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Lower and Upper Solutions for System of Differential Equations Involving Homeomorphism and Nonlinear Boundary Conditions

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Abstract. We study the existence of solution to the system of differential equations $(\phi(u'))' = f(t, u, u')$ with nonlinear boundary conditions

$$g(u(0), u, u') = 0, \quad h(u'(1), u, u') = 0,$$

where $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g, h : \mathbb{R}^n \times C([0, 1], \mathbb{R}^n) \times C([0, 1], \mathbb{R}^n) \to \mathbb{R}^n$ are continuous, $\phi : \prod_{i=1}^n (-a_i, a_i) \to \mathbb{R}^n$, $0 < a_i \leq +\infty$, $\phi(s) = (\phi_1(s_1), \ldots, \phi_n(s_n))$ and $\phi_i : (-a_i, a_i) \to \mathbb{R}$ is a one dimensional regular or singular homeomorphism. Our proofs are based on the concept of the lower and upper solutions.

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1. Introduction

We consider the following system of differential equations

$$(\phi(u'))' = f(t, u, u'), \tag{1}$$

subject to nonlinear boundary conditions of the following type

$$g(u(0), u, u') = 0, \quad h(u'(1), u, u') = 0, \tag{2}$$

where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g, h : \mathbb{R}^n \times C([0,1], \mathbb{R}^n) \times C([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ are continuous and $\phi : \prod_{i=1}^n (-a_i, a_i) \to \mathbb{R}^n$ given by

$$\phi(s) = (\phi_1(s_1), \ldots, \phi_n(s_n))$$

is such that: $\phi_i : (-a_i, a_i) \to \mathbb{R}$ is a one dimensional increasing homeomorphism with $\phi_i(0) = 0$ and $0 < a_i \leq +\infty$, $i = 1, \ldots, n$. If $a_i = +\infty$ then $\phi_i : \mathbb{R} \to \mathbb{R}$ is a *regular* homeomorphism and if $a_i < +\infty$, ϕ_i is said to be a *singular* homeomorphism, cf. [1,5,10].

Lower and upper solutions are an effective tool in proving the existence of solutions to differential equations. The idea of lower solution comes from Picard [19]. More than twenty years later, Perron [18] and then Müller [16] proved the existence of solutions to the first order Cauchy problem together with their localization between functions α and β, which are ordered $\alpha \leq \beta$. Papers of Scorza Dragoni [21,22], for second order Dirichlet boundary value problem, indicate the important role played by the functions α and β, which we call today lower and upper solutions, and form the core of the method. Later, Nagumo [17] enlarged the class of functions to be considered as lower and upper solutions: the differential inequalities must be satisfied between the functions α and β and not, as in Scorza Dragoni's papers, for larger sets of values of u'. Since then, the theory of lower and upper solutions has been constantly developing, see, for instance [6].

There are many results regarding the existence of solutions, using lower and upper solutions, for two-points boundary value problems for second-order differential equations. We refer to monographs [7,20] and the literature included therein for a detailed description of these results. Since then, using the method of lower and upper solutions, the mentioned results have been generalized for problems with non-local boundary conditions [4,8,13], non-linear second-order coupled systems [15], non-linear boundary conditions or equations involving homeomorphisms [2,11,20].

The aim of this paper is to give sufficient conditions for the existence of solutions to the problem (1)–(2), using the method of lower and upper solutions. To the best of our knowledge this approach has never been used before in the context of the systems of differential equations involving homeomorphism and nonlinear boundary conditions (although there are known results for this type of systems of equations using other methods, cf. [9,12]). Firstly, in Sect. 2, we consider an auxiliary problem, which from the Schauder fixed point theorem has a solution, under certain additional assumptions (Theorem 2.1). The concept of lower and upper solutions for a system of equations is described in Sect. 3 (Definition 3.1), cf. [14]. In Sect. 4, we obtain the existence and localization of a solution to the problem (1)–(2) (Theorem 4.1). In our approach we used some ideas from the paper [3]. Our results cover some recent ones derived in [23]. The scalar equation with the nonlinear BCs considered in [23] are a special case of the problem (1)–(2) (see Sect. 4). Despite the fact that the main results in [23] are based on the extension of Mawhin's continuation theorem for quasi-linear operators, the assumptions made there allow us to find the lower and upper solutions for this particular case of the problem (1)–(2) (Corollary 4.1). In Sect. 4, we also present an example which illustrates an application of Theorem 4.1.

2. Auxiliary Problem

Denote by $C^1([0,1],\mathbb{R}^n)$ the Banach space of all continuously differentiable functions $u:[0,1]\to\mathbb{R}^n$ endowed with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

The following assumptions will be needed throughout the paper:

(A1) $f:[0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous;
(A2) $\phi=(\phi_1,\ldots,\phi_n):\prod_{i=1}^n(-a_i,a_i)\to\mathbb{R}^n$ and ϕ_i is one dimensional increasing homeomorphism with $\phi_i(0)=0$ and $0<a_i\le+\infty$, $i=1,\ldots,n$;
(A3) $g,h:\mathbb{R}^n\times C([0,1],\mathbb{R}^n)\times C([0,1],\mathbb{R}^n)\to\mathbb{R}^n$ are continuous.

For the purposes of this section, we introduce the following assumptions:

(S1) f is bounded;
(S2) $\mathcal{A},\mathcal{B}:C^1([0,1],\mathbb{R}^n)\to\mathbb{R}^n$ are continuous and bounded not necessarily linear functionals with $\mathcal{B}\left(C^1([0,1],\mathbb{R}^n)\right)\subset\prod_{i=1}^n(-a_i,a_i)$.

Consider the following auxiliary problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0)=\mathcal{A}(u), \quad u'(1)=\mathcal{B}(u). \tag{3}$$

Note that the solutions to (3) are exactly the fixed points of the operator $T:C^1([0,1],\mathbb{R}^n)\to C^1([0,1],\mathbb{R}^n)$ defined as

$$T(u)(t) = \mathcal{A}(u) + \int_0^t \phi^{-1}\left(\phi(\mathcal{B}(u)) - \int_s^1 f(z,u(z),u'(z))\,dz\right)ds. \tag{4}$$

We have

$$(T(u))'(t) = \phi^{-1}\left(\phi(\mathcal{B}(u)) - \int_t^1 f(s,u(s),u'(s))\,ds\right).$$

If A,B,M are bounds for $\mathcal{A},\mathcal{B},f$, respectively, then for any $u\in C^1([0,1],\mathbb{R}^n)$ and $t\in[0,1]$ we obtain

$$|T(u)(t)| \le A + \phi^{-1}\left(\phi(B)+M\right) := R, \tag{5}$$

and

$$|(T(u))'(t)| \le \phi^{-1}\left(\phi(B)+M\right) := R' \le R. \tag{6}$$

Clearly, the functions $T(u)$, $(T(u))'$ are continuous and, from (5) and (6), the operator T is well defined. Moreover, from (5) and (6), the fact that ϕ is a homeomorphism and the Lebesgue Dominated Convergence Theorem, T is continuous.

Denote by B_R the ball of radius R centered at the origin in the space $C^1([0,1],\mathbb{R}^n)$. To prove that T is completely continuous it is sufficient to observe that the image of B_R under T is relatively compact. From (5) and (6), $T(B_R)$ is uniformly bounded in $C^1([0,1],\mathbb{R}^n)$. Observe that, for every $u \in B_R$, we have

$$\left| \int_{t_2}^{t_1} f(r, u(r), u'(r))\, dr \right| \leq M \left| t_1 - t_2 \right|.$$

As ϕ is a homeomorphism, one can see that $\{(T(u))' : u \in B_R\}$ is equicontinuous on $[0,1]$. The fact that $\{(T(u))' : u \in B_R\}$ is uniformly bounded implies that $\{T(u) : u \in B_R\}$ is equicontinuous on $[0,1]$. Hence T is completely continuous.

Consequently, from the Schauder fixed point theorem, the operator $T : B_R \to B_R$ defined in (4) has a fixed point and the following theorem holds true.

Theorem 2.1. *Let Assumptions* $(A1)$, $(A2)$, $(S1)$ *and* $(S2)$ *be satisfied. Then the problem* (3) *has at least one solution.*

Remark 2.1. Note that in the case where ϕ is a singular homomorphism, then Theorem 2.1 holds under assumptions (A1), (A2) and (S2), i.e. the condition (S1) is not necessary to prove this Theorem.

Remark 2.2. Theorem 2.1 holds true if (A1) is replaced by the assumption that f is a Carathéodory function. In case when f is an L^1–Carathéodory function, one can also skip condition (S1).

3. Lower and Upper Solutions

By a solution to the problem (1)–(2) we mean a function $u : [0,1] \to \mathbb{R}^n$, $u = (u_1, \ldots, u_n) \in C^1([0,1],\mathbb{R}^n)$ with $|u_i'(t)| < a_i$ on $[0,1]$ for each i and $\phi(u') \in C^1([0,1],\mathbb{R}^n)$, which satisfies (1) and (2) on $[0,1]$.

Let $\alpha, \beta \in C^1([0,1],\mathbb{R}^n)$ with $|\alpha_i'(t)| < a_i$, $|\beta_i'(t)| < a_i$ on $[0,1]$, $i = 1, 2, \ldots, n$ and $\phi(\alpha'), \phi(\beta') \in C^1([0,1],\mathbb{R}^n)$.

Definition 3.1. We say that α is a lower solution and β is an upper solution to problem (1)–(2), if

$$\alpha_i(t) \leq \beta_i(t) \text{ and } \alpha_i'(t) \leq \beta_i'(t), \quad t \in [0,1],\ i \in \{1, \ldots, n\},$$

and for each $i \in \{1, \ldots, n\}$

$$\left(\phi_i(\alpha_i')\right)'(t) \geq f(t, u_1, \ldots, u_n, v_1, \ldots, v_{i-1}, \alpha_i'(t), v_{i+1} \ldots, v_n),$$
$$\left(\phi_i(\beta_i')\right)'(t) \leq f(t, u_1, \ldots, u_n, v_1, \ldots, v_{i-1}, \beta_i'(t), v_{i+1} \ldots, v_n),$$

whenever $\alpha_j(t) \leq u_j \leq \beta_j(t)$, $j \in \{1, \ldots, n\}$, and $\alpha_j'(t) \leq v_j \leq \beta_j'(t)$, $j \in \{1, \ldots, n\} \backslash \{i\}$, and

$$g_i(\alpha(0), \nu, \xi) \leq 0, \quad h_i(\alpha'(1), \nu, \xi) \leq 0,$$

$$g_i(\beta(0),\nu,\xi)\geq 0,\quad h_i(\beta'(1),\nu,\xi)\geq 0,$$

whenever $\alpha_j(t)\leq \nu_j(t)\leq \beta_j(t)$, $\alpha_j'(t)\leq \xi_j(t)\leq \beta_j'(t)$, $j\in\{1,\ldots,n\}$.

Remark 3.1. In the case, where $n=1$ and α, β are constant Definition 3.1 takes the following form.

We say that α is a lower solution and β is an upper solution to problem (1)–(2), if $\alpha\leq\beta$ and

$$f(t,u,0)=0,$$

whenever $\alpha\leq u\leq\beta$, $t\in[0,1]$, and

$$g(\alpha,\nu,0)\leq 0,\quad g(\beta,\nu,0)\geq 0,\quad h(0,\nu,0)=0,$$

whenever $\alpha\leq\nu\leq\beta$.

Remark 3.2. Observe that in the case when $n=1$ if g and h are nonincreasing functions with respect to their second and third arguments, then the assumptions on g and h in Definition 3.1 can be written as

$$g(\alpha(0),\alpha,\alpha')\leq 0,\quad h(\alpha'(1),\alpha,\alpha')\leq 0,$$

and

$$g(\beta(0),\beta,\beta')\geq 0,\quad h(\beta'(1),\beta,\beta')\geq 0.$$

Remark 3.3. In the case when f is a Carathéodory function, the definition of the lower and upper solutions can be weakened in the standard way.

4. Main Results

Now we shall prove that the existence of a couple of lower and upper solutions implies the existence and localization of a solution to (1)–(2).

Theorem 4.1. *Let Assumptions* (A1)–(A3) *be satisfied. Suppose that there exist a couple* (α,β) *of lower and upper solutions to* (1)–(2)*. Then problem* (1)–(2) *has at least one solution* u *such that* $\alpha\leq u\leq\beta$ *and* $\alpha'\leq u'\leq\beta'$.

Proof. For each $i\in\{1,\ldots,n\}$ we define the continuous and bounded functions $\zeta_i,\eta_i:[0,1]\times\mathbb{R}\to\mathbb{R}$ by

$$\zeta_i(t,x)=\begin{cases}\alpha_i(t), & x<\alpha_i(t)\\ x, & \alpha_i(t)\leq x\leq\beta_i(t)\\ \beta_i(t), & x>\beta_i(t)\end{cases},$$

$$\eta_i(t,y)=\begin{cases}\alpha_i'(t), & y<\alpha_i'(t)\\ y, & \alpha_i'(t)\leq y\leq\beta_i'(t)\\ \beta_i'(t), & y>\beta_i'(t)\end{cases}$$

and consider the modified problem

$$(\phi_i(u_i'))'=f_i^*(t,u,u'),\quad u_i(0)=\mathcal{A}_i^*(u),\quad u_i'(1)=\mathcal{B}_i^*(u),\tag{7}$$

where

$$f_i^*(t,x,y) = f_i(t,\zeta_1(t,x_1),\ldots,\zeta_n(t,x_n),\eta_1(t,y_1),\ldots,\eta_n(t,y_n))$$

and $\mathcal{A}^*,\mathcal{B}^* : C^1([0,1],\mathbb{R}^n) \to \mathbb{R}^n$ are given by

$$\mathcal{A}_i^*(v) = \zeta_i(0, v_i(0) - g_i(v(0),v,v')), \quad \mathcal{B}_i^*(v) = \eta_i(1, v_i'(1) - h_i(v'(1),v,v')).$$

Clearly, f^* is bounded and $\mathcal{B}_i^*(v) \in (-a_i,a_i)$.

First let us observe that Theorem 2.1 implies that problem (7) has at least one solution.

Now we shall show that if u is a solution to (7), then

$$\alpha_i'(t) \le u_i'(t) \le \beta_i'(t), \tag{8}$$

for all $t \in [0,1]$ and $i \in \{1,\ldots,n\}$. Suppose to the contrary that for some i we have $u_i' \not\le \beta_i'$. From (7), we have

$$u_i'(1) = \eta_i(1, u_i'(1) - h_i(u'(1),u,u')) \in [\alpha_i'(1), \beta_i'(1)],$$

which means that there exists $t_0 \in (0,1]$ such that $u_i'(t_0) = \beta_i'(t_0)$ and $u_i'(t) > \beta_i'(t)$ for all $t \in [t_0-\delta, t_0)$. By the monotonicity of ϕ_i, $\phi_i(u_i'(t)) > \phi_i(\beta_i'(t))$ for all $t \in [t_0-\delta,t_0)$. Now, knowing that u is a solution of (7) and using Definition 3.1, we reach a contradiction. Indeed, for $t \in [t_0-\delta,t_0)$, we obtain

$$\begin{aligned}
&\phi_i(\beta_i'(t)) - \phi_i(u_i'(t)) \\
&\quad = \int_t^{t_0} \Big((\phi_i(u_i'))'(s) - (\phi_i(\beta_i'))'(s) \Big)\, ds \\
&\quad = \int_t^{t_0} \big(f_i^*(s, u_1(s),\ldots,u_n(s), u_1'(s),\ldots,\beta_i'(s),\ldots,u_n'(s)) \\
&\qquad - (\phi_i(\beta_i'(s)))' \Big)\, ds \ge 0.
\end{aligned}$$

The proof that $\alpha'(t) \le u'(t)$, $t \in [0,1]$, is similar.

Now, let us prove that if u is a solution to (7), then

$$\alpha_i(t) \le u_i(t) \le \beta_i(t), \tag{9}$$

for all $t \in [0,1]$ and $i \in \{1,\ldots,n\}$. From (8), we obtain

$$\alpha_i(t) - \alpha_i(0) \le u_i(t) - u_i(0) \le \beta_i(t) - \beta_i(0). \tag{10}$$

Moreover, we have

$$u_i(0) = \zeta_i(0, u_i(0) - g_i(u(0),u,u')) \in [\alpha_i(0), \beta_i(0)]. \tag{11}$$

Consequently, (10) and (11) imply (9).

Now, observe that

$$\alpha_i'(1) \le u_i'(1) - h_i(u'(1),u,u') \le \beta_i'(1), \tag{12}$$

$i \in \{1,\dots,n\}$. Suppose to the contrary that $u_i'(1) - h_i(u'(1),u,u') < \alpha_i'(1)$. Then, by the definition of η_i, $u_i'(1) = \alpha_i'(1)$ and thus, from Definition 3.1, one has

$$\alpha_i'(1) > u_i'(1) - h_i(u'(1),u,u') = \alpha_i'(1) - h_i(\alpha'(1),u,u') \geq \alpha_i'(1),$$

a contradiction. In a similar way, it can be shown that $u_i'(1) - h_i(u'(1),u,u') \leq \beta_i'(1)$.

Finally, notice that

$$\alpha_i(0) \leq u_i(0) - g_i(u(0),u,u') \leq \beta_i(0), \tag{13}$$

$i \in \{1,\dots,n\}$. Indeed, if we suppose to the contrary that $u_i(0) - g_i(u(0),u,u') < \alpha_i(0)$, then, by the definition of ζ_i, $u_i(0) = \alpha_i(0)$ and, from Definition 3.1, we reach a contradiction

$$\alpha_i(0) > u_i(0) - g_i(u(0),u,u') = \alpha_i(0) - g_i(\alpha(0),u,u') \geq \alpha_i(0).$$

It is now easy to see that a solution to (7) is also a solution to (1) satisfying (9) and (8). Moreover, from (12) and (13) we obtain that this solution satisfies the boundary conditions (2). □

Example 4.1. Consider the system of differential equations

$$\begin{cases} (\phi_1(u_1'))' = (u_2'-3)^3\, e^{u_1+u_2} + \sin\left[\dfrac{\pi}{3}(u_1'-2t)\right], & t \in [0,1] \\ (\phi_2(u_2'))' = (u_2'-3)^2\left(u_1^2 + u_2^2 + (u_1')^2\right), \end{cases} \tag{14}$$

with $\phi_i : \mathbb{R} \to \mathbb{R}$ an increasing homeomorphism such that $\phi_i(0) = 0$, subject to the nonlinear functional boundary conditions

$$\begin{cases} 3\,u_1(0) + \frac{1}{2}\max_{t\in[0,1]} u_2(t) = 0, \\ 4\,u_1(0) - u_2(0) + \max_{t\in[0,1]} u_2'(t) = 0, \\ 6\,u_1'(1) - 16 + 2\int_0^1 u_1(t)\,dt + \int_0^{\frac{1}{3}} u_2'(t)\,dt = 0, \\ u_1'(1) - u_2'(1) + \frac{1}{10}\int_0^1 u_2^2(t)\,dt = 0. \end{cases} \tag{15}$$

Obviously, problem (14)–(15) can be seen as a particular case of (1)–(2) with

$$f_1(t,u_1,u_2,v_1,v_2) = (v_2-3)^3\, e^{u_1+u_2} + \sin\left[\frac{\pi}{3}(v_1-2t)\right],$$

$$f_2(t,u_1,u_2,v_1,v_2) = (v_2-3)^2\left(u_1^2 + u_2^2 + v_1^2\right),$$

and

$$g_1(z_1,z_2,\nu_1,\nu_2,\xi_1,\xi_2) = 3\,z_1 + \tfrac{1}{2}\max_{t\in[0,1]} \nu_2(t),$$

$$g_2(z_1,z_2,\nu_1,\nu_2,\xi_1,\xi_2) = 4\,z_1 - z_2 + \max_{t\in[0,1]} \xi_2(t),$$

$$h_1(z_1,z_2,\nu_1,\nu_2,\xi_1,\xi_2) = 6\,z_1 - 16 + 2\int_0^1 \nu_1(t)\,dt + \int_0^{\frac{1}{3}} \xi_2(t)\,dt,$$

$$h_2(z_1,z_2,\nu_1,\nu_2,\xi_1,\xi_2)=z_1-z_2+\tfrac{1}{10}\int_0^1 \nu_2^2(t)\,dt.$$

Observe that the couple $\alpha(t)=(t^2-1,3\,t)$, $\beta(t)=(3\,t,3\,t+1)$, $t\in[0,1]$, are a lower and an upper solution for problem (14)–(15). Indeed, let $t\in[0,1]$ and $\alpha_j(t)\le u_j\le\beta_j(t)$, $j=1,2$. Since $\alpha_2'(t)=\beta_2'(t)=3$, for $\alpha_2'(t)\le v_2\le\beta_2'(t)$ we have

$$\left(\phi_1(\alpha_1')\right)'(t)\ge 0=(3-3)^3\,e^{u_1+u_2}+\sin\left[\frac{\pi}{3}(2t-2t)\right]=f_1(t,u_1,u_2,\alpha_1'(t),v_2),$$

and

$$\left(\phi_1(\beta_1')\right)'(t)=0=(3-3)^3\,e^{u_1+u_2}+\sin\left[\frac{\pi}{3}(3-2t)\right]=f_1(t,u_1,u_2,\beta_1'(t),v_2),$$

while for $\alpha_1'(t)\le v_1\le\beta_1'(t)$ we get

$$\left(\phi_2(\alpha_2')\right)'(t)\ge 0=(3-3)^2\,(u_1^2+u_2^2+v_1^2)=f_2(t,u_1,u_2,v_1,\alpha_2'(t)),$$

and

$$\left(\phi_2(\beta_2')\right)'(t)=0=(3-3)^2\,(u_1^2+u_2^2+v_1^2)=f_2(t,u_1,u_2,v_1,\beta_2'(t)).$$

Moreover, if $\alpha_j(t)\le\nu_j(t)\le\beta_j(t)$, $\alpha_j'(t)\le\xi_j(t)\le\beta_j'(t)$, $j=1,2$, then

$$g_1(\alpha_1(0),\alpha_2(0),\nu_1,\nu_2,\xi_1,\xi_2)=-3+\tfrac{1}{2}\max_{t\in[0,1]}\nu_2(t)=-3+2=-1\le 0,$$

$$g_2(\alpha_1(0),\alpha_2(0),\nu_1,\nu_2,\xi_1,\xi_2)=-4+\max_{t\in[0,1]}\xi_2(t)=-4+3=-1\le 0,$$

$$g_1(\beta_1(0),\beta_2(0),\nu_1,\nu_2,\xi_1,\xi_2)=\tfrac{1}{2}\max_{t\in[0,1]}\nu_2(t)=2\ge 0,$$

$$g_2(\beta_1(0),\beta_2(0),\nu_1,\nu_2,\xi_1,\xi_2)=-1+\max_{t\in[0,1]}\xi_2(t)=-1+3=2\ge 0,$$

$$h_1(\alpha_1'(1),\alpha_2'(1),\nu_1,\nu_2,\xi_1,\xi_2)=12-16+2\int_0^1\nu_1(t)\,dt+\int_0^{\frac{1}{3}}\xi_2(t)\,dt\le 0,$$

$$h_2(\alpha_1'(1),\alpha_2'(1),\nu_1,\nu_2,\xi_1,\xi_2)=2-3+\tfrac{1}{10}\int_0^1\nu_2^2(t)\,dt\le-\tfrac{13}{45}<0,$$

$$h_1(\beta_1'(1),\beta_2'(1),\nu_1,\nu_2,\xi_1,\xi_2)=18-16+2\int_0^1\nu_1(t)\,dt+1\ge\tfrac{5}{3}>0,$$

$$h_2(\beta_1'(1),\beta_2'(1),\nu_1,\nu_2,\xi_1,\xi_2)=\tfrac{1}{10}\int_0^1\nu_2^2(t)\,dt\ge 0.$$

Therefore, Theorem 4.1 ensures that problem (14)–(15) has at least one solution (u_1,u_2) such that for $t\in[0,1]$

$$t^2-1\le u_1(t)\le 3\,t,\quad 3t\le u_2(t)\le 3t+1,$$

and

$$2\,t\le u_1'(t)\le 3,\quad u_2'(t)=3.$$

It is worth mentioning that the couple $\alpha(t) = (t^2 - 1, 3\,t)$, $\beta(t) = (3\,t, 3\,t + 1)$, are a lower and an upper solution to problem (14)–(15) involving any homeomorphism with $a_i > 3$.

We finish this paper by considering the scalar case of the differential equation (1) with the homeomorphism $\phi : (-a, a) \to \mathbb{R}$, $0 < a \leq +\infty$, and the boundary conditions

$$g(u(0), G(u')) = 0, \quad h(u'(1), H(u)) = 0, \tag{16}$$

where the functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $G, H : C([0,1]) \to \mathbb{R}$ are continuous functionals and map bounded sets into bounded sets, cf. [23]. Moreover, assume that the following conditions hold:

(H1) there exist $M_1, M_2 \in (-a, a)$, $M_1 \leq 0 \leq M_2$, such that for all $y \in \mathbb{R}$ we have

$$h(M_1, y) \leq 0, \quad h(M_2, y) \geq 0;$$

(H2) there exist $m_1, m_2 \in \mathbb{R}$, $m_1 \leq m_2$, such that for all $y \in \mathbb{R}$ we have

$$g(m_1, y) \leq 0, \quad g(m_2, y) \geq 0;$$

(H3) for each $t \in [0,1]$ and each $x \in [m_1 + M_1, m_2 + M_2]$, we have

$$f(t, x, M_1) \leq 0, \quad f(t, x, M_2) \geq 0.$$

Now, we are in a position to establish an existence result in the line of those in [23], as a consequence of Theorem 4.1. In this case, the lower and upper solutions are straight lines.

Corollary 4.1. *Assume that conditions* $(A1)$, $(A2)$ *and* $(H1)$–$(H3)$ *hold. Then problem* (1)–(16) *has at least one solution* u *such that*

$$m_1 + M_1 \leq u(t) \leq m_2 + M_2, \quad M_1 \leq u'(t) \leq M_2, \quad t \in [0,1].$$

Proof. Note that problem (1)–(16) is a particular case of problem (1)–(2) and, under assumptions $(H1)$, $(H2)$ and $(H3)$, the functions α and β defined as

$$\alpha(t) = M_1\,t + m_1, \quad \beta(t) = M_2\,t + m_2,$$

are a couple of lower and upper solutions for problem (1)–(2). Therefore the conclusion follows from Theorem 4.1. □

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Declarations

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