CROSSED PRODUCTS IN WEAK CONTEXTS

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Abstract. We define the general notion of crossed products in a weak context, which generalizes the ones defined by Blattner, Cohen and Montgomery, Doi and Takeuchi in the context of Hopf algebras and the one given by Brzeziński. Also, the crossed products obtained by the authors, for weak Hopf algebras living in a symmetric monoidal category and weak C-cleft extensions associated to weak entwined structures, are particular instances of this theory.

MSC: 18D05, 16W30.

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1 Introduction

Hopf crossed products, that is smash products where the multiplication is twisted by a cocycle σ, were introduced independently by Blattner, Cohen and Montgomery [7] and Doi and Takeuchi [16], as a generalization of group crossed products to the context of Hopf algebras living in a category of vector spaces over a field K. These objects, which play an important role in the theory of extensions of Hopf algebras, are constructed in the following way: Let H be a Hopf algebra with unit η_H, product μ_H, counit ε_H and coproduct δ_H. Suppose that $\varphi_A : H\otimes A \to A$ is a weak action of H on the K-algebra A and let $\sigma : H\otimes H \to A$ be a K-linear map. In the vector space $A\otimes H$, denoted by $A\sharp_\sigma H$, define the product (possible non-associative)

$$\mu_{A\sharp_\sigma H} = (\mu_A\otimes H)\circ(\mu_A\otimes\sigma_H^A)\circ(A\otimes\psi_H^A\otimes H)$$

where

$$\sigma_H^A = (\sigma\otimes\mu_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H),$$

$$\psi_H^A = (\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A),$$

μ_A is the product of A and c is the flip. If $A\sharp_\sigma H$ is associative with $\eta_A\otimes\eta_H$ as unity morphism, we call $A\sharp_\sigma H$ a crossed product. A necessary and sufficient conditions that $A\sharp_\sigma H$ be a crossed product was found by Blattner, Cohen and Montgomery [Corollary 4.6,[7]], and by Doi and Takeuchi [Lemma 10, [16]]. The result is the following: $A\sharp_{\sigma}H$ is a crossed product if and only if σ is normal ($\sigma\circ(\eta_H\otimes H)=\varepsilon_H\otimes\eta_A=\sigma\circ(H\otimes\eta_A)$), σ satisfy the twisted module condition

$$\mu_A\circ(\varphi_A\otimes A)\circ(H\otimes\varphi_A\otimes A)\circ(H\otimes H\otimes c_{A,A})\circ(H\otimes H\otimes\sigma_A\otimes A)\circ(\delta_{H\otimes H}\otimes A)=\mu_A\circ(A\otimes\varphi_A)\circ(\sigma_H^A\otimes A),$$

and the cocycle condition

$$\partial_4(\sigma_A)\wedge\partial_2(\sigma_A)=\partial_1(\sigma_A)\wedge\partial_3(\sigma_A),$$

where the morphisms ∂_i are defined by

$$\partial_1=\varphi_A\circ(H\otimes\sigma_A),\ \ \partial_2=\sigma_A\circ(\mu_H\otimes H),$$

$$\partial_3=\sigma_A\circ(H\otimes\mu_H),\ \ \partial_4=\sigma_A\otimes\varepsilon_H,$$

and $\wedge$ denotes the usual convolution in $Hom_{\mathcal{C}}(H\otimes H\otimes H,A)$.

A more general notion of crossed product was introduced by Brzeziński in [8], as follows: Let A be a K-algebra and V a vector space equipped with a distinguished morphism $\eta_V:K\to V$. Given maps $\psi_V^A:V\otimes A\to A\otimes V$ and $\sigma_V^A:V\otimes V\to A\otimes V$, the object $A\sharp V$, whose underlaying vector space is $A\otimes V$, endowed with the product

$$\mu_{A\sharp V}=(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)$$

is called a crossed product if it is associative with $\eta_A\otimes\eta_H$ as identity. In this case, to ensure that the product of $A\sharp V$ is associative and unitary, the morphisms ψ_V^A (the twisting morphism) and σ_V^A (the cocycle) must satisfy the following suitable conditions: The twisting morphism is compatible with the the algebra structure of A, $\psi_V^A\circ(\eta_V\otimes A)=A\otimes\eta_V$, σ_V^A is normal ($\sigma_V^A\circ(\eta_V\otimes V)=\eta_A\otimes V=\sigma_V^A\circ(V\otimes\eta_V)$), and it is a cocycle which satisfies the twisted module condition, that is:

$$(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\sigma_V^A\otimes V)=(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\sigma_V^A),$$

$$(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\sigma_V^A\otimes A)=(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\psi_V^A).$$

As a particular instance of the crossed product constructed in the last paragraph, we obtain the crossed products defined by Blattner, Cohen, Montgomery, Doi and Takeuchi. Also, the twisted tensor products or matched pairs, studied by Cap, Schichl, Vanzura and Tambara [14], [26] are examples of the Brzeziński's crossed products. On the other hand, this notion of crossed products is needed in the theory of braided Hopf crossed products developed by J.A. Gucccione and J.J. Guccione in [18], which includes the classical type (Blattner-Cohen-Montgomery, Doi-Takeuchi) and the automorphism Ore extensions type. Finally, Brzeziński's theory can be generalized, in an straightforward form, to the context of braided monoidal categories, and then, we obtain as examples, the crossed products by braided groups defined by Majid and Bespalov in [21],[6].

Unfortunately, all these constructions are not valid when we want to extend the theory of crossed products to more general Hopf structures like weak Hopf algebras or weak entwined structures. The aim of this paper, inspired by the work of Brzeziński's and by our own papers [2], [3], [4], [5], is to introduce a general notion of crossed products that includes the crossed products described in the previous paragraphs and also the new crossed products that arises in weak contexts such as weak Hopf algebras or weak entwining structures.

This paper is organized as follows: in Section 2, for an algebra A and an object V, living in a strict monoidal category with equalizers and coequalizers, we introduce the notion of crossed product system and we prove that the product induced by it is associative if satisfies the twisted and the cocycle conditions. In Section 3 we obtain that the notion of weak C-cleft extension, introduced by us in [4], provides an example of crossed product system satisfying the twisted and the cocycle conditions. As a consequence, the crossed product defined in [4] is a particular instance of the product induced by a crossed product system. This crossed products are deeply connected with Galois theory as we can see in the intrinsic characterization of weak cleftness in terms of weak C-Galois extensions obtained in [5](see also [1] for the Hopf algebra case in braided categories). In Section 4, we define the notion of crossed product system with unity and we prove the main result of this paper, that is Theorem 4.6. As a particular case of this Theorem, we obtain Brzeziński's characterization of crossed products and, of course, the classical characterizations related in the first paragraph of this Introduction. Finally, in Section 5, we apply our theory to the context of weak Hopf algebras in a symmetric monoidal category with split idempotents, obtaining that our construction is valid to develop a theory of crossed products for weak Hopf algebras. The final example of this section is especially interesting because we prove that for all morphism of weak Hopf algebras with coalgebra splitting, it is possible to obtain a crossed product system that is also an example of the cleft theory developed in Section 3.

2 Crossed product systems

Throughout the paper $\mathcal{C}$ denotes a strict monoidal category with tensor product $\otimes$ and base object K. Given objects A, B, D and a morphism $f : B \rightarrow D$, we write $A \otimes f$ for $id_A \otimes f$ and $f \otimes A$ for $f \otimes id_A$. Also we assume that $\mathcal{C}$ admits equalizers and coequalizers. It is an easy exercise to prove that, under these conditions, all idempotent splits, i.e., for every morphism $\nabla_Y : Y \rightarrow Y$, such that $\nabla_Y = \nabla_Y \circ \nabla_Y$, there exist an object Z and morphisms $i_Y : Z \rightarrow Y$ and $p_Y : Y \rightarrow Z$ satisfying $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$.

We assume that the reader is familiar with the notions of algebra, coalgebra, module and comodule. Unless otherwise explicitly established, we assume that algebras are associative with unity and the coalgebras coassociative with counity. Given an algebra A and a coalgebra C, we let $\eta_A : K \rightarrow A$, $\mu_A : A \otimes A \rightarrow A$, $\varepsilon_D : D \rightarrow K$, and $\delta_D : D \rightarrow D \otimes D$ denote the unity, the product, the counity, and the coproduct respectively. Given two algebras A and B, $f : A \rightarrow B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if $\mathcal{C}$ is braided with braiding c, given A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is also an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. If D and E are coalgebras, $f : D \rightarrow E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. If $\mathcal{C}$ is braided with braiding c, given D, E coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Definition 2.1 An algebra A and and object V together with two morphisms

$$\psi_V^A : V \otimes A \rightarrow A \otimes V, \quad \sigma_V^A : V \otimes V \rightarrow A \otimes V$$

is called a crossed product system if the following equalities hold:

(a1) $(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\psi_V^A\otimes A)=\psi_V^A\circ(V\otimes\mu_A)$,

(a2) $(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)$,

(a3) $\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\nabla_{A\otimes V}\otimes V)=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)$.

where the morphism $\nabla_{A\otimes V}:A\otimes V\to A\otimes V$, is defined by

$$\nabla_{A\otimes V}=(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(A\otimes V\otimes\eta_A).$$

In what follows we denote the crossed product systems by $(A,V,\psi_V^A,\sigma_V^A)$.

For example, if for ψ_V^A the equality $\eta_A\otimes V=\psi_V^A\circ(V\otimes\eta_A)$ holds, then $\nabla_{A\otimes V}=id_{A\otimes V}$ and therefore $(A,V,\psi_V^A,\sigma_V^A)$ is a crossed product system for all morphism $\sigma_V^A:V\otimes V\to A\otimes V$.

Remark 2.2 Note that if $(A,V,\psi_V^A,\sigma_V^A)$ is a crossed product system, the morphism $\nabla_{A\otimes V}$ is idempotent. Let $p_{A\otimes V}:A\otimes V\to A\times V$, $i_{A\otimes V}:A\times V\to A\otimes V$ be the morphisms such that $i_{A\otimes V}\circ p_{A\otimes V}=\nabla_{A\otimes V}$, $p_{A\otimes V}\circ i_{A\otimes V}=id_{A\times V}$ where $A\times V$ represents the image of the idempotent morphism $\nabla_{A\otimes V}$. Composing with $p_{A\otimes V}$ in (a2) and (a3) we obtain the following:

(a2') $p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\nabla_{A\otimes V})=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)$,

(a3') $p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\nabla_{A\otimes V}\otimes V)=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)$.

Similarly, composing with $i_{A\otimes V}$ in (a2) and (a3) we obtain

(a2") $(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes i_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes i_{A\otimes V})$,

(a3") $\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(i_{A\otimes V}\otimes V)=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(i_{A\otimes V}\otimes V)$.

Proposition 2.3 *Let $(A,V,\psi_V^A,\sigma_V^A)$ be a crossed product system. The following identities hold*

(b1) $\nabla_{A\otimes V}\circ(\eta_A\otimes V)=\psi_V^A\circ(V\otimes\eta_A)$.

(b2) $(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)$.

(b3) $\nabla_{A\otimes V}\circ\psi_V^A=\psi_V^A$.

(b4) $(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\nabla_{A\otimes V}\otimes A)=(\mu_A\otimes V)\circ(A\otimes\psi_V^A)$.

Proof. The proof is a straightforward consequence of the definition of $\nabla_{A\otimes V}$. □

Definition 2.4 We will say that a crossed product system satisfies the twisted condition if the following equality holds

$$p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\sigma_V^A\otimes A)=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\psi_V^A)$$

or, equivalently,

$$\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\sigma_V^A\otimes A)=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\psi_V^A)$$

Definition 2.5 We will say that a crossed product system satisfies the cocycle condition if the following equality holds

$$p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\sigma_V^A\otimes V)=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\sigma_V^A)$$

or, equivalently,

$$\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\sigma_V^A\otimes V)=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\sigma_V^A).$$

Definition 2.6 Let $(A,V,\psi_V^A,\sigma_V^A)$ be a crossed product system. The product

$$\mu_{A\times V}:A\times V\otimes A\times V\to A\times V,$$

defined by

$$\mu_{A\times V}=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\mu_A\otimes V)\circ(A\otimes A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V}),$$

is called the product induced by $(A,V,\psi_V^A,\sigma_V^A)$.

Proposition 2.7 *Let $(A,V,\psi_V^A,\sigma_V^A)$ be a crossed product system satisfying the twisted and the cocycle conditions. The product induced by $(A,V,\psi_V^A,\sigma_V^A)$ is associative.*

Proof. Composing $\mu_{A\times V}\circ(\mu_{A\times V}\otimes A\times V)$ with $p_{A\otimes V}\otimes p_{A\otimes V}\otimes p_{A\otimes V}$ we obtain the following:

$$\begin{aligned}
&\mu_{A\times V}\circ(\mu_{A\times V}\otimes A\times V)\circ(p_{A\otimes V}\otimes p_{A\otimes V}\otimes p_{A\otimes V})\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\mu_A\otimes V)\circ(A\otimes A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ(\nabla_{A\otimes V}\otimes A\otimes V)\circ\\
&\quad(\mu_A\otimes V\otimes A\otimes V)\circ(A\otimes\mu_A\otimes V\otimes A\otimes V)\circ(A\otimes A\otimes\sigma_V^A\otimes A\otimes V)\circ(A\otimes\psi_V^A\otimes V\otimes A\otimes V)\circ\\
&\quad(\nabla_{A\otimes V}\otimes\nabla_{A\otimes V}\otimes\nabla_{A\otimes V})\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ\\
&\quad(A\otimes[(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ(\sigma_V^A\otimes A\otimes V)])\circ\\
&\quad(\mu_A\otimes V\otimes V\otimes A\otimes V)\circ(A\otimes\psi_V^A\otimes V\otimes A\otimes V)\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ\\
&\quad(A\otimes[(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\nabla_{A\otimes V}\otimes V)\circ(\mu_A\otimes V\otimes V)\circ(A\otimes\psi_V^A\otimes V)\circ(\sigma_V^A\otimes A\otimes V)])\circ\\
&\quad(\mu_A\otimes V\otimes V\otimes A\otimes V)\circ(A\otimes\psi_V^A\otimes V\otimes A\otimes V)\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ\\
&\quad(A\otimes[(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\nabla_{A\otimes V}\otimes V)\circ(\mu_A\otimes V\otimes V)\circ(A\otimes\sigma_V^A\otimes V)\circ(\psi_V^A\otimes V\otimes V)\circ\\
&\quad(V\otimes\psi_V^A\otimes V)])\circ\\
&\quad(\mu_A\otimes V\otimes V\otimes A\otimes V)\circ(A\otimes\psi_V^A\otimes V\otimes A\otimes V)\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ
\end{aligned}$$

$$(A\otimes[(\mu_A\otimes V)\circ(A\otimes\mu_A\otimes V)\circ(A\otimes A\otimes\sigma_V^A)\circ(A\otimes\sigma_V^A\otimes V)\circ(\psi_V^A\otimes V\otimes V)\circ(V\otimes\psi_V^A\otimes V)])\circ$$

$$(\mu_A\otimes V\otimes V\otimes A\otimes V)\circ(A\otimes\psi_V^A\otimes V\otimes A\otimes V)$$

$$= p_{A\otimes V}\circ(\mu_A\otimes V)\circ$$

$$(A\otimes[(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})\circ(A\otimes\mu_A\otimes V)\circ(A\otimes A\otimes\sigma_V^A)\circ(A\otimes\sigma_V^A\otimes V)\circ(\psi_V^A\otimes V\otimes V)\circ$$

$$(V\otimes\psi_V^A\otimes V)])\circ$$

$$(\mu_A\otimes V\otimes V\otimes A\otimes V)\circ(A\otimes\psi_V^A\otimes V\otimes A\otimes V)$$

$$= p_{A\otimes V}\circ(\mu_A\otimes V)\circ$$

$$(\mu_A\otimes[\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\sigma_V^A)])\circ$$

$$(A\otimes\mu_A\otimes V\otimes V\otimes V)\circ(A\otimes A\otimes\psi_V^A\otimes V\otimes V)\circ(A\otimes\psi_V^A\otimes\psi_V^A\otimes V)$$

$$= p_{A\otimes V}\circ(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ(A\otimes V\otimes\mu_A\otimes V)\circ(A\otimes V\otimes\mu_A\otimes\sigma_V^A)\circ$$

$$(A\otimes V\otimes A\otimes\psi_V^A\otimes V)$$

$$= \mu_{A\times V}\circ(A\times V\otimes\mu_{A\times V})\circ(p_{A\otimes V}\otimes p_{A\otimes V}\otimes p_{A\otimes V}).$$

In the last computations, the first equality follows by definition, the second one by (a2) and (b4), the third one by (a3) and (b2), the fourth one by the twisted condition, the fifth one by (a3) and (b2), the sixth one by (b2), the seventh one by the cocycle condition, the eighth one by (a1) and finally, the ninth one by (a2) and (b4).

Therefore, $\mu_{A\times V}\circ(\mu_{A\times V}\otimes A\times V)=\mu_{A\times V}\circ(A\times V\otimes\mu_{A\times V})$. □

3 Weak cleft extensions and crossed product systems

Weak entwining structures have been introduced by Caenepeel and De Groot [13] as a generalization of entwining structures defined by Brzezinski and Majid [9], [10]. They introduce the so-called entwining structures, consisting of an algebra A, a coalgebra C, and an entwining $\psi : C\otimes A\rightarrow A\otimes C$ satisfying four technical conditions which have been replaced by weaker axioms in the definition of Caenepeel and De Groot. The definition in the monoidal setting is the following:

Definition 3.1 A weak entwining structure on $\mathcal{C}$ consists of a triple (A, C, ψ), where A is an algebra, C a coalgebra, and $\psi : C\otimes A\rightarrow A\otimes C$ a morphism satisfying the relations

(c1) $\psi\circ(C\otimes\mu_A)=(\mu_A\otimes C)\circ(A\otimes\psi)\circ(\psi\otimes A)$,

(c2) $(A\otimes\delta_C)\circ\psi=(\psi\otimes C)\circ(C\otimes\psi)\circ(\delta_C\otimes A)$,

(c3) $\psi\circ(C\otimes\eta_A)=(e_{RR}\otimes C)\circ\delta_C$,

(c4) $(A\otimes\varepsilon_C)\circ\psi=\mu_A\circ(e_{RR}\otimes A)$,

where $e_{RR} : C\rightarrow A$ is the morphism defined by $e_{RR}=(A\otimes\varepsilon_C)\circ\psi\circ(C\otimes\eta_A)$. The morphism ψ is called entwining.

In the definition of entwining structure the morphism $e_{RR} = \eta_A \otimes \varepsilon_C$ and, obviously, any entwining structure is a weak entwining structure. Moreover, a weak entwining structure is an entwining structure if and only if $e_{RR} = \eta_A \otimes \varepsilon_C$.

Definition 3.2 Let (A, C, ψ) be a weak entwining structure in $\mathcal{C}$. We denote by $\mathcal{M}_A^C(\psi)$ the category whose objects are triples (M, ϕ_M, ρ_M), where (M, ϕ_M) is a right A-module (i.e. $\phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A)$, $id_M = \phi_M \circ (M \otimes \eta_A)$), (M, ρ_M) is a right C-comodule (i. e. $(\rho_M \otimes C) \circ \rho_M = (M \otimes \delta_C) \circ \rho_M$, $(M \otimes \varepsilon_C) \circ \rho_M = id_M$), and

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes A).$$

The objects of $\mathcal{M}_A^C(\psi)$ will be called weak entwined modules and a morphism in $\mathcal{M}_A^C(\psi)$ is a morphism of A-modules and C-comodules. If (A, C, ψ) is an entwining structure then we find the category of entwined modules introduced by Brzeziński in [9].

3.3 We have the following (see [4]). Let (A, C, ψ) be a weak entwining structure such that there exists a coaction ρ_A satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. If for all $(M, \phi_M, \rho_M) \in \mathcal{M}_A^C(\psi)$, we denote by M_C the equalizer of ρ_M and $\zeta_M = (\phi_M \otimes C) \circ (M \otimes (\rho_A \circ \eta_A))$ and by i_C^M the injection of M_C in M, then:

- *i)* The triple $(A_C, \eta_{A_C}, \mu_{A_C})$ is an algebra in $\mathcal{C}$, where $\eta_{A_C} : K \to A_C$ and $\mu_{A_C} : A_C \otimes A_C \to A_C$ are the factorizations of η_A and $\mu_A \circ (i_C^A \otimes i_C^A)$ respectively, through the equalizer i_C^A.
- *ii)* The pair (M_C, ϕ_{M_C}) is a right A_C-module, where $\phi_{M_C} : M_C \otimes A_C \to M_C$ is the factorization of $\phi_M \circ (i_C^M \otimes i_C^A)$ through the equalizer i_C^M.

Definition 3.4 Let (A, C, ψ) be a weak entwining structure and suppose that (A, ρ_A) is a right C-comodule. By $Reg^{WR}(C, A)$ we denote the set of morphisms $h \in Hom_{\mathcal{C}}(C, A)$ such that there exists a morphism $h^{-1} \in Hom_{\mathcal{C}}(C, A)$ (the left weak inverse of h) satisfying $h^{-1} \wedge h = e_{RR}$.

Let A be an algebra and C be a coalgebra in $\mathcal{C}$. By $Reg(C, A)$ we denote the set of morphisms $h : C \to A$ such that there exists a morphism $h^{-1} : C \to A$ (the inverse of h) satisfying $h^{-1} \wedge h = h \wedge h^{-1} = \varepsilon_C \otimes \eta_A = \eta_A \circ \varepsilon_C$. Of course, if (A, C, ψ) is an entwining structure in $\mathcal{C}$ $e_{RR} = \varepsilon_C \otimes \eta_A$ and $Reg(C, A) \subset Reg^{WR}(C, A)$.

Remark 3.5 Suppose that (A, C, ψ) is a weak entwining structure such that there exists a coaction ρ_A satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Then if $h \in Hom_{\mathcal{C}}(C, A)$ is a morphism of right C-comodules $h \wedge e_{RR} = h$.

Definition 3.6 Let (A, C, ψ) be a weak entwining structure and suppose that $(A, \mu_A, \rho_A) \in \mathcal{M}_A^C(\psi)$. We will say that $A_C \hookrightarrow A$ is a weak C-cleft extension if there exists a morphism $h \in Reg^{WR}(C, A)$ of right C-comodules, called weak cleaving morphism, such that

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge h^{-1})$$

where $\zeta_A = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \eta_A))$ is the morphism defined in 3.3.

Observe that, if $A_C \hookrightarrow A$ is a weak C-cleft extension with weak cleaving morphism h, the morphism $g = e_{RR} \wedge h^{-1}$ verifies $g \wedge h = e_{RR}$, $e_{RR} \wedge g = g$ and $\psi \circ (C \otimes g) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge g)$. Then, as a consequence, we can suppose without loss of generality that $e_{RR} \wedge h^{-1} = h^{-1}$.

The definition of weak C-cleft extension was introduced in [4] and is a generalization of the one used by Brzeziński [9] (see [15], [16], [17], [22] for the classical definitions) in the context of entwined modules but changing $Reg(C,A)$ by $Reg^{WR}(C,A)$ and adding a new condition.

The explanation and the conceptual meaning of the last definition appear if we link it with Galois theory. An old result in this theory says that if $B \subset A$ is a finite Galois extension of fields with Galois group H, then A/B has a normal basis, i.e. there exists $a \in A$ such that the set $\{x.a \; ; \; x \in H\}$ is a basis for A over B. The notion of normal basis for extensions, associated to Hopf algebras in categories of modules over a commutative ring, was introduced by Kreimer and Takeuchi in [19] and in [16] Doi and Takeuchi characterized the H-Galois extensions with normal basis in terms of H-cleft extensions. Recently, in the work of Brzeziński [9] we can find a more general formulation of these last results in the context of entwining structures. In [5], we formulate the definition of weak C-Galois extension with normal basis for a weak entwining structure living in a strict monoidal category with equalizers and coequalizers and we characterize this extensions using the notion of cleftness introduced in Definition 3.6. Of course, as a particular instances, we recover the results described in this paragraph.

Remarks 3.7 *i)* Let $A_C \hookrightarrow A$ be a weak C-cleft extension with weak cleaving morphism h. Then, the entwining ψ is completely determined in the following form:

$$\psi = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A).$$

ii) Let (A, C, ψ) be an entwined structure and suppose that $(A, \mu_A, \rho_A) \in \mathcal{M}_A^C(\psi)$. If $h \in Reg(C,A)$ is a morphism of right C-comodules we have that

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ h^{-1} = \zeta_A \circ (e_{RR} \wedge h^{-1}).$$

Then, as a consequence, a C-cleft extension for an entwining structure is a weak C-cleft extension.

3.8 Let $A_C \hookrightarrow A$ be a weak C-cleft extension. The morphism

$$q_C^A = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A : A \to A$$

factors through the equalizer i_C^A (see [4]). Therefore, there exists a morphism $p_C^A : A \to A_C$ such that $i_C^A \circ p_C^A = q_C^A$.

On the other hand, the morphism $\varphi_A : C \otimes A \to A$ defined by

$$\varphi_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes A)$$

factors through the equalizer i_C^A. Moreover, if φ'_A is the factorization of φ_A, we have $\varphi'_A = p_C^A \circ \mu_A \circ (h \otimes A)$ and the morphism $\varphi_{A_C} = \varphi'_A \circ (C \otimes i_A^C) : C \otimes A_C \to A_C$ verifies $\mu_{A_C} \circ (\varphi'_A \otimes \varphi_{A_C}) \circ (C \otimes \psi \otimes A_C) \circ (\delta_C \otimes i_A^C \otimes A_C) = \varphi_{A_C} \circ (C \otimes \mu_{A_C})$ [Proposition 1.15 of [4]].

Finally, (see [Proposition 1.17 of [4]]) the morphism $\sigma_A : C \otimes C \to A$ defined by $\sigma_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes h)$ factors through the equalizer i_C^A. If σ_{A_C} is the factorization of σ_A, then $\sigma_{A_C} = p_C^A \circ \mu_A \circ (h \otimes h)$.

Lemma 3.9 *Let $A_C \hookrightarrow A$ be a weak C-cleft extension. The following identities hold*

(d1) $\mu_A \circ (A \otimes e_{RR}) \circ \rho_A = id_A$.

(d2) $\mu_A \circ (q_C^A \otimes h) \circ \rho_A = id_A$.

(d3) $\rho_A \circ \mu_A = (\mu_A \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A)$.

(d4) $\mu_A \circ (i_C^A \otimes h) = \mu_A \circ (q_C^A \otimes A) \circ (\mu_A \otimes h) \circ (i_C^A \otimes (\rho_A \circ h))$.

Proof. (d1) We have

$$\mu_A \circ (A \otimes e_{RR}) \circ \rho_A = \mu_A \circ (A \otimes A \otimes \varepsilon_C) \circ (A \otimes \psi) \circ (\rho_A \otimes \eta_A) =$$

$$(A \otimes \varepsilon_C) \circ \rho_A \circ \mu_A \circ (A \otimes \eta_A) = id_A.$$

(d2) This equality follows from (d1). Indeed:

$$\mu_A \circ (q_C^A \otimes h) \circ \rho_A = \mu_A \circ ((\mu_A \circ (A \otimes h^{-1}) \circ \rho_A) \otimes h) \circ \rho_A =$$

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \rho_A = \mu_A \circ (A \otimes e_{RR}) \circ \rho_A = id_A.$$

(d3) Using the condition of weak entwined module for A and (d2) we have

$$\begin{aligned}
&(\mu_A \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A) \\
&= (\mu_A \otimes C) \circ ((\mu_A \circ (q_C^A \otimes h) \circ \rho_A) \otimes \psi) \circ (\rho_A \otimes A) \\
&= (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\rho_A \otimes A) \\
&= \rho_A \circ \mu_A.
\end{aligned}$$

(d4) This equality is a consequence of the following computations:

$$\begin{aligned}
&\mu_A \circ (q_C^A \otimes A) \circ (\mu_A \otimes h) \circ (i_C^A \otimes (\rho_A \circ h)) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (\mu_A \otimes h^{-1} \otimes A) \circ (A \otimes \psi \otimes A) \circ ((\rho_A \circ i_C^A) \otimes ((A \otimes h) \circ \rho_A \circ h)) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (\mu_A \otimes h^{-1} \otimes h) \circ (\mu_A \otimes \psi \otimes A) \circ (i_C^A \otimes (\rho_A \circ \eta_A) \otimes (\rho_A \circ h)) \\
&= \mu_A \circ (\mu_A \otimes (\mu_A \circ (h^{-1} \otimes h))) \circ (A \otimes (\rho_A \circ \mu_A) \otimes C) \circ (i_C^A \otimes \eta_A \otimes (\rho_A \circ h)) \\
&= \mu_A \circ (i_C^A \otimes (\mu_A \circ (A \otimes e_{RR}) \circ \rho_A \circ h)) \\
&= \mu_A \circ (i_C^A \otimes h).
\end{aligned}$$

In the previous series of equalities, the first an the third ones follow from the weak entwined module condition for A. In the second one we used the definition of i_C^A and in the fourth one we applied the right C-comodule condition for h. Finally, the fifth one follows by (d1). □

3.10 Let $A_C \hookrightarrow A$ be a weak C-cleft extension with weak cleaving morphism h. The left A_C-module and right C-comodule ($\varphi_{A_C \otimes C} = \mu_{A_C} \otimes C$, $\rho_{A_C \otimes C} = A_C \otimes \delta_C$) morphisms

$$\omega_A : A_C \otimes C \to A, \qquad \omega'_A : A \to A_C \otimes C,$$

defined by $\omega_A = \mu_A \circ (i_C^A \otimes h)$ and $\omega'_A = (p_C^A \otimes C) \circ \rho_A$ satisfy the equality $\omega_A \circ \omega'_A = id_A$ because $\omega_A \circ \omega'_A = \mu_A \circ (A \otimes e_{RR}) \circ \rho_A = id_A$. As a consequence, the morphism $\Omega_A = \omega'_A \circ \omega_A$ is an idempotent morphism and we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[dr, "\omega'_A"] & \\
A_C\otimes C \arrow[ur, "\omega_A"] \arrow[rr, "\Omega_A"] \arrow[dr, "r_A"'] & & A_C\otimes C \\
 & A_C\times C \arrow[ur, "s_A"'] &
\end{tikzcd}$$

where $r_A\circ s_A = id_{A_C\times C}$. Therefore, the morphism $b_A = r_A\circ\omega'_A$ is an isomorphism of right C-comodules and left A_C-modules with inverse $b_A^{-1} = \omega_A\circ s_A$. The module and comodule structures of $A_C\times C$ are the ones induced by the isomorphism b_A and they are equal to

$$\varphi_{A_C\times C} = r_A\circ(\mu_{A_C}\otimes C)\circ(A_C\otimes s_A), \qquad \rho_{A_C\times C} = (r_A\otimes C)\circ(A_C\otimes\delta_C)\circ s_A,$$

respectively.

Also, b_A is an isomorphism of algebras with

$$\eta_{A_C\times C} = b_A\circ\eta_A, \quad \mu_{A_C\times C} = b_A\circ\mu_A\circ(b_A^{-1}\otimes b_A^{-1}).$$

Under these conditions, the product $\mu_{A_C\times C}$ can be identified in the following way (see [4])

$$\mu_{A_C\times C} = r_A\circ(\mu_{A_C}\otimes C)\circ(\mu_{A_C}\otimes\sigma_C^{A_C})\circ(A_C\otimes\psi_C^{A_C}\otimes C)\circ(s_A\otimes s_A)$$

where

$$\psi_C^{A_C} = (\varphi'_A\otimes C)\circ(C\otimes\psi)\circ(\delta_C\otimes i_C^A) = (p_C^A\otimes C)\circ\rho_A\circ\mu_A\circ(h\otimes i_C^A) = \omega'_A\circ\mu_A\circ(h\otimes i_C^A),$$

$$\sigma_C^{A_C} = (\varphi'_A\otimes C)\circ(C\otimes\psi)\circ(\delta_C\otimes h) = (p_C^A\otimes C)\circ\rho_A\circ\mu_A\circ(h\otimes h) = \omega'_A\circ\mu_A\circ(h\otimes h).$$

The product described in the previous paragraph is an example of product associated to a crossed product system. For to prove this assertion we need previously the following technical lemma.

Lemma 3.11 *Let $A_C\hookrightarrow A$ be a weak C-cleft extension with weak cleaving morphism h. The following equalities hold*

(e1) $p_C^A\circ\mu_A\circ(i_C^A\otimes A) = \mu_{A_C}\circ(A_C\otimes p_C^A)$.

(e2) $\Omega_A = (\mu_{A_C}\otimes C)\circ(A\otimes\psi_C^{A_C})\circ(A_C\otimes C\otimes\eta_{A_C})$.

where p_C^A is the morphism introduced in 3.8 and $\psi_C^{A_C}$, Ω_A are the morphisms defined in 3.10.

Proof. (e1) Composing with i_C^A we obtain

$$\begin{aligned}
&i_C^A\circ p_C^A\circ\mu_A\circ(i_C^A\otimes A)\\
&= \mu_A\circ(\mu_A\otimes h^{-1})\circ(A\otimes\psi)\circ((\rho_A\circ i_C^A)\otimes A)\\
&= \mu_A\circ(\mu_A\otimes h^{-1})\circ(\mu_A\otimes\psi)\circ(i_C^A\otimes(\rho_A\circ\eta_A)\otimes A)
\end{aligned}$$

$$
\begin{aligned}
&= \mu_A \circ (A \otimes [\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \psi) \circ ((\rho_A \circ \eta_A) \otimes A)]) \circ (i_C^A \otimes A) \\
&= \mu_A \circ (i_C^A \otimes q_C^A) \\
&= i_C^A \circ \mu_{A_C} \circ (A_C \otimes p_C^A).
\end{aligned}
$$

In the previous equalities, the second follows by definition of i_C^A, the third one by associativity of μ_A and the fifth one by definition of μ_{A_C}. In the first and in the fourth ones we used the weak entwining module condition for A.

Therefore, $p_C^A \circ \mu_A \circ (i_C^A \otimes A) = \mu_{A_C} \circ (A_C \otimes p_C^A)$.

(e2) We have

$$
\begin{aligned}
&\Omega_A \\
&= ((p_C^A \circ \mu_A) \otimes C) \circ (A \otimes \psi) \circ ((\rho_A \circ i_C^A) \otimes h) \\
&= ((p_C^A \circ \mu_A) \otimes C) \circ (\mu_A \otimes \psi) \circ (i_C^A \otimes (\rho_A \circ \eta_A) \otimes h) \\
&= ((p_C^A \circ \mu_A) \otimes C) \circ (i_C^A \otimes [\rho_A \circ \mu_A \circ (\eta_A \otimes h)]) \\
&= ((\mu_{A_C} \circ (A_C \otimes p_C^A)) \otimes C) \circ (A_C \otimes (\rho_A \circ h)) \\
&= ((\mu_{A_C} \circ (A_C \otimes p_C^A)) \otimes C) \circ (A_C \otimes [\rho_A \circ \mu_A \circ (h \otimes \eta_A)]) \\
&= (\mu_{A_C} \otimes C) \circ (A_C \otimes [((p_C^A \circ \mu_A) \otimes C) \circ (A \otimes \psi) \otimes ((\rho_A \circ h) \otimes A)]) \circ (A_C \otimes C \otimes \eta_A) \\
&= (\mu_{A_C} \otimes C) \circ (A \otimes \psi_C^{A_C}) \circ (A_C \otimes C \otimes \eta_{A_C}).
\end{aligned}
$$

In these equalities we applied the properties used in the proof of (e1). Observe that the fourth equality follows by (e1). □

Remark 3.12 *i)* Lemma 3.11 implies that $\Omega_A = \nabla_{A_C \otimes C}$ and then the object $A_C \times C$ obtained in 3.10 using Ω_A is an example of the one defined in section 1.

ii) Note that the following equality

$$(\text{e3}) \quad q_C^A \circ \mu_A \circ (q_C^A \otimes A) = \mu_A \circ (q_C^A \otimes q_C^A)$$

is a consequence of (e1).

Proposition 3.13 *Let $A_C \hookrightarrow A$ be a weak C-cleft extension. If $\psi_C^{A_C}$, $\sigma_C^{A_C}$ are the morphisms defined in 3.10, $(A_C, C, \psi_C^{A_C}, \sigma_C^{A_C})$ is a crossed product system satisfying the cocycle and twisted conditions. As a consequence, the product $\mu_{A_C \times C}$ defined in 3.10 is the one induced by the crossed product system $(A_C, C, \psi_C^{A_C}, \sigma_C^{A_C})$.*

Proof. Let $\psi_C^{A_C}$, $\sigma_C^{A_C}$ are the morphisms defined in 3.10. Then

$$
\begin{aligned}
&(\mu_{A_C} \otimes C) \circ (A_C \otimes \psi_C^{A_C}) \circ (\psi_C^{A_C} \otimes A_C) \\
&= (p_C^A \otimes C) \circ (\mu_A \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A) \circ ((\mu_A \circ (h \otimes i_C^A)) \otimes i_C^A)
\end{aligned}
$$

$$
\begin{aligned}
&= (p_C^A \otimes C) \circ \rho_A \circ \mu_A \circ ((\mu_A \circ (h \otimes i_C^A)) \otimes i_C^A) \\
&= (p_C^A \otimes C) \circ \rho_A \circ \mu_A \circ (h \otimes (\mu_A \circ (i_C^A \otimes i_C^A))) \\
&= \psi_C^{A_C} \circ (C \otimes \mu_{A_C}).
\end{aligned}
$$

The first equality follows by (e1), the second one by (d3), the third one by the associativity on μ_A and the fourth one by the definition of μ_{A_C}.

On the other hand,

$$
\begin{aligned}
& (\mu_{A_C} \otimes C) \circ (A_C \otimes \sigma_C^{A_C}) \circ (\psi_C^{A_C} \otimes C) \circ (C \otimes \nabla_{A_C \otimes C}) \\
&= (p_C^A \otimes C) \circ (\mu_A \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A) \circ (\mu_A \otimes h) \circ (A \otimes (q_C^A \circ \mu_A) \otimes C) \circ \\
& \quad (h \otimes i_C^A \otimes (\rho_A \circ h)) \\
&= (p_C^A \otimes C) \circ \rho_A \circ \mu_A \circ (h \otimes [\mu_A \circ (q_C^A \otimes h) \circ (\mu_A \otimes C) \circ (i_C^A \otimes (\rho_A \circ h))]) \\
&= (p_C^A \otimes C) \circ \rho_A \circ \mu_A \circ (h \otimes [\mu_A \circ (i_C^A \otimes h)]) \\
&= (p_C^A \otimes C) \circ \rho_A \circ \mu_A \circ ((\mu_A \circ (h \otimes i_C^A)) \otimes h) \\
&= (p_C^A \otimes C) \circ (\mu_A \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A) \circ ((\mu_A \circ (h \otimes i_C^A)) \otimes h) \\
&= (p_C^A \otimes C) \circ ((\mu_A \circ (A \otimes (\mu_A \circ (q_C^A \otimes h) \circ \rho_A))) \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A) \circ \\
& \quad ((\mu_A \circ (h \otimes i_C^A)) \otimes h) \\
&= ((p_C^A \circ \mu_A \circ ((q_C^A \circ \mu_A) \otimes h) \circ (A \otimes \rho_A)) \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((q_C^A \otimes h) \circ (\rho_A \circ \mu_A \circ (h \otimes i_C^A)) \otimes h) \\
&= \nabla_{A_C \otimes C} \circ (\mu_{A_C} \otimes C) \circ (A_C \otimes \sigma_C^{A_C}) \circ (\psi_C^{A_C} \otimes C).
\end{aligned}
$$

Here we used (e1) in the first equality. The second one follows by (d3), the third one by (d4) and the fourth one by the associativity of μ_A. In the fifth one we used (d3) and in the sixth one (d2). The seventh one is a consequence of (e3) and the eighth one follows by (e1).

Finally,

$$
\begin{aligned}
& (\mu_{A_C} \otimes C) \circ (A_C \otimes \sigma_C^{A_C}) \circ (\nabla_{A_C \otimes C} \otimes C) \\
&= ((p_C^A \circ \mu_A) \otimes C) \circ (i_C^A \otimes ((\mu_A \otimes C) \circ (q_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes A))) \circ (\rho_A \otimes A) \circ (h \otimes h))) \\
&= ((p_C^A \circ \mu_A) \otimes C) \circ (i_C^A \otimes (\rho_A \circ \mu_A \circ (h \otimes h))) \\
&= ((p_C^A \circ \mu_A) \otimes C) \circ (i_C^A \otimes (((\mu_A \circ (q_C^A \otimes h) \circ \rho_A) \otimes C) \circ \rho_A \circ \mu_A \circ (h \otimes h))) \\
&= (\mu_{A_C} \otimes C) \circ (A_C \otimes ((p_C^A \otimes C) \circ \rho_A \circ h)) \circ (\mu_{A_C} \otimes C) \circ (A_C \otimes p_C^A \otimes C) \circ (A_C \otimes (\rho_A \circ \mu_A \circ (h \otimes h))) \\
&= \nabla_{A_C \otimes C} \circ (\mu_{A_C} \otimes C) \circ (A_C \otimes \sigma_C^{A_C}).
\end{aligned}
$$

In the last computations, the first equality follows by (e1), the second one by (d3) and the third one by (d2). In the fourth and the fifth ones we used the weak entwined module condition for A and (e1).

Therefore,

$$\nabla_{A_C\otimes C}\circ(\mu_{A_C}\otimes C)\circ(A_C\otimes\sigma_C^{A_C})\circ(\nabla_{A_C\otimes C}\otimes C)=\nabla_{A_C\otimes C}\circ(\mu_{A_C}\otimes C)\circ(A_C\otimes\sigma_C^{A_C})$$

and then, $(A_C, C, \psi_C^{A_C}, \sigma_C^{A_C})$ is a crossed product system.

On the other hand, $(A_C, C, \psi_C^{A_C}, \sigma_C^{A_C})$ satisfies the cocycle condition because:

$$\begin{aligned}
&\nabla_{A_C\otimes C}\circ(\mu_A\otimes C)\circ(A\otimes\sigma_C^{A_C})\circ(\sigma_C^{A_C}\otimes C)\\
&=\omega'_A\circ\mu_A\circ((i_C^A\circ\mu_{A_C}\circ(p_C^A\otimes p_C^A))\otimes h)\circ(A\otimes(\rho_A\circ\mu_A))\circ(A\otimes h\otimes A)\circ((\rho_A\circ\mu_A)\otimes A)\circ(h\otimes h\otimes h)\\
&=\omega'_A\circ\mu_A\circ(q_C^A\otimes h)\circ((\mu_A\otimes C)\circ(q_C^A\otimes(\rho_A\circ\mu_A))\circ(A\otimes h\otimes A)\circ(\rho_A\otimes A))\circ(\mu_A\otimes A)\circ(h\otimes h\otimes h)\\
&=\omega'_A\circ\mu_A\circ(q_C^A\otimes h)\circ\rho_A\circ\mu_A\circ(\mu_A\otimes A)\circ(h\otimes h\otimes h)\\
&=\omega'_A\circ\mu_A\circ(\mu_A\otimes A)\circ(h\otimes h\otimes h)\\
&=\omega'_A\circ\mu_A\circ(q_C^A\otimes h)\circ\rho_A\circ\mu_A\circ(A\otimes(\mu_A\circ(q_C^A\otimes h)\circ\rho_A))\circ(A\otimes\mu_A)\circ(h\otimes h\otimes h)\\
&=\omega'_A\circ\mu_A\circ(q_C^A\otimes h)\circ((\mu_A\otimes C)\circ(q_C^A\otimes(\rho_A\circ\mu_A))\circ(A\otimes h\otimes A)\circ(\rho_A\otimes A))\circ(\mu_A\otimes h)\circ(A\otimes q_C^A\otimes C)\circ\\
&\quad(A\otimes(\rho_A\circ\mu_A))\circ(h\otimes h\otimes h)\\
&=\omega'_A\circ\mu_A\circ((i_C^A\circ\mu_A)\otimes h)\circ(p_C^A\otimes p_C^A\otimes C)\circ(A\otimes(\rho_A\circ(\mu_A\circ(h\otimes A))))\circ((\rho_A\circ\mu_A)\otimes h)\circ\\
&\quad(A\otimes(i_C^A\circ p_C^A)\otimes C)\circ(A\otimes(\rho_A\circ\mu_A))\circ(h\otimes h\otimes h)\\
&=\nabla_{A_C\otimes C}\circ(\mu_A\otimes C)\circ(A\otimes\sigma_C^{A_C})\circ(\psi_C^{A_C}\otimes C)\circ(A\otimes\sigma_C^{A_C}).
\end{aligned}$$

In the last computations, the first and the eighth equalities follows by definition. The second and the seventh ones follows by (e1) and the third and the sixth ones by (d3). In the fourth and fifth ones we used (d2).

Finally, the proof for the twisted condition is similar to the one used for the cocycle property but changing the morphism $h\otimes h\otimes h$ by $h\otimes h\otimes i_C^A$. □

4 Crossed product systems with unity

Definition 4.1 Let $(A, V, \psi_V^A, \sigma_V^A)$ be a crossed product system. If there exists a morphism $\eta_V : K\to V$ satisfying

$$\nabla_{A\otimes V}\circ(A\otimes\eta_V)=\psi_V^A\circ(\eta_V\otimes A)$$

the crossed product system is called a crossed product system with unity.

Definition 4.2 We will say that a crossed product system with unity $(A, V, \psi_V^A, \sigma_V^A)$ is normal if the following equalities hold

$$p_{A\otimes V}\circ\sigma_V^A\circ(\eta_V\otimes V)=p_{A\otimes V}\circ(\eta_A\otimes V)=p_{A\otimes V}\circ\sigma_V^A\circ(V\otimes\eta_V)$$

or, equivalently,

$$\nabla_{A\otimes V}\circ\sigma_V^A\circ(\eta_V\otimes V)=\nabla_{A\otimes V}\circ(\eta_A\otimes V)=\nabla_{A\otimes V}\circ\sigma_V^A\circ(V\otimes\eta_V)$$

Proposition 4.3 *A crossed product system with unity is normal if and only if*

$$p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes\eta_V\otimes V)=p_{A\otimes V}=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes V\otimes\eta_V)$$

Proof. Suppose that $(A,V,\psi_V^A,\sigma_V^A)$ is a normal crossed product system. Using (a3), (b2) and the normality condition we have the following:

$$\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes\eta_V\otimes V)=(\mu_A\otimes V)\circ(A\otimes(\nabla_{A\otimes V}\circ\sigma_V^A\circ(\eta_V\otimes V))=$$

$$\mu_A\circ(A\otimes(\nabla_{A\otimes V}\circ(\eta_A\otimes V))=\nabla_{A\otimes V},$$

$$\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes V\otimes\eta_V)=(\mu_A\otimes V)\circ(A\otimes(\nabla_{A\otimes V}\circ\sigma_V^A\circ(V\otimes\eta_V))=$$

$$\mu_A\circ(A\otimes(\nabla_{A\otimes V}\circ(\eta_A\otimes V))=\nabla_{A\otimes V}.$$

Therefore,

$$p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes\eta_V\otimes V)=p_{A\otimes V}=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes V\otimes\eta_V).$$

Conversely, composing in $p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes\eta_V\otimes V)=p_{A\otimes V}$ with $\eta_A\otimes V$ we have

$$p_{A\otimes V}\circ\sigma_V^A\circ(\eta_V\otimes V)=p_{A\otimes V}\circ(\eta_A\otimes V).$$

Finally, if we compose in $p_{A\otimes V}=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes V\otimes\eta_V)$ with $\eta_A\otimes V$, using (b2) we obtain

$$p_{A\otimes V}\circ(\eta_A\otimes V)=p_{A\otimes V}\circ\sigma_V^A\circ(V\otimes\eta_V).\quad\square$$

Remark 4.4 Proposition 4.3 shows that, for all normal crossed product system with unity, we have the following equalities:

$$\nabla_{A\otimes V}=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes(\sigma_V^A\circ(\eta_V\otimes V)))=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes(\sigma_V^A\circ(V\otimes\eta_V))).$$

Example 4.5 Let $A_C\hookrightarrow A$ be a weak C-cleft extension with weak cleaving morphism h. Suppose that there exists a morphism $\eta_C:K\to C$ verifying the equality $\eta_A=h\circ\eta_C$. In these conditions, the crossed product system $(A_C,C,\psi_C^{A_C},\sigma_C^{A_C})$, obtained in 3.13, has unity η_C and satisfies the normal condition.

Theorem 4.6 *Let A be an algebra, V an object and $\eta_V:K\to V$ a morphism. The following are equivalent:*

- *i) There exists an idempotent morphism $\nabla_{A\otimes V}:A\otimes V\to A\otimes V$, with image $A\times V$ and factorization $\nabla_{A\otimes V}=i_{A\otimes V}\circ p_{A\otimes V}$, satisfying $(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)$, the object $A\times V$ is an algebra with unit $\eta_{A\times V}=p_{A\otimes V}\circ(\eta_A\otimes\eta_V)$, and the product $\mu_{A\times V}$ such that*

$$\text{(f1)}\quad i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(A\otimes\eta_V))\otimes p_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V).$$

- *ii) There exist morphisms $\psi_V^A:V\otimes A\to A\otimes V$, $\sigma_V^A:V\otimes V\to A\otimes V$ such that $(A,V,\psi_V^A,\sigma_V^A)$ is a normal crossed product system with unity which satisfies the twisted and cocycle conditions.*

Proof. *ii)* $\Longrightarrow$ *i)* Let $\psi_V^A : V\otimes A\to A\otimes V$, $\sigma_V^A : V\otimes V\to A\otimes V$ be morphisms such that $(A, V, \psi_V^A, \sigma_V^A)$ is a normal crossed product system with unity which satisfies the twisted and the cocycle conditions. The morphism $\nabla_{A\otimes V} : A\otimes V\to A\otimes V$, defined by

$$\nabla_{A\otimes V} = (\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(A\otimes V\otimes\eta_A),$$

is idempotent and the equality $(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V}) = \nabla_{A\otimes V}\circ(\mu_A\otimes V)$ holds (see (b2)).

If $\mu_{A\times V}$ is the induced product, we have (f1). Indeed:

$$\begin{aligned}
&i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(A\otimes\eta_V))\otimes p_{A\otimes V})\\
&=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)))\circ((\nabla_{A\otimes V}\circ(A\otimes\eta_V))\otimes\nabla_{A\otimes V})\\
&=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)))\circ(A\otimes\eta_V\otimes A\otimes V)\\
&=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes\eta_V\otimes V)))\\
&=\nabla_{A\otimes V}\circ(\mu_A\otimes V).
\end{aligned}$$

In these computations, the first equality follows by definition of the induced product, the second one by (b4) and (a2), the third one by the unity condition and (a3). The last equality is obtained using the normality condition and (b2).

On the other hand, by 2.7, we obtain that $\mu_{A\times V}$ is associative. Then, for to finish the first part of this proof we only need to show that

$$\mu_{A\times V}\circ(A\times V\otimes\eta_{A\times V}) = id_{A\times V} = \mu_{A\times V}\circ(\eta_{A\times V}\otimes A\times V).$$

Indeed, composing the morphism $\mu_{A\times V}\circ(A\times V\otimes\eta_{A\times V})$ with $p_{A\otimes V}$ we obtain the following:

$$\begin{aligned}
&\mu_{A\times V}\circ(p_{A\otimes V}\otimes\eta_{A\times V})\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ(\nabla_{A\otimes V}\otimes(\nabla_{A\otimes V}\circ(\eta_A\otimes\eta_V))\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes V\otimes\eta_V)\\
&=p_{A\otimes V}.
\end{aligned}$$

The first equality follows by definition of the induced product, the second one by (b4) and (a3). The last equality is obtained using the normality condition.

Also, if we compute $\mu_{A\times V}\circ(\eta_{A\times V}\otimes p_{A\otimes V})$ we have

$$\begin{aligned}
&\mu_{A\times V}\circ(\eta_{A\times V}\otimes p_{A\otimes V})\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ((\nabla_{A\otimes V}\circ(\eta_A\otimes\eta_V)\otimes\nabla_{A\otimes V})\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(\eta_V\otimes A\otimes V)\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ((\nabla_{A\otimes V}\circ(A\otimes\eta_V))\otimes V)\\
&=p_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(A\otimes\eta_V\otimes V)\\
&=p_{A\otimes V}.
\end{aligned}$$

The first equality follows from the definition of the induced product, the second one by (a2)and (b4), the third one by the unity condition $\nabla_{A\otimes V}\circ(A\otimes\eta_V)=\psi_V^A\circ(\eta_V\otimes A)$, the fourth one by (a3). Finally, in the last equality, we used the normality of the crossed product system.

Therefore, we obtain $\mu_{A\times V}\circ(A\times V\otimes\eta_{A\times V})=id_{A\times V}=\mu_{A\times V}\circ(\eta_{A\times V}\otimes A\times V)$, and then $A\times V$ is an algebra.

i) $\Longrightarrow$ *ii)* Assume that there exists an idempotent morphism $\nabla_{A\otimes V}:A\otimes V\to A\otimes V$, with image $A\times V$ and factorization $\nabla_{A\otimes V}=i_{A\otimes V}\circ p_{A\otimes V}$, satisfying $(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)$, the object $A\times V$ is an algebra with unit $\eta_{A\times V}=p_{A\otimes V}\circ(\eta_A\otimes\eta_V)$ and such that (f1) holds. Define morphisms

$$\psi_V^A:V\otimes A\to A\otimes V,\quad \psi_V^A=i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes(p_{A\otimes V}\circ(A\otimes\eta_V))),$$

$$\sigma_V^A:V\otimes V\to A\otimes V,\quad \sigma_V^A=i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes(p_{A\otimes V}\circ(\eta_A\otimes V))).$$

Using the equality $p_{A\otimes V}\circ i_{A\otimes V}=id_{A\otimes V}$, it is immediate to show that these morphisms satisfy $\nabla_{A\otimes V}\circ\psi_V^A=\psi_V^A$, and $\nabla_{A\otimes V}\circ\sigma_V^A=\sigma_V^A$. Also, the equality $p_{A\otimes V}\circ i_{A\otimes V}=id_{A\otimes V}$ and (f1) implies that

$$\text{(f2)}\quad i_{A\otimes V}\circ\mu_{A\times V}\circ(p_{A\otimes V}\otimes(p_{A\otimes V}\circ(A\otimes\eta_V)))=(\mu_A\otimes V)\circ(A\otimes\psi_V^A),$$

$$\text{(f3)}\quad i_{A\otimes V}\circ\mu_{A\times V}\circ(p_{A\otimes V}\otimes(p_{A\otimes V}\circ(\eta_A\otimes V)))=(\mu_A\otimes V)\circ(A\otimes\sigma_V^A).$$

Moreover, by $(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)$ we have

$$\nabla_{A\otimes V}=(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(A\otimes V\otimes\eta_A)$$

and trivially $\psi_V^A\circ(\eta_V\otimes A)=\nabla_{A\otimes V}\circ(A\otimes\eta_V)$, or equivalently, $(A,V,\psi_V^A,\sigma_V^A)$ has unity.

It remains to check that $(A,V,\psi_V^A,\sigma_V^A)$ is a normal crossed product system which satisfy the twisted and cocycle conditions.

To prove (a1) compute

$$\begin{aligned}
&(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\psi_V^A\otimes A)\\
&=i_{A\otimes V}\circ\mu_{A\times V}\circ(\mu_{A\times V}\otimes A\otimes V)\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes(p_{A\otimes V}\circ(A\otimes\eta_V))\otimes(p_{A\otimes V}\circ(A\otimes\eta_V)))\\
&=i_{A\otimes V}\circ\mu_{A\times V}\circ(A\otimes V\otimes\mu_{A\times V})\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes(p_{A\otimes V}\circ(A\otimes\eta_V))\otimes(p_{A\otimes V}\circ(A\otimes\eta_V)))\\
&=i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes(p_{A\otimes V}\circ(\mu_A\otimes\eta_V)))\\
&=\psi_V^A\circ(V\otimes\mu_A).
\end{aligned}$$

The first equality follows from $p_{A\otimes V}\circ i_{A\otimes V}=id_{A\otimes V}$, the second one by the associativity of $\mu_{A\times V}$, the third one by (f1) and finally the fourth one by definition.

Using the same arguments and the equalities $(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)$, $\nabla_{A\otimes V}\circ\sigma_V^A=\sigma_V^A$, we obtain the proof for (a2) and (a3). Indeed, we have

$$\begin{aligned}
&(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\nabla_{A\otimes V})\\
&=i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes[\mu_{A\times V}\circ((p_{A\otimes V}\circ(A\otimes\eta_V))\otimes(p_{A\otimes V}\circ(\eta_A\otimes V)))\circ\nabla_{A\otimes V}])\\
&=i_{A\otimes V}\circ\mu_{A\times V}\circ((p_{A\otimes V}\circ(\eta_A\otimes V))\otimes p_{A\otimes V})
\end{aligned}$$

$$= i_{A\otimes V} \circ \mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes [\mu_{A\times V} \circ ((p_{A\otimes V} \circ (A \otimes \eta_V)) \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))])$$
$$= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V),$$

and

$$\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nabla_{A\otimes V} \otimes V)$$
$$= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nabla_{A\otimes V} \otimes V)$$
$$= i_{A\otimes V} \circ \mu_{A\times V} \circ (p_{A\otimes V} \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))$$
$$= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A)$$
$$= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A).$$

On the other hand, the equalities $\mu_{A\times V} \circ (A\times V \otimes \eta_{A\times V}) = id_{A\times V} = \mu_{A\times V} \circ (\eta_{A\times V} \otimes A\times V)$, imply
$$p_{A\otimes V} \circ \sigma_V^A \circ (\eta_V \otimes V) = p_{A\otimes V} \circ (\eta_A \otimes V) = p_{A\otimes V} \circ \sigma_V^A \circ (V \otimes \eta_V)$$
and then the crossed product system with unity $(A, V, \psi_V^A, \sigma_V^A)$ is normal.

To prove the twisted condition compute

$$\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A)$$
$$= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A)$$
$$= i_{A\otimes V} \circ \mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes [\mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes (p_{A\otimes V} \circ (A \otimes \eta_V)))])$$
$$= i_{A\otimes V} \circ \mu_{A\times V} \circ ([\mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))] \otimes (p_{A\otimes V} \circ (A \otimes \eta_V)))$$
$$= (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A)$$
$$= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A).$$

The first equality follows by $(\mu_A \otimes V) \circ (A \otimes \nabla_{A\otimes V}) = \nabla_{A\otimes V} \circ (\mu_A \otimes V)$ and $\nabla_{A\otimes V} \circ \sigma_V^A = \sigma_V^A$, the second one by $p_{A\otimes V} \circ i_{A\otimes V} = id_{A\otimes V}$, (f3) and (f1). The third one follows by the associativity of $\mu_{A\times V}$ and the fourth one by (f2) and (f1). The last one follows by $(\mu_A \otimes V) \circ (A \otimes \nabla_{A\otimes V}) = \nabla_{A\otimes V} \circ (\mu_A \otimes V)$ and $\nabla_{A\otimes V} \circ \psi_V^A = \psi_V^A$.

Finally, one verifies the cocycle condition by the same arguments used in the proof of the twisted condition. Indeed:

$$\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A)$$
$$= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A)$$
$$= i_{A\otimes V} \circ \mu_{A\times V} \circ ([\mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))] \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))$$
$$= i_{A\otimes V} \circ \mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes [\mu_{A\times V} \circ ((p_{A\otimes V} \circ (\eta_A \otimes V)) \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))])$$
$$= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V)$$
$$= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V). \ \square$$

Remarks 4.7 *i)* Note that if there exists an idempotent morphism $\nabla_{A\otimes V} : A \otimes V \to A \otimes V$, with image $A \times V$ and factorization $\nabla_{A\otimes V} = i_{A\otimes V} \circ p_{A\otimes V}$, satisfying $(\mu_A \otimes V) \circ (A \otimes \nabla_{A\otimes V}) = \nabla_{A\otimes V} \circ (\mu_A \otimes V)$ and the object $A \times V$ is an algebra with unit $\eta_{A\times V} = p_{A\otimes V} \circ (\eta_A \otimes \eta_V)$ and such that (f1) holds, the product $\mu_{A\times V}$ is the one induced by the crossed product system $(A, V, \psi_V^A, \sigma_V^A)$, where ψ_V^A and σ_V^A are the morphism defined in the proof of the last theorem. Using the usual arguments, one can verify this assertion computing:

$$\begin{aligned}
&\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \mu_A \otimes V) \circ (A \otimes A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \circ (\nabla_{A\otimes V} \otimes \nabla_{A\otimes V}) \\
&= (\mu_A \otimes V) \circ (\mu_A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \\
&= i_{A\otimes V} \circ \mu_{A\times V} \circ (p_{A\otimes V} \otimes [\mu_{A\times V} \circ ((p_{A\otimes V} \circ (A \otimes \eta_V)) \otimes (p_{A\otimes V} \circ (\eta_A \otimes V)))]) \\
&= i_{A\otimes V} \circ \mu_{A\times V} \circ (p_{A\otimes V} \otimes p_{A\otimes V})
\end{aligned}$$

Therefore,

$$\mu_{A\times V} = p_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \mu_A \otimes V) \circ (A \otimes A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \circ (i_{A\otimes V} \otimes i_{A\otimes V}).$$

ii) If $\mathcal{C}$ is the category of vector spaces over a field K and $\nabla_{A\otimes V} = id_{A\otimes V}$, Theorem 4.6 is the result proved by Brzeziński in [8]. In this situation $A \times V = A \otimes V$ and Brzeziński's Proposition describe conditions which allow to built an algebra structure on a tensor product of an algebra A and a vector space V.

iii) The referee has pointed out the paper of Wisbauer [27], where the author proves a result such that with a weak modification in the conditions has a strong similarity with Theorem 4.6.

5 Weak Hopf algebras and crossed product systems

Weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [23]) are generalizations of Hopf algebras that were defined by Böhm, Nill and Szlachányi in [11], [12]. The axioms are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. The main motivation for studying weak Hopf algebras comes from quantum field theory and operator algebras.

Let $\mathcal{C}$ be a strict symmetric monoidal category with split idempotents. Bellow we collect the definition an basic properties of weak Hopf algebras.

Definition 5.1 A weak Hopf algebra H in $\mathcal{C}$ is by definition an algebra (H, η_H, μ_H) and coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(g1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

(g2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(g3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$

$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(g4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called antipode of H) satisfying:

(g4-1) $\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.

(g4-2) $\mu_H \circ (\lambda_H \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

(g4-3) $\mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \circ \delta_H = \lambda_H$.

Axioms (g2) and (g3) above are the weaker version to the usual bialgebra axioms of δ_H being a unit preserving map and ε_H being an algebra homomorphism. Axioms (g4-1), (g4-2) and (g4-3) generalize the properties of the antipode in a Hopf algebra with respect to the counit ε_H. Observe that in the definition of Hopf algebra, (g2-g4) are replaced by the conditions

(g2') $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$,

(g3') $\delta_H \circ \eta_H = \eta_H \otimes \eta_H$,

(g4') There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ satisfying:

$$\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_H.$$

Therefore, a Hopf algebra is always a weak Hopf algebra. Then, a weak Hopf algebra is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving and if and only if the counit is a homomorphism of algebras.

If H is a weak Hopf algebra, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If we define the morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H.$$

it is straightforward to show (see [11]) that they are idempotent and Π_H^L, Π_H^R satisfy the equalities:

$$\Pi_H^L = \mu_H \circ (H \otimes \lambda_H) \circ \delta_H, \quad \Pi_H^R = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H.$$

Moreover, we have that (see [13])

$$\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L,$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R.$$

Also it is easy to show the formulas:

$$\Pi_H^L=\overline{\Pi}_H^R\circ\lambda_H=\lambda_H\circ\overline{\Pi}_H^L,\quad \Pi_H^R=\overline{\Pi}_H^L\circ\lambda_H=\lambda_H\circ\overline{\Pi}_H^R,$$
$$\Pi_H^L\circ\lambda_H=\Pi_H^L\circ\Pi_H^R=\lambda_H\circ\Pi_H^R,\quad \Pi_H^R\circ\lambda_H=\Pi_H^R\circ\Pi_H^L=\lambda_H\circ\Pi_H^L.$$

Finally, if λ_H is bijective (for example, when H is finite), we can find the equalities:

$$\overline{\Pi}_H^L=\mu_H\circ(H\otimes\lambda_H^{-1})\circ c_{H,H}\circ\delta_H,\quad \overline{\Pi}_H^R=\mu_H\circ(\lambda_H^{-1}\otimes H)\circ c_{H,H}\circ\delta_H.$$

A morphism between weak Hopf algebras H and B is a morphism $f:H\to B$ which is both algebra and coalgebra morphism. If $f:H\to B$ is a weak Hopf algebra morphism, then $\lambda_B\circ f=f\circ\lambda_H$ [Proposition 1.4 [2]].

Definition 5.2 Let H be a weak Hopf algebra and A an algebra. By a weak action of H on A we mean a morphism $\varphi_A:H\otimes A\to A$ in $\mathcal{C}$ such that the following equalities hold:

(h1) $\varphi_A\circ(\eta_H\otimes A)=id_A$,

(h2) $\mu_A\circ((\varphi_A\circ(H\otimes\eta_A))\otimes A)=\varphi_A\circ(\Pi_H^L\otimes A)$,

(h3) $\mu_A\circ c_{A,A}\circ((\varphi_A\circ(H\otimes\eta_A))\otimes A)=\varphi_A\circ(\overline{\Pi}_H^L\otimes A)$,

(h4) $\mu_A\circ(\varphi_A\otimes\varphi_A)\circ(H\otimes c_{H,A}\otimes A)\circ(\delta_H\otimes A\otimes A)=\varphi_A\circ(H\otimes\mu_A)$,

Note that the equality (h2) implies

$$\text{(h5)}\quad \varphi_A\circ(H\otimes\eta_A)=\varphi_A\circ(\Pi_H^L\otimes\eta_A),$$

or, equivalently,

$$\text{(h5}'\text{)}\quad \varphi_A\circ(H\otimes\eta_A)=\varphi_A\circ(\overline{\Pi}_H^L\otimes\eta_A).$$

Note that, if H is a Hopf algebra, replacing (h3-4) by

$$\text{(h3}'\text{)}\quad \varphi_A\circ(H\otimes\eta_A)=\varepsilon_H\otimes\eta_A.$$

we obtain the classical definition of weak action (see [7]).

Lemma 5.3 *Let H be a weak Hopf algebra and A an algebra. Given a weak action $\varphi_A:H\otimes A\to A$, the morphism $\psi_H^A=(\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A):H\otimes A\to A\otimes H$ satisfies* (a1) *and then the morphism $\nabla_{A\otimes H}=(\mu_A\otimes H)\circ(A\otimes\psi_H^A)\circ(A\otimes H\otimes\eta_A)$ is idempotent. Moreover, the unity condition holds, i.e., $\nabla_{A\otimes H}\circ(A\otimes\eta_H)=\psi_H^A\circ(\eta_H\otimes A)$.*

Proof. By (h4) we have

$$(\mu_A\otimes H)\circ(A\otimes\psi_H^A)\circ(\psi_H^A\otimes A)$$
$$=((\mu_A\circ(\varphi_A\otimes\varphi_A)\circ(H\otimes c_{H,A}\otimes A)\circ(\delta_H\otimes A\otimes A))\otimes H)\circ(H\otimes A\otimes c_{H,A})\circ$$
$$(H\otimes c_{H,A}\otimes A)\circ(\delta_H\otimes A\otimes A)$$

$$= ((\varphi_A \circ (H \otimes \mu_A)) \otimes H) \circ (H \otimes A \otimes c_{H,A}) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$$

$$= \psi_H^A \circ (H \otimes \mu_A),$$

and, as a consequence, $\nabla_{A\otimes H}$ is idempotent.
On the other hand, using (h3) we obtain the unity condition. Indeed:

$$\psi_H^A \circ (\eta_H \otimes A)$$

$$= (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes A)$$

$$= ((\varphi_A \circ (\overline{\Pi}_H^L \otimes A)) \otimes H) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes A)$$

$$= ((\mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes A)) \otimes H) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes A)$$

$$= (\mu_A \otimes H) \circ (A \otimes \varphi_A \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes (\delta_H \circ \eta_H) \otimes \eta_A)$$

$$= \nabla_{A\otimes H} \circ (A \otimes \eta_H). \quad \square$$

5.4 Let H be a weak Hopf algebra and A be an algebra. Assume that there are a weak action $\varphi_A : H \otimes A \to A$ and a morphism $\sigma_A : H \otimes H \to A$ such that for ψ_H^A and

$$\sigma_H^A = (\sigma_A \otimes \mu_H) \circ \delta_{H\otimes H} : H \otimes H \to A \otimes H,$$

(a2) and (a3) hold. Then, by 5.3, we obtain that $(A, H, \psi_H^A, \sigma_H^A)$ is a crossed product system with unity.

Remark 5.5 Let H be a weak Hopf algebra and A an algebra. If $\varphi_A : H \otimes A \to A$ is a weak action of H on A, considering (h3), we have

$$(\varphi_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes A \otimes H)$$

$$= ((\varphi_A \circ c_{A,H} \circ c_{H,A}) \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes A \otimes H)$$

$$= ((\varphi_A \circ c_{A,H}) \otimes \mu_H) \circ (A \otimes (\delta_H \circ \eta_H) \otimes H)$$

$$= ((\varphi_A \circ c_{A,H}) \otimes H) \circ (A \otimes ((\overline{\Pi}_H^L \otimes H) \circ \delta_H))$$

$$= (\mu_A \circ (A \otimes (\varphi_A \circ (H \otimes \eta_A)))) \otimes H) \circ (A \otimes \delta_H)$$

$$= \nabla_{A\otimes H}.$$

Therefore, if (A, φ_A) is also a left H-module (i.e., (A, φ_A) satisfies $\varphi_A \circ (\eta_H \otimes A) = id_A$ and $\varphi_A \circ (H \otimes \varphi_A) = \varphi_A \circ (\mu_H \otimes A)$, the image of $\nabla_{A\otimes H}$, denoted by $A \times H$, is the tensor product of A and H in the representation category of H (the category of left H-modules), denoted by $Rep(H)$. In [23] and [24] it is possible to find a detailed construction of a non-strict monoidal structure in $Rep(H)$ for a weak Hopf algebra living in a category of vector spaces. This construction can be extended without any difficulty to the general categorical case of this paper, i.e., for a weak Hopf algebra in a strict symmetric monoidal category. In the following lines we give a brief resume of the monoidal structure of $Rep(H)$.

For two left H-modules (M,φ_M), (N,φ_N) the tensor product is defined as object as the image $M\times N$ of $\nabla_{M\otimes N}=\varphi_{M\otimes N}\circ(\eta_H\otimes M\otimes N):M\otimes N\rightarrow M\otimes N$ where $\varphi_{M\otimes N}:H\otimes M\otimes N\rightarrow M\otimes N$ is defined by $\varphi_{M\otimes N}=(\varphi_M\otimes\varphi_N)\circ(H\otimes c_{H,M}\otimes N)\circ(\delta_H\otimes M\otimes N)$. As a consequence, $M\times N$ is a left H-module with the following action:

$$\varphi_{M\times N}=p_{M\otimes N}\circ\varphi_{M\otimes N}\circ(H\otimes i_{M\otimes N})$$

where $p_{M\otimes N}$ and $i_{M\otimes N}$ are the morphisms such that $p_{M\otimes N}\circ i_{M\otimes N}=\nabla_{M\otimes N}$ and $i_{M\otimes N}\circ p_{M\otimes N}=id_{M\times N}$

The base object is $H_L=Im(\Pi_H^L)$ or, equivalently, the equalizer of δ_H and $\zeta_H^1=(H\otimes\Pi_H^L)\circ\delta_H$ or the equalizer of δ_H and $\zeta_H^2=(H\otimes\overline{\Pi}_H^R)\circ\delta_H$. The structure of left H-module for H_L is the one derived of the following morphism

$$\varphi_{H_L}=p_L\circ\mu_H\circ(H\otimes i_L),$$

where $p_L:H\rightarrow H_L$ and $i_L:H_L\rightarrow H$ are the morphism such that $\Pi_H^L=i_L\circ p_L$ and $p_L\circ i_L=id_{H_L}$.

The unit constrains are:

$$l_M=\varphi_M\circ(i_L\otimes M)\circ i_{H_L\otimes M}:H_L\times M\rightarrow M,$$

$$r_M=\varphi_M\circ c_{M,H}\circ(M\otimes(\overline{\Pi}_H^L\circ i_L))\circ i_{M\otimes H_L}:M\times H_L\rightarrow M.$$

These morphisms are isomorphisms with inverses:

$$l_M^{-1}=p_{H_L\otimes M}\circ(p_L\otimes\varphi_M)\circ((\delta_H\circ\eta_H)\otimes M):M\rightarrow H_L\times M,$$

$$r_M^{-1}=p_{M\otimes H_L}\circ(\varphi_M\otimes p_L)\circ(H\otimes c_{H,M})\circ((\delta_H\circ\eta_H)\otimes M):M\rightarrow M\times H_L.$$

If M, N, P are objects in the category $Rep(H)$, the associativity constrains are defined by

$$a_{M,N,P}=p_{(M\times N)\otimes P}\circ(p_{M\otimes N}\otimes P)\circ(M\otimes i_{N\otimes P})\circ i_{M\otimes(N\times P)}:M\times(N\times P)\rightarrow(M\times N)\times P$$

where the inverse is the morphism

$$a_{M,N,P}^{-1}=p_{M\otimes(N\times P)}\circ(M\otimes p_{N\otimes P})\circ(i_{M\otimes N}\otimes P)\circ i_{(M\times N)\otimes P}:(M\times N)\times P\rightarrow M\times(N\times P).$$

If $\gamma:M\rightarrow M'$ and $\phi:N\rightarrow N'$ are morphisms in the category, then

$$\gamma\times\phi=p_{M'\times N'}\circ(\gamma\otimes\phi)\circ i_{M\otimes N}:M\times N\rightarrow M'\times N'$$

is a morphism in $Rep(H)$ and

$$(\gamma'\times\phi')\circ(\gamma\times\phi)=(\gamma'\circ\gamma)\times(\phi'\circ\phi),$$

where $\gamma':M'\rightarrow M''$ and $\phi':N'\rightarrow N''$ are morphisms in $Rep(H)$.

Theorem 5.6 *Let H be a weak Hopf algebra and A an algebra. Assume that there are a weak action $\varphi_A:H\otimes A\rightarrow A$ and a morphism $\sigma_A:H\otimes H\rightarrow A$ such that $(A,H,\psi_H^A=(\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A),\sigma_H^A=(\sigma_A\otimes\mu_H)\circ\delta_{H\otimes H})$ is a crossed product system with unity. Then, $A\times H$ with the induced product $\mu_{A\times H}$ and unit $\eta_{A\times H}=p_{A\otimes H}\circ(\eta_A\otimes\eta_H)$, is an algebra if and only if $(A,H,\psi_H^A,\sigma_H^A)$ is normal and satisfies the twisted and cocycle conditions.*

Proof. Suppose that $A \times H$ is an algebra with unity $\eta_{A\times H} = p_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ and product $\mu_{A\times H}$. Under these conditions we know that $\mu_{A\times H} \circ (A \times H \otimes \eta_{A\times H}) = id_{A\times H}$ and then

$$
\begin{aligned}
& p_{A\otimes H} \\
&= \mu_{A\times H} \circ (p_{A\times H} \otimes (p_{A\times H} \circ (\eta_A \otimes \eta_H))) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (A \otimes ((\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nabla_{A\otimes H}))) \circ (\nabla_{A\otimes H} \otimes \eta_A \otimes \eta_H) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H) \circ (\nabla_{A\otimes H} \otimes \eta_A \otimes \eta_H) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nabla_{A\otimes H} \otimes \eta_H) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (A \otimes H \otimes \eta_H).
\end{aligned}
$$

In the last computations, the second equality is simply the definition of $\mu_{A\times H}$, while the third one follows by (a2) and (b2). In the fourth one we used the idempotent character of $\nabla_{A\otimes H}$ and the fifth one follows by (a3) and (b2).

Also, using the equality $\mu_{A\times H} \circ (\eta_{A\times H} \otimes A \times H) = id_{A\times H}$ we obtain

$$
\begin{aligned}
& p_{A\otimes H} \\
&= \mu_{A\times H} \circ ((p_{A\times H} \circ (\eta_A \otimes \eta_H)) \otimes p_{A\times H}) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (A \otimes ((\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H))) \circ ((\nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)) \otimes \nabla_{A\otimes H}) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H) \circ ((\psi_H^A \circ (\eta_H \otimes \eta_A)) \otimes A \otimes H) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ ((\psi_H^A \circ (\eta_H \otimes A)) \otimes H) \\
&= p_{A\times H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ ((\nabla_{A\otimes H} \circ (A \otimes \eta_H)) \otimes H) \\
&= p_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (A \otimes \eta_H \otimes H).
\end{aligned}
$$

As in the first computations of this proof, the second equality is simply the definition of $\mu_{A\times H}$. The third equality follows from the idempotent character of $\nabla_{A\otimes H}$ and by (a2), while the fourth one follows by (a1). In the fifth one we used the unity condition and finally, the sixth one follows by (a3).

Therefore, we have

$$
p_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (A \otimes \eta_H \otimes H) = p_{A\otimes H} = p_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (A \otimes H \otimes \eta_H)
$$

and, by 4.3, we obtain that $(A, H, \psi_H^A, \sigma_H^A)$ is normal.

By 4.6, to derive the twisted and cocycle conditions we only need to show (f1). Indeed, using the normality and the properties of $(A, H, \psi_H^A, \sigma_H^A)$ we have

$$
\begin{aligned}
& i_{A\otimes H} \circ \mu_{A\times H} \circ ((p_{A\times H} \circ (A \otimes \eta_H)) \otimes p_{A\times H}) \\
&= \nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H) \circ ((\psi_H^A \circ (\eta_H \otimes A)) \otimes \nabla_{A\otimes H}) \\
&= \nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (\eta_H \otimes (\nabla_{A\otimes H} \circ (\mu_A \otimes H)))
\end{aligned}
$$

$$= \nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes (\sigma_H^A \circ (\eta_H \otimes H))) \circ \nabla_{A\otimes H} \circ (\mu_A \otimes H)$$

$$= \nabla_{A\otimes H} \circ (\mu_A \otimes H).$$

Conversely, by Theorem 4.6, if $(A, H, \psi_H^A, \sigma_H^A)$ is normal and satisfies the twisted and cocycle conditions we have that $A \times H$, with the induced product $\mu_{A\times H}$ and unit $\eta_{A\times H} = p_{A\otimes H} \circ (\eta_A \otimes \eta_H)$, is an algebra. □

Remark 5.7 Let H be a weak Hopf algebra and A an algebra. Assume that there are a weak action $\varphi_A : H \otimes A \to A$ and a morphism $\sigma_A : H \otimes H \to A$ such that $(A, H, \psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A), \sigma_H^A = (\sigma_A \otimes \mu_H) \circ \delta_{H\otimes H})$ is a crossed product system with unity. Note that

$$\begin{aligned}
&p_{A\otimes H} \circ \sigma_H^A \circ (H \otimes \eta_H)\\
&= p_{A\otimes H} \circ (\sigma_A \otimes \mu_H) \circ \delta_{H\otimes H} \circ (H \otimes \eta_H)\\
&= p_{A\otimes H} \circ (\sigma_A \otimes H) \circ (H \otimes ((\Pi_H^R \otimes H) \circ \delta_H)) \circ \delta_H\\
&= p_{A\otimes H} \circ ((\sigma_A \circ (H \otimes \Pi_H^R) \circ \delta_H) \otimes H) \circ \delta_H,
\end{aligned}$$

and

$$\begin{aligned}
&p_{A\otimes H} \circ \sigma_H^A \circ (\eta_H \otimes H)\\
&= p_{A\otimes H} \circ (\sigma_A \otimes \mu_H) \circ \delta_{H\otimes H} \circ (\eta_H \otimes H)\\
&= p_{A\otimes H} \circ ((\sigma_A \circ c_{H,H}) \otimes H) \circ (H \otimes ((\overline{\Pi}_H^L \otimes H) \circ \delta_H)) \circ \delta_H\\
&= p_{A\otimes H} \circ ((\sigma_A \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \circ \delta_H) \otimes H) \circ \delta_H.
\end{aligned}$$

Then, $(A, H, \psi_H^A, \sigma_H^A)$ is normal if and only if

$$p_{A\otimes H} \circ ((\sigma_A \circ (H \otimes \Pi_H^R) \circ \delta_H) \otimes H) \circ \delta_H = p_{A\otimes H} \circ (\eta_A \otimes H) =$$
$$p_{A\otimes H} \circ ((\sigma_A \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \circ \delta_H) \otimes H) \circ \delta_H.$$

On the other hand, the twisted condition is equivalent to

(i1) $$\begin{aligned}
&p_{A\otimes H}\circ\\
&([\mu_A \circ (\varphi_A \otimes A) \circ (A \otimes \varphi_A \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma_A \otimes A) \circ (\delta_{H\otimes H} \otimes A)] \otimes c_{H,A})\circ\\
&(H \otimes H \otimes \mu_H \otimes A) \circ (\delta_{H\otimes H} \otimes A)\\
=&p_{A\otimes H} \circ ([\mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A)] \otimes c_{H,A}) \circ (H \otimes H \otimes \mu_H \otimes A) \circ (\delta_{H\otimes H} \otimes A),
\end{aligned}$$

and the cocycle condition can be viewed in the following form:

(i2) $$\begin{aligned}
&p_{A\otimes H} \circ (\partial_4(\sigma_A) \wedge \partial_2(\sigma_A) \otimes \mu_H) \circ (H \otimes H \otimes H \otimes H \otimes \mu_H) \circ \delta_{H\otimes H\otimes H}\\
=&p_{A\otimes H} \circ (\partial_1(\sigma_A) \wedge \partial_3(\sigma_A) \otimes \mu_H) \circ (H \otimes H \otimes H \otimes H \otimes \mu_H) \circ \delta_{H\otimes H\otimes H},
\end{aligned}$$

where

$$\partial_1 = \varphi_A \circ (H \otimes \sigma_A), \qquad \partial_2 = \sigma_A \circ (\mu_H \otimes H),$$

$$\partial_3 = \sigma_A \circ (H \otimes \mu_H), \qquad \partial_4 = \sigma_A \otimes \varepsilon_H,$$

and $\wedge$ denotes the usual convolution in $Hom_{\mathcal{C}}(H \otimes H \otimes H, A)$.

When H is a Hopf algebra, the normal condition for $(A, H, \psi_H^A, \sigma_H^A)$ is equivalent to

$$\sigma_A \circ (\eta_H \otimes H) = \sigma_A \circ (H \otimes \eta_H) = \eta_A \otimes \varepsilon_H$$

because $\Pi_H^R = \overline{\Pi}_H^L = \eta_H \otimes \varepsilon_H$ and $\nabla_{A\otimes H} = id_{A\otimes H}$. Then, we have that $(A, H, \psi_H^A, \sigma_H^A)$ is normal if and only if σ_A is normal in the classical sense (see [7]).

Also, composing with $A \otimes \varepsilon_H$ in (i1) and (i2) we obtain that $(A, H, \psi_H^A, \sigma_H^A)$ satisfy the twisted condition if and only if

$$\mu_A \circ (\varphi_A \otimes A) \circ (H \otimes \varphi_A \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma_A \otimes A) \circ (\delta_{H\otimes H} \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A),$$

and satisfies the cocycle condition if and only if $\partial_4(\sigma_A) \wedge \partial_2(\sigma_A) = \partial_1(\sigma_A) \wedge \partial_3(\sigma_A)$. Therefore, in the Hopf algebra case, $(A, H, \psi_H^A, \sigma_H^A)$ satisfy the twisted and the cocycle conditions if and only if σ_A is a twisted cocycle (see also [7] for the definition).

As a consequence, Theorem 5.6 is a generalization of the results obtained by Blattner, Cohen and Montgomery [7] and Doi and Takeuchi [16] in the study of crossed products in a category of vector spaces. Also, if $\mathcal{C}$ is a braided category whose underlying monoidal category is of vector spaces, using a similar computations, we obtain the conditions described by Majid [21] which allow to built an algebra structure in the tensor product of an algebra A and a Hopf algebra H. In these cases the algebra $A \times H$ was denoted by $A\sharp_{\sigma_A} H$ (the crossed product of A and H).

Example 5.8 Let H, B be weak Hopf algebras in a strict symmetric monoidal category $\mathcal{C}$ with split idempotents. Let $g : B \to H$ be a morphism of weak Hopf algebras and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. If we define $\rho_B : B \to B \otimes H$ and the entwining $\psi : H \otimes B \to B \otimes H$ by

$$\rho_B = (B \otimes g) \circ \delta_B, \quad \psi = (B \otimes \mu_H) \circ (c_{H,B} \otimes H) \circ (H \otimes \rho_B)$$

we have that (B, H, ψ) is a weak entwining structure where $e_{RR} = \Pi_B^R \circ f$.

The morphism $q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \to B$ is an idempotent in $\mathcal{C}$ [Proposition 2.1, [3]]. As a consequence, there exist an epimorphism p_H^B, a monomorphism i_H^B and an object B_H such that the diagram

$$\begin{tikzcd} B \arrow[rr, "q_H^B"] \arrow[dr, "p_H^B"'] & & B \\ & B_H \arrow[ur, "i_H^B"'] & \end{tikzcd}$$

commutes and $p_H^B \circ i_H^B = id_{B_H}$. Also,

$$\begin{tikzcd} B_H \arrow[r, "i_H^B"] & B \arrow[r, shift left, "\rho_B"] \arrow[r, shift right, "(B \otimes \overline{\Pi}_H^R) \circ \rho_B"'] & B \otimes H \end{tikzcd}$$

is an equalizer diagram in $\mathcal{C}$, and therefore, the algebra defined by the equalizer of ρ_B and ζ_B in 3.3 is the same that the one defined by the equalizer of ρ_B and $(B\otimes \overline{\Pi}_H^R)\circ \rho_B$ because, in this situation, $(B\otimes \overline{\Pi}_H^R)\circ \rho_B=\zeta_B$.

Moreover, $f\in Reg^{WR}(H,B)$ because for $f^{-1}=\lambda_B\circ f$ we obtain that $f^{-1}\wedge f=\Pi_B^R\circ f=e_{RR}$ and $B_H\hookrightarrow B=(B,\mu_B,\rho_B)$ is a weak H-cleft extension [Example 1.1, [4]].

On the other hand, let $\varphi_{B_H}:H\otimes B_H\rightarrow B_H$ be the morphism defined in 3.8. The morphism φ_{B_H} satisfies

(j1) $\varphi_{B_H}\circ (\eta_H\otimes B_H)=id_{B_H}$,

(j2) $\mu_{B_H}\circ ((\varphi_{B_H}\circ (H\otimes \eta_{B_H}))\otimes B_H)=\varphi_{B_H}\circ (\Pi_H^L\otimes B_H)$,

(j3) $\mu_{B_H}\circ c_{B_H,B_H}\circ ((\varphi_{B_H}\circ (H\otimes \eta_{B_H}))\otimes B_H)=\varphi_{B_H}\circ (\overline{\Pi}_H^L\otimes B_H)$,

(j4) $\varphi_{B_H}\circ (H\otimes \mu_{B_H})=\mu_{B_H}\circ (\varphi_{B_H}\otimes \varphi_{B_H})\circ (H\otimes c_{H,B_H}\otimes B_H)\circ (\delta_H\otimes B_H\otimes B_H)$,

(j5) $\varphi_{B_H}\circ (H\otimes \eta_{B_H})=\varphi_{B_H}\circ (\Pi_H^L\otimes \eta_{B_H})$,

and then φ_{B_H} is a weak action of H on B_H. Moreover, if f is a morphism of algebras (B_H,φ_{B_H}) is a left H-module (see [Proposition 2.5, [3]])

By 3.10 we know that the left B_H-module and right H-comodule ($\varphi_{B_H\otimes H}=\mu_{B_H}\otimes H$, $\rho_{B_H\otimes H}=B_H\otimes \delta_H$) morphisms $\omega_B: B_H\otimes H\rightarrow B$, $\omega_B^{\prime}: B\rightarrow B_H\otimes H$, defined by $\omega_B=\mu_B\circ (i_H^B\otimes f)$ and $\omega_B^{\prime}=(p_H^B\otimes H)\circ \rho_B$ satisfy the equality $\omega_B\circ \omega_B^{\prime}=id_B$. As a consequence, the morphism $\Omega_B=\omega_B^{\prime}\circ \omega_B$ is an idempotent, and we have a commutative diagram

$$\begin{tikzcd}
 & B \arrow[dr, "\omega_B^{\prime}"] & \\
B_H\otimes H \arrow[ur, "\omega_B"] \arrow[rr, "\Omega_B"] \arrow[dr, "r_B"'] & & B_H\otimes H \\
 & B_H\times H \arrow[ur, "s_B"'] &
\end{tikzcd}$$

where $r_B\circ s_B=id_{B_H\times H}$. Therefore, the morphism $b_B=r_B\circ \omega_B^{\prime}$ is an isomorphism of right H-comodules and left B_H-modules with inverse $b_B^{-1}=\omega_B\circ s_B$. The module and comodule structures of $B_H\times H$ are the ones induced by the isomorphism b_B and they are equal to $\varphi_{B_H\times H}=r_B\circ (\mu_{B_H}\otimes H)\circ (B_H\otimes s_B)$, $\rho_{B_H\times H}=(r_B\otimes H)\circ (B_H\otimes \delta_H)\circ s_B$, respectively. Also, b_B is an isomorphism of algebras with $\eta_{B_H\times H}=b_B\circ \eta_B$, $\mu_{B_H\times H}=b_B\circ \mu_B\circ (b_B^{-1}\otimes b_B^{-1})$. Under these conditions, the unity $\eta_{B_H\times H}$ and the product $\mu_{B_H\times H}$ can be identified in the following way (see [Theorem 2.8, [3]])

$$\eta_{B_H\times H}=r_B\circ (\eta_{B_H}\otimes \eta_H),$$

$$\mu_{B_H\times H}=r_B\circ (\mu_{B_H}\otimes H)\circ (\mu_{B_H}\otimes \sigma_H^{B_H})\circ (B_H\otimes \psi_H^{B_H}\otimes H)\circ (s_B\otimes s_B),$$

where

$$\psi_H^{B_H}=(\varphi_B^{\prime}\otimes H)\circ (H\otimes \psi)\circ (\delta_H\otimes i_H^B)=\omega_B^{\prime}\circ \mu_B\circ (f\otimes i_H^B)=(\varphi_{B_H}\otimes H)\circ (H\otimes c_{H,B_H})\circ (\delta_H\otimes B_H),$$

$$\sigma_H^{B_H}=(\varphi_B^{\prime}\otimes H)\circ (H\otimes \psi)\circ (\delta_H\otimes f)=\omega_B^{\prime}\circ \mu_B\circ (f\otimes f)=(\sigma_{B_H}\otimes \mu_H)\circ \delta_{H\otimes H},$$

being $\sigma_{B_H} = p_H^B \circ \mu_B \circ (f \otimes f)$ the morphism defined in 3.8.

Moreover, the idempotent morphism Ω_B satisfies

$$\Omega_B = (\varphi_{B_H} \otimes \mu_H) \circ (B_H \otimes c_{H,B_H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes B_H \otimes H) =$$

$$(\mu_{B_H} \otimes H) \circ (B_H \otimes \psi_H^{B_H}) \circ (B_H \otimes H \otimes \eta_{B_H}) = \nabla_{B_H \otimes H},$$

and we have (b2) and the following equality:

$$\text{(k1)} \quad (\mu_{B_H} \otimes H) \circ (B \otimes \sigma_{B_H}) \circ ((\nabla_{B_H \otimes H} \circ (B_H \otimes \eta_H)) \otimes H) = \nabla_{B_H \otimes H}.$$

Indeed, using the equalities (e3), $\mu_B \circ (B \otimes (\Pi_B^L \circ f \circ g)) \circ \delta_B = id_B$ and $g \circ f = id_H$ we obtain (k1):

$$\begin{aligned}
&(\mu_{B_H} \otimes H) \circ (B \otimes \sigma_{B_H}) \circ ((\nabla_{B_H \otimes H} \circ (B_H \otimes \eta_H)) \otimes H) \\
&= (p_H^B \otimes H) \circ (\mu_B \otimes H) \circ ((\mu_B \circ (B \otimes (\Pi_B^L \circ f \circ g)) \circ \delta_B) \otimes f \otimes \mu_H) \circ (B \otimes c_{H,H} \otimes H) \circ (B \otimes g \otimes H \otimes H) \circ \\
&\quad (\delta_B \otimes \delta_H) \circ (i_H^B \otimes H) \\
&= (p_H^B \otimes H) \circ \mu_{B \otimes H} \circ (B \otimes g \otimes B \otimes g) \circ ((\delta_B \circ i_H^B) \otimes ((f \otimes f) \circ \delta_H)) \\
&= \nabla_{B_H \otimes H}.
\end{aligned}$$

Then, since (a1), (b2), (k1) and the unity condition are satisfied, the equality (f1) is valid for the product $\mu_{B_H \times H}$. Indeed:

$$\begin{aligned}
&i_{B_H \otimes H} \circ \mu_{B_H \times H} \circ ((p_{B_H \otimes H} \circ (B_H \otimes \eta_H)) \otimes p_{B_H \otimes H}) \\
&= \nabla_{B_H \otimes H} \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H}) \circ (B_H \otimes \psi_H^{B_H} \otimes H) \circ ((\psi_H^{B_H} \circ (\eta_H \otimes B_H)) \otimes \nabla_{B_H \otimes H}) \\
&= \nabla_{B_H \otimes H} \circ (\mu_{B_H} \otimes H) \circ (B_H \otimes \sigma_{B_H}) \circ (\psi_H^{B_H} \otimes H) \circ (H \otimes \nabla_{B_H \otimes H}) \circ (\eta_H \otimes \mu_{B_H} \otimes H) \\
&= \nabla_{B_H \otimes H} \circ (\mu_{B_H} \otimes H) \circ (B_H \otimes \sigma_{B_H}) \circ ((\nabla_{B_H \otimes H} \circ (B_H \otimes \eta_H)) \otimes H) \circ \nabla_{B_H \otimes H} \circ (\mu_{B_H} \otimes H) \\
&= \nabla_{B_H \otimes H} \circ \nabla_{B_H \otimes H} \circ \nabla_{B_H \otimes H} \circ (\mu_{B_H} \otimes H) \\
&= \nabla_{B_H \otimes H} \circ (\mu_{B_H} \otimes H).
\end{aligned}$$

Therefore, by Theorem 4.6, we have that $(B_H, H, \psi_H^{B_H}, \sigma_H^{B_H})$ is an example of crossed product system with unity satisfying the normality, twisted and cocycle conditions. Also, is an example of the theory developed in section 3 for crossed product systems associated to a weak cleft extension.

Acknowledgements

The authors have been supported by Ministerio de Educación, Xunta de Galicia and by FEDER, Projects: MTM2007-62427, MTM2006-14908-CO2-01, PGIDT07PXB322079PR.

The authors wish to express their gratitude to the referee for his interesting suggestions, remarks and for sending us the reference [27]. Also we would like to thank R. Street for showing us the connections of our work with [20] and [25].

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