caso de álgebras de Leibniz. Por ejemplo, los resultados clásicos sobre subálgebras de Cartan [2, 46] y el teorema de Engel [7] se han determinado para el caso de álgebras de Leibniz.

El análogo de la descomposición de Levi-Malcev para álgebras de Leibniz fue probado por D.W. Barnes [11], que afirma que cualquier álgebra de Leibniz se descompone en una suma semidirecta de su radical soluble y un álgebra de Lie semisimple. La parte semisimple se puede describir a partir de los ideales de Lie simples, por lo tanto, el problema principal de la descripción de las álgebras de Leibniz de dimensión finita consiste en el estudio de las álgebras de Leibniz solubles.

Nuestro objetivo es probar el teorema de Mubarakzjanov para el caso de álgebras de Leibniz. En el siguiente teorema se extiende dicha afirmación para el caso de Leibniz.

Teorema 1.2.2. *Sea R un álgebra de Leibniz soluble y N su nilradical. Entonces la dimensión del espacio vectorial complementario a N no es más grande que el número maximal de derivaciones nil-independientes de N.*

Usando este método, en el siguiente teorema se clasifican las álgebras de Leibniz solubles con nilradical nulo-filiforme.

Teorema 1.2.6. *Sea R un álgebra de Leibniz soluble cuyo nilradical es NF_n. Entonces existe una base $\{e_1, e_2, \dots, e_n, x\}$ del álgebra R tal que la tabla de* *multiplicación de R con respecto a esta base tiene la siguiente forma:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\ [x, e_1] = e_1, & \\ [e_i, x] = -ie_i, & 1 \leq i \leq n. \end{cases}$$

Este fue el primer paso para la investigación de las álgebras de Leibniz solubles con nilradical dado. Además, esta clasificación se extiende al caso en que el nilradical sea una suma directa de ideales nulo-filiformes y el espacio vectorial complementario del nilradical sea de dimensión uno.

En la Sección 1.3 se investigan las álgebras de Leibniz solubles cuyo nilradical es el álgebra de Lie de matrices triangulares superiores. Dado que en el trabajo [55] se estudian las álgebras de Lie solubles con nilradical triangular, nosotros reducimos nuestro estudio a las álgebras de Leibniz que no son de Lie. En el siguiente corolario, se presentan algunas propiedades de la matriz de los operadores de las multiplicaciones por la derecha e izquierda para las álgebras de Leibniz solubles de dimensiones mínimas posibles con nilradical triangular.

Corolario 1.3.3. *Para un álgebra de Leibniz del conjunto $L(n, 1)$, las matrices de los operadores de las multiplicaciones por la izquierda y por la derecha, $A = (a_{ij,pq})$ y $B = (b_{ij,pq})$, tienen las siguientes propiedades:*

(1) El número máximo de elementos fuera de la diagonal de la matriz A es $n-1$;

(2) El número máximo de elementos fuera de la diagonal de la matriz B es $n+1$.

En el Teorema 1.3.4 se demuestra que las álgebras de Leibniz solubles de dimensiones máximas posibles con nilradical triangular son álgebras de Lie. Además, se establece la clasificación de las álgebras de Leibniz solubles de dimensiones bajas con nilradicales triangulares.

Teorema 1.3.4. *Un álgebra de Leibniz soluble del conjunto $L(n, n-1)$ es un álgebra de Lie.*

En el Capítulo 2 consideramos las álgebras de Leibniz solubles con nilradical filiforme naturalmente graduado. En los trabajos [4,52] se estudian las álgebras de Lie solubles con nilradicales filiformes naturalmente graduados. Se establece que la dimensión del espacio vectorial complementario es igual a 1 o 2. En los teoremas siguientes se clasifican las álgebras de Leibniz, que no son de Lie, con nilradicales $n_{n,1}$ y Q_{2n}, y cuyo espacio vectorial complementario tiene dimensión 1.

Teorema 2.1.5. *Cualquier álgebra de Leibniz soluble de dimensión* $(n+1)$ *con nilradical* $n_{n,1}$ *es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$$R_{n+1,1}(0,0,1), \quad R_{n+1,1}(0,1,0), \quad R_{n+1,1}(1,1,0), \quad R_{n+1,1}(1,0,0).$$

Teorema 2.1.6. *Cualquier álgebra de Leibniz soluble de dimensión* $(2n+1)$ *con nilradical* Q_{2n} *es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$$R_{2n+1,1}(0,0,1), \quad R_{2n+1,1}(0,1,0), \quad R_{2n+1,1}(1,1,0), \quad R_{2n+1,1}(1,0,0).$$

En el caso de que el espacio vectorial complementario sea de dimensión 2 se establece que no existen álgebras de Leibniz, que no sean de Lie, con nilradicales $n_{n,1}$ y Q_{2n}.

Es bien conocido que existen dos clases de álgebras de Leibniz filiformes naturalmente graduadas, F_n^1 y F_n^2. En los Teoremas 2.2.2, 2.2.3, 2.3.3 y 2.3.4 se clasifican las álgebras de Leibniz solubles con nilradicales F_n^1 y F_n^2.

Teorema 2.2.2. *Cualquier álgebra de Leibniz soluble de dimensión* $(n+1)$ *con nilradical* F_n^1 *es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$$R_1, \quad R_2(\alpha), \quad R_3, \quad R_4, \quad R_5(\alpha_4, \dots, \alpha_n).$$

Además, el primer parámetro que no se anula $\{\alpha_4, \dots, \alpha_n\}$ *en las álgebras* $R_5(\alpha_4, \dots, \alpha_n)$*, puede ajustar la escala a 1.*

Teorema 2.2.3. *Cualquier álgebra de Leibniz soluble de dimensión* $(n+2)$ *con nilradical* F_n^1 *es isomorfa a un álgebra con la siguiente tabla de multiplicación:*

$$\begin{array}{lll} [e_i, e_1] = e_{i+1}, \quad 2 \le i \le n-1, & [e_1, x] = e_1, & \\ [e_i, y] = e_i, \quad 2 \le i \le n, & [e_i, x] = (i-1)e_i, & 2 \le i \le n, \\ & [x, e_1] = -e_1. & \end{array}$$

Teorema 2.3.3. *Cualquier álgebra de Leibniz soluble de dimensión* $(n+1)$ *con nilradical* F_n^2 *es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$$L_1(\alpha), \quad L_2(\alpha), \quad L_3, \quad L_4(\alpha), \quad L_5(\alpha), \quad L_6(\alpha_3, \alpha_4, \dots, \alpha_n, \lambda, \delta).$$

En el álgebra $L_6(\alpha_3, \alpha_4, \dots, \alpha_n, \lambda, \delta)$ *el primer parámetro que no se anula* $\{\alpha_3, \alpha_4, \dots, \alpha_n, \lambda\}$ *puede ajustar la escala a 1.*

Teorema 2.3.4. *Cualquier álgebra de Leibniz soluble de dimensión* $(n+2)$ *con nilradical* F_n^2 *es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$$L_1 : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n-1, \\ [e_1, x] = e_1, & [x, e_1] = -e_1, \\ [e_2, y] = -[y, e_2] = e_2, & [e_i, x] = (i-1)e_i, \quad 3 \leq i \leq n, \end{cases}$$

$$L_2 : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n-1, \\ [e_1, x] = e_1, & [x, e_1] = -e_1, \\ [e_2, y] = e_2, & [e_i, x] = (i-1)e_i, \quad 3 \leq i \leq n. \end{cases}$$

El último capítulo de la tesis se dedica a la investigación de las álgebras de Leibniz correspondientes a las álgebras de Lie de tipo diamante.

Toda álgebra de Leibniz L que no sea de Lie contiene un ideal no trivial (a partir de ahora denotado por I), que es el subespacio generado por los cuadrados de los elementos del álgebra L. Además, este ideal está contenido en el anulador por la derecha de L, esto es $[L, I] = 0$. Obsérvese también que el ideal I es el ideal minimal con la propiedad de que el álgebra cociente L/I es un álgebra de Lie; esta álgebra cociente se llama la liezación del álgebra de Leibniz L y se dice que es el álgebra de Lie correspondiente al álgebra de Leibniz L.

Un álgebra de Leibniz L con álgebra de Lie correspondiente de tipo diamante $\overline{\mathfrak{D}} = L/I$ se puede descomponer en una suma directa de espacios vectoriales $L = \mathfrak{D} \oplus I$, donde $\mathfrak{D}$ es la preimagen de $\overline{\mathfrak{D}}$ por el homomorfismo natural $\varphi \colon L \to \overline{\mathfrak{D}}$. Claramente, el I ideal puede considerarse como un $\overline{\mathfrak{D}}$-módulo de Leibniz. Teniendo en cuenta que el ideal I está contenido en el anulador por la derecha del álgebra L, las multiplicaciones en L se determinan a partir de los productos $[\mathfrak{D}, \mathfrak{D}]$ y $[I, \overline{\mathfrak{D}}]$. Dado que I es un módulo de Leibniz sobre el álgebra de Lie $\overline{\mathfrak{D}}$, se tiene que el producto $[I, \overline{\mathfrak{D}}]$ corresponde a un elegido $\overline{\mathfrak{D}}$-módulo de Lie a la derecha. Así, el principal problema de la descripción de las álgebras de Leibniz con álgebra de Lie correspondiente $\overline{\mathfrak{D}}$ y con el ideal I elegido por un $\overline{\mathfrak{D}}$-módulo a la derecha específico consiste en identificar el producto $[\mathfrak{D}, \mathfrak{D}]$.

Para un álgebra de Lie dada de tipo diamante $\mathfrak{D}$ de dimensión 4 construimos el así llamado *módulo Fock* sobre $\mathfrak{D}$, el espacio lineal $\mathbb{F}[x]$ de polinomios sobre x ($\mathbb{F}$ denota un cuerpo algebraicamente cerrado de característica cero) con la siguiente acción.

Definición 3.2.1. *El espacio lineal* $\mathbb{F}[x]$ *es llamado el* $\mathfrak{D}$*-módulo Fock si existe una acción* $(\mathbb{F}[x], \mathfrak{D}) \mapsto \mathbb{F}[x]$*, la cual verifica lo siguiente:*

$$\begin{array}{lcl} (p(x), \overline{1}) & \mapsto & p(x), \\ (p(x), \overline{x}) & \mapsto & xp(x), \\ (p(x), \overline{\frac{\partial}{\partial x}}) & \mapsto & \frac{\partial}{\partial x}(p(x)), \\ (p(x), \overline{e}) & \mapsto & -x\frac{\partial(p(x))}{\partial x}. \end{array}$$

para cualquier $p(x) \in \mathbb{F}[x]$.

En el siguiente teorema, se clasifican las álgebras de Leibniz de dimensión infinita correspondientes al módulo de Fock de las álgebras de Lie de tipo diamante.

Teorema 3.2.2. *El álgebra de Leibniz* L *con condiciones* $L/I \cong \mathfrak{D}$*, e* I *es el* $\mathfrak{D}_{\mathbb{C}}$*-módulo de Fock, admite una base*

$$\{\overline{1}, \overline{x}, \overline{\frac{\partial}{\partial x}},\ \overline{e},\ x^t \mid t \in \mathbb{N} \cup \{0\}\}$$

tal que la tabla de multiplicaciones en esta base tiene la siguiente forma:

$$\begin{array}{ll} [\overline{e}, \overline{x}] = \overline{x}, & [\overline{x}, \overline{e}] = -\overline{x}, \\ [\overline{e}, \overline{\frac{\partial}{\partial x}}] = -\overline{\frac{\partial}{\partial x}}, & [\overline{\frac{\partial}{\partial x}}, \overline{e}] = \overline{\frac{\partial}{\partial x}}, \\ [\overline{x}, \overline{\frac{\partial}{\partial x}}] = \overline{1}, & [\overline{\frac{\partial}{\partial x}}, \overline{x}] = -\overline{1}, \\ [x^t, \overline{1}] = x^t, & [x^t, \overline{x}] = x^{t+1}, \\ [x^t, \overline{\frac{\partial}{\partial x}}] = tx^{t-1}, & [x^t, \overline{e}] = -tx^t, \end{array}$$

donde los productos omitidos son iguales a cero.

En la Sección 3.3 estudiamos las álgebras de Leibniz cuya álgebra de Lie correspondiente es el álgebra de Lie de tipo diamante $\mathfrak{D}$ de dimensión cuatro y el ideal I es una de sus representaciones de Leibniz indescomponibles de $\mathfrak{D}$ de dimensión finita, las cuales están descritas en [26]. En este artículo hay cuatro $\mathfrak{D}$-módulos indescomponibles: U_n^1, U_n^2, W_n^1 y W_n^2.

Teorema 3.3.1. *Sea un álgebra de Leibniz compleja arbitraria con el álgebra de Lie correspondiente $\overline{\mathfrak{D}}_{\mathbb{C}}$, y el ideal I asociado definido como U_n^1 $\overline{\mathfrak{D}}_{\mathbb{C}}$-módulo. Entonces admite una base*

$$\{J, P_+, P_-, T, v_0^0, v_{2k}^0, v_{2k-1}^1, v_0^2, v_{2k}^2\}_{k=1,\dots,n/2},$$

donde n es par, y la tabla de multiplicación $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ tiene la siguiente forma:

- $n = 4s$

$$\begin{cases} [J, P_+] = -iP_+, & [J, P_-] = iP_-, \quad [P_+, P_-] = -2iT, \\ [P_+, J] = iP_+, & [P_-, J] = -iP_-, \quad [P_-, P_+] = 2iT + 2\alpha_1 v_{2s}^2, \\ [J, T] = \alpha_1 v_{2s}^2 & [J, J] = \alpha_2 v_{2s}^2, \quad [P_+, P_+] = \alpha_3 v_{2s-2}^2, \\ [P_-, P_-] = \alpha_4 v_{2s+2}^2. & \end{cases}$$

- $n = 4s - 2$

$$\begin{cases} [J, P_+] = -iP_+, & [P_+, J] = iP_+ + 2is\beta_1 v_{2s-2}^2, \\ [J, P_-] = iP_-, & [P_-, J] = -iP_- - 2is\beta_1 v_{2s}^2, \\ [P_+, P_-] = -2iT, & [P_-, P_+] = 2iT + 2\beta_2 v_{2s-1}^1, \\ [J, J] = \beta_1 v_{2s-1}^1, & [J, T] = \beta_2 v_{2s-1}^1, \\ [P_+, P_+] = \beta_3 v_{2s-3}^1, & [P_-, P_-] = \beta_4 v_{2s+1}^1, \\ [P_+, T] = 2is\beta_2 v_{2s-2}^2, & [T, P_+] = -i(2s\beta_2 - (s-1)\beta_3) v_{2s-2}^2, \\ [P_-, T] = -2is\beta_2 v_{2s}^2, & [T, P_-] = i(4s\beta_2 - (s-1)\beta_4) v_{2s}^2. \end{cases}$$

donde $\alpha_i,\ \beta_i \in \mathbb{C},\ 1 \le i \le 4.$

Teorema 3.3.2. *Sea un álgebra de Leibniz compleja arbitraria con el álgebra de Lie correspondiente $\overline{\mathfrak{D}}_{\mathbb{C}}$, y el ideal I asociado definido como U_n^2 $\overline{\mathfrak{D}}_{\mathbb{C}}$-módulo de Leibniz. Entonces admite una base*

$$\{J, P_+, P_-, T, v_{2k-1}^0, v_0^1, v_{2k}^1, v_{2k-1}^2\}_{k=1,\dots,n/2},$$

donde n es par, y la tabla de multiplicación $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ tiene la siguiente forma:

- $n = 4s$

$$\begin{cases} [J,P_+] = -iP_+, \\ [P_+,J] = iP_+ + i(2s+1)\gamma_1 v^2_{2s-1}, \\ [J,P_-] = iP_-, \\ [P_-,J] = -iP_- - i(2s+1)\gamma_1 v^2_{2s+1}, \\ [P_+,P_-] = -2iT, \\ [P_-,P_+] = 2iT + 2\gamma_2 v^1_{2s}, \\ [J,J] = \gamma_1 v^1_{2s}, \\ [J,T] = \gamma_2 v^1_{2s}, \\ [P_+,P_+] = \gamma_3 v^1_{2s-2}, \\ [P_-,P_-] = \gamma_4 v^1_{2s+2}, \\ [P_+,T] = i(2s+1)\gamma_2 v^2_{2s-1}, \\ [T,P_+] = -i((2s+1)\gamma_2 - \dfrac{(2s-1)\gamma_3}{2})v^2_{2s-1}, \\ [P_-,T] = -i(2s+1)\gamma_2 v^2_{2s+1}, \\ [T,P_-] = i(2(2s+1)\gamma_2 - \dfrac{(2s-1)\gamma_4}{2})v^2_{2s+1}, \end{cases}$$

- $n = 4s-2$

$$\begin{cases} [J,P_+] = -iP_+, & [J,P_-] = iP_-, & [P_+,P_-] = -2iT, \\ [P_+,J] = iP_+, & [P_-,J] = -iP_-, & [P_-,P_+] = 2iT + 2\delta_1 v^2_{2s-1}, \\ [J,T] = \delta_1 v^2_{2s-1} & [J,J] = \delta_2 v^2_{2s-1}, & [P_+,P_+] = \delta_3 v^2_{2s-3}, \\ [P_-,P_-] = \delta_4 v^2_{2s+1}, \end{cases}$$

donde $\gamma_i,\ \delta_i \in \mathbb{C},\ 1 \leq i \leq 4.$

Teorema 3.3.4. *Sea L un álgebra de Leibniz compleja con el álgebra de Lie correspondiente de tipo diamante $\overline{\mathfrak{D}}_{\mathbb{C}}$, y el ideal I asociado definido como un $\mathfrak{D}_{\mathbb{C}}$-módulo de Leibniz por las representaciones indescomponibles W^1_n o W^2_n.*

Entonces $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ *tiene la siguiente forma:*

$$
\begin{aligned}
[J, P_+] &= -[P_+, J] = -iP_+,\\
[J, P_-] &= -[P_-, J] = iP_-,\\
[P_+, P_-] &= -[P_-, P_+] = -2iT.
\end{aligned}
$$

Según el teorema de Ado, dada cualquier álgebra de Lie compleja $\mathfrak{g}$ de dimensión finita, existe un álgebra de matrices isomorfa a $\mathfrak{g}$. De esta manera, toda álgebra de Lie compleja de dimensión finita puede representarse como una subálgebra de Lie del álgebra lineal general compleja $\mathfrak{gl}(n, \mathbb{C})$, formada por todas las matrices complejas $n \times n$, para algún $n \in \mathbb{N}$. Nosotros consideramos el siguiente invariante valorado entero de $\mathfrak{g}$:

$$\mu(\mathfrak{g}) = \min\{\, \dim(M) \mid M \text{ es un } \mathfrak{g}\text{-módulo fiel}\}.$$

Se deduce de la demostración del teorema de Ado que $\mu(\mathfrak{g})$ puede acotarse por una función que depende solo de n. Este valor también es igual al valor minimal n tal que $\mathfrak{gl}(\mathbb{C}, n)$ contiene una subálgebra isomorfa a $\mathfrak{g}$:

$$\widehat{\mu}(\mathfrak{g}) = \min\{n \in \mathbb{N} \mid \exists \text{ subálgebra de } \mathfrak{gl}(\mathbb{C}, n) \text{ isomorfa a } \mathfrak{g}\}.$$

Dada un álgebra de Lie $\mathfrak{g}$, una representación de $\mathfrak{g}$ in $\mathbb{C}^n$ es un homomorfismo de álgebras de Lie $f\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(\mathbb{C}, n)$. El entero natural n se llama la dimensión de esta representación. Consideramos representaciones fieles porque tales representaciones nos permiten identificar un álgebra de Lie dada con su imagen por la representación, que es una subálgebra de Lie de $\mathfrak{gl}(\mathbb{C}, n)$. Las representaciones también se pueden definir usando espacios vectoriales arbitrarios V de dimensión n (véase [29]). En tal caso, una representación sería un homomorfismo de álgebras de Lie de $\mathfrak{g}$ al álgebra de Lie $\mathfrak{gl}(V)$ de endomorfismos del espacio vectorial V, el cual es llamado un $\mathfrak{g}$-módulo. Sin embargo, basta con considerar representaciones en $\mathbb{C}^n$ porque siempre existe un $n \in \mathbb{N}$ tal que V es isomorfo a $\mathbb{C}^n$.

Muchos trabajos se dedican a encontrar el valor $\mu(\mathfrak{g})$ para varias álgebras de Lie de dimensión finita. En [17], se encuentra el valor de $\mu(\mathfrak{g})$ para álgebras de Lie abelianas y álgebras de Heisenberg, y además, se estima el valor de $\mu(\mathfrak{g})$ para álgebras de Lie filiformes.

En la Sección 3.4 encontramos la representación fiel minimal del álgebra de Lie general compleja de tipo diamante $\mathfrak{D}_m(\mathbb{C})$ de dimensión $(2m+2)$, la cual es isomorfa a una subálgebra del álgebra de Lie lineal especial $\mathfrak{sl}(m+2,\mathbb{C})$. Luego se construyen álgebras de Leibniz con el álgebra de Lie correspondiente de tipo general de diamante y con el ideal generado por los cuadrados de los elementos en esta representación fiel.

Proposición 3.4.1. *Sea $\mathfrak{D}_m(\mathbb{C})$ un álgebra de Lie de tipo general de diamante de dimensión $(2m+2)$ con la base*

$$\{J, P_1^+, P_2^+, \ldots, P_m^+, Q_1^-, Q_2^-, \ldots, Q_m^-, T\}.$$

Entonces su representación fiel minimal está dada por la correspondencia

$$\varphi : \theta J + \sum_{k=1}^{m} \alpha_k P_k^+ + \sum_{k=1}^{m} \beta_k Q_k^- + \delta T \longmapsto$$
$$\begin{pmatrix} \frac{im}{m+2}\theta & \alpha_m & \alpha_{m-1} & \ldots & \alpha_2 & \alpha_1 & -\frac{i}{2}\delta \\ 0 & -\frac{2i}{m+2}\theta & a_1 & \ldots & 0 & 0 & \beta_m \\ 0 & 0 & -\frac{2i}{m+2}\theta & \ldots & 0 & 0 & \beta_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\frac{2i}{m+2}\theta & a_1 & \beta_2 \\ 0 & 0 & 0 & \ldots & 0 & -\frac{2i}{m+2}\theta & \beta_1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \frac{im}{m+2}\theta \end{pmatrix}.$$

Teorema 3.4.2. *Sea L un álgebra de Leibniz arbitraria con el álgebra de Lie correspondiente $\mathfrak{D}_m(\mathbb{C})$ y el ideal I asociado definido como un $\mathfrak{D}_m(\mathbb{C})$-módulo. Entonces existe una base*

$$\{J, P_1^+, P_2^+, \ldots, P_m^+, Q_1^-, Q_2^-, \ldots, Q_m^-, T, X_1, X_2, \ldots, X_{m+2}\}$$

de L tal que

$$[\mathfrak{D}_m(\mathbb{C}), \mathfrak{D}_m(\mathbb{C})] \subseteq \mathfrak{D}_m(\mathbb{C}).$$

Finalmente, en la Sección 3.5 encontramos una representación fiel de $\mathfrak{D}_m(\mathbb{R})$ que es isomorfa a una subálgebra del álgebra de Lie simpléctica $\mathfrak{sp}(2m+2,\mathbb{R})$. También investigamos las álgebras de Leibniz construidas por esta representación de álgebras de Lie de tipo general de diamante.

Proposición 3.5.1. *Sea $\mathfrak{D}_m(\mathbb{R})$ un álgebra de Lie real de tipo general de diamante de dimensión $(2m+2)$ con la base*

$$\{J, P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m, T\}.$$

Entonces es isomorfa a una subálgebra de $\mathfrak{sp}(2m+2, \mathbb{R})$ vía la aplicación

$$\psi : aJ + \sum_{k=1}^{m} b_k P_k + \sum_{k=1}^{m} c_k Q_k + dT \longmapsto$$

$$\left(\begin{array}{ccccc|ccccc}
0 & b_1 & b_2 & \ldots & b_m & c_m & \ldots & c_2 & c_1 & 2d \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & -a & c_1 \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & -a & 0 & c_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -a & \ldots & 0 & 0 & c_m \\
\hline
0 & 0 & 0 & \ldots & a & 0 & \ldots & 0 & 0 & -b_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & a & \ldots & 0 & 0 & \ldots & 0 & 0 & -b_2 \\
0 & a & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & -b_1 \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0
\end{array}\right).$$

Teorema 3.5.2. *Sea un álgebra de Leibniz real arbitraria con el álgebra de Lie correspondiente $\mathfrak{D}_m$, y el ideal I asociado definido como un $\mathfrak{D}_m$-módulo. Entonces admite una base*

$$\{J, P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m, T, X_1, X_2, \ldots, X_{2m+2}\}$$

tal que la tabla de multiplicación $[\mathfrak{D}_m, \mathfrak{D}_m]$ tiene la siguiente forma:

$$\left\{\begin{array}{ll}
[J, J] = a_1 X_{2m+2}, & [J, P_k] = -[P_k, J] = Q_k, \\
[J, Q_k] = -[Q_k, J] = -P_k, & [P_k, Q_k] = -[Q_k, P_k] = T, \\
[P_k, P_s] = [Q_k, Q_s] = b_{k,s} X_{2m+2}, & [P_k, Q_s] = [Q_k, P_s] = c_{k,s} X_{2m+2},
\end{array}\right.$$

con las restricciones

$$b_{k,s} = -b_{s,k}, \qquad c_{k,s} = c_{s,k}, \quad 1 \leq k, s \leq m, \ k \neq s.$$

Introduction

Leibniz algebras were introduced and investigated at the beginning of the 90s of the past century by J.-L. Loday [37–40]. Earlier, such algebraic structures were considered by Bloh who called them D-algebras [13]. In studying the properties of the homology of Lie algebras, Loday observed that the antisymmetry of the product was not needed to prove the derivation property defined on chains. This motivated him to introduce the notion of right (equivalently, left) Leibniz algebra, which is a nonassociative algebra with the right (equivalently, left) multiplication operator being a derivation. Leibniz algebras generalize Lie algebras in natural way.

It is already well known, there exist three different types of Lie algebras: the semisimple, the solvable, and those which are neither semisimple nor solvable. So, determining the classification of Lie algebras, in general, is equivalent to revealing the classification of each of these three types. However, by the Levi-Malcev Theorem, which is a combination of the results formulated firstly by Levi [34] in 1905, and later by Malcev [41] in 1945: any finite-dimensional Lie algebra over a field of characteristic zero can be expressed as a semidirect sum (the Levi-Malcev decomposition) of a semisimple subalgebra (called the Levi factor) and its radical (its maximal solvable ideal). It reduces the task of classifying all the Lie algebras to obtaining the classification of the semisimple and of solvable Lie algebras.

The classification of semisimple Lie algebras was completely solved by the well-known Cartan Theorem: any semisimple complex or real Lie algebra can be decomposed into a direct sum of ideals which are simple subalgebras being mutually orthogonal with respect to the Cartan-Killing form. So, the problem of classifying semisimple Lie algebras is then equivalent to that of classifying all non-isomorphic simple Lie algebras; and the classification of simple Lie algebras was already obtained by Killing, Cartan and others in the last decade of the 19$^{\text{th}}$ century (see [25]). Hence, it can be admitted that the problem of the classification of semisimple Lie algebras is at present totally solved. Indeed, mainly Killing and Cartan, although other authors also worked in this subject, classified simple Lie algebras in five different classes (the so-called simple classical Lie algebras): the algebras belonging to the linear special group, those odd orthogonal algebras, the even orthogonal algebras, the symplectic algebras, plus five Lie algebras having no relation among them and not belonging to any of the previous classes.

With respect to the classification of solvable Lie algebras, in spite of the first attempts by Lie [35, 36] and Bianchi [12], it can be said that Dozias was one of the first authors who faced this problem seriously, in 1963: she classified in her Ph.D. thesis the solvable Lie algebras of dimensions less than 6 over the field of the real numbers [28]. In that same year Mubarakzjanov (see [42–44] and [48] too) also classified these algebras up to dimension 6 over the field of the real numbers.

Partially due to the difficulties of obtaining a complete classification of the solvable Lie algebras, some authors considered the idea of a classification of solvable extensions of certain classes of Lie algebras. In particular, the most relevant were the analysis of all solvable, non-nilpotent algebras with a given nilradical. So, Rubin and Winternitz started a research line ([4,5,31,45,50–55, 57]) which has been dealing with the classification of solvable Lie algebras with a given nilradical, such as the filiform Lie algebras, the abelian algebras (also called 1-step), the Heisenberg algebras, the algebras of strictly upper triangular matrices, and so on (for arbitrary finite dimensions). The investigation of solvable Lie algebras with some special types of nilradical comes from different problems in physics and was the subject of various papers [3, 5, 16, 24, 27] and many other references given therein.

Malcev [41] had already reduced in 1945 the classification of complex solvable Lie algebras to the classification of one subset, the nilpotent Lie algebras. To do it, Malcev defined a particular type of algebra, that he called splittable algebra, whose structure is completely determined from its maximal nilpotent ideal (called nilradical) and proved that an arbitrary solvable Lie algebra is contained in a unique minimal splittable algebra. The relation between an algebra and its splittings led him to the construction of all solvable Lie algebras with a given splitting. Mubarakzjanov proved that the dimension of the splittable algebra does not exceed the number of nil-independent derivations of the nilradical [42]. So, in this way, the classification of all solvable Lie algebras had been reduced to the classification of the nilpotent Lie algebras. A lot of progress has been made in the classifications of nilpotent Lie algebras. With respect to the classification of nilpotent Lie algebras, many attempts have been made on this topic, and lots of lists of algebras have been published with bigger or less fortune.

During the last 25 years the theory of Leibniz algebras has been actively investigated and numerous papers have been devoted to the study of these algebras. Ayupov and Omirov classified the three-dimensional complex Leibniz algebras in 1999 [8]. Then, they had begun to investigate the nilpotent Leibniz algebras. At the present time the nilpotent Leibniz algebras whose dimension is less than five are classified. The classification finite-dimensional of complex nilpotent Lie algebras is already a complicated problem. Due to lack of antisymmetry the problem of classifying complex nilpotent Leibniz algebras is more difficult.

A lot of progress has been made in the study of other classifications concerning some particular properties of nilpotent Leibniz algebras. Omirov considered the graded nilpotent Leibniz algebras. In [9], the authors classified naturally graded null-filiform and filiform Leibniz algebras. After, many works were devoted to the study of the characteristic sequence of the nilpotent Leibniz algebras [19–23].

Simple and semisimple Leibniz algebras were defined by Dzhumadil'daev [1]. In the paper [30] the authors investigated the semisimple Leibniz algebras and they showed that the splitting theorem (Cartan Theorem) for semisimple Leibniz algebras is not true in general case.

In fact, many results of the theory of Lie algebras are extended to Leibniz algebras case. For instance, the classical results on Cartan subalgebras [2, 46] and Engel's theorem [7] are established in Leibniz algebras case.

The analogue of the Levi-Malcev decomposition for Leibniz algebras was proved by D.W. Barnes [11], that asserts that any Leibniz algebra decomposes into a semidirect sum of its solvable radical and a semisimple Lie algebra. The semisimple part can be described from the simple Lie ideals, therefore, the main problem of the description of the finite-dimensional Leibniz algebras consists of the study of the solvable Leibniz algebras.

Our goal is to prove the Mubarakzjanov theorem for the Leibniz algebras case. In Theorem 1.2.2 the assertion for Leibniz case is extended. In Theorem 1.2.6 solvable Leibniz algebras with null-filiform nilradical are classified.

It is a first step for the investigation of solvable Leibniz algebras with given nilradical. Moreover, this classification is extended to the case when the nilradical is a direct sum of null-filiform ideals and the complementary vector space of the nilradical has one dimension.

In Section 1.3 solvable Leibniz algebras whose nilradical is the Lie algebra of upper triangular matrices are investigated. Since in the work [55] solvable Lie algebras with triangular nilradical are studied, we reduce our study to non-Lie Leibniz algebras. In Corollary 1.3.3, some properties of the matrix of right and left multiplication operators for solvable Leibniz algebras of minimum possible dimensions with triangular nilradical are presented. In Theorem 1.3.4 it is proved that solvable Leibniz algebras of maximum possible dimensions with triangular nilradical are Lie algebras. Furthermore, the classification of the low-dimensional solvable Leibniz algebras with triangular nilradicals is established.

In Chapter 2 we consider solvable Leibniz algebras with naturally graded filiform nilradical. In the works [4, 52] solvable Lie algebras with naturally graded filiform nilradical are described. In Theorems 2.1.5–2.1.6 the non-Lie Leibniz algebras with $n_{n,1}$ and Q_{2n} nilradicals whose complementary vector space has dimension 1 are classified. In the case of complementary vector space of dimension two it is established that non-Lie Leibniz algebras with $n_{n,1}$ and Q_{2n} nilradicals do not exist. It is well known that there are two classes of naturally graded filiform Leibniz algebras F_n^1 and F_n^2. In Theorems 2.2.2, 2.2.3, 2.3.3 and 2.3.4, solvable Leibniz algebras with nilradical F_n^1 and F_n^2 are classified.

Every non-Lie Leibniz algebra L contains a non-trivial ideal (from now on denoted by I), which is the subspace spanned by the squares of elements of the algebra L. Moreover, this ideal is contained in the right annihilator of L, that is $[L, I] = 0$. Note also that the ideal I is the minimal ideal with the property that the quotient algebra L/I is a Lie algebra (the quotient algebra is called liezation of the Leibniz algebra L).

One of the approaches to the investigation of Leibniz algebras is a description of such algebras whose quotient algebra with respect to the ideal I is a given Lie algebra [6, 18, 47, 49].

The map $I \times L/I \to I$ defined as $(i, \overline{x}) \mapsto [i, x]$ endows I with a structure of L/I-module. Considering the direct sum of vector spaces $Q(L) := L/I \oplus I$, the operation $(-,-)$ defines a Leibniz algebra structure on $Q(L)$ with multiplication

$$[\overline{x},\overline{y}]=\overline{[x,y]},\quad [\overline{x},i]=[x,i],\quad [i,\overline{x}]=0,\quad [i,j]=0,\quad x,y\in L,\ i,j\in I.$$

Therefore, for a given Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module M, we can construct a Leibniz algebra $L=\mathfrak{g}\oplus M$ by the above construction.

The last chapter of the thesis is devoted to the investigation of Leibniz algebras corresponding to Diamond Lie algebras. Actually, for a Leibniz algebra L corresponding to the Diamond Lie algebra $\overline{\mathfrak{D}}=L/I$ we decompose it into direct sum of vector spaces $L=\mathfrak{D}\oplus I$, where $\mathfrak{D}$ is the preimage of $\overline{\mathfrak{D}}$ under the natural homomorphism $\varphi\colon L\to\overline{\mathfrak{D}}$. Clearly, the ideal I can be considered as a Leibniz $\overline{\mathfrak{D}}$-module. Taking into account that the ideal I is contained in the right annihilator of the algebra L, the multiplications in L are determined by the products $[\mathfrak{D},\mathfrak{D}]$ and $[I,\overline{\mathfrak{D}}]$. Since I is a Leibniz module over the Lie algebra $\overline{\mathfrak{D}}$, then the product $[I,\overline{\mathfrak{D}}]$ corresponds to a chosen right Lie $\overline{\mathfrak{D}}$-module. Thus, the main problem of the description of Leibniz algebras corresponding to the Lie algebra $\overline{\mathfrak{D}}$ and with the ideal I chosen by specific right $\overline{\mathfrak{D}}$-module consists of identifying the product $[\mathfrak{D},\mathfrak{D}]$.

For a given four-dimensional Diamond Lie algebra $\mathfrak{D}$ we construct the so-called *Fock module* over $\mathfrak{D}$, the linear space $\mathbb{F}[x]$ of polynomials on x ($\mathbb{F}$ denotes an algebraically closed field of characteristic zero) with the action which is introduced in Section 3.2. In Theorem 3.2.2 infinite-dimensional Leibniz algebras corresponding to the Fock module of Diamond Lie algebras are classified.

In Section 3.3 we study Leibniz algebras whose corresponding Lie algebra is the four-dimensional Diamond Lie algebra $\mathfrak{D}$ and the ideal I is one of its finite-dimensional indecomposable Leibniz representations of $\mathfrak{D}$ which are described in the work [26]. In this paper there are four indecomposable $\mathfrak{D}$-modules: U_n^1, U_n^2, W_n^1 and W_n^2.

According Ado's Theorem, given any finite-dimensional complex Lie algebra $\mathfrak{g}$, there exists a matrix algebra isomorphic to $\mathfrak{g}$. In this way, every finite-dimensional complex Lie algebra can be represented as a Lie subalgebra of the complex general linear algebra $\mathfrak{gl}(n,\mathbb{C})$, formed by all the complex $n\times n$ matrices, for some $n\in\mathbb{N}$. We consider the following integer valued invariant of $\mathfrak{g}$:

$$\mu(\mathfrak{g})=\min\{\dim(M)\mid M \text{ is a faithful } \mathfrak{g}\text{-module}\}.$$

It follows from the proof of Ado's Theorem that $\mu(\mathfrak{g})$ can be bounded by a function depending only on n. This value is also equal to the minimal value n such that $\mathfrak{gl}(\mathbb{C}, n)$ contains a subalgebra isomorphic to $\mathfrak{g}$:

$$\widehat{\mu}(\mathfrak{g}) = \min\{n \in \mathbb{N} \mid \exists \text{ subalgebra of } \mathfrak{gl}(\mathbb{C}, n) \text{ isomorphic to } \mathfrak{g}\}.$$

Given a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ in $\mathbb{C}^n$ is a homomorphism of Lie algebra $f \colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(\mathbb{C}, n)$. The natural integer n is called the dimension of this representation. We consider faithful representations because such representations allow us to identify a given Lie algebra with its image under the representation, which is a Lie subalgebra of $\mathfrak{gl}(\mathbb{C}, n)$. Representations can be also defined by using arbitrary n-dimensional vector spaces V (see [29]). In this case, a representation would be a homomorphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of the vector space V, which is called $\mathfrak{g}$-module. However, it is sufficient to consider representations on $\mathbb{C}^n$ because there always exists a unique $n \in \mathbb{N}$ such that V is isomorphic to $\mathbb{C}^n$.

Many works are devoted to finding the value $\mu(\mathfrak{g})$ of several finite-dimensional Lie algebras. In [17] the value of $\mu(\mathfrak{g})$ for abelian Lie algebras and Heisenberg algebras is found, moreover, the value of $\mu(\mathfrak{g})$ for filiform Lie algebras is estimated.

In Section 3.4 we find a minimal faithful representation of the $(2m+2)$-dimensional complex general Diamond Lie algebra $\mathfrak{D}_m(\mathbb{C})$, which is isomorphic to a subalgebra of the special linear Lie algebra $\mathfrak{sl}(m+2, \mathbb{C})$. Then we construct Leibniz algebras with corresponding general Diamond Lie algebra and the ideal generated by the squares of elements in these faithful representations.

Finally, in Section 3.5 we find a faithful representation of $\mathfrak{D}_m$ which is isomorphic to a subalgebra of the symplectic Lie algebra $\mathfrak{sp}(2m+2, \mathbb{R})$. We also investigate the Leibniz algebras constructed by this representation of general Diamond Lie algebras.

Chapter 1

Solvable Leibniz algebras with null-filiform and triangular nilradicals

In this chapter we put the first steps to describing solvable Leibniz algebras with given nilradicals. It is known that any solvable Leibniz algebra can decomposed in sum of the nilradical and its complementary vector space. For the solvable Lie algebras Mubarakzjanov offered the method in which he said that the dimension of the complementary space is not greater than the maximal number of nil-independent derivations of the nilradical. Our goal is to show the validity of the Mubarakzjanov's method for Leibniz algebras. Using this method the solvable Leibniz algebras with null-filiform nilradicals are classified. Moreover, we classify the minimal dimensional solvable Leibniz algebras whose nilradical is equal to the sum of null-filiform algebras. In the last section we describe solvable Leibniz algebras with triangular nilradicals. Furthermore, we establish that a solvable Leibniz algebra of maximal possible dimension with a given triangular nilradical is a Lie algebra.

1.1 Basic results from the theories of Lie and Leibniz algebras

In this section we give necessary definitions and preliminary results.

Definition 1.1.1 ([15]). *An algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is called a Lie algebra if its multiplication (denoted by $(x,y) \mapsto [x,y]$) satisfies the identities:*
(1) $[x,x]=0$,
(2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$,
for all x,y,z *in* $\mathfrak{g}$.

The product $[x,y]$ is called the bracket of x and y. Identity (2) is called the Jacobi identity.

Definition 1.1.2. *An algebra L over a field $\mathbb{K}$ is called a Leibniz algebra if for any $x,y,z \in L$, the Leibniz identity*

$$[[x,y],z]=[[x,z],y]+[x,[y,z]]$$

is satisfied, where $[-,-]$ is the multiplication in L.

For the shortness, instead Leibniz identity $[[x,y],z]=[[x,z],y]+[x,[y,z]]$ we will use below the notation $\{x,y,z\}$.

For a Leibniz algebra L we consider the following derived and lower central series:

$$\text{(i)} \qquad L^{(1)}=L, \qquad L^{(n+1)}=[L^{(n)},L^{(n)}], \qquad n>1;$$
$$\text{(ii)} \qquad L^{1}=L, \qquad L^{n+1}=[L^{n},L], \qquad n>1.$$

Definition 1.1.3. *An algebra L is called solvable (nilpotent) if there exists $s \in \mathbb{N}$ ($k \in \mathbb{N}$, respectively) such that $L^{(s)}=0$ ($L^{k}=0$, respectively). The minimal number s (respectively, k) with such property is called index of solvability (respectively, of nilpotency) of the algebra L.*

Evidently, the index of nilpotency of an n-dimensional algebra is not greater than $n+1$.

Definition 1.1.4. *An n-dimensional Leibniz algebra is called null-filiform if* $\dim L^{i}=n+1-i,\ 1 \leq i \leq n+1$.

Evidently, any null-filiform Leibniz algebra has maximal index of nilpotency.

Theorem 1.1.5 ([7]). *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra:*

$$NF_n: \quad [e_i,e_1]=e_{i+1}, \qquad 1 \leq i \leq n-1,$$

where $\{e_1,e_2,\dots,e_n\}$ is a basis of the algebra.

Actually, a nilpotent Leibniz algebra is null-filiform if it is a one-generated algebra. Note, that this notion has no sense in Lie algebras case, because they are at least two-generated.

Definition 1.1.6. *A Leibniz algebra L is said to be filiform if* $\dim L^i = n-i$, *for* $2 \le i \le n$, *where* $n = \dim L$.

Definition 1.1.7. *Given a filiform Leibniz algebra L, put* $L_i = L^i/L^{i+1},\ 1 \le i \le n-1$, *and* $\operatorname{gr} L = L_1 \oplus L_2 \oplus \dots L_{n-1}$. *Then* $[L_i, L_j] \subseteq L_{i+j}$ *and we obtain the graded algebra* $\operatorname{gr} L$. *If* $\operatorname{gr} L$ *and L are isomorphic, denoted by* $\operatorname{gr} L \cong L$, *we say that the algebra L is naturally graded.*

Thanks to [56] it is known two types of naturally graded filiform Lie algebras. Moreover the second class appears only in the case of even dimension.

Theorem 1.1.8. *Any complex naturally graded filiform Lie algebra is isomorphic to one of the following non-isomorphic algebras:*

$$n_{n,1} : [e_i, e_1] = -[e_1, e_i] = e_{i+1}, \qquad 2 \le i \le n-1.$$

$$Q_{2n} : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \le i \le n. \end{cases}$$

In the following theorem we present the classification of naturally graded filiform non-Lie Leibniz algebras.

Theorem 1.1.9 ([10]). *Any complex n-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non-isomorphic algebras:*

$$F_n^1 : \ [e_i, e_1] = e_{i+1}, \quad 2 \le i \le n-1, \quad F_n^2 : \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1. \end{cases}$$

Definition 1.1.10. *The (unique) maximal nilpotent ideal of a Leibniz algebra is called the nilradical of the algebra.*

A derivation for Leibniz algebras is defined as usual.

Definition 1.1.11. *A linear map $d\colon L \to L$ is called a derivation of L if*

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for any} \ \ x, y \in L.$$

The space of all derivations is denoted by $\operatorname{Der}(L)$.

For an arbitrary element $x \in L$, we consider the right multiplication operator $R_x \colon L \to L$ defined by $R_x(z) = [z, x]$. Right multiplication operators are derivations of the algebra L and are called *inner derivations*. The set $R(L) = \{R_x \mid x \in L\}$ is a Lie algebra with respect to the commutator and the following identity holds:

$$R_x R_y - R_y R_x = R_{[y,x]} \,.$$

The *right annihilator* of a Leibniz algebra L, denoted by $\mathrm{Ann}_r(L)$, is $\mathrm{Ann}_r(L) = \{x \in L \mid [y, x] = 0 \text{ for all } y \in L\}$. The *left annihilator* of a Leibniz algebra L, denoted by $\mathrm{Ann}_l(L)$, is $\mathrm{Ann}_l(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$. The *center* of a Leibniz algebra L, denoted by $\mathsf{Center}(L)$, is $\mathsf{Center}(L) = \mathrm{Ann}_r(L) \cap \mathrm{Ann}_l(L) = \{x \in L \mid [y, x] = [x, y] = 0 \text{ for all } y \in L\}$.

Definition 1.1.12 ([42])**.** *Let $d_1, d_2, \ldots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \ldots, d_n$ are said to be nil-independent if*

$$\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n$$

is not nilpotent for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, which are not all zero.

In other words, if for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ there exists a natural number k such that $(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_2)^k = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

The classical Engel's theorem for Lie algebras has the following analogue for Leibniz algebras.

Theorem 1.1.13. *A Leibniz algebra L is nilpotent if and only if R_x is nilpotent for any $x \in L$.*

Similar to the case of finite-dimensional Lie algebras we have the following theorem.

Theorem 1.1.14. *A Leibniz algebra L is solvable if and only if L^2 is a nilpotent Leibniz algebra.*

Further, we will use Lie's theorem for proving the main result.

Theorem 1.1.15 ([32] Lie's theorem)**.** *If L is a solvable Lie algebra of linear transformations in a finite-dimensional vector space V over an algebraically closed field of characteristic 0, then the matrices of L can be taken in simultaneous triangular form.*

Let us consider the finite-dimensional Lie algebra $T(n)$ of the upper-triangular $n \times n$ matrices with $n \geq 3$ over the field of the complex numbers. The products of the basis elements $\{N_{ij} \mid 1 \leq i < j \leq n\}$ of $T(n)$, where N_{ij} is a matrix with the only non-zero entry at i-th row and j-th column equal to 1, can be computed by

$$[N_{ij}, N_{kl}] = \delta_{jk} N_{il} - \delta_{il} N_{kj}.$$

For a natural number f let $G(n, f)$ be a set of solvable Lie algebras of dimension $\frac{1}{2}n(n-1)+f$ with nilradical $T(n)$. Let $Q = \langle X^1, X^2, \dots, X^f \rangle$, where Q is the complementary vector space of the nilradical $T(n)$ to an algebra from $G(n, f)$.

Denote

$$[N_{ij}, X^\alpha] = \sum_{1 \leq q-p < n} a^\alpha_{ij,pq} N_{pq}, \quad [X^\alpha, X^\beta] = \sum_{1 \leq q-p < n} \sigma^{\alpha\beta}_{pq} N_{pq}, \tag{1.1.1}$$

where $1 \leq \alpha, \beta \leq f$, and $a^\alpha_{ij,pq}, \sigma^{\alpha\beta}_{pq} \in \mathbb{C},\ p < q \leq n$.

Let N be a vector column

$$(N_{12}, N_{23}, \dots, N_{(n-1)n}, N_{13}, N_{24}, \dots, N_{(n-2)n}, \dots, N_{1n})^T.$$

Then we have

$$R_{X^\alpha}(N) = A^\alpha N,$$

where $A^\alpha = (a^\alpha_{ij,pq}),\ 1 \leq i < j \leq n,\ 1 \leq p < q \leq n$, are $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ complex matrices.

The following lemma provides some information about the structure of the matrices above.

Lemma 1.1.16 ([55]). *The structure matrices $A^\alpha = (a^\alpha_{ij,pq}),\ 1 \leq i < j \leq n,\ 1 \leq p < q \leq n$, have the following properties:*

(i) They are upper triangular;

(ii) The only off-diagonal matrix elements that do not vanish identically and cannot be annulled by a redefinition of the elements X^α are:

$$a^\alpha_{12,2n}, \qquad a^\alpha_{i(i+1),1n} \ \ (2 \leq i \leq n-2), \qquad a^\alpha_{(n-1)n,1(n-1)};$$

(iii) The diagonal elements $a^{\alpha}_{i(i+1),i(i+1)}$, $1 \le i \le n-1$, *are free to vary. The other diagonal elements satisfy*

$$a^{\alpha}_{ik,ik} = \sum_{p=i}^{k-1} a^{\alpha}_{p(p+1),p(p+1)}, \quad k > i+1.$$

Lemma 1.1.17 ([55]). *The maximal number of non-nilpotent elements is*

$$f_{\max} = n-1.$$

1.2 Solvable Leibniz algebras with null-filiform nilradical

Let R be a solvable Leibniz algebra. Then it can be decomposed into the form $R = N \oplus Q$, where N is the nilradical and Q is the complementary vector space. Since the square of a solvable algebra is a nilpotent ideal and the finite sum of nilpotent ideals is a nilpotent ideal too, then the ideal R^2 is nilpotent, i.e. $R^2 \subseteq N$ and consequently, $Q^2 \subseteq N$.

Lemma 1.2.1. *Let* $x \in Q$ *be such that the operator* $\mathcal{R}_{x|_N}$ *is nilpotent. Then the subspace* $V = \langle x + N \rangle$ *is a nilpotent ideal of the algebra* R.

Proof. Since $R^2 \subseteq N$, V is an ideal. We argue that it is nilpotent. If $a \in N$, then $\mathcal{R}_{a|_N}$ is a nilpotent operator. Let us suppose that there exists $k \in \mathbb{N}$ such that $\left(\mathcal{R}_{a|_N}\right)^k = 0$, then $\left(\mathcal{R}_{a|_V}\right)^{k+1} = 0$. Hence $\mathcal{R}_{a|_V}$ is nilpotent. V is an ideal of the solvable Leibniz algebra R, then $\mathrm{Inn}(V)$ is a solvable Lie algebra of $\mathrm{End}(V)$, and so by Lie's theorem there exists a basis such that $\mathcal{R}_{a|_V}$ and $\mathcal{R}_{x|_V}$ are upper triangular; moreover, $\mathcal{R}_{a|_V}$ is nilpotent, which means that $\mathcal{R}_{a|_V}$ has zero diagonal elements. On the other hand, by assumption, $\mathcal{R}_{x|_N}$ is nilpotent, then with a similar argument as the previous one, there exists $s \in \mathbb{N}$ such that $\left(\mathcal{R}_{x|_N}\right)^s = 0$, then $\left(\mathcal{R}_{x|_V}\right)^{s+1} = 0$. Summarizing, $\mathcal{R}_{a|_V}$ and $\mathcal{R}_{x|_V}$ are nilpotent and upper triangular, hence $\mathcal{R}_{a|_V} + \mathcal{R}_{x|_V}$ is nilpotent. Thus, by Engel's theorem, V is a nilpotent ideal. □

Theorem 1.2.2. *Let* R *be a solvable Leibniz algebra and* N *its nilradical. Then the dimension of the complementary vector space to* N *is not greater than the maximal number of nil-independent derivations of* N.

Proof. We assert that every $\mathcal{R}_{x|_N}$, $x \in Q$, is a non-nilpotent outer derivation of N. Indeed, if there exists some $x \in Q$ such that the operator $\mathcal{R}_{x|_N}$ is nilpotent, then the subspace $V = \langle x + N \rangle$ is a nilpotent ideal of the algebra R by Lemma 1.2.1, contradicting the maximality condition of N.

Let $\{x_1, \dots, x_m\}$ be a basis of Q. Then the operators $\mathcal{R}_{x_1|_N}, \dots, \mathcal{R}_{x_m|_N}$ are nil-independent, since if for some scalars $\{\alpha_1, \dots, \alpha_m\} \in \mathbb{C} \setminus \{0\}$ we have that $\left(\sum_{i=1}^{m} \alpha_i \mathcal{R}_{x_i|_N}\right)^k = 0$, then $\mathcal{R}^k_{y|_N}$, where $y = \sum_{i=1}^{m} \alpha_i x_i$. Hence $y = 0$, and so $\alpha_i = 0$ for $i = 1, \dots, m$.

Therefore, we see that the dimension of Q is bounded by the maximal number of nil-independent derivations of the nilradical N. Moreover, similar to the case of Lie algebras, for a solvable Leibniz algebra R we also have the inequality $\dim N \geq \dfrac{\dim R}{2}$. □

From Theorem 1.2.2 we conclude the following properties of derivations of null-filiform Leibniz algebras.

Proposition 1.2.3. *Any derivation of the algebra NF_n has the following matrix form:*

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & 2a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & 3a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & na_1 \end{pmatrix}.$$

Proof. The proof is carried out by checking the derivation property on the algebra NF_n. □

Corollary 1.2.4. *The maximal number of nil-independent derivations of the n-dimensional null-filiform Leibniz algebra NF_n is 1.*

Proof. Let

$$D_i = \begin{pmatrix} a_1^i & a_2^i & a_3^i & \dots & a_n^i \\ 0 & 2a_1^i & a_2^i & \dots & a_{n-1}^i \\ 0 & 0 & 3a_1^i & \dots & a_{n-2}^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & na_1^i \end{pmatrix}, \quad i = 1, 2, \dots, p,$$

be derivations of NF_n. If $p > 1$, then $\left(D_i - \frac{a_1^i}{a_1^1}D_1\right)^n = 0$ with non-trivial scalars. Hence $\{D_1, D_2, \ldots, D_p\}$ are not nil-independent. □

Corollary 1.2.5. *The dimension of a solvable Leibniz algebra with nilradical NF_n is equal to $n+1$.*

Proof. Let us assume that the solvable Leibniz algebra is decomposed as $R = NF_n \oplus Q$. By Corollary 1.2.4 and Theorem 1.2.2 we have $\dim Q = 1$. □

From now on, the method of classification of algebras used in this thesis is based on the following procedure.

CLASSIFICATION PROCEDURE

The classification algorithm of Leibniz algebras in fixed dimension consist of the following steps:

- **finding a basis in which the multiplication table of an algebra have the most convenient form;**
- **to reduce the study of general basis transformations to the simple ones;**
- **to find relations between parameters (structural constants) in initial and transformed bases;**
- **present the list of pairwise non-isomorphic algebras such that any algebra with the considered conditions is isomorphic to an algebra of the presented list.**

Now we apply this classification algorithm for solvable Leibniz algebras with given nilradicals.

Theorem 1.2.6. *Let R be a solvable Leibniz algebra whose nilradical is NF_n. Then there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\ [x, e_1] = e_1, & \\ [e_i, x] = -ie_i, & 1 \leq i \leq n. \end{cases}$$

Proof. According to Theorem 1.1.5 and Corollary 1.2.5 there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$ such that all products of elements of the basis, except for the products $[e_i, x]$ which can be derived from the equalities $[e_{i+1}, x] = \big[[e_i, e_1], x\big] = \big[e_i, [e_1, x]\big] + \big[[e_i, x], e_1\big]$, $1 \le i \le n-1$, have the following form:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\ [x, e_1] = \sum\limits_{i=1}^{n} \alpha_i e_i, \\ [e_1, x] = \sum\limits_{i=1}^{n} \beta_i e_i, \\ [x, x] = \sum\limits_{i=1}^{n} \gamma_i e_i, \end{cases}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of NF_n and $\{x\}$ is a basis of Q.

Now we consider the following two possible cases.

Case 1. Let $\alpha_1 \neq 0$. Then taking the change of basis:

$$e_i' = \frac{1}{\alpha_1} \sum_{j=i}^{n} \alpha_{j-i+1} e_j, \quad 1 \le i \le n, \qquad x' = \frac{1}{\alpha_1} x,$$

we can assume that $[x, e_1] = e_1$ and other products by redesignation of parameters can be assumed not changed.

From the products

$$0 = \big[x, [x, x]\big] = \Big[x, \sum_{i=1}^{n} \gamma_i e_i\Big] = \sum_{i=1}^{n} \gamma_i [x, e_i] = \gamma_1 e_1,$$

we can deduce that $\gamma_1 = 0$.

On the other hand, from the Leibniz identity $\{x, e_1, x\}$ we get $\beta_1 [x, e_1] = [e_1, x] - \sum\limits_{i=3}^{n} \gamma_{i-1} e_i$, i.e. $\beta_1 e_1 = \sum\limits_{i=1}^{n} \beta_i e_i - \sum\limits_{i=3}^{n} \gamma_{i-1} e_i$.

Comparing the coefficients at the elements of the basis, we obtain $\beta_2 = 0$ and $\gamma_i = \beta_{i+1}$ for $2 \le i \le n-1$. From the equality $\{e_1, e_1, x\}$ we derive that $\beta_1 = -1$.

Thus, we have

$$[e_1, x] = -e_1 + \sum_{i=3}^{n} \beta_i e_i, \qquad [x, x] = \sum_{i=2}^{n-1} \beta_{i+1} e_i + \gamma_n e_n.$$

Now we are going to prove the following identity

$$[e_i, x] = -ie_i + \sum_{j=i+2}^{n} \beta_{j-i+1} e_j, \tag{1.2.1}$$

for $1 \le i \le n$. We have seen that (1.2.1) is true for $i = 1$. Assume that 1.2.1 holds for each i, $1 \le i < k \le n$. Then

$$\begin{aligned}
[e_k, x] &= \big[[e_{k-1}, e_1], x\big] = \big[e_{k-1}, [e_1, x]\big] + \big[[e_{k-1}, x], e_1\big] \\
&= [e_{k-1}, -e_1] + \big[-(k-1)e_{k-1} + \sum_{j=k+1}^{n} \beta_{j-k+2} e_j, e_1\big] \\
&= -e_k - (k-1)e_k + \sum_{j=k+1}^{n} \beta_{j-k+2}[e_j, e_1] = -ke_k + \sum_{j=k+2}^{n} \beta_{j-k+1} e_j.
\end{aligned}$$

By induction, we see that indeed (1.2.1) holds for all i, $1 \le i \le n$.

Thus, the multiplication table of the algebra R has the form:

$$\begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\
[x, e_1] = e_1, & \\
[e_i, x] = -ie_i + \sum\limits_{j=i+2}^{n} \beta_{j-i+1} e_j, & 1 \le i \le n, \\
[x, x] = \sum\limits_{i=2}^{n-1} \beta_{i+1} e_i + \gamma_n e_n. &
\end{cases} \tag{1.2.2}$$

Let us take the change of basis:

$$e_i' = e_i + \sum_{j=i+2}^{n} A_{j-i+1} e_j, \ 1 \le i \le n, \qquad x' = \sum_{i=2}^{n-1} A_{i+1} e_i + B_n e_n + x,$$

where parameters A_i, B_n are as follows

$$A_3 = \frac{1}{2}\beta_3, \quad A_4 = \frac{1}{3}\beta_4, \quad A_i = \frac{1}{i-1}\Big(\sum_{j=3}^{i-2} A_{i-j+1}\beta_j + \beta_i\Big), \quad (5 \le i \le n),$$

$$B_n = \frac{1}{n}\Big(\sum_{j=3}^{n-1} A_{n-j+2}\beta_j + \gamma_n\Big).$$

Then taking into account the multiplication table (1.2.2) we compute the products in the new basis

$$
\begin{aligned}
[e_i', e_1'] &= [e_i + \sum_{j=i+2}^{n} A_{j-i+1} e_j, e_1] \\
&= e_{i+1} + \sum_{j=i+3}^{n} A_{j-i} e_j = e_{i+1}', \qquad\qquad 1 \leq i \leq n-1, \\
[x', e_1'] &= [\sum_{i=2}^{n-1} A_{i+1} e_i + B_n e_n + x, e_1] = \sum_{i=3}^{n} A_i e_i + [x, e_1] \\
&= e_1 + \sum_{i=3}^{n} A_i e_i = e_1', \\
[x', x'] &= [\sum_{i=2}^{n-1} A_{i+1} e_i + B_n e_n + x, x] = \sum_{i=2}^{n-1} A_{i+1} [e_i, x] + B_n [e_n, x] + [x, x] \\
&= \sum_{i=2}^{n-1} A_{i+1} \big(-i e_i + \sum_{j=i+2}^{n} \beta_{j-i+1} e_j\big) - n B_n e_n + \sum_{i=2}^{n-1} \beta_{i+1} e_i + \gamma_n e_n \\
&= -\sum_{i=2}^{n-1} i A_{i+1} e_i + \sum_{i=2}^{n-3} A_{i+1} \sum_{j=i+2}^{n-1} \beta_{j-i+1} e_j + \sum_{i=2}^{n-1} \beta_{i+1} e_i \\
&\qquad + \sum_{i=2}^{n-2} A_{i+1} \beta_{n-i+1} e_n - B_n e_n + \gamma_n e_n \\
&= \sum_{i=2}^{n-1} (-i A_{i+1} + \beta_{i+1}) e_i + \sum_{i=4}^{n-1} \sum_{j=3}^{i-1} A_{i-j+2} \beta_j e_i \\
&\qquad + \big(-n B_n + \gamma_n + \sum_{i=2}^{n-1} A_{i+1} \beta_{n-i+1}\big) e_n \\
&= (-2A_3 + \beta_3) e_2 + (-3A_4 + \beta_4) e_3 \\
&\qquad + \sum_{i=4}^{n-1} \sum_{j=3}^{i-1} (-i A_{i+1} + \beta_{i+1} + A_{i-j+2} \beta_j) e_i = 0,
\end{aligned}
$$

$$\begin{aligned}
[e_1', x'] &= \Big[e_1 + \sum_{i=3}^{n} A_i e_i, x\Big] = [e_1, x] + \sum_{i=3}^{n} A_i [e_i, x] \\
&= -e_1 + \sum_{i=3}^{n} \beta_i e_i + \sum_{i=3}^{n} A_i \Big(-ie_i + \sum_{j=i+2}^{n} \beta_{j-i+1} e_j\Big) \\
&= -e_1 + \sum_{i=3}^{n} \beta_i e_i - \sum_{i=3}^{n} iA_i e_i + \sum_{i=3}^{n} A_i \sum_{j=i+2}^{n} \beta_{j-i+1} e_j \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i - \sum_{i=3}^{n} (i-1) A_i e_i + \sum_{i=3}^{n} \beta_i e_i + \sum_{i=3}^{n} \Big(\sum_{j=3}^{i-2} A_{i-j+1} b_j\Big) e_i \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \big(-(i-1)A_i + \beta_i\big) e_i + \sum_{i=5}^{n} \sum_{j=3}^{i-2} A_{i-j+1} \beta_j e_i \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + (-2A_3 + \beta_3) e_3 + (-3A_4 + \beta_4) e_4 \\
&+ \sum_{i=5}^{n} \sum_{j=3}^{i-2} \big(-(i-1)A_i + \beta_i + A_{i-j+1}\beta_j\big) e_i = -e_1 - \sum_{i=3}^{n} A_i e_i = -e_1'.
\end{aligned}$$

By means of similar computations as in equations (1.2.1) we deduce that $[e_i', x'] = -ie_i',\ 1 \leq i \leq n$.

Finally, we obtain the multiplication table of the algebra R given in the assertion of the theorem.

Case 2. Let $\alpha_1 = 0$. Then from the equalities $\big[e_1, [e_1, x]\big] = -\big[e_1, [x, e_1]\big]$ and $0 = \big[x, [x, x]\big]$, we get $\beta_1 = 0$ and $\gamma_1 = 0$, respectively.

Thus, we have the following products:

$$\begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\
[x, e_1] = \sum_{i=2}^{n} \alpha_i e_i, & \\
[e_1, x] = \sum_{i=2}^{n} \beta_i e_i, & \\
[x, x] = \sum_{i=2}^{n} \gamma_i e_i. &
\end{cases}$$

In a similar way as for the equations (1.2.1), we can prove the equality: $[e_i, x] = \sum\limits_{j=i+1}^{n} \beta_{j-i+1} e_j$. Consequently, we have $[e_i, x] \in \langle\{e_{i+1}, e_{i+2}, \dots, e_n\}\rangle$, i.e. $R^i \subseteq \langle\{e_i, e_{i+1}, \dots, e_n\}\rangle$. Thus, $R^{n+1} = 0$ which contradicts the assumption of non-nilpotency of the algebra R. This implies that, in the case of $\alpha_1 = 0$, there is no non-nilpotent solvable Leibniz algebra with nilradical NF_n. □

Now we are going to clarify the situation when the nilradical is a direct sum of two null-filiform ideals of the nilradical.

Theorem 1.2.7. *Let R be a solvable Leibniz algebra such that $R = NF_k \oplus NF_s + Q$, where $NF_k \oplus NF_s$ is the nilradical of R, NF_k and NF_s are ideals of the nilradical and $\dim Q = 1$. Then NF_k and NF_s are also ideals of the algebra R.*

Proof. Let $\{e_1, e_2, \dots, e_k\}$ be a basis of NF_k, $\{f_1, f_2, \dots, f_s\}$ a basis of NF_s and $\{x\}$ a basis of Q. We can assume, without loss of generality, that $k \geq s$.

By Theorem 1.1.5 we have that $\{e_2, e_3, \dots, e_k, f_2, f_3, \dots, f_s\} \subseteq \mathrm{Ann}_r(R)$ and the following products:

$$[e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq k-1, \qquad [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1.$$

Let us introduce the notations:

$$\begin{cases} [x, e_1] = \sum\limits_{i=1}^{k} \alpha_i e_i + \sum\limits_{i=1}^{s} \beta_i f_i, & [x, f_1] = \sum\limits_{i=1}^{k} \delta_i e_i + \sum\limits_{i=1}^{s} \gamma_i f_i, \\ [e_1, x] = \sum\limits_{i=1}^{k} \lambda_i e_i + \sum\limits_{i=1}^{s} \sigma_i f_i, & [f_1, x] = \sum\limits_{i=1}^{k} \tau_i e_i + \sum\limits_{i=1}^{s} \mu_i f_i, \\ [x, x] = \sum\limits_{i=1}^{k} \rho_i e_i + \sum\limits_{i=1}^{s} \xi_i f_i. & \end{cases}$$

From the products

$$0 = \big[x, [e_1, f_1]\big] = \big[[x, e_1], f_1\big] - \big[[x, f_1], e_1\big] = \sum_{i=2}^{s} \beta_{i-1} f_i - \sum_{i=2}^{k} \delta_{i-1} e_i,$$

we obtain $\beta_i = 0,\ 1 \leq i \leq s-1$ and $\delta_i = 0,\ 1 \leq i \leq k-1$.

The equalities $\big[e_1,[e_1,x]\big] = -\big[e_1,[x,e_1]\big]$ and $\big[f_1,[f_1,x]\big] = -\big[f_1,[x,f_1]\big]$ imply that $\lambda_1 = -\alpha_1$, $\mu_1 = -\gamma_1$.

From the equalities $0 = \big[e_1,[x,x]\big] = \rho_1 e_2$ and $0 = \big[f_1,[x,x]\big] = \xi_1 f_2$, we get $\rho_1 = \xi_1 = 0$.

In a similar way as in the proof of Theorem 1.2.6, the following equalities can be proved:

$$[e_i,x] = -i\alpha_1 e_i + \sum_{j=i+1}^{k} \lambda_{j-i+1} e_j, \qquad 2 \le i \le k,$$

$$[f_i,x] = -i\gamma_1 f_i + \sum_{j=i+1}^{s} \mu_{j-i+1} f_j, \qquad 2 \le i \le s.$$

Summarizing, we have obtained the following multiplication table for the algebra R:

$$\begin{cases} [e_i,e_1] = e_{i+1}, & 1 \le i \le k-1,\\ [f_i,f_1] = f_{i+1}, & 1 \le i \le s-1,\\ [x,e_1] = \sum\limits_{i=1}^{k} \alpha_i e_i + \beta_s f_s, & \\ [x,f_1] = \delta_k e_k + \sum\limits_{i=1}^{s} \gamma_i f_i, & \\ [e_1,x] = -\alpha_1 e_1 + \sum\limits_{i=2}^{k} \lambda_i e_i + \sum\limits_{i=1}^{s} \sigma_i f_i, & \\ [f_1,x] = \sum\limits_{i=1}^{k} \tau_i e_i - \gamma_1 f_1 + \sum\limits_{i=2}^{s} \mu_i f_i, & \\ [e_i,x] = -i\alpha_1 e_i + \sum\limits_{j=i+1}^{k} \lambda_{j-i+1} e_j, & 2 \le i \le k,\\ [f_i,x] = -i\gamma_1 e_i + \sum\limits_{j=i+1}^{s} \mu_{j-i+1} f_j, & 2 \le i \le s,\\ [x,x] = \sum\limits_{i=2}^{k} \rho_i e_i + \sum\limits_{i=2}^{s} \xi_i f_i. & \end{cases} \tag{1.2.3}$$

Below, we analyze the different cases that can appear in terms of the possible values of α_1 and γ_1.

Case 1. Let $\alpha_1 = \gamma_1 = 0$. Then the multiplication table (1.2.3) implies $[e_i, x] \in \langle\{e_{i+1}, e_{i+2}, \dots, e_k\}\rangle$, $[f_i, x] \in \langle\{f_{i+1}, f_{i+2}, \dots, f_s\}\rangle$, $[e_1, x] \in \langle\{e_2, e_3, \dots, e_k, f_1, f_2, \dots, f_s\}\rangle$ and $[f_1, x] \in \langle\{e_1, e_2, \dots, e_k, f_2, f_3, \dots, f_s\}\rangle$. The above facts mean that the algebra R is nilpotent, so we get a contradiction with the assumption of non-nilpotency of R. Therefore, this case is impossible.

Case 2. Let $\alpha_1 \neq 0$ and $\gamma_1 = 0$. Using the following change of basis:

$$e_1' = \frac{1}{\alpha_1}\Big(\sum_{i=1}^{k}\alpha_i e_i + \beta_s f_s\Big), \quad e_i' = \frac{1}{\alpha_1}\sum_{j=i}^{k}\alpha_{j-i+1}e_j, \ 2 \le i \le k, \quad x' = \frac{1}{\alpha_1}x,$$

we assume that

$$[x, e_1] = e_1.$$

From the identity $\{x, x, e_1\}$ we have that

$$e_1 = \sum_{i=2}^{k}\rho_i[e_i, e_1] - [e_1, x] = \sum_{i=3}^{k}\rho_{i-1}e_i + e_1 - \sum_{i=2}^{k}\lambda_i e_i - \sum_{i=1}^{s}\sigma_i f_i.$$

Consequently, $\lambda_2 = \sigma_i = 0$ for $1 \le i \le s$ and $\rho_i = \lambda_{i+1}$ for $2 \le i \le k-1$.

From the identity $\{f_1, x, e_1\}$ we conclude that $0 = \big[[f_1, x], e_1\big] = \sum\limits_{i=2}^{k}\tau_{i-1}e_i \Rightarrow \tau_i = 0, \ 1 \le i \le k-1$.

From the identity $\{x, x, f_1\}$

$$\begin{aligned}
0 &= \sum_{i=3}^{s}\xi_{i-1}f_i - \sum_{i=2}^{s}\gamma_i[f_i, x] + \delta_k[e_k, x] \\
&= \sum_{i=3}^{s}\xi_{i-1}f_i - \sum_{i=2}^{s}\gamma_i\Big(\sum_{j=i+1}^{s}\mu_{j-i+1}f_j\Big) - k\delta_k e_k \\
&= \sum_{i=3}^{s}\xi_{i-1}f_i - \sum_{i=3}^{s}\Big(\sum_{j=3}^{i}\gamma_{j-1}\mu_{i-j+2}\Big)f_i - k\delta_k e_k \\
&= \sum_{i=3}^{s}\Big(\xi_{i-1} - \sum_{j=3}^{i}\gamma_{j-1}\mu_{i-j+2}\Big)f_i - k\delta_k e_k.
\end{aligned}$$

By comparison of coefficients at the elements of the basis we deduce that:

$$\xi_i = \sum_{j=3}^{i+1} \gamma_{j-1}\mu_{i-j+3}, \quad 2 \le i \le s-1 \ \text{ and } \ \delta_k = 0.$$

Now we consider the following change of basis:

$$f_1' = f_1 + \frac{\tau_k}{k} e_k, \qquad f_i' = f_i, \ 2 \le i \le s.$$

Then we obtain

$$[f_1', x] = [f_1 + \frac{\tau_k}{k} e_k, x] = \sum_{i=2}^{s} \mu_i f_i + \tau_k e_k - \tau_k e_k = \sum_{i=2}^{s} \mu_i f_i = \sum_{i=2}^{s} \mu_i f_i'$$

and

$$[x, f_1'] = [x, f_1 + \frac{\tau_k}{k} e_k] = [x, f_1] = \sum_{i=2}^{s} \gamma_i f_i = \sum_{i=2}^{s} \gamma_i f_i'.$$

Thus, we have the following multiplication table of the algebra R:

$$\begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \le i \le k-1, \\
[f_i, f_1] = f_{i+1}, & 1 \le i \le s-1, \\
[x, e_1] = e_1, & \\
[x, f_1] = \sum\limits_{i=2}^{s} \gamma_i f_i, & \\
[e_1, x] = -e_1 + \sum\limits_{i=2}^{k} \lambda_i e_i, & \\
[f_1, x] = \sum\limits_{i=2}^{s} \mu_i f_i, & \\
[e_i, x] = -ie_i + \sum\limits_{j=i+2}^{k} \lambda_{j-i+1} e_j, & 2 \le i \le k, \\
[f_i, x] = \sum\limits_{j=i+1}^{s} \mu_{j-i+1} f_j, & 2 \le i \le s, \\
[x, x] = \sum\limits_{i=2}^{k} \rho_i e_i + \sum\limits_{i=2}^{s} \xi_i f_i. &
\end{cases}$$

From the above multiplication table the following inclusions can be immediately derived:

$$[x, NF_k] \subseteq NF_k, \quad [NF_k, x] \subseteq NF_k, \quad [x, NF_s] \subseteq NF_s, \quad [NF_s, x] \subseteq NF_s.$$

This completes the proof of the assertion established in the theorem for this case.

Case 3. Let $\alpha_1 = 0$ and $\gamma_1 \neq 0$. Due to symmetry of Cases 2 and 3, the proof of the assertion of the theorem follows by applying similar arguments as in Case 2.

Case 4. Let $\alpha_1 \neq 0$ and $\gamma_1 \neq 0$. Consider the following change of basis:

$$e_1' = \frac{1}{\alpha_1}\Big(\sum_{i=1}^{k} \alpha_i e_i + \beta_s f_s\Big), \quad e_i' = \frac{1}{\alpha_1}\sum_{j=i}^{k} \alpha_{j-i+1} e_j, \quad 2 \le i \le k,$$

$$f_1' = \frac{1}{\gamma_1}\Big(\sum_{i=1}^{s} \gamma_i f_i + \delta_k e_k\Big), \quad f_i' = \frac{1}{\gamma_1}\sum_{j=i}^{k} \gamma_{j-i+1} f_j, \quad 2 \le i \le s, \quad x' = \frac{1}{\alpha_1}x.$$

Then we derive

$$[x', e_1'] = [\frac{1}{\alpha_1}x, \frac{1}{\alpha_1}(\sum_{i=1}^{k} \alpha_i e_i + \beta_s f_s)] = \frac{1}{\alpha_1^2}\alpha_1[x, e_1] = \frac{1}{\alpha_1}[x, e_1] = e_1',$$

$$[x', f_1'] = [\frac{1}{\alpha_1}x, \frac{1}{\gamma_1}(\sum_{i=1}^{s} \gamma_i f_i + \delta_k e_k)] = \frac{1}{\alpha_1\gamma_1}\gamma_1[x, f_1] = \frac{\gamma_1}{\alpha_1} f_1'.$$

From the identity $\{x, x, e_1\}$ we deduce:

$$e_1 = \sum_{i=2}^{k} \rho_i [e_i, e_1] - [e_1, x] = \sum_{i=3}^{k} \rho_{i-1} e_i + \alpha_1 e_1 - \sum_{i=2}^{k} \lambda_i e_i - \sum_{i=1}^{s} \sigma_i f_i.$$

Therefore, $\alpha_1 = 1, \lambda_1 = -1,\ \lambda_2 = \sigma_i = 0,\ 1 \le i \le s$ and $\rho_i = \lambda_{i+1},\ 2 \le i \le k-1$.

Expanding the identity $\{x, x, f_1\}$ we derive the equalities:

$$\Big(\frac{\gamma_1}{\alpha_1}\Big)^2 f_1 = \sum_{i=2}^{s} \xi_i [f_i, f_1] - \frac{\gamma_1}{\alpha_1}[f_1, x] = \sum_{i=3}^{s} \xi_{i-1} f_i - \frac{\gamma_1}{\alpha_1}\sum_{i=1}^{s} \mu_i f_i - \frac{\gamma_1}{\alpha_1}\sum_{i=1}^{k} \tau_i e_i$$

from which we have $\mu_1 = -\frac{\gamma_1}{\alpha_1},\ \mu_2 = \tau_i = 0,\ 1 \leq i \leq k$ and $\xi_i = \frac{\gamma_1}{\alpha_1}\mu_{i+1},\ 2 \leq i \leq s-1$.

Finally, we obtain the following products of basis elements in the algebra R:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq k-1, \quad [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1, \\ [x, e_1] = e_1, & [x, f_1] = \dfrac{\gamma_1}{\alpha_1} f_1, \\ [e_1, x] = -e_1 + \displaystyle\sum_{i=3}^{k} \lambda_i e_i, & [f_1, x] = -\dfrac{\gamma_1}{\alpha_1} f_1 + \displaystyle\sum_{i=3}^{s} \mu_i f_i, \\ [x, x] = \displaystyle\sum_{i=2}^{k} \rho_i e_i + \sum_{i=2}^{s} \xi_i f_i. \end{cases}$$

These products are sufficient in order to check the inclusions:

$$[x, NF_k] \subseteq NF_k, \quad [NF_k, x] \subseteq NF_k, \quad [x, NF_s] \subseteq NF_s, \quad [NF_s, x] \subseteq NF_s.$$

Thus, the ideals NF_k and NF_s of the nilradical are also ideals of the algebra. □

Now we are going to describe solvable Leibniz algebras with nilradical $NF_k \oplus NF_s$ and with one-dimensional complementary vector space. Due to Theorem 1.2.7 we can assume that NF_k and NF_s are ideals of the algebra.

Theorem 1.2.8. *Let R be a solvable Leibniz algebra such that $R = NF_k \oplus NF_s + Q$, where $NF_k \oplus NF_s$ is the nilradical of R and $\dim Q = 1$. Let us assume that $\{e_1, e_2, \dots, e_k\}$ is a basis of NF_k, $\{f_1, f_2, \dots, f_s\}$ is a basis of NF_s and $\{x\}$ is a basis of Q. Then the algebra R is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R(\alpha): \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1, \\ [x, e_1] = e_1, & [x, f_1] = \alpha f_1, \quad \alpha \neq 0, \\ [e_i, x] = -ie_i, \quad 1 \leq i \leq k, & [f_i, x] = -i\alpha f_i, \quad 1 \leq i \leq s, \end{cases}$$

$R(\beta_2, \beta_3, \dots, \beta_s, \gamma):$

$$\begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1, \\ [x, e_1] = e_1, & [f_i, x] = \displaystyle\sum_{j=i+1}^{s} \beta_{j-i+1} f_j, \ 1 \leq i \leq s, \\ [e_i, x] = -ie_i, \quad 1 \leq i \leq k, & [x, x] = \gamma f_s. \end{cases}$$

In the second family of algebras the first non-zero element of the vector $(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ *can be assumed to be equal to 1.*

Proof. Firstly, we note that the algebras $NF_k + Q$ and $NF_s + Q$ are not simultaneously nilpotent. Indeed, if they are both nilpotent, then we have:

$$\begin{aligned}
[e_i, e_1] &\in \langle\{e_{i+1}, \ldots, e_k\}\rangle, \quad 1 \leq i \leq k-1,\\
[x, e_1] &\in \langle\{e_2, e_3, \ldots, e_k\}\rangle,\\
[e_i, x] &\in \langle\{e_{i+1}, \ldots, e_k\}\rangle, \quad 1 \leq i \leq k-1,\\
[f_i, f_1] &\in \langle\{f_{i+1}, \ldots, f_s\}\rangle, \quad 1 \leq i \leq s-1,\\
[x, f_1] &\in \langle\{f_2, f_3, \ldots, f_s\}\rangle,\\
[f_j, x] &\in \langle\{f_{j+1}, \ldots, f_s\}\rangle, \quad 2 \leq i \leq s-1
\end{aligned}$$

From the equalities $0 = \big[e_1, [x, x]\big]$, $0 = \big[f_1, [x, x]\big]$ we conclude that:

$$[x, x] \in \langle\{e_2, e_3, \ldots, e_k, f_2, f_3, \ldots, f_s\}\rangle.$$

Therefore, $R^2 \subseteq \{e_2, e_3, \ldots, e_k, f_2, f_3, \ldots, f_s\}$. Moreover, we have $R^i \subseteq \{e_i, e_{i+1}, \ldots, e_k, f_i, f_{i+1}, \ldots, f_s\}$, which implies that $R^{\max\{k,s\}+1} = \{0\}$. Thus, we have a contradiction to the assumption that R is not nilpotent. Hence, the algebras $NF_k + Q$ and $NF_s + Q$ cannot be both nilpotent.

Without loss of generality, we can assume that algebra $NF_k + Q$ is non-nilpotent.

We take the quotient algebra by ideal NF_s, then $R/NF_s \cong \overline{NF_k} + \overline{Q}$. Thanks to Theorem 1.2.6, the structure of the algebra $\overline{NF_k} + \overline{Q}$ is known. Namely,

$$\begin{cases}
[\overline{e_i}, \overline{e_1}] = \overline{e_{i+1}}, & 1 \leq i \leq k-1,\\
[\overline{x}, \overline{e_1}] = \overline{e_1}, &\\
[\overline{e_i}, \overline{x}] = -i\overline{e_i}, & 1 \leq i \leq k.
\end{cases} \tag{1.2.4}$$

Using the fact that NF_k and NF_s are ideals of R and having in mind the multiplication table (1.2.4), we have that:

$$\begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \le i \le k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \le i \le s-1, \\ [x, e_1] = e_1, & [x, f_1] = \sum_{i=1}^{s} \alpha_i f_i, \\ [e_i, x] = -ie_i, \quad 1 \le i \le k, & [f_1, x] = \sum_{i=1}^{s} \beta_i f_i, \\ & [x, x] = \sum_{i=1}^{s} \gamma_i f_i. \end{cases} \tag{1.2.5}$$

If $\alpha_1 \neq 0$, then in a similar way as the Case 1 of Theorem 1.2.6 we obtain the family of algebras $R(\alpha)$, where $\alpha \neq 0$.

The fact that two algebras in the family $R(\alpha)$ with different values of parameter α are not isomorphic can be easily determined by a general change of basis and considering the expansion of the product $[x', f_1']$ in both bases.

Now consider $\alpha_1 = 0$. Then by the change of basis

$$x' = x - (\alpha_2 f_1 + \alpha_3 f_2 + \cdots + \alpha_s f_{s-1}),$$

we can suppose $[x, f_1] = 0$.

From the identity $\{f_1, f_1, x\}$ we get $\beta_1 = 0$.

Similarly to the proof of Equations (1.2.1), we can prove that

$$[f_i, x] = \sum_{m=i+1}^{s} \beta_{m-i+1} f_j, \ 1 \le i \le s.$$

The identity $\{x, f_1, x\}$ implies the following chain of equalities:

$$0 = -\big[[x, x], f_1\big] = -\sum_{m=3}^{s} \gamma_{m-1} f_m \, .$$

Consequently, $\gamma_i = 0, \ 2 \le i \le s-1$.

Thus, we obtain the products of the family $R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$

$$\begin{cases} [f_i, f_1] = f_{i+1}, & 1 \le i \le s-1, \\ [f_i, x] = \sum_{m=i+1}^{s} \beta_{m-i+1} f_m, & 1 \le i \le s, \\ [x, x] = \gamma_s f_s. \end{cases}$$

Now we are going to study the isomorphism inside the family $R(\beta_2, \beta_3, \dots, \beta_s, \gamma)$.

Taking into account that, under general basis transformation, the products (1.2.5) should not be changed, we conclude that it is sufficient to take the following change of basis:

$$f'_i = A_1^{i-1} \sum_{j=i}^{s} A_{j-i+1} f_j, \quad (A_1 \neq 0), \qquad 1 \leq i \leq s, \qquad x' = x.$$

Then we have

$$[f'_1, x'] = \sum_{i=1}^{s} A_i [f_i, x] = \sum_{i=1}^{s-1} A_i \Big(\sum_{j=i+1}^{s} \beta_{j-i+1} f_j \Big) = \sum_{i=2}^{s} \Big(\sum_{j=1}^{i-1} A_j B_{i-j+1} \Big) f_i.$$

On the other hand

$$[f'_1, x'] = \sum_{i=2}^{s} \beta'_i f'_i = \sum_{i=1}^{s-1} A_1^i \beta'_{i+1} \Big(\sum_{j=1}^{s-i} A_j f_{i+j} \Big) = \sum_{i=2}^{s} \Big(\sum_{j=1}^{i-1} A_1^j A_{i-j} \beta'_{j+1} \Big) f_i.$$

Comparing coefficients at the elements of the basis we deduce that:

$$\sum_{i=1}^{k-1} A_i \beta_{k-i+1} = \sum_{i=1}^{k-1} A_1^i A_{k-i} \beta'_{i+1}, \qquad k = 2, 3, \dots, s.$$

From these systems of equations it follows:

$$\beta'_i = \frac{\beta_i}{A_1^{i-1}}, \qquad 2 \leq i \leq s.$$

If we consider

$$\gamma'_s A_1^s f_s = \gamma'_s f'_s = [x', x'] = [x, x] = \gamma_s f_s,$$

then we obtain

$$\gamma'_s = \frac{\gamma_s}{A_1^s}.$$

It is easy to see that by choosing an adequate value for the parameter A_1, then the first non-zero element of the vector $(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ can be assumed to be equal to 1.

Therefore, two algebras $R(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ and $R(\beta'_2, \beta'_3, \dots, \beta'_s, \gamma')$ with different set of parameters are not isomorphic.

For given parameters α and $\beta_2, \beta_3, \dots, \beta_s, \gamma$, the algebras $R(\alpha)$ and $R(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ are not isomorphic because

$$k + s = \dim R(\alpha)^2 \neq \dim R(\beta_2, \beta_3, \dots, \beta_s, \gamma)^2 = k + s - 1.$$

□

Remark 1.2.9. *In the case when all the coefficients* $(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ *are equal to zero we have the split algebra* $(NF_k + Q) \oplus NF_s$*. Therefore, in the non-split case, we can always assume that* $(\beta_2, \beta_3, \dots, \beta_s, \gamma) \neq (0, 0, 0, \dots, 0)$*.*

Now, by an induction process, we are going to generalize Theorem 1.2.8 to the case when the nilradical is a direct sum (greater than 2) of several copies of null-filiform ideals.

Theorem 1.2.10. *Let* R *be a solvable Leibniz algebra such that* $R = NF_{n_1} \oplus NF_{n_2} \oplus \dots \oplus NF_{n_s} + Q$*, where* $NF_{n_1} \oplus NF_{n_2} \oplus \dots \oplus NF_{n_s}$ *is the nilradical of* R *and* $\dim Q = 1$*. There exist* $p, q \in \mathbb{N}$ *with* $p \neq 0$ *and* $p + q = s$*, a basis* $\{e^i_1, e^i_2, \dots, e^i_{n_i}\}$ *of* NF_{n_i}*, for* $1 \leq i \leq p$*, a basis* $\{f^k_1, f^k_2, \dots, f^k_{n_k}\}$ *of* $NF_{n_{p+k}}$*, for* $1 \leq k \leq q$*, and a basis* $\{x\}$ *of* Q *such that the multiplication table of the algebra* R *is given by:*

$$R_{p,q}: \begin{cases} [e^j_i, e^j_1] = e^j_{i+1}, & 1 \leq i \leq n_j - 1, \\ [f^k_i, f^k_1] = f^k_{i+1}, \quad 1 \leq i \leq n_k - 1, & \\ [x, e^j_1] = \delta^j e^j_1, & \delta^j \neq 0 \\ [f^k_i, x] = \displaystyle\sum_{m=i+1}^{n_k} \beta^k_{m-i+1} f^k_m, \quad 1 \leq i \leq n_k, & \\ [e^j_i, x] = -i\delta^j e^j_i, & 1 \leq i \leq n_j, \\ [x, x] = \displaystyle\sum_{m=1}^{k} \gamma^m f_{n_m}, & \end{cases} \tag{1.2.6}$$

where $1 \leq j \leq p$, $1 \leq k \leq q$ *and* $\delta^1 = 1$*. Moreover, the first non-zero component of the vectors* $(\beta^k_2, \beta^k_3, \dots, \beta^k_{n_k}, \gamma^k)$ *can be assumed to be equal to 1. Moreover, the algebras are pairwise non-isomorphic.*

Proof. By induction on s:

If $s=1$, then $p=1,\ q=0$, so $R_{1,0}$ is the algebra given in Theorem 1.2.6.

If $s=2$, then we have two cases: either $p=2,\ q=0$ or $p=1,\ q=1$, which were considered in Theorem 1.2.7. Namely, we have two families of algebras: $R(\alpha)$, which corresponds to $R_{2,0}$, and $R(\beta_2,\beta_3,\dots,\beta_s,\gamma)$, which corresponds to $R_{1,1}$.

Let us assume that the theorem is true for s and we shall prove it for $s+1$.

Let $R = NF_{n_1} \oplus NF_{n_2} \oplus \cdots \oplus NF_{n_s} \oplus NF_{n_{s+1}} + Q$. We consider the quotient algebra by $NF_{n_{s+1}}$, i.e. $R/NF_{n_{s+1}} \cong \overline{NF_{n_1}} \oplus \overline{NF_{n_2}} \oplus \cdots \oplus \overline{N_{nF_s}} + \overline{Q}$. Then we get the multiplication table given in (1.2.6).

Note that the multiplication table for the algebra R can be obtained from (1.2.6) by adding the products

$$
\begin{aligned}
[e_i^{s+1}, e_1^{s+1}] &= e_{i+1}^{s+1}, & 1 \le i \le n_{s+1}-1,\\
[x, e_1^{s+1}] &= \sum_{m=1}^{n_{s+1}} \alpha_m^{s+1} e_m^{s+1}, &\\
[e_1^{s+1}, x] &= \sum_{m=1}^{n_{s+1}} \beta_m^{s+1} e_m^{s+1}, &\\
[x, x] &= \sum_{m=1}^{n_{s+1}} \gamma_m^{s+1} e_m^{s+1}. &
\end{aligned}
$$

If $\alpha_1^{s+1} \neq 0$, then in an analogous way as in proof of Theorem 1.2.6, we deduce that

$$
\begin{aligned}
[e_i^{s+1}, e_1^{s+1}] &= e_{i+1}^{s+1}, & 1 \le i \le n_{s+1}-1,\\
[x, e_1^{s+1}] &= \alpha_{s+1}^{s+1} e_1^{s+1}, &\\
[e_i^{s+1}, x] &= -i\alpha^{s+1} e_i^{s+1}, & 1 \le i \le n_{s+1}.
\end{aligned}
$$

Therefore we get the algebra $R_{p+1,q}$.

If $\alpha_1^{s+1}=0$, then by similar arguments as in Theorem 1.2.8, we obtain

$$\begin{aligned}
[e_i^{s+1}, e_1^{s+1}] &= e_{i+1}^{s+1}, && 1 \le i \le n_{s+1}-1,\\
[e_i^{s+1}, x] &= \sum_{m=i+1}^{n_{s+1}} \beta_{m-i+1}^{s+1} f_m^{s+1}, && 1 \le i \le n_{s+1},\\
[x, x] &= \sum_{m=1}^{k} \gamma^m f_{n_m} + \gamma^{s+1} f_{n_{s+1}}^{s+1}.
\end{aligned}$$

Setting $f_{i-1}^{q+1}=e_{i-1}^{s+1}$ we get the family of algebras $R_{p,q+1}$.

The proof that two algebras of the family $R_{p,q}$ with different values of parameters are not isomorphic is carrying out in a similar way as in the proof of Theorem 1.2.8. □

In fact, due to Theorem 1.2.2, the complementary vector space, in the case when the nilradical of a solvable Leibniz algebra is a direct sum of s copies of null-filiform ideals, has dimension not grater than s. By taking direct sum of ideals $NF_i + Q_i$ and $NF_k \oplus \cdots \oplus NF_s$, where $1 \le i \le k-1,\ k \le s$, we can construct a solvable Leibniz algebra whose nilradical is $NF_1 \oplus \cdots \oplus NF_s$ and whose complementary vector space is k-dimensional for each k ($k \le s$).

1.3 Solvable Leibniz algebras with triangular nilradical

We denote by $L(n, f)$ a set of all non-nilpotent solvable Leibniz algebras with nilradical $T(n)$ and a complementary vector space $\langle X^1, X^2, \dots, X^f\rangle$.

Using notations (1.1.1) we have

$$R_{X^\alpha}(N) = A^\alpha N, \qquad L_{X^\alpha}(N) = B^\alpha N,$$

where $A^\alpha = (a_{ij,pq}^\alpha)$ and $B^\alpha = (b_{ij,pq}^\alpha),\ 1 \le i < j \le n,\ 1 \le p < q \le n.$

Since the proof of the assertions concerning the elements of the matrix A^α in Lemma 1.1.16 uses only the property of derivation, one can check that it obviously extends to our case of Leibniz algebras. For the matrix B^α however, we have the next result.

Lemma 1.3.1. *The following relations hold:*

$$b_{ij,pq}^\alpha = -a_{ij,pq}^\alpha, \quad i+1<j, \quad (p,q) \ne (1,n).$$

Proof. From Lemma 1.1.16 we conclude

$$[N_{12}, X^\alpha] = a^\alpha_{12,12}N_{12} + a^\alpha_{12,2n}N_{2n},$$

$$[N_{i(i+1)}, X^\alpha] = a^\alpha_{i(i+1),i(i+1)}N_{i(i+1)} + a^\alpha_{i(i+1),1n}N_{1n}, \qquad 2 \le i \le n-2,$$

$$[N_{(n-1)n}, X^\alpha] = a^\alpha_{(n-1)n,(n-1)n}N_{(n-1)n} + a^\alpha_{(n-1)n,1(n-1)}N_{1(n-1)},$$

$$[N_{ij}, X^\alpha] = \sum_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)}N_{ij}, \qquad i+1 < j.$$

It is easy to see that $[X^\alpha, N_{12}] + [N_{12}, X^\alpha]$ belongs to the right annihilator of the algebra of $L(n, f)$. From the chain of equalities

$$0 = [N_{12}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]] = [N_{12}, \sum_{i=3}^{n-1} b^\alpha_{12,2i}N_{2i} + (a^\alpha_{12,2n} + b^\alpha_{12,2n})N_{2n}]$$
$$= \sum_{i=3}^{n-1} b^\alpha_{12,2i}N_{1i} + (a^\alpha_{12,2n} + b^\alpha_{12,2n})N_{1n},$$

we deduce $b^\alpha_{12,2j} = 0,\ 3 \le j \le n-1$, and $b^\alpha_{12,2n} = -a^\alpha_{12,2n}$.

Similarly, from

$$0 = [N_{1i}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]] = [N_{1i}, \sum_{j=i+1}^{n} b^\alpha_{ij}N_{ij}] = \sum_{j=i+1}^{n} b^\alpha_{ij}N_{1j}, \quad i > 2,$$

we derive $b^\alpha_{12,ij} = 0,\ 2 < i < j \le n$.

From the equality

$$0 = [N_{i(i+1)}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]], \quad i \ge 2,$$

we get

$$b^\alpha_{12,12} = -a^\alpha_{12,12}, \quad b^\alpha_{12,1i} = 0,\ 3 \le i \le n-1.$$

Therefore, we obtain

$$[X^\alpha, N_{12}] = -a^\alpha_{12,12}N_{12} - a^\alpha_{12,2n}N_{2n} + b^\alpha_{12,1n}N_{1n}.$$

Applying analogous argumentations as we used above for the products with $k \geq 2$,

$$\begin{array}{l}
[N_{1k}, [X^{\alpha}, N_{i(i+1)}] + [N_{i(i+1)}, X^{\alpha}]], \\
[N_{i(i+1)}, [X^{\alpha}, N_{i(i+1)}] + [N_{i(i+1)}, X^{\alpha}]], \quad 2 \leq i \leq n-2, \\
[N_{1k}, [X^{\alpha}, N_{(n-1)n}] + [N_{(n-1)n}, X^{\alpha}]], \\
[N_{i(i+1)}, [X^{\alpha}, N_{(n-1)n}] + [N_{(n-1)n}, X^{\alpha}]], \\
[N_{1k}, [X^{\alpha}, N_{ij}] + [N_{ij}, X^{\alpha}]], \\
[N_{i(i+1)}, [X^{\alpha}, N_{ij}] + [N_{ij}, X^{\alpha}]], \quad 1 < j-i < n-1, \\
[N_{1k}, [X^{\alpha}, N_{1n}] + [N_{1n}, X^{\alpha}]], \\
[N_{i(i+1)}, [X^{\alpha}, N_{1n}] + [N_{1n}, X^{\alpha}]],
\end{array}$$

we obtain

$$\begin{aligned}
[X^{\alpha}, N_{i(i+1)}] &= -a^{\alpha}_{i(i+1),i(i+1)} N_{i(i+1)} + b^{\alpha}_{i(i+1),1n} N_{1n}, \quad 2 \leq i \leq n-2, \\
[X^{\alpha}, N_{(n-1)n}] &= -a^{\alpha}_{(n-1)n,(n-1)n} N_{(n-1)n} - a^{\alpha}_{(n-1)n,1(n-1)} N_{1(n-1)} \\
&\quad + \ b^{\alpha}_{(n-1)n,1n} N_{1n}, \\
[X^{\alpha}, N_{ij}] &= -\sum_{p=i}^{j-1} a^{\alpha}_{p(p+1),p(p+1)} N_{ij} + b^{\alpha}_{ij,1n} N_{1n}, \quad 1 < j-i < n-1, \\
[X^{\alpha}, N_{1n}] &= b^{\alpha}_{1n,1n} N_{1n}.
\end{aligned}$$

From the chain of equalities

$$\begin{aligned}
[X^{\alpha}, N_{1n}] &= [X^{\alpha}, [N_{12}, N_{2n}]] = [[X^{\alpha}, N_{12}], N_{2n}] - [[X^{\alpha}, N_{2n}], N_{12}] \\
&= [-a^{\alpha}_{12,12} N_{12} - a^{\alpha}_{12,2n} N_{2n} + b^{\alpha}_{12,1n} N_{1n}, N_{2n}] \\
&\quad - [-\sum_{p=2}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{2n} + b^{\alpha}_{2n,1n} N_{1n}, N_{12}] \\
&= -a^{\alpha}_{12,12} N_{1n} - \sum_{p=2}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{1n} = -\sum_{p=1}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{1n},
\end{aligned}$$

we get $[X^{\alpha}, N_{1n}] = -\sum\limits_{p=1}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{1n}$.

By induction on j we will prove

$$[X^\alpha, N_{i(i+j)}] = -\sum_{p=i}^{i+j-1} a^\alpha_{p(p+1),p(p+1)} N_{i(i+j)}, \quad j-i \geq 2. \tag{1.3.1}$$

The base of induction ensures the equalities

$$[X^\alpha, N_{i(i+2)}] = [X^\alpha, [N_{i(i+1)}, N_{(i+1)(i+2)}]] = [[X^\alpha, N_{i(i+1)}], N_{(i+1)(i+2)}]$$
$$-[[X^\alpha, N_{(i+1)(i+2)}], N_{i(i+1)}] = -\sum_{p=i}^{i+1} a^\alpha_{p(p+1),p(p+1)} N_{i(i+2)}, \quad 1 \leq i \leq n-2.$$

Let us suppose that (1.3.1) holds for j and we will show it for $j+1$. For $i+j+1 \leq n-1$ we have

$$\begin{aligned}
[X^\alpha, N_{i(i+j+1)}] &= [X^\alpha, [N_{i(i+j)}, N_{(i+j)(i+j+1)}]] \\
&= [[X^\alpha, N_{i(i+j)}], N_{(i+j)(i+j+1)}] - [[X^\alpha, N_{(i+j)(i+j+1)}], N_{i(i+j)}] \\
&= [-\sum_{p=i}^{i+j-1} a^\alpha_{p(p+1),p(p+1)} N_{i(i+j)}, N_{(i+j)(i+j+1)}] \\
&- [-a^\alpha_{(i+j)(i+j+1),(i+j)(i+j+1)} N_{(i+j)(i+j+1)} \\
&+ b^\alpha_{(i+j)(i+j+1),1n} N_{1n}, N_{i(i+j)}] = -\sum_{p=i}^{i+j} a^\alpha_{p(p+1),p(p+1)} N_{i(i+j+1)}.
\end{aligned}$$

The following chain of equalities complete the proof of equality (1.3.1)

$$\begin{aligned}
[X^\alpha, N_{in}] &= [X^\alpha, [N_{i(n-1)}, N_{(n-1)n}]] = [[X^\alpha, N_{i(n-1)}], N_{(n-1)n}] \\
&- [[X^\alpha, N_{(n-1)n}], N_{i(n-1)}] = [-\sum_{p=i}^{n-2} a^\alpha_{p(p+1),p(p+1)} N_{i(n-1)}, N_{(n-1)n}] \\
&- [-a^\alpha_{(n-1)n,(n-1)n} N_{(n-1)n} - a^\alpha_{(n-1)n,1(n-1)} N_{1(n-1)} \\
&+ b^\alpha_{(n-1)n,1n} N_{1n}, N_{i(n-1)}] = -\sum_{p=i}^{n-1} a^\alpha_{p(p+1),p(p+1)} N_{in}.
\end{aligned}$$

Therefore, we obtain

$$[X^\alpha, N_{12}] = -a^\alpha_{12,12}N_{12} - a^\alpha_{12,2n}N_{2n} + b^\alpha_{12,1n}N_{1n},$$

$$[X^\alpha, N_{i(i+1)}] = -a^\alpha_{i(i+1),i(i+1)}N_{i(i+1)} + b^\alpha_{i(i+1),1n}N_{1n},\ 2 \le i \le n-2,$$

$$[X^\alpha, N_{(n-1)n}] = -a^\alpha_{(n-1)n,(n-1)n}N_{(n-1)n} - a^\alpha_{(n-1)n,1(n-1)}N_{1(n-1)} + b^\alpha_{(n-1)n,1n}N_{1n},$$

$$[X^\alpha, N_{ij}] = -\sum_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)}N_{ij},\ j > i+1.$$

Comparison of the above products with notations in (1.1.1) completes the proof of lemma. □

Moreover, we could clarify the product $[X^\alpha, X^\beta]$.

Lemma 1.3.2. *For* $1 \le \alpha, \beta \le n$ *we have* $[X^\alpha, X^\beta] = \sigma^{\alpha\beta}N_{1n}$ *for some* $\sigma^{\alpha\beta} \in \mathbb{C}$.

Proof. Consider

$$\begin{aligned}[N_{12}, [X^\alpha, X^\beta]] &= [[N_{12}, X^\alpha], X^\beta] - [[N_{12}, X^\beta], X^\alpha] \\ &= [a^\alpha_{12,12}N_{12} + a^\alpha_{12,2n}N_{2n}, X^\beta] - [a^\beta_{12,12}N_{12} + a^\beta_{12,2n}N_{2n}, X^\alpha] \\ &= a^\alpha_{12,12}(a^\beta_{12,12}N_{12} + a^\beta_{12,2n}N_{2n}) + a^\alpha_{12,2n}(\sum_{p=2}^{n-1} a^\beta_{p(p+1),p(p+1)}N_{2n}) \\ &- a^\beta_{12,12}(a^\alpha_{12,12}N_{12} + a^\alpha_{12,2n}N_{2n}) - a^\beta_{12,2n}(\sum_{p=2}^{n-1} a^\alpha_{p(p+1),p(p+1)}N_{2n}) \\ &= (a^\alpha_{12,12}a^\beta_{12,2n} - a^\beta_{12,12}a^\alpha_{12,2n} - \sum_{p=2}^{n-1} a^\alpha_{p(p+1),p(p+1)}a^\beta_{12,2n} \\ &+ \sum_{p=2}^{n-1} a^\beta_{p(p+1),p(p+1)}a^\alpha_{12,2n})N_{2n}.\end{aligned}$$

On the other hand,

$$[N_{12}, [X^\alpha, X^\beta]] = [N_{12}, \sum_{1 \le q-p < n} \sigma^{\alpha\beta}_{pq}N_{pq}] = \sum_{i=3}^{n} \sigma^{\alpha\beta}_{2i}N_{1i}.$$

Comparing coefficients at the basis elements we derive

$$\sigma_{2i}^{\alpha\beta} = 0, \quad 3 \le i \le n.$$

For $2 \le i \le n-2$ we consider the chain of equalities

$$\begin{aligned}
&[N_{i(i+1)}, [X^\alpha, X^\beta]] = [[N_{i(i+1)}, X^\alpha], X^\beta] - [[N_{i(i+1)}, X^\beta], X^\alpha] \\
&= a^\alpha_{i(i+1),i(i+1)}(a^\beta_{i(i+1),i(i+1)} N_{i(i+1)} + a^\beta_{i(i+1),1n} N_{1n}) \\
&+ a^\alpha_{i(i+1),1n} \sum_{p=1}^{n-1} a^\beta_{p(p+1),p(p+1)} N_{1n} - a^\beta_{i(i+1),1n} \sum_{p=1}^{n-1} a^\alpha_{p(p+1),p(p+1)} N_{1n} \\
&- a^\beta_{i(i+1),i(i+1)}(a^\alpha_{i(i+1),i(i+1)} N_{i(i+1)} + a^\alpha_{i(i+1),1n} N_{1n}) \\
&= (a^\alpha_{i(i+1),i(i+1)} a^\beta_{i(i+1),1n} + a^\alpha_{i(i+1),1n} \sum_{p=1}^{n-1} a^\beta_{p(p+1),p(p+1)} \\
&- a^\beta_{i(i+1),i(i+1)} a^\alpha_{i(i+1),1n} - a^\beta_{i(i+1),1n} \sum_{p=1}^{n-1} a^\alpha_{p(p+1),p(p+1)}) N_{1n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[N_{i(i+1)}, [X^\alpha, X^\beta]] &= [N_{i(i+1)}, \sum_{k=1}^{i-1} \sigma^{\alpha\beta}_{ki} N_{ki} + \sum_{j=i+2}^{n} \sigma^{\alpha\beta}_{(i+1)j} N_{(i+1)j}] \\
&= -\sum_{k=1}^{i-1} \sigma^{\alpha\beta}_{ki} N_{k(i+1)} + \sum_{j=i+2}^{n} \sigma^{\alpha\beta}_{(i+1)j} N_{ij}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma^{\alpha\beta}_{ki} = \sigma^{\alpha\beta}_{js} = 0, \quad & 1 \le k \le i-1, \quad 2 \le i \le n-2, \\
& 3 \le j \le n-1, \quad j+1 \le s \le n
\end{aligned}$$

and

$$[X^\alpha, X^\beta] = \sigma^{\alpha\beta}_{1(n-1)} N_{1(n-1)} + \sigma^{\alpha\beta}_{1n} N_{1n}.$$

Similar arguments for the products

$$[N_{(n-1)n}, [X^\alpha, X^\beta]]$$

yield $\sigma^{\alpha\beta}_{1(n-1)} = 0$, which completes the proof of the lemma. For convenience let us omit the lower indexes of $\sigma^{\alpha\beta}_{1n}$. □

From Leibniz identity $\{X^\alpha, N_{i(i+1)}, X^\alpha\}$, for $1 \le i \le n-1$, we obtain the restrictions:

$$a^\alpha_{i(i+1),i(i+1)}(a^\alpha_{i(i+1),1n} + b^\alpha_{i(i+1),1n}) = 0, \quad 2 \le i \le n-2,$$
$$a^\alpha_{12,12} b^\alpha_{12,1n} = a^\alpha_{(n-1)n,(n-1)n} b^\alpha_{(n-1)n,1n} = 0.$$

Let us resume the obtained products of the basis elements. For $1 \le \alpha \le f$ we have

$$\begin{cases}
[N_{12}, X^\alpha] = a^\alpha_{12,12} N_{12} + a^\alpha_{12,2n} N_{2n}, \\
[N_{i(i+1)}, X^\alpha] = a^\alpha_{i(i+1),i(i+1)} N_{i(i+1)} + a^\alpha_{i(i+1),1n} N_{1n}, \qquad 2 \le i \le n-2, \\
[N_{(n-1)n}, X^\alpha] = a^\alpha_{(n-1)n,(n-1)n} N_{(n-1)n} + a^\alpha_{(n-1)n,1(n-1)} N_{1(n-1)}, \\
[N_{ij}, X^\alpha] = \sum\limits_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)} N_{ij}, \qquad j > i+1, \\
[X^\alpha, N_{12}] = -a^\alpha_{12,12} N_{12} - a^\alpha_{12,2n} N_{2n} + b^\alpha_{12,1n} N_{1n}, \\
[X^\alpha, N_{i(i+1)}] = -a^\alpha_{i(i+1),i(i+1)} N_{i(i+1)} + b^\alpha_{i(i+1),1n} N_{1n}, \qquad 2 \le i \le n-2, \\
[X^\alpha, N_{(n-1)n}] = -a^\alpha_{(n-1)n,(n-1)n} N_{(n-1)n} - a^\alpha_{(n-1)n,1(n-1)} N_{1(n-1)} \\
\qquad + \ b^\alpha_{(n-1)n,1n} N_{1n}, \\
[X^\alpha, N_{ij}] = -\sum\limits_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)} N_{ij}, \qquad j > i+1, \\
[X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{1n},
\end{cases}$$

with restrictions on parameters:

$$a^\alpha_{i(i+1),i(i+1)}(a^\alpha_{i(i+1),1n} + b^\alpha_{i(i+1),1n}) = 0, \qquad 2 \le i \le n-2,$$
$$a^\alpha_{12,12} b^\alpha_{12,1n} = a^\alpha_{(n-1)n,(n-1)n} b^\alpha_{(n-1)n,1n} = 0.$$

Note that for solvable non-Lie Leibniz algebras of the set $L(n,f)$ the following equality holds

$$[X^\gamma, N_{1n}] = [N_{1n}, X^\gamma] = 0, \quad 1 \le \gamma \le f. \tag{1.3.2}$$

Indeed, if we assume the contrary, then taking into account that $[X^\gamma, N_{1n}] = -[N_{1n}, X^\gamma]$ we can assume $[X^\gamma, N_{1n}] \neq 0$ for some $\gamma \in \{1, \dots, f\}$.

Simplifying the following products using the Leibniz identity

$$[X^\gamma, [N_{12}, X^\alpha] + [X^\alpha, N_{12}]], \qquad [X^\gamma, [N_{i(i+1)}, X^\alpha] + [X^\alpha, N_{i(i+1)}]],$$
$$[X^\gamma, [N_{(n-1)n}, X^\alpha] + [X^\alpha, N_{(n-1)n}]], \qquad [X^\gamma, [X^\alpha, X^\beta] + [X^\beta, X^\alpha]],$$
$$[X^\gamma, [X^\alpha, X^\alpha]],$$

we obtain

$$b^\alpha_{12,1n} = b^\alpha_{(n-1)n,1n} = \sigma^{\alpha\alpha} = 0, \quad b^\alpha_{i(i+1),1n} = -a^\alpha_{i(i+1),1n}, \quad \sigma^{\alpha\beta} = -\sigma^{\beta\alpha}.$$

Thus we get a Lie algebra, which is a contradiction.

Corollary 1.3.3. *For a Leibniz algebra of the set $L(n,1)$, the matrices of the left and right multiplication operators, $A = (a_{ij,pq})$ and $B = (b_{ij,pq})$, have the following properties:*

(1) The maximum number of off-diagonal elements of the matrix A is $n-1$;

(2) The maximum number of off-diagonal elements of the matrix B is $n+1$.

Theorem 1.3.4. *A solvable Leibniz algebra of the set $L(n, n-1)$ is a Lie algebra.*

Proof. Making suitable change of basis we can assume that operator R_{X^1} acts as follows

$$\begin{aligned}
&[N_{12}, X^1] = N_{12} + a^1_{12,2n} N_{2n}, & \\
&[N_{i(i+1)}, X^1] = a^1_{i(i+1),1n} N_{1n}, & 2 \leq i \leq n-2, \\
&[N_{(n-1)n}, X^1] = a^1_{(n-1)n,1(n-1)} N_{1(n-1)}, & \\
&[N_{1j}, X^1] = N_{1j}, & j > 2.
\end{aligned}$$

Since $[N_{1n}, X^1] = N_{1n}$, then from Equation (1.3.2) it follows that the algebra is a Lie algebra. □

So, we present a description of solvable Leibniz algebras with nilradical $T(n)$. Moreover, in the case of maximal possible dimension we show that this algebra is a Lie algebra.

Now we give an illustration for low dimensions of solvable Leibniz algebras with nilradical $T(n)$. Note that the Lie algebra $T(3)$ is nothing else but the Heisenberg algebra $\mathfrak{h}_1$. Solvable Leibniz algebras with Heisenberg nilradical were described in [14]. Therefore, we give the description of solvable Leibniz algebras with nilradical $T(4)$.

We know that the complementary vector space to the nilradical $T(4)$ has dimension less than four. In case when dimension of the complementary space is equal to 3 we obtain a Lie algebra (see Theorem 1.3.4), which falls into the classification already obtained in [55]. So, we will consider the dimension of the complementary vector space to be equal to 1 and 2.

The Leibniz algebras $L(4,1)$.

From previous section we have that the algebra $L(4,1)$ admits a basis $\{N_{12}, N_{23}, N_{34}, N_{13}, N_{24}, N_{14}, X\}$ in which the multiplication table has the following form:

$$\begin{cases} [N_{12}, X] & = & a_{12,12}N_{12} + a_{12,24}N_{24}, \\ [X, N_{12}] & = & -a_{12,12}N_{12} - a_{12,24}N_{24} + b_{12,14}N_{14}, \\ [N_{23}, X] & = & a_{23,23}N_{23} + a_{23,14}N_{14}, \\ [X, N_{23}] & = & -a_{23,23}N_{23} + b_{23,14}N_{14}, \\ [N_{34}, X] & = & -(a_{12,12} + a_{23,23})N_{34} + a_{34,13}N_{13}, \\ [X, N_{34}] & = & (a_{12,12} + a_{23,23})N_{34} - a_{34,13}N_{13} + b_{34,14}N_{14}, \\ [N_{13}, X] & = & -[X, N_{13}] = (a_{12,12} + a_{23,23})N_{13}, \\ [N_{24}, X] & = & -[X, N_{24}] = -a_{12,12}N_{24}, \\ [X, X] & = & \sigma_{14}N_{14}, \end{cases} \tag{1.3.3}$$

where

$$a_{12,12}b_{12,14} = a_{23,23}(a_{23,14} + b_{23,14}) = (a_{12,12} + a_{23,23})b_{34,14} = 0.$$

Since $L(4,1)$ is a non-nilpotent Leibniz algebra we conclude $(a_{12,12}, a_{23,23}) \neq (0,0)$.

Case 1. Let $a_{12,12} = 0$. Then $a_{23,23} \neq 0, b_{23,14} = -a_{23,14}$ and $b_{34,14} = 0$.

Taking the change of basis as follows:

$$X' = \frac{1}{a_{23,23}}X, \quad N'_{23} = N_{23} + \frac{a_{23,14}}{a_{23,23}}N_{14}, \quad N'_{34} = N_{34} - \frac{a_{34,13}}{2a_{23,23}}N_{13}$$

the multiplication (1.3.3) transforms into

$$\begin{aligned}
&[N_{12}, X] = a_{12,24}N_{24}, && [X, N_{12}] = -a_{12,24}N_{24} + b_{12,14}N_{14},\\
&[N_{23}, X] = -[X, N_{23}] = N_{23}, && [N_{34}, X] = -[X, N_{34}] = -N_{34},\\
&[N_{13}, X] = -[X, N_{13}] = N_{13}, && [X, X] = \sigma_{14}N_{14},
\end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

Case 2. If $a_{12,12} \neq 0$, then $b_{12,14} = 0$. Taking by the scaling $X' = \frac{1}{a_{12,12}}X$, we can assume $a_{12,12} = 1$.

Subcase 2.1. Let $a_{23,23} = 0$. Then $b_{34,14} = 0$.

Applying the change of basis

$$N'_{12} = N_{12} + \frac{a_{12,24}}{2}N_{24}, \quad N'_{34} = N_{34} - \frac{a_{34,13}}{2}N_{13}$$

the products (1.3.3) simplify to the following:

$$\begin{aligned}
&[N_{12}, X] = -[X, N_{12}] = N_{12}, && [N_{34}, X] = -[X, N_{34}] = -N_{34},\\
&[N_{13}, X] = -[X, N_{13}] = N_{13}, && [N_{24}, X] = -[X, N_{24}] = -N_{24},\\
&[N_{23}, X] = a_{23,14}N_{14}, && [X, N_{23}] = b_{23,14}N_{14},\\
&[X, X] = \sigma_{14}N_{14},
\end{aligned}$$

where $(a_{23,14} + b_{23,14}, \sigma_{14}) \neq (0, 0)$.

Subcase 2.2. Let $a_{23,23} \neq 0$. Then $b_{23,14} = -a_{23,14}$.

Subcase 2.2.1. Let $a_{23,23} = -1$. Then substituting

$$N'_{23} = N_{23} - a_{23,14}N_{14}, \quad N'_{12} = N_{12} + \frac{a_{12,24}}{2}N_{24}$$

we derive to an algebra with the following multiplication table:

$$\begin{aligned}
&[N_{12}, X] = -[X, N_{12}] = N_{12}, && [N_{23}, X] = [X, N_{23}] = -N_{23},\\
&[N_{34}, X] = a_{34,13}N_{13}, && [X, N_{34}] = -a_{34,13}N_{13} + b_{34,14}N_{14},\\
&[N_{24}, X] = -[X, N_{24}] = -N_{24}, && [X, X] = \sigma_{14}N_{14},
\end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

Note that by permuting the indexes of the basis elements of the above algebra one obtains an algebra from Case 1.

Subcase 2.2.2. Let $a_{23,23} \neq -1$. Then $b_{34,14} = 0$.
Setting

$$N'_{12} = N_{12} + \frac{a_{12,24}}{2} N_{24}, \quad N'_{23} = N_{23} + \frac{a_{23,14}}{a_{23,23}} N_{14},$$

$$N'_{34} = \sigma_{14}(N_{34} - \frac{a_{34,13}}{2(1 + a_{23,23})} N_{13}), \quad N'_{24} = \sigma_{14} N_{24}, \quad N'_{14} = \sigma_{14} N_{14},$$

we get an algebra with the following table of multiplications:

$$\begin{array}{ll} [N_{23}, X] = -[X, N_{23}] = a_{23,23} N_{23}, & [N_{12}, X] = -[X, N_{12}] = N_{12}, \\ [N_{34}, X] = -[X, N_{34}] = -(1 + a_{23,23}) N_{34}, & [N_{24}, X] = -[X, N_{24}] = -N_{24}, \\ [N_{13}, X] = -[X, N_{13}] = (1 + a_{23,23}) N_{13}, & [X, X] = N_{14}, \end{array}$$

where $(1 + a_{23,23}) a_{23,23} \neq 0$.

Non-isomorphisms of obtained algebras can be easily established by considering the dimensions of derived series of the algebras.

Thus, the following theorem is proved.

Theorem 1.3.5. *An arbitrary non-Lie Leibniz algebra of the set $L(4,1)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$L_1 : \begin{array}{ll} [N_{12}, X] = a_{12,24} N_{24}, & [X, N_{12}] = -a_{12,24} N_{24} + b_{12,14} N_{14}, \\ [N_{23}, X] = -[X, N_{23}] = N_{23}, & [N_{34}, X] = -[X, N_{34}] = -N_{34}, \\ [N_{13}, X] = -[X, N_{13}] = N_{13}, & [X, X] = \sigma_{14} N_{14}, \end{array}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

$$L_2 : \begin{array}{ll} [N_{12}, X] = -[X, N_{12}] = N_{12}, & [N_{34}, X] = -[X, N_{34}] = -N_{34} \\ [N_{13}, X] = -[X, N_{13}] = N_{13}, & [N_{24}, X] = -[X, N_{24}] = -N_{24}, \\ [N_{23}, X] = a_{23,14} N_{14}, & [X, N_{23}] = b_{23,14} N_{14}, \\ [X, X] = \sigma_{14} N_{14}, & \end{array}$$

where $(a_{23,14} + b_{23,14}, \sigma_{14}) \neq (0, 0)$.

$$L_3 : \begin{array}{rl} [N_{23}, X] & = -[X, N_{23}] = a_{23,23} N_{23}, \\ [N_{12}, X] & = -[X, N_{12}] = N_{12}, \\ [N_{34}, X] & = -[X, N_{34}] = -(1 + a_{23,23}) N_{34}, \\ [N_{24}, X] & = -[X, N_{24}] = -N_{24}, \\ [N_{13}, X] & = -[X, N_{13}] = (1 + a_{23,23}) N_{13}, \\ [X, X] & = N_{14}, \end{array}$$

where $(1 + a_{23,23})a_{23,23} \neq 0$.

The Leibniz algebras $L(4, 2)$.

Classification of Leibniz algebras in this set is presented in the following theorem.

Theorem 1.3.6. *An arbitrary non-Lie Leibniz algebra of the set* $L(4, 2)$ *admits a basis* $\{N_{12}, N_{23}, N_{34}, N_{13}, N_{24}, N_{14}, X^1, X^2\}$ *in which the multiplication table has the following form:*

$$
\begin{array}{ll}
[N_{12}, X^1] = -[X^1, N_{12}] = N_{12}, & [N_{34}, X^1] = -[X^1, N_{34}] = -N_{34}, \\
[N_{13}, X^1] = -[X^1, N_{13}] = N_{13}, & [N_{24}, X^1] = -[X^1, N_{24}] = -N_{24}, \\
[N_{23}, X^2] = -[X^2, N_{23}] = N_{23}, & [N_{34}, X^2] = -[X^2, N_{34}] = -N_{34}, \\
[N_{13}, X^2] = -[X^2, N_{13}] = N_{13}, & [X^1, X^1] = \sigma^{11} N_{14}, \\
[X^2, X^2] = \sigma^{22} N_{14}, & [X^1, X^2] = \sigma^{12} N_{14}, \\
[X^2, X^1] = \sigma^{21} N_{14}. &
\end{array}
$$

Proof. From Lemmas 1.3.1 and 1.3.2 we have

$$
\begin{aligned}
&[N_{12}, X^1] = a^1_{12,12} N_{12} + a^1_{12,24} N_{24}, \\
&[X^1, N_{12}] = -a^1_{12,12} N_{12} - a^1_{12,24} N_{24} + b^1_{12,14} N_{14}, \\
&[N_{23}, X^1] = a^1_{23,23} N_{23} + a^1_{23,14} N_{14}, \\
&[X^1, N_{23}] = -a^1_{23,23} N_{23} + b^1_{23,14} N_{14}, \\
&[N_{34}, X^1] = -(a^1_{12,12} + a^1_{23,23}) N_{34} + a^1_{34,13} N_{13}, \\
&[X^1, N_{34}] = (a^1_{12,12} + a^1_{23,23}) N_{34} - a^1_{34,13} N_{13} + b^1_{34,14} N_{14}, \\
&[N_{13}, X^1] = -[X^1, N_{13}] = (a^1_{12,12} + a^1_{23,23}) N_{13}, \\
&[N_{24}, X^1] = -[X^1, N_{24}] = -a^1_{12,12} N_{24}, \\
&[N_{12}, X^2] = a^2_{12,12} N_{12} + a^2_{12,24} N_{24}, \\
&[X^2, N_{12}] = -a^2_{12,12} N_{12} - a^2_{12,24} N_{24} + b^2_{12,14} N_{14}, \\
&[N_{23}, X^2] = a^2_{23,23} N_{23} + a^2_{23,14} N_{14}, \\
&[X^2, N_{23}] = -a^2_{23,23} N_{23} + b^2_{23,14} N_{14}, \\
&[N_{34}, X^2] = -(a^2_{12,12} + a^2_{23,23}) N_{34} + a^2_{34,13} N_{13}, \\
&[X^2, N_{34}] = (a^2_{12,12} + a^2_{23,23}) N_{34} - a^2_{34,13} N_{13} + b^2_{34,14} N_{14}, \\
&[N_{13}, X^2] = -[X^2, N_{13}] = (a^2_{12,12} + a^2_{23,23}) N_{13}, \\
&[N_{24}, X^2] = -[X^2, N_{24}] = -a^2_{12,12} N_{24},
\end{aligned}
$$

with the restrictions

$$a^1_{12,12}b^1_{12,14} = a^1_{23,23}(a^1_{23,14} + b^1_{23,14}) = (a^1_{12,12} + a^1_{23,23})b^1_{34,14} = 0,$$
$$a^2_{12,12}b^2_{12,14} = a^2_{23,23}(a^2_{23,14} + b^2_{23,14}) = (a^2_{12,12} + a^2_{23,23})b^2_{34,14} = 0.$$

Taking the change of basis

$$X^{1'} = \frac{a^2_{23,23}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^1 - \frac{a^1_{23,23}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^2,$$

$$X^{2'} = -\frac{a^2_{12,12}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^1 + \frac{a^1_{12,12}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^2,$$

we deduce

$$\begin{aligned}
&[N_{23}, X^1] = a^1_{23,14}N_{14},\\
&[N_{12}, X^1] = -[X^1, N_{12}] = N_{12} + a^1_{12,24}N_{24}\\
&[X^1, N_{23}] = b^1_{23,14}N_{14},\\
&[N_{34}, X^1] = -[X^1, N_{34}] = -N_{34} + a^1_{34,13}N_{13},\\
&[N_{12}, X^2] = a^2_{12,24}N_{24},\\
&[N_{23}, X^2] = -[X^2, N_{23}] = N_{23} + a^2_{23,14}N_{14},\\
&[N_{13}, X^1] = -[X^1, N_{13}] = N_{13},\\
&[X^2, N_{12}] = -a^2_{12,24}N_{24} + b^2_{12,14}N_{14},\\
&[N_{24}, X^1] = -[X^1, N_{24}] = -N_{24},\\
&[N_{34}, X^2] = -[X^2, N_{34}] = -N_{34} + a^2_{34,13}N_{13},\\
&[N_{13}, X^2] = -[X^2, N_{13}] = N_{13}.
\end{aligned}$$

Applying the Leibniz identity to the following triples of elements:

$\{N_{12}, X^1, X^2\}$, $\{N_{23}, X^1, X^2\}$, $\{N_{34}, X^1, X^2\}$, $\{X^1, N_{23}, X^2\}$, $\{X^2, N_{12}, X^1\}$

we get

$$a^2_{12,24} = a^1_{23,14} = a^1_{34,13} = a^2_{34,13} = b^1_{23,14} = b^2_{12,14} = 0.$$

Finally, taking the basis transformation:

$$N'_{12} = N_{12} + \frac{a^1_{12,24}}{2}N_{24}, \quad N'_{23} = N_{23} + a^2_{23,14}N_{14},$$

we obtain the multiplication table listed in the assertion of the theorem. □

Chapter 2

Solvable Leibniz algebras with naturally graded filiform nilradicals

All solvable Lie algebras whose nilradical is naturally graded filiform Lie algebra $n_{n,1}$ are classified in [52]. Further, solvable Lie algebras whose nilradical is naturally graded filiform Lie algebra Q_{2n} are classified in [4].

We give the classifications of solvable non-Lie Leibniz algebras whose nilradical are naturally graded filiform Lie and naturally graded filiform non-Lie Leibniz algebras, separately.

2.1 Solvable Leibniz algebras with naturally graded filiform Lie nilradicals

It is proved that the dimension of a solvable Lie algebra whose nilradical is isomorphic to an n-dimensional naturally graded filiform Lie algebra is not greater than $n+2$. Below, we present its classification.

In order to agree with the multiplication tables of algebras in Theorems 1.1.8 and 1.1.9, we make the following change of basis in the classification of [52]:

$$e'_i = e_{n+1-i}, \quad 1 \le i \le n, \qquad\qquad x = -f.$$

We also use different notation to denote the algebras that appear in [52].

Theorem 2.1.1 ([52]). *There are three types of solvable Lie algebras of dimension $(n+1)$ whose nilradical is isomorphic to $\mathfrak{n}_{n,1}$ $(n \geq 4)$. The isomorphism classes in the basis $\{e_1, \dots, e_n, x\}$ are represented by the following algebras:*

$$S_{n+1}(\alpha,\beta): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = \big((i-2)\alpha + \beta\big)\, e_i, & 2 \leq i \leq n, \\ [e_1, x] = -[x, e_1] = \alpha e_1. \end{cases}$$

The mutually non-isomorphic algebras of this type are $S_{n+1,1}(\beta) = S_{n+1}(1,\beta)$ (depending on the value of β, in this case there are three different classes, $\beta = 0$, $\beta = n-2$ and $\beta \notin \{0, n-2\}$) and $S_{n+1,2} = S_{n+1}(0,1)$.

$$S_{n+1,3}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = (i-1)\, e_i, & 2 \leq i \leq n, \\ [e_1, x] = -[x, e_1] = e_1 + e_2. \end{cases}$$

$$S_{n+1,4}(\alpha_3, \alpha_4, \dots, \alpha_{n-1}): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = e_i + \sum\limits_{l=i+2}^{n} \alpha_{l+1-i}\, e_l, \; 2 \leq i \leq n, \end{cases}$$

where at least one α_i satisfies $\alpha_i \neq 0$ and the first non-vanishing parameter $\{\alpha_3, \dots, \alpha_{n-1}\}$ can be assumed to be equal to 1.

Theorem 2.1.2 ([52]). *There exists only one class of solvable Lie algebras of dimension $n+2$ with nilradical $\mathfrak{n}_{n,1}$. It is represented by a basis $\{e_1, e_2, \dots, e_n, x, y\}$ and the Lie brackets are*

$$S_{n+2}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = (i-2)\, e_i, & 2 \leq i \leq n, \\ [e_1, x] = -[x, e_1] = e_1, & \\ [e_i, y] = -[y, e_i] = e_i, & 2 \leq i \leq n. \end{cases}$$

Now we recall the classification given in [4] after the following change of basis:

$$e_1' = -e_1, \qquad x' = -Y_1, \qquad y' = -Y_2.$$

Concerning solvable Lie algebras with nilradical Q_{2n} we present the following proposition.

Proposition 2.1.3 ([4]). *Any solvable Lie algebra of dimension 2n+1 with nilradical isomorphic to Q_{2n} is isomorphic to one of the following algebras:*

$$Q_{2n+1,1}(\alpha):\left\{\begin{array}{ll}
[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\le i\le 2n-2,\\
[e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\le i\le n,\\
[e_1,x]=-[x,e_1]=e_1, & \\
[e_i,x]=-[x,e_i]=(i-2+\alpha)\,e_i, & 2\le i\le 2n-1,\\
[e_{2n},x]=-[x,e_{2n}]=(2n-3-2\alpha)\,e_{2n}. &
\end{array}\right.$$

$$Q_{2n+1,2}:\left\{\begin{array}{ll}
[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\le i\le 2n-2,\\
[e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\le i\le n,\\
[e_1,x]=-[x,e_1]=e_1+\varepsilon\,e_{2n}, & \varepsilon=0,1,\\
[e_i,x]=-[x,e_i]=(i-n)\,e_i, & 2\le i\le 2n-1,\\
[e_{2n},x]=-[x,e_{2n}]=e_{2n}. &
\end{array}\right.$$

$$Q_{2n+1,3}(\alpha):\left\{\begin{array}{ll}
[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\le i\le 2n-2,\\
[e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\le i\le n,\\
[e_{2+i},x]=-[x,e_{2+i}]=e_{2+i}+\displaystyle\sum_{k=2}^{\lfloor\frac{2n-3-i}{2}\rfloor}\alpha^{2k+1}\,e_{2k+1+i}, & \\
 & 0\le i\le 2n-6,\\
[e_{2n-i},x]=-[x,e_{2n-i}]=e_{2n-i}, & i=1,2,3,\\
[e_{2n},x]=-[x,e_{2n}]=2\,e_{2n}. &
\end{array}\right.$$

Proposition 2.1.4 ([4]). *For any $n\ge 3$ there is unique $(2n+2)$-dimensional solvable Lie algebra having a nilradical isomorphic to Q_{2n}:*

$$\left\{\begin{array}{ll}
[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\le i\le 2n-2,\\
[e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\le i\le n,\\
[e_i,x]=-[x,e_i]=i\,e_i, & 1\le i\le 2n-1,\\
[e_{2n},x]=-[x,e_{2n}]=(2n+1)\,e_{2n}, & \\
[e_i,y]=-[y,e_i]=e_i, & 1\le i\le 2n-1,\\
[e_{2n},y]=-[y,e_{2n}]=2\,e_{2n}. &
\end{array}\right.$$

It is not difficult to see that if R is a solvable non-Lie Leibniz algebra with nilradical isomorphic to either the algebra $n_{n,1}$ or Q_{2n}, then the dimension of R is also not greater than $n+2$ and $2n+2$, respectively.

Let $n_{n,1}$ or Q_{2n} be the nilradical of a solvable Leibniz algebra R. Since the ideal $I = \langle\{[x,x] \mid x \in R\}\rangle$ is contained in $\mathrm{Ann}_r(R)$, then I is abelian, hence it is contained in the nilradical. Taking into account the multiplication in $n_{n,1}$ (respectively, in Q_{2n}) we conclude that $I = \langle\{e_n\}\rangle$.

Since an $(n+1)$-dimensional algebra R is solvable, then the quotient algebra R/I is also a solvable Lie algebra. Below we consider the case of nilradical isomorphic to $n_{n,1}$ The lists of tables of multiplications of R are given in Theorems 2.1.1 and 2.1.2.

Case $\boldsymbol{n_{n,1}}$. Let us assume that R has dimension $n+1$, then the multiplication table in R consists the same products as in $S_{n+1,i}$, $(i = 1,2,3,4)$, except the following products:

$$\begin{array}{lll} [e_1,x] = \alpha_1 e_1 + \gamma_4 e_n, & [e_2,x] = \beta_1 e_2 + \gamma_5 e_n, & \\ [x,e_1] = -\alpha_1 e_1 + \gamma_1 e_n, & [x,e_2] = -\beta_1 e_2 + \gamma_2 e_n, & [x,x] = \gamma_3 e_n, \end{array}$$

where $(\gamma_1+\gamma_4, \gamma_2+\gamma_5, \gamma_3) \neq (0,0,0)$.

Taking the change of basis

$$e_1' = \alpha_1\, e_1 + \gamma_4\, e_n, \qquad e_2' = \beta_1\, e_2 + \gamma_5\, e_n$$

we can assume $\gamma_4 = \gamma_5 = 0$, i.e., $[e_1,x] = \alpha\, e_1$ and $[e_2,x] = \beta_1\, e_2$.

It is not difficult to see that, for the omitted products, the antisymmetric identity holds, i.e.

$$\begin{cases} [e_i,e_1] = -[e_1,e_i] = e_{i+1}, & 2 \le i \le n-1, \\ [e_i,x] = -[x,e_i], & 3 \le i \le n. \end{cases}$$

We have $[e_n,x] = 0$, because of $0 = [x,e_n] = -[e_n,x]$.

Consider

$$0 = [x,e_n] = \big[x,[e_{n-1},e_1]\big] = \big[[x,e_{n-1}],e_1\big] - \big[[x,e_1],e_{n-1}\big] = -(n-2+\beta)\, e_n.$$

In the list of Theorem 2.1.1 only the algebra $S_{n+1,1}(\beta)$ is representative of the class for which the equality $[e_n,x] = 0$ holds. This class corresponds to $\beta = 2-n$.

Therefore, in the case of $\dim R = n+1$ whose nilradical is $n_{n,1}$, we have the following family:

$$R_{n+1,1}(\gamma_1,\gamma_2,\gamma_3): \begin{cases} [e_i,e_1] = -[e_1,e_i] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1,x] = e_1, & \\ [x,e_1] = -e_1 + \gamma_1 e_n, & \\ [e_2,x] = (2-n)\, e_2, & \\ [x,e_2] = (n-2)\, e_2 + \gamma_2 e_n, & \\ [e_i,x] = -[x,e_i] = (i-n)\, e_i, & 3 \le i \le n-1, \\ [x,x] = \gamma_3 e_n, & \end{cases}$$

where $(\gamma_1,\gamma_2,\gamma_3) \neq (0,0,0)$.

Applying the Leibniz identity and the multiplication table of the algebra in Theorem 2.1.2, we conclude that solvable non-Lie Leibniz algebras of dimension $n+2$ with nilradical $n_{n,1}$ do not exist.

Theorem 2.1.5. *Any $(n+1)$-dimensional solvable Leibniz algebra with nilradical $n_{n,1}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R_{n+1,1}(0,0,1), \qquad R_{n+1,1}(0,1,0), \qquad R_{n+1,1}(1,1,0), \qquad R_{n+1,1}(1,0,0).$$

Proof. We consider the general change of generator basis elements in the family $R_{n+1,1}(\gamma_1,\gamma_2,\gamma_3)$:

$$e_1' = \sum_{i=1}^{n} A_i\, e_i, \qquad e_2' = \sum_{i=1}^{n} B_i\, e_i, \qquad x' = D\, x + \sum_{i=1}^{n} C_i\, e_i,$$

where $(A_1B_2 - B_1A_2)D \neq 0$.

Using $[e_i', e_1'] = e_{i+1}',\ 2 \le i \le n-1$ and an induction, we obtain

$$e_i' = A_1^{i-3} \sum_{j=i}^{n} (A_1 B_{j+2-i} - B_1 A_{j+2-i})\, e_j, \quad 3 \le i \le n.$$

From the equalities

$$0 = [e_3', e_2'] = B_1 \sum_{j=4}^{n} (A_1 B_{j-2} - B_1 A_{j-2})\, e_j$$

we have $B_1 = 0$.

Consider the multiplications

$$\begin{aligned}[e'_1, x'] &= A_1 D\, e_1 - D\sum_{i=2}^{n-1} A_i(n-i)\, e_i + \sum_{i=3}^{n}(A_{i-1}C_1 - A_1C_{i-1})\, e_i \\ &= A_1De_1 - A_2D(n-2)\, e_2 + \sum_{i=3}^{n-1}\big(A_{i-1}C_1 - A_1C_{i-1} - (n-i)A_iD\big)\, e_i \\ &+ (A_{n-1}C_1 - A_1C_{n-1})\, e_n.\end{aligned}$$

On the other hand, we have

$$[e'_1, x'] = e'_1 = \sum_{i=1}^{n} A_i\, e_i.$$

Comparing the coefficients at the basis elements we derive:

$$\begin{aligned}& D = 1, \quad A_2 = 0, \quad A_{i+1} = \frac{A_1C_i - A_iC_1}{i-n-1}, \qquad 2 \le i \le n-2, \\ & A_n = A_1C_{n-1} - A_{n-1}C_n.\end{aligned}$$

From the equalities

$$\begin{aligned}-(n-2)\sum_{i=2}^{n} B_ie_i &= -(n-2)e'_2 = [e'_2, x'] = \big[\sum_{i=2}^{n} B_ie_i, x + \sum_{i=1}^{n} C_ie_i\big] \\ &= -\sum_{i=2}^{n-1} B_i(n-i)e_i + C_1\sum_{i=3}^{n} B_{i-1}e_i \\ &= -B_2(n-2)e_2 + \sum_{i=3}^{n-1}\big(B_{i-1}C_1 - B_i(n-i)\big)e_i + B_{n-1}C_1e_n,\end{aligned}$$

we deduce the following restrictions:

$$B_i = (-1)^i \frac{B_2C_1^{i-2}}{(i-2)!}, \qquad 3 \le i \le n.$$

In an analogous way, comparing coefficients at the basis element e_n in the equalities, we obtain:

$$\gamma'_3 A_1^{n-2} B_2\, e_n = \gamma'_3\, e'_n = [x', x'] = (\gamma_3 + C_1\gamma_1 + C_2\gamma_2)\, e_n,$$

and thus

$$\gamma_3' = \frac{\gamma_3 + C_1\gamma_1 + C_2\gamma_2}{A_1^{n-2}B_2}.$$

With a similar argument, we obtain

$$-e_1' + A_1^{n-2}B_2\gamma_1'\, e_n = -e_1' + \gamma_1'\, e_n' = [x', e_1'] = -e_1' + A_1\gamma_1\, e_n,$$

and

$$-(n-2)\, e_2' + A_1^{n-2}B_2\gamma_2'\, e_n = (n-2)\, e_2' + \gamma_2'\, e_n' = [x', e_2'] = (n-2)\, e_2' + B_2\gamma_2\, e_n.$$

Hence

$$\gamma_1' = \frac{\gamma_1}{A_1^{n-3}B_2} \qquad \text{and} \qquad \gamma_2' = \frac{\gamma_2}{A_1^{n-2}}.$$

Now we shall consider the possible cases of the parameters $\{\gamma_1, \gamma_2, \gamma_3\}$.

Case 1. Let $\gamma_1 = 0$. Then $\gamma_1' = 0$.

If $\gamma_2 = 0$, then $\gamma_2' = 0$ and $\gamma_3' = \frac{\gamma_3}{A_1^{n-2}B_2} \neq 0$. Putting $B_2 = \frac{\gamma_3}{A_1^{n-2}}$, then we have $\gamma_3' = 1$, and thus the algebra is $R_{n+1,1}(0,0,1)$.

If $\gamma_2 \neq 0$, then putting $A_1 = \sqrt[n-2]{\gamma_2}$ and $C_2 = -\frac{\gamma_3}{\gamma_2}$, we get $\gamma_2' = 1$ and $\gamma_3' = 0$, i.e. we obtain the algebra $R_{n+1,1}(0,1,0)$.

Case 2. Let $\gamma_1 \neq 0$. Then setting $B_2 = \frac{\gamma_1}{A_1^{n-3}}$ and $C_1 = -\frac{\gamma_3 + C_2\gamma_2}{\gamma_1}$, we get:

$$\gamma_1' = 1, \qquad \gamma_2' = \frac{\gamma_2}{A_1^{n-2}}, \qquad \gamma_3' = 0.$$

If $\gamma_2 \neq 0$, then putting $A_1 = \sqrt[n-2]{\gamma_2}$ we have $\gamma_2' = 1$, and thus we obtain the algebra $R_{n+1,1}(1,1,0)$.

If $\gamma_2 = 0$, then we get the algebra $R_{n+1,1}(1,0,0)$. □

Case Q_{2n}. Applying the Leibniz identity, we easily conclude that the solvable non-Lie Leibniz algebras with nilradical Q_{2n} exist only in the case of $\dim R = 2n+1$ and they are isomorphic to $Q_{2n+1,1}(\alpha)$ with $\alpha = \frac{2n-3}{2}$. Thus, we have

$$R_{2n+1,1}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i\, e_{2n}, & 2 \le i \le n, \\ [e_1, x] = e_1, & \\ [x, e_1] = -e_1 + \gamma_1\, e_n, & \\ [e_2, x] = \dfrac{2n-3}{2}\, e_2, & \\ [x, e_2] = -\dfrac{2n-3}{2}\, e_2 + \gamma_2\, e_n, & \\ [e_i, x] = -[x, e_i] = \dfrac{2n+2i-7}{2}\, e_i, & 3 \le i \le 2n-1, \\ [x, x] = \gamma_3\, e_n, & \end{cases}$$

where $(\gamma_1, \gamma_2, \gamma_3) \neq (0,0,0)$.

Theorem 2.1.6. *Any $(2n+1)$-dimensional solvable Leibniz algebra with nilradical Q_{2n} is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R_{2n+1,1}(0,0,1), \quad R_{2n+1,1}(0,1,0), \quad R_{2n+1,1}(1,1,0), \quad R_{2n+1,1}(1,0,0).$$

Proof. We make the following change of generator basis elements:

$$e'_1 = \sum_{i=1}^{2n} a_i\, e_i, \qquad e'_2 = \sum_{i=1}^{2n} b_i\, e_i, \qquad x' = d\, x + \sum_{i=1}^{2n} c_i\, e_i,$$

where $(a_1 b_2 - b_1 a_2) d \neq 0$, while the other elements of the new basis (i.e. $e'_i, 3 \le i \le 2n$) are obtained as products of the above elements.

The multiplication table in this new basis implies the following restrictions on the coefficients:

$$a_2 = b_1 = 0 \quad d_1 = 1,$$

$$a_i = \frac{2(a_1 c_{i-1} - c_1 a_{i-1})}{2n+2i-9}, \quad 3 \le i \le 2n-1, \quad a_{2n} = \sum_{k=2}^{2n-2} (-1)^{k+1} c_k a_{2n-k+1},$$

$$b_i = (-1)^i \frac{c_1^{i-2} b_2}{(i-2)!}, \quad 4 \le i \le 2n-1, \quad b_{2n} = \frac{2}{2n-3} \sum_{k=2}^{2n-1} (-1)^{k+1} c_k b_{2n-k+1}.$$

Calculating the new parameters we obtain:

$$\gamma_1' = \frac{\gamma_1}{a_1^{2n-4}b_2^2}, \quad \gamma_2' = \frac{\gamma_2}{a_1^{2n-3}b_2}, \quad \gamma_3' = \frac{\gamma_3 + c_1\gamma_1 + c_2\gamma_2}{a_1^{2n-3}b_2^2}.$$

Case 1. Let $\gamma_1 = 0$, then $\gamma_1' = 0$,

If $\gamma_2 = 0$, then $\gamma_2' = 0$ and $\gamma_3' = \frac{\gamma_3}{a_1^{n-2}b_2} \neq 0$. Putting $b_2 = \sqrt{\frac{\gamma_3}{a_1^{2n-3}}}$, then we have that $\gamma_3' = 1$ and the algebra $R_{2n+1,1}(0,0,1)$.

If $\gamma_2 \neq 0$, then putting $b_2 = \frac{\gamma_2}{a_1^{2n-3}}$ and $c_2 = -\frac{\gamma_3}{\gamma_2}$ we get $\gamma_2' = 1$, and $\gamma_3' = 0$, i.e. we obtain the algebra $R_{2n+1,1}(0,1,0)$.

Case 2. Let $\gamma_1 \neq 0$, then assuming $b_2 = \frac{\sqrt{\gamma_1}}{a_1^{n-2}}$ and $c_1 = -\frac{\gamma_3 + c_2\gamma_2}{\gamma_1}$ we get:

$$\gamma_1' = 1, \quad \gamma_2' = \frac{\sqrt{\gamma_1}\gamma_2}{a_1^{n-1}} \quad \gamma_3' = 0.$$

If $\gamma_2 \neq 0$, then setting $a_1 = \sqrt[n-1]{\sqrt{\gamma_1}\gamma_2}$, we derive $\gamma_2' = 1$, and thus we obtain the algebra $R_{2n+1,1}(1,1,0)$.

If $\gamma_2 = 0$, then we get the algebra $R_{2n+1,1}(1,0,0)$.

□

2.2 Solvable Leibniz algebras with nilradical F_n^1

In the following proposition we describe the derivations of the algebra F_n^1.

Proposition 2.2.1. *Any derivation of the algebra F_n^1 has the following matrix form:*

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{n-1} & \alpha_n \\
0 & \alpha_1+\alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{n-1} & \beta \\
0 & 0 & 2\alpha_1+\alpha_2 & \alpha_3 & \dots & \alpha_{n-2} & \alpha_{n-1} \\
0 & 0 & 0 & 3\alpha_1+\alpha_2 & \dots & \alpha_{n-3} & \alpha_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & (n-1)\alpha_1+\alpha_2
\end{pmatrix}.$$

Proof. Let d be a derivation of the algebra F_n^1. We set

$$d(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \qquad d(e_2) = \sum_{i=1}^{n} \beta_i e_i.$$

From the equality

$$0 = d([e_1, e_2]) = [d(e_1), e_2] + [e_1, d(e_2)] = \beta_1 e_3,$$

we get $\beta_1 = 0$.

Further, we have

$$d(e_3) = d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] = (2\alpha_1 + \alpha_2) e_3 + \sum_{i=3}^{n-1} \alpha_i e_{i+1}.$$

On the other hand,

$$d(e_3) = d([e_2, e_1]) = [d(e_2), e_1] + [e_2, d(e_1)] = (\alpha_1 + \beta_2) e_3 + \sum_{i=3}^{n-1} \beta_i e_{i+1}.$$

Therefore, $\beta_2 = \alpha_1 + \alpha_2, \ \beta_i = \alpha_i, \ \ 3 \leq i \leq n-1$.

Applying the property of derivation to the products $[e_i, e_1] = e_{i+1}$ and by an induction on i, it is easy to get that the following equalities for $3 \leq i \leq n$:

$$d(e_i) = \big((i-1)\alpha_1 + \alpha_2\big) e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n.$$

□

From Proposition 2.2.1 we conclude that the number of nil-independent outer derivations of the algebra F_n^1 is equal to two. Therefore, we have that any solvable Leibniz algebra whose nilradical is F_n^1 has dimension either $n+1$ or $n+2$.

Below, we present the description of such Leibniz algebras when their dimension is equal to $n+1$.

Theorem 2.2.2. *An arbitrary* $(n{+}1)$*-dimensional solvable Leibniz algebra with nilradical* F_n^1 *is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R_1 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [x, e_1] = -e_1 - e_2, & \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-1)e_i, & 2 \leq i \leq n, \end{cases}$$

$$R_2(\alpha): \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-1+\alpha)\, e_i, & 2 \leq i \leq n, \end{cases}$$

$$R_3: \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-n)\, e_i, & 2 \leq i \leq n, \\ [x, x] = e_n. & \end{cases}$$

$$R_4: \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1 + e_n, & \\ [e_i, x] = (i+1-n)\, e_i, & 2 \leq i \leq n, \\ [x, x] = -e_{n-1}. & \end{cases}$$

$$R_5(\alpha_4, \dots, \alpha_n): \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, x] = e_2 + \displaystyle\sum_{i=4}^{n-1} \alpha_i\, e_i & \\ [e_i, x] = e_i + \displaystyle\sum_{j=i+2}^{n} \alpha_{j-i+2}\, e_j, & 2 \leq i \leq n. \end{cases}$$

Moreover, the first non-vanishing parameter $\{\alpha_4, \dots, \alpha_n\}$ in the algebras $R_5(\alpha_4, \dots, \alpha_n)$, can be scaled to 1.

Proof. From Theorem 1.1.9 and arguments after Proposition 2.2.1 we deduce that there exists a basis $\{e_1, e_2, \dots, e_n, x\}$ such that the multiplication table of the algebra F_n^1 is completed with the products coming from $\mathcal{R}_{x|_{F_n^1}}(e_i),\ 1 \leq$

$i \leq n$, i.e.

$$[e_1, x] = \sum_{i=1}^{n} \alpha_i \, e_i, \qquad [e_2, x] = (\alpha_1 + \alpha_2) \, e_2 + \sum_{i=3}^{n-1} \alpha_i \, e_i + \beta \, e_n,$$

$$[e_i, x] = \big((i-1)\alpha_1 + \alpha_2\big) \, e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} \, e_j, \quad 3 \leq i \leq n.$$

We denote the remaining products as follows:

$$[x, e_1] = \sum_{i=1}^{n} \beta_i \, e_i, \qquad [x, e_2] = \sum_{i=1}^{n} \gamma_i \, e_i, \qquad [x, x] = \sum_{i=1}^{n} \delta_i \, e_i.$$

From the chain of equalities

$$\begin{aligned} 0 = [x, e_3] &= \big[x, [e_2, e_1]\big] = \big[[x, e_2], e_1\big] - \big[[x, e_1], e_2\big] = \big[[x, e_2], e_1\big] \\ &= (\gamma_1 + \gamma_2) \, e_3 + \sum_{i=4}^{n} \gamma_{i-1} \, e_i, \end{aligned}$$

we conclude that $\gamma_2 = -\gamma_1, \ \gamma_i = 0, \ 3 \leq i \leq n-1$.

Since $\gamma_1 e_3 = \big[e_1, [x, e_2]\big] = \big[[e_1, x], e_2\big] - \big[[e_1, e_2], x\big] = 0$, then $\gamma_1 = 0$.

The identity $\{e_1, x, e_1\}$ implies $\beta_1 = -\alpha_1$.

Applying the Leibniz identity to the elements of the form $\{x, x, e_2\}$ and $\{x, e_2, x\}$, we conclude:

$$\begin{cases} \big((n-1)\alpha_1 + \alpha_2\big)\gamma_n = 0, \\ (n-2)\alpha_1 \gamma_n = 0. \end{cases}$$

Note that $\gamma_n = 0$ $\big($otherwise $\alpha_1 = \alpha_2 = 0$ and then we get a contradiction with the non-nilpotency of the derivation D (see Proposition 2.2.1)$\big)$.

Now we are going to consider the possible cases of the parameters α_1 and α_2.

Case 1. Let $\alpha_1 \neq 0$.

Case 1.1. Let $\alpha_1 \neq \beta_2$. Then taking the following change of basis:

$$x' = -\frac{1}{\alpha_1} x, \qquad e_1' = e_1 - \frac{1}{\alpha_1} \sum_{i=2}^{n} \beta_i \, e_i,$$

$$e_i' = -\frac{1}{\alpha_1} \big((-\alpha_1 + \beta_2) \, e_i + \sum_{j=i+1}^{n} \beta_{j-i+2} \, e_j\big), \qquad 2 \leq i \leq n,$$

we obtain

$$[e_1,e_1]=e_3, \qquad [e_1,x]=\sum_{i=1}^{n}\mu_i\, e_i, \qquad [e_i,e_1]=e_{i+1}, \qquad 2\le i\le n-1,$$
$$[x,e_1]=e_1, \qquad [e_2,x]=\sum_{i=1}^{n}\eta_i\, e_i, \qquad [x,e_2]=0, \qquad [x,x]=\sum_{i=1}^{n}\theta_i\, e_i.$$

The equalities

$$0=\big[[e_1,e_2],x\big]=\big[e_1,[e_2,x]\big]+\big[[e_1,x],e_2\big]=\big[e_1,\sum_{i=1}^{n}\eta_i\, e_i\big]=\eta_1\, e_3,$$

imply $\eta_1=0$.

Consider

$$\begin{aligned}[e_3,x]&=\big[[e_1,e_1],x\big]=\big[e_1,[e_1,x]\big]+\big[[e_1,x],e_1\big]\\ &=\mu_1\, e_3+(\mu_1+\mu_2)\, e_3+\sum_{i=3}^{n-1}\mu_i\, e_{i+1}=(2\mu_1+\mu_2)\, e_3+\sum_{i=3}^{n-1}\mu_i\, e_{i+1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}[e_3,x]&=\big[[e_2,e_1],x\big]=\big[e_2,[e_1,x]\big]+\big[[e_2,x],e_1\big]=\mu_1\, e_3+\eta_2\, e_3+\sum_{i=3}^{n-1}\eta_i\, e_{i+1}\\ &=(\mu_1+\eta_2)\, e_3+\sum_{i=3}^{n-1}\eta_i\, e_{i+1}.\end{aligned}$$

The comparison of both right-hand sides implies:

$$\eta_2=\mu_1+\mu_2, \qquad \eta_i=\mu_i, \quad 3\le i\le n-1,$$

this means:

$$[e_2,x]=(\mu_1+\mu_2)\, e_2+\sum_{i=3}^{n-1}\mu_i\, e_i+\eta_n\, e_n \ \text{ and } \ [e_3,x]=(2\mu_1+\mu_2)\, e_3+\sum_{i=3}^{n-1}\mu_i\, e_{i+1}.$$

Now we shall prove the following equalities by an induction on i:

$$[e_i,x]=\big((i-1)\mu_1+\mu_2\big)\, e_i+\sum_{j=i+1}^{n}\mu_{j-i+2}\, e_j, \quad 3\le i\le n. \tag{2.2.1}$$

Obviously, the equality holds for $i = 3$. Let us assume that the equality holds for $3 < i < n$, and we shall prove it for $i + 1$:

$$\begin{aligned}[e_{i+1}, x] &= \big[[e_i, e_1], x\big] = \big[e_i, [e_1, x]\big] + \big[[e_i, x], e_1\big] \\ &= \mu_1\, e_{i+1} + \big((i-1)\mu_1 + \mu_2\big)\, e_{i+1} + \sum_{j=i+2}^{n} \mu_{j-i+1}\, e_j \\ &= (i\mu_1 + \mu_2)\, e_{i+1} + \sum_{j=i+2}^{n} \mu_{j-i+1}\, e_j;\end{aligned}$$

so the induction proves the equalities (2.2.1) for any $i,\ 3 \leq i \leq n$.

Applying the Leibniz identity to the triples of elements $\{e_1, x, e_1\}$, $\{e_1, x, x\}$, $\{x, e_1, x\}$, we deduce that:

$$\mu_1 = -1, \qquad \mu_2 = \theta_1 = 0, \qquad \theta_i = \mu_{i+1}, \quad 2 \leq i \leq n-1.$$

Below, we summarize the multiplication table of the algebra

$$\begin{cases} [e_1, e_1] = e_3 & \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, x] = -e_1 + \sum\limits_{i=3}^{n} \mu_i\, e_i, & \\ [e_2, x] = -e_2 + \sum\limits_{i=3}^{n-1} \mu_i\, e_i + \eta_n\, e_n, & \\ [e_i, x] = -(i-1)\, e_i + \sum\limits_{j=i+1}^{n} \mu_{j-i+2}\, e_j, & 3 \leq i \leq n, \\ [x, e_1] = e_1, & [x, x] = \sum\limits_{i=2}^{n-1} \mu_{i+1}\, e_i + \theta_n\, e_n. \end{cases}$$

Let us take the change of basis in the following form:

$$e_1' = e_1 + \sum_{i=3}^{n} A_i e_i, \qquad e_2' = e_2 + \sum_{i=3}^{n} A_i e_i,$$

$$e_i' = e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j, \qquad 3 \le i \le n,$$
$$x' = \sum_{i=2}^{n-1} A_{i+1} e_i + B e_n + x,$$

where

$$A_3 = \mu_3, \qquad A_i = \frac{1}{(i-2)}\Big(\mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2}\Big), \quad 4 \le i \le n, \quad \text{and}$$
$$B = \frac{1}{n-1}\Big(\theta_n + \sum_{j=3}^{n} A_j \mu_{n-j+3}\Big).$$

Then

$$[x', e_1'] = \Big[\sum_{i=2}^{n-1} A_{i+1} e_i + B e_n + x, e_1\Big] = e_1 + \sum_{i=3}^{n} A_i e_i = e_1',$$
$$[e_1', x'] = [e_1, x] + \sum_{i=3}^{n} A_i [e_i, x]$$
$$= -e_1 + \sum_{i=3}^{n} \mu_i e_i + \sum_{i=3}^{n} A_i \Big(-(i-1)e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \Big)$$
$$= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \mu_i e_i - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=3}^{n} A_i \Big(\sum_{j=i+1}^{n} \mu_{j-i+2} e_j \Big)$$
$$= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \mu_i e_i - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=4}^{n} \Big(\sum_{j=3}^{i-1} A_j \mu_{i-j+2} \Big) e_i$$
$$= -e_1 - \sum_{i=3}^{n} A_i e_i + (\mu_3 - A_3) e_3$$
$$+ \sum_{i=4}^{n} \Big(-A_i(i-2) + \mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2} \Big) e_i$$
$$= -e_1 - \sum_{i=3}^{n} A_i e_i = -e_1',$$

$$\begin{aligned}
[e_2', x'] &= [e_2, x] + \sum_{i=3}^{n} A_i [e_i, x] \\
&= -e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n + \sum_{i=3}^{n} A_i \Big(-(i-1)e_i + \sum_{j=i+1}^{n} \mu_{j-i+2}\, e_j \Big) \\
&= -e_2 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n - \sum_{i=3}^{n} A_i (i-2) e_i \\
&\quad + \sum_{i=3}^{n} A_i \Big(\sum_{j=i+1}^{n} \mu_{j-i+2}\, e_j \Big) \\
&= -e_2 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n - \sum_{i=3}^{n} A_i (i-2) e_i \\
&\quad + \sum_{i=4}^{n} \Big(\sum_{j=3}^{i-1} A_j \mu_{i-j+2} \Big)\, e_i \\
&= -e_2 - \sum_{i=3}^{n} A_i e_i + (\mu_3 - A_3) e_3 \\
&\quad + \sum_{i=4}^{n-1} \Big(-A_i(i-2) + \mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2} \Big)\, e_i \\
&\quad + \Big(\eta_n - (n-2) A_n + \sum_{i=3}^{n-1} A_i \mu_{n-i+2} \Big)\, e_n = -e_2' + \eta'\, e_n',
\end{aligned}$$

$$\begin{aligned}
[x', x'] &= \sum_{i=2}^{n-1} A_{i+1} [e_i, x] + B[e_n, x] + [x, x] \\
&= \sum_{i=2}^{n-1} A_{i+1} \Big(-(i-1)e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \Big) \\
&\quad - B(n-1) e_n + \sum_{i=2}^{n-1} \mu_{i+1} e_i + \theta_n e_n
\end{aligned}$$

$$= -\sum_{i=2}^{n-1} A_{i+1}(i-1)e_i + \sum_{i=2}^{n-1} \mu_{i+1}e_i - B(n-1)e_n + \theta_n e_n$$
$$+ \sum_{i=2}^{n-1} A_{i+1}\Big(\sum_{j=i+1}^{n} \mu_{j-i+2}e_j\Big)$$
$$= -\sum_{i=2}^{n-1} A_{i+1}(i-1)e_i + \sum_{i=2}^{n-1} \mu_{i+1}e_i - B(n-1)e_n + \theta_n e_n$$
$$+ \sum_{i=3}^{n} \Big(\sum_{j=3}^{i} A_j\mu_{i-j+3}\Big) e_i$$
$$= (\mu_3 - A_3)e_2 + \sum_{i=3}^{n-1}\Big(- A_{i+1}(i-1) + \mu_{i+1} + \sum_{j=3}^{i} A_j\mu_{i-j+3}\Big) e_i$$
$$+ \Big(- B(n-1) + \theta_n + \sum_{j=3}^{n} A_j\mu_{n-j+3}\Big) e_n = 0.$$

With a similar induction as the given for Equations (2.2.1), it is easy to check that the following equalities hold:

$$[e_i, x] = -(i-1)\, e_i, \qquad 3 \leq i \leq n.$$

Thus, we obtain the following multiplication table:

$$\begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [x, e_1] = e_1, & [e_1, x] = -e_1, & \\ [e_2, x] = -e_2 + \eta e_n, & [e_i, x] = -(i-1)e_i, & 3 \leq i \leq n. \end{cases}$$

If $\eta \neq 0$, then by taking the change of basis

$$e_2' = e_2 + \frac{\eta}{n-2} e_n,$$

we get $\eta' = 0$.

Finally, by applying the change of basis $x' = -x$ and $e_1' = e_1 - e_2$, we get the algebra R_1.

Case 1.2. Let $\alpha_1 = \beta_2$. Then by taking the following change of basis:

$$e_1' = e_1 - e_2, \qquad e_i' = e_i, \qquad 2 \leq i \leq n,$$

we can assume that the multiplication table is the following

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [x, e_1] = -\alpha_1 e_1 + \sum\limits_{i=3}^{n} \beta_i e_i, & \\ [e_1, x] = \alpha_1 e_1 + (\alpha_n - \beta) e_n, & \\ [e_2, x] = (\alpha_1 + \alpha_2) e_2 + \sum\limits_{i=3}^{n-1} \alpha_i e_i + \beta e_n, & \\ [e_i, x] = \big((i-1)\alpha_1 + \alpha_2\big) e_i + \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, & 3 \le i \le n, \\ [x, x] = \sum\limits_{i=1}^{n} \delta_i e_i. & \end{cases}$$

Now, by taking

$$x' = \frac{1}{\alpha_1} x - \frac{1}{\alpha_1} \sum_{i=2}^{n-1} \beta_{i+1} e_i,$$

and renaming the parameters, we get

$$\mathcal{F} : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1 + \beta e_n, & \\ [e_2, x] = (1 + \alpha_2) e_2 + \sum\limits_{i=3}^{n-1} \alpha_i e_i + \lambda e_n, & \\ [e_i, x] = (i - 1 + \alpha_2) e_i + \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, & 3 \le i \le n, \\ [x, x] = \sum\limits_{i=1}^{n} \delta_i e_i. & \end{cases}$$

Making the change of basis

$$x' = x, \qquad e_1' = e_1, \qquad e_i' = e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j, \qquad 2 \le i \le n,$$

where

$$A_3 = -\alpha_3, \qquad A_i = -\frac{1}{i-1}\Big(\alpha_i + \sum_{j=3}^{i-1} A_j\alpha_{i-j+2}\Big), \qquad 4 \le i \le n-1,$$

$$A_n = -\frac{1}{n-2}\Big(\lambda + \sum_{j=3}^{n-1} A_j\alpha_{n-j+2}\Big),$$

and applying the Leibniz identity, we obtain the family of algebras

$$F(\alpha,\beta,\gamma): \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1 + \beta e_n, & \\ [e_i, x] = (i-1+\alpha)e_i, & 2 \le i \le n, \\ [x, x] = -\beta e_{n-1} + \gamma e_n. & \end{cases}$$

Below, we shall investigate the isomorphism inside the family. For this purpose we consider the general change of generator basis elements in the family $F(\alpha,\beta,\gamma)$,

$$e_1' = \sum_{i=1}^{n} A_i e_i, \qquad e_2' = \sum_{i=1}^{n} B_i e_i, \qquad x' = Cx + \sum_{i=1}^{n} P_i e_i.$$

Then we obtain in the new basis $\{e_1', e_2', \dots, e_n', x'\}$ the behavior of the parameters with the following expressions:

$$\alpha' = \alpha, \;\; \beta' = \frac{A_1\beta + (n-2+\alpha)A_n}{A_1^{n-2}B_2}, \;\; \gamma' = \frac{\gamma A_1 + (n-1+\alpha)(P_nA_1 - P_1A_n)}{A_1^{n-3}B_2}.$$

Case 1.2.1. $\alpha \neq 2-n$. Taking

$$A_n = -\frac{A_1\beta}{n-2+\alpha},$$

we get $\beta' = 0$.

Case 1.2.1.1. $\alpha \neq 1-n$. Putting

$$P_n = \frac{-\gamma A_1 + (n-1+\alpha)P_1A_n}{(n-1+\alpha)A_1},$$

we have $\gamma' = 0$ and the family $R_2(\alpha)$ with $\alpha \in \mathbb{C}\backslash\{2-n, 1-n\}$.

Case 1.2.1.2. $\alpha = 1-n$. Then

$$\gamma' = \frac{\gamma}{A_1^{n-4}B_2}.$$

If $\gamma \neq 0$, then setting $B_2 = \frac{\gamma}{A_1^{n-4}}$, we derive $\gamma' = 1$ and thus, the algebra R_3 is obtained.

If $\gamma = 0$, then we have the algebra $R_2(\alpha)$ with $\alpha = 1-n$.

Case 1.2.2. $\alpha = 2-n$. Then

$$\beta' = \frac{\beta}{A_1^{n-3}B_2}, \qquad \gamma' = \frac{\gamma A_1 + P_n A_1 - P_1 A_n}{A_1^{n-3}B_2}.$$

Setting $P_n = \frac{-\gamma A_1 + P_1 A_n}{A_1}$, we have $\gamma' = 0$.

If $\beta \neq 0$, then taking $B_2 = \frac{\beta}{A_1^{n-3}}$, we get $\beta' = 1$ and hence we obtain the algebra R_4.

If $\beta = 0$, then we have the algebra $R_2(\alpha)$ with $\alpha = 2-n$.

Applying a general change of basis, it is easy to check that any algebra of the family $F(\alpha, \beta, \gamma)$ is not isomorphic to the algebra R_1.

Case 2. Let $\alpha_1 = 0, \alpha_2 \neq 0$. Then making the following change of basis

$$x' = x - \sum_{i=2}^{n-1} \beta_{i+1} e_i,$$

we can assume that $[x, e_1] = \beta_2 e_2$.

From the identity

$$\big[x, [x, e_1]\big] = \big[[x, x], e_1\big] - \big[[x, e_1], x\big],$$

we derive

$$0 = \sum_{i=3}^{n} \delta_{i-1} e_i - \beta_2 [e_2, x] = \sum_{i=3}^{n} \delta_{i-1} e_i - \beta_2 \Big(\sum_{i=2}^{n-1} \alpha_i e_i + \beta e_n\Big),$$

consequently, $\beta_2 = 0, \ \delta_i = 0, \ 2 \leq i \leq n-1$.

Making the change of basis

$$x' = x - \frac{\delta}{\alpha_2}\, e_n,$$

one can assume $[x,x]=0$.

Summarizing, we obtain the following multiplication table of the algebra in this case

$$\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \qquad 2\leq i\leq n-1,\\ [e_1,x]=\displaystyle\sum_{i=2}^{n}\alpha_i\, e_i, & [e_2,x]=\displaystyle\sum_{i=2}^{n-1}\alpha_i\, e_i+\beta\, e_n,\\ [e_i,x]=\displaystyle\sum_{j=i}^{n}\alpha_{j-i+2}\, e_j, & \qquad 3\leq i\leq n.\end{cases}$$

Now we shall study the behavior of the parameters in this family of algebras under the general change of basis of the form

$$\begin{cases} e_1'=\displaystyle\sum_{i=1}^{n}A_ie_i,\\ e_2'=(A_1+A_2)\, e_2+\displaystyle\sum_{j=3}^{n-1}A_j\, e_j+Be_n,\\ e_i'=A_1^{i-2}\Big((A_1+A_2)\, e_i+\displaystyle\sum_{j=i+1}^{n}A_{j-i+2}\, e_j\Big), \quad 3\leq i\leq n,\\ x'=\displaystyle\sum_{i=1}^{n}B_ie_i+B_{n+1}x,\end{cases}$$

where $A_1(A_1+A_2)B_{n+1}\neq 0$.

Then the equalities

$$0=[x',e_1']=\Big[\sum_{i=1}^{n}B_ie_i+B_{n+1}x, A_1e_1\Big]=A_1\left((B_1+B_2)e_3+\sum_{i=4}^{n}B_{i-1}e_i\right)$$

imply $B_1=-B_2,\ B_i=0,\ 3\leq i\leq n-1.$

Now we shall express the product $[e_1', x']$ as a linear combination of the basis $\{e_1, e_2, \dots, e_n, x\}$, namely:

$$
\begin{aligned}
[e_1', x'] &= \Big[\sum_{i=1}^{n} A_i e_i, B_1 e_1 + B_{n+1} x\Big] \\
&= B_1\Big((A_1 + A_2)e_3 + \sum_{i=4}^{n} A_{i-1} e_i\Big) \\
&\quad + B_{n+1}\left(A_1 \sum_{i=2}^{n} \alpha_i e_i + A_2\left(\sum_{i=2}^{n-1} \alpha_i e_i + \beta e_n\right) + \sum_{i=3}^{n} A_i \sum_{j=i}^{n} \alpha_{j-i+2} e_j\right) \\
&= B_1(A_1 + A_2)e_3 + \sum_{i=4}^{n} B_1 A_{i-1} e_i + B_{n+1} A_1 \sum_{i=2}^{n} \alpha_i e_i \\
&\quad + B_{n+1} A_2 \sum_{i=2}^{n-1} \alpha_i e_i + B_{n+1} A_2 \beta e_n + B_{n+1} \sum_{i=3}^{n} \sum_{j=3}^{i} A_j \alpha_{i-j+2} e_i \\
&= B_{n+1}(A_1 + A_2)\alpha_2\, e_2 + \Big((A_1 + A_2)(B_1 + B_{n+1}\alpha_3) + B_{n+1} A_3 \alpha_2\Big)\, e_3 \\
&\quad + \sum_{i=4}^{n-1} \left(B_1 A_{i-1} + B_{n+1}(A_1 + A_2)\alpha_i + \sum_{j=3}^{i} B_{n+1} A_j \alpha_{i-j+2}\right) e_i \\
&\quad + \left(B_1 A_{n-1} + B_{n+1}\Big(A_1 \alpha_n + A_2 \beta + \sum_{i=3}^{n} A_i \alpha_{n-i+2}\Big)\right) e_n.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
[e_1', x'] = \sum_{i=2}^{n} \alpha_i' e_i' &= \alpha_2'((A_1 + A_2)e_2 + \sum_{j=3}^{n-1} A_j e_j + B e_n) \\
&\quad + \sum_{i=3}^{n} \alpha_i' A_1^{i-2} \left((A_1 + A_2)e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_2'(A_1+A_2)e_2 + \big(A_1(A_1+A_2)\alpha_3' + A_3\alpha_2'\big)e_3 \\
&\quad + \sum_{i=4}^{n-1}\Big(\alpha_i' A_1^{i-2}(A_1+A_2) + \sum_{j=3}^{i} A_1^{i-j}A_j\alpha_{i-j+2}'\Big)e_i \\
&\quad + (B\alpha_2' + \alpha_n' A_1^{n-2}(A_1+A_2) + \sum_{j=3}^{n-1} A_1^{n-j}A_j\alpha_{n-j+2}')e_n.
\end{aligned}
$$

Comparing coefficients at the basis elements in both right-hand sides, we obtain the following relations:

$$
\begin{aligned}
&\alpha_2'(A_1+A_2) = B_{n+1}(A_1+A_2)\alpha_2, \\
&A_1(A_1+A_2)\alpha_3' + A_3\alpha_2' = (A_1+A_2)(B_1+B_{n+1}\alpha_3) + B_{n+1}A_3\alpha_2, \\
&\alpha_i' A_1^{i-2}(A_1+A_2) + \sum_{j=3}^{i} A_1^{i-j}A_j\alpha_{i-j+2}' = B_1A_{i-1} + B_{n+1}(A_1+A_2)\alpha_i \\
&\qquad\qquad + \sum_{j=3}^{i} B_{n+1}A_j\alpha_{i-j+2}, \qquad 4 \le i \le n-1, \\
&A_1^{n-2}\alpha_n'(A_1+A_2) + \sum_{i=3}^{n-1} A_1^{n-i}A_i\alpha_{n-i+2}' + B\alpha_2' = B_1A_{n-1} \\
&\qquad\qquad + B_{n+1}\Big(A_1\alpha_n + A_2\beta + \sum_{i=3}^{n} A_i\alpha_{n-i+2}\Big).
\end{aligned}
$$

The simplification of these relations implies the following identities:

$$
\begin{aligned}
\alpha_2' &= B_{n+1}\alpha_2, \\
\alpha_3' &= \frac{B_1 + \alpha_3 B_{n+1}}{A_1}, \\
\alpha_i' &= \frac{B_{n+1}\alpha_i}{A_1^{i-2}}, \qquad 4 \le i \le n-1, \\
\alpha_n' &= \frac{(\alpha_n A_1 + \beta A_2 + \alpha_2 A_n - \alpha_2 B)B_{n+1}}{A_1^{n-2}(A_1+A_2)}.
\end{aligned}
$$

Analogously, considering the product $[e'_2, x']$, we get:

$$\begin{aligned}
[e'_2, x'] &= \Big[(A_1 + A_2)e_2 + \sum_{j=3}^{n-1} A_j e_j + B e_n, B_1 e_1 + B_{n+1} x\Big] \\
&= B_1\Big((A_1 + A_2)e_3 + \sum_{i=4}^{n} A_{i-1} e_i\Big) \\
&\quad + B_{n+1}\left((A_1 + A_2)\left(\sum_{i=2}^{n-1} \alpha_i e_i + \beta e_n\right)\right. \\
&\quad \left. + \sum_{i=3}^{n-1} A_i \sum_{j=i}^{n} \alpha_{j-i+2} e_j + B\alpha_2 e_n\right) \\
&= B_1(A_1 + A_2)e_3 + \sum_{i=4}^{n} B_1 A_{i-1} e_i + B_{n+1}(A_1 + A_2) \sum_{i=2}^{n-1} \alpha_i e_i \\
&\quad + B_{n+1}(A_1 + A_2)\beta e_n + B_{n+1} \sum_{i=3}^{n} \sum_{j=3}^{i-1} A_j \alpha_{i-j+2} e_i + B B_{n+1} \alpha_2 e_n \\
&= B_{n+1}(A_1 + A_2)\alpha_2\, e_2 + \Big((A_1 + A_2)(B_1 + B_{n+1}\alpha_3) + B_{n+1} A_3 \alpha_2\Big)\, e_3 \\
&\quad + \sum_{i=4}^{n-1} \Big(B_1 A_{i-1} + B_{n+1}(A_1 + A_2)\alpha_i + \sum_{j=3}^{i} B_{n+1} A_j \alpha_{i-j+2}\Big)\, e_i \\
&\quad + \left(B_1 A_{n-1} + B_{n+1}\Big((A_1 + A_2)\beta + B\alpha_2 + \sum_{i=3}^{n-1} A_i \alpha_{n-i+2}\Big)\right)\, e_n.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[e'_2, x'] &= \sum_{i=2}^{n-1} \alpha'_i e'_i + \beta' e'_n = \alpha'_2((A_1 + A_2)e_2 + \sum_{j=3}^{n-1} A_j e_j + B e_n) \\
&\quad + \sum_{i=3}^{n-1} \alpha'_i A_1^{i-2} \Big((A_1 + A_2)e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j\Big) + \beta' A_1^{n-2}(A_1 + A_2) e_n
\end{aligned}$$

$$= \alpha_2'(A_1+A_2)e_2 + \big(A_1(A_1+A_2)\alpha_3' + A_3\alpha_2'\big)e_3 + \sum_{i=4}^{n-1}\Big(\alpha_i' A_1^{i-2}(A_1+A_2) + \sum_{j=3}^{i} A_1^{i-j}A_j\alpha_{i-j+2}'\Big)e_i + \Big(B\alpha_2' + \beta' A_1^{n-2}(A_1+A_2) + \sum_{i=3}^{n-1} A_1^{i-j}A_j\alpha_{i-j+2}'\Big)e_n.$$

So we have

$$B_1A_{n-1} + B_{n+1}\Big((A_1+A_2)\beta + B\alpha_2 + \sum_{i=3}^{n-1} A_i\alpha_{n-i+2}\Big) = B\alpha_2' + \beta' A_1^{n-2}(A_1+A_2) + \sum_{i=3}^{n-1} A_1^{i-j}A_j\alpha_{i-j+2}',$$

from which it follows

$$\beta' = \frac{\beta B_{n+1}(A_1+A_2)}{A_1^{n-2}(A_1+A_2)},$$

and

$$[x',x'] = -\frac{(\beta B_1 - \alpha_n B_1 - \alpha_2 B_n)B_{n+1}}{A_1^{n-2}(A_1+A_2)}e_n.$$

Since $[x',x']=0$, then $B_n = \dfrac{\beta B_1 - \alpha_n B_1}{\alpha_2}$.

Setting $B_{n+1} = 1/\alpha_2$ and $B_1 = -\alpha_3/\alpha_2$, we derive $\alpha_2' = 1,\ \alpha_3' = 0$.

If $\beta = 0$, then $\beta' = 0$ and putting $B = \dfrac{\alpha_n - \alpha_2 A_1}{\alpha_2}$ we have $\alpha_n' = 0$. Hence we obtain the algebra $R_5(\alpha_4,\dots,\alpha_{n-1},0)$.

If $\beta \neq 0$, then choosing

$$A_1 = \sqrt[n-2]{\frac{\beta}{\alpha_2}}, \qquad A_2 = -\frac{A_1\alpha_n}{\beta},$$

we obtain $\beta' = 1,\ \alpha_n' = 0$ and the algebra $R_5(\alpha_4,\dots,\alpha_{n-1},1)$. □

In the next theorem the classification of $(n+2)$-dimensional solvable Leibniz algebras with nilradical F_n^1 is given.

Theorem 2.2.3 ([33]). *An arbitrary $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^1 is isomorphic to an algebra with the following multiplication table:*

$$\begin{array}{lll} [e_i,e_1]=e_{i+1}, & 2\le i\le n-1, & [e_1,x]=e_1, \\ [e_i,y]=e_i, & 2\le i\le n, & [e_i,x]=(i-1)e_i, \quad 2\le i\le n, \\ & & [x,e_1]=-e_1. \end{array}$$

2.3 Solvable Leibniz algebras with nilradical F_n^2

In this section we describe solvable Leibniz algebras with nilradical F_n^2.

Proposition 2.3.1. *An arbitrary derivation of the algebra F_n^2 has the following matrix form:*

$$D=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{n-1} & \alpha_n \\ 0 & \beta & 0 & 0 & \dots & 0 & \gamma \\ 0 & 0 & 2\alpha_1 & \alpha_3 & \dots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 0 & 0 & 3\alpha_1 & \dots & \alpha_{n-3} & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & (n-1)\alpha_1 \end{pmatrix}.$$

Proof. The proof follows by straightforward calculations in a similar way as the proof of Proposition 2.2.1. □

It is easy to check that the number of nil-independent derivations of the algebra F_n^2 is equal to 2.

Corollary 2.3.2. *The dimension of a solvable Leibniz algebra with nilradical F_n^2 is either $n+1$ or $n+2$.*

Theorem 2.3.3. *An $(n+1)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$L_1(\alpha):\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1, \\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, & 3\le i\le n, \\ [x,e_1]=e_1, & [x,x]=\alpha\, e_2, & \alpha\in\{0,1\}. \end{cases}$$

$$L_2(\alpha):\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leq i\leq n,\\ [x,e_1]=e_1, & [e_2,x]=\alpha\,e_2, \quad \alpha\neq 0.\end{cases}$$

$$L_3:\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leq i\leq n,\\ [x,e_1]=e_1, & [e_2,x]=(1-n)\,e_2+e_n.\end{cases}$$

$$L_4(\alpha):\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leq i\leq n,\\ [x,e_1]=e_1, & [e_2,x]=-\alpha\,e_2, \quad \alpha\neq 1,\\ [x,e_2]=\alpha\,e_2.\end{cases}$$

$$L_5(\alpha):\begin{cases}[e_1,e_1]=e_3,\\ [e_1,x]=-e_1-\alpha\,e_2, \quad \alpha\in\{0,1\}, \quad [e_2,x]=-e_2,\\ [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [e_i,x]=-(i-1)\,e_i, \quad 3\leq i\leq n,\\ [x,e_1]=e_1+\alpha\,e_2, \quad [x,e_2]=e_2.\end{cases}$$

$$L_6(\alpha_3,\alpha_4,\dots,\alpha_n,\lambda,\delta):\begin{cases}[e_1,e_1]=e_3,\\ [e_1,x]=\sum\limits_{i=3}^{n}\alpha_i\,e_i, \quad [e_2,x]=e_2,\\ [e_i,e_1]=e_{i+1}, \quad 3\leq i\leq n-1,\\ [e_i,x]=\sum\limits_{j=i+1}^{n}\alpha_{j-i+2}\,e_j, \quad 3\leq i\leq n-1,\\ [x,x]=\lambda\,e_n, \quad [x,e_2]=\delta\,e_2, \quad \delta\in\{0,-1\}.\end{cases}$$

In the algebra $L_6(\alpha_3,\alpha_4,\dots,\alpha_n,\lambda,\delta)$ the first non-vanishing parameter $\{\alpha_3,\alpha_4,\dots,\alpha_n,\lambda\}$ can be scaled to 1.

Proof. Let R be a solvable Leibniz algebra satisfying the conditions of the theorem, then there exists a basis $\{e_1,e_2,\dots,e_n,x\}$, such that $\{e_1,e_2,\dots,e_n\}$ is the standard basis of F_n^2, and for non nilpotent outer derivations of the algebra F_n^2, we have that $[e_i,x]=\mathcal{R}_{x_{|F_n^2}}(e_i),\ 1\leq i\leq n.$

Due to Proposition 2.3.1 we can assume that

$$[e_1,x]=\sum_{i=1}^{n}\alpha_i\, e_i, \qquad [e_2,x]=\beta_2\, e_2+\beta_n\, e_n,$$
$$[e_i,x]=(i-1)\alpha_1\, e_i+\sum_{j=i+1}^{n}\alpha_{j-i+2}\, e_j, \quad 3\le i\le n.$$

Let us introduce the following notations:

$$[x,e_1]=\sum_{i=1}^{n}\gamma_i\, e_i, \qquad [x,e_2]=\sum_{i=1}^{n}\delta_i\, e_i, \qquad [x,x]=\sum_{i=1}^{n}\lambda_i\, e_i.$$

Considering the Leibniz identity for the triples of elements $\{e_1,x,x\}$, $\{e_1,x,e_1\}$, $\{x,e_2,e_1\}$, we obtain $\lambda_1=0$, $\gamma_1=-\alpha_1$ and $[x,e_2]=\delta_2e_2+\delta_ne_n$. By setting $e_2'=\delta_2e_2+\delta_ne_n$, we can assume that $[x,e_2]=\delta e_2$.

Now we distinguish the following possible cases

Case 1. Let $\alpha_1\neq 0$. Then the following change of basis

$$e_1'=e_1+\frac{1}{\gamma_1}\sum_{j=3}^{n}\gamma_je_j, \quad e_2'=e_2, \quad e_i'=e_i+\frac{1}{\gamma_1}\sum_{j=i+1}^{n}\gamma_{j-i+2}\, e_j, \quad 3\le i\le n,$$
$$x'=\frac{1}{\gamma_1}\, x,$$

implies $[x',e_1']=e_1'+\gamma e_2'$ (where $\gamma=\frac{\gamma_2}{\gamma_1}$) and the rest of products remains unchanging.

From the equalities:

$$e_1+\gamma(1+\delta)\, e_2=\big[x,[x,e_1]\big]=\big[[x,x],e_1\big]-\big[[x,e_1],x\big]$$
$$=\sum_{i=4}^{n}\lambda_{i-1}\, e_i-\sum_{i=1}^{n}\alpha_i\, e_i-\gamma\beta_2\, e_2-\gamma\beta_n\, e_n,$$

we deduce

$$\alpha_1=-1,\ \alpha_3=0,\ \alpha_2=-\gamma(1+\delta+\beta_2),\ \lambda_i=\alpha_{i+1},\ \ 3\le i\le n-2, \quad \text{and}$$
$$\lambda_{n-1}=\alpha_n+\gamma\beta_n.$$

In addition, if we take the following change of basis:

$$e_1' = e_1 + \sum_{i=4}^{n} A_i\, e_i, \quad e_2' = e_2, \quad e_i' = e_i + \sum_{j=i+2}^{n} A_{j-i+2}\, e_j, \quad 3 \le i \le n,$$

$$x' = \sum_{i=3}^{n-1} A_{i+1}\, e_i + B\, e_n + x,$$

where $A_j = \frac{1}{j-2}\alpha_j,\ j = 4, 5, \quad A_i = \frac{1}{i-2}\big(\alpha_i + \sum\limits_{j=4}^{i-2} A_j \alpha_{i-j+2}\big), \quad 6 \le i \le n$ and $B = \frac{1}{n-1}\big(\lambda_n + \sum\limits_{j=4}^{n-1} A_j \alpha_{i-j+3}\big)$, then we have

$$[e_1', x'] = -e_1' + \alpha_2\, e_2', \qquad [e_2', x'] = \beta_2\, e_2' + \beta_n\, e_n',$$
$$[e_i', x'] = -(i-1)\, e_i', \quad 3 \le i \le n, \qquad [x', x'] = \lambda_2\, e_2' + \gamma\beta_n\, e_{n-1}'.$$

Finally, we obtain the following multiplication table of the algebra R:

$$\begin{cases} [e_1, x] = -e_1 - \gamma(1 + \delta + \beta_2)\, e_2, & [e_2, x] = \beta_2\, e_2 + \beta_n\, e_n, \\ [x, e_1] = e_1 + \gamma\, e_2, & [e_i, x] = -(i-1)\, e_i, \qquad 3 \le i \le n, \\ [x, e_2] = \delta\, e_2, & [x, x] = \lambda_2\, e_2 + \gamma\beta_n\, e_{n-1}. \end{cases}$$

Considering the Leibniz identity for the triples of elements $\{x, x, e_2\}$, $\{x, x, x\}$, $\{x, e_1, x\}$, we obtain:

$$\delta\beta_n = \delta(\delta + \beta_2) = \delta\lambda_2 = \gamma\delta(\delta + \beta_2) = 0.$$

Notice that if $e_2 \in \mathrm{Ann}_r(R)$, then $\dim \mathrm{Ann}_r(R) = n - 1$ and if $e_2 \notin \mathrm{Ann}_r(R)$, then $\dim \mathrm{Ann}_r(R) = n - 2$.

Now we analyze the following possible subcases:

Case 1.1. Let $e_2 \in \mathrm{Ann}_r(R)$. Then $\delta = 0$ and making the change $e_1' = e_1 + \gamma e_2$ we can assume $[x, e_1] = e_1$.

In this case, we must consider two additional subcases:

Case 1.1.1. Let $e_2 \in \mathsf{Center}(R)$. Then $\dim \mathsf{Center}(R) = 1$ and $\beta_2 = \beta_n = 0$. We have two options: if $\lambda_2 = 0$, then we get the split algebra $L_1(0)$; if $\lambda_2 \neq 0$, then we obtain the algebra $L_1(1)$ (by scaling the basis).

Case 1.1.2. Let $e_2 \notin \mathsf{Center}(R)$. Then $\dim \mathsf{Center}(R) = 0$ and $(\beta_2, \beta_n) \neq (0, 0)$.

Let us take the following general change of basis:

$$e'_1=\sum_{i=1}^{n}A_i\,e_i,\ e'_2=\sum_{i=1}^{n}B_i\,e_i,\ e'_i=A_1^{i-2}\Big(A_1e_i+\sum_{j=i+1}^{n}A_{j-i+2}\,e_i\Big),\ \ 3\le i\le n,$$
$$x'=\sum_{i=1}^{n}C_i\,e_i+C_{n+1}\,x,$$

where $(A_1B_2-A_2B_1)C_{n+1}\neq 0$.

From $0=[e'_2,e'_1]=[e'_2,e'_2]$, we obtain that $B_1=B_i=0,\ 3\le i\le n-1$, i.e. $e'_2=B_2\,e_2+B_n\,e_n$ and $A_1B_2\neq 0$.

The equalities

$$e'_1=[x',e'_1]=A_1C_1\,e_3+\sum_{i=4}^{n}A_1C_{i-1}\,e_i+A_1C_{n+1}\,e_1,$$

imply that

$$C_{n+1}=1,\qquad A_2=0,\qquad A_3=A_1C_1,\qquad A_i=A_1C_{i-1},\quad 4\le i\le n.$$

Similarly, from

$$\begin{aligned}B_2\beta'_2\,e_2+(B_n\beta'_2+\beta'_nA_1^{n-1})\,e_n&=\beta'_2\,e'_2+\beta'_n\,e'_n=[e'_2,x']\\&=B_2\beta_2\,e_2+\big(B_2\beta_n-(n-1)B_n\big)\,e_n,\end{aligned}$$

and

$$\begin{aligned}\lambda'_2B_2\,e_2+\lambda'_2B_n\,e_n=\lambda'_2\,e'_2=[x',x']&=(\lambda_2+C_2\beta_2)\,e_2+(C_1^2-2C_3)\,e_3\\&+\sum_{i=4}^{n-1}\big(C_1C_{i-1}-(i-1)C_i\big)\,e_i+\big(C_1C_{n-1}-(n-1)C_n+C_2\beta_n\big)\,e_n,\end{aligned}$$

we obtain $C_i=\dfrac{1}{(i-1)!}C_1^{i-1},\ 3\le i\le n-1$ and

$$\beta'_2=\beta_2,\qquad \beta'_n=\frac{B_2\beta_n-B_n(\beta_2+n-1)}{A_1^{n-1}},\qquad \lambda'_2=\frac{\lambda_2+\beta_2C_2}{B_2},$$
$$\lambda'_2B_n=C_1C_{n-1}-(n-1)C_n+C_2\beta_n.$$

Now we need to distinguish two subcases:

Case 1.1.2.1. Let $\beta_2 = 1-n$. Putting $C_2 = -\frac{\lambda_2}{1-n}$, $C_n = \frac{C_1C_{n-1}+C_2\beta_n}{n-1}$, we get $\lambda_2' = 0$ and $\beta_n' = \frac{B_2\beta_n}{A_1^{n-1}}$.

If $\beta_n = 0$, then we get the algebra $L_2(\alpha)$ for $\alpha = 1-n$.

If $\beta_n \neq 0$, then making $A_1 = \sqrt[n-1]{\beta_n B_2}$, we obtain $\beta_n' = 1$ and the algebra L_3.

Case 1.1.2.2. Let $\beta_2 \neq 1-n$. Taking the change $B_n = \frac{B_2\beta_n}{\beta_2+n-1}$, we obtain $\beta_n = 0$. Since $\beta_2 \neq 0$, we set $C_2 = -\frac{\lambda_2}{\beta_2}$, $C_n = \frac{C_1C_{n-1}+C_2\beta_n}{n-1}$ and we get $\lambda_2 = 0$, i.e., the algebra $L_2(\alpha)$ is obtained, for $\alpha \notin \{1-n, 0\}$.

Case 1.2. Let $e_2 \notin \mathrm{Ann}_r(R)$. Then $\delta \neq 0$ and $\beta_2 = -\delta, \beta_n = \lambda_2 = 0$.

Let us consider the general change of basis in the following form:

$$e_1' = \sum_{i=1}^{n} A_i\, e_i, \qquad e_2' = \sum_{i=1}^{n} B_i\, e_i,$$
$$e_i' = A_1^{i-2}\big(A_1\, e_i + \sum_{j=i+1}^{n} A_{j-i+2}\, e_j\big),\ 3 \leq i \leq n, \quad x' = \sum_{i=1}^{n} C_i\, e_i + C_{n+1}\, x,$$

where $(A_1B_2 - A_2B_1)C_{n+1} \neq 0$.

Then from $0 = [e_2', e_1'] = [e_2', e_2']$, we derive that $B_1 = B_i = 0,\ 3 \leq i \leq n-1$, i.e. $e_2' = B_2\, e_2 + B_n\, e_n$ and $A_1B_2 \neq 0$.

Similarly, from the equations:

$$e_1' + \gamma'\, e_2' = [x', e_1'] = A_1C_{n+1}\, e_1 + C_{n+1}(A_1\gamma + A_2\delta)\, e_2 + A_1C_1\, e_3 + \sum_{i=4}^{n} A_1C_{i-1}\, e_i,$$

and

$$\delta'(B_2\, e_2 + B_n\, e_n) = \delta'\, e_2' = [x', e_2'] = B_2\delta\, e_2,$$

we obtain

$$C_{n+1} = 1, \qquad A_3 = A_1C_1, \qquad A_i = A_1C_{i-1}, \qquad 4 \leq i \leq n-1,$$
$$\gamma' = \frac{A_1\gamma + A_2(\delta-1)}{B_2}, \qquad A_1C_{n-1} = A_n + \gamma' B_n, \quad \delta' = \delta, \qquad \delta' B_n = 0.$$

Now we distinguish the following possible subcases:

Case 1.2.1. Let $\delta \neq 1$. Then by the substitution $A_2 = -\dfrac{A_1\gamma}{\delta-1}$, $A_n = A_1C_{n-1}$ into the above conditions, we get $\gamma' = 0$ and the algebra $L_4(\alpha)$.

Case 1.2.2. Let $\delta = 1$. Then $B_n = 0$. If $\gamma = 0$, then $\gamma' = 0$. If $\gamma \neq 0$, then by putting $B_2 = A_1\gamma$ and $A_n = A_1 C_{n-1} - B_n$, we get $\gamma' = 1$. Thus, the algebras $L_5(\alpha),\ \alpha \in \{0, 1\}$ are obtained.

Case 2. Let $\alpha_1 = 0$. Then $\beta_2 \neq 0$ and by replacing x by $x' = \dfrac{1}{\beta_2}\,x$, we can assume $[e_2, x'] = e_2 + \beta_n e_n$.

Under these conditions, the multiplication table of the algebra R has the form:

$$\begin{cases} [e_1, x] = \sum\limits_{i=2}^{n} \alpha_i\, e_i, & [e_2, x] = e_2 + \beta_n\, e_n, \\ [x, e_1] = \sum\limits_{i=2}^{n} \gamma_i\, e_i, & [e_i, x] = \sum\limits_{j=i+1}^{n} \alpha_{j-i+2}\, e_j, \quad 3 \leq i \leq n-1, \\ [x, e_2] = \delta\, e_2, & [x, x] = \sum\limits_{i=2}^{n} \lambda_i\, e_i. \end{cases}$$

Making the transformation $x' = x - \gamma_3\, e_1 - \sum\limits_{i=3}^{n-1} \gamma_{i+1}\, e_i$, we can assume $[x, e_1] = \gamma\, e_2$.

Similarly as above, we obtain the conditions:

$$\gamma(\delta + 1) = \alpha_2\delta - \gamma = \beta_n\delta = \delta(\delta + 1) = \lambda_2\delta = 0.$$

Now we distinguish the following subcases depending on the possible values of the parameter δ:

Case 2.1. Let $\delta \neq 0$. Then $\dim \operatorname{Ann}_r(R) = n - 2$ and $\beta_n = \lambda_2 = 0,\ \delta = -1,\ \alpha_2 = -\gamma$. By means of the change of the basis element $e_1' = e_1 + \gamma e_2$, we can suppose that $[x', e_1] = 0$.

Taking the general change of basis as in the above considered cases, we derive the following conditions for the parameters

$$\alpha_i' = \frac{\alpha_i}{A_1^{i-2}}, \quad 3 \leq i \leq n, \qquad \lambda_n' = \frac{\lambda_n}{A_1^{n-1}}.$$

Consequently, we deduce the algebra $L_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, -1)$.

Case 2.2. Let $\delta = 0$. Then $\dim \operatorname{Ann}_r(R) = n - 1$ and $\gamma = 0$. Taking the change of basis $e_2' = e_2 + \beta_n\, e_n$ and $x' = x - \lambda_2\, e_2$, we can assume that

$[e_2,x]=e_2,\ [x,x]=\lambda_n e_n$. Therefore, we have the products

$$[e_1,x]=\sum_{i=2}^{n}\alpha_i e_i,\quad [e_2,x]=e_2,\quad [e_i,x]=\sum_{j=i+1}^{n}\alpha_{j-i+2}e_j,\quad 3\le i\le n-1,$$
$$[x,x]=\lambda_n e_n.$$

Applying similar arguments to general transformation of bases, we have

$$\alpha_2'=0,\qquad \alpha_i'=\frac{\alpha_i}{A_1^{i-2}},\ \ 3\le i\le n,\qquad \lambda_n'=\frac{\lambda_n}{A_1^{n-1}}.$$

Thus, we obtain the algebra $L_6(\alpha_3,\alpha_4,\ldots,\alpha_n,\lambda,0)$. □

Theorem 2.3.4. *An arbitrary $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is isomorphic to one of the following non-isomorphic algebras:*

$$L_1:\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [e_1,x]=e_1, & [x,e_1]=-e_1, & \\ [e_2,y]=-[y,e_2]=e_2, & [e_i,x]=(i-1)e_i, & 3\le i\le n,\end{cases}$$

$$L_2:\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [e_1,x]=e_1, & [x,e_1]=-e_1, & \\ [e_2,y]=e_2, & [e_i,x]=(i-1)e_i, & 3\le i\le n.\end{cases}$$

Proof. Let

$$\mathcal{R}_{x_{|F_n^2}}=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{n-1} & \alpha_n\\ 0 & \beta & 0 & 0 & \ldots & 0 & \gamma\\ 0 & 0 & 2\alpha_1 & \alpha_3 & \ldots & \alpha_{n-2} & \alpha_{n-1}\\ 0 & 0 & 0 & 3\alpha_1 & \ldots & \alpha_{n-3} & \alpha_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & 0 & (n-1)\alpha_1\end{pmatrix}$$

and

$$\mathcal{R}_{y_{|F_n^2}}=\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-1} & \lambda_n\\ 0 & \mu & 0 & 0 & \ldots & 0 & \nu\\ 0 & 0 & 2\lambda_1 & \lambda_3 & \ldots & \lambda_{n-2} & \lambda_{n-1}\\ 0 & 0 & 0 & 3\lambda_1 & \ldots & \lambda_{n-3} & \lambda_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & 0 & (n-1)\lambda_1\end{pmatrix}$$

be two nil-independent outer derivations of the algebra F_n^2.

Taking the change of the basis elements x, y

$$x' = \frac{\mu}{\alpha_1\mu - \gamma_1\beta}x - \frac{\beta}{\alpha_1\mu - \gamma_1\beta}y,$$

$$y' = -\frac{\gamma_1}{\alpha_1\mu - \gamma_1\beta}x + \frac{\alpha_1}{\alpha_1\mu - \gamma_1\beta}y,$$

we can assume that $\alpha_1 = \mu = 1,\ \lambda_1 = \beta = 0$.

Thus, we have the products:

$$\begin{array}{lll} [e_1, x] = e_1 + \sum\limits_{i=2}^{n} \alpha_i\, e_i, & [e_2, x] = \gamma\, e_n, & \\ [e_i, x] = (i-1)\, e_i + \sum\limits_{j=i+1}^{n} \alpha_{j-i+2}\, e_j, & & 3 \leq i \leq n, \\ [e_1, y] = \sum\limits_{i=2}^{n} \lambda_i\, e_i, & [e_2, y] = e_2 + \nu\, e_n, & \\ [e_i, y] = \sum\limits_{j=i+1}^{n} \lambda_{j-i+2}\, e_j, & & 3 \leq i \leq n. \end{array}$$

Applying similar arguments and changes of bases which we have used in Theorem 2.3.3, we obtain isomorphism classes of algebras whose representative algebras are L_1 and L_2. □

Remark 2.3.5. *In fact, the algebra $L_1 = I_1 \oplus J_1$, where $I_1 = NF_{n-1} + \langle x \rangle$ and $J_1 = \langle e_2, y \rangle$, verifies that I_1 is a solvable Leibniz algebra with nilradical NF_{n-1} and J_1 is a two-dimensional solvable Lie algebra. The algebra $L_2 = I_2 \oplus J_2$, where $I_2 = NF_{n-1} + \langle x \rangle$ and $J_2 = \langle e_2, y \rangle$, verifies that J_2 is a two-dimensional solvable non-Lie Leibniz algebra. Thus, from Theorem 2.3.4, we conclude that any $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is split.*

Chapter 3

Leibniz algebras corresponding to the Diamond Lie algebras

3.1 Preliminary definitions and results

The real Diamond Lie algebra $\mathfrak{D}_{\mathbb{R}}$ is a four-dimensional Lie algebra with basis $\{J, P_1, P_2, T\}$ and non-zero relations:

$$[J, P_1] = P_2, \quad [J, P_2] = -P_1, \quad [P_1, P_2] = T.$$

The complexification of the Diamond Lie algebra: $\mathfrak{D}_{\mathbb{C}} = \mathfrak{D} \otimes_{\mathbb{R}} \mathbb{C}$ displays the following (complex) basis: $\{P_+ = P_1 - iP_2,\ P_- = P_1 + iP_2,\ T,\ J\}$, where i is the imaginary unit, whose nonzero commutators are

$$[J, P_+] = iP_+, \quad [J, P_-] = -iP_-, \quad [P_+, P_-] = 2iT.$$

If we change the basis

$$J' = -iJ, \quad P_1' = P_+, \quad P_2' = P_-, \quad T' = 2iT$$

then we can assume that

$$[J, P_1] = P_1, \quad [J, P_2] = -P_2, \quad [P_1, P_2] = T. \tag{3.1.1}$$

In the paper [26] the authors construct for any $n \in \mathbb{N}$ a $(3n+3)$-dimensional Lie module V_n over the algebra $\mathfrak{D}$, which is endowed with a basis $\{v_k^j\}_{k=0,\dots,n}^{j=0,1,2}$.

They use an action $\mathfrak{D} \cdot V_n$ (the action $V_n \cdot \mathfrak{D}$ evidently is defined by the anti-symmetrical law).

In the Leibniz algebra $L = \mathfrak{D} \oplus V_n$ we shall identify the action $V_n \cdot \mathfrak{D}$ with the product $[V_n, D]$. In order to have compatibility instead the action $\mathfrak{D} \cdot V_n$ we will use $V_n \cdot \mathfrak{D}$ as follows:

$$\begin{array}{lll} v_k^j \cdot J = -i/2(n-2k)v_k^j, & & 0 \leq k \leq n, \quad j = 0,1,2, \\ v_k^j \cdot P_+ = (k-n-1)v_{k-1}^{j+1}, & & 0 < k \leq n, \quad j = 0,1, \\ v_0^j \cdot P_+ = 0, & v_k^2 \cdot P_+ = 0, & 0 < k \leq n, \quad j = 0,1, \\ v_k^j \cdot P_- = -(k+1)v_{k+1}^{j+1}, & & 0 \leq k < n, \quad j = 0,1, \\ v_n^j \cdot P_- = 0, & v_k^2 \cdot P_- = 0, & 0 \leq k < n, \quad j = 0,1, \\ v_k^0 \cdot T = i/2(n-2k)v_k^2, & v_k^j \cdot T = 0, & 0 \leq k \leq n, \quad j = 1,2. \end{array} \tag{3.1.2}$$

In the next proposition two different decompositions of the module V_n are presented.

Proposition 3.1.1 ([26]). *Let V_n be the $\mathfrak{D}$-module constructed above and let us denote by $\lfloor x \rfloor$ integer part of x. It is verified that:*

- *If $n = 0$, then V_n decomposes in the direct sum of three trivial one-dimensional modules.*

- *If $n = 2j$, $j \in N$, $j \geq 1$, then V_n decomposes into the direct sum $V_n = U_n^1 \oplus U_n^2$ of two modules, respectively, of dimension $3(n/2) + 2$ and $3(n/2)+1$, given by $U_n^1 = \mathsf{Span}\{v_0^0, v_{2k}^0, v_{2k-1}^1, v_0^2, v_{2k}^2\}_{k=1,\dots,\lfloor n/2 \rfloor}$ and $U_n^2 = \mathsf{Span}\{v_{2k-1}^0, v_0^1, v_{2k}^1, v_{2k-1}^2\}_{k=1,\dots,\lfloor n/2 \rfloor}$.*

- *If $n = 2j+1$, $j \in N$, $j \geq 0$, then V_n decomposes into the direct sum $V_n = W_n^1 \oplus W_n^2$ of two modules of equal dimension $3n/2$, respectively, given by $W_n^1 = \mathsf{Span}\{v_{2k}^0, v_{2k+1}^1, v_{2k}^2\}_{k=0,\dots,\lfloor n/2 \rfloor}$ and $W_n^2 = \mathsf{Span}\{v_{2k+1}^0, v_{2k}^1, v_{2k+1}^2\}_{k=0,\dots,\lfloor n/2 \rfloor}$.*

In the paper [26] it is proved that the terms of the above decompositions are indecomposable $\mathfrak{D}$-modules.

Theorem 3.1.2. *The modules U_n^1, U_n^2, W_n^1 and W_n^2 are indecomposable $\mathfrak{D}$-modules.*

Now we define a $(2m+2)$-dimensional general Diamond Lie algebra $\mathfrak{D}_m$ with basis $\{J, P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m, T\}$ and whose multiplication table has the next form

$$[J, P_k] = Q_k, \qquad [J, Q_k] = -P_k, \qquad [P_k, Q_k] = T, \qquad 1 \leq k \leq m.$$

The complexification (for which we shall keep the same symbol $\mathfrak{D}_m(\mathbb{C})$) of the Diamond Lie algebra: $\mathfrak{D}_m \otimes_R \mathbb{C}$ displays the following (complex) basis:

$$P_k^+ = P_k - iQ_k, \qquad Q_k^- = P_k + iQ_k, \qquad T, \quad J, \qquad 1 \leq k \leq m,$$

where i is the imaginary unit, whose nonzero commutators are

$$[J, P_k^+] = iP_k^+, \quad [J, Q_k^-] = -iQ_k^-, \quad [P_k^+, Q_k^-] = 2iT, \qquad 1 \leq k \leq m. \quad (3.1.3)$$

The Ado's theorem from Lie algebras theory states that every finite-dimensional complex Lie algebra can be represented as a matrix Lie algebra, formed by matrices. However, that result does not specify which is the minimal order of the matrices involved in such representations. In [17], the value of the minimal order of the matrices for abelian Lie algebras and Heisenberg algebras $\mathfrak{h}_m$, defined on a $(2m+1)$-dimensional vector space with basis $X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z$, and brackets $[X_i, Y_i] = Z$, is found. For abelian Lie algebras of dimension n the minimal order is $\lceil 2\sqrt{n-1} \rceil$.

Lemma 3.1.3 ([17]). *For the Heisenberg Lie algebras $\mathfrak{h}_m$, the minimal faithful matrix representation has order equal to $m+2$.*

3.2 Leibniz algebras constructed by Fock representation of the Diamond Lie algebra

It is known that if we denote by $\overline{x}$ the operator associated to the position and by $\frac{\overline{\partial}}{\partial x}$ the one associated to the momentum (acting for instance on the space V of differentiable functions on a single variable), then $[\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}_V$. Thus we can identify the subalgebra generated by $\overline{1}, \overline{x}$ and $\frac{\overline{\partial}}{\partial x}$ with the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ whose multiplication table in the basis $\{\overline{1}, \overline{x}, \frac{\overline{\partial}}{\partial x}\}$ has a unique non-zero product $[\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}$.

For a given Heisenberg algebra $\mathfrak{h}_1$ this explanation gives rise to the so-called *Fock module* over $\mathfrak{h}_1$, the linear space $\mathbb{F}[x]$ of polynomials on x ($\mathbb{F}$ denotes an algebraically closed field of characteristic zero) with the action induced by

$$\begin{array}{lcl} (p(x), \overline{1}) & \mapsto & p(x) \\ (p(x), \overline{x}) & \mapsto & xp(x) \\ (p(x), \overline{\frac{\partial}{\partial x}}) & \mapsto & \frac{\partial}{\partial x}(p(x)) \end{array} \tag{3.2.1}$$

for any $p(x) \in \mathbb{F}[x]$.

In this section we define the Fock module over the algebra $\mathfrak{D}$. For the algebra $\mathfrak{D}$ we have the existence of a basis $\{J, P_1, P_2, T\}$ with the multiplication table (3.1.1).

Let us introduce new notations for the basis elements of $\mathfrak{D}$:

$$\overline{e} := J, \quad \overline{x} := P_1, \quad \overline{\frac{\partial}{\partial x}} := P_2, \quad \overline{1} := T.$$

The action of the linear space $\mathbb{F}[x]$ on $\{\overline{1}, \overline{x}, \overline{\frac{\partial}{\partial x}}\}$ is induced by the action (3.2.1). Further, we need to define the action on $\overline{e}$.

We set

$$(1, \overline{e}) = \lambda + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n.$$

Now we consider the action on $1 \in \mathbb{F}[x]$, the elements $\overline{e}, \overline{\frac{\partial}{\partial x}} \in \mathfrak{D}$:

$$\begin{aligned} 0 = (1, [\overline{e}, \overline{\frac{\partial}{\partial x}}]) &= ((1, \overline{e}), \overline{\frac{\partial}{\partial x}}) - ((1, \overline{\frac{\partial}{\partial x}}), \overline{e}) \\ &= (\lambda + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n, \overline{\frac{\partial}{\partial x}}) = \sum_{i=1}^{n} i\lambda_i x^{i-1}. \end{aligned}$$

Therefore, $(1, \overline{e}) = \lambda$.

From the equalities

$$x = (1, \overline{x}) = (1, [\overline{e}, \overline{x}]) = ((1, \overline{e}), \overline{x}) - ((1, \overline{x}), \overline{e}) = (\lambda, \overline{x}) - (x, \overline{e}) = \lambda x - (x, \overline{e}),$$

we derive $(x, \overline{e}) = (\lambda - 1)x$.

By induction one can prove the equality:

$$(x^t, \overline{e}) = (\lambda - t)x^t, \quad t \in \mathbb{N} \cup \{0\}.$$

Taking the change of basis $\overline{e}' := \overline{e} - \lambda\overline{1}$ we can assume that $(x^t, \overline{e}) = -tx^t,\ t \geq 0$.

Therefore, the action on $\overline{e}$ is defined as follows:

$$(p(x), \overline{e}) \mapsto -x\frac{\partial(p(x))}{\partial x}.$$

So, we are ready to define a notion of Fock module for Diamond Lie algebras.

Definition 3.2.1. *The linear space $\mathbb{F}[x]$ is called the Fock $\mathfrak{D}$-module if an action $(\mathbb{F}[x], \mathfrak{D}) \mapsto \mathbb{F}[x]$ is defined as follows:*

$$\begin{array}{lcl} (p(x), \overline{1}) & \mapsto & p(x), \\ (p(x), \overline{x}) & \mapsto & xp(x), \\ (p(x), \frac{\overline{\partial}}{\partial x}) & \mapsto & \frac{\partial}{\partial x}(p(x)), \\ (p(x), \overline{e}) & \mapsto & -x\frac{\partial(p(x))}{\partial x}. \end{array} \qquad (3.2.2)$$

for any $p(x) \in \mathbb{F}[x]$.

The main result of this section consists of the classification of Leibniz algebras, whose corresponding Lie algebra is the complex Diamond Lie algebra $\mathfrak{D}$ and the ideal I is the Fock $\mathfrak{D}$-module.

Theorem 3.2.2. *A Leibniz algebra L with conditions $L/I \cong \mathfrak{D}$, and I is the Fock $\mathfrak{D}_{\mathbb{C}}$-module, admits a basis*

$$\{\overline{1}, \overline{x}, \frac{\overline{\partial}}{\partial x},\ \overline{e},\ x^t \mid t \in \mathbb{N} \cup \{0\}\}$$

such that the multiplication table in this basis has the following form:

$$\begin{array}{ll} [\overline{e}, \overline{x}] = \overline{x}, & [\overline{x}, \overline{e}] = -\overline{x}, \\ [\overline{e}, \frac{\overline{\partial}}{\partial x}] = -\frac{\overline{\partial}}{\partial x}, & [\frac{\overline{\partial}}{\partial x}, \overline{e}] = \frac{\overline{\partial}}{\partial x}, \\ [\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}, & [\frac{\overline{\partial}}{\partial x}, \overline{x}] = -\overline{1}, \\ [x^t, \overline{1}] = x^t, & [x^t, \overline{x}] = x^{t+1}, \\ [x^t, \frac{\overline{\partial}}{\partial x}] = tx^{t-1}, & [x^t, \overline{e}] = -tx^t, \end{array}$$

where the omitted products are equal to zero.

Proof. Taking into account the action (3.2.2), we conclude that $\{\overline{1}, \overline{x}, \frac{\overline{\partial}}{\partial x}, \ \overline{e}, \ x^t \mid t \in \mathbb{N} \cup \{0\}\}$ is a basis of L and

$$[x^t, \overline{1}] = x^t, \quad [x^t, \overline{x}] = x^{t+1}, \quad [x^t, \frac{\overline{\partial}}{\partial x}] = tx^{t-1}, \quad [x^t, \overline{e}] = -tx^t.$$

Let us denote

$$[\frac{\overline{\partial}}{\partial x}, \overline{1}] = q(x), \quad [\overline{1}, \overline{1}] = r(x), \quad [\overline{x}, \overline{1}] = p(x), \quad [\overline{e}, \overline{1}] = m(x).$$

Taking the following change of basis elements:

$$\frac{\overline{\partial}}{\partial x}' = \frac{\overline{\partial}}{\partial x} - q(x), \quad \overline{1}' = \overline{1} - r(x), \quad \overline{x}' = \overline{x} - p(x), \quad \overline{e}' = \overline{e} - m(x),$$

we obtain

$$[\overline{x}, \overline{1}] = 0, \quad [\frac{\overline{\partial}}{\partial x}, \overline{1}] = 0, \quad [\overline{1}, \overline{1}] = 0, \quad [\overline{e}, \overline{1}] = 0.$$

Consequently, $\overline{1} \in \mathrm{Ann}_r(\mathfrak{D})$.

The chain of equalities $[[\mathfrak{D}, \mathfrak{D}], \overline{1}] = [\mathfrak{D}, [\mathfrak{D}, \overline{1}]] + [[\mathfrak{D}, \overline{1}], \mathfrak{D}] = 0$, imply

$$[\overline{1}, \overline{x}] = [\overline{1}, \frac{\overline{\partial}}{\partial x}] = [\overline{1}, \overline{e}] = [\overline{x}, \overline{x}] = [\frac{\overline{\partial}}{\partial x}, \frac{\overline{\partial}}{\partial x}] = [\overline{e}, \overline{e}] = 0,$$

$$[\overline{x}, \frac{\overline{\partial}}{\partial x}] = -[\frac{\overline{\partial}}{\partial x}, \overline{x}] = \overline{1}, \quad [\overline{e}, \overline{x}] = -[\overline{x}, \overline{e}] = \overline{x}, \quad [\frac{\overline{\partial}}{\partial x}, \overline{e}] = -[\overline{e}, \frac{\overline{\partial}}{\partial x}] = \frac{\overline{\partial}}{\partial x}.$$

□

3.3 Leibniz algebras corresponding to the four-dimensional complex Diamond Lie algebra with the indecomposable modules

In this section we are going to describe Leibniz algebras L such that $L/I \cong \mathfrak{D}_{\mathbb{C}}$, where $\mathfrak{D}_{\mathbb{C}}$ is the four-dimensional complex Diamond Lie algebra and the ideal I is identified as right $\mathfrak{D}_{\mathbb{C}}$-modules $U_n^1, \ U_n^2, \ W_n^1$ and W_n^2.

In order to have the compatibility with the above representations, in the law of the Diamond algebra we use the following multiplication table (we make the change of basis $P'_+ = P_-$, $P'_- = P_+$):

$$[J, P_+] = -iP_+, \quad [J, P_-] = iP_-, \quad [P_+, P_-] = -2iT.$$

3.3.1 Leibniz algebras whose ideal I is the $\mathfrak{D}_{\mathbb{C}}$-module U_n^1

Let $\{v_0^0,\, v_{2k}^0,\, v_{2k-1}^1,\, v_0^2,\, v_{2k}^2\}_{k=1,\dots,n/2}$ be the basis of module U_n^1 chosen in Proposition 3.1.1, with even n. Then from Proposition 3.1.1 and the action (3.1.2) we have the following products in the Leibniz algebra L:

$$\begin{cases} [v_{2k}^0, J] = \frac{i}{2}(n-4k)v_{2k}^0, & k=0,\dots,\frac{n}{2},\\ [v_{2k-1}^1, J] = \frac{i}{2}(n-4k+2)v_{2k-1}^1, & k=1,\dots,\frac{n}{2},\\ [v_{2k}^2, J] = \frac{i}{2}(n-4k)v_{2k}^2, & k=0,\dots,\frac{n}{2},\\ [v_{2k}^0, P_+] = (n-2k+1)v_{2k-1}^1, & k=1,\dots,\frac{n}{2},\\ [v_{2k-1}^1, P_+] = (n-2k+2)v_{2k-2}^2, & k=1,\dots,\frac{n}{2},\\ [v_{2k}^0, P_-] = (2k+1)v_{2k+1}^1, & k=0,\dots,\frac{n}{2}-1,\\ [v_{2k-1}^1, P_-] = 2kv_{2k}^2, & k=1,\dots,\frac{n}{2},\\ [v_{2k}^0, T] = -\frac{i}{2}(n-4k)v_{2k}^2, & k=0,\dots,\frac{n}{2}. \end{cases} \tag{3.3.1}$$

Theorem 3.3.1. *An arbitrary complex Leibniz algebra with corresponding Lie algebra $\overline{\mathfrak{D}}_{\mathbb{C}}$, and the ideal I associated as $\overline{\mathfrak{D}}_{\mathbb{C}}$-module defined by* (3.3.1), *admits a basis $\{J, P_+, P_-, T, v_0^0, v_{2k}^0, v_{2k-1}^1, v_0^2, v_{2k}^2\}_{k=1,\dots,n/2}$, where n is even, and the multiplication table $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ has the following form:*

- $n = 4s$

$$\begin{cases} [J,P_+] = -iP_+, & [J,P_-] = iP_-, & [P_+,P_-] = -2iT,\\ [P_+,J] = iP_+, & [P_-,J] = -iP_-, & [P_-,P_+] = 2iT + 2\alpha_1 v_{2s}^2,\\ [J,T] = \alpha_1 v_{2s}^2 & [J,J] = \alpha_2 v_{2s}^2, & [P_+,P_+] = \alpha_3 v_{2s-2}^2,\\ [P_-,P_-] = \alpha_4 v_{2s+2}^2. \end{cases}$$

- $n = 4s-2$

$$\begin{cases} [J,P_+] = -iP_+, & [P_+,J] = iP_+ + 2is\beta_1 v_{2s-2}^2,\\ [J,P_-] = iP_-, & [P_-,J] = -iP_- - 2is\beta_1 v_{2s}^2,\\ [P_+,P_-] = -2iT, & [P_-,P_+] = 2iT + 2\beta_2 v_{2s-1}^1,\\ [J,J] = \beta_1 v_{2s-1}^1, & [J,T] = \beta_2 v_{2s-1}^1,\\ [P_+,P_+] = \beta_3 v_{2s-3}^1, & [P_-,P_-] = \beta_4 v_{2s+1}^1,\\ [P_+,T] = 2is\beta_2 v_{2s-2}^2, & [T,P_+] = -i(2s\beta_2 - (s-1)\beta_3)v_{2s-2}^2,\\ [P_-,T] = -2is\beta_2 v_{2s}^2, & [T,P_-] = i(4s\beta_2 - (s-1)\beta_4)v_{2s}^2. \end{cases}$$

where $\alpha_i,\ \beta_i \in \mathbb{C},\ 1 \le i \le 4$.

Proof. We will consider two cases $n = 4s$ and $n = 4s - 2$.

Let us introduce notation

$$[J, J] = a_0^0 v_0^0 + \sum_{k=1}^{n/2} a_{2k}^0 v_{2k}^0 + \sum_{k=1}^{n/2} a_{2k-1}^1 v_{2k-1}^1 + a_0^2 v_0^2 + \sum_{k=1}^{n/2} a_{2k}^2 v_{2k}^2.$$

Case 1. Let $n = 4s$. Taking the following change of basis:

$$J' = J + \frac{ia_0^0}{2s} v_0^0 + \sum_{k=1}^{s-1} \frac{ia_{2k}^0}{2s-2k} v_{2k}^0 + \sum_{k=s+1}^{2s} \frac{ia_{2k}^0}{2s-2k} v_{2k}^0 + \sum_{k=1}^{2s} \frac{ia_{2k-1}^1}{2s-2k+1} v_{2k-1}^1$$
$$+ \frac{ia_0^2}{2s} v_0^2 + \sum_{k=1}^{s-1} \frac{ia_{2k}^2}{2s-2k} v_{2k}^2 + \sum_{k=s+1}^{2s} \frac{ia_{2k}^2}{2s-2k} v_{2k}^2,$$

we can assume that $[J, J] = a_{2s}^0 v_{2s}^0 + a_{2s}^2 v_{2s}^2$.

Lifting from the quotient Lie algebra $\overline{\mathfrak{D}}$ to the Leibniz algebra L we have

$$[J, P_+] = -iP_+ + b_0^0 v_0^0 + \sum_{k=1}^{2s} b_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} b_{2k-1}^1 v_{2k-1}^1 + b_0^2 v_0^2 + \sum_{k=1}^{2s} b_{2k}^2 v_{2k}^2,$$
$$[J, P_-] = iP_- + c_0^0 v_0^0 + \sum_{k=1}^{2s} c_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} c_{2k-1}^1 v_{2k-1}^1 + c_0^2 v_0^2 + \sum_{k=1}^{2s} c_{2k}^2 v_{2k}^2,$$
$$[P_+, P_-] = -2iT + d_0^0 v_0^0 + \sum_{k=1}^{2s} d_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} d_{2k-1}^1 v_{2k-1}^1 + d_0^2 v_0^2 + \sum_{k=1}^{2s} d_{2k}^2 v_{2k}^2.$$

Making the change of basis elements as follows:

$$P'_+ = P_+ + ib_0^0 v_0^0 + \sum_{k=1}^{2s} ib_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} ib_{2k-1}^1 v_{2k-1}^1 + ib_0^2 v_0^2 + \sum_{k=1}^{2s} ib_{2k}^2 v_{2k}^2,$$
$$P'_- = P_- - ic_0^0 v_0^0 - \sum_{k=1}^{2s} ic_{2k}^0 v_{2k}^0 - \sum_{k=1}^{2s} ic_{2k-1}^1 v_{2k-1}^1 - ic_0^2 v_0^2 - \sum_{k=1}^{2s} ic_{2k}^2 v_{2k}^2,$$
$$T' = T + \tfrac{i}{2}(d_0^0 v_0^0 + \sum_{k=1}^{2s} d_{2k}^0 v_{2k}^0 + (d_1^1 + ib_0^0) v_1^1$$
$$+ \sum_{k=2}^{2s} (d_{2k-1}^1 + i(2k-1) b_{2k-2}^0) v_{2k-1}^1 + d_0^2 v_0^2 + \sum_{k=1}^{2s} (d_{2k}^2 + 2ikb_{2k-1}^1) v_{2k}^2),$$

we derive the products

$$[J, P_+] = -iP_+, \quad [J, P_-] = iP_-, \quad [P_+, P_-] = -2iT.$$

Considering the chain of equalities

$$\begin{aligned} P_+ &= i[J, P_+] = [J, [P_+, J]] = [[J, P_+], J] - [[J, J], P_+], \\ P_- &= -i[J, P_-] = [J, [P_-, J]] = [[J, P_-], J] - [[J, J], P_-], \end{aligned}$$

we conclude

$$[P_+, J] = iP_+ + ia_{2s}^0(2s+1)v_{2s-1}^1 \text{ and } [P_-, J] = -iP_- - ia_{2s}^0(2s+1)v_{2s+1}^1.$$

We set

$$[P_+, P_+] = q_0^0 v_0^0 + \sum_{k=1}^{2s} q_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} q_{2k-1}^1 v_{2k-1}^1 + q_0^2 v_0^2 + \sum_{k=1}^{2s} q_{2k}^2 v_{2k}^2,$$

$$[P_-, P_-] = l_0^0 v_0^0 + \sum_{k=1}^{2s} l_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} l_{2k-1}^1 v_{2k-1}^1 + l_0^2 v_0^2 + \sum_{k=1}^{2s} l_{2k}^2 v_{2k}^2,$$

$$[J, T] = r_0^0 v_0^0 + \sum_{k=1}^{2s} r_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} r_{2k-1}^1 v_{2k-1}^1 + r_0^2 v_0^2 + \sum_{k=1}^{2s} r_{2k}^2 v_{2k}^2.$$

Applying the Leibniz identity to the following triples of elements we get further constraints on the structure constants

Leibniz identity		Constraint	
$\{P_+, J, P_+\}$	$\Longrightarrow$	$a_{2s}^0 = q_0^0 = q_0^2 = q_{2k-1}^1 = 0,$	$1 \leq k \leq 2s,$
		$q_{2k}^0 = q_{2k}^2 = 0,$	$k \neq s-1,\ s > 1$
$\{P_-, J, P_-\}$	$\Longrightarrow$	$l_0^0 = l_0^2 = l_{2k-1}^1 = 0,$	$1 \leq k \leq 2s,$
		$l_{2k}^0 = l_{2k}^2 = 0 = 0,$	$k \neq s+1.$

Thus, we obtain

$$[J, J] = a_{2s}^2 v_{2s}^2, \qquad [P_+, J] = iP_+, \quad [P_-, J] = -iP_-,$$

$$[P_+, P_+] = q_{2s-2}^0 v_{2s-2}^0 + q_{2s-2}^2 v_{2s-2}^2, \quad [P_-, P_-] = l_{2s+2}^0 v_{2s+2}^0 + l_{2s+2}^2 v_{2s+2}^2.$$

Note that for $s=1$, we have $[P_+,P_+]=q_0^0v_0^0+q_0^2v_0^2$, which agrees with the case $s>1$.

Similarly, we derive the following constraints:

Leibniz identity		Constraint
$\{J,T,J\}$	$\Longrightarrow$	$[J,T]=r_{2s}^0v_{2s}^0+r_{2s}^2v_{2s}^2,$
$\{P_+,J,P_-\}$	$\Longrightarrow$	$[T,J]=0,$
$\{J,P_+,T\}$	$\Longrightarrow$	$[P_+,T]=i(2s+1)r_{2s}^0v_{2s-1}^1,$
$\{J,P_-,T\}$	$\Longrightarrow$	$[P_-,T]=-i(2s+1)r_{2s}^0v_{2s+1}^1.$

Taking into account the product $[P_+,P_-]=-2iT$ in the following chain of equalities

$$-2i[J,T]=[J,[P_+,P_-]]=[[J,P_+],P_-]-[[J,P_-],P_+]=-i[P_+,P_-]-i[P_-,P_+],$$

we deduce $[P_-,P_+]=2iT+2r_{2s}^0v_{2s}^0+2r_{2s}^2v_{2s}^2$.

In order to identify the products $[T,P_+]$ and $[T,P_-]$, we introduce the notations:

$$[T,P_+]=m_0^0v_0^0+\sum_{k=1}^{2s}m_{2k}^0v_{2k}^0+\sum_{k=1}^{2s}m_{2k-1}^1v_{2k-1}^1+m_0^2v_0^2+\sum_{k=1}^{2s}m_{2k}^2v_{2k}^2,$$

$$[T,P_-]=t_0^0v_0^0+\sum_{k=1}^{2s}t_{2k}^0v_{2k}^0+\sum_{k=1}^{2s}t_{2k-1}^1v_{2k-1}^1+t_0^2v_0^2+\sum_{k=1}^{2s}t_{2k}^2v_{2k}^2.$$

In a similar way as above we obtain

Leibniz identity		Constraint
$\{T,P_+,J\}$	$\Longrightarrow$	$[T,P_+]=m_{2s-1}^1v_{2s-1}^1,$
$\{T,P_-,J\}$	$\Longrightarrow$	$[T,P_-]=t_{2s+1}^1v_{2s+1}^1,$
$\{P_+,P_-,T\}$	$\Longrightarrow$	$[T,T]=-s(2s+1)r_{2s}^0v_{2s}^2,$
$\{P_+,P_+,T\}$	$\Longrightarrow$	$q_{2s-2}^0=-(s+1)(2s+1)r_{2s}^0,$
$\{P_+,P_+,P_-\}$	$\Longrightarrow$	$m_{2s-1}^1=-i/2(2s+1)(2s^2+s+1)r_{2s}^0,$
$\{P_-,P_-,T\}$	$\Longrightarrow$	$l_{2s+2}^0=-(2s+1)(s+1)r_{2s}^0,$
$\{T,P_+,P_-\}$	$\Longrightarrow$	$t_{2s+1}^1=-i/2(2s+1)(2s^2+s+3)r_{2s}^0.$

From these restrictions, we derive

$$\begin{aligned}
[P_+, P_+] &= -(s+1)(2s+1)r_{2s}^0 v_{2s-2}^0 + q_{2s-2}^2 v_{2s-2}^2,\\
[T, P_+] &= -i/2(2s+1)(2s^2+s+1)r_{2s}^0 v_{2s-1}^1,\\
[P_-, P_-] &= -(2s+1)(s+1)r_{2s}^0 v_{2s+2}^0 + l_{2s+2}^2 v_{2s+2}^2,\\
[T, P_-] &= -i/2(2s+1)(2s^2+s+3)r_{2s}^0 v_{2s+1}^1.
\end{aligned}$$

Finally, if we apply the Leibniz identity to the triple of elements $\{P_-, P_+, P_-\}$, we obtain $r_{2s}^0 = 0$. Thus, by assuming $(r_{2s}^2, a_{2s}^2, q_{2s-2}^2, l_{2s+2}^2) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, we get the first family of the theorem.

Case 2. Let $n = 4s - 2$. Taking the change in the following form:

$$\begin{aligned}
J' = J + \frac{ia_0^0}{2s-1}v_0^0 + \sum_{k-1}^{2s-1} \frac{ia_{2k}^0}{2s-2k-1}v_{2k}^0 + \sum_{k=1}^{s-1} \frac{ia_{2k-1}^1}{2s-2k}v_{2k-1}^1\\
+ \sum_{k=s+1}^{2s-1} \frac{ia_{2k-1}^1}{2s-2k}v_{2k-1}^1 + \frac{ia_0^2}{2s-1}v_0^2 + \sum_{k=1}^{2s-1} \frac{ia_{2k}^2}{2s-2k-1}v_{2k}^2,
\end{aligned}$$

we can assume that $[J, J] = a_{2s-1}^1 v_{2s-1}^1$.

Analogously to the previous case, we can deduce the products

$$[J, P_+] = -iP_+, \qquad [J, P_-] = iP_-, \qquad [P_+, P_-] = -2iT.$$

Verifying the Leibniz identity on triples of elements we have the following restrictions:

Leibniz identity		Constraint
$\{J, J, P_+\}$	$\Longrightarrow$	$[P_+, J] = iP_+ + 2isa_{2s-1}^1 v_{2s-2}^2,$
$\{J, J, P_-\}$	$\Longrightarrow$	$[P_-, J] = -iP_- - 2isa_{2s-1}^1 v_{2s}^2.$

We put

$$[P_+,P_+]=q_0^0v_0^0+\sum_{k=1}^{2s-1}q_{2k}^0v_{2k}^0+\sum_{k=1}^{2s-1}q_{2k-1}^1v_{2k-1}^1+q_0^2v_0^2+\sum_{k=1}^{2s-1}q_{2k}^2v_{2k}^2,$$

$$[P_-,P_-]=l_0^0v_0^0+\sum_{k=1}^{2s-1}l_{2k}^0v_{2k}^0+\sum_{k=1}^{2s-1}l_{2k-1}^1v_{2k-1}^1+l_0^2v_0^2+\sum_{k=1}^{2s-1}l_{2k}^2v_{2k}^2,$$

$$[J,T]=r_0^0v_0^0+\sum_{k=1}^{2s-1}r_{2k}^0v_{2k}^0+\sum_{k=1}^{2s-1}r_{2k-1}^1v_{2k-1}^1+r_0^2v_0^2+\sum_{k=1}^{2s-1}r_{2k}^2v_{2k}^2.$$

From the Leibniz identity, we have

Leibniz identity		Constraint
$\{P_+,J,P_+\}$	$\Longrightarrow$	$[P_+,P_+]=q_{2s-3}^1v_{2s-3}^1,$
$\{P_-,J,P_-\}$	$\Longrightarrow$	$[P_-,P_-]=l_{2s+1}^1v_{2s+1}^1,$
$\{J,J,T\}$	$\Longrightarrow$	$[J,T]=r_{2s-1}^1v_{2s-1}^1,$
$\{P_+,J,P_-\}$	$\Longrightarrow$	$[T,J]=0,$
$\{J,P_+,T\}$	$\Longrightarrow$	$[P_+,T]=2isr_{2s-1}^1v_{2s-2}^2,$
$\{J,P_-,T\}$	$\Longrightarrow$	$[P_-,T]=-2isr_{2s-1}^1v_{2s}^2,$
$\{J,P_+,P_-\}$	$\Longrightarrow$	$[P_-,P_+]=2iT+2r_{2s-1}^1v_{2s-1}^1.$

Setting

$$[T,P_+]=m_0^0v_0^0+\sum_{k=1}^{2s-1}m_{2k}^0v_{2k}^0+\sum_{k=1}^{2s-1}m_{2k-1}^1v_{2k-1}^1+m_0^2v_0^2+\sum_{k=1}^{2s-1}m_{2k}^2v_{2k}^2,$$

$$[T,P_-]=t_0^0v_0^0+\sum_{k=1}^{2s-1}t_{2k}^0v_{2k}^0+\sum_{k=1}^{2s-1}t_{2k-1}^1v_{2k-1}^1+t_0^2v_0^2+\sum_{k=1}^{2s-1}t_{2k}^2v_{2k}^2,$$

and applying the Leibniz identity to the following triples of elements:

$$\{T,P_+,J\},\quad \{T,P_-,J\},\quad \{P_+,P_-,T\},$$
$$\{P_+,P_+,P_-\},\quad \{T,P_+,P_-\},\quad \{P_-,P_+,P_-\},$$

we derive

$$\begin{aligned}
[T, P_+] &= (i(s-1)q^1_{2s-3} - 2isr^1_{2s-1})v^2_{2s-2},\\
[T, P_-] &= (4isr^1_{2s-1} - i(s-1)l^1_{2s-2})v^2_{2s},\\
[T, T] &= 0.
\end{aligned}$$

Finally, by denoting $(a^1_{2s-1}, r^1_{2s-1}, q^1_{2s-3}, l^1_{2s+1}) = (\beta_1, \beta_2, \beta_3, \beta_4)$, we obtain the second family. □

3.3.2 Leibniz algebras whose ideal I is the $\mathfrak{D}_{\mathbb{C}}$-module U_n^2

Suppose that the ideal I is defined as a Leibniz $\mathfrak{D}_{\mathbb{C}}$-module by the irreducible representation U_n^2 and $\{v^0_{2k-1}, v^1_0, v^1_{2k}, v^2_{2k-1}\}_{k=1,\dots,n/2}$ for even n is the basis of I chosen as in Proposition 3.1.1. Then the products $[I, \mathfrak{D}_{\mathbb{C}}]$ have the following form:

$$\begin{cases}
[v^0_{2k-1}, J] = \frac{i}{2}(n-4k+2)v^0_{2k-1}, & k = 1, \dots, \frac{n}{2},\\
[v^1_{2k}, J] = \frac{i}{2}(n-4k)v^1_{2k}, & k = 0, \dots, \frac{n}{2},\\
[v^2_{2k-1}, J] = \frac{i}{2}(n-4k+2)v^2_{2k-1}, & k = 1, \dots, \frac{n}{2},\\
[v^0_{2k-1}, P_+] = (n-2k+2)v^1_{2k-2}, & k = 1, \dots, \frac{n}{2},\\
[v^1_{2k}, P_+] = (n-2k+1)v^2_{2k-1}, & k = 1, \dots, \frac{n}{2},\\
[v^0_{2k-1}, P_-] = 2kv^1_{2k}, & k = 1, \dots, \frac{n}{2},\\
[v^1_{2k}, P_-] = (2k+1)v^2_{2k+1}, & k = 0, \dots, \frac{n}{2} - 1,\\
[v^0_{2k-1}, T] = -i/2(n-4k+2)v^2_{2k-1}, & k = 1, \dots, \frac{n}{2}.
\end{cases}$$

Theorem 3.3.2. *An arbitrary complex Leibniz algebra with corresponding Lie algebra $\overline{\mathfrak{D}}_{\mathbb{C}}$ and with the ideal I defined as a Leibniz $\overline{\mathfrak{D}}_{\mathbb{C}}$-module U_n^2 admits a basis $\{J, P_+, P_-, T, v^0_{2k-1}, v^1_0, v^1_{2k}, v^2_{2k-1}\}_{k=1,\dots,n/2}$, where n is even, and such that the multiplication table $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ has the following form:*

- $n = 4s$

$$\begin{cases} [J,P_+] = -iP_+, \\ [P_+,J] = iP_+ + i(2s+1)\gamma_1 v^2_{2s-1}, \\ [J,P_-] = iP_-, \\ [P_-,J] = -iP_- - i(2s+1)\gamma_1 v^2_{2s+1}, \\ [P_+,P_-] = -2iT, \\ [P_-,P_+] = 2iT + 2\gamma_2 v^1_{2s}, \\ [J,J] = \gamma_1 v^1_{2s}, \\ [J,T] = \gamma_2 v^1_{2s}, \\ [P_+,P_+] = \gamma_3 v^1_{2s-2}, \\ [P_-,P_-] = \gamma_4 v^1_{2s+2}, \\ [P_+,T] = i(2s+1)\gamma_2 v^2_{2s-1}, \\ [T,P_+] = -i((2s+1)\gamma_2 - \dfrac{(2s-1)\gamma_3}{2})v^2_{2s-1}, \\ [P_-,T] = -i(2s+1)\gamma_2 v^2_{2s+1}, \\ [T,P_-] = i(2(2s+1)\gamma_2 - \dfrac{(2s-1)\gamma_4}{2})v^2_{2s+1}, \end{cases}$$

- $n = 4s-2$

$$\begin{cases} [J,P_+] = -iP_+, & [J,P_-] = iP_-, & [P_+,P_-] = -2iT, \\ [P_+,J] = iP_+, & [P_-,J] = -iP_-, & [P_-,P_+] = 2iT + 2\delta_1 v^2_{2s-1}, \\ [J,T] = \delta_1 v^2_{2s-1} & [J,J] = \delta_2 v^2_{2s-1}, & [P_+,P_+] = \delta_3 v^2_{2s-3}, \\ [P_-,P_-] = \delta_4 v^2_{2s+1}, \end{cases}$$

where $\gamma_i,\ \delta_i \in \mathbb{C},\ 1 \le i \le 4.$

Proof. Let us denote

$$[J,J] = \sum_{k=1}^{n/2} a^0_{2k-1} v^0_{2k-1} + a^1_0 v^1_0 + \sum_{k=1}^{n/2} a^1_{2k} v^1_{2k} + \sum_{k=1}^{n/2} a^2_{2k-1} v^2_{2k-1}.$$

In a similar way to the proof of Theorem 3.3.1 we will consider the cases $n = 4s$ and $n = 4s - 2$.

Case 1. Let $n = 4s$. Taking the change of element J as follows:

$$J' = J + \sum_{k=1}^{2s} \frac{ia_{2k-1}^0}{2s-2k+1} v_{2k-1}^0 + \frac{ia_0^1}{2s} v_0^1 + \sum_{k=1}^{s-1} \frac{ia_{2k}^1}{2s-2k} v_{2k}^1 + \sum_{k=s+1}^{2s} \frac{ia_{2k}^1}{2s-2k} v_{2k}^1 + \sum_{k=1}^{2s} \frac{ia_{2k-1}^2}{2s-2k+1} v_{2k-1}^2,$$

we can assume that $[J, J] = a_{2s}^1 v_{2s}^1$.

Applying similar arguments as in the proof of Theorem 3.3.1, we derive

$$[J, P_+] = -iP_+, \quad [J, P_-] = iP_-, \quad [P_+, P_-] = -2iT.$$

We set

$$[P_+, P_+] = \sum_{k=1}^{2s} q_{2k-1}^0 v_{2k-1}^0 + q_0^1 v_0^1 + \sum_{k=1}^{2s} q_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} q_{2k-1}^2 v_{2k-1}^2,$$
$$[P_-, P_-] = \sum_{k=1}^{2s} l_{2k-1}^0 v_{2k-1}^0 + l_0^1 v_0^1 + \sum_{k=1}^{2s} l_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} l_{2k-1}^2 v_{2k-1}^2,$$
$$[J, T] = \sum_{k=1}^{2s} r_{2k-1}^0 v_{2k-1}^0 + r_0^1 v_0^1 + \sum_{k=1}^{2s} r_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} r_{2k-1}^2 v_{2k-1}^2.$$

Considering the Leibniz identity to the following triples of elements:

$$\{J, P_+, J\}, \ \{J, P_-, J\}, \ \{P_+, J, P_+\}, \ \{P_-, J, P_-\}, \ \{J, J, T\}, \ \{P_+, J, P_-\},$$

we deduce restrictions which imply the expressions for the products

$$[P_+, J] = iP_+ + ia_{2s}^1(2s+1)v_{2s-1}^2, \quad [P_-, J] = -iP_- - ia_{2s}^1(2s+1)v_{2s+1}^2,$$
$$[P_+, P_+] = q_{2s-2}^1 v_{2s-2}^1, \quad [P_-, P_-] = l_{2s+2}^1 v_{2s+2}^1,$$
$$[J, T] = r_{2s}^1 v_{2s}^1, \quad [T, J] = 0.$$

Moreover, we have

Leibniz identity		Constraint
$\{J, P_+, T\}$	$\Longrightarrow$	$[P_+, T] = ir^1_{2s}(2s+1)v^2_{2s-1}$,
$\{J, P_-, T\}$	$\Longrightarrow$	$[P_-, T] = -ir^1_{2s}(2s+1)v^2_{2s+1}$,
$\{J, P_+, P_-\}$	$\Longrightarrow$	$[P_-, P_+] = 2iT + 2r^1_{2s}v^1_{2s}$.

We also denote

$$[T, P_+] = \sum_{k=1}^{2s} m^0_{2k-1}v^0_{2k-1} + m^1_0 v^1_0 + \sum_{k=1}^{2s} m^1_{2k}v^1_{2k} + \sum_{k=1}^{2s} m^2_{2k-1}v^2_{2k-1},$$
$$[T, P_-] = \sum_{k=1}^{2s} t^0_{2k-1}v^0_{2k-1} + t^1_0 v^1_0 + \sum_{k=1}^{2s} t^1_{2k}v^1_{2k} + \sum_{k=1}^{2s} t^2_{2k-1}v^2_{2k-1}.$$

Applying the Leibniz identity to the triples $\{T, P_+, J\}$, $\{T, P_-, J\}$, $\{P_+, P_-, P_-\}$, we obtain

$$[T, P_+] = m^0_{2s-1}v^0_{2s-1} + m^2_{2s-1}v^2_{2s-1},$$
$$[T, P_-] = t^0_{2s+1}v^0_{2s+1} + t^2_{2s+1}v^2_{2s+1},$$
$$[T, T] = 0.$$

Finally, we have

Leibniz identity		Constraint
$\{P_+, P_+, P_-\}$	$\Longrightarrow$	$m^0_{2s-1} = 0$,
		$m^2_{2s-1} = 1/2i(2s-1)q^1_{2s-2} - i(2s+1)r^1_{2s}$,
$\{T, P_+, P_-\}$	$\Longrightarrow$	$t^0_{2s+1} = 0$,
$\{P_-, P_+, P_-\}$	$\Longrightarrow$	$t^2_{2s+1} = 2i(2s+1)r^1_{2s} - 1/2i(2s-1)l^1_{2s+2}$.

Denoting the parameters $(a^1_{2s}, r^1_{2s}, q^1_{2s-2}, l^1_{2s+2})$ by $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, we obtain the first family of the theorem.

Case 2. Let $n = 4s-2$. The second family of the theorem is obtained by applying similar arguments as in the previous case. □

3.3.3 Leibniz algebras whose ideal I is the Leibniz $\mathfrak{D}_{\mathbb{C}}$-module either W_n^1 or W_n^2

Lemma 3.3.3. *Let L be a Leibniz algebra such that $L/I \cong \overline{\mathfrak{D}}_{\mathbb{C}}$, where $\overline{\mathfrak{D}}_{\mathbb{C}}$ is the Diamond Lie algebra and I is its right $\overline{\mathfrak{D}}_{\mathbb{C}}$-module. If there exists a basis $\{X_1, X_2, \dots, X_n\}$ of I such that $[X_k, J] = \alpha_k X_k,\ \alpha_k \notin \{-2i, 0, 2i\}$, for $1 \leq k \leq n$, where i is the imaginary unit, then $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ has the following form:*

$$[J, P_+] = -[P_+, J] = -iP_+,$$
$$[J, P_-] = -[P_-, J] = iP_-,$$
$$[P_+, P_-] = -[P_-, P_+] = -2iT,$$

Proof. Here we shall use the multiplication table (3.1.1) of the complex Diamond Lie algebra. Let us assume that $[J, J] = \sum\limits_{k=1}^{n} m_k X_k$. Then by setting $J' := J - \sum\limits_{k=1}^{n} \frac{m_k}{\alpha_k} X_k$, we can assume that $[J, J] = 0$.

Let us denote

$$[J, P_+] = -iP_+ + \sum_{k=1}^{n} q_k X_k, \quad [J, P_-] = -P_- + \sum_{k=1}^{n} r_k X_k.$$

Taking the following basis transformation:

$$J' = J, \quad P'_+ = P_+ - \sum_{k=1}^{n} q_k X_k, \quad P'_- = P_- + \sum_{k=1}^{n} r_k X_k, \quad T' = \frac{i}{2}[P'_+, P'_-],$$

we can assume that

$$[J, P_+] = -iP_+, \quad [J, P_-] = -P_-, \quad [P_+, P_-] = -2iT.$$

Applying the Leibniz identity for the triples $\{J, J, P_+\}$, $\{J, J, P_-\}$ we derive

$$[P_+, J] = -[J, P_+], \quad [P_-, J] = -[J, P_-].$$

We put

$$[J, T] = \sum_{k=1}^{n} t_k X_k.$$

Considering the Leibniz identity for the triple $\{J, J, T\}$ and taking into account the condition $\alpha_i \neq 0$, we get $[J, T] = 0$.

Similarly, from the Leibniz identity for the following triples we obtain:

$$\begin{cases} \{J, P_+, P_-\}, & \Rightarrow \ [P_+, P_-] = -[P_+, P_-], \\ \{J, P_+, T\}, & \Rightarrow \ [P_+, T] = 0, \\ \{J, P_-, T\}, & \Rightarrow \ [P_-, T] = 0, \\ \{P_+, P_-, T\}, & \Rightarrow \ [T, T] = 0, \\ \{P_+, J, P_-\}, & \Rightarrow \ [T, J] = 0. \end{cases}$$

Taking into account the condition of proposition $\alpha_k \notin \{-2i, 0, 2i\}$ in the Leibniz identity for the triples:

$$\{P_+, J, P_+\}, \ \{P_-, J, P_-\}, \ \{P_+, P_+, P_-\}, \ \{P_-, P_-, P_+\}$$

we obtain the products

$$[P_+, P_+] = [P_-, P_-] = [T, P_+] = [T, P_-] = 0,$$

which complete the proof of the lemma. □

Let L be a Leibniz algebra such that the ideal I is defined as a Leibniz $\mathfrak{D}$-module by indecomposable Lie representation W_n^1 of the algebra $\mathfrak{D}$ [26]. Then one can assume that $I = \mathsf{Span}\{v_{2k}^0, v_{2k+1}^1, v_{2k}^2\}_{k=0,\dots,\lfloor n/2 \rfloor}$ where n is odd and

$$\begin{cases} [v_{2k}^0, J] = \frac{i}{2}(n-4k)v_{2k}^0, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v_{2k+1}^1, J] = \frac{i}{2}(n-4k-2)v_{2k-1}^1, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v_{2k}^2, J] = \frac{i}{2}(n-4k)v_{2k}^2, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v_{2k}^0, P_+] = (n-2k+1)v_{2k-1}^1, & k = 1, \dots, \lfloor n/2 \rfloor, \\ [v_{2k+1}^1, P_+] = (n-2k)v_{2k}^2, & k = 1, \dots, \lfloor n/2 \rfloor, \\ [v_{2k}^0, P_-] = (2k+1)v_{2k+1}^1, & k = 0, \dots, \lfloor (n-1)/2 \rfloor, \\ [v_{2k+1}^1, P_-] = (2k+2)v_{2k+2}^2, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v_{2k}^0, T] = -i/2(n-4k)v_{2k}^2, & k = 0, \dots, \lfloor n/2 \rfloor \end{cases} \tag{3.3.2}$$

Let us define the ideal I as a Leibniz $\mathfrak{D}$-module by the indecomposable Lie representation W_n^2 of the algebra $\mathfrak{D}$ [26]. Then one can assume $I = \mathsf{Span}\{v_{2k+1}^0, v_{2k}^1, v_{2k+1}^2\}_{k=0,\dots,\lfloor n/2 \rfloor}$, where n is odd and

$$\begin{cases} [v^0_{2k+1}, J] = \frac{i}{2}(n-4k-2)v^0_{2k+1}, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v^1_{2k}, J] = \frac{i}{2}(n-4k)v^1_{2k}, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v^2_{2k+1}, J] = \frac{i}{2}(n-4k-2)v^2_{2k+1}, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v^0_{2k+1}, P_+] = (n-2k)v^1_{2k}, & k = 1, \dots, \lfloor n/2 \rfloor, \\ [v^1_{2k}, P_+] = (n-2k+1)v^2_{2k-1}, & k = 1, \dots, \lfloor n/2 \rfloor, \\ [v^0_{2k+1}, P_-] = (2k+2)v^1_{2k+2}, & k = 0, \dots, \lfloor (n-1)/2 \rfloor, \\ [v^1_{2k}, P_-] = (2k+1)v^2_{2k+1}, & k = 0, \dots, \lfloor n/2 \rfloor, \\ [v^0_{2k+1}, T] = -i/2(n-4k-2)v^2_{2k+1}, & k = 0, \dots, \lfloor n/2 \rfloor. \end{cases} \tag{3.3.3}$$

Theorem 3.3.4. *Let L be a complex Leibniz algebra with corresponding Diamond Lie algebra $\overline{\mathfrak{D}}_{\mathbb{C}}$ and the ideal I is defined as a Leibniz $\mathfrak{D}_{\mathbb{C}}$-module by indecomposable Lie representation either W^1_n or W^2_n. Then $[\mathfrak{D}_{\mathbb{C}}, \mathfrak{D}_{\mathbb{C}}]$ has the following form:*

$$[J, P_+] = -[P_+, J] = -iP_+,$$
$$[J, P_-] = -[P_-, J] = iP_-,$$
$$[P_+, P_-] = -[P_-, P_+] = -2iT.$$

Proof. The proof of the theorem follows from the products (3.3.2)–(3.3.3) and Lemma 3.3.3. □

3.4 Leibniz algebras constructed by a minimal faithful representation of the general Diamond Lie algebras

In this section we are going to study Leibniz algebras L such that $L/I \cong \mathfrak{D}_m(\mathbb{C})$ and the $\mathfrak{D}_m(\mathbb{C})$-module I is a minimal faithful representation, that is, the action $I \times \mathfrak{D}_m(\mathbb{C}) \to I$ gives rise to a minimal faithful representation of $\mathfrak{D}_m(\mathbb{C})$. Moreover, this representation factorizes through $\mathfrak{sl}(m+2, \mathbb{C})$.

Proposition 3.4.1. *Let $\mathfrak{D}_m(\mathbb{C})$ be a $(2m+2)$-dimensional general Diamond Lie algebra with the basis*

$$\{J, P^+_1, P^+_2, \dots, P^+_m, Q^-_1, Q^-_2, \dots, Q^-_m, T\}.$$

Then its minimal faithful representation is given by the correspondence

$$\varphi : \theta J + \sum_{k=1}^{m} \alpha_k P_k^+ + \sum_{k=1}^{m} \beta_k Q_k^- + \delta T \longmapsto$$
$$\begin{pmatrix} \frac{im}{m+2}\theta & \alpha_m & \alpha_{m-1} & \dots & \alpha_2 & \alpha_1 & -\frac{i}{2}\delta \\ 0 & -\frac{2i}{m+2}\theta & a_1 & \dots & 0 & 0 & \beta_m \\ 0 & 0 & -\frac{2i}{m+2}\theta & \dots & 0 & 0 & \beta_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{2i}{m+2}\theta & a_1 & \beta_2 \\ 0 & 0 & 0 & \dots & 0 & -\frac{2i}{m+2}\theta & \beta_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{im}{m+2}\theta \end{pmatrix}.$$

Proof. Consider the linear map $\varphi\colon \mathfrak{D}_m(\mathbb{C}) \to \mathfrak{sl}(m+2,\mathbb{C})$ given by

$$\varphi(J) = \frac{im}{m+2} e_{1,1} - \sum_{s=2}^{m+1} \frac{2i}{m+2} e_{s,s} + \frac{im}{m+2} e_{m+2,m+2},$$
$$\varphi(T) = -\frac{i}{2} e_{1,m+2}, \qquad \varphi(P_k^+) = e_{1,m+2-k},$$
$$\varphi(Q_k^-) = e_{m+2-k,m+2}, \qquad 1 \le k \le m,$$

where $e_{i,j}$ is the matrix whose (i,j)-th entry is a 1 and all others 0's.

By checking $[\varphi(x), \varphi(y)] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for all $x, y \in \mathfrak{D}_m(\mathbb{C})$, we verify that φ is an injective homomorphism of algebras. It is easy to see that $\mathfrak{D}_m \setminus J \cong \mathfrak{h}_m$. By Lemma 3.1.3 we obtain that it is minimal. □

Let us denote by $V = \mathbb{C}^{m+2}$ the natural $\varphi(\mathfrak{D}_m(\mathbb{C}))$-module and endow it with a $\mathfrak{D}_m(\mathbb{C})$-module structure, $V \times \mathfrak{D}_m(\mathbb{C}) \to V$, given by

$$(x, e) := x\varphi(e),$$

where $x \in V$ and $e \in \mathfrak{D}_m(\mathbb{C})$.

Then the action of $\mathfrak{D}_m(\mathbb{C})$ on $V = \langle X_1, X_2, \ldots, X_{m+2}\rangle$ is given as follows:

$$\begin{cases} (X_1, J) = \frac{im}{m+2} X_1, & \\ (X_k, J) = -\frac{2i}{m+2} X_k, & 2 \leq k \leq m+1, \\ (X_{m+2}, J) = \frac{im}{m+2} X_{m+2}, & \\ (X_1, P_k^+) = X_{m+2-k}, & 1 \leq k \leq m, \\ (X_{m+2-k}, Q_k^-) = X_{m+2}, & 1 \leq k \leq m, \\ (X_1, T) = -\frac{i}{2} X_{m+2}, & \end{cases} \tag{3.4.1}$$

and the remaining products in the action being zero.

Now we investigate Leibniz algebras L such that $L/I \cong \mathfrak{D}_m(\mathbb{C})$ and $I = V$ as a $\mathfrak{D}_m(\mathbb{C})$-module.

Theorem 3.4.2. *Let L be an arbitrary Leibniz algebra with corresponding Lie algebra $\mathfrak{D}_m(\mathbb{C})$ and the ideal I associated as $\mathfrak{D}_m(\mathbb{C})$-module defined by* (3.4.1). *Then there exists a basis*

$$\{J, P_1^+, P_2^+, \ldots, P_m^+, Q_1^-, Q_2^-, \ldots, Q_m^-, T, X_1, X_2, \ldots, X_{m+2}\}$$

of L such that

$$[\mathfrak{D}_m(\mathbb{C}), \mathfrak{D}_m(\mathbb{C})] \subseteq \mathfrak{D}_m(\mathbb{C}).$$

Proof. Here we shall use the multiplication table (3.1.3) of the complex Diamond Lie algebra. Let us assume that

$$[J, J] = \sum_{k=1}^{m+2} \delta_k X_k.$$

Then by setting

$$J' := J + \frac{i(m+2)\delta_1}{m} X_1 - \sum_{k=2}^{m+1} \frac{i(m+2)\delta_i}{2} X_i + \frac{i(m+2)\delta_{m+2}}{m} X_{m+2},$$

we can assume that $[J, J] = 0$.

Let us denote

$$[J, P_k^+] = iP_k^+ + \sum_{s=1}^{m+2} \alpha_{k,s} X_s, \quad [J, Q_k^-] = -iQ_k^- + \sum_{s=1}^{m+2} \beta_{k,s} X_s, \quad 1 \leq k \leq m.$$

Taking the following basis transformation:

$$J' = J, \qquad P_k^{+\prime} = P_k^+ - \sum_{s=1}^{m+2} i\alpha_{k,s}X_s, \qquad Q_k^{-\prime} = Q_k^- + \sum_{k=2}^{m+2} i\beta_{k,s}X_s,$$
$$T' = -i/2[P_1^{+\prime}, Q_1^{-\prime}], \qquad 1 \leq k \leq m,$$

we can assume that

$$[J, P_k^+] = iP_k^+, \quad [J, Q_k^-] = -iQ_k^-, \quad [P_1^+, Q_1^-] = 2iT, \quad 1 \leq k \leq m.$$

By applying the Leibniz identity to the triples $\{J, J, P_k^+\}$, $\{J, J, Q_k^-\}$, we derive

$$[P_k^+, J] = -[J, P_k^+], \qquad [Q_k^-, J] = -[J, Q_k^-], \qquad 1 \leq k \leq m.$$

Considering the Leibniz identity for the triples we get the following constraints.

Leibniz identity		Constraints	
$\{P_1^+, J, Q_1^-\}$	$\Longrightarrow$	$[T, J] = 0,$	
$\{J, T, J\}$	$\Longrightarrow$	$[J, T] = 0,$	
$\{J, P_k^+, Q_s^-\}$	$\Longrightarrow$	$[P_k^+, Q_s^-] = -[Q_s^-, P_k^+],$	$1 \leq k, s \leq m,$
$\{P_k^+, J, Q_s^-\}$	$\Longrightarrow$	$[P_k^+, Q_s^-] = 0,$	$1 \leq k, s \leq m, \ k \neq s,$
$\{P_k^+, J, Q_k^-\}$	$\Longrightarrow$	$[P_k^+, Q_k^-] = 2iT,$	$2 \leq k \leq m,$
$\{P_k^+, J, P_s^+\}$	$\Longrightarrow$	$[P_k^+, P_s^+] = 0,$	$1 \leq k, s \leq m,$
$\{Q_k^-, J, Q_s^-\}$	$\Longrightarrow$	$[Q_k^-, Q_s^-] = 0,$	$1 \leq k, s \leq m,$
$\{J, P_k^+, T\}$	$\Longrightarrow$	$[P_k^+, T] = 0,$	$1 \leq k \leq m,$
$\{J, Q_k^-, T\}$	$\Longrightarrow$	$[Q_k^-, T] = 0$	$1 \leq k \leq m,$
$\{P_k^+, P_k^+, Q_k^-\}$	$\Longrightarrow$	$[T, P_k^+] = 0,$	$1 \leq k \leq m,$
$\{Q_k^-, Q_k^-, P_k^+\}$	$\Longrightarrow$	$[T, Q_k^-] = 0,$	$1 \leq k \leq m.$

□

3.5 Leibniz algebras constructed by a faithful representation of the general Diamond algebra which is isomorphic to a subalgebra of $\mathfrak{sp}(2m+2,\mathbb{R})$

In this section we are going to study Leibniz algebras L such that $L/I \cong \mathfrak{D}_m(\mathbb{R})$ and the $\mathfrak{D}_m(\mathbb{R})$-module I is a faithful representation. Moreover, this representation factorizes through $\mathfrak{sp}(2m+2,\mathbb{R})$.

Proposition 3.5.1. *Let $\mathfrak{D}_m(\mathbb{R})$ be a $(2m+2)$-dimensional general real Diamond Lie algebra with the basis $\{J, P_1, P_2, \dots, P_m, Q_1, Q_2, \dots, Q_m, T\}$. Then it is isomorphic to a subalgebra of $\mathfrak{sp}(2m+2,\mathbb{R})$ via the map*

$$\psi : aJ + \sum_{k=1}^{m} b_k P_k + \sum_{k=1}^{m} c_k Q_k + dT \longmapsto$$
$$\left(\begin{array}{ccccc|ccccc}
0 & b_1 & b_2 & \dots & b_m & c_m & \dots & c_2 & c_1 & 2d \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -a & c_1 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & -a & 0 & c_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & -a & \dots & 0 & 0 & c_m \\
\hline
0 & 0 & 0 & \dots & a & 0 & \dots & 0 & 0 & -b_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & a & \dots & 0 & 0 & \dots & 0 & 0 & -b_2 \\
0 & a & 0 & \dots & 0 & 0 & \dots & 0 & 0 & -b_1 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0
\end{array}\right).$$

Proof. Consider the linear map $\psi\colon \mathfrak{D}_m \to \mathfrak{sp}(2m+2,\mathbb{R})$ given by

$$\psi(J) = -\sum_{s=2}^{m+1} e_{k,2m+3-k} + \sum_{s=m+2}^{2m+1} e_{k,2m+3-k},$$
$$\psi(T) = 2e_{1,2m+2}, \qquad \psi(P_k) = e_{1,1+k} - e_{2m+2-k,2m+2},$$
$$\psi(Q_k) = e_{1,2m+2-k} + e_{k+1,2m+2}, \qquad 1 \leq k \leq m.$$

By checking $[\psi(x), \psi(y)] = \psi(x)\psi(y) - \psi(y)\psi(x)$ for all $x, y \in \mathfrak{D}_m$, we verify that ψ is an injective homomorphism of algebras. □

Let us denote by $V = \mathbb{R}^{2m+2}$ the natural $\psi(\mathfrak{D}_m)$-module and endow it with a $\mathfrak{D}_m$-module structure, $V \times \mathfrak{D}_m \to V$, given by

$$(x, e) := x\psi(e),$$

where $x \in V$ and $e \in \mathfrak{D}_m$.

Then the action of $\mathfrak{D}_m$ on $V = \langle X_1, X_2, \ldots, X_{2m+2} \rangle$ is given below:

$$\begin{cases} (X_k, J) = -X_{2m+3-k}, & 2 \le k \le m+1, \\ (X_k, J) = X_{2m+3-k}, & m+2 \le k \le 2m+1, \\ (X_1, P_k) = X_{k+1}, & 1 \le k \le m, \\ (X_{2m+2-k}, P_k) = -X_{2m+2}, & 1 \le k \le m, \\ (X_1, Q_k) = X_{2m+2-k}, & 1 \le k \le m, \\ (X_{k+1}, Q_k) = X_{2m+2}, & 1 \le k \le m, \\ (X_1, T) = 2X_{2m+2}, & 1 \le k \le m, \end{cases} \tag{3.5.1}$$

and the remaining products in the action being zero.

Theorem 3.5.2. *An arbitrary real Leibniz algebra with corresponding Lie algebra $\mathfrak{D}_m$, and the I ideal associated as $\mathfrak{D}_m$-module defined by* (3.5.1)*, admits a basis $\{J, P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m, T, X_1, X_2, \ldots, X_{2m+2}\}$ such that the multiplication table $[\mathfrak{D}_m, \mathfrak{D}_m]$ has the following form:*

$$\begin{cases} [J, J] = a_1 X_{2m+2}, & [J, P_k] = -[P_k, J] = Q_k, \\ [J, Q_k] = -[Q_k, J] = -P_k, & [P_k, Q_k] = -[Q_k, P_k] = T, \\ [P_k, P_s] = [Q_k, Q_s] = b_{k,s} X_{2m+2}, & [P_k, Q_s] = [Q_k, P_s] = c_{k,s} X_{2m+2}, \end{cases}$$

with the restrictions

$$b_{k,s} = -b_{s,k}, \qquad c_{k,s} = c_{s,k}, \quad 1 \le k, s \le m, \ k \ne s.$$

Proof. Let us assume that

$$[J, J] = \sum_{k=1}^{2m+2} \delta_k X_k, \qquad [J, T] = \sum_{k=1}^{2m+2} \rho_k X_k$$

Then by setting

$$J' = J - \frac{1}{2}\rho_{2m+2}X_1 + \sum_{k=2}^{m+1} \delta_{2m+3-k}X_k - \sum_{k=m+2}^{2m+1} \delta_{2m+3-k}X_k,$$

and considering the Leibniz identity for the triple $\{J, T, J\}$, we get

$$[J, J] = \delta_{2m+2}X_{2m+2}, \qquad [J, T] = \rho_1 X_1.$$

Let us suppose

$$[J, P_k] = Q_k + \sum_{s=1}^{2m+2} \lambda_{k,s}X_s, \quad [J, Q_k] = -P_k + \sum_{s=1}^{2m+2} \mu_{k,s}X_s, \quad 1 \leq k \leq m.$$

Taking the following basis transformation:

$$J' = J, \qquad P'_k = P_k - \sum_{s=1}^{2m+2} \mu_{k,s}X_s,$$

$$Q'_k = Q_k + \sum_{k=1}^{2m+2} \lambda_{k,s}X_s, \qquad T' = [P'_1, Q'_1], \qquad 1 \leq k \leq m,$$

we can assume that

$$[J, P_k] = Q_k, \qquad [J, Q_k] = -P_k, \qquad [P_1, Q_1] = T, \qquad 1 \leq k \leq m.$$

By applying the Leibniz identity to the triples $\{J, J, P_k\}$, $\{J, J, Q_k\}$ we derive

$$[P_k, J] = -[J, P_k], \qquad [Q_k, J] = -[J, Q_k], \qquad 1 \leq k \leq m.$$

The verification of the Leibniz identity leads to the following restrictions.

Leibniz identity		Constraints	
$\{J, P_k, T\}$	$\Longrightarrow$	$[Q_k, T] = \rho_1 X_{k+1},$	$1 \leq k \leq m,$
$\{J, Q_k, T\}$	$\Longrightarrow$	$[P_k, T] = -\rho_1 X_{2m+2-k},$	$1 \leq k \leq m,$
$\{P_1, T, Q_1\}$	$\Longrightarrow$	$[T, T] = 0,$	

We set

$$\begin{cases} [P_j,Q_j]=T+\sum\limits_{t=1}^{2m+2}\beta_{j,t}X_t, & [Q_k,P_k]=-T+\sum\limits_{t=1}^{2m+2}\gamma_{k,t}X_t,\\ [P_k,P_s]=\sum\limits_{t=1}^{2m+2}\eta_{k,s,t}X_t, & [Q_k,Q_s]=\sum\limits_{t=1}^{2m+2}\theta_{k,s,t}X_t,\\ [P_k,Q_s]=\sum\limits_{t=1}^{2m+2}\nu_{k,s,t}X_t,\ k\neq s, & [Q_k,P_s]=\sum\limits_{t=1}^{2m+2}\xi_{k,s,t}X_t,\ k\neq s, \end{cases}$$

where $2\leq j\leq m,\ 1\leq k,s\leq m$.

From the Leibniz identity for the triples $\{P_k,P_s,T\}$, $\{Q_k,Q_s,T\}$ and $\{J,P_k,Q_k\}$ we obtain

$$\eta_{k,k,1}=\theta_{k,k,1}=\frac{1}{2}\rho_1,\quad \eta_{k,s,1}=\theta_{k,s,1}=0,\quad \theta_{k,k,t}=-\eta_{k,k,t},$$

$$1\leq k,s\leq m,\quad k\neq s,\quad 2\leq t\leq 2m+2.$$

Analogously, from $\{J,P_k,Q_s\},\{J,Q_k,Q_s\}$ and $\{J,P_k,P_s\}$, we get

$$\begin{aligned} &[P_k,P_s]=-[Q_s,Q_k], && [P_k,Q_s]=[P_s,Q_k],\\ &[Q_k,P_s]=[Q_s,P_k], && 1\leq k,s\leq m,\ k\neq s. \end{aligned} \tag{3.5.2}$$

By applying the Leibniz identity to the triples $\{P_1,J,Q_1\}$, $\{P_1,P_1,Q_1\}$ and $\{Q_1,P_1,Q_1\}$, we have

$$\begin{aligned} [T,J]&=\sum_{s=2}^{2m+2}2\eta_{1,1,s}X_s,\\ [T,P_1]&=\frac{3}{2}\rho_1X_{2m+1}+\eta_{1,1,2}X_{2m+2},\\ [T,Q_1]&=-\frac{3}{2}\rho_1X_2. \end{aligned}$$

By the next identities $\{Q_1,J,P_1\}$ and $\{P_1,J,P_1\}$ we deduce

$$\gamma_{1,s}=-4\eta_{1,1,2m+3-s},\quad \gamma_{1,k}=4\eta_{1,1,2m+3-k},$$

$$\eta_{1,1,2m+2}=\gamma_{1,1}=\gamma_{1,2m+2}=\eta_{1,1,t}=0,$$

with $2\leq s\leq m+1,\ \ m+2\leq k\leq 2m+1,\ \ 2\leq t\leq 2m+1$.

Hence, we have

$$\begin{cases} [T,J]=0, & [Q_1,P_1]=-T,\\ [P_1,P_1]=\frac{1}{2}\rho_1X_1, & [Q_1,Q_1]=\frac{1}{2}\rho_1X_1,\\ [T,P_1]=\frac{3}{2}\rho_1X_{2m+1}, & [T,Q_1]=-\frac{3}{2}\rho_1X_2,\end{cases}$$

By using the next Leibniz identity for $\{P_k,P_k,Q_k\}$, $\{Q_k,P_k,Q_k\}$ we get

$$[T,P_k]=\frac{3}{2}\rho_1X_{2m+2-k}-\beta_{k,1}X_{k+1}+(\beta_{k,2m+2-k}+\eta_{k,k,k+1})X_{2m+2},\quad 2\le k\le m.$$

$$[T,Q_k]=-\frac{3}{2}\rho_1X_{k+1}-\beta_{k,1}X_{2m+2-k}$$

By applying the Leibniz identity to the triples of elements $\{P_k,J,Q_k\}$ and $\{Q_k,J,P_k\}$, we get

$$\beta_{k,s}=\gamma_{k,s}=-2\eta_{k,k,2m+3-s},\qquad \beta_{k,t}=\gamma_{k,t}=2\eta_{k,k,2m+3-t},\qquad \eta_{k,k,2m+2}=0,$$

where $2\le k\le m,\quad 2\le s\le m+1,\quad m+2\le t\le 2m+1.$

By the next Leibniz identity applied to $\{P_k,J,P_k\}$ and $\{T,P_k,Q_k\}$, we have

$$\gamma_{k,1}=\beta_{k,1}=0,\qquad \eta_{k,k,s}=0,\qquad 2\le k\le m,\quad 2\le s\le 2m+1.$$

So, we have

$$\begin{cases} [P_k,Q_k]=-[Q_k,P_k]=T,\\ [P_k,P_k]=[Q_k,Q_k]=\frac{1}{2}\rho_1X_1,\\ [T,P_k]=\frac{3}{2}\rho_1X_{2m+2-k},\\ [T,Q_k]=-\frac{3}{2}\rho_1X_{k+1},\end{cases}$$

where $2\le k\le m$.

By verifying the Leibniz identity on elements, we obtain the following restrictions.

Leibniz identity		Constraints	
$\{P_k,P_s,T\}$	$\Longrightarrow$	$\eta_{k,s,1}=0,$	$1\le k,s\le m,\ k\ne s,$
$\{Q_k,Q_s,T\}$	$\Longrightarrow$	$\theta_{k,s,1}=0,$	$1\le k,s\le m,\ k\ne s,$
$\{P_k,Q_s,T\}$	$\Longrightarrow$	$\nu_{k,s,1}=0,$	$1\le k,s\le m,\ k\ne s,$
$\{Q_k,P_s,T\}$	$\Longrightarrow$	$\xi_{k,s,1}=0,$	$1\le k,s\le m,\ k\ne s.$

By applying the Leibniz identity to the triples $\{P_k, P_s, J\}$, $\{Q_k, Q_s, J\}$, we get

$$[[P_k, P_s], J] = -[Q_k, P_s] - [P_k, Q_s], \qquad [[Q_k, Q_s], J] = [Q_k, P_s] + [P_k, Q_s],$$

it follows that

$$[[P_k, P_s], J] = -[[Q_k, Q_s], J],$$

hence

$$\begin{array}{ll} \xi_{k,s,2m+2} = -\nu_{k,s,2m+2}, & \\ \theta_{k,s,t} = -\eta_{k,s,t}, & 1 \leq k, s \leq m, \quad 2 \leq t \leq 2m+1, \quad k \neq s. \end{array}$$

and

$$\begin{array}{ll} \nu_{k,s,t} + \xi_{k,s,t} = -\eta_{k,s,2m+3-t} & 2 \leq t \leq m+1, \\ \nu_{k,s,t} + \xi_{k,s,t} = \eta_{k,s,2m+3-t} & m+2 \leq t \leq 2m+1. \end{array}$$

Let us consider the identity

$$[[Q_k, P_s], J] = [Q_k, [P_s, J]] + [[Q_k, J], P_s] = -[Q_k, Q_s] + [P_k, P_s]$$

We have that $\theta_{k,s,2m+2} = \eta_{k,s,2m+2}$ and

$$\begin{array}{ll} \xi_{k,s,t} = -2\eta_{k,s,2m+3-t}, & 2 \leq t \leq m+1, \\ \xi_{k,s,t} = 2\eta_{k,s,2m+3-t}, & m+2 \leq t \leq m+1, \end{array}$$

and

$$\begin{array}{ll} \nu_{k,s,t} = \eta_{k,s,2m+3-t}, & 2 \leq t \leq m+1, \\ \nu_{k,s,t} = -\eta_{k,s,2m+3-t}, & m+2 \leq t \leq 2m+1. \end{array}$$

Analogously, by applying the Leibniz identity to the triple $\{P_k, Q_s, J\}$, we get

$$\begin{array}{ll} \nu_{k,s,t} = -2\eta_{k,s,2m+3-t}, & 2 \leq t \leq m+1, \\ \nu_{k,s,t} = 2\eta_{k,s,2m+3-t}, & m+2 \leq t \leq 2m+1. \end{array}$$

We get that $\nu_{k,s,t} = 0, \ 1 \leq k, s \leq m, \ 2 \leq t \leq 2m+1, \ k \neq s$. It implies that $\eta_{k,s,t} = \xi_{k,s,t} = \theta_{k,s,t} = 0$ for $1 \leq k, s \leq m, \ 2 \leq t \leq 2m+1, \ k \neq s$.

Hence, we have

$$\begin{array}{ll} [P_k, P_s] = [Q_k, Q_s] = \eta_{k,s,2m+2} X_{2m+2}, & 1 \leq k, s \leq m, \quad k \neq s, \\ [P_k, Q_s] = [Q_k, P_s] = \nu_{k,s,2m+2} X_{2m+2}, & 1 \leq k, s \leq m, \quad k \neq s. \end{array}$$

By equation (3.5.2) we have the following restrictions

$$\eta_{k,s,2m+2} = -\eta_{s,k,2m+2}, \qquad \nu_{k,s,2m+2} = \nu_{s,k,2m+2}, \qquad 1 \leq k, s \leq m,\ k \neq s.$$

Finally, we apply the Leibniz identity to the triple $\{P_k, P_k, P_s\}$, with $k \neq s$, and we obtain $\rho_1 = 0$. We denote again $(\delta_{2m+2}, \eta_{k,s,2m+2}, \nu_{k,s,2m+2}) = (a_1, b_{k,s}, c_{k,s})$. $\square$

Bibliography

[1] Abdykassymova S., Dzhumadil'daev A.S. Simple Leibniz algebras of rank 1. The Abstract Presented to the IX Internal Conference of the representation Theory of Algebras. Beijing, 2000 China, p. 17–18.

[2] Albeverio S.A., Ayupov S.A., Omirov B.A. Cartan subalgebras, weight spaces, and criterion of solvability of finite-dimensional Leibniz algebras. Rev. Mat. Complut. **19**, 2006, p. 183–195.

[3] Ancochea Bermúdez J.M., Campoamor-Stursberg R., García Vergnolle L. Indecomposable Lie algebras with nontrivial Levi decomposition cannot have filiform radical. Int. Math. Forum **1**(5-8), 2006, p. 309–316.

[4] Ancochea Bermúdez J.M., Campoamor-Stursberg R., García Vergnolle L. Solvable Lie algebras with naturally graded nilradicals and their invariants. J. Phys. A **39**(6), 2006, p. 1339–1355.

[5] Ancochea Bermúdez J.M., Campoamor-Stursberg R., García Vergnolle L. Classification of Lie algebras with naturally graded quasi-filiform nilradicals. J. Geom. Phys. **61**(11), 2011, p. 2168–2186.

[6] Ayupov Sh.A., Camacho L.M., Khudoyberdiyev A.Kh., Omirov B.A. Leibniz algebras associated with representations of filiform Lie algebras. J. Geom. Phys. **98**, 2015, p. 181–195.

[7] Ayupov Sh.A., Omirov B.A. On Leibniz algebras. Algebra and Operator Theory (Tashkent, 1997), Kluwer Academic Publishers, Dordrecht, 1998, p. 1–12.

[8] Ayupov Sh.A., Omirov B.A. On 3-dimensional Leibniz algebras. Uzbek Math. J. **1**, 1999, p. 9–14.

[9] Ayupov Sh.A., Omirov B.A. On a description of component in the set of nilpotent Leibniz algebras containing the algebra of maximal nilindex, and classification of graded filiform Leibniz algebras. Computer Algebra in Scientific Computing CASC. Springer, 2000, p. 146–155.

[10] Ayupov Sh.A., Omirov B.A. On some classes of nilpotent Leibniz algebras. Sib. Math. J. **42**(1), 2001, p. 15–24.

[11] Barnes D.W. On Levi's theorem for Leibniz algebras. Bull. Aust. Math. Soc. **86**(2), 2012, p. 184–185.

[12] Bianchi L. Lezioni sulla teoria dei gruppi continui finite di trasformazioni. Enrico Spoerri Editore, Pisa, 1918, p. 550–557.

[13] Bloh A. A generalization of the concept of a Lie algebra. Sov. Math. Dokl. **165**(3), 1965, p. 1450–1452.

[14] Bosko-Dunbar L., Dunbar J.D., Hird J.T., Stagg K. Solvable Leibniz Algebras with Heisenberg nilradical. Comm. Algebra **43**(6), 2015, p. 2272–2281.

[15] Bourbaki N. Lie groups and Lie algebras. Chapters 1–3. (English summary), Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998.

[16] Boyko V., Patera J., Popovych R. Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements. Linear Algebra Appl. **428**, 2008, p. 834–854.

[17] Burde D. On a refinement of Ado's Theorem. Arch. Math. (Basel) **70**, 1998, p. 118–127.

[18] Calderón A.J., Camacho L.M., Omirov B.A. Leibniz algebras of Heisenberg type. J. Algebra **452**, 2016, p. 427–447.

[19] Camacho L.M., Cañete E.M., Gómez J.R., Omirov B.A. 3-filiform Leibniz algebras of maximum length, whose naturally graded algebras are Lie algebras. Linear and Multilinear Algebra **59**, 2011, p. 1039–1058.

[20] Camacho L.M., Cañete E.M., Gómez J.R., Omirov B.A. Quasi-filiform Leibniz algebras of maximum length. Sib. Math. J. **52**(5), 2011, p. 840–853.

[21] Camacho L.M., Gómez J.R., González A.R., Omirov B.A. Naturally graded quasi-filiform Leibniz algebras. J. Symbolic. Comput. **44**, 2009, p. 527–539.

[22] Camacho L.M., Gómez J.R., Navarro R.M., Omirov B.A. Classification of some nilpotent class of Leibniz superalgebras. Acta Math. Sinica (english series) **26**(5), 2009, p. 799–816.

[23] Camacho L.M., Gómez J.R., Omirov B.A. On the filiform Leibniz algebras of maximal length. ICM. Madrid. 2006, p. 63–64.

[24] Campoamor-Stursberg R. Solvable Lie algebras with an $\mathbb{N}$-graded nilradical of maximal nilpotency degree and their invariants. J. Phys. A **43**(18), 2010, p. 195–202.

[25] Cartan E. Sur la structure des groupes de transformations finis et continus. Thèse, Nony, Paris, and Oeuvres Completes, Partie I. Tome 1, 1894.

[26] Casati P., Minniti S., Salari V. Indecomposable representations of the Diamond Lie algebra. J. Math. Phys. **51**, 2010, 033515, 20 pp.

[27] Conti D., Fernández M., Santisteban J.A. Solvable Lie algebras are not that hypo. Transform. Groups **16**(1), 2011, p. 51–69.

[28] Dozias J. Sur les algèbres de Lie résolubles, réelles, de dimension inférieure ou égale à 5. Thèse de 3e Cycle, Falculté des Sciences de Paris, 1963.

[29] Fulton W., Harris J. Representation Theory. A First Course. Springer-Verlag, New York, 1991.

[30] Gómez-Vidal S., Omirov B.A., Khudoyberdiyev A.Kh. Some remarks on semisimple Leibniz algebras. J. Algebra **410**, 2014, p. 526–540.

[31] Graaf W.A. Classification of solvable Lie algebras. Experimental Mathematics **14**(1), 2005, p. 15–25.

[32] Jacobson N. Lie algebras. Interscience Tracts in Pure and Applied Mathematics, no. **10**, Wiley, New York-London, 1962.

[33] Ladra M., Masutova K.K., Omirov B.A. Solvable Leibniz algebra with non-Lie and non-split naturally graded filiform nilradical and its rigidity. Linear Algebra Appl. **507**, 2016, p. 513–517.

[34] Levi E.E. Sulla struttura dei gruppi finiti e continui. Atti. Accad. Sci. Torino **40**, 1905, p. 551–565.

[35] Lie S., Engel F. Theorie der Transformationsgruppen. vol. **3**, B.G. Teubner, Leipzig, 1893, p. 713–732.

[36] Lie S., Scheffers G. Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen. B.G. Teubner, Leipzig, 1893, p. 550–610.

[37] Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. Enseign. Math. (2) **39**(3-4), 1993, p. 269–293.

[38] Loday J.-L., Frabetti A., Goichot F. Dialgebras and related operads. Lecture Notes in Math., vol. **1763**. Springer, Berlin, 2001.

[39] Loday J.-L., Pirashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology. Math. Ann. **296**(1), 1993, p. 139–158.

[40] Loday J.-L., Pirashvili T. Leibniz representations of Leibniz algebras. J. Algebra **181**, 1996, p. 414–425.

[41] Malcev A.I. On solvable Lie algebras (in Russian). Izvestiya Akademii Nauk SSSR. Seriya matematicheskaya **9**, 1945, p. 329–356.

[42] Mubarakzjanov G.M. On solvable Lie algebras (in Russian). Izv. Vysš. Učehn. Zaved. Matematika **1963**(1), 1963, p. 114–123.

[43] Mubarakzjanov G.M. Classification of real structures of Lie algebras of fifth order (in Russian). Izv. Vysš. Učehn. Zaved. Matematika **1963**(3), 1963, p. 99–106.

[44] Mubarakzjanov G.M. Classification of solvable Lie algebras of sixth order with a non-nilpotent basis elements (in Russian). Izv. Vysš. Učehn. Zaved. Matematika **1963**(4), 1963, p. 104–116.

[45] Ndogmo J.C., Winternitz P. Solvable Lie algebras with abelian nilradicals. J. Phys. A: Math. Gen. **27**, 1994, p. 405–423.

[46] Omirov B.A. Conjugacy of Cartan subalgebras of complex finite dimensional Leibniz algebras. J. Algebra **302**, 2006, p. 887–896.

[47] Omirov B.A., Rakhimov I.S. On Lie-like complex filiform Leibniz algebras. Bull. Aust. Math. Soc. **79**(3), 2009, p. 391–404.

[48] Patera J., Sharp R.T., Winternitz P., Zassenhaus H. Invariants of real low dimensional Lie algebras. J. Math. Phys. **17**, 1976, p. 986–994.

[49] Rakhimov I.S., Omirov B.A., Turdibaev R.M. On description of Leibniz algebras corresponding to sl_2. Algebr. Represent. Theory **16**(5), 2013, p. 1507–1519.

[50] Rubin J., Winternitz P. Solvable Lie algebras with Heisenberg ideals. J. Phys. A: Math. Gen. **26**(5), 1993, p. 1123–1138.

[51] Šnobl L., Karásek D. Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras. Linear Algebra Appl. **432**(7), 2010, p. 1836–1850.

[52] Šnobl L., Winternitz P. A class of solvable Lie algebras and their Casimir invariants. J. Phys. A **38**(12), 2005, p. 2687–2700.

[53] Šnobl L., Winternitz P. All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency $n-1$. J. Phys. A: Math. Theor. **42**(10), 2009, p. 105–201.

[54] Šnobl L., Winternitz P. Solvable Lie algebras with Borel nilradicals. J. Phys. A **45**(9), 2012, 095202, 18 pp.

[55] Tremblay S., Winternitz P. Solvable Lie algebras with triangular nilradicals. J. Phys. A: Math. Gen. **31**(2), 1998, p. 789–806.

[56] Vergne M. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes. Bull. Soc. Math. France **98**, 1970, p. 81–116.

[57] Wang Y., Lin J., Deng S. Solvable Lie algebras with quasifiliform nilradicals. Comm. Algebra **36**(11), 2008, p. 4052–4067.