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Conformally Einstein Lorentzian Lie Groups: The Non-solvable Case

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Abstract

We describe all left-invariant Bach-flat Lorentz metrics on non-solvable four-dimensional Lie groups, showing that they are conformally Einstein if and only if they are locally conformally flat. As a consequence we show the existence of strictly Bach-flat left-invariant metrics on $SU(2) \times \mathbb{R}$ whose restriction to $SU(2)$ may be positive definite, Lorentzian or degenerate.

Keywords Conformally Einstein · Bach tensor · C-space · Lie group

Mathematics Subject Classification 53C50 · 83D05 · 53C25 · 53C18

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1 Introduction

Einstein metrics are central in Geometry and Physics, since they are critical for the Einstein–Hilbert functional. The Einstein condition amounts to a generically overdetermined PDE system which is not yet well understood. Among the different simplifications usually considered in the construction of Einstein metrics, focusing on left-invariant metrics on Lie groups provides a substantial simplification of the analysis, since the Einstein field equations reduce to a purely algebraic system which is nevertheless rather involved. Another approach, based on the possibility of conformally deforming a metric to obtain an Einstein one (see Brinkmann 1924), is the main concern of our work.

A pseudo-Riemannian manifold (M, g) is *conformally Einstein* if there is an Einstein representative of the conformal class $[g]$. The quadratic curvature functional given by the L^2-norm of the Weyl tensor, $g \mapsto \int_M \|W\|^2 \,\mathrm{dvol}_g$, is conformally invariant in dimension four and thus so is its gradient. Critical metrics of the functional above have vanishing *Bach tensor*

$$\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \tfrac{1}{2} W[\rho], \tag{1}$$

where ρ denotes the Ricci tensor, $W[\rho]_{ij} = W_{iajb}\rho^{ab}$, and div_k denotes the divergence operator acting on the k^{th}-argument of the corresponding tensor field (see Bach 1921). Locally conformally flat metrics are trivially Bach-flat. Moreover since the Weyl tensor is trace-free and Einstein metrics have divergence-free Weyl tensor, it immediately follows from (1) that Einstein metrics are Bach-flat. Due to the conformal invariance, so are conformally Einstein metrics in dimension four. On the other hand, if a conformal metric $\overline{g} = e^{-2\sigma} g$ is Einstein, then $\overline{\operatorname{div}}\, \overline{W} = 0$, and a straightforward calculation shows that $\operatorname{div}_4 W - W(\cdot, \cdot, \cdot, \nabla\sigma) = 0$. More generally (M, g) is a *conformal C-space* if there is a vector field X such that (see Gover and Nagy 2007)

$$\operatorname{div}_4 W - W(\cdot, \cdot, \cdot, X) = 0. \tag{2}$$

It now follows that the vanishing of (1) and (2) are necessary conditions to be conformally Einstein in dimension four.

As a matter of notation we say that a Bach-flat metric is *strict* if it is not conformally Einstein. Moreover, a conformally Einstein metric is *non-trivial* if it is neither Einstein, nor locally conformally flat, nor a plane wave.

Four-dimensional Lie algebras are semi-direct extensions of low-dimensional semi-simple Lie algebras and solvable Lie algebras, since any real Lie algebra is a semi-direct product of the radical and a semi-simple subalgebra (Knapp 2002). Hence connected and simply connected four-dimensional Lie groups are isomorphic to the product Lie groups $SU(2)\times\mathbb{R}$, $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$, or to one of the solvable semi-direct extensions of the three-dimensional unimodular Lie groups $\widetilde{E}(2)$, $E(1,1)$, H_3 and $\mathbb{R}^3$ corresponding to the simply connected Euclidean, Poincaré, Heisenberg and Abelian Lie groups, respectively. As a step toward the description of conformally Einstein four-dimensional Lorentzian Lie groups (see Anderson and Torre 2020), in this paper we prove the following

Theorem 1.1 *A four-dimensional non-solvable Lie group equipped with a left-invariant Lorentzian metric is conformally Einstein if and only if it is locally conformally flat.*

Hence, non-trivial conformally Einstein Lorentzian Lie groups must be solvable, and we refer to Calviño-Louzao et al. (2024a, b, 2025) for a description of such metrics in solvable Lie groups. Clearly the non-solvable Lie group $SU(2) \times \mathbb{R}$ admits a left-invariant locally conformally flat metric such that it is locally isometric to the product $\mathbb{S}^3 \times \mathbb{R}$. In addition, there is also a family of non-symmetric left-invariant Lorentz metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ which are locally conformally flat and whose restriction to $\widetilde{SL}(2, \mathbb{R})$ is degenerate (cf. Remark 4.1). We refer to Calvaruso and Zaeim (2014) for more information on locally conformally flat homogeneous four-manifolds. Also, note that any Einstein Lorentz Lie group must be solvable (Calvaruso and Zaeim 2013; Otero-Casal 2022).

Since the L^2-norm of the Weyl tensor is the action underlying the theories of conformal gravity, Bach-flat metrics constitute the background metrics in the theory (Mannheim 2006, 2012). Up to the best of our knowledge, the first strictly Bach-flat Lorentzian metrics were described by H.-J. Schmidt in Schmidt (1984). Strictly Bach-flat metrics are rather exceptional in the Riemannian setting. Indeed, in the non-symmetric homogeneous setting they may only occur as left-invariant metrics on solvable Lie groups (Abbena et al. 2013; Calviño-Louzao et al. 2019; LeBrun and Nurowski 2019). In sharp contrast we show that they are quite abundant in the Lorentzian framework (see also Demaret et al. 1999; Liu et al. 2013 for some non-necessarily homogeneous examples).

We give a complete description of all such left-invariant metrics on $SU(2) \times \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$. A remarkable fact is that all Bach-flat non-solvable four-dimensional Lorentzian Lie groups have Petrov type III (the Weyl curvature operator acting on the space of two-forms, $\mathcal{W} : \Lambda^2 \to \Lambda^2$, is three-step nilpotent), unless they are locally conformally flat. Moreover, we emphasize the existence of one-parameter families of Bach-flat left-invariant metrics both on $SU(2) \times \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$.

The components of the Bach tensor are rather involved polynomials in the structure constants. For each case we show that Bach-flatness is determined by three polynomial equations $\mathfrak{P}_1 = 0$, $\mathfrak{P}_2 = 0$, and $\mathfrak{P}_3 = 0$ on the structure constants, whose roots can be explicitly given in most cases. A key observation to obtain such characterization is that one may express the Bach tensor in terms of the polynomials $\mathfrak{P}_i$, $i = 1, 2, 3$.

Since the solutions of a system of polynomial equations depend on the ideal generated by the polynomials, we approach the problem of solving the polynomial equations given by the components of the Bach tensor by trying to find “simpler polynomials" in the corresponding ideal. Hence we use the theory of Gröbner bases (see Cox et al. 2015) which provides a useful computational technique to obtain such simpler polynomials. As a technical fact, since Gröbner bases are very sensitive with respect to the monomial orderings under consideration, we specify the variables in each calculation, where we always use the lexicographical order with respect to such variables except in one case which will be clearly indicated. A basic observation from the computational point of view is that the polynomial equations determining the components of the Bach tensor can be thought of as polynomials on the structure constants, or as polynomials on the structure constants together with the polynomials $\mathfrak{P}_i$ introduced above (i.e., treating the $\mathfrak{P}_i$'s as variables).

It is worth emphasizing here that although the Lie groups under consideration are products $G \times \mathbb{R}$ with $G = SU(2)$ or $G = \widetilde{SL}(2, \mathbb{R})$, the Lorentzian left-invariant metrics are not the product ones and there are essentially three different cases to be considered since the restriction of the metric to G may be positive definite, or degenerate (Calvaruso and Castrillón 2016). In each case, we first characterize the left-invariant Lorentzian Bach-flat metrics and then show that none of them is a conformal C-space, unless it is locally conformally flat, from which the proof of Theorem 1.1 follows.

Section 2 covers the case when the Lorentzian metric restricts to a positive-definite inner product on $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. For these left-invariant metrics, given by (3), the Bach-flat property is characterized in Theorem 2.2, from where it follows that they are realized on $SU(2) \times \mathbb{R}$. We show that they are conformal C-spaces if and only if they are locally conformally flat, thus proving the corresponding part of Theorem 1.1. The case when the Lorentzian metric restricts to a Lorentzian inner product on $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$ is more involved since the structure operator of $\mathfrak{g}_3$ may have non-trivial Jordan normal form, as discussed at the beginning of Sect. 3. The diagonalizable case, considered in Sect. 3.1, is quite parallel to Sect. 2 and we skip some of the details already covered in the previous discussion. These Bach-flat metrics are realized on $SU(2) \times \mathbb{R}$ or $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$. The cases when the structure operator has a complex root or a double root of the minimal polynomial are considered in the subsequent Sects. 3.2, 3.3, providing families of Bach-flat metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ which are not C-spaces. In Sect. 3.4 we show the non-existence of Bach-flat metrics when the structure operator has a triple root of its minimal polynomial. Finally the case when the Lorentzian metric is degenerate on $\mathfrak{g}_3$ is considered in Sect. 4 where we describe all such left-invariant Bach-flat metrics, which are realized on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$.

The computation of the Gröbner bases in this paper was performed with the computer algebra software MATHEMATICA 12.0. We provide a complementary file with the calculations of the Gröbner bases, which also contains the expression of the Bach tensor of each of the left-invariant metrics in terms of three polynomials $\mathfrak{P}_1$, $\mathfrak{P}_2$, and $\mathfrak{P}_3$ that will determine the Bach-flatness condition. This complementary file is available at `https://zenodo.org/records/13881729`

2 Direct Extensions with Riemannian Lie Groups $\widetilde{SL}(2, \mathbb{R})$ or $SU(2)$

Let $\langle \cdot, \cdot \rangle$ be a Lorentzian inner product on $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$ whose restriction to the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$ is of Riemannian signature. In this case the structure operator of $\mathfrak{g}_3$, defined in Milnor (1976), is self-adjoint and diagonalizable. As a consequence, there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, with e_4 timelike, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, such that the Lie algebra is given by

$$\mathfrak{g}_R \begin{cases} [e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \\ [e_1, e_4] = \gamma_1 \lambda_2 e_2 + \gamma_2 \lambda_3 e_3, \quad [e_2, e_4] = -\gamma_1 \lambda_1 e_1 + \gamma_3 \lambda_3 e_3, \\ [e_3, e_4] = -\gamma_2 \lambda_1 e_1 - \gamma_3 \lambda_2 e_2, \end{cases} \tag{3}$$

for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, where $\lambda_1\lambda_2\lambda_3 \neq 0$ determine the Lie algebra $\mathfrak{su}(2)$ if they have the same sign and $\mathfrak{sl}(2, \mathbb{R})$ otherwise.

Remark 2.1 A straightforward calculation shows that $\mathfrak{g}_R$ is never Einstein. Moreover, it is locally conformally flat if and only if $\lambda_1 = \lambda_2 = \lambda_3$, and the same holds for being locally symmetric. Note that any homogeneous locally conformally flat structure with diagonalizable Ricci operator is locally symmetric (Calvaruso and Zaeim 2014).

The characterization of the Bach-flatness of $\mathfrak{g}_R$ is then given as follows.

Theorem 2.2 *A left-invariant metric $\mathfrak{g}_R$ is Bach-flat if and only if it is either locally conformally flat (whenever $\lambda_1 = \lambda_2 = \lambda_3$, in which case the Ricci operator is diagonalizable) or, otherwise, it is isomorphically homothetic to a metric determined by Equation (3), where $\lambda_1 = 1$ and the remaining parameters satisfy the equations $\{\mathfrak{P}_i = 0\}$, for the polynomials*

$$\begin{aligned} \mathfrak{P}_1 &= \gamma_1^2 + \delta\gamma_2^2 + \delta\gamma_3^2 - 1, \\ \mathfrak{P}_2 &= (\lambda_2 - \lambda_3)(2\lambda_2 + 2\lambda_3 - 1)\gamma_2^2 + (\lambda_3 - 1)(\lambda_2 - 2\lambda_3 - 2)\gamma_3^2 \\ &\quad - \delta\lambda_3(\lambda_2 - 2\lambda_3 + 1), \\ \mathfrak{P}_3 &= 3(\lambda_2 - \lambda_3)\gamma_2^2 - 3(\lambda_3 - 1)\big\{2(\lambda_3 + 1)(\lambda_2^2 - 1) + \lambda_2\big\}\gamma_3^2 \\ &\quad + \delta\lambda_3\big\{2(3\lambda_3 - 2)\lambda_2^2 - (4\lambda_3 + 1)\lambda_2 + 3\big\}, \end{aligned} \tag{4}$$

with $\delta = 1$. Moreover, in the latter case, the Bach-flat metric is always strict and it is realized on the product Lie group $SU(2) \times \mathbb{R}$.

We note that the parameter δ, introduced in the previous theorem and used in what follows, has the purpose of facilitating the solution of the case in Sect. 3.1 proceeding exactly as in the present section. In the case at hand $\delta = 1$, while in Sect. 3.1 it takes the value $\delta = -1$.

On the other hand, since $\lambda_1 \neq 0$, we may consider the orthogonal basis $\hat{e}_i = \frac{1}{\lambda_1} e_i$ and assume $\lambda_1 = 1$ from now on, working in the homothetic class of the initial metric.

Remark 2.3 Explicit solutions to Equation (4) can be described as follows. We assume $\lambda_2 \neq 1 \neq \lambda_3$, since otherwise the space is locally conformally flat (see the theorem's proof below). A direct calculation using (4) shows that the parameters γ_1, γ_2, and γ_3

are determined by λ_2 and λ_3 as follows:

$$\begin{aligned}
\gamma_1^2 &= \frac{\lambda_2\big\{\lambda_2\lambda_3(4\lambda_2-\lambda_3+11)-6\lambda_2(\lambda_2+1)-\lambda_3(\lambda_3-4)-5\lambda_3^3\big\}}{3(\lambda_3-1)(\lambda_2-\lambda_3)\big\{2(\lambda_2+\lambda_3)(\lambda_2\lambda_3+1)+2\lambda_2^2+2\lambda_3^2+3\lambda_2\lambda_3\big\}},\\
\gamma_2^2 &= \frac{\delta\lambda_3\big\{\lambda_2\lambda_3(\lambda_2-4\lambda_3-11)+\lambda_2(\lambda_2-4)+6\lambda_3(\lambda_3+1)+5\lambda_2^3\big\}}{3(\lambda_2-1)(\lambda_2-\lambda_3)\big\{2(\lambda_2+\lambda_3)(\lambda_2\lambda_3+1)+2\lambda_2^2+2\lambda_3^2+3\lambda_2\lambda_3\big\}},\\
\gamma_3^2 &= \frac{\delta\lambda_2\lambda_3\big\{\lambda_2\lambda_3(6\lambda_2+6\lambda_3-11)-\lambda_2(4\lambda_2-1)-\lambda_3(4\lambda_3-1)+5\big\}}{3(\lambda_2-1)(\lambda_3-1)\big\{2(\lambda_2+\lambda_3)(\lambda_2\lambda_3+1)+2\lambda_2^2+2\lambda_3^2+3\lambda_2\lambda_3\big\}}.
\end{aligned} \tag{5}$$

Note that the denominators in (5) do not vanish if the metric is strictly Bach-flat. Indeed, since $\lambda_2 \neq 1 \neq \lambda_3$, the denominators vanish at the zeros of the polynomial

$$\mathbf{p} = (\lambda_2-\lambda_3)\big\{2(\lambda_2+\lambda_3)(\lambda_2\lambda_3+1)+2\lambda_2^2+2\lambda_3^2+3\lambda_2\lambda_3\big\}.$$

Computing a Gröbner basis for the ideal $\mathcal{I} = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathbf{p}\}\rangle$ in the polynomial ring $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \lambda_3, \lambda_2]$, one gets 13 polynomials including

$$\mathbf{g}_1 = \lambda_2^3(\lambda_2-1)^2(\lambda_2+1)^2 \quad \text{and} \quad \mathbf{g}_2 = \lambda_2^2\big\{\lambda_2^2(\lambda_2+1)(3\lambda_2-5)+4\lambda_3\big\},$$

which cannot vanish simultaneously.

Since the expressions in Equation (5) must be non-negative, after analyzing the corresponding inequalities it follows that $\lambda_2 > 0$ and also $\lambda_3 > 0$. As a consequence, these metrics are realized on the product Lie group $SU(2)\times\mathbb{R}$. Moreover, setting $\lambda_1 = 1$, for each $\lambda_2 > 0$, $\lambda_2 \neq 1$, there exist infinitely many values of $\lambda_3 \in (0, +\infty) \setminus \{1\}$ which provide strictly Bach-flat metrics determined by (5).

Proof of Theorem 2.2 The proof is organized as follows. First of all we show that all the components of the Bach tensor can be expressed in terms of the polynomials $\mathfrak{P}_i$, $i = 1, 2, 3$ (Step 1 below). Secondly we eliminate the possibilities $\lambda_2 = 1$ or $\lambda_3 = 1$ (Step 2) in order to further investigate the polynomial system $\{\mathfrak{P}_i = 0\}$. Finally we show in Step 3 that the Bach-flat metrics are strict, since they cannot be C-spaces unless they are locally conformally flat ($\lambda_1 = \lambda_2 = \lambda_3$).

It is worth pointing out here that Step 3 strongly depends on the result of Step 1, since we may replace the polynomial system $\{\mathfrak{B}_{ij} = 0\}$ given by the components of the Bach tensor by the much simpler system $\{\mathfrak{P}_i = 0\}$.

Step 1: expression of the Bach tensor in terms of the polynomials $\mathfrak{P}_i$. A key observation for characterizing the Bach-flatness of the metrics $\mathfrak{g}_R$ is that we are able to express the corresponding Bach tensor in terms of the polynomials in Equation (4). In what follows we will work with the components of the Bach tensor both substituting the polynomials $\mathfrak{P}_i$ by their expressions in (4) and by considering those polynomials as variables. To differentiate these two possibilities, we rename each polynomial $\mathfrak{P}_i$

as $\mathfrak{Q}_i$. By a direct calculation one may check that the Bach tensor is determined by

$$\begin{aligned}
&\mathfrak{B}_{11} = \tfrac{1}{144}\widetilde{\mathfrak{B}}_{11}, \quad \mathfrak{B}_{12} = \tfrac{\gamma_2\gamma_3}{24}\widetilde{\mathfrak{B}}_{12},\ \mathfrak{B}_{13} = -\tfrac{\gamma_1\gamma_3}{24}\widetilde{\mathfrak{B}}_{13},\ \mathfrak{B}_{14} = \tfrac{\gamma_3}{24}\widetilde{\mathfrak{B}}_{14},\\
&\mathfrak{B}_{23} = -\tfrac{\gamma_1\gamma_2}{24}\widetilde{\mathfrak{B}}_{23},\ \mathfrak{B}_{24} = -\tfrac{\gamma_2}{24}\widetilde{\mathfrak{B}}_{24},\ \mathfrak{B}_{33} = -\tfrac{1}{144}\widetilde{\mathfrak{B}}_{33},\ \mathfrak{B}_{34} = \tfrac{\gamma_1}{24}\widetilde{\mathfrak{B}}_{34},\\
&\mathfrak{B}_{44} = -\tfrac{1}{48}\widetilde{\mathfrak{B}}_{44}, \quad \mathfrak{B}_{22} = -\mathfrak{B}_{11} - \delta(\mathfrak{B}_{33} - \mathfrak{B}_{44}),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\widetilde{\mathfrak{B}}_{11} &= 24(\lambda_2 - 1)(3\lambda_2^3 + 2\lambda_2^2 + 2\lambda_2 + 5)\boldsymbol{\mathfrak{Q}_1^2} + 18\boldsymbol{\mathfrak{Q}_2^2}\\
&\quad -3\delta(24\lambda_2^2 + 4\lambda_2 - 8\lambda_3 + 1)\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_2} + 25\delta\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_3}\\
&\quad -2\delta\Big\{6(8\gamma_3^2\lambda_2^4 - 4\gamma_3^2\lambda_2^3\lambda_3 - \delta\lambda_2^2\lambda_3^2) - 24(\gamma_2^2 + \gamma_3^2 - \delta)\lambda_3^3 + 4\delta\lambda_2^2\lambda_3\\
&\quad +4(6\gamma_2^2 + 3\gamma_3^2 - 17\delta)\lambda_2\lambda_3^2 + 6(2\gamma_2^2 + 25\gamma_3^2 - 2\delta)\lambda_3^2\\
&\quad -(12\gamma_2^2 + 87\gamma_3^2 + 17\delta)\lambda_2\lambda_3\\
&\quad +3(25\gamma_2^2 + 33\gamma_3^2)\lambda_2 - 3(25\gamma_2^2 - 8\gamma_3^2 - 25\delta)\lambda_3 - 198\gamma_3^2\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad +6\Big\{4\gamma_3^2(2\lambda_2^2 + 2\lambda_3^2 - \lambda_2\lambda_3) + 4(\gamma_2^2 + \gamma_3^2)\lambda_2 - 4(\gamma_2^2 - \delta)\lambda_3 - 25\gamma_3^2\Big\}\boldsymbol{\mathfrak{Q}_2}\\
&\quad -18\gamma_3^2\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{12} &= 2\delta(\lambda_3^2 - \lambda_2)(8\lambda_2^2 + 5\lambda_2 + 8)\boldsymbol{\mathfrak{Q}_1} - (8\lambda_3^2 - 8\lambda_2 - 9)\boldsymbol{\mathfrak{Q}_2} + 3\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{13} &= 2\delta(8\lambda_2^4 - 4\lambda_2^3 + \lambda_2^2\lambda_3 - \lambda_2^2 + 4\lambda_2\lambda_3 - 8\lambda_3)\boldsymbol{\mathfrak{Q}_1} - (8\lambda_2^2 - 8\lambda_3 - 9)\boldsymbol{\mathfrak{Q}_2} + 3\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{14} &= 2\delta(8\lambda_2^4 - 4\lambda_2^3\lambda_3 - \lambda_2^2\lambda_3^2 - 4\lambda_2^3 + 2\lambda_2^2\lambda_3 - 4\lambda_2\lambda_3^2 - \lambda_2^2 + 8\lambda_3^2 - 4\lambda_2\lambda_3)\boldsymbol{\mathfrak{Q}_1}\\
&\quad -(8\lambda_2^2 + 8\lambda_3^2 - 4\lambda_2\lambda_3 - 9)\boldsymbol{\mathfrak{Q}_2} + 3\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{23} &= 2\delta(8\lambda_2^3\lambda_3 - 4\lambda_2^2\lambda_3 + \lambda_2^2 - \lambda_2\lambda_3 + 4\lambda_2 - 8)\boldsymbol{\mathfrak{Q}_1} - (8\lambda_2\lambda_3 + 1)\boldsymbol{\mathfrak{Q}_2} - 3\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{24} &= 2\delta(8\lambda_2^2\lambda_3^2 - 4\lambda_2^2\lambda_3 - 4\lambda_2\lambda_3^2 - \lambda_2^2 - \lambda_3^2 + 2\lambda_2\lambda_3 - 4\lambda_2 - 4\lambda_3 + 8)\boldsymbol{\mathfrak{Q}_1}\\
&\quad -(8\lambda_3^2 - 4\lambda_3 - 1)\boldsymbol{\mathfrak{Q}_2} + 3\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{33} &= 24\delta(\lambda_2 - 1)^2(\lambda_2^2 + \lambda_2 + 1)\boldsymbol{\mathfrak{Q}_1^2} + 6\delta\boldsymbol{\mathfrak{Q}_2^2}\\
&\quad +3(16\lambda_3^2 - 8\lambda_2^2 + 4\lambda_2 + 3)\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_2} + 11\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_3}\\
&\quad -8\Big\{12(\gamma_3^2 - \delta)\lambda_2^4 - 12(\gamma_2^2 + \gamma_3^2 - \delta)\lambda_3^4 + 6(\gamma_3^2 - \delta)\lambda_2\lambda_3^3 - 6(\gamma_3^2 - 2\delta)\lambda_2^3\\
&\quad +6(\gamma_2^2 - \delta)\lambda_3^3 - \delta\lambda_2^2\lambda_3 + 2\delta\lambda_2\lambda_3^2 - \delta\lambda_2\lambda_3 - 6(\gamma_2^2 - 2\delta)\lambda_2 + 12(\gamma_2^2 - \delta)\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad -6\delta\Big\{8(\gamma_2^2 + \gamma_3^2 - \delta)\lambda_3^2 - 8(\gamma_3^2 - \delta)\lambda_2^2 + (\gamma_2^2 + 9\gamma_3^2 - \delta) - 4\delta\lambda_2\Big\}\boldsymbol{\mathfrak{Q}_2}\\
&\quad -18\delta(\gamma_2^2 + \gamma_3^2 - \delta)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{34} &= 16\delta(\lambda_2 - 1)^2(\lambda_2^2 + \lambda_2 + 1)\boldsymbol{\mathfrak{Q}_1} - (8\lambda_2^2 - 4\lambda_2 - 1)\boldsymbol{\mathfrak{Q}_2} + 3\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{44} &= 24\delta(\lambda_2 - 1)^2(\lambda_2^2 + \lambda_2 + 1)\boldsymbol{\mathfrak{Q}_1^2} + 6\delta\boldsymbol{\mathfrak{Q}_2^2} - 3(8\lambda_2^2 - 4\lambda_2 - 1)\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_2} + 9\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_3}\\
&\quad +4\delta(\lambda_2 - 1)^2(8\lambda_2^2 - \lambda_3^2 - 4\lambda_2\lambda_3 + 8\lambda_2 - 4\lambda_3 + 8)\boldsymbol{\mathfrak{Q}_1}\\
&\quad -2(8\lambda_2^2 + 8\lambda_3^2 - 4\lambda_2\lambda_3 - 4\lambda_2 - 4\lambda_3 - 1)\boldsymbol{\mathfrak{Q}_2} + 6\boldsymbol{\mathfrak{Q}_3}.
\end{aligned}$$

Step 2: case $\lambda_2 = 1$ or $\lambda_3 = 1$ (locally conformally flat metrics). First, note that if $\lambda_2 = \lambda_3 = 1$, then $\mathfrak{g}_R$ is locally conformally flat and locally symmetric (see Remark 2.1). Now, assuming $\mathfrak{g}_R$ is Bach-flat and taking $\lambda_2 = 1$, $\lambda_3 \neq 1$, we have $\mathfrak{B}_{34} = \frac{1}{4}\gamma_1(\gamma_2^2 + \gamma_3^2)(\lambda_3 - 1)^2 = 0$. If $\gamma_2 = \gamma_3 = 0$, then $\mathfrak{B}_{44} = \frac{\delta}{6}(\lambda_3 - 1)^2\lambda_3^2 = 0$, which is a contradiction since $\lambda_3 \neq 1$. If $\gamma_1 = 0$, then

$$\begin{aligned}\mathfrak{B}_{14} &= \ \ \tfrac{\delta}{12}\gamma_3(\lambda_3 - 1)^2\big\{8\delta(\gamma_2^2 + \gamma_3^2)(\lambda_3^2 + \lambda_3 + 1) - 8\lambda_3^2 - 4\lambda_3 - 3\big\},\\ \mathfrak{B}_{24} &= -\tfrac{\delta}{12}\gamma_2(\lambda_3 - 1)^2\big\{8\delta(\gamma_2^2 + \gamma_3^2)(\lambda_3^2 + \lambda_3 + 1) - 8\lambda_3^2 - 4\lambda_3 - 3\big\}.\end{aligned}$$

Excluding the case $\gamma_2 = \gamma_3 = 0$, the vanishing of the above expressions implies

$$8\delta(\gamma_2^2 + \gamma_3^2)(\lambda_3^2 + \lambda_3 + 1) - 8\lambda_3^2 - 4\lambda_3 - 3 = 0,$$

which allows us to express γ_2^2 in terms of γ_3 and λ_3. Using this expression for γ_2^2, a long but direct calculation leads to

$$\mathfrak{B}_{44} = -\tfrac{\delta(\lambda_3-1)^2(8\lambda_3+3)}{128(\lambda_3^2+\lambda_3+1)} = 0 \quad \text{and} \quad \mathfrak{B}_{33} = -\tfrac{\delta(8\lambda_3^5-47\lambda_3^4-27\lambda_3^3+40\lambda_3^2+17\lambda_3+9)}{128(\lambda_3^2+\lambda_3+1)^2} = 0,$$

from which one easily obtains that there is no solution if $\lambda_3 \neq 1$. The case $\lambda_3 = 1$, $\lambda_2 \neq 1$ is obtained in an analogous way.

Step 3: case $\lambda_2 \neq 1$, $\lambda_3 \neq 1$ (strictly Bach-flat metrics). Equation (6), which gives the components of the Bach tensor of $\mathfrak{g}_R$ in terms of the polynomials in (4), clearly implies that the vanishing of those polynomials is a sufficient condition for the Bach-flatness of $\mathfrak{g}_R$. In what follows, we will show that it is also a necessary condition assuming that g_R is Bach-flat, and that the resulting metrics are always strictly Bach-flat. We will make use of the theory of Gröbner bases.

As indicated before, the components of the Bach tensor can be thought of as polynomials on the structure constants (i.e., replacing the polynomials $\mathfrak{Q}_i$ by the expressions of the polynomials $\mathfrak{P}_i$ in Equation (4)), or as polynomials on the structure constants together with the $\mathfrak{Q}_i$'s (i.e., taking the $\mathfrak{Q}_i$'s as variables which satisfy the relations $\mathfrak{Q}_i - \mathfrak{P}_i = 0$). This distinction will play an essential role in the calculation of the different Gröbner bases below. We will distinguish these two possibilities indicating the corresponding polynomial ring in each case.

(i) The vanishing of $\mathfrak{P}_1$. We consider the polynomial ring $\mathbb{R}[\lambda_3, \lambda_2, \gamma_1, \gamma_2, \gamma_3]$ and the ideal $\mathcal{I} = \langle\{\mathfrak{B}_{ij}\}\rangle$ generated by the components of the Bach tensor. Computing a Gröbner basis of $\mathcal{I}$ we get 91 polynomials, one of them being

$$\mathbf{g} = \gamma_1\gamma_2\gamma_3(\gamma_1^2 + \delta\gamma_2^2 - 1)^2(\lambda_2 - 1)\mathfrak{P}_1.$$

Next we show that necessarily $\mathfrak{P}_1 = 0$, analyzing the cases $\gamma_i = 0$, $i = 1, 2, 3$, and $\gamma_1^2 + \delta\gamma_2^2 = 1$ (recall that we are assuming $\lambda_2 \neq 1$ by Step 2).

If $\gamma_1 = 0$, the computation of a new Gröbner basis for the ideal $\mathcal{I}_1 = \langle\mathcal{I} \cup \{\gamma_1\}\rangle$ leads to 27 polynomials including

$$\mathbf{g}_{11} = \delta(\delta\gamma_2^2 + \delta\gamma_3^2 - 1)(\lambda_2 - 1)^2\lambda_2^2.$$

Since this polynomial must vanish and $\gamma_1 = 0$, we get $\mathfrak{P}_1 = \delta\gamma_2^2 + \delta\gamma_3^2 - 1 = 0$.

If $\gamma_2 = 0$, we proceed as above but considering the ideal $\mathcal{I}_2 = \langle \mathcal{I} \cup \{\gamma_2\}\rangle$ in $\mathbb{R}[\lambda_2, \lambda_3, \gamma_1, \gamma_2, \gamma_3]$ to obtain 27 polynomials, among which we have

$$\mathbf{g}_{21} = (\gamma_1^2 + \delta\gamma_3^2 - 1)(\lambda_3 - 1)^2\lambda_3^2.$$

Hence $\mathfrak{P}_1 = \gamma_1^2 + \delta\gamma_3^2 - 1 = 0$.

For the two remaining cases, we work again in $\mathbb{R}[\lambda_3, \lambda_2, \gamma_1, \gamma_2, \gamma_3]$. If $\gamma_3 = 0$ and $\gamma_2 \neq 0$ we take the ideal $\mathcal{I}_3 = \langle \mathcal{I} \cup \{\gamma_3\}\rangle$ which leads to 41 polynomials which include

$$\mathbf{g}_{31} = \gamma_2^3(\gamma_1^2 + \delta\gamma_2^2 - 1)(\lambda_2 - 1)^2\lambda_2^2.$$

Thus $\mathfrak{P}_1 = \gamma_1^2 + \delta\gamma_2^2 - 1 = 0$.

Finally, if $\gamma_1^2 + \delta\gamma_2^2 = 1$ and $\gamma_1\gamma_2\gamma_3 \neq 0$, we compute a Gröbner basis for the ideal $\mathcal{I}_4 = \langle \mathcal{I} \cup \{\gamma_1^2 + \delta\gamma_2^2 - 1\}\rangle$, obtaining a set which contains 46 polynomials, one of them being

$$\mathbf{g}_{41} = \gamma_2^2\gamma_3^2(\lambda_2 - 1)\lambda_2^3 \neq 0.$$

Hence, there is no solution in this case.

(ii) The vanishing of $\mathfrak{P}_2$ and $\mathfrak{P}_3$. In this case, we work in the polynomial ring $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \lambda_3, \lambda_2]$ and consider $\mathcal{I} = \langle \{\mathfrak{B}_{ij}, \mathfrak{Q}_1, \mathfrak{Q}_2 - \mathfrak{P}_2, \mathfrak{Q}_3 - \mathfrak{P}_3\}\rangle$. Computing a Gröbner basis for this ideal we obtain 27 polynomials which include

$$\mathbf{g}_1 = \mathfrak{Q}_2^2 \quad \text{and} \quad \mathbf{g}_2 = \mathfrak{Q}_3^2.$$

Hence, necessarily $\mathfrak{Q}_2 = \mathfrak{Q}_3 = 0$, or equivalently, $\mathfrak{P}_2 = \mathfrak{P}_3 = 0$.

(iii) The Bach-flat metrics are always strict. To finish the proof, we will show that the metrics $\mathfrak{g}_R$ determined by the solutions of the system of polynomial equations $\{\mathfrak{P}_i = 0\}$ are strictly Bach-flat. We recall that $\lambda_2 \neq 1$ and $\lambda_3 \neq 1$ from Step 2. In particular, we will see that, in this setting, a non-locally conformally flat Bach-flat metric $\mathfrak{g}_R$ is never a conformal C-space, showing that $\mathfrak{C} = \operatorname{div}_4 W - W(\cdot, \cdot, \cdot, X) \neq 0$ for any non-null vector field X on the Lie group which, at the neutral element of the group, can be expanded as $X = \sum_\alpha \boldsymbol{X}_\alpha e_\alpha$. The expressions of the components $\mathfrak{C}_{ijk}$ are very lengthy, so we do not include them here for the sake of clarity. They can be obtained after a long but direct calculation. Due to the complexity of the components $\mathfrak{C}_{ijk}$ we are not able to get a Gröbner basis directly. Thus, our strategy consists in reducing the number of variables as follows. Assuming the metric is Bach-flat, i.e., the polynomials $\mathfrak{P}_i$ in Equation (4) vanish, a long but straightforward calculation shows that

$$\begin{aligned}
8\mathfrak{C}_{143} &= 2\gamma_1(\lambda_2 - 1)(2\lambda_2 - \lambda_3 + 2)\boldsymbol{X}_1 - 2\delta\gamma_1\gamma_3\lambda_2(2\lambda_2 - \lambda_3 - 1)\boldsymbol{X}_4 \\
&\quad -\delta(\lambda_2 - 1)(\lambda_3^2 - 5\lambda_2\lambda_3 - 2\lambda_2 - 4\lambda_3 - 2)\gamma_2\gamma_3^2 \\
&\quad +\lambda_2(\lambda_3^2 - 5\lambda_2\lambda_3 + \lambda_2 + 2\lambda_3 + 1)\gamma_2, \\
-16\mathfrak{C}_{241} &= 4\gamma_3(\lambda_2 - \lambda_3)(2\lambda_2 + 2\lambda_3 - 1)\boldsymbol{X_2} + 4\delta\gamma_2\gamma_3\lambda_3(\lambda_2 - 2\lambda_3 + 1)\boldsymbol{X}_4 \\
&\quad -3(\lambda_2 - \lambda_3)\gamma_1\gamma_2^2 + (\lambda_3 - 1)(2\lambda_2 + 5)(\lambda_2 - 2\lambda_3 - 2)\gamma_1\gamma_3^2
\end{aligned}$$

$$\begin{aligned}
&+3\delta\lambda_3(2\lambda_2\lambda_3-\lambda_2-1)\gamma_1,\\
16\delta\mathfrak{C}_{341} = {}& 4\gamma_3(\lambda_2-\lambda_3)(2\lambda_2+2\lambda_3-1)\boldsymbol{X_3}-4\gamma_1\gamma_3\lambda_2(2\lambda_2-\lambda_3-1)\boldsymbol{X}_4\\
&-(2\lambda_2\lambda_3^2-4\lambda_2^2\lambda_3-10\lambda_2^2-2\lambda_3^2+5\lambda_2\lambda_3-3\lambda_2+2\lambda_3+10)\gamma_2\gamma_3^2\\
&+3(\lambda_2-\lambda_3)\gamma_2^3-3\delta\lambda_3(\lambda_2-1)(2\lambda_2+1)\gamma_2. \qquad (7)
\end{aligned}$$

Next we show that the vanishing of the coefficients of $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{X}_3$ in the expressions above does not produce conformal C-spaces. To do this, we analyze the cases $\gamma_1\gamma_3 = 0$ and $\mathbf{q} = (\lambda_2-\lambda_3)(2\lambda_2-\lambda_3+2)(2\lambda_2+2\lambda_3-1) = 0$ separately. We fix the polynomial ring $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \lambda_2, \lambda_3, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4]$.

We start with the case $\gamma_1\gamma_3 = 0$. Let $\mathcal{I}_1 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}_{ijk}, \gamma_1\gamma_3\}\rangle$. Computing a Gröbner basis for this ideal we get 114 polynomials which include

$$\mathbf{g}_{11} = (\lambda_3+1)^2(\lambda_3-1)\lambda_2\lambda_3^2 \quad\text{and}\quad \mathbf{g}_{12} = -(2\lambda_3^3+3\lambda_3^2-\lambda_3-2\lambda_2-2)\lambda_2\lambda_3.$$

Since $(\lambda_3-1)\lambda_2\lambda_3 \neq 0$, necessarily $\lambda_3 = -1$, and thus $\mathbf{g}_{12} = -2\lambda_2^2 \neq 0$, so there is no conformal C-space in this case.

If $\mathbf{q} = 0$, we consider the ideal $\mathcal{I}_2 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}_{ijk}, \mathbf{q}\}\rangle$. The computation of a Gröbner basis for $\mathcal{I}_2$ leads to 83 polynomials, among which we have

$$\mathbf{g}_{21} = (\lambda_3-2)(2\lambda_3-1)(\lambda_3-1)\lambda_3^2 \quad\text{and}\quad \mathbf{g}_{22} = (2\lambda_3^3-3\lambda_3^2+2\lambda_2-3\lambda_3+2)\lambda_3.$$

Since $(\lambda_3-1)\lambda_3 \neq 0$, then either $\lambda_3 = 2$ or $\lambda_3 = \frac{1}{2}$, and in any of these two cases $\mathbf{g}_{22} \neq 0$. Hence the conformal C-space condition does not hold.

Now, since $\gamma_1\gamma_3\mathbf{q} \neq 0$, we can eliminate $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{X}_3$ from the expressions in Equation (7) and this allows us to eliminate those variables from the polynomials $\mathfrak{C}_{ijk}$. Thus, after removing the denominators when necessary, we obtain new polynomials $\mathfrak{C}'_{ijk}$, in this case in the polynomial ring $\mathbb{R}[\lambda_3, \lambda_2, \gamma_1, \gamma_2, \gamma_3, \boldsymbol{X}_4]$. We consider the ideal $\mathcal{I}_3 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}'_{ijk}\}\rangle$ and compute a Gröbner basis, obtaining 64 polynomials. A detailed analysis of the basis shows that the polynomial

$$\mathbf{g}_{31} = \gamma_3^4(\lambda_2-1)^2\boldsymbol{X}_4^2(\boldsymbol{X}_4^2+1)\Big\{4\boldsymbol{X}_4^2\left((\delta^2-\delta+3)\gamma_3^2+(\gamma_3^2-2)^2+25(\gamma_3^2-\delta)^2\boldsymbol{X}_4^2\right)+1\Big\}$$

belongs to the basis and therefore $\boldsymbol{X}_4 = 0$, since $\gamma_3(\lambda_2-1) \neq 0$ and the last polynomial does not vanish.

The condition $\boldsymbol{X}_4 = 0$ is crucial to obtaining a Gröbner basis in the general situation. Thus, in the polynomial ring $\mathbb{R}[\lambda_3, \lambda_2, \gamma_1, \gamma_2, \gamma_3, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4]$, we compute a Gröbner basis for the ideal $\mathcal{I}_4$ generated by $\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}_{ijk}, \boldsymbol{X}_4\}$ and obtain 104 polynomials, among which we have

$$\mathbf{g}_{41} = \gamma_3(\lambda_2-1)(\boldsymbol{X}_1^2+\boldsymbol{X}_2^2+\delta\boldsymbol{X}_3^2) \quad\text{and}\quad \mathbf{g}_{42} = \gamma_3(\lambda_2-1)(\gamma_3^2-4\boldsymbol{X}_2^2-4\delta\boldsymbol{X}_3^2).$$

Hence, since $\gamma_3(\lambda_2-1) \neq 0$, it follows that the system $\mathbf{g}_{41} = \mathbf{g}_{42} = 0$ is incompatible, and thus no conformal C-space can be obtained. □

Remark 2.4 A long but direct calculation shows that under the conditions in Equation (4) the Weyl curvature operator acting on the space of two-forms is three-step nilpotent and thus isotropic, i.e., $\|W\|^2 = 0$. Moreover the Ricci operator has nonzero real or complex eigenvalues depending on the possible values of λ_2 and λ_3, and the real eigenvalues may have the same or opposite signs.

3 Direct Extensions with Lorentzian Lie Groups $\widetilde{SL}(2, \mathbb{R})$ or $SU(2)$

In order to describe all left-invariant Lorentz metrics on $\widetilde{SL}(2, \mathbb{R})$ or $SU(2)$ we follow the approach in Rahmani (1992). We consider the structure operator $L(X \times Y) = [X, Y]$, where the vector-cross product $\langle X \times Y, Z\rangle = \det(X, Y, Z)$ is now defined with respect to a Lorentzian inner product on $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. As in the Riemannian case, unimodularity of the underlying Lie group translates into self-adjointness of L which, however, may have non-trivial Jordan normal form as follows (see, for example, Rahmani 1992). Therefore, one must consider the following possibilities:

Ia. *L is diagonalizable*. Hence, there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where we assume e_3 to be timelike, such that $L(e_i) = \lambda_i e_i$.

Ib. *L has complex eigenvalues*. Then, there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where we assume e_3 to be timelike, such that

$$L = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}, \quad \beta \neq 0\,.$$

II. *L has a double root of its minimal polynomial*. Then, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ such that

$$L = \begin{pmatrix} \lambda_1 & 0 & 0 \\ \varepsilon & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \varepsilon = \pm 1, \quad \text{where} \quad \langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1\,.$$

III. *L has a triple root of its minimal polynomial*. Then, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ such that

$$L = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \quad \text{where} \quad \langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1\,.$$

The special linear Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ admits inner products realizing all the possibilities above. On the contrary, Lorentzian inner products on $\mathfrak{g}_3 = \mathfrak{su}(2)$ are necessarily of type Ia. Next we consider all the possible Lorentzian inner products in this setting separately and show the existence of strictly Bach-flat metrics in all cases but III.

3.1 The Structure Operator *L* is Diagonalizable

There exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$, with e_3 timelike, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, such that the Lie algebra is determined by

$$\mathfrak{g}_{L.Ia} \begin{cases} [e_1, e_2] = -\lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1, \\ [e_1, e_4] = \gamma_1\lambda_2 e_2 + \gamma_2\lambda_3 e_3, \qquad [e_2, e_4] = -\gamma_1\lambda_1 e_1 + \gamma_3\lambda_3 e_3, \\ [e_3, e_4] = \gamma_2\lambda_1 e_1 + \gamma_3\lambda_2 e_2, \end{cases} \tag{8}$$

for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, where $\lambda_1\lambda_2\lambda_3 \neq 0$ determines the Lie algebra $\mathfrak{su}(2)$ if $\varepsilon_1\lambda_1$, $\varepsilon_2\lambda_2$ and $\varepsilon_3\lambda_3$ have the same sign, or $\mathfrak{sl}(2, \mathbb{R})$ otherwise, where $\varepsilon_k = \langle e_k, e_k\rangle$.

In this case, a direct calculation shows that the Bach tensor is determined by the expressions obtained in Equation (6) with $\delta = -1$. Moreover, proceeding exactly as in Sect. 2, we obtain the following characterization of the Bach-flatness of metrics (8).

Theorem 3.1 *A left-invariant metric $\mathfrak{g}_{L.Ia}$ is Bach-flat if and only if it is either locally conformally flat (whenever $\lambda_1 = \lambda_2 = \lambda_3$, in which case the Ricci operator is diagonalizable) or, otherwise, it is isomorphically homothetic to a metric determined by Equation* (8)*, where $\lambda_1 = 1$ and the remaining parameters satisfy the equations $\{\mathfrak{P}_i = 0\}$, for the polynomials*

$$\begin{aligned} \mathfrak{P}_1 &= \gamma_1^2 - \gamma_2^2 - \gamma_3^2 - 1, \\ \mathfrak{P}_2 &= (\lambda_2 - \lambda_3)(2\lambda_2 + 2\lambda_3 - 1)\gamma_2^2 + (\lambda_3 - 1)(\lambda_2 - 2\lambda_3 - 2)\gamma_3^2 \\ &\quad + \lambda_3(\lambda_2 - 2\lambda_3 + 1), \\ \mathfrak{P}_3 &= 3(\lambda_2 - \lambda_3)\gamma_2^2 - 3(\lambda_3 - 1)\big\{2(\lambda_3 + 1)(\lambda_2^2 - 1) + \lambda_2\big\}\gamma_3^2 \\ &\quad - \lambda_3\big\{2(3\lambda_3 - 2)\lambda_2^2 - (4\lambda_3 + 1)\lambda_2 + 3\big\}. \end{aligned} \tag{9}$$

Moreover, in the latter case, the Bach-flat metric is always strict and it is realized on the product Lie group $SU(2) \times \mathbb{R}$ or $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$.

Remark 3.2 Since $\lambda_1 \neq 0$, we consider the orthogonal basis $\hat{e}_i = \frac{1}{\lambda_1} e_i$ and assume $\lambda_1 = 1$. Proceeding as in Remark 2.3 the parameters γ_1, γ_2 and γ_3 are given by

$$\begin{aligned} \gamma_1^2 &= \frac{\lambda_2\big\{\lambda_2\lambda_3(4\lambda_2 - \lambda_3 + 11) - 6\lambda_2(\lambda_2 + 1) - \lambda_3(\lambda_3 - 4) - 5\lambda_3^3\big\}}{3(\lambda_3 - 1)(\lambda_2 - \lambda_3)\big\{2(\lambda_2 + \lambda_3)(\lambda_2\lambda_3 + 1) + 2\lambda_2^2 + 2\lambda_3^2 + 3\lambda_2\lambda_3\big\}}, \\ \gamma_2^2 &= \frac{-\lambda_3\big\{\lambda_2\lambda_3(\lambda_2 - 4\lambda_3 - 11) + \lambda_2(\lambda_2 - 4) + 6\lambda_3(\lambda_3 + 1) + 5\lambda_2^3\big\}}{3(\lambda_2 - 1)(\lambda_2 - \lambda_3)\big\{2(\lambda_2 + \lambda_3)(\lambda_2\lambda_3 + 1) + 2\lambda_2^2 + 2\lambda_3^2 + 3\lambda_2\lambda_3\big\}}, \\ \gamma_3^2 &= \frac{-\lambda_2\lambda_3\big\{\lambda_2\lambda_3(6\lambda_2 + 6\lambda_3 - 11) - \lambda_2(4\lambda_2 - 1) - \lambda_3(4\lambda_3 - 1) + 5\big\}}{3(\lambda_2 - 1)(\lambda_3 - 1)\big\{2(\lambda_2 + \lambda_3)(\lambda_2\lambda_3 + 1) + 2\lambda_2^2 + 2\lambda_3^2 + 3\lambda_2\lambda_3\big\}}, \end{aligned} \tag{10}$$

where the denominators are different from zero if the metric is strictly Bach-flat.

The non-negativity of the expressions in Equation (10) implies that at least one of λ_2 and λ_3 must be negative. Besides, there exist solutions for the other combinations of signs of λ_2 and λ_3, so these metrics are realized on the product Lie group $SU(2) \times \mathbb{R}$

or $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$. Moreover, setting $\lambda_1=1$, for any nonzero value $\lambda_2\in\mathbb{R}\setminus[\frac{4}{5},\frac{5}{4}]$, $\lambda_2\neq 1$, there exist an infinite number of $\lambda_3\in(-\infty,1)\setminus\{0\}$ which provide strictly Bach-flat metrics after taking the remaining parameters $\gamma_1,\gamma_2,\gamma_3$ as in (10).

A direct calculation, as in Sect. 2, shows that the Weyl curvature operator acting on the space of two-forms is three-step nilpotent in the non-locally conformally flat case, and thus $\|W\|^2=0$. Moreover, the Ricci operator has two real and two complex eigenvalues.

3.2 The Structure Operator *L* has Complex Eigenvalues

If the structure operator L is of type Ib, then there exists an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ of $\mathfrak{g}=\mathfrak{g}_3\times\mathfrak{r}$, with e_3 timelike, where $\mathfrak{g}_3=\operatorname{span}\{e_1,e_2,e_3\}$ and $\mathfrak{r}=\operatorname{span}\{e_4\}$, such that the Lie algebra structure is given by

$$\mathfrak{g}_{L.Ib}\begin{cases}[e_1,e_2]=-\beta e_2-\alpha e_3,\quad [e_1,e_3]=-\alpha e_2+\beta e_3,\quad [e_2,e_3]=\lambda e_1,\\ [e_1,e_4]=(\alpha^2+\beta^2)(\gamma_1 e_2+\gamma_2 e_3),\\ [e_2,e_4]=-(\gamma_1\alpha-\gamma_2\beta)\lambda e_1+\gamma_3\beta e_2+\gamma_3\alpha e_3,\\ [e_3,e_4]=(\gamma_2\alpha+\gamma_1\beta)\lambda e_1+\gamma_3\alpha e_2-\gamma_3\beta e_3,\end{cases}\tag{11}$$

for certain $\gamma_1,\gamma_2,\gamma_3\in\mathbb{R}$, where $\alpha\in\mathbb{R}$, $\beta\neq 0$, and $\lambda\neq 0$. In this case the three-dimensional unimodular Lie algebra corresponds to $\mathfrak{sl}(2,\mathbb{R})$.

Remark 3.3 A straightforward calculation shows that a metric $\mathfrak{g}_{L.Ib}$ is never Einstein, nor locally conformally flat, nor locally symmetric.

The characterization of the Bach-flatness of $\mathfrak{g}_{L.Ib}$ is given as follows.

Theorem 3.4 *A left-invariant metric $\mathfrak{g}_{L.Ib}$ is Bach-flat if and only if it corresponds to a metric determined by Equation (11) with parameters satisfying the equations $\{\mathfrak{P}_i=0\}$, where*

$$\begin{aligned}\mathfrak{P}_1&=(\alpha^2-\beta^2)(\gamma_1^2-\gamma_2^2)-\gamma_3^2-4\alpha\beta\gamma_1\gamma_2-1,\\ \mathfrak{P}_2&=-2\beta^2(\alpha\lambda-4\beta^2)(\gamma_1^2-\gamma_2^2)+(2\lambda^2-\alpha\lambda-\alpha^2+11\beta^2)\gamma_3^2\\ &\quad-2\beta\big\{(\alpha^2-\beta^2)\lambda-4\alpha(\alpha^2+3\beta^2)\big\}\gamma_1\gamma_2+\alpha\lambda-\alpha^2+11\beta^2,\\ \mathfrak{P}_3&=6\alpha\beta^2\lambda^3(\gamma_1^2-\gamma_2^2)-3\big\{2\lambda^4-\alpha\lambda^3-(3\alpha^2-5\beta^2)\lambda^2+2(\alpha^2+\beta^2)^2\big\}\gamma_3^2\\ &\quad+6\beta(\alpha^2-\beta^2)\lambda^3\gamma_1\gamma_2-3\alpha\lambda^3+(\alpha^2+\beta^2)\lambda^2+8\alpha(\alpha^2+\beta^2)\lambda-6(\alpha^2+\beta^2)^2.\end{aligned}\tag{12}$$

Moreover, the Bach-flat metric is always strict.

Remark 3.5 Solutions to Equation (12) are more involved than in the previous cases. However the parameters γ_1, γ_2, and γ_3 can be obtained from appropriate values of α, $\beta\neq 0$, and $\lambda\neq 0$ as in Remarks 2.3 and 3.2, although the explicit solutions in the general case are unmanageable.

We illustrate the situation in the particular case with $\alpha=0$. Working at the homothetical level, one may simplify the expressions in Equation (12) as follows. Rescaling the basis $\hat{e}_i = \frac{1}{\beta} e_i$ such that the initial metric $\mathfrak{g}_{L.Ib}$ is homothetic to the corresponding one given by (11) with parameters $(\hat{\lambda}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) = (\frac{\lambda}{\beta}, \frac{\alpha}{\beta}, 1, \beta\gamma_1, \beta\gamma_2, \gamma_3)$, Equation (12) becomes

$$
\begin{aligned}
&\gamma_1^2 - \gamma_2^2 + \gamma_3^2 + 1 = 0,\\
&8\gamma_1^2 - 8\gamma_2^2 + 2\lambda\gamma_1\gamma_2 + (2\lambda^2 + 11)\gamma_3^2 + 11 = 0,\\
&6\lambda^3\gamma_1\gamma_2 + 3(2\lambda^4 + 5\lambda^2 + 2)\gamma_3^2 - \lambda^2 + 6 = 0.
\end{aligned}
$$

A straightforward calculation shows that strictly Bach-flat metrics are obtained for any $\lambda \in \mathbb{R} \setminus \left(-\frac{\sqrt{15}}{5}, \frac{\sqrt{15}}{5}\right)$, with γ_1, γ_2, and γ_3 given by

$$
\gamma_1^2 = \frac{-4\lambda^2 + \sqrt{(\lambda^2+1)(25\lambda^2+81)\lambda^2}}{3(\lambda^2+1)}, \quad \gamma_2 = \frac{-\left(3(\lambda^2+1)\gamma_1^2 + 8\lambda^2\right)\gamma_1}{(5\lambda^2+9)\lambda},
$$
$$
\gamma_3^2 = \frac{5\lambda^2 - 3}{3(\lambda^2+1)},
$$

which provides a one-parameter family of type Ib strictly Bach-flat metrics whose structure operator has purely imaginary eigenvalues.

A direct calculation from the characteristic polynomial of the Ricci operator shows that there are two real and two complex Ricci curvatures, the former ones having opposite signs.

Proof of Theorem 3.4 We proceed as in the proof of Theorem 2.2. First of all we show that all the components of the Bach tensor can be expressed in terms of the polynomials $\mathfrak{P}_i$ given by (12), $i = 1, 2, 3$ (Step 1 below). Then we show in Step 2 that the Bach-flat metrics are strict, since they cannot be C-spaces. We emphasize that, as in the proof of Theorem 2.2, Step 2 strongly depends on the result of Step 1, since we may replace the polynomial system $\{\mathfrak{B}_{ij} = 0\}$ given by the components of the Bach tensor by the much simpler system $\{\mathfrak{P}_i = 0\}$.

Step 1: expression of the Bach tensor in terms of the polynomials $\mathfrak{P}_i$. We express the components of the Bach tensor of metrics $\mathfrak{g}_{L.Ib}$ in terms of the polynomials in Equation (12), and again we rename each polynomial $\mathfrak{P}_i$ as $\mathfrak{Q}_i$. A long but direct calculation allows us to check that the Bach tensor of metrics $\mathfrak{g}_{L.Ib}$ is determined by

$$
\begin{aligned}
&\mathfrak{B}_{11} = \tfrac{-1}{72}\widetilde{\mathfrak{B}}_{11},\ \mathfrak{B}_{12} = \tfrac{\gamma_3}{24}\widetilde{\mathfrak{B}}_{12},\ \mathfrak{B}_{13} = \tfrac{-\gamma_3}{24}\widetilde{\mathfrak{B}}_{13},\ \mathfrak{B}_{14} = \tfrac{-\gamma_3}{24}\widetilde{\mathfrak{B}}_{14},\\
&\mathfrak{B}_{23} = \tfrac{-1}{96}\widetilde{\mathfrak{B}}_{23},\ \mathfrak{B}_{24} = \tfrac{1}{24}\widetilde{\mathfrak{B}}_{24},\ \mathfrak{B}_{33} = \tfrac{1}{1152}\widetilde{\mathfrak{B}}_{33},\ \mathfrak{B}_{34} = \tfrac{-1}{24}\widetilde{\mathfrak{B}}_{34},\\
&\mathfrak{B}_{44} = \tfrac{1}{48}\widetilde{\mathfrak{B}}_{44},\ \mathfrak{B}_{22} = -\mathfrak{B}_{11} + \mathfrak{B}_{33} - \mathfrak{B}_{44},
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
\widetilde{\mathfrak{B}}_{11} = {}& 3\Big\{20\lambda^4 - 12\alpha\lambda^3 + \beta^2\lambda^2 + 4\alpha(\alpha^2 - 3\beta^2)\lambda - 12(\alpha^2 + 3\beta^2)^2\Big\}\boldsymbol{\mathfrak{Q}_1^2} - 9\boldsymbol{\mathfrak{Q}_2^2}\\
&+12(\alpha\lambda - 3\alpha^2 - 9\beta^2)\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_2} - 4\boldsymbol{\mathfrak{Q}_1\mathfrak{Q}_3}
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigl\{24\alpha\beta^2\lambda(2\lambda^2-\alpha\lambda-\alpha^2+11\beta^2)(\gamma_1^2-\gamma_2^2)-3\Bigl(32\lambda^4-28\alpha\lambda^3\\
&+(31\alpha^2+191\beta^2)\lambda^2+6\alpha(\alpha^2-15\beta^2)\lambda-41\alpha^4+103\beta^4-66\alpha^2\beta^2\Bigr)\gamma_3^2\\
&+24\beta\lambda\Bigl((\alpha^2-\beta^2)\lambda(2\lambda-\alpha)+4\alpha^4-6\beta^4+22\alpha^2\beta^2\Bigr)\gamma_1\gamma_2-24\alpha\lambda^3\\
&+(17\alpha^2+5\beta^2)\lambda^2-2\alpha(49\alpha^2+\beta^2)\lambda+105(\alpha^2+\beta^2)^2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\\
&-\Bigl\{24\alpha\beta^2\lambda(\gamma_1^2-\gamma_2^2)-3(25\lambda^2-4\alpha\lambda-4(3\alpha^2-5\beta^2))\gamma_3^2\\
&+24\beta(\alpha^2-\beta^2)\gamma_1\gamma_2\lambda-12\alpha\lambda\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}+9\gamma_3^2\boldsymbol{\mathfrak{Q}}_{\mathbf{3}},\\
\widetilde{\mathfrak{B}}_{12}=&\Bigl\{2\beta\Bigl(16\alpha\lambda^3+(9\alpha^2+35\beta^2)\lambda^2+4\alpha(3\alpha^2+11\beta^2)\lambda+8(\alpha^2+\beta^2)(\alpha^2+3\beta^2)\Bigr)\gamma_1\\
&+2\Bigl(8(\alpha^2-\beta^2)\lambda^3-\alpha(3\alpha^2-23\beta^2)\lambda^2+(\alpha^2-3\beta^2)(3\alpha^2+11\beta^2)\lambda\\
&-8\alpha(\alpha^2+\beta^2)(\alpha^2+3\beta^2)\Bigr)\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\\
&+\Bigl\{\beta\Bigl(9\lambda^2+16\alpha\lambda+8(\alpha^2+\beta^2)\Bigr)\gamma_1+\Bigl(9\alpha\lambda^2+8(\alpha^2-\beta^2)\lambda-8\alpha(\alpha^2+\beta^2)\Bigr)\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}\\
&+3(\beta\gamma_1+\alpha\gamma_2)\boldsymbol{\mathfrak{Q}}_{\mathbf{3}},\\
\widetilde{\mathfrak{B}}_{13}=&\Bigl\{2\Bigl(8(\alpha^2-\beta^2)\lambda^3-\alpha(3\alpha^2-23\beta^2)\lambda^2+(3\alpha^4-33\beta^4+2\alpha^2\beta^2)\lambda\\
&-8\alpha(\alpha^2+\beta^2)(\alpha^2+3\beta^2)\Bigr)\gamma_1-2\beta\Bigl(16\alpha\lambda^3+(9\alpha^2+35\beta^2)\lambda^2\\
&+4\alpha(3\alpha^2+11\beta^2)\lambda+8(\alpha^2+\beta^2)(\alpha^2+3\beta^2)\Bigr)\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\\
&+\Bigl\{\Bigl(9\alpha\lambda^2+8(\alpha^2-\beta^2)\lambda-8\alpha(\alpha^2+\beta^2)\Bigr)\gamma_1-\beta\Bigl(9\lambda^2+16\alpha\lambda+8(\alpha^2+\beta^2)\Bigr)\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}\\
&+3(\alpha\gamma_1-\beta\gamma_2)\boldsymbol{\mathfrak{Q}}_{\mathbf{3}},\\
\widetilde{\mathfrak{B}}_{14}=&\Bigl\{(6\alpha^2-94\beta^2)\lambda^2-4\alpha(3\alpha^2-5\beta^2)\lambda+6\alpha^4-138\beta^4-4\alpha^2\beta^2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\\
&-\Bigl\{9\lambda^2-4(3\alpha^2-5\beta^2)\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}-3\boldsymbol{\mathfrak{Q}}_{\mathbf{3}},\\
\widetilde{\mathfrak{B}}_{23}=&\,12\beta\Bigl\{2\alpha\lambda^2+3(\alpha^2-2\beta^2)\lambda-3\alpha(\alpha^2+17\beta^2)\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}^{\mathbf{2}}-9\beta(\lambda+8\alpha)\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}\\
&+\Bigl\{2\alpha\beta\Bigl(32\lambda^4-16\alpha\lambda^3-(16\alpha^2+9\beta^2)\lambda^2-18\alpha(\alpha^2-\beta^2)\lambda\\
&+18(\alpha^4-9\beta^4+16\alpha^2\beta^2)\Bigr)(\gamma_1^2-\gamma_2^2)+\beta\Bigl(18\lambda^3-169\alpha\lambda^2+(119\alpha^2+27\beta^2)\lambda\\
&-328\alpha(\alpha^2+3\beta^2)\Bigr)\gamma_3^2+2\Bigl(16(\alpha^2-\beta^2)(2\lambda-\alpha)\lambda^3+\beta^2(39\alpha^2+25\beta^2)\lambda^2\\
&+8\alpha(2\alpha^4+2\beta^4+13\alpha^2\beta^2)\lambda-8(4\alpha^6+12\beta^6+11\alpha^4\beta^2+127\alpha^2\beta^4)\Bigr)\gamma_1\gamma_2\\
&-\beta\Bigl(328\alpha(\alpha^2+3\beta^2)-25(7\alpha^2+3\beta^2)\lambda-49\alpha\lambda^2-32\lambda^3\Bigr)\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\\
&-\Bigl\{4\alpha\beta\Bigl(\lambda^2-9(\alpha^2-\beta^2)\Bigr)(\gamma_1^2-\gamma_2^2)+164\alpha\beta\gamma_3^2+4\Bigl((\alpha^2-\beta^2)\lambda^2+8\alpha^4+8\beta^4\\
&+52\alpha^2\beta^2\Bigr)\gamma_1\gamma_2-4\beta(4\lambda-41\alpha)\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}-12\Bigl\{\alpha\beta(\gamma_1^2-\gamma_2^2)+(\alpha^2-\beta^2)\gamma_1\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{3}},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathfrak{B}}_{24}=&\Bigl\{4\beta\Bigl(4\lambda^4-2\alpha\lambda^3-\beta^2\lambda^2+2\alpha(\alpha^2+\beta^2)\lambda-4(\alpha^2+\beta^2)(\alpha^2+3\beta^2)\Bigr)\gamma_1\\
&+\Bigl(16\alpha\lambda^4-8(2\alpha^2+\beta^2)\lambda^3-4\alpha\beta^2\lambda^2-8(\alpha^2+\beta^2)(2\alpha^2+3\beta^2)\lambda\\
&+16\alpha(\alpha^2+\beta^2)(\alpha^2+3\beta^2)\Bigr)\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\\
&-\Bigl\{\beta\Bigl(\lambda^2+8(\alpha^2+\beta^2)\Bigr)\gamma_1+\Bigl(\alpha\lambda^2+4(\alpha^2+\beta^2)(\lambda-2\alpha)\Bigr)\gamma_2\Bigr\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}\\
&-3(\beta\gamma_1+\alpha\gamma_2)\boldsymbol{\mathfrak{Q}}_{\mathbf{3}},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathfrak{B}}_{33} = & \Big\{192\lambda^4 - 192\alpha\lambda^3 - 72\beta^2\lambda^2 - 192\alpha(\alpha^2+3\beta^2)\lambda + 48(\alpha^2-3\beta^2)(4\alpha^2+27\beta^2)\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}^{\mathbf{2}} \\
& -168\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}^{\mathbf{2}} - \Big\{579\lambda^2 - 12\alpha\lambda - 48(4\alpha^2 - 51\beta^2)\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}} - 113\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\boldsymbol{\mathfrak{Q}}_{\mathbf{3}} \\
& +\Big\{48\beta^2\Big(16\lambda^4 - 26\alpha\lambda^3 + (\alpha^2 + 88\beta^2)\lambda^2 - 126\alpha\beta^2\lambda + 9(\alpha^2 + 5\beta^2)(\alpha^2 + 7\beta^2)\Big)\gamma_1^2 \\
& +48\Big(16\alpha^2\lambda^4 - 2\alpha(4\alpha^2-9\beta^2)\lambda^3 - \beta^2(5\alpha^2+92\beta^2)\lambda^2 + 2\alpha(4\alpha^4+67\beta^4+8\alpha^2\beta^2)\lambda \\
& -16\alpha^6 - 363\beta^6 - 89\alpha^4\beta^2 - 220\alpha^2\beta^4\Big)\gamma_2^2 + 6\Big(144\lambda^4 - 108\alpha\lambda^3 \\
& -(157\alpha^2 - 1359\beta^2)\lambda^2 + 18\alpha(\alpha^2 - 35\beta^2)\lambda + 103\alpha^4 + 3919\beta^4 - 298\alpha^2\beta^2\Big)\gamma_3^2 \\
& +48\beta\Big(32\alpha\lambda^4 - 2(17\alpha^2 - 9\beta^2)\lambda^3 + 3\alpha(31\alpha^2 + 89\beta^2)\lambda^2 \\
& -18(\alpha^4 - 6\beta^4 + 9\alpha^2\beta^2)\lambda + 36\alpha\beta^2(11\alpha^2 + 23\beta^2)\Big)\gamma_1\gamma_2 + 618\alpha^4 + 23514\beta^4 \\
& -524\alpha^3\lambda + 5930\beta^2\lambda^2 + 768\lambda^4 + 4\alpha\lambda(\beta^2 - 84\lambda^2) - 2\alpha^2(894\beta^2 + 263\lambda^2)\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}} \\
& -\Big\{48\beta^2\Big(\lambda^2+9\alpha\lambda-9(\alpha^2 + 3\beta^2)\Big)\gamma_1^2 + 48\Big(\alpha^2\lambda^2-9\alpha\beta^2\lambda+8\alpha^4+35\beta^4+25\alpha^2\beta^2\Big)\gamma_2^2 \\
& +216\Big(\alpha\lambda + \alpha^2 - 9\beta^2\Big)\gamma_3^2 + 48\beta\Big(2\alpha\lambda^2 + 9(\alpha^2 - \beta^2)\lambda - 36\alpha(\alpha^2 + 2\beta^2)\Big)\gamma_1\gamma_2 \\
& +24(9\alpha^2 - 81\beta^2 - \alpha\lambda + 2\lambda^2)\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}} - 144\Big\{(\beta\gamma_1 + \alpha\gamma_2)^2 + \gamma_3^2 + 1\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{3}}, \\
\widetilde{\mathfrak{B}}_{34} = & \Big\{4\Big(4\alpha\lambda^4 - 2(2\alpha^2 + \beta^2)\lambda^3 - \alpha\beta^2\lambda^2 - 2(\alpha^2 + \beta^2)(2\alpha^2 + 3\beta^2)\lambda \\
& +4\alpha(\alpha^2 + \beta^2)(\alpha^2 + 3\beta^2)\Big)\gamma_1 - 4\beta\Big(4\lambda^4 - 2\alpha\lambda^3 - \beta^2\lambda^2 + 2\alpha(\alpha^2 + \beta^2)\lambda \\
& -4(\alpha^2 + \beta^2)(\alpha^2 + 3\beta^2)\Big)\gamma_2\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}} \\
& -\Big\{(\alpha\lambda^2 + 4(\alpha^2 + \beta^2)(\lambda - 2\alpha))\gamma_1 - \beta(\lambda^2 + 8(\alpha^2 + \beta^2))\gamma_2\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}} - 3(\alpha\gamma_1 - \beta\gamma_2)\boldsymbol{\mathfrak{Q}}_{\mathbf{3}}, \\
\widetilde{\mathfrak{B}}_{44} = & 6\Big\{4\lambda^4 - 4\alpha\lambda^3 - 3\beta^2\lambda^2 - 4(\alpha^3 + 3\alpha\beta^2)\lambda + 4(\alpha^2 + 3\beta^2)^2\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}^{\mathbf{2}} + 6\boldsymbol{\mathfrak{Q}}_{\mathbf{2}}^{\mathbf{2}} \\
& -3\Big\{\lambda^2 + 4\alpha\lambda - 8(\alpha^2 + 3\beta^2)\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}} - 9\boldsymbol{\mathfrak{Q}}_{\mathbf{1}}\boldsymbol{\mathfrak{Q}}_{\mathbf{3}} \\
& +4\Big\{8\lambda^4 - 12\alpha\lambda^3 + (3\alpha^2 + \beta^2)\lambda^2 - 2\alpha(\alpha^2 + \beta^2)\lambda + 3\alpha^4 - 69\beta^4 - 2\alpha^2\beta^2\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{1}} \\
& -2\Big\{\lambda^2 + 8\alpha\lambda - 4(3\alpha^2 - 5\beta^2)\Big\}\boldsymbol{\mathfrak{Q}}_{\mathbf{2}} - 6\boldsymbol{\mathfrak{Q}}_{\mathbf{3}}.
\end{aligned}
$$

Observe that the components of the Bach tensor above may be thought of as polynomials on the structure constants $\{\alpha, \beta, \lambda, \gamma_1, \gamma_2, \gamma_3\}$, when the polynomials $\mathfrak{Q}_i$ are replaced by the corresponding expressions of the polynomials $\mathfrak{P}_i$ in Equation (12), or they may be considered as polynomials on the structure constants together with the $\mathfrak{Q}_i$'s, when the $\mathfrak{Q}_i$'s are considered as variables satisfying $\mathfrak{Q}_i - \mathfrak{P}_i = 0$. We will always indicate the corresponding polynomial ring to distinguish these two possibilities.

Step 2: strictly Bach-flat metrics. Clearly, if the polynomials $\mathfrak{P}_i$ in Equation (12) vanish then the metric is Bach-flat. Next, we show that the converse is also true and that, moreover, the resulting metrics are always strictly Bach-flat.

(i) The vanishing of $\mathfrak{P}_1$. In view of Equation (13) we distinguish the cases $\gamma_3 = 0$ and $\gamma_3 \neq 0$, since this parameter directly influences the vanishing of several of the components.

If $\gamma_3 = 0$ we first notice that $\gamma_1 = \gamma_2 = 0$ does not allow Bach-flat metrics. Indeed, considering the ideal $\mathcal{I}_1 = \langle\{\mathfrak{B}_{ij}, \gamma_3, \gamma_1, \gamma_2\}\rangle$ in the polynomial ring

$\mathbb{R}[\lambda, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3]$ and computing a Gröbner basis we get 17 polynomials including

$$\mathbf{g}_1 = \beta^3(\alpha^2 + \beta^2)^2,$$

which does not vanish. Therefore, we may assume that $\gamma_1^2 + \gamma_2^2 \neq 0$ and take auxiliary variables φ_1, φ_2, and φ_3 which indicate that β, $\alpha^2 + \beta^2$, and $\gamma_1^2 + \gamma_2^2$ are not zero by means of the polynomials $\beta\varphi_1 - 1$, $(\alpha^2 + \beta^2)\varphi_2 - 1$, and $(\gamma_1^2 + \gamma_2^2)\varphi_3 - 1$, respectively. We consider $\mathcal{I}_1' = \langle\{\mathfrak{B}_{ij}, \gamma_3, \beta\varphi_1 - 1, (\alpha^2+\beta^2)\varphi_2 - 1, (\gamma_1^2+\gamma_2^2)\varphi_3 - 1\}\rangle$ in the polynomial ring $\mathbb{R}[\lambda, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3, \varphi_1, \varphi_2, \varphi_3]$. We emphasize that this is the only case where we use the graded reverse lexicographical order to compute the corresponding Gröbner basis for the ideal $\mathcal{I}_1'$. Thus we obtain 217 polynomials, one of them being

$$\mathbf{g}_1' = \mathfrak{P}_1,$$

so necessarily $\mathfrak{P}_1 = 0$.

If $\gamma_3 \neq 0$, the vanishing of the Bach tensor is determined by the vanishing of the polynomials $\widetilde{\mathfrak{B}}_{ij}$ in Equation (13). Computing a Gröbner basis for the ideal $\mathcal{I}_2$ generated by $\{\widetilde{\mathfrak{B}}_{ij}\}$ in $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \lambda, \alpha, \beta]$ we obtain 225 polynomials including

$$\mathbf{g}_2 = \alpha\beta^2\lambda^2\,\mathfrak{P}_1,$$

so to ensure the vanishing of $\mathfrak{P}_1$ it is sufficient to show that if $\alpha = 0$ then $\mathfrak{P}_1 = 0$, since $\lambda\beta \neq 0$. To do this, we compute a new Gröbner basis for $\mathcal{I}_2' = \langle \mathcal{I}_2 \cup \{\alpha\}\rangle$ in the polynomial ring $\mathbb{R}[\lambda, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3]$ obtaining 23 polynomials, among which we find

$$\mathbf{g}_2' = -\beta^5(\gamma_3^2 + 1)^2\,\mathfrak{P}_1,$$

which implies $\mathfrak{P}_1 = 0$.

(ii) The vanishing of $\mathfrak{P}_2$ and $\mathfrak{P}_3$. Working in the polynomial ring $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \lambda, \alpha, \beta, \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3]$ we compute a Gröbner basis for the ideal $\mathcal{I}$ generated by $\{\mathfrak{B}_{ij}, \mathfrak{Q}_1, \mathfrak{Q}_2 - \mathfrak{P}_2, \mathfrak{Q}_3 - \mathfrak{P}_3\}$. Thus we obtain 65 polynomials including

$$\mathbf{g}_1 = \mathfrak{Q}_2^3 \quad \text{and} \quad \mathbf{g}_2 = \mathfrak{Q}_3^2.$$

Hence $\mathfrak{Q}_2 = \mathfrak{Q}_3 = 0$, or equivalently, $\mathfrak{P}_2 = \mathfrak{P}_3 = 0$.

(iii) The Bach-flat metrics are always strict. In the rest of the proof we will show that a Bach-flat metric $\mathfrak{g}_{L.Ib}$ is never a conformal C-space and therefore these metrics are strictly Bach-flat. Let $\mathfrak{C} = \operatorname{div}_4 W - W(\cdot, \cdot, \cdot, X)$, with X a vector field on the Lie group which, at the neutral element of the group, can be expanded as $X = \sum_\alpha \boldsymbol{X}_\alpha e_\alpha$. The expressions of the components $\mathfrak{C}_{ijk}$ are very lengthy, so we do not include them here for the sake of clarity. Also, due to the complexity of those components, we are not able to get a Gröbner basis directly. Thus, as in the previous sections, our strategy consists in reducing the number of variables. We proceed as follows. Assuming the metric is Bach-flat, i.e., the polynomials $\mathfrak{P}_i$ in Equation (12) vanish, a long but straightforward calculation shows that

$$-8\mathfrak{C}_{231} = 2(\alpha^2 + \beta^2)\gamma_3\Big\{\big((\lambda - \alpha)\gamma_1 - 3\beta\gamma_2\big)\boldsymbol{X}_2 - \big(3\beta\gamma_1 + (\lambda - \alpha)\gamma_2\big)\boldsymbol{X}_3\Big\}$$

$$\begin{aligned}
&+2\alpha\beta^2(\lambda^2+2\beta^2)(\gamma_1^2-\gamma_2^2)-\big(4\lambda^3-3\alpha\lambda^2-\alpha(\alpha^2+19\beta^2)\big)\gamma_3^2\\
&+2\beta\big((\alpha^2-\beta^2)\lambda^2+5\alpha^4+3\beta^4+12\alpha^2\beta^2\big)\gamma_1\gamma_2-\alpha(\lambda^2-\alpha^2-19\beta^2),\\
8\mathfrak{C}_{234}={}&2\Big\{(\lambda-\alpha)(2\alpha\lambda+\alpha^2+\beta^2)\gamma_1-\beta\big(2(\lambda-\alpha)\lambda+3(\alpha^2+\beta^2)\big)\gamma_2\Big\}\boldsymbol{X}_2\\
&-2\Big\{\beta\big(2(\lambda-\alpha)\lambda+3(\alpha^2+\beta^2)\big)\gamma_1+(\lambda-\alpha)(2\alpha\lambda+\alpha^2+\beta^2)\gamma_2\Big\}\boldsymbol{X}_3\\
&-\gamma_3\Big\{\alpha\big((\alpha^2-7\beta^2)\lambda^2-2\alpha(\alpha^2+\beta^2)\lambda+(\alpha^2+\beta^2)^2\big)(\gamma_1^2-\gamma_2^2)\\
&+16\alpha\beta^2(\gamma_3^2+1)-2\beta\big((5\alpha^2-3\beta^2)\lambda^2-2\alpha(\alpha^2+\beta^2)\lambda\\
&-3(\alpha^2+\beta^2)^2\big)\gamma_1\gamma_2\Big\},
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
24\mathfrak{C}_{232}={}&6(\alpha^2+\beta^2)\gamma_3\big((\lambda-\alpha)\gamma_1-3\beta\gamma_2\big)\boldsymbol{X}_1\\
&-6\Big\{(\lambda-\alpha)(2\alpha\lambda+\alpha^2+\beta^2)\gamma_1-\beta\big(2(\lambda-\alpha)\lambda+3(\alpha^2+\beta^2)\big)\gamma_2\Big\}\boldsymbol{X}_4\\
&+4\Big\{\big((\alpha^2-\beta^2)\lambda^2+\alpha(\alpha^2+\beta^2)\lambda-2(\alpha^2+\beta^2)^2\big)(\gamma_1^2-\gamma_2^2)\\
&+4\beta^2\gamma_3^2-2\beta\lambda(2\alpha\lambda+\alpha^2+\beta^2)\gamma_1\gamma_2+2((\lambda-\alpha)\lambda+2\beta^2)\Big\}\boldsymbol{X}_3\\
&-3\gamma_3\lambda\Big\{\beta(8\alpha\lambda+\alpha^2-3\beta^2)\gamma_1+\big((3\alpha^2-5\beta^2)\lambda-\alpha(3\alpha^2+7\beta^2)\big)\gamma_2\Big\},\\
-24\mathfrak{C}_{233}={}&6(\alpha^2+\beta^2)\gamma_3\big(3\beta\gamma_1+(\lambda-\alpha)\gamma_2\big)\boldsymbol{X}_1\\
&-6\Big\{\beta(2(\lambda-\alpha)\lambda+3(\alpha^2+\beta^2))\gamma_1+(\lambda-\alpha)(2\alpha\lambda+\alpha^2+\beta^2)\gamma_2\Big\}\boldsymbol{X}_4\\
&+4\Big\{2(\alpha\beta^2\lambda-\alpha^4-3\beta^4)(\gamma_1^2-\gamma_2^2)+(\lambda+\alpha)\lambda\gamma_3^2\\
&+(3\lambda-\alpha)\lambda+2\beta((\alpha^2-\beta^2)\lambda-8\alpha\beta^2)\gamma_1\gamma_2\Big\}\boldsymbol{X}_2\\
&-3\gamma_3\lambda(((3\alpha^2-5\beta^2)\lambda-\alpha(3\alpha^2+7\beta^2))\gamma_1-\beta(8\alpha\lambda+\alpha^2-3\beta^2)\gamma_2).
\end{aligned}\tag{15}$$

One easily checks that if $\beta\lambda(\alpha^2+\beta^2)\gamma_3(\gamma_1^2+\gamma_2^2)(\lambda-\alpha)(\lambda-4\alpha)\neq 0$, or equivalently, $\gamma_3(\gamma_1^2+\gamma_2^2)(\lambda-\alpha)(\lambda-4\alpha)\neq 0$, then we can clear $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$ in terms of the structure constants $\{\lambda,\alpha,\beta,\gamma_1,\gamma_2,\gamma_3\}$ from $\mathfrak{C}_{231}=\mathfrak{C}_{234}=0$ in Equation (14). Once determined $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$, the same condition allows us to clear $\boldsymbol{X}_1$ and $\boldsymbol{X}_4$ in terms of the structure constants from $\mathfrak{C}_{232}=\mathfrak{C}_{233}=0$ in Equation (15). To see that we can assume $\gamma_3(\gamma_1^2+\gamma_2^2)(\lambda-\alpha)(\lambda-4\alpha)\neq 0$ we show that the vanishing of any of its factors does not produce conformal C-spaces. Starting with the factor γ_3, the computation of a Gröbner basis for the ideal $\langle\{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{C}_{ijk},\gamma_3\}\rangle$ in $\mathbb{R}[\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3,\boldsymbol{X}_4,\alpha,\beta,\gamma_1,\gamma_2,\gamma_3,\lambda]$ gives 4 polynomials containing $\beta\neq 0$, so the space cannot be a conformal C-space one. For the other three factors, taking $\mathbf{Q}_1=\{\gamma_1,\gamma_2\}$, $\mathbf{Q}_2=\{\lambda-\alpha\}$ and $\mathbf{Q}_3=\{\lambda-4\alpha\}$, we consider the polynomial ring $\mathbb{R}[\alpha,\beta,\gamma_1,\gamma_2,\gamma_3,\lambda]$ and compute a Gröbner basis for the ideal generated by $\{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathbf{Q}_i\}$. The basis obtained contains the polynomial $\gamma_3^2 + 1 \neq 0$ (for $i = 1$), $\beta^2\{(5\lambda^2 + \beta^2)\gamma_3^2 + \lambda^2 + \beta^2\} \neq 0$ (for $i = 2$) and $(9\lambda^2 + 16\beta^2)\gamma_3^2 + \lambda^2 + 16\beta^2 \neq 0$ (for $i = 3$), so the metric is not Bach-flat.

Hence we can clear $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{X}_3$, and $\boldsymbol{X}_4$ using the expressions in Equations (14) and (15) and this allows us to eliminate those variables in the polynomials $\mathfrak{C}_{ijk}$. Thus, after removing the denominators when necessary, we obtain new polynomials $\mathfrak{C}'_{ijk}$, in this case in the variables $\{\gamma_1, \gamma_2, \gamma_3, \alpha, \lambda, \beta\}$. At this point a key observation is that

$$\mathbf{q} = (\lambda + 2\alpha)\big\{5\lambda^3 - 6\alpha\lambda^2 - 3(\alpha^2 - 3\beta^2)\lambda + 4\alpha(\alpha^2 - 9\beta^2)\big\}$$

must necessarily vanish. This condition was obtained after a detailed analysis of several Gröbner bases and it can be easily checked as follows. Suppose that $\mathbf{q} \neq 0$. Hence $\lambda^3\mathbf{q}$ is also different from zero and we introduce an auxiliary variable φ to indicate it by means of the polynomial $\lambda^3\mathbf{q}\,\varphi - 1$. In the polynomial ring $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \alpha, \lambda, \beta, \varphi]$ we consider $\mathcal{I}_1 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}'_{ijk}, \lambda^3\mathbf{q}\,\varphi - 1\}\rangle$ and computing a Gröbner basis for this ideal we observe that it reduces to $\{1\}$, which implies that no conformal C-space exists if $\mathbf{q} \neq 0$.

Finally, if $\mathbf{q} = 0$, we analyze the vanishing of the two factors in $\mathbf{q}$ separately, showing that neither of these two cases may occur since the space would not be Bach-flat. If $\lambda + 2\alpha = 0$, we consider the ideal $\mathcal{I}_2 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \lambda + 2\alpha\}\rangle$ in $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \alpha, \lambda, \beta]$ and the computation of a Gröbner basis gives a set of 34 polynomials containing $\beta^2\big\{(9\lambda^2 + 4\beta^2)\gamma_3^2 + \lambda^2 + 4\beta^2\big\} \neq 0$. Hence, the metric is not Bach-flat.

If $\mathbf{q}_1 = 5\lambda^3 - 6\alpha\lambda^2 - 3(\alpha^2 - 3\beta^2)\lambda + 4\alpha(\alpha^2 - 9\beta^2)$ vanishes, we proceed as with the polynomial $\mathbf{q}$ above. We consider $\mathbf{q}_2 = \big(2(\lambda + \alpha)^2 + \alpha^2 + 3\beta^2\big)\gamma_3^2 + 3(\alpha^2 + \beta^2)$ and introduce an auxiliary variable φ to reflect that $\mathbf{q}_2$ never vanishes by means of the polynomial $\mathbf{q}_2\,\varphi - 1$. Computing a Gröbner basis for the ideal $\mathcal{I}_3 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathbf{q}_1, \mathbf{q}_2\,\varphi - 1\}\rangle$ in $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \alpha, \lambda, \beta, \varphi]$ we obtain 10 polynomials which include $\beta^2 \neq 0$. Hence, we conclude that the metric is not Bach-flat.

Therefore metrics $\mathfrak{g}_{L.Ib}$ given by Equation (11) satisfying $\{\mathfrak{P}_i = 0\}$ are always strictly Bach-flat. □

Remark 3.6 As a computational remark, we emphasize that using the polynomials $\{\mathfrak{P}_i\}$ instead of the components of the Bach tensor $\{\mathfrak{B}_{ij}\}$ is essential in showing the strictness of the Bach-flat metrics, since a direct use of the components of the Bach tensor in order to compute Gröbner bases has been useless in our work. Finally, a long but direct calculation shows that under the conditions in Equation (12) the Weyl curvature operator acting on the space of two-forms is three-step nilpotent and hence $\|W\|^2 = 0$.

3.3 The Structure Operator *L* has a Double Root of Its Minimal Polynomial

If L is of type II, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$, with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and

$\mathfrak{r} = \operatorname{span}\{u_4\}$, such that the Lie algebra is given by

$$\mathfrak{g}_{L.II} \begin{cases} [u_1, u_2] = \lambda_2 u_3, \qquad [u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \qquad [u_2, u_3] = \lambda_1 u_2, \\ [u_1, u_4] = \lambda_1 \gamma_1 u_1 + \varepsilon \gamma_1 u_2 + \gamma_2 \lambda_2 u_3, \ [u_2, u_4] = -\gamma_1 \lambda_1 u_2 + \gamma_3 \lambda_2 u_3, \\ [u_3, u_4] = -\gamma_3 \lambda_1 u_1 - (\gamma_2 \lambda_1 + \varepsilon \gamma_3) u_2, \end{cases} \tag{16}$$

with $\varepsilon^2 = 1$, for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, where $\lambda_1 \lambda_2 \neq 0$. The underlying unimodular Lie algebra corresponds to $\mathfrak{sl}(2, \mathbb{R})$.

Remark 3.7 A straightforward calculation shows that a metric $\mathfrak{g}_{L.II}$ is never Einstein, nor locally conformally flat, nor locally symmetric.

The Bach-flatness of metrics $\mathfrak{g}_{L.II}$ can be characterized as follows.

Theorem 3.8 *A left-invariant metric $\mathfrak{g}_{L.II}$ is Bach-flat if and only if it corresponds to a metric determined by Equation* (16) *with parameters satisfying the equations $\{\mathfrak{P}_i = 0\}$, where*

$$\begin{aligned} \mathfrak{P}_1 &= \gamma_1^2 - 2\gamma_2\gamma_3 + 1, \\ \mathfrak{P}_2 &= \lambda_1(\lambda_1 - \lambda_2)\gamma_2 + \varepsilon(4\lambda_1 + 3\lambda_2)\gamma_3, \\ \mathfrak{P}_3 &= 16\varepsilon\gamma_3^2 - (17\lambda_1 + 28\lambda_2)\gamma_2\gamma_3 + 14\lambda_2. \end{aligned} \tag{17}$$

Moreover, the Bach-flat metric is always strict.

Remark 3.9 Equation (17) can be explicitly solved by taking

$$\gamma_1^2 = -\frac{\lambda_1(12\lambda_1 + 5\lambda_2)}{3(4\lambda_1^2 + 4\lambda_2^2 + 7\lambda_1\lambda_2)}, \quad \gamma_2^2 = -\frac{\varepsilon(\gamma_1^2 + 1)(4\lambda_1 + 3\lambda_2)}{2\lambda_1(\lambda_1 - \lambda_2)} \quad \text{and} \quad \gamma_3 = \frac{\gamma_1^2 + 1}{2\gamma_2}. \tag{18}$$

Note that the above expressions are well defined since $\{\mathfrak{P}_i = 0\}$ is not satisfied for $\lambda_2 = \lambda_1$ or $\gamma_2 = 0$. Moreover, since the expressions of γ_1^2 and γ_2^2 in Equation (18) must be non-negative, it follows that strictly Bach-flat metrics are obtained whenever $\lambda_1 \in \left(0, -\frac{5}{12}\lambda_2\right]$ with $\lambda_2 < 0$, if $\varepsilon = 1$, or $\lambda_1 \in \left[-\frac{5}{12}\lambda_2, 0\right)$ with $\lambda_2 > 0$, if $\varepsilon = -1$.

Proof of Theorem 3.8 Once again we express the Bach tensor of metrics $\mathfrak{g}_{L.II}$ in terms of the polynomials in Equation (17) and, as in the previous cases, we rename each polynomial $\mathfrak{P}_i$ as $\mathfrak{Q}_i$. A long by direct calculation allows us to check that the Bach tensor of metrics $\mathfrak{g}_{L.II}$ are determined by

$$\begin{aligned} &\mathfrak{B}_{11} = \tfrac{1}{64260}\widetilde{\mathfrak{B}}_{11}, \qquad \mathfrak{B}_{12} = \tfrac{-1}{451584}\widetilde{\mathfrak{B}}_{12}, \ \mathfrak{B}_{13} = \tfrac{-\gamma_1}{9408}\widetilde{\mathfrak{B}}_{13}, \\ &\mathfrak{B}_{14} = \tfrac{-1}{588}\widetilde{\mathfrak{B}}_{14}, \qquad \mathfrak{B}_{22} = \tfrac{\gamma_3^2}{84}\widetilde{\mathfrak{B}}_{22}, \qquad \mathfrak{B}_{23} = \tfrac{-\gamma_1\gamma_3\lambda_1}{84}\widetilde{\mathfrak{B}}_{23}, \\ &\mathfrak{B}_{24} = \tfrac{\gamma_3(\lambda_2 - \lambda_1)}{84}\widetilde{\mathfrak{B}}_{24}, \ \mathfrak{B}_{34} = \tfrac{1}{2}\widetilde{\mathfrak{B}}_{34}, \qquad \mathfrak{B}_{44} = \tfrac{1}{1176}\widetilde{\mathfrak{B}}_{44}, \\ &\mathfrak{B}_{33} = -2\mathfrak{B}_{12} - \mathfrak{B}_{44}, \end{aligned} \tag{19}$$

where the polynomials are

$$
\begin{aligned}
\widetilde{\mathfrak{B}}_{11} &= 112455\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2}^2 - 257040\varepsilon\lambda_1^3\boldsymbol{\mathfrak{Q}_1}^2 + 19584\gamma_2\gamma_3\boldsymbol{\mathfrak{Q}_2}^2 - 1190\varepsilon\boldsymbol{\mathfrak{Q}_3}^2\lambda_1 \\
&\quad +714(315\gamma_2\lambda_1\lambda_2 + 1223\varepsilon\gamma_3\lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + 5712\varepsilon\lambda_1(15\lambda_1 - 7\lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3} \\
&\quad -2\Big\{9\gamma_2(391\lambda_1 + 15\lambda_2) - 15337\varepsilon\gamma_3\Big\}\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3} \\
&\quad -1428\Big\{\varepsilon\lambda_1\lambda_2(720\lambda_1 - 407\lambda_2) + 3\gamma_3^2(645\lambda_1^2 + 859\lambda_2^2 + 1316\lambda_1\lambda_2)\Big\}\boldsymbol{\mathfrak{Q}_1} \\
&\quad -18\Big\{2\gamma_2\lambda_2(238\lambda_1 - 15(3\gamma_2\gamma_3 + 7)\lambda_2) + 1767\varepsilon\gamma_3\lambda_2 + 2238\gamma_3^3\Big\}\boldsymbol{\mathfrak{Q}_2} \\
&\quad +170\Big\{18\gamma_2^2\lambda_2^2(\lambda_1 - \lambda_2) + \varepsilon\lambda_2(89\lambda_1 + 9\lambda_2) - 3\gamma_3^2(194\lambda_1 + 163\lambda_2)\Big\}\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{12} &= 53376\gamma_3^2\boldsymbol{\mathfrak{Q}_2}^2 + 4(288\lambda_1^2 + 57\lambda_2^2 - 688\lambda_1\lambda_2)\boldsymbol{\mathfrak{Q}_3}^2 \\
&\quad -2352\gamma_3(96\lambda_1^2 + 7\lambda_2^2 - 68\lambda_1\lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + 147\lambda_2(64\lambda_1^2 + 21\lambda_2^2 - 176\lambda_1\lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3} \\
&\quad -16\gamma_3(240\lambda_1 + 677\lambda_2)\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3} \\
&\quad -1029\lambda_2^2\Big\{128\lambda_1^2 - (84\gamma_2\gamma_3 - 42)\lambda_2^2 + (669\gamma_2\gamma_3 - 352)\lambda_1\lambda_2\Big\}\boldsymbol{\mathfrak{Q}_1} \\
&\quad -336\gamma_3\lambda_2\Big\{(1701\gamma_2\gamma_3 - 256)\lambda_1 + 6(162\gamma_2\gamma_3 - 97)\lambda_2\Big\}\boldsymbol{\mathfrak{Q}_2} \\
&\quad -28\lambda_2\Big\{768\lambda_1^2 - 6(102\gamma_2\gamma_3 - 115)\lambda_2^2 + (2817\gamma_2\gamma_3 - 2144)\lambda_1\lambda_2\Big\}\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{13} &= 1312\gamma_3\boldsymbol{\mathfrak{Q}_2}^2 - 147(32\lambda_1^2 - 3\lambda_2^2 + 4\lambda_1\lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + 2(32\lambda_1 - 221\lambda_2)\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3} \\
&\quad +147\Big\{\gamma_2\lambda_1(\lambda_1 - \lambda_2)(16\lambda_1^2 - 3\lambda_2^2 + 4\lambda_1\lambda_2) - 9\varepsilon\gamma_3\lambda_2^3\Big\}\boldsymbol{\mathfrak{Q}_1} \\
&\quad -14\lambda_2\Big\{5(171\gamma_2\gamma_3 - 32)\lambda_1 + 162(2\gamma_2\gamma_3 - 1)\lambda_2\Big\}\boldsymbol{\mathfrak{Q}_2} \\
&\quad -56\Big\{\gamma_2\lambda_1(8\lambda_1^2 - 5\lambda_1\lambda_2 - 3\lambda_2^2) - 7\varepsilon\gamma_3\lambda_2^2\Big\}\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{14} &= 82\gamma_3\boldsymbol{\mathfrak{Q}_2}^2 - 294\lambda_1(\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + (4\lambda_1 - 46\lambda_2)\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3} + 147\gamma_2\lambda_1^2(\lambda_1 - \lambda_2)^2\boldsymbol{\mathfrak{Q}_1} \\
&\quad -28\lambda_2\Big\{5(6\gamma_2\gamma_3 - 1)\lambda_1 + 3(5\gamma_2\gamma_3 - 3)\lambda_2\Big\}\boldsymbol{\mathfrak{Q}_2} \\
&\quad -14\Big\{\gamma_2(\lambda_1 - \lambda_2)(2\lambda_1^2 - 2\lambda_2^2 - \lambda_1\lambda_2) - 7\varepsilon\gamma_3\lambda_2^2\Big\}\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{22} &= 147\lambda_1^2(\lambda_1^2 - \lambda_2^2)\boldsymbol{\mathfrak{Q}_1} + 6\gamma_3(2\lambda_1^2 + 2\lambda_2^2 - 7\lambda_1\lambda_2)\boldsymbol{\mathfrak{Q}_2} - 2(\lambda_1 - \lambda_2)(5\lambda_1^2 - 2\lambda_2^2)\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{23} &= 21\lambda_1^2(\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_1} - 18\gamma_3(2\lambda_1 - 3\lambda_2)\boldsymbol{\mathfrak{Q}_2} + 2(\lambda_1 - \lambda_2)(\lambda_1 + 2\lambda_2)\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{24} &= 21\lambda_1^2(\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_1} - 12\gamma_3(3\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_2} + 2(\lambda_1 - \lambda_2)(\lambda_1 + 2\lambda_2)\boldsymbol{\mathfrak{Q}_3}, \\
\widetilde{\mathfrak{B}}_{34} &= \gamma_1\gamma_3\lambda_1(\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_2}, \\
\widetilde{\mathfrak{B}}_{44} &= 27\gamma_3^2\boldsymbol{\mathfrak{Q}_2}^2 + 3(\lambda_1 - \lambda_2)^2\boldsymbol{\mathfrak{Q}_3}^2 - 882\gamma_3\lambda_1(\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + 18\gamma_3(\lambda_1 - \lambda_2)\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3} \\
&\quad +84\gamma_3(\lambda_1 - \lambda_2)(7\lambda_1 - 2\lambda_2)\boldsymbol{\mathfrak{Q}_2} - 56\lambda_2(\lambda_1 - \lambda_2)^2\boldsymbol{\mathfrak{Q}_3}.
\end{aligned}
$$

Using the expressions in Equation (19) it is clear that the vanishing of the polynomials $\mathfrak{P}_i$ in Equation (17) implies the vanishing of the Bach tensor. Next, we show that the converse also holds and that, moreover, the resulting metrics are always strictly Bach-flat.

We consider the polynomial ring $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \lambda_1, \lambda_2, \varepsilon]$, i.e., viewing the components of the Bach tensor as polynomials on the structure constants and the $\mathfrak{Q}_i$'s, which are taken as variables satisfying the relations $\mathfrak{Q}_i - \mathfrak{P}_i = 0$. Computing a Gröbner basis for $\mathcal{I}_1 = \langle\{\mathfrak{B}_{ij}, \mathfrak{Q}_1 - \mathfrak{P}_1, \mathfrak{Q}_2 - \mathfrak{P}_2, \mathfrak{Q}_3 - \mathfrak{P}_3, \varepsilon^2 - 1\}\rangle$ we obtain 179 polynomials which include

$$\begin{aligned}
\mathbf{g}_{11} &= \mathfrak{Q}_2(\lambda_1 - \lambda_2)^2\lambda_1\lambda_2^2,\\
\mathbf{g}_{12} &= \mathfrak{Q}_3(\lambda_1 - \lambda_2)^2\lambda_1\lambda_2^2,\\
\mathbf{g}_{13} &= -\lambda_1\Big\{40572\mathfrak{Q}_1\lambda_1^3\lambda_2 - 47(\lambda_1 - \lambda_2)^2\mathfrak{Q}_3^2 - 2\big(1449\mathfrak{Q}_1\lambda_1^3 - 697(\lambda_1 - \lambda_2)^2\lambda_2\big)\mathfrak{Q}_3\Big\}.
\end{aligned}$$

Note that if $\lambda_2 = \lambda_1$, then $\mathfrak{B}_{14} = -\frac{3}{2}\gamma_3^3\lambda_1^2$ and $\mathfrak{B}_{14} = 0$ implies $\gamma_3 = 0$, in which case $\mathfrak{B}_{11} = -\varepsilon(4\gamma_1^4 + 5\gamma_1^2 + 1)\lambda_1^3 \neq 0$ preventing the metric from being Bach-flat. Thus the vanishing of $\mathbf{g}_{11}$, $\mathbf{g}_{12}$, and $\mathbf{g}_{13}$ clearly implies $\mathfrak{Q}_1 = \mathfrak{Q}_2 = \mathfrak{Q}_3 = 0$, or equivalently, $\mathfrak{P}_1 = \mathfrak{P}_2 = \mathfrak{P}_3 = 0$.

Finally we show that the Bach-flat metrics are always strict. Let us consider $\mathfrak{C} = \operatorname{div}_4 W - W(\cdot,\cdot,\cdot,X)$, with X a vector field on the Lie group which, at the neutral element of the group, can be expanded as $X = \sum_\alpha \boldsymbol{X}_\alpha u_\alpha$. The expressions of the components $\mathfrak{C}_{ijk}$ are very lengthy, so we do not include them here for the sake of clarity. In the polynomial ring $\mathbb{R}[\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4, \gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2, \varepsilon]$ we take the ideal $\mathcal{I}_2 = \langle\{\varepsilon^2 - 1, \mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}_{ijk}\}\rangle$. Computing a Gröbner basis we obtain 40 polynomials, one of them being

$$\mathbf{g}_2 = (2\lambda_1 + \lambda_2)(4\lambda_1 + 5\lambda_2)\lambda_2^2.$$

A direct checking shows that if $\lambda_2 = -2\lambda_1$ or $\lambda_2 = -\frac{4}{5}\lambda_1$ then the polynomial system $\{\mathfrak{P}_i = 0\}$ does not hold. Thus, a Bach-flat metric $\mathfrak{g}_{L.II}$ is never a conformal C-space and therefore these metrics are strictly Bach-flat. ◻

Remark 3.10 A long but direct calculation shows that under the conditions in Equation (17) the Weyl curvature operator acting on the space of two-forms is three-step nilpotent and thus $\|W\|^2 = 0$. Moreover, the Ricci operator has two real and two complex roots, since the discriminant of its characteristic polynomial is negative. Furthermore, the two real eigenvalues have opposite signs.

3.4 The Structure Operator *L* has a Triple Root of Its Minimal Polynomial

If the structure operator is of type III, then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$, with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, such that the corresponding Lie algebra is determined by

$$\mathfrak{g}_{L.III}\begin{cases}
[u_1, u_2] = u_1 + \lambda u_3, \qquad [u_1, u_3] = -\lambda u_1, \qquad [u_2, u_3] = \lambda u_2 + u_3,\\
[u_1, u_4] = \gamma_1\lambda u_1 + \gamma_2\lambda^2 u_3,\\
[u_2, u_4] = \gamma_3 u_1 - (\gamma_1 - \gamma_2)\lambda u_2 - (\gamma_1 - \gamma_2 - \gamma_3\lambda)u_3,\\
[u_3, u_4] = -\gamma_3\lambda u_1 - \gamma_2\lambda^2 u_2 - \gamma_2\lambda u_3,
\end{cases}$$

for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, where $\lambda \neq 0$. Again, the underlying unimodular Lie algebra corresponds to $\mathfrak{sl}(2, \mathbb{R})$.

In this case, a direct calculation shows that $\mathfrak{B}_{14} = -\frac{15}{4}\gamma_2^3\lambda^5$. Therefore, the Bach-flatness condition implies $\gamma_2 = 0$, and this leads to $\mathfrak{B}_{23} = -\frac{1}{4}(\gamma_1^4 + 5\gamma_1^2 + 4)\lambda^3 \neq 0$. Hence, we conclude that no Bach-flat metric may exist if the structure operator has a triple root of its minimal polynomial.

4 Direct Extensions with Degenerate Lie Groups $\widetilde{SL}(2, \mathbb{R})$ or $SU(2)$

Let $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$ be a four-dimensional Lie algebra with a Lorentzian inner product $\langle\cdot,\cdot\rangle$ which restricts to a degenerate inner product on the subalgebra $\mathfrak{g}_3$. Let $\mathfrak{g}_3' = [\mathfrak{g}_3, \mathfrak{g}_3]$ be the derived subalgebra of $\mathfrak{g}_3$. Left-invariant metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$ arise when $\dim \mathfrak{g}_3' = 3$. In this case, $\mathfrak{g}_3' = \mathfrak{g}_3$ and we consider the pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \times \operatorname{span}\{u_4\}$ with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle$ and $\mathrm{ad}_{u_3} : \mathfrak{g}_3 \to \mathfrak{g}_3$. Since $\mathfrak{g}_3' = \mathfrak{g}_3$, ad_{u_3} must be of rank 2 and, apart from 0, it must have either two real eigenvalues or two conjugate complex eigenvalues. Moreover, writing $u_3 = [x_1, x_2]$, $x_1, x_2 \in \mathfrak{g}_3$, we have $\mathrm{ad}_{u_3} = \mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} - \mathrm{ad}_{x_2} \circ \mathrm{ad}_{x_1}$, which implies $\mathrm{tr}(\mathrm{ad}_{u_3}) = 0$. Thus, ad_{u_3} may be diagonalizable with opposite real eigenvalues, it may have purely imaginary complex eigenvalues, or it may be three-step nilpotent. While the latter case does not support any Bach-flat metric, they exist in the other two possibilities as follows.

4.1 ad_{u_3} has Real Eigenvalues $\{0, \lambda, -\lambda\}$, with $\lambda \neq 0$

In this case, let v_1 and v_2 be unit eigenvectors, i.e., $[v_1, u_3] = \lambda v_1$ and $[v_2, u_3] = -\lambda v_2$. The Jacobi identity implies $[v_1, v_2] \in \operatorname{span}\{u_3\}$. Thus, rescaling u_3 if necessary, we get a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$, with $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_3, v_4\rangle = 1$, $\langle v_1, v_2\rangle = \kappa \neq \pm 1$, where $\mathfrak{g}_3 = \operatorname{span}\{v_1, v_2, v_3\}$ and $\mathfrak{r} = \operatorname{span}\{v_4\}$, such that the corresponding structure of the Lie algebra is given by

$$\mathfrak{g}_{D.i} \begin{cases} [v_1, v_2] = v_3, & [v_1, v_3] = \lambda v_1, \qquad\qquad [v_1, v_4] = \gamma_1 v_1 + \gamma_2 v_3, \\ [v_2, v_3] = -\lambda v_2, & [v_2, v_4] = -\gamma_1 v_2 + \gamma_3 v_3, [v_3, v_4] = \gamma_3 \lambda v_1 + \gamma_2 \lambda v_2, \end{cases} \tag{20}$$

for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, where $\lambda \neq 0$, which are realized on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$.

Remark 4.1 A straightforward calculation shows that a metric $\mathfrak{g}_{D.i}$ is neither Einstein nor locally symmetric. Moreover, it is locally conformally flat if and only if $\kappa = \gamma_1 = \gamma_2 = \gamma_3 = 0$, in which case the Ricci operator has eigenvalues $\pm\lambda$ and $\pm\lambda\sqrt{-1}$. Moreover $\ell = v_3$ is a null geodesic vector field such that the manifold is Kundt and all scalar curvature invariants of order one, two or three vanish identically. We refer to Coley et al. (2009) for more information on Kundt spacetimes with vanishing scalar curvature invariants.

As in the previous cases, the Bach tensor of metrics $\mathfrak{g}_{D.i}$ can be expressed in terms of three polynomials such that Bach-flatness is characterized as follows.

Theorem 4.2 *A left-invariant metric $\mathfrak{g}_{D.i}$ is Bach-flat if and only if it is either locally conformally flat (whenever $\kappa = \gamma_1 = \gamma_2 = \gamma_3 = 0$) or, otherwise, it corresponds* *to a metric determined by Equation* (20) *with parameters satisfying the equations* $\{\mathfrak{P}_i = 0\}$, *where*

$$\begin{aligned} \mathfrak{P}_1 &= 3(\kappa^2-1)\gamma_1 + 2\kappa, \\ \mathfrak{P}_2 &= 24\gamma_1^2 - 8\lambda\gamma_2\gamma_3 + 9\kappa\gamma_1 + 6, \\ \mathfrak{P}_3 &= 24\kappa\gamma_1^2 - 4\lambda(\gamma_2^2+\gamma_3^2) + 17\gamma_1. \end{aligned} \tag{21}$$

Moreover, in the latter case, the Bach-flat metric is always strict.

Remark 4.3 Explicit solutions to Equation (21) can be given as follows. Note that $\{\mathfrak{P}_i = 0\}$ does not hold if $\gamma_2\gamma_3 = 0$. Hence, setting $\gamma_2 \neq 0$, for any value of the parameter $\kappa \in \mathbb{R} \setminus (-9, 9)$ the remaining parameters $\gamma_1, \gamma_3, \lambda$ are determined by

$$\begin{aligned} \gamma_1 &= \frac{2\kappa}{3(1-\kappa^2)}, \quad \gamma_3 = \frac{\gamma_2\kappa(17-\kappa^2) \pm \sqrt{\gamma_2^2(\kappa^2-81)(\kappa^2-1)^2}}{7\kappa^2+9}, \\ \lambda &= \frac{7\kappa^2+9}{12\gamma_2\gamma_3(\kappa^2-1)^2}, \end{aligned}$$

thus providing a two-parameter family of strictly Bach-flat metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$.

Proof of Theorem 4.2 We proceed as in the previous theorems. First of all we show that all the components of the Bach tensor can be expressed in terms of the polynomials $\mathfrak{P}_i$ given by (21). In the non-locally conformally flat case, we show that the corresponding Bach-flat metrics are strict, since they cannot be C-spaces. Moreover, as in the previous situations, we sometimes replace the polynomial system $\{\mathfrak{B}_{ij} = 0\}$ given by the components of the Bach tensor by the much simpler system $\{\mathfrak{P}_i = 0\}$, and consider the components of the Bach tensor as polynomials on the structure constants together with the $\mathfrak{Q}_i$'s (i.e., taking the $\mathfrak{Q}_i$'s as variables satisfying the relations $\mathfrak{Q}_i - \mathfrak{P}_i = 0$).

A long but direct calculation shows that the components of the Bach tensor of any metric $\mathfrak{g}_{D.i}$ are determined by

$$\begin{aligned} &\mathfrak{B}_{12} = \tfrac{-\lambda^2}{864(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{12}, \ \mathfrak{B}_{13} = \tfrac{-\lambda^3}{18(\kappa^2-1)}\widetilde{\mathfrak{B}}_{13}, \quad \mathfrak{B}_{14} = \tfrac{\lambda^2}{432(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{14}, \\ &\mathfrak{B}_{22} = \tfrac{-\lambda^2}{864(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{22}, \ \mathfrak{B}_{23} = \tfrac{\lambda^3}{18(\kappa^2-1)}\widetilde{\mathfrak{B}}_{23}, \quad \mathfrak{B}_{24} = \tfrac{-\lambda^2}{432(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{24}, \\ &\mathfrak{B}_{33} = \tfrac{\lambda^3}{48(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{33}, \quad \mathfrak{B}_{34} = \tfrac{\lambda^2}{288(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{34}, \ \mathfrak{B}_{44} = \tfrac{-\lambda}{432(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{44}, \\ &\mathfrak{B}_{11} = 2\kappa\mathfrak{B}_{12} - \mathfrak{B}_{22} + 2(\kappa^2-1)\mathfrak{B}_{34}, \end{aligned} \tag{22}$$

where, renaming the polynomials $\mathfrak{P}_i$ as $\mathfrak{Q}_i$, one has

$$\begin{aligned} \widetilde{\mathfrak{B}}_{12} = {} & 48(21\kappa^2 + 64\gamma_1\kappa - 24)\boldsymbol{\mathfrak{Q}_1}^3 - 48\kappa(7\kappa^2+3)\boldsymbol{\mathfrak{Q}_1}^2\boldsymbol{\mathfrak{Q}_2} - 432(\kappa^2-1)\boldsymbol{\mathfrak{Q}_1}^2\boldsymbol{\mathfrak{Q}_3} \\ & + \Big\{1920(\gamma_2^2+\gamma_3^2)\lambda + 243\kappa^5 - 2538\kappa^3 + 2603\kappa - 12192\gamma_1\Big\}\boldsymbol{\mathfrak{Q}_1}^2 \\ & + 9\kappa(\kappa^2-1)^2(3\kappa^2+2)\boldsymbol{\mathfrak{Q}_2}^2 + 45\kappa(\kappa^2-1)^2\boldsymbol{\mathfrak{Q}_3}^2 \\ & - 6(27\kappa^6 - 159\kappa^4 - 40\kappa^2 + 12)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} - 6\kappa(36\kappa^4 - 167\kappa^2 - 29)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3} \\ & + 18(\kappa^2-1)^2(4\kappa^2+1)\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3} + (288\kappa^4 - 940\kappa^2 - 768\gamma_1\kappa + 684)\boldsymbol{\mathfrak{Q}_1} \end{aligned}$$

$$
\begin{aligned}
&-48\kappa(\kappa^2-1)(2\kappa^2-3)\boldsymbol{\mathfrak{Q}_2}+24(\kappa^2-1)(5\kappa^2-3)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{13} &= 16\gamma_2\boldsymbol{\mathfrak{Q}_1}^2+\Big\{9\gamma_2\kappa^3+9\gamma_3\kappa^2+4(12\gamma_1\gamma_3-11\gamma_2)\kappa+6(8\gamma_1\gamma_2+\gamma_3)\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad -3(\kappa^2-1)(\gamma_2\kappa+\gamma_3)(\kappa\boldsymbol{\mathfrak{Q}_2}+\boldsymbol{\mathfrak{Q}_3}),\\
\widetilde{\mathfrak{B}}_{14} &= 128\gamma_2\boldsymbol{\mathfrak{Q}_1}^3+8\Big\{9\gamma_2\kappa^3+9\gamma_3\kappa^2+4(12\gamma_1\gamma_3-25\gamma_2)\kappa+48\gamma_1\gamma_2+70\gamma_3\Big\}\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad -24(\kappa^2-1)(\gamma_2\kappa+\gamma_3)(\kappa\boldsymbol{\mathfrak{Q}_2}+\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}\\
&\quad -\Big\{9\gamma_2\kappa^4-288\gamma_3\kappa^3-314\gamma_2\kappa^2+16(192\gamma_1\gamma_2+79\gamma_3)\kappa-1536\gamma_1\gamma_3-351\gamma_2\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad +3\kappa(\kappa^2-1)(\gamma_2\kappa^2-32\gamma_3\kappa+63\gamma_2)\boldsymbol{\mathfrak{Q}_2}+3(\kappa^2-1)(55\gamma_2\kappa^2-32\gamma_3\kappa+9\gamma_2)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{22} &= 480(3\kappa+8\gamma_1)\boldsymbol{\mathfrak{Q}_1}^3-480\kappa(\kappa\boldsymbol{\mathfrak{Q}_2}+\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad +\Big\{768\gamma_3^2\kappa(\kappa^2-1)\lambda+81\kappa^2(5\kappa^2-36)-960\gamma_1\kappa+11520\gamma_1^2-2749\Big\}\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad +45(\kappa^2-1)^2(\kappa^2\boldsymbol{\mathfrak{Q}_2}^2+\boldsymbol{\mathfrak{Q}_3}^2+2\kappa\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3})-6\kappa(45\kappa^4-207\kappa^2+2)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2}\\
&\quad -6(45\kappa^4-153\kappa^2-52)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3}\\
&\quad +16\Big\{3\gamma_3^2(9\kappa^6-44\kappa^4+29\kappa^2+6)\lambda-117\kappa^3+419\kappa-624\gamma_1\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad -48\kappa(\kappa^2-1)(3\gamma_3^2(\kappa^2-1)^2\lambda-13\kappa)\boldsymbol{\mathfrak{Q}_2}-(\kappa^2-1)(144\gamma_3^2(\kappa^2-1)^2\lambda-624\kappa)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{23} &= 16\gamma_3\boldsymbol{\mathfrak{Q}_1}^2+\Big\{9\gamma_3\kappa^3+9\gamma_2\kappa^2+(48\gamma_1\gamma_2-44\gamma_3)\kappa+6(8\gamma_1\gamma_3+\gamma_2)\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad -3(\kappa^2-1)(\gamma_3\kappa+\gamma_2)(\kappa\boldsymbol{\mathfrak{Q}_2}+\boldsymbol{\mathfrak{Q}_3}),\\
\widetilde{\mathfrak{B}}_{24} &= 128\gamma_3\boldsymbol{\mathfrak{Q}_1}^3+8\Big\{9\gamma_3\kappa^3+9\gamma_2\kappa^2+(48\gamma_1\gamma_2-100\gamma_3)\kappa+48\gamma_1\gamma_3+70\gamma_2\Big\}\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad -24(\kappa^2-1)(\gamma_3\kappa+\gamma_2)(\kappa\boldsymbol{\mathfrak{Q}_2}+\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}\\
&\quad -\Big\{9\gamma_3\kappa^4-288\gamma_2\kappa^3-314\gamma_3\kappa^2+16(192\gamma_1\gamma_3+79\gamma_2)\kappa-1536\gamma_1\gamma_2-351\gamma_3\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad +3\kappa(\kappa^2-1)(\gamma_3\kappa^2-32\gamma_2\kappa+63\gamma_3)\boldsymbol{\mathfrak{Q}_2}+3(\kappa^2-1)(55\gamma_3\kappa^2-32\gamma_2\kappa+9\gamma_3)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{33} &= 64\kappa\boldsymbol{\mathfrak{Q}_1}^2+(9\kappa^4-50\kappa^2+512\gamma_1\kappa-23)\boldsymbol{\mathfrak{Q}_1}-(3\kappa^5+26\kappa^3-29\kappa)\boldsymbol{\mathfrak{Q}_2}\\
&\quad -(21\kappa^4-10\kappa^2-11)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{34} &= 96(3\kappa+8\gamma_1)\boldsymbol{\mathfrak{Q}_1}^3-96\kappa(\kappa\boldsymbol{\mathfrak{Q}_2}+\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad +(81\kappa^4-810\kappa^2-1344\gamma_1\kappa+2304\gamma_1^2-3)\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad +9(\kappa^2-1)^2(\kappa^2\boldsymbol{\mathfrak{Q}_2}^2+\boldsymbol{\mathfrak{Q}_3}^2+2\kappa\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3})\\
&\quad -2\kappa(27\kappa^4-162\kappa^2+7)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2}-2(27\kappa^4-126\kappa^2+35)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3}\\
&\quad -4\Big\{128(\gamma_2^2+\gamma_3^2)\lambda-27\kappa^3-45\kappa-448\gamma_1\Big\}\boldsymbol{\mathfrak{Q}_1}-4\kappa^2(9\kappa^2+23)\boldsymbol{\mathfrak{Q}_2}\\
&\quad +4\kappa(9\kappa^2-41)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{44} &= 207\boldsymbol{\mathfrak{Q}_1}^3-3(23\kappa\boldsymbol{\mathfrak{Q}_2}-7\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}^2+\Big\{96(\gamma_2^2+\gamma_3^2)\lambda+81\kappa^3-567\kappa-504\gamma_1\Big\}\boldsymbol{\mathfrak{Q}_1}^2\\
&\quad +9(\kappa^2-1)^2(\kappa\boldsymbol{\mathfrak{Q}_2}^2+\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3})-(54\kappa^4-243\kappa^2-96\gamma_1\kappa+73)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2}\\
&\quad -3(9\kappa^3-13\kappa+32\gamma_1)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3}\\
&\quad -6\Big\{128\gamma_1(\gamma_2^2+\gamma_3^2)\lambda+3\kappa^2-224\gamma_1\kappa-256\gamma_1^2+111\Big\}\boldsymbol{\mathfrak{Q}_1}\\
&\quad +2(3\kappa^3+181\kappa-96\gamma_1)\boldsymbol{\mathfrak{Q}_2}-6(23\kappa^2+32\gamma_1\kappa+33)\boldsymbol{\mathfrak{Q}_3}.
\end{aligned}
$$

In view of the expressions in Equation (22) it is clear that the metric is Bach-flat if the polynomials $\mathfrak{P}_i$ in Equation (21) vanish. Next we show that, in the non-locally conformally flat case, the converse also holds and that, moreover, the resulting metrics are always strictly Bach-flat.

Note that since $\lambda \neq 0$, if the metric is Bach-flat, then the vanishing of the Bach tensor is determined by the vanishing of the polynomials $\widetilde{\mathfrak{B}}_{ij}$ in Equation (22). Considering the components of the Bach tensor as polynomials on the structure constants together with the $\mathfrak{Q}_i$'s (i.e., taking the $\mathfrak{Q}_i$'s as variables satisfying the relations $\mathfrak{Q}_i - \mathfrak{P}_i = 0$), we work in the polynomial ring $\mathbb{R}[\kappa, \lambda, \gamma_1, \gamma_2, \gamma_3, \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3]$ and compute a Gröbner basis for the ideal $\mathcal{I}_1 = \langle\{\widetilde{\mathfrak{B}}_{ij}, \mathfrak{Q}_1 - \mathfrak{P}_1, \mathfrak{Q}_2 - \mathfrak{P}_2, \mathfrak{Q}_3 - \mathfrak{P}_3\}\rangle$. Thus we get 20 polynomials, among which we have

$$\begin{aligned}&\mathbf{g}_{11} = \mathfrak{Q}_1, \quad \mathbf{g}_{12} = \mathfrak{Q}_3,\\ &\mathbf{g}_{13} = \kappa\mathfrak{Q}_2, \; \mathbf{g}_{14} = \gamma_1\mathfrak{Q}_2, \quad \text{and} \quad \mathbf{g}_{15} = (\gamma_2^2 + \gamma_3^2)\lambda\mathfrak{Q}_2.\end{aligned}$$

Hence $\mathfrak{Q}_1 = \mathfrak{Q}_2 = \mathfrak{Q}_3 = 0$ (equivalently, $\mathfrak{P}_1 = \mathfrak{P}_2 = \mathfrak{P}_3 = 0$), or otherwise $\kappa = \gamma_1 = \gamma_2 = \gamma_3 = 0$, in which case the metric is locally conformally flat (see Remark 4.1).

Finally we show that, in the non-locally conformally flat case, the Bach-flat metrics determined by $\mathfrak{P}_1 = \mathfrak{P}_2 = \mathfrak{P}_3 = 0$ are always strict. The expressions of the components of $\mathfrak{C} = \operatorname{div}_4 W - W(\cdot, \cdot, \cdot, X)$, with X a vector field on the Lie group which, at the neutral element of the group, can be expanded as $X = \sum_\alpha \boldsymbol{X}_\alpha v_\alpha$, are very lengthy, so we do not include them here for the sake of clarity. Note that the components $\mathfrak{C}_{ijk}$ are not polynomials since they contain the factor $\kappa^2 - 1$ in the denominators. We take $\mathfrak{C}'_{ijk} = (\kappa^2 - 1)\mathfrak{C}_{ijk}$ to get polynomials in $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \kappa, \lambda, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4]$. Now, computing a Gröbner basis for the ideal $\mathcal{I}_2 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}'_{ijk}\}\rangle$ we get 19 polynomials, one of them being

$$\mathbf{g}_2 = (4\boldsymbol{X}_3^2 + 1)(1000\boldsymbol{X}_3^2 + 49).$$

Since this polynomial does not vanish we conclude that Bach-flat metrics $\mathfrak{g}_{D.i}$ determined by (21) are never conformal C-spaces and therefore these metrics are strictly Bach-flat. □

Remark 4.4 A long but direct calculation shows that under the conditions in Equation (21), the Weyl curvature operator acting on the space of two-forms is three-step nilpotent, and thus $\|W\|^2 = 0$. Moreover, the Ricci operator has two real and two complex eigenvalues, where the real roots are different from zero and have the same sign.

4.2 ad_{u_3} has Complex Eigenvalues $\{0, \beta\sqrt{-1}, -\beta\sqrt{-1}\}$, with $\beta \neq 0$

In this case we consider unit vectors v_1 and v_2 such that $[v_1, u_3] = \beta v_2$ and $[v_2, u_3] = -\beta v_1$. The Jacobi identity implies $[v_1, v_2] \in \operatorname{span}\{u_3\}$. Thus, rescaling u_3 if necessary, we get a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \times \mathfrak{r}$, with $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_3, v_4\rangle = 1$, $\langle v_1, v_2\rangle = \kappa \neq \pm 1$, where $\mathfrak{g}_3 = \operatorname{span}\{v_1, v_2, v_3\}$ and $\mathfrak{r} = \operatorname{span}\{v_4\}$, such that the Lie algebra is determined by

$$\mathfrak{g}_{D.ii} \begin{cases} [v_1, v_2] = v_3, & [v_1, v_3] = \beta v_2, \qquad [v_1, v_4] = \gamma_1 v_2 + \gamma_2 v_3, \\ [v_2, v_3] = -\beta v_1, & [v_2, v_4] = -\gamma_1 v_1 + \gamma_3 v_3, [v_3, v_4] = \gamma_2\beta v_1 + \gamma_3\beta v_2, \end{cases} \tag{23}$$

for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, where $\beta \neq 0$. These metrics are realized on $SU(2) \times \mathbb{R}$ if $\beta < 0$, while for values of $\beta > 0$ Equation (23) determines left-invariant metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$.

Remark 4.5 A straightforward calculation shows that a metric $\mathfrak{g}_{D.ii}$ is never Einstein, nor locally conformally flat, nor locally symmetric.

Bach-flatness of metrics $\mathfrak{g}_{D.ii}$ is characterized as follows.

Theorem 4.6 *A left-invariant metric $\mathfrak{g}_{D.ii}$ is Bach-flat if and only if it corresponds to a metric determined by Equation (23) with parameters satisfying the equations $\{\mathfrak{P}_i = 0\}$, where*

$$\begin{aligned} \mathfrak{P}_1 &= 3(\kappa^2 - 1)\gamma_1 - 2, \\ \mathfrak{P}_2 &= 24\gamma_1^2 + 4\beta(\gamma_2^2 + \gamma_3^2) + 9\gamma_1, \\ \mathfrak{P}_3 &= 24\gamma_1^2 + 8\beta\kappa\gamma_2\gamma_3 + 17\gamma_1. \end{aligned} \tag{24}$$

Moreover, the Bach-flat metric is always strict and it is realized on the product Lie group $SU(2) \times \mathbb{R}$.

Remark 4.7 Equation (24) can be explicitly solved as follows. Since $\{\mathfrak{P}_i = 0\}$ is not satisfied when $\gamma_2 = \gamma_3 = 0$, equations $\mathfrak{P}_1 = \mathfrak{P}_2 = 0$ lead to

$$\gamma_1 = \frac{2}{3(\kappa^2 - 1)} \quad \text{and} \quad \beta = -\frac{9\kappa^2 + 7}{6(\gamma_2^2 + \gamma_3^2)(\kappa^2 - 1)^2}, \tag{25}$$

and thus equation $\mathfrak{P}_3 = 0$ becomes

$$(17\kappa^2 - 1)(\gamma_2^2 + \gamma_3^2) - 2\gamma_2\gamma_3\kappa(9\kappa^2 + 7) = 0.$$

If $17\kappa^2 - 1 \neq 0$ then setting $\gamma_2 \neq 0$ and $\kappa \in \mathbb{R} \setminus \left(-\frac{1}{9}, \frac{1}{9}\right)$, $\kappa \neq \pm\frac{1}{\sqrt{17}}$, $\kappa \neq \pm 1$, the corresponding Bach-flat metrics are determined by Equation (25) together with

$$\gamma_3 = \frac{\gamma_2\kappa\left(9\kappa^2 + 7\right) \pm \sqrt{\gamma_2^2(\kappa^2 - 1)^2(81\kappa^2 - 1)}}{17\kappa^2 - 1}.$$

If $17\kappa^2 - 1 = 0$, then Bach-flat metrics (23) are determined by

$$\kappa = \pm\frac{1}{\sqrt{17}}, \quad \beta = -\frac{17}{12(\gamma_2^2 + \gamma_3^2)} \quad \text{and} \quad \gamma_1 = -\frac{17}{24},$$

where either γ_2 or γ_3 vanishes.

Note that since $\beta < 0$ as shown in (25), all the Bach-flat metrics given by (23)-(24) are realized in $SU(2) \times \mathbb{R}$.

Proof of Theorem 4.6 Next we firstly show that all the components of the Bach tensor can be expressed in terms of the polynomials $\mathfrak{P}_i$ given by (24). Secondly we show that none of the metrics is a C-space, and thus they are strictly Bach-flat. As in the previous cases we rename each polynomial $\mathfrak{P}_i$ in (24) as $\mathfrak{Q}_i$. A long but direct calculation shows that the Bach tensor of metrics $\mathfrak{g}_{D.ii}$ is determined by

$$\begin{aligned}
&\mathfrak{B}_{11} = \tfrac{\beta^2}{288(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{11},\ \mathfrak{B}_{12} = \tfrac{\beta^2}{288(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{12},\ \mathfrak{B}_{13} = \tfrac{\beta^3}{6(\kappa^2-1)}\widetilde{\mathfrak{B}}_{13},\\
&\mathfrak{B}_{14} = \tfrac{-\beta^2}{144(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{14},\ \mathfrak{B}_{23} = \tfrac{-\beta^3}{6(\kappa^2-1)}\widetilde{\mathfrak{B}}_{23},\quad \mathfrak{B}_{24} = \tfrac{\beta^2}{144(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{24},\\
&\mathfrak{B}_{33} = \tfrac{-\beta^3}{144(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{33},\ \mathfrak{B}_{34} = \tfrac{-\beta^2}{288(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{34},\ \mathfrak{B}_{44} = \tfrac{\beta}{432(\kappa^2-1)^2}\widetilde{\mathfrak{B}}_{44},\\
&\mathfrak{B}_{22} = -\mathfrak{B}_{11} + 2\kappa\mathfrak{B}_{12} + 2(\kappa^2-1)\mathfrak{B}_{34},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
\widetilde{\mathfrak{B}}_{11} = {} & 32(3\boldsymbol{\mathfrak{Q}_1} + 5\boldsymbol{\mathfrak{Q}_2} + 5\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}^2 + 4\Big\{64\gamma_3^2(\kappa^2-1)\beta + 27\kappa^2 - 48\gamma_1(20\gamma_1+1) + 86\Big\}\boldsymbol{\mathfrak{Q}_1}^2\\
& -15(\kappa^2-1)^2(\boldsymbol{\mathfrak{Q}_2}^2 + \boldsymbol{\mathfrak{Q}_3}^2 + 2\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3}) - 2(18\kappa^4 - 15\kappa^2 - 163)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2}\\
& +2(33\kappa^2 + 127)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3} + 8\Big\{2\gamma_3^2(\kappa^2-1)(15\kappa^2+17)\beta + 33\kappa^2 + 96\gamma_1 + 35\Big\}\boldsymbol{\mathfrak{Q}_1}\\
& -12(\kappa^2-1)\Big\{\big(4\gamma_3^2(\kappa^2-1)^2\beta - \kappa^2 + 5\big)\boldsymbol{\mathfrak{Q}_2} + \big(4\gamma_3^2(\kappa^2-1)^2\beta + 5\kappa^2 - 1\big)\boldsymbol{\mathfrak{Q}_3}\Big\},\\
\widetilde{\mathfrak{B}}_{12} = {} & 16\kappa(9\boldsymbol{\mathfrak{Q}_2} + 7\boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}^2 - 2\kappa(1536\gamma_1^2 + 728\gamma_1 + 73)\boldsymbol{\mathfrak{Q}_1}^2 - 15\kappa(\kappa^2-1)^2\boldsymbol{\mathfrak{Q}_2}^2\\
& -3\kappa(3\kappa^4 - 4\kappa^2 - 1)\boldsymbol{\mathfrak{Q}_3}^2 + 2\kappa(26\kappa^2 + 24\gamma_1 + 119)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2}\\
& +2\kappa(65\kappa^2 + 72\gamma_1 + 63)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3} - 6\kappa(4\kappa^4 - 7\kappa^2 + 2)\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3}\\
& -2(24\gamma_2\gamma_3(16\gamma_1+5)\beta + (1152\gamma_1^3 + 408\gamma_1^2 + 1135\gamma_1 + 134)\kappa)\boldsymbol{\mathfrak{Q}_1}\\
& -2(24\gamma_2\gamma_3\beta - 70\kappa^3 + (72\gamma_1^2 + 3\gamma_1 + 68)\kappa)(\boldsymbol{\mathfrak{Q}_2} + \boldsymbol{\mathfrak{Q}_3}),\\
\widetilde{\mathfrak{B}}_{13} = {} & \Big\{\gamma_2(16\gamma_1+5)\kappa + \gamma_3(16\gamma_1-1)\Big\}\boldsymbol{\mathfrak{Q}_1} - (\kappa^2-1)(\gamma_2\kappa + \gamma_3)(\boldsymbol{\mathfrak{Q}_2} + \boldsymbol{\mathfrak{Q}_3}),\\
\widetilde{\mathfrak{B}}_{14} = {} & 8\Big\{\gamma_2(16\gamma_1+5)\kappa + \gamma_3(16\gamma_1+23)\Big\}\boldsymbol{\mathfrak{Q}_1}^2 - 8(\kappa^2-1)(\gamma_2\kappa + \gamma_3)(\boldsymbol{\mathfrak{Q}_2} + \boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}\\
& +8\Big\{9\gamma_3\kappa^2 - 2\gamma_2(32\gamma_1+1)\kappa + \gamma_3(128\gamma_1+49)\Big\}\boldsymbol{\mathfrak{Q}_1}\\
& -(\kappa^2-1)\Big\{(63\gamma_3\kappa^2 - 32\gamma_2\kappa + \gamma_3)\boldsymbol{\mathfrak{Q}_2} + (9\gamma_3\kappa^2 - 32\gamma_2\kappa + 55\gamma_3)\boldsymbol{\mathfrak{Q}_3}\Big\},\\
\widetilde{\mathfrak{B}}_{23} = {} & \Big\{\gamma_3(16\gamma_1+5)\kappa + \gamma_2(16\gamma_1-1)\Big\}\boldsymbol{\mathfrak{Q}_1} - (\kappa^2-1)(\gamma_3\kappa + \gamma_2)(\boldsymbol{\mathfrak{Q}_2} + \boldsymbol{\mathfrak{Q}_3}),\\
\widetilde{\mathfrak{B}}_{24} = {} & 8\Big\{\gamma_3(16\gamma_1+5)\kappa + \gamma_2(16\gamma_1+23)\Big\}\boldsymbol{\mathfrak{Q}_1}^2 - 8(\kappa^2-1)(\gamma_3\kappa + \gamma_2)(\boldsymbol{\mathfrak{Q}_2} + \boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}\\
& +8\Big\{9\gamma_2\kappa^2 - 2\gamma_3(32\gamma_1+1)\kappa + \gamma_2(128\gamma_1+49)\Big\}\boldsymbol{\mathfrak{Q}_1}\\
& -(\kappa^2-1)\Big\{(63\gamma_2\kappa^2 - 32\gamma_3\kappa + \gamma_2)\boldsymbol{\mathfrak{Q}_2} + (9\gamma_2\kappa^2 - 32\gamma_3\kappa + 55\gamma_2)\boldsymbol{\mathfrak{Q}_3}\Big\},\\
\widetilde{\mathfrak{B}}_{33} = {} & 320\boldsymbol{\mathfrak{Q}_1}^2 + 64(3\kappa^2 + 24\gamma_1 + 10)\boldsymbol{\mathfrak{Q}_1} - 3(\kappa^2-1)\Big\{(29\kappa^2+3)\boldsymbol{\mathfrak{Q}_2} + (11\kappa^2+21)\boldsymbol{\mathfrak{Q}_3}\Big\},\\
\widetilde{\mathfrak{B}}_{34} = {} & 96(\boldsymbol{\mathfrak{Q}_2} + \boldsymbol{\mathfrak{Q}_3})\boldsymbol{\mathfrak{Q}_1}^2 - 4(576\gamma_1^2 + 304\gamma_1 + 41)\boldsymbol{\mathfrak{Q}_1}^2 - 9(\kappa^2-1)^2(\boldsymbol{\mathfrak{Q}_2}^2 + \boldsymbol{\mathfrak{Q}_3}^2 + 2\boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3})\\
& +8(5\kappa^2 + 27)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + 16(7\kappa^2 + 9)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3} - 32(96\gamma_1^2 + 76\gamma_1 + 11)\boldsymbol{\mathfrak{Q}_1}\\
& +4(23\kappa^2 + 9)\boldsymbol{\mathfrak{Q}_2} + 4(41\kappa^2 - 9)\boldsymbol{\mathfrak{Q}_3},\\
\widetilde{\mathfrak{B}}_{44} = {} & 101\boldsymbol{\mathfrak{Q}_1}^2\boldsymbol{\mathfrak{Q}_2} + 35\boldsymbol{\mathfrak{Q}_1}^2\boldsymbol{\mathfrak{Q}_3} - 6(352\gamma_1^2 + 128\gamma_1 + 9)\boldsymbol{\mathfrak{Q}_1}^2 - 9(\kappa^2-1)^2(\boldsymbol{\mathfrak{Q}_2}^2 + \boldsymbol{\mathfrak{Q}_2}\boldsymbol{\mathfrak{Q}_3})\\
& +(45\kappa^2 + 96\gamma_1 + 263)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_2} + (27\kappa^2 + 96\gamma_1 + 113)\boldsymbol{\mathfrak{Q}_1}\boldsymbol{\mathfrak{Q}_3}\\
& -12(384\gamma_1^3 + 400\gamma_1^2 + 32\gamma_1 - 3)\boldsymbol{\mathfrak{Q}_1} - 2(117\kappa^2 - 96\gamma_1 - 61)\boldsymbol{\mathfrak{Q}_2}
\end{aligned}$$

$$+2(99\kappa^2+96\gamma_1+133)\mathfrak{Q}_3.$$

Using the expressions in Equation (26), it is clear that the vanishing of the polynomials $\mathfrak{P}_i$ in Equation (24) implies the vanishing of the Bach tensor. Next we show that the converse also holds and that, moreover, the resulting metrics are always strictly Bach-flat. Since $\beta \neq 0$, the Bach-flatness of the metric is given by the vanishing of the polynomials $\widetilde{\mathfrak{B}}_{ij}$ in Equation (26). Once again, we consider the components of the Bach tensor as polynomials on the structure constants and the $\mathfrak{Q}_i$'s, which are taken as variables which satisfy the relations $\mathfrak{Q}_i - \mathfrak{P}_i = 0$. Let $\mathcal{I}_1 = \langle\{\widetilde{\mathfrak{B}}_{ij}, \mathfrak{Q}_1 - \mathfrak{P}_1, \mathfrak{Q}_2 - \mathfrak{P}_2, \mathfrak{Q}_3 - \mathfrak{P}_3\}\rangle$ in the polynomial ring $\mathbb{R}[\kappa, \beta, \gamma_1, \gamma_2, \gamma_3, \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3]$. We compute a Gröbner basis for this ideal and get 13 polynomials which include

$$\mathbf{g}_{11} = \mathfrak{Q}_1, \quad \mathbf{g}_{12} = \mathfrak{Q}_2, \quad \text{and} \quad \mathbf{g}_{13} = \mathfrak{Q}_3.$$

Therefore $\mathfrak{Q}_1 = \mathfrak{Q}_2 = \mathfrak{Q}_3 = 0$, or equivalently, $\mathfrak{P}_1 = \mathfrak{P}_2 = \mathfrak{P}_3 = 0$.

Finally, we show that the Bach-flat metrics are always strict. We use the fact that $\beta \neq 0$ introducing an auxiliary variable $\bar{\beta}$ and considering the polynomial $\beta\bar{\beta} - 1$. The expressions of the components of $\mathfrak{C} = \operatorname{div}_4 W - W(\cdot,\cdot,\cdot,X)$, with X a vector field on the Lie group which, at the neutral element of the group, can be expanded as $X = \sum_\alpha X_\alpha v_\alpha$, are very lengthy, so we do not include them here for the sake of clarity. Note that the components $\mathfrak{C}_{ijk}$ are not polynomials since they contain the factor $\kappa^2 - 1$ in the denominators. We take $\mathfrak{C}'_{ijk} = (\kappa^2 - 1)\mathfrak{C}_{ijk}$ to get polynomials in $\mathbb{R}[X_1, X_2, X_3, X_4, \gamma_1, \gamma_2, \gamma_3, \kappa, \bar{\beta}, \beta]$. Computing a Gröbner basis for the ideal $\mathcal{I}_2 = \langle\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{C}'_{ijk}, \beta\bar{\beta} - 1\}\rangle$ we get 9 polynomials, one of them being

$$\mathbf{g}_2 = 9\kappa^2 + 1.$$

Since $\mathbf{g}_2$ does not vanish we conclude that a Bach-flat metric $\mathfrak{g}_{D.ii}$ is never a conformal C-space and therefore these metrics are strictly Bach-flat.

Remark 4.8 A long but direct calculation shows that under the conditions in Equation (24), the Weyl curvature operator acting on the space of two-forms is three-step nilpotent. Hence $\|W\|^2 = 0$ and moreover, the discriminant of the Ricci operator is negative, which shows that it has two real and two complex roots. Furthermore, the real eigenvalues are nonzero and may have the same or opposite signs.

4.3 ad_{u_3} is Three-Step Nilpotent

As in the previous cases, we consider a Jordan basis $\{u_1, u_2, u_3\}$ for ad_{u_3} such that $\mathrm{ad}_{u_3}(u_1) = u_2$ and $\mathrm{ad}_{u_3}(u_2) = u_3$. After normalizing $\{u_1, u_2\}$ and rescaling u_3, one has a basis $\{v_1, v_2, v_3\}$ of $\mathfrak{sl}(2,\mathbb{R})$ such that $[v_1, v_2] = \alpha v_1 + \beta v_3$, $[v_1, v_3] = -v_2$, and $[v_2, v_3] = \alpha v_3$, for some parameters α, β with $\alpha \neq 0$. Moreover, the inner product is determined by the nonzero components $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = 1$, $\langle v_1, v_2\rangle = \kappa$, with $\kappa^2 \neq 1$. Hence the left-invariant metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ in this case are determined by the Lie brackets

$$\mathfrak{g}_{D.iii}\begin{cases}[v_1,v_2]=\alpha v_1+\beta v_3,\ [v_1,v_3]=-v_2,[v_1,v_4]=\gamma_1 v_1+\gamma_2 v_2+\frac{\beta\gamma_1}{\alpha}v_3,\\ [v_3,v_4]=\gamma_3 v_2-\gamma_1 v_3,[v_2,v_3]=\alpha v_3,[v_2,v_4]=-\alpha\gamma_3 v_1-(\alpha\gamma_2+\beta\gamma_3)v_3,\end{cases}$$

where $\{v_i\}$ is a basis of the Lie algebra with $\langle v_1,v_1\rangle=\langle v_2,v_2\rangle=\langle v_3,v_4\rangle=1$, and $\langle v_1,v_2\rangle=\kappa$, with $\alpha\neq 0$, $\kappa^2\neq 1$, and $\gamma_1,\gamma_2,\gamma_3,\beta\in\mathbb{R}$.

Theorem 4.9 *A left-invariant metric $\mathfrak{g}_{D.iii}$ is never Bach-flat.*

Proof A long but straightforward calculation shows that the Bach tensor satisfies

$$8\mathfrak{B}_{13}-8\kappa\mathfrak{B}_{23}+4(\kappa^2-1)(\alpha-2\gamma_3)\mathfrak{B}_{33}=5\alpha^3$$

which, together with $\alpha\neq 0$, prevents the space from being Bach-flat. □

5 Conclusions

Besides the foundational work of Schmidt (1984), the results above exhibit a source of Bach-flat metrics on both Lie groups $SU(2)\times\mathbb{R}$ and $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$.

We characterize all the Bach-flat left-invariant Lorentz metrics on non-solvable four-dimensional Lie groups. Generically, these are families of metrics depending on two-parameters. This situation is in sharp contrast with the solvable case, where strictly Bach-flat metrics are rather exceptional (Calviño-Louzao et al. 2024a, b, 2025), and with the Riemannian situation (Abbena et al. 2013; Calviño-Louzao et al. 2019).

Conformally Einstein left-invariant Lorentz metrics on non-solvable Lie groups of dimension four are locally conformally flat. Moreover, they have a diagonalizable Ricci operator (in which case they are locally symmetric) or otherwise their Ricci operator has real and complex eigenvalues $\pm\lambda$, $\pm\lambda\sqrt{-1}$ (see also Calvaruso and Zaeim 2014). Apart from the locally conformally flat metrics in $SU(2)\times\mathbb{R}$ and $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ considered in Sects. 2 and 3.1, no other left-invariant Bach-flat Lorentzian metric on non-solvable Lie groups may be symmetric.

Since plane waves are conformally Einstein (see Leistner 2006), no left-invariant Lorentz metric on $SU(2)\times\mathbb{R}$ or $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ may be a plane wave. Moreover, all Bach-flat metrics on those Lie groups which are not locally conformally flat have Petrov type III, which shows that none of them is locally conformal to a pp-wave.

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