The zero-Hopf bifurcations in the Kolmogorov systems of degree 3 in $\mathbb{R}^3$

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Abstract

In this work we study the periodic orbits which bifurcate from all zero-Hopf bifurcations that an arbitrary Kolmogorov system of degree 3 in $\mathbb{R}^3$ can exhibit. The main tool used is the averaging theory.

Keywords: Lotka–Volterra system, Kolmogorov systems, zero-Hopf bifurcation, limit cycle

1. Introduction and statement of the main results

Lotka-Volterra systems were initially proposed, independently, by Alfred J. Lotka in 1925 [1] and Vito Volterra in 1926 [2], both in the context of competing species. These Lotka-Volterra systems are differential systems of the form

$$\dot{x} = xP(x, y), \qquad \dot{y} = yQ(x, y),$$

where P and Q are polynomials of degree 1. Later on the *Lotka-Volterra systems* were generalized and considered on arbitrary dimension $n \geq 2$, i.e.

$$\dot{x}_i = x_i P_i(x_1, \ldots, x_n),$$

where P_i are polynomials of degree 1. Finally in 1936 Andrei Kolmogorov [3] extended those systems to arbitrary degree, i.e. the polynomials P_i can have any degree. These last systems are now called *Kolmogorov systems*.

The Lotka-Volterra and Kolmogorov systems have been used for modelling many natural phenomena, such as the time evolution of conflicting species in biology [4], chemical reactions [5], plasma physics [6], hydrodynamics [7], and many other phenomena as social science and economics [8].

We want to study the limit cycles of the Kolmogorov systems of degree 3 in $\mathbb{R}^3$ which bifurcate in the zero-Hopf bifurcations of the singular points (a, b, c) which are not on the invariant planes $x = 0$, $y = 0$ and $z = 0$ of the Kolmogorov system

$$\dot{x} = xP(x, y, z), \quad \dot{y} = yQ(x, y, z), \quad \dot{z} = zR(x, y, z),$$

with P, Q and R polynomials of degree 2. Doing the scaling $(x, y, z) \to (x/a, y/b, z/c)$ we can assume without loss of generality that $(a, b, c) = (1, 1, 1)$. Therefore it is sufficient to study the limit cycles which can bifurcate

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from the singular point $(1,1,1)$ of the system

$$
\begin{aligned}
\dot{x} &= x\left(a_1(x-1)+a_2(y-1)+a_3(z-1)+a_4(x-1)^2+a_5(x-1)(y-1)\right.\\
&\qquad \left.+\,a_6(x-1)(z-1)+a_7(y-1)^2+a_8(y-1)(z-1)+a_9(z-1)^2\right),\\
\dot{y} &= y\left(b_1(x-1)+b_2(y-1)+b_3(z-1)+b_4(x-1)^2+b_5(x-1)(y-1)\right.\\
&\qquad \left.+\,b_6(x-1)(z-1)+b_7(y-1)^2+b_8(y-1)(z-1)+b_9(z-1)^2\right),\\
\dot{z} &= z\left(c_1(x-1)+c_2(y-1)+c_3(z-1)+c_4(x-1)^2+c_5(x-1)(y-1)\right.\\
&\qquad \left.+\,c_6(x-1)(z-1)+c_7(y-1)^2+c_8(y-1)(z-1)+c_9(z-1)^2\right),
\end{aligned}
\tag{1.1}
$$

when this singular point is a *zero-Hopf equilibrium*, i.e. when the eigenvalues of the linear part of the system at $(1,1,1)$ are of the form 0 and $\pm\beta i$ with $\beta>0$. Here the dot denotes derivative with respect to the time t.

Limit cycles, i.e. isolated periodic orbits in the set of all periodic orbits of a differential system, play an important role in the qualitative theory of differential equations. The behavior of many real-world oscillatory systems have been modelized by limit cycles, see for instance the famous limit cycle of van der Pol [9]. The study of limit cycles was initiated by Poincaré [10]. A big interest in their study was motivated by the famous 16th Hilbert problem [11, 12, 13]. Limit cycles are also studied in dimension higher than two, see for instance [14].

In the next result we characterize when the singular point $(1,1,1)$ is zero-Hopf.

Proposition 1.1. *The singular point $(1,1,1)$ of system (1.1) is zero-Hopf if and only if one of the following sets of conditions hold, with $\gamma=a_3b_3(b_2-a_1)-a_2b_3^2+a_3^2b_1$ and $\beta>0$:*

(i) $\gamma=0$, $c_3=-a_1-b_2$,

$$c_1=\frac{1}{\gamma}\left(a_1^3b_3-a_1^2a_3b_1-a_1\left(a_3b_1b_2-b_3\left(2a_2b_1+\beta^2\right)\right)-b_1(a_2(a_3b_1-b_2b_3)+a_3\left(\beta^2+b_2^2\right))\right) \text{ and}$$

$$c_2=\frac{1}{\gamma}\left(a_1^2a_2b_3+a_1a_2(b_2b_3-a_3b_1)+a_2^2b_1b_3-a_3b_2\left(\beta^2+b_2^2\right)+a_2\left(b_3\left(\beta^2+b_2^2\right)-2a_3b_1b_2\right)\right).$$

(ii) $\gamma\neq0$, $a_3b_3\neq0$, $a_2=\dfrac{a_3b_2}{b_3}$, $b_1=\dfrac{a_1b_3}{a_3}$, $c_3=-a_1-b_2$ *and* $c_2=-\dfrac{(a_1+b_2)^2+a_3c_1+\beta^2}{b_3}$.

(iii) $\gamma\neq0$, $b_3\neq0$, $a_1=a_2=a_3=0$, $c_2=-\dfrac{b_2^2+\beta^2}{b_3}$ *and* $c_3=-b_2$.

(iv) $\gamma\neq0$, $a_3\neq0$, $b_1=b_2=b_3=0$, $c_1=-\dfrac{a_1^2+\beta^2}{a_3}$ *and* $c_3=-a_1$.

(v) $\gamma\neq0$, $b_1\neq0$, $a_2=-\dfrac{a_1^2+\beta^2}{b_1}$, $a_3=b_3=c_3=0$ *and* $b_2=-a_1$.

Proposition 1.1 is proved in section 2.

In Theorem 3 of [15] are provided sufficient conditions in order that the Kolmogorov systems (1.1) under conditions (i) exhibit a zero-Hopf bifurcation from which two limit cycles bifurcate, the kind of stability or inestability of these limit cycles is also provided.

In this paper we use the averaging theory of first order for studying the limit cycles bifurcating from the zero-Hopf bifurcations of the Kolmogorov systems (1.1) under conditions (ii)–(v).

Our main result concerning the Kolmogorov systems (1.1) under the conditions (ii) is the following. The expressions of A_i, with $i=0,\dots,4$, K_1 and N are defined in Appendix B.

Theorem 1.2. *If $a_3b_3\neq0$, $N\neq0$, $a_2=a_3b_2/b_3$, $b_1=a_1b_3/a_3$, $c_3=-a_1-b_2$, $c_2=-((a_1+b_2)^2+a_3c_1+\beta^2)/b_3$, $A_1\neq0$, $A_2\neq0$, $A_3\neq0$,and $A_0A_4(A_1A_2-A_0A_3)>0$, then the Kolmogorov system (1.1) has two limit cycles bifurcating from the zero-Hopf equilibrium point $(1,1,1)$. Moreover the following statements hold.*

(a) If $K_1 > 0$, $A_2A_3(A_0A_3 - A_1A_2)N < 0$ *and* $|2A_0A_3 - A_1A_2| < \sqrt{K_1}$, *then the two limit cycles have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(b) If $b_3A_2N > 0$, $b_3A_3(A_0A_3 - A_1A_2) > 0$ *and*

- *either* $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 - \sqrt{K_1}) < 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) < 0$,
- *or* $K_1 \leq 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2) < 0$;

or if $b_3A_2N < 0$, $b_3A_3(A_0A_3 - A_1A_2) < 0$ *and*

- *either* $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 - \sqrt{K_1}) > 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) > 0$,
- *or* $K_1 \leq 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2) > 0$;

then one limit cycle is local repeller, and the other is a local attractor.

(c) If $b_3A_2N > 0$, $b_3A_3(A_0A_3 - A_1A_2) > 0$, $K_1 > 0$ *and* $|2A_0A_3 - A_1A_2| < \sqrt{K_1}$; *or if* $A_2A_3(A_0A_3 - A_1A_2)N < 0$ *and*

- *either* $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 - \sqrt{K_1}) > 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) > 0$,
- *or* $K_1 \leq 0$ *and* $b_3A_1(2A_0A_3 - A_1A_2)N > 0$;

then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(d) If $b_3A_2N < 0$, $b_3A_3(A_0A_3 - A_1A_2) < 0$, $K_1 > 0$ *and* $|2A_0A_3 - A_1A_2| < \sqrt{K_1}$; *or if* $A_2A_3(A_0A_3 - A_1A_2)N < 0$ *and*

- *either* $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 - \sqrt{K_1}) < 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) < 0$,
- *or* $K_1 \leq 0$ *and* $b_3A_1(2A_0A_3 - A_1A_2)N < 0$;

then one limit cycle is a local attractor, and the other is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(e) If $K_1 < 0$, $A_2A_3(A_0A_3 - A_1A_2)N < 0$ *and* $2A_0A_3 = A_1A_2$; *then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.*

The main result concerning the Kolmogorov systems (1.1) under the conditions (iii) is the following. The expressions of B_i with $i = 0, ..., 4$, and K_2 are given in Appendix B.

Theorem 1.3. *If* $b_3 \neq 0$, $a_1 = a_2 = a_3 = 0$, $c_2 = -(b_2^2 + \beta^2)/b_3$, $c_3 = -b_2$, $B_1 \neq 0$, $B_2 \neq 0$, $B_3 \neq 0$ *and* $B_0B_4(B_1B_2 - B_0B_3) > 0$, *then the Kolmogorov system* (1.1) *has two limit cycles bifurcating from the zero-Hopf equilibrium point* $(1, 1, 1)$. *Moreover the following statements hold.*

(a) If $K_2 > 0$, $B_2B_3(B_0B_3 - B_1B_2) > 0$ *and* $|B_1B_2 - 2B_0B_3| < \sqrt{K_2}$; *then the two limit cycles are unstable and have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(b) If $B_2 < 0$, $B_3(B_0B_3 - B_1B_2) > 0$, $K_2 > 0$ *and* $|B_1B_2 - 2B_0B_3| < \sqrt{K_2}$; *or if* $B_2B_3(B_0B_3 - B_1B_2) > 0$, $K_2 > 0$, $B_1 < 0$ *and* $B_1B_2 - 2B_0B_3 < -\sqrt{K_2}$; *then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(c) If $B_2 < 0$, $B_3(B_0B_3 - B_1B_2) > 0$ *and*

- *either* $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) < 0$ *and* $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) < 0$,
- *or* $K_2 \leq 0$ *and* $B_1(B_1B_2 - 2B_0B_3) < 0$;

or if $B_2 > 0$, $B_3(B_0B_3 - B_1B_2) < 0$, $K_2 > 0$, $B_1 < 0$ *and* $B_1B_2 - 2B_0B_3 < -\sqrt{K_2}$; *then one limit cycle is a local attractor and the other limit cycle is a local repeller.*

(d) If $B_2 > 0$, $B_3(B_0B_3 - B_1B_2) < 0$, $K_2 > 0$ and $|B_1B_2 - 2B_0B_3| < \sqrt{K_2}$; or if $B_2B_3(B_0B_3 - B_1B_2) > 0$ and

- *either $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) < 0$ and $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) < 0$,*
- *or $K_2 \leq 0$ and $B_1(B_1B_2 - 2B_0B_3) < 0$,*

then one limit cycle is a local attractor and the other limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(e) $B_2B_0 < 0$, $K_2 < 0$ and $B_1B_2 = 2B_0B_3$; then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.

Theorems 1.2 and 1.3 are proved in section 2. Examples showing that the conditions provided by both theorems are non-empty are given in section 3.

Kolmogorov systems (1.1) under conditions (iv) are the same as under conditions (iii) but interchanging the variables x and y, so if we change the conditions $b_3 \neq 0$, $a_1 = a_2 = a_3 = 0$, $c_2 = -(b_2^2 + \beta^2)/b_3$, $c_3 = -b_2$ into $a_3 \neq 0$, $b_1 = b_2 = b_3 = 0$, $c_1 = -(a_1^2 + \beta^2)/a_3$, $c_3 = -a_1$, and redefine the constants B_i for $i = 0, ..., 4$ as it is indicated in Appendix B the same Theorem 1.3 holds.

At last our main result concerning the Kolmogorov systems (1.1) under the conditions (v) is the following, with the expressions of D_i, for $i = 0, ..., 4$, and K_4 given in Appendix B.

Theorem 1.4. *If $b_1 \neq 0$, $a_3 = b_3 = c_3 = 0$, $a_2 = -(a_1^2 + \beta^2)/b_1$, $b_2 = -a_1$, $D_1 \neq 0$, $D_2 \neq 0$, $D_3 \neq 0$ and $D_0D_4(D_1D_2 - D_0DB_3) > 0$, then the Kolmogorov system (1.1) has two limit cycles bifurcating from the zero-Hopf equilibrium point $(1, 1, 1)$. Moreover the following statements hold.*

(a) If $K_4 > 0$, $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$ and $|D_1D_2 - 2D_0D_3| < \sqrt{K_4}$; then the two limit cycles are unstable and have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(b) If $b_1D_2(a_1c_1 + b_1c_2) < 0$, $b_1D_3(D_0D_3 - D_1D_2) > 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) < 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) < 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) < 0$;*

or if $b_1D_2(a_1c_1 + b_1c_2) > 0$, $b_1D_3(D_0D_3 - D_1D_2) < 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) > 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) > 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) > 0$;*

then one limit cycle is a local repeller, and the other is a local attractor.

(c) If $b_1D_2(a_1c_1 + b_1c_2) < 0$, $b_1D_3(D_0D_3 - D_1D_2) > 0$, $K_4 > 0$ and $|D_1D_2 - 2D_0D_3| < \sqrt{K_4}$; or if $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) > 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) > 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) > 0$;*

then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(d) If $b_1D_2(a_1c_1 + b_1c_2) > 0$, $b_1D_3(D_0D_3 - D_1D_2) < 0$ $K_4 > 0$ and $|D_1D_2 - 2D_0D_3| < \sqrt{K_4}$; or if $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$ and

- *either* $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) < 0$ *and* $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) < 0$,
- *or* $K_4 \leq 0$ *and* $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) < 0$;

then one limit cycle is a local attractor, and the other is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(e) *If* $D_2D_3(a_1c_1 + b_1c_2)(D_0D_2 - D_1D_2) > 0$, $K_4 < 0$ *and* $D_1D_2 = 2D_0D_3$; *then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.*

Theorem 1.4 is proved in section 2. In section 3 can be found examples showing that the conditions provided by this theorem are non-empty.

All the necessary computations for proving our results have been made with the algebraic manipulator Mathematica 12.0.0.0 (for a Mac OS X x86) in a computer MacBook Air of 2019. The computations done with Mathematica were verified for families (i) and (ii) also with the software Maple.

2. Proof of results

Proof of Proposition 1.1. We want to characterize when the singular point $(1, 1, 1)$ of system (1.1) is a zero-Hopf equilibrium. At first, through the change of variables $(x, y, z) \rightarrow (x+1, y+1, z+1)$, we translate the point $(1, 1, 1)$ to the origin of coordinates, obtaining the system:

$$\begin{aligned}
\dot{x} &= (1+x)(a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6xz + a_7y^2 + a_8yz + a_9z^2),\\
\dot{y} &= (1+y)(b_1x + b_2y + b_3z + b_4x^2 + b_5xy + b_6xz + b_7y^2 + b_8yz + b_9z^2),\\
\dot{z} &= (1+z)(c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2).
\end{aligned} \tag{2.1}$$

In order that the origin of system (2.1) can exhibit a zero-Hopf bifurcation we must require that the eigenvalues of the linear part of the system at the origin be of the form 0 and $\pm\beta i$ with $\beta > 0$. We compute the characteristic polynomial and require that it has the form $\lambda(\lambda^2 + \beta^2)$. Solving the resultant equation we get the five solutions given in (i)–(v). □

Proof of Theorem 1.2. We consider system (1.1) under conditions (ii) of Proposition 1.1, and we proceed to study the limit cycles bifurcating from the zero-Hopf equilibrium point, applying the averaging theory of first order, summarized in Theorem A.1 of Appendix A. To do so we perturb the parameters a_2, b_1, c_2 and c_3 which define the zero-Hopf equilibrium under the assumption (ii) as follows

$$a_2 = \frac{a_3b_2}{b_3} + \varepsilon a_{21}, \quad b_1 = \frac{a_1b_3}{a_3} + \varepsilon b_{11}, \quad c_2 = -\frac{(a_1+b_2)^2 + a_3c_1 + \beta^2}{b_3} + \varepsilon c_{21}, \quad c_3 = -a_1 - b_2 + \varepsilon c_{31},$$

where ε is a small parameter and $\beta > 0$.

We write the linear part of system (2.1) at the origin in its real Jordan normal form

$$J = \begin{pmatrix} 0 & -\beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

The variables of the system having its linear part in the real Jordan normal form are (X, Y, Z). Then system (1.1) under conditions (ii) of Proposition 1.1 becomes of the form

$$\begin{aligned}
\dot{X} &= -\beta Y + O(\varepsilon),\\
\dot{Y} &= \beta X + O(\varepsilon),\\
\dot{Z} &= O(\varepsilon).
\end{aligned}$$

The complete explicit expression of this system is given in system $(\dot{X},\dot{Y},\dot{Z})$ in file ss[[2]].

We note that there are infinitely many linear changes of variables for writing the linear part of system (2.1) at the origin in its real Jordan normal form. This forces to choose some of the entries (denoted by y_i for $i=1,\dots,9$ in file ss[[2]]) of the changing matrix. Thus in the file ss[[2]] we choose $y_1=y_7=1$ and $y_2=0$ in order to fix a unique changing matrix.

We note that the choice of the matrix J given in (2.2) instead of the matrix

$$J=\begin{pmatrix} 0 & 0 & 0\\ 0 & -\beta & 0\\ \beta & 0 & 0\end{pmatrix}$$

is irrelevant. If we choose this last expression for the matrix in its real Jordan form, then instead of doing the change to cylindrical coordinates $(X,Y,Z)\to(r\cos\theta,r\sin\theta,Z)$, we must do the change $(X,Y,Z)\to(X,r\cos\theta,r\sin\theta)$. This change to cylindrical coordinates is necessary in order to arrive to write the differential system in the normal form for applying the averaging theory described in Theorem A.1.

Now we want to write the system in such a way that conditions of Theorem A.1 are satisfied. For this we write the system in cylindrical coordinates by means of the change of variables $(X,Y,Z)\to(r\cos\theta,r\sin\theta,Z)$ obtaining system $(\dot{r},\dot{\theta},\dot{Z})$ of file ss[[2]].

In order to study the periodic solutions in a neigborhood of the origin, i.e. in a neigborhood of the zero-Hopf equilibrium, we do the scaling $(r,Z)\to(\varepsilon R,\varepsilon Z)$, where $\varepsilon>0$ is the same parameter used before. We obtain system $(\dot{R},\dot{Z})$ of file ss[[2]].

We take the variable θ as the new independent variable and so we obtain the system

$$R'=\varepsilon F_{11}+O(\varepsilon^2),\quad Z'=\varepsilon F_{12}+O(\varepsilon^2), \tag{2.3}$$

with coefficients F_{11} and F_{12} given in the file ss[[2]].

Note that system (2.3) is in the normal form (A.1), so we can apply the averaging theory with $T=2\pi$, $x=(R,Z)$, $t=\theta$ and $\varepsilon R(\theta,x,\varepsilon)=O(\varepsilon^2)$. The functions F_{11}, F_{12} and R are $\mathcal{C}^2$ in x and 2π-periodic in θ. Applying Theorem A.1 we compute the averaging function of first order $f_1=(f_{11}(R,Z),f_{12}(R,Z))$, and we obtain

$$f_{11}=\frac{\pi R(A_0+A_1Z)}{a_3^2b_3\beta^5},\qquad f_{12}=-\frac{\pi(A_2Z+A_3Z^2+A_4R^2)}{a_3^2b_3\beta^5N}, \tag{2.4}$$

and A_i, for $i=0,...,4$, K_1 and N are given in Appendix B.

We look for the isolated solutions of the equation $(f_{11}(R,Z),f_{12}(R,Z))=(0,0)$, and we obtain, appart from the origin, $(R_1,Z_1)=(0,-A_2/A_3)$ and $(R_2,Z_2)=\left(\pm\sqrt{A_0(A_1A_2-A_0A_3)}/(A_1\sqrt{A_4}),-A_0/A_1\right)$. We consider always the positive expression of R_2, i.e. we consider the positive sign if $A_1>0$ and the negative sign if $A_1<0$.

We compute the Jacobian matrix of f_1, which is

$$\begin{pmatrix} \dfrac{\pi(A_0+A_1Z)}{a_3^2b_2\beta^5} & \dfrac{\pi RA_1}{a_3^2b_3\beta^5}\\[2ex] -\dfrac{2\pi RA_4}{a_3^2b_3\beta^5N} & -\dfrac{\pi(A_2+2A_3Z)}{a_3^2b_3\beta^5N}\end{pmatrix},$$

and its determinant is $\pi^2(-2A_1A_4R^2+(A_0+A_1Z)(A_2+2A_3Z))/(a_3^4b_3^2\beta^{10}N)$. Evaluating the determinant at the solution (R_1,Z_1) we get that it is equal to $\pi^2A_2(A_0A_3-A_1A_2)/(a_3^4b_3^2A_3\beta^{10}N)$, and at the solutions (R_2,Z_2) we get that it is equal to $2\pi^2A0(A_1A_2-A_0A_3)/(a_3^4b_3^2A_1\beta^{10}N)$.

From the hypothesis considered these determinants are nonzero, therefore it follows from Theorem A.1 that for ε sufficiently small system (2.3) has two 2π-periodic solutions $(R_1(\theta,\varepsilon),Z_1(\theta,\varepsilon))$ and $(R_2(\theta,\varepsilon),Z_2(\theta,\varepsilon))$ such that $(R_j(\theta,\varepsilon),Z_j(\theta,\varepsilon))\to(R_j,Z_j)$ for $j=1,2$ when $\varepsilon\to 0$.

Moreover the Jacobian matrix evaluated at the solution (R_1, Z_1) has eigenvalues equal to $\pi A_2/(a_3^2 b_3 \beta^5 N)$ and $\pi(A_0A_3 - A_1A_2)/(a_3^2 b_3 A_3 \beta^5)$. Since the eigenvalues of the Jacobian matrix evaluated at the solutions provide the stability of the fixed point corresponding to the Poincaré map defined in a neighborhood of the solution, if $A_2 b_3 N > 0$ and $A_3 b_3(A_0A_3 - A_1A_2) > 0$, then the fixed point of the Poincaré map has an unstable manifold of dimension two, and the corresponding periodic solution is unstable and has an unstable manifold of dimension three, which is equivalent to say that is a repelling periodic orbit. If $A_2 b_3 N < 0$ and $A_3 b_3(A_0A_3 - A_1A_2) < 0$, then the fixed point of the Poincaré map has a stable manifold of dimension two, and the associated periodic solution is stable and has a stable manifold of dimension three, which is equivalent to say that is a attracting periodic orbit. Finally, if $A_2A_3N(A_0A_3 - A_1A_2) < 0$, then the fixed point of the Poincaré application is a saddle point with a stable manifold of degree one and an unstable manifold of degree one, and the corresponding periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

On the other hand, the Jacobian matrix evaluated at (R_2, Z_2) has eigenvalues equal to $\pi(2A_0A_3 - A_1A_2 \pm \sqrt{K_1})/(2a_3^2 b_3 A_1 \beta^5 N)$, and so its stability is as follows. If $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) > 0$ and $b_3A_1N(2A_0A_3 - A_1A_2 - \sqrt{K_1}) > 0$ or if $K_1 < 0$ and $b_3A_1N(2A_0A_3 - A_1A_2) > 0$, then the fixed point of the Poincaré map has an unstable manifold of dimension two, and the periodic solution is unstable and has an unstable manifold of dimension three. If $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) < 0$ and $b_3A_1N(2A_0A_3 - A_1A_2\sqrt{K_1}) < 0$ or if $K_1 < 0$ and $b_3A_1N(2A_0A_3 - A_1A_2) < 0$, then the fixed point of the Poincaré map has an unstable manifold of dimension two, and the periodic solution is stable and has a stable manifold of dimension three. If $K_1 > 0$ and $-\sqrt{K_1} < 2A_0A_3 - A_1A_2 < \sqrt{K_1}$, then the fixed point of the Poincaré map is a saddle point with a stable manifold of degree one and an unstable manifold of degree one, and the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders. If $K_1 < 0$ and $A_1A_2 = 2A_0A_3$, the fixed point of the Poincaré map asociated with the periodic orbit is linearly stable, and we cannot decide about the stability of the periodic orbit.

Combining the above information of the eigenvalues of the Jacobian matrix for both (R_1, Z_1) and (R_2, Z_2) we get statements (a)–(e) in the theorem.

Now we shall go back through the changes of variables and we obtain two periodic solutions, for $j = 1, 2$, $(x_j(t,\varepsilon), y_j(t,\varepsilon), z_j(t,\varepsilon))$ bifurcating from $(1,1,1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x_j(t,\varepsilon), y_j(t,\varepsilon), z_j(t,\varepsilon)) = (1,1,1) + O(\varepsilon)$ for $j = 1, 2$. This completes the proof of the theorem. □

Proof of Theorem 1.3. We consider system (1.1) under conditions (iii) of Proposition 1.1. In order to study the zero-Hopf bifurcation we perturb the parameters a_1, a_2, a_3, c_2 and c_3 which define the zero-Hopf equilibrium under conditions (iii) as follows

$$a_1 = \varepsilon a_{11}, \quad a_2 = \varepsilon a_{21}, \quad a_3 = \varepsilon a_{31}, \quad c_2 = -\frac{b_2^2 + \beta^2}{b_3} + \varepsilon c_{21}, \quad c_3 = -b_2 + \varepsilon c_{31},$$

where ε is a parameter to be taken sufficiently small.

We write the lineal part of system (2.1) at the origin in its real Jordan normal form, and the associated system becomes system $(\dot{X}, \dot{Y}, \dot{Z})$ of file ss[[3]]. Then we write the system in cylindrical coordinates obtaining system $(\dot{r}, \dot{\theta}, \dot{Z})$ of file ss[[3]], and we do the scaling $(r, Z) \to (\varepsilon R, \varepsilon Z)$ obtaining system $(\dot{R}, \dot{Z})$ in file ss[[3]].

As in the proof of Theorem 1.2 in order to apply Theorem A.1, we take the variable θ as the new independent variable obtaining a system

$$R' = \varepsilon F_{11} + O(\varepsilon^2), \quad Z' = \varepsilon F_{12} + O(\varepsilon^2) \tag{2.5}$$

which coefficients F_{11} and F_{12} are given in the file ss[[3]]. The averaged function of first order $f_1 = (f_{11}(R,Z), f_{12}(R,Z))$ is

$$f_{11} = \frac{\pi R(B_0 + B_1 Z)}{b_3^2 \beta^5}, \qquad f_{12} = \frac{\pi(B_2 Z + B_3 Z^2 + B_4 R^2)}{b_3^2 \beta^5},$$

with B_i, for $i = 0, ..., 4$, and K_2 given in Appendix B.Solving the equation $(f_{11}(R, Z), f_{12}(R, Z)) = (0, 0)$ we obtain two solutions $(R_1, Z_1) = (0, -B_2/B_3)$ and $(R_2, Z_2) = \left(\pm\sqrt{B_0(B_1B_2 - B_0B_3)}/(B_1\sqrt{B_4}), -B_0/B_1\right)$. Again we consider always the positive expression of R_2.

We compute the Jacobian matrix of f_1 and we get

$$\begin{pmatrix} \dfrac{\pi(B_0 + B_1 Z)}{b_3^2\beta^5} & \dfrac{\pi R B_1}{b_3^2\beta^5} \\ \dfrac{2\pi R B_4}{b_3^2\beta^5} & \dfrac{\pi(B_2 + 2B_3 Z)}{b_3^2\beta^5} \end{pmatrix},$$

whose determinant is $\pi^2(-2B_1B_4R^2 + (B_0 + B_1Z)(B_2 + 2B_3Z))/(b_3^4\beta^{10})$. The determinant at the solution (R_1, Z_1) is $\pi^2B_2(B_1B_2 - B_0B_3)/(b_3^4B_3\beta^{10})$, and at the solutions (R_2, Z_2) is $2\pi^2B_0(B_0B_3 - B_1B_2)/(b_3^4B_1\beta^{10})$.

From the hypothesis considered these determinants are nonzero, so it follows from Theorem A.1 that for ε sufficiently small, system (2.3) has two solutions $(R_1(\theta, \varepsilon), Z_1(\theta, \varepsilon))$ and $(R_2(\theta, \varepsilon), Z_2(\theta, \varepsilon))$ such that $(R_j(\theta, \varepsilon), Z_j(\theta, \varepsilon)) \to (R_j, Z_j)$ for $j = 1, 2$ when $\varepsilon \to 0$.

The Jacobian matrix evaluated at the solution (R_1, Z_1) has eigenvalues equal to $-\pi B_2/(b_3^2\beta^5)$ and $\pi(B_0B_3 - B_1B_2)/(b_3^2B_3\beta^5)$. We study the stability of the associated periodic orbit which is provided by these eigenvalues. If $B_2 < 0$ and $B_3(B_0B_3 - B_1B_2) > 0$, the associated periodic solution is unstable and has an unstable manifold of dimension three. If $B_2 > 0$ and $B_3(B_0B_3 - B_1B_2) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. Finally if $B_2B_3(B_0B_3 - B_1B_2) > 0$, the periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

On the other hand, the Jacobian matrix evaluated at (R_2, Z_2) has eigenvalues equal to $\pi(B_1B_2 - 2B_0B_3 \pm \sqrt{K_2})/(2b_3^2B_1\beta^5)$, and so if $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) > 0$ and $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) > 0$, or if $K_2 < 0$ and $B_1(B_1B_2 - 2B_0B_3) > 0$, then the associated periodic solution is unstable and has an unstable manifold of dimension three. If $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) < 0$ and $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) < 0$, or if $K_2 < 0$ and $B_1(B_1B_2 - 2B_0B_3) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. If $K_2 > 0$ and $-\sqrt{K_2} < B_1B_2 - 2B_0B_3 < \sqrt{K_2}$, then the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders. If $K_2 < 0$ and $B_1B_2 - 2B_0B_3 = 0$, the fixed point of the Poincaré map associated with the periodic orbit is linearly stable, and we cannot decide about the stability of the periodic orbit.

Combining the above information of the eigenvalues of the Jacobian matrix for both (R_1, Z_1) and (R_2, Z_2) we get statements (a)–(e) of the theorem.

Now we shall go back through the changes of variables and we obtain two periodic solutions,for $j = 1, 2$, $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon))$ bifurcating from $(1, 1, 1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon)) = (1, 1, 1) + O(\varepsilon)$ for $j = 1, 2$. This completes the proof of the theorem. □

Proof of Theorem 1.4. We consider system (1.1) under conditions (v) of Proposition 1.1. In order to study the zero-Hopf bifurcation we perturb the parameters a_2, a_3, b_2, b_3 and c_3 which define the zero-Hopf equilibrium point into the form

$$a_2 = -\frac{a_1^2 + \beta^2}{b_1} + \varepsilon a_{21}, \quad a_3 = \varepsilon a_{31}, \quad b_2 = -a_1 + \varepsilon b_{21}, \quad b_3 = \varepsilon b_{31}, \quad c_3 = \varepsilon c_{31},$$

where ε is a sufficiently small parameter.

We write the system with the linear part at the origin in its real Jordan normal form, then we write it in cylindrical coordinates, and finally we do the scaling $(r, Z) \to (\varepsilon R, \varepsilon Z)$. Thus we obtain respectively systems $(\dot{X}, \dot{Y}, \dot{Z})$, $(\dot{r}, \dot{\theta}, \dot{Z})$ and $(\dot{R}, \dot{Z})$ of file ss[[5]]. Taking θ as the new independent variable we obtain a system in the form

$$R' = \varepsilon F_{11} + O(\varepsilon^2), \quad Z' = \varepsilon F_{12} + O(\varepsilon^2) \tag{2.6}$$

with coefficients F_{11} and F_{12} given in the file ss[[5]]. As in previous proofs we are in conditions to apply Theorem A.1. Now the averaged function of first order $f_1 = (f_{11}(R, Z), f_{12}(R, Z))$ is

$$f_{11} = \frac{\pi R(D_0 + D_1 Z)}{b_1 \beta^5}, \qquad f_{12} = \frac{\pi(D_2 Z + D_3 Z^2 + D_4 R^2)}{b_1(a_1 c_1 + b_1 c_2)\beta^5},$$

with D_i, for $i = 0, ..., 4$, and K_4 given in Appendix B. We look for the solutions of $(f_{11}(R, Z), f_{12}(R, Z)) = (0, 0)$, and we obtain $(R_1, Z_1) = (0, -D_2/D_3)$ and $(R_2, Z_2) = \left(\pm\sqrt{D_0(D_1 D_2 - D_0 D_3)}/(D_1\sqrt{D_4}), -D_0/D_1\right)$, considering the positive expression of R_2.

We compute the Jacobian matrix of f_1 and we get

$$\begin{pmatrix} \dfrac{\pi(D_0 + D_1 Z)}{b_1\beta^5} & \dfrac{\pi R D_1}{b_1\beta^5} \\ \dfrac{2\pi R D_4}{b_1\beta^5(a_1 c_1 + b_1 c_2)} & \dfrac{\pi(D_2 + 2D_3 Z)}{b_1\beta^5(a_1 c_1 + b_1 c_2)} \end{pmatrix},$$

whose determinant is $\pi^2(-2D_1 D_4 R^2 + (D_0 + D_1 Z)(D_2 + 2D_3 Z))/(b_1^2\beta^{10}(a_1 c_1 + b_1 c_2))$. The determinant at the solution (R_1, Z_1) is $\pi^2 D_2(D_1 D_2 - D_0 D_3)/(b_1^2 D_3 \beta^{10}\ (a_1 c_1 + b_1 c_2))$, and at the solution (R_2, Z_2) is $2\pi^2 D_0(D_0 D_3 - D_1 D_2)/(b_1^2 D_1 \beta^{10}(a_1 c_1 + b_1 c_2))$, and both are nonzero by the hypotheses. By Theorem A.1, for ε sufficiently small system (2.6) has two solutions $(R_1(\theta, \varepsilon), Z_1(\theta, \varepsilon))$ and $(R_2(\theta, \varepsilon), Z_2(\theta, \varepsilon))$ such that $(R_j(\theta, \varepsilon), Z_j(\theta, \varepsilon)) \to (R_j, Z_j)$ for $j = 1, 2$ when $\varepsilon \to 0$.

The Jacobian matrix evaluated at the solution (R_1, Z_1) has eigenvalues equal to $-\pi D_2/(b_1\beta^5(a_1 c_1 + b_1 c_2))$ and $\pi(D_0 D_3 - D_1 D_2)/(b_1 D_3 \beta^5)$. We study the stability of the associated periodic orbit which is provided by these eigenvalues. If $b_1 D_2(a_1 c_1 + b_1 c_2) < 0$ and $b_1 D_3(D_0 D_3 - D_1 D_2) > 0$, the associated periodic solution is unstable and has an unstable manifold of dimension three. If $b_1 D_2(a_1 c_1 + b_1 c_2) > 0$ and $b_1 D_3(D_0 D_3 - D_1 D_2) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. Finally if $D_2 D_3(a_1 c_1 + b_1 c_2)(D_0 D_3 - D_1 D_2) > 0$, the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

On the other hand, the Jacobian matrix evaluated at (R_2, Z_2) has eigenvalues equal to $\pi(D_1 D_2 - 2D_0 D_3 \pm \sqrt{K_4})/(2b_1 D_1 \beta^5(a_1 c_1 + b_1 c_2))$. Then if $K_4 > 0$, $b_1 D_1(a_1 c_1 + b_1 c_2)(D_1 D_2 - 2D_0 D_3 + \sqrt{K_4}) > 0$ and $b_1 D_1(a_1 c_1 + b_1 c_2)(D_1 D_2 - 2D_0 D_3 - \sqrt{K_4}) > 0$, or if $K_4 < 0$ and $b_1 D_1(a_1 c_1 + b_1 c_2)(D_1 D_2 - 2D_0 D_3) > 0$, then the associated periodic solution is unstable and has an unstable manifold of dimension three. If $K_4 > 0$, $b_1 D_1(a_1 c_1 + b_1 c_2)(D_1 D_2 - 2D_0 D_3 + \sqrt{K_4}) < 0$ and $b_1 D_1(a_1 c_1 + b_1 c_2)(D_1 D_2 - 2D_0 D_3 - \sqrt{K_4}) < 0$, or if $K_4 < 0$ and $b_1 D_1(a_1 c_1 + b_1 c_2)(D_1 D_2 - 2D_0 D_3) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. If $K_4 > 0$ and $-\sqrt{K_4} < D_1 D_2 - 2D_0 D_3 < \sqrt{K_4}$, then the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders. If $K_4 < 0$ and $D_1 D_2 = 2D_0 D_3$, the fixed point of the Poincaré map associated with the periodic orbit is linearly stable, and we cannot decide about the stability of the periodic orbit.

Combining the above information we get statements (a)–(e) in the theorem, and going back through the changes of variables we obtain two periodic solutions $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon))$ for $j = 1, 2$ bifurcating from $(1, 1, 1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon)) = (1, 1, 1) + O(\varepsilon)$ for $j = 1, 2$. This completes the proof of the theorem. □

3. Examples

3.1. Examples of Theorem 1.2

We give examples showing that the conditions provided by Theorem 1.2 are non-empty. We give only the values of the parameters which are nonzero. System (1.1) with the set of parameters

$$\{a_1 = a_3 = a_5 = b_1 = b_2 = b_3 = -1,\ a_2 = -1 - \varepsilon,\ a_4 = 1/2,\ c_1 = 3/2,\ c_2 = 7/2,\ c_3 = 2,\ c_6 = 4\}$$

has two limit cycles whose type of stability is given in statement (a) of Theorem 1.2.

The following sets of parameters verify, respectively, the four sets of conditions in statement (b) of Theorem 1.2, so for all of them system (1.1) has two limit cycles whose type of stability is given in statement (b):

$$\left\{\begin{matrix} a_1 = a_3 = -1,\ a_2 = -1+\varepsilon,\ a_4 = -7/8,\ a_5 = 1,\ b_1 = b_2 = b_3 = -1,\ c_1 = 7/4,\ c_2 = 37/16, \\ c_3 = 2+5\varepsilon,\ c_6 = 6 \end{matrix}\right\},$$

$$\{a_1 = a_3 = b_1 = b_2 = b_3 = -1,\ a_2 = -1+\varepsilon,\ a_4 = -1/2,\ a_5 = 1,\ c_1 = 3/2,\ c_2 = 7/2,\ c_3 = 2+\varepsilon,\ c_6 = 5\},$$

$$\left\{\begin{matrix} a_1 = a_3 = a_5 = b_2 = -1,\ a_2 = 1+\varepsilon,\ a_4 = -7/8,\ b_1 = 1+\varepsilon,\ b_3 = 1,\ c_1 = 7/4,\ c_2 = -37/16, \\ c_3 = 2-5\varepsilon,\ c_6 = 5 \end{matrix}\right\},$$

$$\{a_1 = a_3 = b_2 = -1,\ a_2 = 1+\varepsilon,\ a_4 = 3/4,\ a_5 = b_1 = b_3 = 1,\ c_2 = -5,\ c_3 = 2-3\varepsilon\}.$$

The following sets of parameters verify, respectively, the three sets of conditions in statement (c) of Theorem 1.2, so for all of them, system (1.1) has two limit cycles whose type of stability is given in statement (c):

$$\{a_1 = a_3 = a_5 = b_1 = b_2 = b_3 = -1,\ a_2 = -1+\varepsilon,\ a_4 = 1/2,\ c_1 = 3/2,\ c_2 = 7/2,\ c_3 = 2+\varepsilon,\ c_6 = 4\},$$

$$\{a_1 = a_5 = b_3 = -1,\ a_2 = -3/4-\varepsilon,\ a_3 = b_1 = 1,\ b_2 = 3/4,\ c_2 = 17/16,\ c_3 = 1/4+\varepsilon,\ c_6 = 2\},$$

$$\{a_1 = a_5 = b_2 = b_3 = c_2 = -1,\ a_2 = -3-\varepsilon,\ a_4 = 1/2,\ b_1 = -1/3,\ c_1 = c_6 = 2,\ c_3 = 2+2\varepsilon\}.$$

The following sets of parameters verify, respectively, the three sets of conditions in statement (d) of Theorem 1.2, so for all of them, system (1.1) has two limit cycles whose type of stability is given in statement (d):

$$\{a_1 = a_3 = a_5 = b_2 = -1,\ a_2 = 1+\varepsilon,\ a_4 = -1/2,\ b_1 = b_3 = 1,\ c_1 = 3/2,\ c_2 = -7/2,\ c_3 = 2-\varepsilon,\ c_6 = 27/8\},$$

$$\{a_2 = -1/4+\varepsilon,\ a_3 = b_2 = -1/2,\ a_4 = 3,\ a_5 = b_3 = -1,\ c_1 = -1/4,\ c_2 = 393/1024,\ c_3 = 1/2+29\varepsilon,\ c_6 = 33\},$$

$$\{a_1 = a_3 = a_5 = b_2 = -1,\ a_2 = 1+\varepsilon,\ a_4 = -2,\ b_1 = b_3 = 1,\ c_1 = 3/2,\ c_2 = -7/2,\ c_3 = 2-\varepsilon, c_6 = -1\}.$$

Finally system (1.1) with the set of parameters

$$\{a_1 = a_3 = b_1 = b_2 = b_3 = b_9 = -1,\ a_2 = -1+\varepsilon,\ c_1 = 3/2,\ c_2 = 7/2,\ c_3 = 2+\varepsilon,\ c_6 = 44/3\}$$

has two limit cycles whose type of stability is given in statement (e) of Theorem 1.2.

3.2. Examples of Theorem 1.3

We give examples showing that the conditions provided by Theorem 1.3 are non-empty. We give only the values of the parameters which are nonzero. System (1.1) with the set of parameters

$$\{a_1 = -2\varepsilon,\ a_2 = \varepsilon,\ a_4 = c_2 = 2,\ a_7 = b_1 = b_2 = b_3 = -1,\ c_3 = 1\}$$

has two limit cycles whose type of stability is given in statement (a) of Theorem 1.3.

The parameters

$$\{a_1 = -2\varepsilon,\ a_2 = \varepsilon,\ a_4 = -7/2,\ a_5 = 4,\ a_7 = b_1 = b_2 = -1,\ b_3 = c_3 = 1,\ c_2 = -2\}$$

verify the first set of conditions in statement (b) of Theorem 1.3, and the parameters

$$\{a_1 = -2\varepsilon,\ a_2 = \varepsilon,\ a_4 = -3,\ a_7 = b_3 = c_3 = 1,\ b_1 = b_2 = -1,\ c_2 = -2\}$$

verify the second set of conditions. For both of them there exist two limit cycles for system (1.1) whose type of stability is given in statement (b).

The following sets of conditions verify, respectively, the three sets of conditions in statement (c) of Theorem 1.3, so for all of them, system (1.1) has two limit cycles whose type of stability is given in statement (c):

$$\{a_2 = -\varepsilon,\ a_4 = 4,\ a_5 = -2,\ a_7 = b_1 = b_2 = b_3 = -1,\ c_2 = 2,\ c_3 = 1\},$$

$$\{a_1 = 4\varepsilon,\ a_3 = -\varepsilon,\ a_4 = -2,\ a_6 = -1,\ a_7 = 1,\ b_1 = b_3 = -1,\ c_2 = 1\},$$

$$\{a_2 = -\varepsilon,\ a_4 = -2,\ a_7 = 1,\ b_1 = b_2 = b_3 = -1,\ c_2 = 2,\ c_3 = 1\}.$$

The following sets of conditions verify, respectively, the three sets of conditions in statement (d) of Theorem 1.3, so for all of them, system (1.1) has two limit cycles whose type of stability is given in statement (d):

$$\{a_1 = 2\varepsilon,\ a_2 = -\varepsilon,\ a_4 = 1/2,\ a_7 = b_1 = b_2 = b_3 = -1,\ c_2 = 2,\ c_3 = 1\},$$

$$\{a_2 = \varepsilon,\ a_4 = -1/16,\ a_5 = -1/2,\ a_7 = b_3 = c_3 = 1,\ b_1 = b_2 = -1,\ c_2 = -2\},$$

$$\{a_1 = -2\varepsilon,\ a_3 = -\varepsilon,\ a_6 = a_7 = b_3 = 1,\ b_1 = c_2 = -1\}.$$

Finally system (1.1) with the parameters

$$\{a_3 = -\varepsilon,\ a_4 = a_7 = b_1 = b_3 = c_9 = -1,\ c_2 = 1\}$$

has two limit cycles whose type of stability is given in statement (e) of Theorem 1.3.

3.3. Examples of Theorem 1.4

We give examples showing that the conditions provided by Theorem 1.4 are non-empty. We give only the values of the parameters which are nonzero. System (1.1) with parameters

$$\{a_1 = a_9 = b_1 = c_1 = -1,\ a_2 = a_4 = 2,\ a_3 = \varepsilon,\ b_2 = 1 + 2\varepsilon\}$$

has two limit cycles whose type of stability is given in statement (a) of Theorem 1.4. The following sets of parameters verify, respectively, the four sets of conditions in statement (b) of Theorem 1.4, so for all of them system (1.1) has two limit cycles whose type of stability is given in statement (b):

$$\{a_1 = b_1 = c_1 = -1,\ a_2 = 2,\ a_3 = -\varepsilon,\ a_4 = -2,\ a_6 = 3,\ a_9 = 1,\ b_2 = 1 - (7/8)\varepsilon\},$$

$$\{a_1 = b_1 = -1,\ a_2 = 2,\ a_3 = \varepsilon,\ a_4 = -2,\ a_6 = -5,\ a_9 = 1,\ b_2 = c_1 = 1\},$$

$$\{a_1 = -1,\ a_2 = -2,\ a_3 = -\varepsilon,\ a_4 = 2,\ a_6 = 3,\ a_9 = -1,\ b_1 = b_2 = c_1 = 1\},$$

$$\{a_1 = a_9 = -1,\ a_2 = -2,\ a_3 = -\varepsilon,\ a_4 = 2,\ a_6 = 5,\ a_9 = -1,\ b_1 = b_2 = c_1 = 1\}.$$

The following sets of parameters verify, respectively, the three sets of conditions in statement (c) of Theorem 1.4, so for all of them system (1.1) has two limit cycles whose type of stability is given in statement (c).

$$\{a_1 = a_9 = b_1 = -1,\ a_2 = 2,\ a_3 = \varepsilon,\ a_4 = 3,\ a_6 = -1,\ a_9 = -1,\ b_2 = c_1 = 1,\ c_9 = 2\},$$

$$\{a_1 = a_9 = -1,\ a_2 = -2,\ a_3 = -\varepsilon,\ a_6 = -1,\ a_9 = b_1 = c_1 = 1,\ b_2 = 1 + (7/8)\varepsilon\},$$

$$\{a_1 = a_9 = c_1 = -1,\ a_2 = -2,\ a_3 = -\varepsilon,\ a_4 = 1,\ a_6 = -1,\ a_9 = b_1 = 1,\ b_2 = 1 - 4\varepsilon,\ c_9 = -1/2\}.$$

The following sets of parameters verify, respectively, the three sets of conditions in statement (d) of Theorem 1.4, so for all of them system (1.1) has two limit cycles whose type of stability is given in statement (d):

$$\{a_1 = -1,\ a_2 = -2,\ a_3 = -\varepsilon,\ a_4 = -3,\ a_6 = b_1 = b_2 = c_1 = 1,\ c_9 = -2\},$$

$$\{a_1 = b_1 = c_1 = -1,\ a_2 = 2,\ a_3 = \varepsilon,\ a_4 = -3,\ a_6 = b_1 = 1,\ b_2 = 1 + 2\varepsilon,\ c_9 = 2\},$$

$$\{a_1 = b_1 = c_1 = -1,\ a_2 = 2,\ a_3 = \varepsilon,\ a_4 = -1,\ a_6 = b_1 = 1,\ b_2 = 1 + 4\varepsilon,\ c_9 = 1/2\}.$$

Finally system (1.1) with parameters

$$\{a_1 = a_6 = b_1 = -1,\ a_2 = 2,\ a_3 = (7/8)\varepsilon,\ b_1 = 1,\ b_2 = c_1 = 1,\ c_9 = 1/2\}$$

has two limit cycles whose type of stability is given in statement (e) of Theorem 1.4.

Now we illustrate the two limit cycles exhibited for the system

$$\begin{aligned}
\dot{x} &= x\left(2 + \frac{1}{2}(x-1)^2 - x - (x-1)(y-1) - z - (1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(3 - x - y - z),\\
\dot{z} &= z\left(\frac{3}{2}(x-1) + \frac{7}{2}(y-1) + 2(z-1) + 4(x-1)(z-1)\right),
\end{aligned} \tag{3.1}$$

which realize the conditions of statement (a) of Theorem 1.2. For this differential system the two real zeros of the averaged functions (2.4) which provide the two limit cycles bifurcating from the equilibrium point $(1, 1, 1)$ of system (3.1), when this point has been translated to the origin of coordinates, are

$$(R, Z) = (0, 4/3), \quad \text{and} \quad (R, Z) = (\sqrt{65/8}), -2).$$

Going back through the changes of variables of the proof of Theorem 1.2, we get that the initial conditions in the coordinates (x, y, z) at time $t = 0$ for the two limit cycles are

$$(0, 4\varepsilon/3, -4\varepsilon/3), \quad \text{and} \quad \left((-3 - 2\sqrt{10/13})\varepsilon, (-1 - 2\sqrt{10/13})\varepsilon, (4 + 7\sqrt{10/13})\varepsilon\right).$$

We compute numerically the two limit cycles starting at these initial conditions for the value $\varepsilon = 5/1000$, see Figure 1.

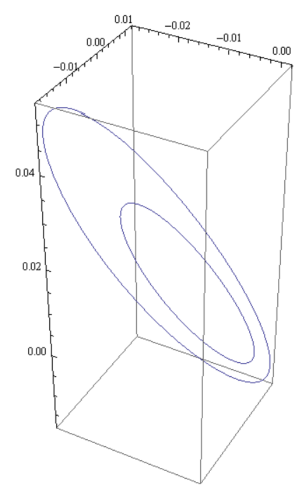

Figure 1: The two limit cycles which have bifurcated from the zero-Hopf equilibrium point $(1, 1, 1)$ of system (3.1) with $\varepsilon = 5/1000$, when this point has been translated to the origin of coordinates

A. Averaging theory

We summarize the averaging theory of first order, which provides sufficient conditions for the existence of periodic orbits for a periodic differential system depending on small parameters. For additional details and the proof of the result stated in this appendix, see [16, 17, 18] and [19, Theorems 11.5, 11.6].

Theorem A.1. *We consider the following differential system*

$$x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{A.1}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable and* D *is an open subset of* $\mathbb{R}^n$. *We define* $f_1 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \int_0^T F_1(s, z) ds, \tag{A.2}$$

and assume that:

1. F_1 *and* R *are locally Lipschitz with respect to* x*;*
2. *for* $a \in D$ *with* $f_1(a) = 0$*, there exists a neighboirhood* V *of* a *such that* $f_1(z) \neq 0$ *for all* $z \in \overline{V} \backslash (a)$ *and* $d_B(f_1, V, 0) \neq 0$*, where* $d_B(f_1, V, 0)$ *is the Brouwer degree.*

Then for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $\varphi(\cdot, \varepsilon)$ *of system* (A.1) *such that* $\varphi(\cdot, \varepsilon) \to a$ *as* $\varepsilon \to 0$*. The kind of stability of the limit cycle is given by the eigenvalues of the Jacobian matrix at the point* a*.*

Note that a sufficient condition for showing that the Brouwer degree of a function f at a point a is nonzero, is that the Jacobian of the function f at a, when it is defined, is nonzero, see [20].

B. Notation

$$
\begin{aligned}
A_0 ={}& a_3\beta^2\left((a_3^2b_{11}-b_3^2a_{21})(a_1(a_1+b_2)+a_3c_1)+a_3\beta^2(a_3b_{11}+b_3c_{31})\right),\\
A_1 ={}& a_1^6\left(-2a_9b_3+2a_3b_9\right)+2a_1^5\left(b_3\left(a_8b_3-4a_9b_2\right)-a_3^2b_6+a_3\left(a_6b_3-b_3b_8+4b_2b_9\right)\right)\\
&+a_3^2\left(2\left(-b_3\left(a_9b_2^2+b_3(a_7b_3-a_8b_2)\right)+a_3^3b_4-a_3^2\left(a_4b_3-b_3b_5+b_2b_6\right)\right.\right.\\
&+a_3b_3\left(a_6b_2-a_5b_3+b_3b_7-b_2b_8\right)+a_3b_2^2b_9\big)c_1^2+c_1\left(4a_3^2b_4-2a_3b_2b_6+a_3b_3\right.\\
&\left(3b_5-2a_4+c_6\right)+b_3\left(a_6b_2+b_2^2+b_3(2b_7-a_5+c_8)+b_2(b_3-b_8-2c_9)\right)\big)\beta^2\\
&+\left(2a_3b_4+b_3(b_5+c_6)\right)\beta^4\big)+a_1a_3\left(-2a_3^3c_1(b_6c_1-2b_2b_4)+a_3^2\left(2c_1\left(-2b_2\right.\right.\right.\\
&\left(a_4b_3-b_3b_5+b_2b_6\right)+\left(a_6b_3-b_3b_8+2b_2b_9\right)c_1\big)+\left(4b_2b_4+(b_3-4b_6)c_1\right)\beta^2\big)\\
&a_3\left(2c_1\left(2b_2b_3(a_6b_2-a_5b_3+b_3b_7-b_2b_8)+2b_2^3b_9+b_3(a_8b_3-2a_9b_2)c_1\right)\right.\\
&+\left(-2a_4b_2b_3-2b_2^2b_6+4b_2b_9c_1+b_2b_3(3b_5+2c_1+c_6)+b_3c_1(3a_6-3b_8-2c_9)\right)\\
&\beta^2+(b_3-2b_6)\beta^4\big)+b_3\left(4b_2b_3(a_8b_2-a_7b_3)c_1+\left(a_6b_2^2+b_2^3+2a_8b_3c_1+b_2b_3\right.\right.\\
&\left(2b_7-a_5+c_8\right)+b_2^2(b_3-b_8-2c_9)\big)\beta^2+(a_6+b_2-b_8-2c_9)\beta^4-4a_9b_2c_1\left(b_2^2\right.\\
&+\left.\beta^2\right)\big)\big)+a_1^4\left(2a_3^3b_4+2a_3\left(3a_6b_2b_3-a_5b_3^2+b_3^2b_7-3b_2b_3b_8+6b_2^2b_9-2a_9b_3c_1\right)\right.\\
&+a_3^2\left(2b_3b_5-2a_4b_3-6b_2b_6+4b_9c_1\right)+a_3(b_3+4b_9)\beta^2-2b_3\left(b_3\left(a_7b_3-3a_8b_2\right)\right.\\
&+\left.2a_9(3b_2^2+\beta^2)\right)\big)+a_1^2\left(4a_3^4b_4c_1+2a_3^3\left(b_2^2b_4-4b_2b_6c_1+c_1(2b_3b_5-2a_4b_3+b_9c_1)\right.\right.\\
&+\left.2b_4\beta^2\right)+a_3\left(2b_2^2b_3(a_6b_2-a_5b_3+b_3b_7-b_2b_8)+2b_2^4b_9-4b_3\left(3a_9b_2^2+b_3\left(a_7b_3\right.\right.\right.\\
&-\left.\left.2a_8b_2\right)\right)c_1+\left(4a_6b_2b_3+b_2^2(3b_3+4b_9)+b_3\left(-4a_9c_1+b_3(2b_7-a_5+c_8)\right)\right.\\
&+\left.b_2b_3(b_3-4(b_8+c_9))\right)\beta^2+(b_3+2b_9)\beta^4\big)+a_3^2\left(-2b_2^3b_6+8b_2b_3(a_6-b_8)c_1\right.\\
&-2b_3c_1\left(2a_5b_3-2b_3b_7+a_9c_1\right)+2b_2^2(b_3b_5+6b_9c_1)+\left(b_2(b_3-6b_6)+4b_9c_1\right.\\
&+\left.b_3(3b_5+c_1+c_6)\right)\beta^2-2a_4b_3(b_2^2+\beta^2)\big)-2b_3\left(a_9(b_2^2+\beta^2)^2-b_2b_3\left(-a_7b_2b_3\right.\right.\\
&+\left.\left.a_8(b_2^2+\beta^2)\right)\right)\big)+a_1^3\left(2a_3\left(b_2b_3\left(3a_6b_2-2a_5b_3+2b_3b_7-3b_2b_8\right)+4b_2^3b_9+2b_3c_1\right.\right.\\
&\left(a_8b_3-3a_9b_2\right)\big)+4a_3^3(b_2b_4-b_6c_1)+a_3\left(3a_6b_3+3b_2b_3-3b_3b_8+8b_2b_9-2b_3c_9\right)\\
&\beta^2+a_3^2\left(-4a_4b_2b_3-6b_2^2b_6+4b_3(a_6-b_8)c_1+4b_2(b_3b_5+3b_9c_1)+(b_3-4b_6)\beta^2\right)\\
&+2b_3\left(-4a_9b_2(b_2^2+\beta^2)+b_3\left(-2a_7b_2b_3+a_8(3b_2^2+\beta^2)\right)\right)\big)\big),\\
A_2 ={}& -2a_3\beta^2\left(\left(a_3^2b_{11}-b_3^2a_{21}\right)\left(a_1(a_1+b_2)+a_3c_1\right)+a_3^2b_{11}\beta^2\right)\Big(\left(a_1(a_1+b_2)+a_3c_1\right)^2\\
&+\left(2a_1^2+2a_1b_2+b_2^2+2a_3c_1\right)\beta^2+\beta^4\Big),\\
A_3 ={}& -2\Big(\left(a_1(a_1+b_2)+a_3c_1\right)^2+\left(2a_1^2+2a_1b_2+b_2^2+2a_3c_1\right)\beta^2+\beta^4\Big)\Big(a_1^6\left(a_3b_9\right.\\
&-\left.a_9b_3\right)+a_1^5\left(b_3\left(a_8b_3-4a_9b_2\right)-a_3^2b_6+a_3\left(a_6b_3-b_3b_8+4b_2b_9\right)\right)\\
&+a_3^2\left(\left(-b_3\left(a_9b_2^2+b_3(a_7b_3-a_8b_2)\right)+a_3^3b_4-a_3^2\left(a_4b_3-b_3b_5+b_2b_6\right)+a_3b_3\right.\right.\\
&\left(a_6b_2-a_5b_3+b_3b_7-b_2b_8\right)+a_3b_2^2b_9\big)c_1^2+\left(b_3\left(a_6b_2-a_5b_3\right)+2a_3^2b_4\right.\\
&+\left.a_3\left(b_3b_5-2a_4b_3-b_2b_6\right)\right)c_1\beta^2+(a_3b_4-a_4b_3)\beta^4\big)+a_1a_3\left(a_3^3c_1\left(2b_2b_4-b_6c_1\right)\right.\\
&+a_3^2\left(c_1\left(-2b_2\left(a_4b_3-b_3b_5+b_2b_6\right)+\left(a_6b_3-b_3b_8+2b_2b_9\right)c_1\right)+2\beta^2\left(b_2b_4\right.\right.\\
&-\left.\left.b_6c_1\right)\right)+a_3\left(c_1\left(2b_2b_3\left(a_6b_2-a_5b_3+b_3b_7-b_2b_8\right)+2b_2^3b_9+b_3c_1\left(a_8b_3\right.\right.\right.\\
&-\left.\left.2a_9b_2\right)\right)+\left(b_2b_3b_5-2a_4b_2b_3-b_2^2b_6+2a_6b_3c_1-b_3b_8c_1+2b_2b_9c_1\right)\beta^2-b_6\beta^4\big)\\
&+b_3\left(2b_2b_3c_1(a_8b_2-a_7b_3)+\left(a_6b_2^2-a_5b_2b_3+a_8b_3c_1\right)\beta^2+a_6\beta^4-2a_9b_2c_1\left(b_2^2\right.\right.\\
&+\left.\left.\beta^2\right)\right)\big)+a_1^4\left(a_3^3b_4+a_3^2\left(b_3b_5-a_4b_3-3b_2b_6+2b_9c_1\right)+a_3\left(3a_6b_2b_3-a_5b_3^2\right.\right.\\
&+\left.b_3^2b_7-3b_2b_3b_8+6b_2^2b_9-2a_9b_3c_1+2b_9\beta^2\right)-b_3\left(b_3\left(a_7b_3-3a_8b_2\right)+2a_9\left(3b_2^2\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\ \beta^2))\big)\big) + a_1^3 \left(2a_3^3 \left(b_2b_4 - b_6c_1\right) + a_3^2 \left(2b_2b_3(b_5 - a_4) - 3b_2^2b6 + 2a6b_3c_1\right.\right.\\
&-\ \left. 2b_3b_8c_1 + 6b_2b_9c_1 - 2b_6\beta^2\right) + a_3 \left(3b_2^2b_3(a_6 - b_8) - 2b_2b_3^2(b_7 - a_5) + 4b_2^3b_9\right.\\
&-\ \left. 6a_9b_2b_3c_1 + 2a_8b_3^2c_1 + \left(2a_6b_3 - b_3b_8 + 4b_2b_9\right)\beta^2\right) + b_3 \left(-4a_9b_2 \left(b_2^2 + \beta^2\right)\right.\\
&+\ \left.\left. b_3 \left(3a_8b_2^2 - 2a_7b_2b_3 + a_8\beta^2\right)\right)\right) + a_1^2 \left(2a_3^4b_4c_1 + a_3^3 \left(b_2^2b_4 - 4b_2b_6c_1 + c_1 \left(2b_3b_5\right.\right.\right.\\
&-\ \left.\left. 2a_4b_3 + b_9c_1\right) + 2b_4\beta^2\right) - a_3^2 \left(b_2^3b_6 + 4b_2b_3c_1(b_8 - a_6) + b_3c_1 \left(2a_5b_3 - 2b_3b_7\right.\right.\\
&+\ \left.\left. a_9c_1\right) - b_2^2(b_3b_5 + 6b_9c_1) + (3b_2b_6 - b_3b_5 - 2b_9c_1)\beta^2 + a_4b_3(b_2^2 + 2\beta^2)\right)\\
&+\ a_3 \left(b_2^2b_3^2b_7 - b_2^3b_3b_8 + b_2^4b_9 - 6a_9b_2^2b_3c_1 + 4a_8b_2b_3^2c_1 - 2a_7b_3^3c_1 + \beta^2 \left(-b_2b_3b_8\right.\right.\\
&+\ \left.\left. 2b_2^2b_9 - 2a_9b_3c_1\right) + b_9\beta^4 - a_5b_3^2(b_2^2 + \beta^2) + a_6b_2b_3(b_2^2 + 3\beta^2)\right) + b_3 \left(-a_9 \left(b_2^2\right.\right.\\
&+\ \left.\left.\left.\left. \beta^2\right)^2 + b_2b_3 \left(-a_7b_2b_3 + a_8(b_2^2 + \beta^2)\right)\right)\right)\right)\Big)\Big),\\
A_4 = &-\left((a_1 + b_2)\left(a_1(a_1 + b_2) + a_3c_1\right)\beta + a_1\beta^2\right)^2 \left(a_3^3b_4 - a_3^2\left(a_4b_3 - b_3b_5 + b_6(a_1 + b_2)\right)\right.\\
&-\ b_3\left(a_1^2a_9 + 2a_1a_9b_2 + a_9b_2^2 - a_1a_8b_3 - a_8b_2b_3 + a_7b_3^2 + a_9\beta^2\right) + a_3\left(a_6b_2b_3\right.\\
&-\ \left.\left. a_5b_3^2 + b_3^2b_7 - b_2b_3b_8 + a_1^2b_9 + b_2^2b_9 + a_1\left(a_6b_3 - b_3b_8 + 2b_2b_9\right) + b_9\beta^2\right)\right),\\
K_1 = &\ A_1^2A_2^2 - 4A_0A_1A_2\left(A_3 + 2A_1\left(\left(a_1(a_1 + b_2) + a_3c_1\right)^2 + \beta^2\left(2a_1^2 + 2a_1b_2 + b_2^2 + 2a_3c_1\right)\right.\right.\\
&+\ \left.\left.\beta^4\right)\right) + 4A_0^2A_3\left(A_3 + 2A_1\left(\beta^2\left(2a_1^2 + 2a_1b_2 + 2a_3c_1 + b_2^2\right) + \left(a_1(a_1 + b_2)\right.\right.\right.\\
&+\ \left.\left.\left. a_3c_1\right)^2 + \beta^4\right)\right),\\
N = &\ (a_1(a_1 + b_2) + a_3c_1)^2 + (2a_1^2 + 2a_1b_2 + b_2^2 + 2a_3c_1)\beta^2 + \beta^4.
\end{aligned}
$$

Under conditions (iii) of Proposition 1.1, the expressions of B_i with $i = 0, ..., 4$ and K_2 are the following.

$$
\begin{aligned}
B_0 &= \beta^2b_3\left((b_1b_2 + b_3c_1)(a_{31}b_2 - a_{21}b_3) + \beta^2(a_{31}b_1 + b_3c_{31})\right),\\
B_1 &= b_3(b_1b_2 + b_3c_1)(2(a_8b_2 - a_7b_3)(b_1b_2 + b_3c_1) + \beta^2(2a_8b_1 + a_6b_2 + b_2^2 + b_3(-a_5\\
&\quad + 2b_7 + c_8) + b_2(b_3 - b_8 - 2c_9))) + \beta^4b_3(b_3(b_5 + c_6) + b_1(a_6 + b_2 - b_8 - 2c_9))\\
&\quad - 2a_9(b_2b_3c_1 + b_1(b_2^2 + \beta^2))^2,\\
B_2 &= 2b_3\beta^2((b_1b_2 + b_3c_1)(a_{21}b_3 - a_{31}b_2) + \beta^2(a_{11}b_3 - a_{31}b_1)),\\
B_3 &= 2(b_3((a_7b_3 - a_8b_2)(b_1b_2 + b_3c_1)^2) - (a_8b_1 + a_6b_2 - a_5b_3)(b_1b_2 + b_3c_1)\beta^2\\
&\quad + (a_4b_3 - a_6b_1)\beta^4) + a_9(b_2b_3c_1 + b_1(b_2^2 + \beta^2))^2),\\
B_4 &= \beta^4(b_3(a_7b_3 - a_8b_2) + a_9(b_2^2 + \beta^2)),\\
K_2 &= B_1^2B_2(8B_0 + B_2) - 4B_0B_1(2B_0 + B_2)B_3 + 4B_0^2B_3^2.
\end{aligned}
$$

On the other hand, under conditions (iv) of Proposition 1.1, we redefine the expressions of B_i with $i = 0, ..., 4$ as follows:

$$
\begin{aligned}
B_0 &= \beta^2a_3(a_1a_2 + a_3c_2)(b_{31}a_1 - b_{11}a_3) + \beta^4a_3(b_{31}a_2 + a_3c_{31}),\\
B_1 &= a_3(a_1a_2 + a_3c_2)\left(2(b_6a_1 - b_4a_3)(a_1a_2 + a_3c_2) + \beta^2(2a_2b_6 + a_1b_8 + a_1^2 + a_3(-b_5\right.\\
&\quad + 2a_4 + c_6) + a_1(a_3 - a_6 - 2c_9))) + \beta^4a_3\left(a_3(a_5 + c_8) + a_2(b_8 + a_1 - a_6\right.\\
&\quad -2c_9)) - 2b_9(a_1a_3c_2 + a_2(a_1^2 + \beta^2))^2,\\
B_2 &= 2a_3\beta^2((a_1a_2 + a_3c_2)(b_{11}a_3 - b_{31}a_1) + \beta^2(b_{21}a_3 - b_{31}a_2)),\\
B_3 &= 2(a_3((a_3b_4 - a_1b_6)(a_1a_2 + a_3c_2)^2) - (b_6a_2 + a_1b_8 - a_3b_5)(a_1a_2 + a_3c_2)\beta^2\\
&\quad + (a_3b_7 - a_2b_8)\beta^4) + b_9(a_1a_3c_2 + a_2(a_1^2 + \beta^2))^2),\\
B_4 &= \beta^4(a_3(a_3b_4 - a_1b_6) + b_9(a_1^2 + \beta^2)),
\end{aligned}
$$

The expression of K_2 is the same as under conditions (iii).

$$
\begin{aligned}
D_0 &= (a_1b_{31} - b_1a_{31})(a_1c_1 + b_1c_2)\beta^2 + (b_1b_{21} + c_1b_{31})\beta^4,\\
D_1 &= (a_1c_1 + b_1c_2)(2(a_1b_9 - a_9b_1)(a_1c_1 + b_1c_2) + (b_1(a_6 + b8) + 2b_9c_1)\beta^2),\\
D_2 &= 2(a_1c_1 + b_1c_2)\beta^2((a_{31}b_1 - a_1b_{31})(a_1c_1 + b_1c_2) + (b_1c_{31} - b_{31}c_1)\beta^2)\\
D_3 &= 2(a_1c_1 + b_1c_2)^2((a_9b_1 - a_1b_9)(a_1c_1 + b_1c_2) + (b_1c_9 - b_9c_1)\beta^2),\\
D_4 &= (a_9b_1 - a_1b_9)(a_1c_1 + b_1c_2)^3 - (a_1c_1 + b_1c_2)\left(a_1^3b_4 + a_1^2b_1(b_5 - a_4) + a_1\left(-a_5b_1^2\right.\right.\\
&\quad \left. + b_1^2b_7 + b_1b_8c_1 + 2b_9c_1^2 - b_1b_6c_2 - b_1c_1c_9\right) - b_1\left(a_7b_1^2 + a_8b_1c_1 + a_9c_1^2\right.\\
&\quad \left.\left. - a_6b_1c_2 - b_9c_1c_2 + b_1c_2c_9\right)\right)\beta^2 + \left(-b_9c_1^3 + a_1^2(b_1c_4 - 2b_4c_1) + a_1b_1\left(a_4c_1\right.\right.\\
&\quad \left. - b_5c_1 - b_4c_2 + b_1c_5\right) + b_1^3c_7 + b_1^2\left(-c_1b_7 + a_4c_2 - c_2c_6 + c_1c_8\right) + b_1c_1\left(-b_8c_1\right.\\
&\quad \left.\left. + b_6c_2 + c_1c_9\right)\right)\beta^4 + (b_1c_4 - b_4c_1)\beta^6,\\
K_4 &= (D_1D_2 - 2D_0D_3)^2 + 8a_1c_1D_0D_1(D_1D_2 - D_0D_3) + 8b_1c_2D_0D_1(D_1D_2 - D_0D_3).
\end{aligned}
$$

Acknowledgements

The first and third authors are partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación (Spain), grant MTM2016-79661-P (European FEDER support included, UE) and the Consellería de Educación, Universidade e Formación Profesional (Xunta de Galicia), grant ED431C 2019/10 with FEDER funds. The first author is also supported by the Ministerio de Educacion, Cultura y Deporte de España, contract FPU17/02125.

The second author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016 - 77278 - P (FEDER) and MDM - 2014 - 0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

The fourth author is partially supported by FCT/Portugal through UID/ MAT/ 04459/2013.

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