Constant Sign Solutions of Two-Point Fourth Order Problems

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Abstract

In this paper we characterize the sign of the Green's function related to the fourth order linear operator $u^{(4)} + M\,u$ coupled with the two point boundary conditions $u(1) = u(0) = u'(0) = u''(0) = 0$. We obtain the exact values on the real parameter M for which the related Green's function is negative in $(0, 1) \times (0, 1)$. Such property is equivalent to the fact that the operator satisfies a maximum principle in the space of functions that satisfy the homogeneous boundary conditions.

When $M > 0$ the best estimate follows from spectral theory. When $M < 0$, we obtain an estimation by studying the disconjugacy properties of the solutions of the homogeneous equation $u^{(4)} + M\,u = 0$. The optimal value is attained by studying the exact expression of the Green's function. Such study allow us to ensure that there is no real parameter M for which the Green's function is positive on $(0, 1) \times (0, 1)$.

Moreover, we obtain maximum principles of this operator when the solutions verify suitable non-homogeneous boundary conditions.

We apply the obtained results, by means of the method of lower and upper solutions, to nonlinear problems coupled with this boundary conditions.

Key words: Fourth order boundary value problem; Maximum Principles; Lower and Upper solutions.

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1 Introduction

Fourth order boundary value problems represents a fundamental tool when describing elastic deflections of bridges [9].

In this paper we describe the set of the real parameters M for which the fourth order boundary value problem

$$L_M\, u(t) \equiv u^{(4)}(t) + M\, u(t) = \sigma(t), \text{ a.e. } t \in [0,1];\ u(1) = u(0) = u'(0) = u''(0) = 0, \tag{1}$$

has nonpositive solutions, for any nonnegative L^1-function σ.

After an immediate change of variables, we obtain analogous results for the adjoint boundary conditions $u(0) = u(1) = u'(1) = u''(1) = 0$. It is not difficult to verify, [2], that such property is equivalent to the fact that the related Green's function is negative on $(0,1) \times (0,1)$.

Such conditions model the deflection of beam which is incrusted and clamped at one side and supported in the other one. This study complete the one done in [7], where the clamped beam conditions $u(0) = u(1) = u'(0) = u'(1) = 0$ have been described. Moreover it continues the work done in [6] when the simply supported case $u(0) = u(1) = u''(1) = u''(1) = 0$ has been characterized.

In this case, we distinguish two situations, depending on the sign of the real parameter M. When it is positive, we deduce the best estimate by means of the spectral theory, see, for instance, [4, **?**, 13]. For negative values of M, we use the classical theory of disconjugacy [8], to obtain an estimate on M that ensures the negativeness of the Green's function. To verify that the given estimation is optimal, by means of the Mathematica Package developed in [5] (see also [4]), we calculate the exact expression of the related Green's function. This study, combined with the spectral theory, is also applied to deduce that we cannot expect positive Green's functions for any value of the real parameter M.

We also attain, in section 5, comparison results when non homogeneous boundary conditions are considered. Section 6 is devoted to deduce existence results for nonlinear boundary value problems by means of the lower and upper solutions method.

2 Preliminaries

In this section, we introduce the definitions and main results that will be used along the paper.

Denote by $I = [0,1]$, $L^1(I)$ be the space of the Lebesgue integrable functions on I, and consider $W^{4,1}(I)$ the usual Sobolev space of C^3-functions, with u''' absolutely continuous in I.

Definition 2.1. *Let $X \subset W^{4,1}(I)$ a given set. We say that the linear operator $L_M : X \longrightarrow L^1(I)$ is* inverse positive *on X if the following property is fulfilled for all $y \in X$*

$$\text{If} \quad L_M\, y(t) \geq 0 \text{ a.e } t \in I \qquad \text{then} \quad y(t) \geq 0 \text{ for all } t \in I.$$

Analogously, the linear operator L_M *is* inverse negative *on* X *if the following property is fulfilled for all* $y \in X$

$$\textit{If} \quad L_M\, y(t) \geq 0 \textit{ a.e } t \in I \qquad \textit{then} \quad y(t) \leq 0 \textit{ for all } t \in I.$$

We refer L_M *as a* strongly inverse positive *(*strongly inverse negative*) operator on* X *if*

$$L_M\, y(t) \geq 0,\ L_M\, y(t) \not\equiv 0 \textit{ a.e } t \in I \qquad \textit{then} \quad y(t) > 0\ (y(t) < 0) \textit{ on } (0,1).$$

In the inverse positive case we say that L_M satisfies an *(strong) anti-maximum principle* in X. In case of inverse negativeness we refer to a *(strong) maximum principle* in X.

If we consider the set

$$X_0 = \{u \in W^{4,1}(I), \quad u(1) = u(0) = u'(0) = u''(0) = 0\}.$$

We have the following properties that characterize the maximum and anti-maximum principle as the constant sign of the related Green's function. They are particular cases of Lemmas 1.6.3 and 1.6.10, and Corollaries 1.6.6 and 1.6.12 in [4]

Proposition 2.2. *If* L_M *is inverse negative (inverse positive) on* X_0 *then* L_M *is invertible on* X_0*.*

Proposition 2.3. *The operator* L_M *is inverse negative (inverse positive) on* X_0 *if and only if the Green's function* g_M*, related to problem* (1)*, is nonpositive (nonnegative) on* $I \times I$*.*

Having in mind the two previous results, to obtain the validity of the comparison results for operator L_M on X_0, we must study the set of parameters for which the related Green's function has constant sign on $I \times I$. In order to make this study, we consider the following condition, introduced in [4, Section 8], for a nonpositive Green's function

(N_g) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (0,1)$ and $k_1,\ k_2 \in L^1(I)$, such that $k_1(s) < k_2(s) < 0$ for a.e. $s \in I$, satisfying

$$\phi(t)\, k_1(s) \leq g_M(t,s) \leq \phi(t)\, k_2(s), \quad \text{for a.e. } (t,s) \in I \times I.$$

Remark 2.4. *We note that if function* g_M *satisfies property* (N_g) *then operator* L_M *is strongly inverse negative on* X_0*.*

Under this assumption, we introduce the following set of parameters M in which the Green's function has constant sign

$$N_L = \{M \in \mathbb{R}, \quad \text{such that} \quad g_M(t,s) \leq 0 \text{ for all } (t,s) \in I \times I\} \tag{2}$$

and

$$P_L = \{M \in \mathbb{R}, \quad \text{such that} \quad g_M(t,s) \geq 0 \text{ for all } (t,s) \in I \times I\}. \tag{3}$$

As a consequence of [4, Theorems 1.8.5 and 1.8.9] we deduce the following topological property of these sets.

Proposition 2.5. *The sets N_L and P_L are (may be empty) real intervals.*

We can describe, as a consequence of [4, Lemma 1.8.25], the set N_L as follows

Proposition 2.6. *Let $\bar{M} \in \mathbb{R}$ be fixed. Suppose that operator $L_{\bar{M}}$ is invertible in X_0, its related Green's function is nonpositive on $I \times I$, it satisfies condition (N_g) and the set N_L, defined in (2), is bounded from below.*

Then $N_L = [\bar{M} - \bar{\mu}, \bar{M} - \lambda_1)$, with $\lambda_1 < 0$ the first eigenvalue of operator $L_{\bar{M}}$, and $\bar{\mu} \geq 0$ is such that $L_{\bar{M}-\bar{\mu}}$ is invertible in X_0 and the related nonpositive Green's function $g_{\bar{M}-\bar{\mu}}$ vanishes at some points of the square $I \times I$.

So, once condition (N_g) is verified, to characterize the right side of the set N_L is enough to find the first eigenvalue of operator L_M.

Moreover, we have the following result concerning both sets

Proposition 2.7. *[4, Theorem 1.8.35] Let $\bar{M} \in \mathbb{R}$ be such that problem* (1) *has a unique solution for $M = \bar{M}$ and the related Green's function $g_{\bar{M}}$ satisfies condition (N_g). If the interval P_L, defined in* (3)*, is nonempty then* $\sup(N_L) = \inf(P_L)$*, with N_L defined in* (2).

Since the infimum of the set N_L is a regular point for the operator $L_{\bar{M}}$, the characterization of the left side of the interval N_L is more difficult to deal with. In this case we will use the theory of disconjugacy that gives us sufficient conditions to ensure that the Green's function has constant sign. This tool has been recently applied in [12] to third order equations with different kind of two-point boundary conditions. We refer to [8] and references therein for an exhaustive study of this theory.

Definition 2.8. *[8, Definition 1 in Page 1] Let $p_k \in C([a,b])$, $k = 1, \ldots, n$. A linear differential equation of n order*

$$L_n\, y(t) := y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \ldots + p_{n-1}(t)\, y'(t) + p_n(t)\, y(t) = 0,$$

is disconjugate *in $[a,b]$ if every nontrivial solution of this equation has less than n zeroes in $[a,b]$, multiple zeroes being counted according to their multiplicity.*

One can see in [8, Page 105] that if operator L_n is disconjugate on $[a,b]$ then, for any $\sigma \in C([a,b])$ and $k \in \{1, \ldots, n-1\}$, the two-point boundary value problem

$$\begin{cases} L_n\, y(t) = \sigma(t),\ t \in [a,b], \\ y^{(i)}(a) = 0,\ i = 0, \ldots, k-1, \\ y^{(j)}(b) = 0,\ j = 0, \ldots, n-k-1, \end{cases}$$

has a unique solution that can be represented in the form

$$y(t) = \int_a^b G(t,s)\sigma(s)ds.$$

By approximation theory, one can verify that the same result holds for any $\sigma \in L^1([a,b])$.

Next result is a direct consequence of [8, Theorem 11 in Page 108] (see also [12]).

Proposition 2.9. *If operator L_n is disconjugate in $[a,b]$, then*

$$(-1)^{n-k} G(t,s) > 0, \quad a < t, s < b.$$

As a direct consequence of this result we deduce the following one

Corollary 2.10. *If operator L_M is disconjugate in I then the Green's function g_M, related to problem* (1)*, satisfies that $g_M(t,s) < 0$, on $(0,1) \times (0,1)$.*

Thus, the concept of disconjugacy implies the negative sign of the studied Green's function. So we are interested in to obtain sufficient conditions that allow us to ensure such property. To this end, we use the concept of Markov system.

Definition 2.11. *[8, Page 87] The functions $y_1, \ldots, y_n \in C^n([a,b])$ are said to form a* Markov system *on a real interval J if the n Wronskians*

$$W_k := W[y_1, \ldots, y_k] = \det \begin{pmatrix} y_1 & \cdots & y_k \\ \vdots & \vdots & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{pmatrix}, \quad k = 1, \ldots, n,$$

are positive throughout J.

Moreover, as a consequence of Theorem 3, Page 94, and Theorems 4 and 5, page 97, in [8], we arrive at the following equivalence between both concepts

Proposition 2.12. *The equation* (1) *has a Markov fundamental system of solutions on $[a,b)$ if and only if it is disconjugate on $[a,b]$.*

3 Study of the sign of the Green's function

This section is devoted to the study of the values of the real parameter M for which the Green's function g_M, related to problem (1), is nonpositive on $I \times I$. As we have seen in Proposition 2.3, this study is equivalent to look for the values of M for which the fourth order operator L_M satisfies a maximum principle on the set X_0.

In order to apply Proposition 2.6 we take $\bar{M} = 0$ and verify condition (N_g) for g_0. To this end, we first study the eigenvalues of operator L_0.

By routine calculations we deduce that equation

$$u^{(4)}(t) = \lambda\, u(t),\ t \in [0,1];\ u(1) = u(0) = u'(0) = u''(0) = 0,$$

has nontrivial solutions if and only if $\lambda_n = -m_n^4 < 0$, being $\{m_n\}_{n \geq 1}$, the increasing sequence of positive (and simple) roots of the following equality

$$\tanh \frac{m}{\sqrt{2}} = \tan \frac{m}{\sqrt{2}}. \tag{4}$$

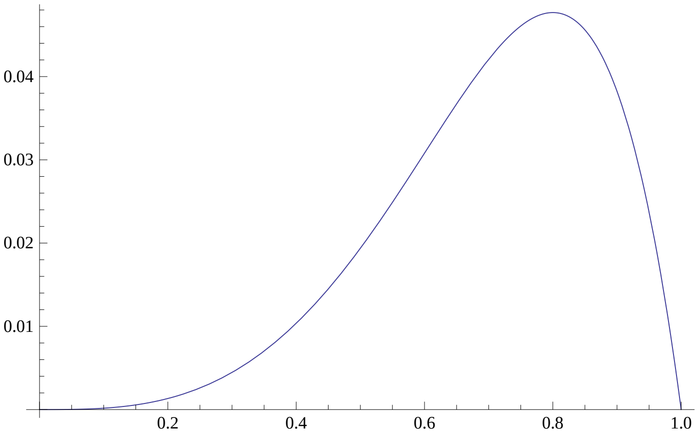

Figure 1: Graphic of Φ_1.

Moreover, the corresponding eigenfunction follows the expression

$$\Phi_n(t) = \frac{1}{\sqrt{2}m_n^3}\left(\cosh\left(\frac{m_n t}{\sqrt{2}}\right)\sin\left(\frac{m_n t}{\sqrt{2}}\right) - \sinh\left(\frac{m_n t}{\sqrt{2}}\right)\cos\left(\frac{m_n t}{\sqrt{2}}\right)\right).$$

It is not difficult to verify that the expression of the Green's function for this situation is given by

$$g_0(t,s) = \frac{1}{6}\begin{cases}(s-1)^3t^3 + (t-s)^3, & 0 \le s \le t \le 1,\\ (s-1)^3 t^3, & 0 \le t \le s \le 1.\end{cases}$$

It is obvious that $g_0(t,s) < 0$ for all $(t,s) \in (0,1)\times(0,1)$. Let's see that condition (N_g) is fulfilled.

Let $m_1 \approx 5.553$ be the smallest positive root of equation (4), which corresponds with $\lambda_1 = -m_1^4 < 0$, the first eigenvalue of operator $u^{(4)}$ on X_0.

It is not difficult to verify (it follows also from classical spectral theory [10, 11, 13]) that $\Phi_1(t) > 0$ for all $t \in (0,1)$.

Now, in order to prove condition (N_g), we use the following properties:

$$\lim_{t\to 0^+}\frac{g_0(t,s)}{\Phi_1(t)} = (s-1)^3 \quad \text{for all } s \in (0,1),$$

and

$$\lim_{t\to 1^-}\frac{g_0(t,s)}{\Phi_1(t)} = \frac{s\,(s-1)^2\,m_1^2}{2\,\sin\left(\frac{m_1}{\sqrt{2}}\right)\sinh\left(\frac{m_1}{\sqrt{2}}\right)} \quad \text{for all } s \in (0,1).$$

As consequence, since $g_0(t,0) = g_0(t,1) = 0$, we can extend function $g_0(t,s)/\Phi_1(t)$ with continuity to the compact interval $I\times I$. So there is a contant

$$k_1 = \min_{(t,s)\in I\times I}\frac{g_0(t,s)}{\Phi_1(t)} < 0.$$

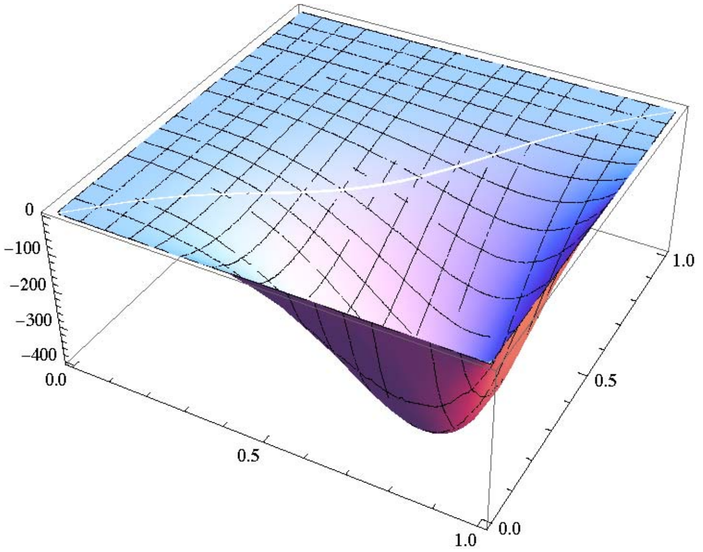

Figure 2: Graphic of $G_M(t,s)$, for $M=(5.553)^4$.

On the other hand, for every $s \in (0,1)$ fixed, we have that function $g_0(\cdot,s)/\Phi_1(\cdot)$ is continuous and strictly negative on I. Thus, due to the continuous dependence of g_0 with respect to s, there exists a continuous function $k_2 : I \to (k_1, 0)$ (in particular it is in $L^1(I)$) such that

$$k_2(s) = \max_{t\in I} \frac{g_0(t,s)}{\Phi_1(t)} < 0.$$

As consequence of the two previous assertions we have that condition (N_g) is fulfilled.

So, from Proposition 2.6 we have that for $M>0$, the set N_L coincides with $(0, m_1^4)$.

Now, we are in a position to study the negative values of the parameter M for which the Green's function g_M is negative on $(0,1)\times(0,1)$.

Since in this case there are not eigenvalues, we must use the disconjugacy property showed in Corollary 2.10. To this end, we will look for a Markov fundamental system of solutions in the line of Proposition 2.12.

Now, let $\epsilon > 0$ be small enough, and denote

$$y_1(t) = z(t+\epsilon),\ y_2(t) = -z'(t+\epsilon),\ y_3(t) = z''(t+\epsilon),\ y_4(t) = -z'''(t+\epsilon),$$

where z is given by the following expression

$$z(t) = \frac{1}{2\,m^3}\left(\sinh m\,t - \sin mt\right).$$

It is immediate to verify that such functions are the unique solutions of the following initial value problems:

$$y_k^{(4)}(t) = m^4\, y_k(t),\ y_k^{(j)}(-\epsilon) = 0,\ j = 0,\ldots,3,\ j \neq 4-k,\ y_k^{(4-k)}(-\epsilon) = (-1)^{k+1}.$$

So, we arrive at the following result

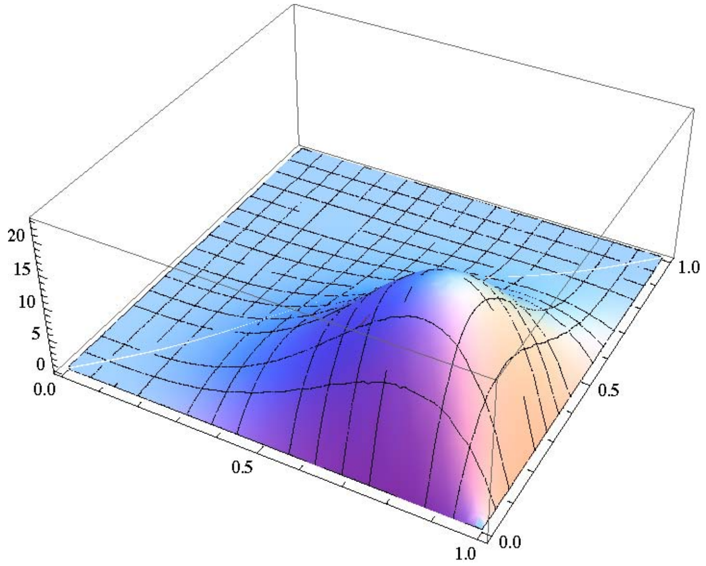

Figure 3: Graphic of $G_M(t,s)$, for $M=(5.554)^4$.

Lemma 3.1. *Equation* $u^{(4)}-m^4u=0$ *is disconjugate on* I *for all* $m\in(0,m_0)$*, where* $m_0\approx 4.730$ *is the first positive root of the algebraic equation*

$$\cos m\cosh m=1.$$

Proof. By construction, the set $\{y_1,y_2,y_3,y_4\}$ forms a fundamental set of solutions of equation $u^{(4)}-m^4u=0$. Let's see that this set is a Markov system on I for all $m\in(0,m_0)$.

Under routine calculations, we obtain that

$$\begin{aligned}
W_1[y_1](t)=W_3[y_1,y_2,y_3](t) &= \frac{1}{2\,m^3}\left(\sinh m\,(t+\epsilon)-\sin m(t+\epsilon)\right),\\
W_2[y_1,y_2](t) &= \frac{1}{2m^4}\left(1-\cos m\,(t+\epsilon)\,\cosh m\,(t+\epsilon)\right),\\
W_4[y_1,y_2,y_3,y_4](t) &= 1.
\end{aligned}$$

Obviously, we have that $W_1[y_1](t)$, $W_3[y_1,y_2,y_3](t)$ and $W_4[y_1,y_2,y_3,y_4](t)$ are strictly positive for all $t>-\epsilon$.

Moreover, by definition of m_0, we have that if $m\in(0,m_0)$ then $W_2[y_1,y_2](t)>0$ for all $t\in(-\epsilon,1-\epsilon)$. Passing to the limit with ϵ tending to zero, we have that the set $\{y_1,y_2,y_3,y_4\}$ is a fundamental Markov system on $[0,1)$.

The result follows from Proposition 2.12. $\square$

As a consequence of the results given in this section we arrive at the following conclusion

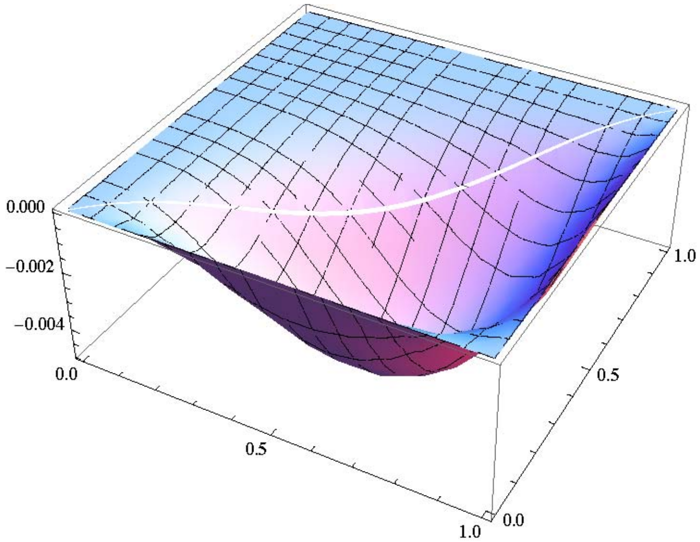

Figure 4: Graphic of $G_M(t,s)$, for $M = -(4.730)^4$.

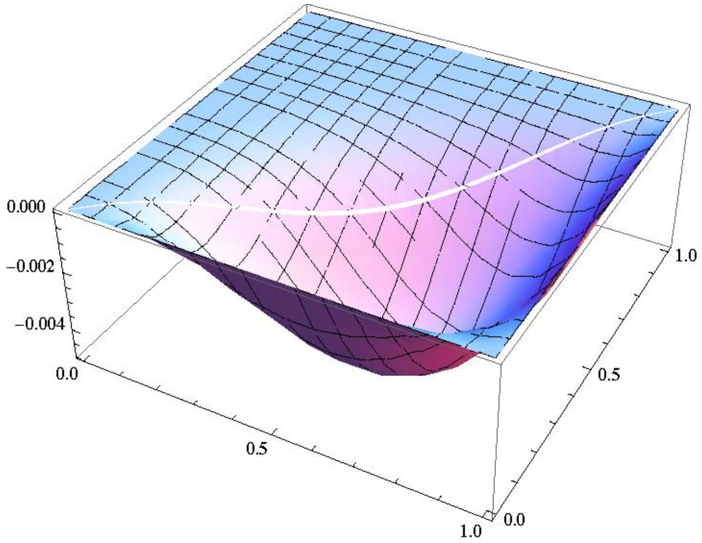

Figure 5: Graphic of $G_M(t,s)$, for $M = -(4.731)^4$.

Theorem 3.2. *If $M \in [-m_0^4, m_1^4)$, then the Green's function g_M, related to problem (1), is such that $g_M(t,s) < 0$ for all $(t,s) \in (0,1) \times (0,1)$.*

Proof. Since $-m_1^4$ is the first eigenvalue of $u^{(4)}$ in X_0, The result follows in $[0, m_1^4)$ from Proposition 2.6. Lemma 3.1 and Corollary 2.10 ensure the validity on $(-m_0^4, 0]$.

Since, from Propositions 2.5 and 2.6, N_M is a closed interval at the left side, we have that $-m_0^4 \in N_M$. □

3.1 Optimal values of N_L and P_L.

In this subsection we characterize the sets N_L and P_L.
Set N_L.

As we have seen along this section, $[-m_0^4, m_1^4) \subset N_L$. Let's see that the equality holds.

First, we notice that, from Proposition 2.6, the Green's function cannot be negative on $(0,1) \times (0,1)$ for all $M \geq m_1^4$. So the right side of the interval is fully described.

Let's see that for any $M < -m_0^4$ the Green's function g_M takes both positive and negative values on $(0,1) \times (0,1)$.

By using the package developed in [5], we deduce that g_M follows the expression (with $M = -m^4$):

$$g_M(t,s) = \begin{cases} G_1(t,s,m), & \text{if } 0 \leq s \leq t \leq 1, \\ G_2(t,s,m), & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

where

$$G_1(t,s,m) \quad = \quad \frac{-e^{m(s-t)} + e^{m(t-s)} - 2\sin(m(t-s))}{4m^3} + G_2(t,s,m),$$

with

$$G_2(t,s,m) = \frac{e^{-m(s+t-1)}\left(e^{m(2s-1)} + 2e^{ms}\sin(m-ms) - e^m\right)\left(e^{2mt} - 2e^{mt}\sin(mt) - 1\right)}{4m^3\left(e^{2m} - 2e^m\sin(m) - 1\right)}.$$

Now, we study the behavior of the function over the diagonal points $(t, 1-t)$ near to the corner point $(1,0)$. As consequence we focus our attention on $f(t,m) := G_1(t, 1-t, m)$ in a neighborhood of $t = 1$.

Thus, we have that $f(1,m) = \frac{\partial f}{\partial t}(1,m) = 0$ for all $m > 0$ and

$$\frac{\partial^2 f}{\partial t^2}(1,m) = \frac{2\left(\cos(m)\cosh(m) - 1\right)}{m(\sinh(m) - \sin(m))}.$$

So, we conclude that $\frac{\partial^2 f}{\partial t^2}(1,m)$ changes its sign from negative to positive in a neighborhood of m_0. Thus, we deduce that g_M cannot be negative on $(0,1) \times (0,1)$ for M on a, small enough, left neighboorhod of $-m_0^4$. Proposition 2.5 ensures that it cannot be negative again for any $M < -m_0^4$.

As a conclusion, we have that $N_L = [-m_0^4, m_1^4)$.

Set P_L.

From Proposition 2.7, we know that g_M cannot be positive on $(0,1) \times (0,1)$ for any $M < -m_0^4$. So we must study the case $M > m_1^4$. We will prove that in such a case the Green's function changes it sign on $(0,1) \times (0,1)$, i.e. P_L is the empty set.

Denoting $M = m^4$, by means of [5] again, we obtain that the Green's function g_M is given by

$$g_M(t,s) = \begin{cases} G_3(t,s,m), & \text{if } 0 \le s \le t \le 1, \\ G_4(t,s,m), & \text{if } 0 \le t \le s \le 1, \end{cases}$$

with

$$G_3(t,s,m) = \frac{e^{-\sqrt{2}m(t-1)}}{2\sqrt{2}m^3\left(\left(e^{\sqrt{2}m}-1\right)\cos\left(\frac{m}{\sqrt{2}}\right) - \left(e^{\sqrt{2}m}+1\right)\sin\left(\frac{m}{\sqrt{2}}\right)\right)} h_1(t,s,m),$$

being

$$\begin{aligned} h_1(t,s,m) = {} & e^{\frac{m(t-s)}{\sqrt{2}}}\left(-e^{\sqrt{2}m(s+t-1)} + e^{\sqrt{2}ms} + e^{\sqrt{2}m(t-1)} - 1\right)\cos\left(\frac{m(s-t-1)}{\sqrt{2}}\right) - \\ & - e^{-\frac{m(s-3t)}{\sqrt{2}}}\left(e^{\sqrt{2}m(s-t-1)} + 1\right)\cos\left(\frac{m(s-t+1)}{\sqrt{2}}\right) + e^{-\frac{m(s-3t)}{\sqrt{2}}}\cos\left(\frac{m(s+t-1)}{\sqrt{2}}\right) + \\ & + e^{\frac{m(s+t-2)}{\sqrt{2}}}\cos\left(\frac{m(s+t-1)}{\sqrt{2}}\right) + e^{-\frac{m(s-3t+2)}{\sqrt{2}}}\sin\left(\frac{m(s-t+1)}{\sqrt{2}}\right) - \\ & - e^{\frac{m(s+t)}{\sqrt{2}}}\sin\left(\frac{m(s-t+1)}{\sqrt{2}}\right) - e^{\frac{m(t-s)}{\sqrt{2}}}\sin\left(\frac{m(s+t-1)}{\sqrt{2}}\right) + \\ & + e^{\frac{m(s+3t-2)}{\sqrt{2}}}\sin\left(\frac{m(s+t-1)}{\sqrt{2}}\right). \end{aligned}$$

Moreover

$$G_4(t,s,m) = \frac{-e^{-\frac{m(s+t-2)}{\sqrt{2}}}}{2\sqrt{2}m^3\left(\left(e^{\sqrt{2}m}-1\right)\cos\left(\frac{m}{\sqrt{2}}\right) - \left(e^{\sqrt{2}m}+1\right)\sin\left(\frac{m}{\sqrt{2}}\right)\right)} h_2(t,s,m)$$

where

$$\begin{aligned} h_2(t,s,m) \quad = \quad & \left(\left(e^{\sqrt{2}m(s-1)}-1\right)\cos\left(\frac{m(s-1)}{\sqrt{2}}\right) - \left(e^{\sqrt{2}m(s-1)}+1\right)\sin\left(\frac{m(s-1)}{\sqrt{2}}\right)\right) \times \\ & \times \left(e^{\sqrt{2}mt}-1\right)\cos\left(\frac{mt}{\sqrt{2}}\right) - \left(e^{\sqrt{2}mt}+1\right)\sin\left(\frac{mt}{\sqrt{2}}\right). \end{aligned}$$

To verify that g_M has not constant sign on $(0,1) \times (0,1)$ we study the values of $h(t,m) := g_M(t,t)$ near to $t = 0$.

In this case we have that, for all $m > 0$ it is satisfied

$$h(0,m) = \frac{\partial h}{\partial t}(0,m) = \frac{\partial^2 h}{\partial t^2}(0,m) = 0,$$

and

$$\frac{\partial^3 h}{\partial t^3}(0,m) = -1.$$

In particular, we decude that g_M takes negative values for all $M > m_1^4$. So it cannot be positive on $(0,1) \times (0,1)$. Now, Proposition 2.7 implies that it tales both positive and negative values for this range of parameters. In consequence $P_L = \emptyset$.

To finalize this section we consider the following boundary value problem

$$u^{(4)}(t) + M\,u(t) = \sigma(t), \text{ a.e. } t \in I;\ u(0) = u(1) = u'(1) = u''(1) = 0. \tag{5}$$

In [4, Section 1.4] one can see that problem (5) is just the adjoint of (1). So, the eigenvalues of both problems coincide and the Green's function g_m^* related to this problem is given by

$$g_M^*(t,s) = g_M(s,t).$$

In particular the two functions take the same values at $I \times I$ and the range of the maximum and anti-maximum principles on the set

$$Y_0 = \{u \in W^{4,1}(I), \quad u(0) = u(1) = u'(1) = u''(1) = 0\}$$

is the same.

4 Non homogeneous boundary conditions

In this section we obtain maximum principles for operator L_M in wider sets than X_0. We are dealing with the case in which suitable inequalities on the boundary of the interval I are allowed. By direct calculation, one can verify the following property

Lemma 4.1. *Let $\sigma \in L^1(I)$ and $a_1, a_2, a_3, b \in \mathbb{R}$. Assume that the boundary value problem*

$$\begin{cases} u^{(4)}(t) + Mu(t) = \sigma(t), \quad t \in I, \\ u(0) = a_1,\ u'(0) = a_2,\ u''(0) = a_3,\ u(1) = b, \end{cases} \tag{6}$$

with $M = \pm m^4$, has only the trivial solution for $\sigma \equiv 0$ and $a_1 = a_2 = a_3 = b = 0$.

Then (6) has a unique solution given by

$$u(t) = \int_0^1 g_M(t,s)\,\sigma(s)\,ds + \sum_{k=1}^{3} a_k\, v_{k,m}(t) + b\, w_m(t),$$

where $v_{k,m}$, $k=1,2,3$, and w_m are defined respectively as the unique solutions of

$$\begin{cases} u^{(4)}(t)+M\,u(t)=0, \quad t\in I,\\ u^{(k-1)}(0)=1,\; u^{(j)}(0)=0,\; j=0,1,2,\; j\neq k-1,\; u(1)=0, \end{cases} \tag{7}$$

and

$$\begin{cases} u^{(4)}(t)+M\,u(t)=0, \quad t\in I,\\ u^{(j)}(0)=0,\; j=0,1,2,\; u(1)=1. \end{cases} \tag{8}$$

First, consider the case $M=0$. It is immediate to verify that $v_{1,0}(t)=1-t^3$, $v_{2,0}(t)=t-t^3$, $v_{3,0}(t)=(t^2-t^3)/2$ and $w_0(t)=t^3$. It is obvious that all of them are strictly positive on $[0,1)$.

Let's see now what happens with $M=-m^4<0$.

Concerning function, $v_{1,m}(t)$, since $v_{1,0}'(1)<0$, we have, by continuity with respect to the parameter m at 0, that there is $\bar{m}$ such that $v_{1,m}(t)>0$ on $[0,1)$ for all $m\in[0,\bar{m})$.

Now, it is not difficult to verify that

$$v_{1,m}'(1)=\frac{m\sin(m)\sinh(m)}{\sin(m)-\sinh(m)}.$$

So, we have that $v_{1,m}'(1)<0$ for all $m\in(0,\pi)$.

Thus, if it changes sign on $(0,1)$ for some $m\in(0,\pi)$, there are $\tilde{m}\in(\bar{m},\pi)$ and $t_0\in(0,1)$ such that $v_{1,\tilde{m}}\geq 0$ on I and $v_{1,\tilde{m}}(t_0)=v_{1,\tilde{m}}'(t_0)=0$. So $v_{1,\tilde{m}}$ must be a nontrivial solution of problem

$$u^{(4)}(t)-\tilde{m}^4u(t)=0, \quad t\in I, u'(0)=u''(0)=u(t_0)=u'(t_0). \tag{9}$$

But this problem has nontrivial solutions if and only if there is $l\in\{1,2,\dots\}$ such that $t_0=l\,\pi/\tilde{m}>1$, a contradiction. In consequence $v_{1,m}(t)>0$ on $[0,1)$ for all $m\in(0,\pi]$.

Let's see that $v_{1,m}$ changes sign for any $m>\pi$. First, since the roots of $m=k\,\pi$, $k=1,2,\dots$, are the (simple) roots of equation $v_{1,m}'(1)=0$, we have that $v_{1,m}$ changes sign for all $m\in(\pi,2\,\pi)$. If there are $\hat{m}>2\,\pi$, with $v_{1,\hat{m}}>0$ on $[0,1)$, we have that

$$(v_{1,\pi}-v_{1,\hat{m}})^{(4)}(t)-\pi^4(v_{1,\pi}-v_{1,\hat{m}})(t)=(\pi^4-\hat{m}^4)\,v_{1,\hat{m}}(t)<0, \quad t\in[0,1).$$

So, since $\pi<m_0$ and $v_{1,\pi}-v_{1,\hat{m}}\in X_0$, the negative sign of the corresponding Green's function implies that $v_{1,\pi}>v_{1,\hat{m}}$ on $(0,1)$.

But, from $v_{1,\pi}'(1)=0$ we deduce that $v_{1,\hat{m}}'(1)=0$. As consequence, $\hat{m}=k\,\pi$ for some $k=3,4,\dots$. So $v_{1,\hat{m}}$ is an eigenfunction of problem (9), with $t_0=1$ and, by classical spectral theory, has $k-1$ simple zeroes on $(0,1)$.

The expression of $v_{2,m}$ is given by

$$v_{2,m}(t)=\frac{\sin(m)\sinh(mt)-\sinh(m)\sin(mt)}{m\,(\sin(m)-\sinh(m))}.$$

In [6] it is proved that this expression is positive on $(0,1)$ if and only if $m \in (0, m_1/\sqrt{2}]$.

To study $v_{3,m}$ we have that if there is a double zero on $(0,1)$ then it must be an eigenfunction of problem

$$u^{(4)}(t) - \tilde{m}^4 u(t) = 0, \quad t \in I, u(0) = u'(0) = u(t_0) = u'(t_0).$$

The eigenvalues of this equation are the roots of

$$\cos(m\, t_0) \cosh(m\, t_0) = 1.$$

In this case $t_0 \in (0,1)$ if and only if $m > m_0$. So, arguing as for $v_{1,m}$, we conclude that $v_{3,m} > 0$ on $(0,1)$ if and only if $m \in (0, m_0]$.

Finally, one can verify that

$$w_m(t) = \frac{\sinh(mt) - \sin(mt)}{\sinh(m) - \sin(m)}$$

is strictly positive on $(0,1]$ for all $m > 0$.

Considering the case $M = m^4 > 0$, we now that the function $v_{1,m}$ is positive for m close enough to 0. Now, since equation

$$u^{(4)}(t) = \lambda\, u(t),\ t \in I, \quad u'(0) = u''(0) = u(t_0) = u'(t_0),$$

has only negative eigenvalues, we conclude that $v_{1,m}$ cannot have double zeroes on $(0,1)$. So, using the fact that

$$v'_{1,m}(1) = \frac{m\left(\cos\left(\sqrt{2}m\right) - \cosh\left(\sqrt{2}m\right)\right)}{2\sqrt{2}\left(\sin\left(\frac{m}{\sqrt{2}}\right)\cosh\left(\frac{m}{\sqrt{2}}\right) - \cos\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right)\right)},$$

we deduce that this function remains positive on $(0,1)$ up to the first eigenvalue m_1.

Similar arguments holds for $v_{2,m}$, $v_{3,m}$ together with

$$v'_{2,m}(1) = \frac{\sinh\left(\frac{m}{\sqrt{2}}\right)\cosh\left(\frac{m}{\sqrt{2}}\right) - \sin\left(\frac{m}{\sqrt{2}}\right)\cos\left(\frac{m}{\sqrt{2}}\right)}{\cos\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right) - \sin\left(\frac{m}{\sqrt{2}}\right)\cosh\left(\frac{m}{\sqrt{2}}\right)},$$

$$v'_{3,m}(1) = -\frac{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2}{2\sqrt{2}m\left(\sin\left(\frac{m}{\sqrt{2}}\right)\cosh\left(\frac{m}{\sqrt{2}}\right) - \cos\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right)\right)}.$$

The positiveness of w_m on $(0,1)$ for m $(0, m_1)$ holds from the expression

$$w'_m(t) = \frac{\sqrt{2}m \sin\left(\frac{mt}{\sqrt{2}}\right)\sinh\left(\frac{mt}{\sqrt{2}}\right)}{\sin\left(\frac{m}{\sqrt{2}}\right)\cosh\left(\frac{m}{\sqrt{2}}\right) - \cos\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right)}.$$

Define the sets

$$X_1 = \{u \in W^{4,1}(I), \quad u(0) = u'(0) = 0, \quad u''(0) \leq 0,\, u(1) \leq 0\}.$$

$$X_2 = \{u \in W^{4,1}(I), \quad u(0) = 0,\ u'(0) \leq 0, \quad u''(0) \leq 0,\, u(1) \leq 0\}.$$

$$X_3 = \{u \in W^{4,1}(I), \quad u(0) \leq 0,\ u'(0) = 0, \quad u''(0) \leq 0,\, u(1) \leq 0\}.$$

$$X_4 = \{u \in W^{4,1}(I), \quad u(0) \leq 0,\ u'(0) \leq 0, \quad u''(0) \leq 0,\, u(1) \leq 0\}.$$

So, as consequence of all the previous results, taking into account that $m_1^4/4 \approx 3,9266 > \pi$, using Lemma 4.1 we obtain the following maximum principles for operator L_M.

Theorem 4.2. *The following assertions hold:*

1. *Operator L_M is strongly inverse negative on X_1 if and only if $M \in [-m_0^4, m_1^4)$.*
2. *Operator L_M is strongly inverse negative on X_2 if and only if $M \in [-m_1^4/4, m_1^4)$.*
3. *Operator L_M is strongly inverse negative on X_3 if and only if $M \in [-\pi^4, m_1^4)$.*
4. *Operator L_M is strongly inverse negative on X_4 if and only if $M \in [-\pi^4, m_1^4)$.*

We can consider a new set, by assuming that the values of the function are the same at the extremes of the interval of definition:

$$X_5 = \{u \in W^{4,1}(I), \quad u(0) = u(1) \leq 0, \quad u'(0) \leq 0, \quad u''(0) \leq 0\}.$$

In this case, using Lemma 4.1 again, we must study the behavior of function $x_m := v_{1,m} + w_m$ on I.

From the previous results, we have that $x_m > 0$ on $(0,1)$ for all $M \in [-\pi^4, m_1^4)$. Since, m_1^4 is the best possible estimate obtained for g_M, we can improve the range of values of M for which operator is strongly inverse negative on X_5 only for $M < 0$.

So, since $x_m(0) = x_m(1) = 1$, and problem (1) has a unique solution for any $M < 0$, we have that the function changes sign on $(0,1)$ if and only if there is $\bar{m}$ and $t_0 \in (0,1)$ such that $u_{\bar{m}}(t_0) = u'_{\bar{m}}(t_0) = 0$. In such a case, $x_{\bar{m}}$ is an eigenfunction of problem (9) and, therefore, $t_0 = k\,\pi/\bar{m}$ for some $k \in \{1, 2, \ldots\}$. Taking into account that $2\,\pi/\bar{m} < 1$ if and only if $\bar{m} > 2\,\pi (> m_0)$, we only need to study the case $k = 1$.

From the fact that ($M = m^4$)

$$x_m(\pi/m) = \frac{\sinh(\pi)(\cos(m) + \cosh(m) - 2) + (\cosh(\pi) - 1)(\sin(m) - \sinh(m))}{2(\sin(m) - \sinh(m))},$$

we have that $x_m > 0$ on $(0,1)$ if and only if $m \in (0, \bar{m}_0)$, where $\bar{m}_0 \approx 4.3474$ is the unique solution of the algebraic equation

$$\frac{\sinh(\pi)}{\cosh(\pi)-1} = \frac{\sinh(m)-\sin(m)}{\cos(m)+\cosh(m)-2}.$$

So we arrive at the following result

Theorem 4.3. *Operator L_M is strongly inverse negative on X_5 if and only if $M \in [-\bar{m}_0^4, m_1^4)$.*

Taking into account that $v_{k,m}(t)$, $k = 1,2,3$, and $w_m(t)$ are the unique solutions of problems (7) and (8) if and only if $v_{k,m}(1-t)$, $k = 1,2,3$, and $w_m(1-t)$ are the unique solutions of

$$\begin{cases} u^{(4)}(t) + M\,u(t) = 0, \quad t \in I, \\ u^{(k-1)}(1) = 1,\ u^{(j)}(1) = 0,\ j = 0,1,2,\ j \neq k-1,\ u(0) = 0, \end{cases}$$

and

$$\begin{cases} u^{(4)}(t) + M\,u(t) = 0, \quad t \in I, \\ u^{(j)}(1) = 0,\ j = 0,1,2,\ u(0) = 1, \end{cases}$$

we can characterize maximum principles for operator L_M on the sets

$$\begin{aligned} Y_1 &= \{u \in W^{4,1}(I), \quad u(1) = u'(1) = 0, \quad u''(1) \le 0,\ u(0) \le 0\}. \\ Y_2 &= \{u \in W^{4,1}(I), \quad u(1) = 0,\ u'(1) \le 0, \quad u''(1) \le 0,\ u(0) \le 0\}. \\ Y_3 &= \{u \in W^{4,1}(I), \quad u(1) \le 0,\ u'(1) = 0, \quad u''(1) \le 0,\ u(0) \le 0\}. \\ Y_4 &= \{u \in W^{4,1}(I), \quad u(1) \le 0,\ u'(1) \le 0, \quad u''(1) \le 0,\ u(0) \le 0\}. \\ Y_5 &= \{u \in W^{4,1}(I), \quad u(1) = u(0) \le 0, \quad u'(1) \le 0, \quad u''(1) \le 0\}. \end{aligned}$$

The result is the following

Theorem 4.4. *The following assertions hold:*

1. *Operator L_M is strongly inverse negative on Y_1 if and only if $M \in [-m_0^4, m_1^4)$.*
2. *Operator L_M is strongly inverse negative on Y_2 if and only if $M \in [-m_1^4/4, m_1^4)$.*
3. *Operator L_M is strongly inverse negative on Y_3 if and only if $M \in [-\pi^4, m_1^4)$.*
4. *Operator L_M is strongly inverse negative on Y_4 if and only if $M \in [-\pi^4, m_1^4)$.*
5. *Operator L_M is strongly inverse negative on Y_5 if and only if $M \in [-\bar{m}_0^4, m_1^4)$.*

5 Nonlinear boundary value problem

In this section we apply the previous results to deduce the existence of solution for the boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \quad \text{a.e} \quad t \in I, \\ u^{(k)}(0) = A_k, \ k = 0, 1, 2, \ j \neq k-1, \ u(1) = B, \end{cases} \tag{10}$$

Here $A_k, \ k = 0, 1, 2$, and B are real numbers, $f : I \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e, it is continuous in the second variable, Lebesgue measurable in the first and, for any $R > 0$ there is $h_R \in L^1(I)$ such that $|f(t,x)| \leq h_R(t)$ for a.e. $t \in I$ and $|x| \leq R$.

We introduce the following sets

$$Z_1 = \{u \in W^{4,1}(I), \quad u(0) = A_0, \ u'(0) = A_1, \ u''(0) \leq A_2, \ u(1) \leq B\}.$$

$$Z_2 = \{u \in W^{4,1}(I), \quad u(0) = A_0, \ u'(0) \leq A_1, \ u''(0) \leq A_2, \ u(1) \leq B\}.$$

$$Z_3 = \{u \in W^{4,1}(I), \quad u(0) \leq A_0, \ u'(0) = A_1, \ u''(0) \leq A_2, \ u(1) \leq B\}.$$

$$Z_4 = \{u \in W^{4,1}(I), \quad u(0) \leq A_0, \ u'(0) \leq A_1, \ u''(0) \leq A_2, \ u(1) \leq B\}.$$

$$Z_5 = \{u \in W^{4,1}(I), \quad u(0) = u(1) \leq A_0 = B, \ u'(0) \leq A_1, \ u''(0) \leq A_2\}.$$

The obtained results are in the framework of the classical monotone iterative techniques [1, 2, 4, 6, 7, 14].

Definition 5.1. *We say that $\alpha_k \in Z_k$, $k \in \{1, 2, 3, 4, 5\}$, is a* lower solution *of problem* (10) *if*

$$\alpha_k^{(4)}(t) \geq f(t, \alpha_k(t)), \quad \text{a.e. } t \in I. \tag{11}$$

We say that β_k, $k \in \{1, 2, 3, 4, 5\}$, is an upper solution *of* (10) *if $-\beta_k \in Z_k$ and*

$$\beta_k^{(4)}(t) \leq f(t, \beta_k(t)), \quad \text{a.e. } t \in I. \tag{12}$$

Now we assume the following condition.

(H_d) There is $M \in \mathbb{R}$ such that $f(t,x) + M\,x \leq f(t,y) + M\,y$ for a.e. $t \in I$ and $\alpha(t) \leq y \leq x \leq \beta(t)$.

The following result ensures the existence of extremal solutions of problem (10) lying between a pair of well ordered lower and upper solutions. It is a particular case of [4, Theorem 1.7.2].

Theorem 5.2. *Suppose that there exist $\alpha_k \leq \beta_k$, $k \in \{1, 2, 3, 4, 5\}$, a pair of well ordered lower and upper solutions respectively for the nonlinear boundary value problem* (10). *Assume that the function f satisfies the condition (H_d) for some $M \in \mathbb{R}$ such that the operator L_M is strongly inverse negative in X_k.*

Then there exist two monotone sequences $\{\beta_m^k\}$ and $\{\alpha_m^k\}$, nonincreasing and nondecreasing respectively, with $\beta_0^k = \beta_k$ and $\alpha_0^k = \alpha_k$, which converge uniformly to the extremal solutions in $[\alpha, \beta]$ of problem (10).

So, we can rewrite the previous result in the following way, depending on the chosen set Z_k.

Corollary 5.3. *Suppose that there exist a pair of functions* $\alpha \leq \beta$ *satisfying* (11) *and* (12) *respectively, together with one of the following additional conditions:*

1. $\alpha(0) = A_0 = \beta(0),\ \alpha'(0) = A_1 = \beta'(0),\ \alpha''(0) \leq A_2 \leq \beta''(0),\ \alpha(1) \leq B \leq \beta(1)$, *and condition* (H_d) *holds for* $M \geq -m_0^4$.
2. $\alpha(0) = A_0 = \beta(0),\ \alpha'(0) \leq A_1 \leq \beta'(0),\ \alpha''(0) \leq A_2 \leq \beta''(0),\ \alpha(1) \leq B \leq \beta(1)$, *and condition* (H_d) *holds for* $M \geq -m_1^4/4$.
3. $\alpha(0) \leq A_0 \leq \beta(0),\ \alpha'(0) \leq A_1 \leq \beta'(0),\ \alpha''(0) \leq A_2 \leq \beta''(0),\ \alpha(1) \leq B \leq \beta(1)$, *and condition* (H_d) *holds for* $M \geq -\pi^4$.
4. $\alpha(0) = \alpha(1) \leq A_0 = B \leq \beta(0) = \beta(1),\ \alpha'(0) \leq A_1 \leq \beta'(0),\ \alpha''(0) \leq A_2 \leq \beta''(0)$, *and condition* (H_d) *holds for* $M \geq -\bar{m}_0^4$.

Then there exist two monotone sequences $\{\beta_m\}$ *and* $\{\alpha_m\}$, *nonincreasing and nondecreasing respectively, with* $\beta_0 = \beta$ *and* $\alpha_0 = \alpha$, *which converge uniformly to the extremal solutions in* $[\alpha, \beta]$ *of problem* (10).

To study the boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \quad \text{a.e} \quad t \in I, \\ u^{(k)}(1) = A_k,\ k = 0, 1, 2,\ j \neq k-1,\ u(0) = B, \end{cases} \tag{13}$$

we introduce the following sets

$$W_1 = \{u \in W^{4,1}(I), \quad u(1) = A_0,\ u'(1) = A_1,\ u''(1) \leq A_2,\ u(0) \leq B\}.$$

$$W_2 = \{u \in W^{4,1}(I), \quad u(1) = A_0,\ u'(1) \leq A_1,\ u''(1) \leq A_2,\ u(0) \leq B\}.$$

$$W_3 = \{u \in W^{4,1}(I), \quad u(1) \leq A_0,\ u'(1) = A_1,\ u''(1) \leq A_2,\ u(0) \leq B\}.$$

$$W_4 = \{u \in W^{4,1}(I), \quad u(1) \leq A_0,\ u'(1) \leq A_1,\ u''(1) \leq A_2,\ u(0) \leq B\}.$$

$$W_5 = \{u \in W^{4,1}(I), \quad u(1) = u(0) \leq A_0 = B,\ u'(1) \leq A_1,\ u''(1) \leq A_2\}.$$

We define a lower and an upper solution as follows

Definition 5.4. *We say that* $\alpha_k \in W_k$, $k \in \{1, 2, 3, 4, 5\}$, *is a* lower solution *of problem* (13) *if it satisfies the inequality* (11).
We say that β_k, $k \in \{1, 2, 3, 4, 5\}$, *is an* upper solution *of* (13) *if* $-\beta_k \in W_k$ *and* (12) *is fulfilled.*

So, as a consequence of Theorem 5.2, with obvious application to problem (13), we obtain the following result, depending on W_k.

Corollary 5.5. *Suppose that there exist a pair of functions* $\alpha \leq \beta$ *satisfying* (11) *and* (12) *respectively, together with one of the following additional conditions:*

1. $\alpha(1) = A_0 = \beta(1),\ \alpha'(1) = A_1 = \beta'(1),\ \alpha''(1) \leq A_2 \leq \beta''(1),\ \alpha(0) \leq B \leq \beta(0)$, *and condition* (H_d) *holds for* $M \geq -m_0^4$.
2. $\alpha(1) = A_0 = \beta(1),\ \alpha'(1) \leq A_1 \leq \beta'(1),\ \alpha''(1) \leq A_2 \leq \beta''(1),\ \alpha(0) \leq B \leq \beta(0)$, *and condition* (H_d) *holds for* $M \geq -m_1^4/4$.
3. $\alpha(1) \leq A_0 \leq \beta(1),\ \alpha'(1) \leq A_1 \leq \beta'(1),\ \alpha''(1) \leq A_2 \leq \beta''(1),\ \alpha(0) \leq B \leq \beta(0)$, *and condition* (H_d) *holds for* $M \geq -\pi^4$.
4. $\alpha(1) = \alpha(0) \leq A_0 = B \leq \beta(1) = \beta(0),\ \alpha'(1) \leq 0 \leq \beta'(1),\ \alpha''(1) \leq A_2 \leq \beta''(1)$, *and condition* (H_d) *holds for* $M \geq -\bar{m}_0^4$.

Then there exist two monotone sequences $\{\beta_m\}$ *and* $\{\alpha_m\}$, *nonincreasing and nondecreasing respectively, with* $\beta_0 = \beta$ *and* $\alpha_0 = \alpha$, *which converge uniformly to the extremal solutions in* $[\alpha, \beta]$ *of problem* (13).

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