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An automatic method for generating multiple alignment alternatives for a railway bypass

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ABSTRACT

This paper deals with the problem of designing a bypass on a railway line. Based on a geometrical model capable of determining automatically the need of major structures (bridges, tunnels, overpasses and underpasses), the optimal design of a railway bypass is formulated in the framework of Mixed Integer Non Linear Programming (MINLP), and it is solved with a numerical algorithm which provides different layout alternatives that are optimal solutions (local minima) from the economic point of view. The proposed method is tested on a case study with the aim of showing its practical usefulness as a support tool for engineers in order to accomplish the complex and time-consuming task to generate a set of initial alternatives for the design of a railway bypass.

1. Introduction

The design of a new layout for a road or a railway line is a very complex process in which a great number of factors should be taken into account by the engineers. In general, the final alignment is selected among several potential alternatives by comparing them using multi-criteria analysis techniques. These alternatives are usually defined by the engineers from scratch as a result of applying their own expertise to each particular case, trying to obtain the best possible solution satisfying all the requirements previously established by all sorts of regulations (technical, environmental, etc.). Numerical optimization can be a very useful tool to support the engineers in order to help them with this time-consuming task.

The problem of designing a linear transport infrastructure can be considered as finding an optimal geometry for a given geographic information (Akhmet et al., 2022). It is a hot topic in the field of civil engineering, and due its complexity, it is usually performed in two consecutive stages: (i) determining the 2D horizontal alignment (Lee et al., 2009, Bosurgi and D'Andrea, 2012, Mondal et al., 2015, Bosurgi et al., 2016, Casal et al., 2017, Sushma and Maji, 2020) and (ii) finding the optimal vertical alignment for a 2D layout already given (Hare et al., 2014, Monnet et al., 2020, Momo et al., 2022, and Sushma et al., 2022). In practice, engineers address this task as an iterative process by repeating these stages until a satisfactory 3D alignment is achieved. The 3D design, considering the two alignments simultaneously, has also been addressed in the literature, both for highways (Jong and Schonfeld, 2003, Kim et al., 2007, Li et al., 2013, Hirpa et al., 2016, Pushak et al., 2016, Vázquez-Méndez et al., 2018) and railway lines (Li et al., 2016, Li et al., 2017, Ghoreishi et al., 2019, Pu et al., 2019a,b, Zhang et al., 2020, Song et al., 2020, 2022a, Song et al., 2022b). In addition to the previous classification on the type of alignment, these papers can be also classified according to other many criteria. For example, the recently work of Gao et al. (2022) contains a table where many papers are characterized by the factors (economical, environmental, geological...) included in the optimization process, and by the algorithm(s) used for its numerical resolution. The hypotheses assumed to simplify the problem are also another key aspect. For example, almost all papers take into account transition curves in the vertical alignment, but transition horizontal curves are used in fewer works (Casal et al., 2017, Vázquez-Méndez et al., 2018, Ghoreishi et al., 2019, Vázquez-Méndez et al., 2021a,b). Other differentiating feature is how to deal with some important design variables, like the number of curves and/or the number of slope changes, radii, and the horizontal and/or vertical intersection points (IPs). Regarding to the number of curves and slope changes, they are fixed in advance (Jong and Schonfeld, 2003, Kim et al., 2007, Li et al., 2013, Hirpa et al., 2016, Casal et al., 2017, Ghoreishi et al., 2019, Vázquez-Méndez et al.,

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2021a) or included in the optimization process (Li et al., 2016, Li et al., 2017, Pu et al., 2019a,b, Sushma and Maji, 2020, Vázquez-Méndez et al., 2021b, Song et al., 2022b, Gao et al., 2022). Additionally, fixed radius for all the horizontal curves are considered in some papers (Jong and Schonfeld, 2003, Gao et al., 2022). With respect to IPs, some works do not make any additional hypotheses on their location (Hirpa et al., 2016, Casal et al., 2017, Vázquez-Méndez et al., 2018, 2021a,b), but many others assume that they are located in orthogonal sections or on nodes/centroids of a given grid (Jong and Schonfeld, 2003, Li et al., 2013, Li et al., 2016, Pushak et al., 2016, Li et al., 2017, Ghoreishi et al., 2019, Pu et al., 2019a, Gao et al., 2022). Finally, other very important aspect is how to determine promising corridors where to look for the optimal solution. Mondal et al. (2015), Casal et al. (2017) and Vázquez-Méndez et al. (2021a) directly assume a given initial handmade layout. Other works determine the corridors from geological information in a previous step, either with the adaptation of several path-finding algorithms (Pushak et al., 2016), or with a Distance Transform (DT) method (Li et al., 2016, Li et al., 2017, Pu et al., 2019a,b, Song et al., 2020). In the latter, the optimal layout is obtained by using a well-known optimization method (PSO, NOMAD, GA...) to optimize costs along the corridor obtained in the previous step. Additionally, recent works used multi-start methods (Vázquez-Méndez et al., 2021b), motion planning algorithms Sushma and Maji (2020) and deep learning methods (Gao et al., 2022) to directly search the optimal layout without first selecting any corridor. According to all previous aspects, the framework of the current paper is the study of horizontal and vertical alignments simultaneously, considering transition curves in both alignments, minimizing the main infrastructure costs, guaranteeing the main technical (safety) and ecological constraints, using gradient-type techniques, assuming that the number of curves and the number of slope changes are only upper bounded, without making pre-assumptions about radii and IPs, and searching the optimal layout without first preselecting any corridor.

The design of railway bypasses has an important particularity: the connection with the former layout is of great importance. In a previous work, Vázquez-Méndez et al. (2021a) presented a mathematical model for automatic design an urban railway bypass. This model turned out to be a useful tool for engineers working in this field, but it has some flaws which limit its practical application. Basically, the weakness are due to the facts that no structures were considered, the number of curves and slope changes must be fixed in advance, a handmade initial alignment is necessary to run the algorithm, and the output is only one layout. In this paper we present a new method that fixes all these flaws.

In order to define a set of possible alternatives for a specific layout with an appropriate economical cost estimation it is very important to include in the design the main types of structures. Some of these structures (specifically, bridges and tunnels) are defined along the longitudinal axis of the horizontal alignment, and they have been already considered in the literature (Kim et al., 2007, Li et al., 2013, Li et al., 2016, Ghoreishi et al., 2019, Pu et al., 2019a, Zhang et al., 2020, Song et al., 2022a, Song et al., 2022b). However, other necessary structures, such as overpasses and underpasses, should be also taken into account in the design. This fact is even more relevant in areas with complex orography or zones in which there are significant number of linear infrastructures that potentially may intersect with the new alignment. The geometrical definition of both overpasses and underpasses depends not only on the new horizontal alignment but also on the traversed infrastructures. Assuming that no layout modification is considered for the existing infrastructures, the aim is to keep clear the new railway of level crossing and assure a minimum vertical distance between both, making possible safely crossings. As far as we know, costs of overpasses and underpasses were not yet included to date in any geometrical model previously proposed for 3D layout optimization.

In this paper, an automatic method to support the engineers with the complex task of generating a set of initial alternatives for a new railway bypass is suggested. They are (sub)optimal from an economic viewpoint and can be used as a starting point for a subsequent refinement process carried out by the engineers if any aspects of some proposed options should be reviewed. The original or refined (if necessary) layout alternatives will serve as basis for the selection process in which additional decision factors can be considered for choosing the final layout.

The method described in this paper is implemented for the specific case of a railway bypass avoiding a forbidden area, but it could be easily modified for more generic cases involving other linear infrastructures. At the early stage of this new approach, some sections of the current layout (tangents and/or circular curves) must be chosen on both sides of the area to be circumvented. In the next section, we propose a method for generating automatically some interesting alignments for a bypass linking two given sections. Then, the set of initial alternatives is obtained by applying this method to link all the possible combinations of sections chosen at the early stage.

To develop the method, we start from the geometrical model proposed in Vázquez-Méndez et al. (2021a) and improve it to include the capability to automatically identify and compute the economic costs of the four aforementioned structures: bridges, tunnels, overpasses and underpasses. From this new model, taking into account main infrastructure costs, technical constraints and all restrictions on passage, the optimal design of a railway bypass is formulated in the framework of Mixed Integer Non Linear Programming (MINLP). The formulated problem is *non standard*, in the sense that the dimension of the continuous variable depends on the integer one. After carefully analyzing its particularities, an algorithm is proposed for the numerical resolution of the problem, which provides the different layout alternatives, being all optimal solutions (local minima) from the economic point of view.

In order to assess its practical usefulness, the proposed method is applied to a real case problem consisting on the design of a bypass to the A Coruña-Lugo railway line surrounding the urban area of Parga (Spain). This case study serves to show that the method may represent a very helpful tool for engineers in order to define multiple design alternatives through a more systematic method, reducing the time needed to accomplish this task and assuring that all the potential layouts proposed are (sub)optimal in terms of economical cost.

2. Materials and methods

2.1. Mathematical model

In this section we present the method for the automatic generation of multiple (sub)optimal alternatives of a bypass design properly linking two given sections –tangents or circular curves– of an existing railway line. First we recall the geometrical model (and notation) proposed in Vázquez-Méndez et al. (2021a). Next, we detail a method for the automatic identification of major structures and give a new formulation for the optimal design of a railway bypass. Finally, we propose a numerical algorithm to solve this new formulation that leads to the advertised method.

2.1.1. Geometrical model

Any horizontal alignment (HA) with N symmetric curves linking two given sections is uniquely determined by a vector $\mathbf{x}^N \in \mathbb{R}^{d_N}$ defining the horizontal intersection points (HIP) $\mathbf{v}_i \in \mathbb{R}^2$, and the radii $R_i \geq R_{\min} > 0$ and angles $\omega_i \geq 0$ of the circular curves (see Casal et al., 2017). The dimension of this vector is $d_N = 4N - 2$ if both connection sections are tangent segments, $d_N = 4N$ if both are circular curves, and $d_N = 4N - 1$ if one is a tangent segment and the other a circular curve. In a similar way, any vertical alignment (VA) with M slope changes is determined by a vector $\mathbf{y}^M \in \mathbb{R}^{3M-2}$ defining the vertical intersection points (VIP) $s_j \in (0,1)$, the gradients $m_j \in [-m_{\max}, m_{\max}]$, and the parameters $K_{vj} \geq K_{v\min}$ giving the vertical transition (parabolic) curves. Therefore, any bypass with N curves and M slope changes linking two given sections is uniquely determined by

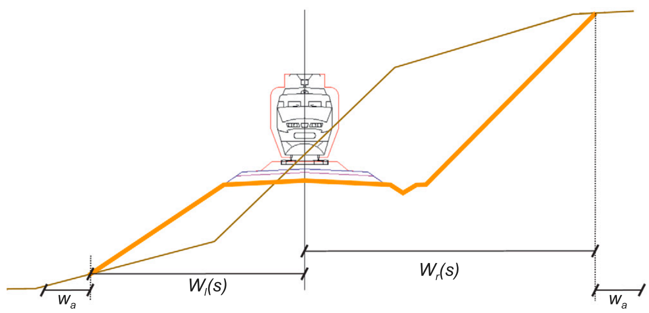

Fig. 1. Shape of transition (cut and fill) cross section.

a vector $\mathbf{u}^{N,M} = (\mathbf{x}^N, \mathbf{y}^M) \in \mathbb{R}^{d_N+3M-2}$ (hereafter decision vector), and for each decision vector $\mathbf{u}^{N,M}$ a parametrization of the central axis of the bypass can be computed (see Vázquez-Méndez et al., 2021a for details)

$$\begin{array}{rccl} \sigma_{\mathbf{u}^{N,M}}: & [0, L(\mathbf{x}^N)] & \longmapsto & \mathbb{R}^3 \\ & s & \longmapsto & \sigma_{\mathbf{u}^{N,M}}(s) = (\sigma_1(s), \sigma_2(s), \sigma_3(s)), \end{array} \tag{1}$$

where $L(\mathbf{x}^N)$ represents the length of the HA.

Further, by defining the cross-section at each $s \in [0, L(\mathbf{x}^N)]$, the left $W_l(s)$ and right $W_r(s)$ afar width from the central axis can be computed by intersecting the cross-section with the ground elevation (see Fig. 1). From these two functions, the parametrization of the central axis (1), and the shape of the cross-section, it is straightforward to compute the parametrization of the surface that defines the railway linking the two sections to be connected (see Vázquez-Méndez et al., 2018)

$$\begin{array}{rccl} \tau_{\mathbf{u}^{N,M}}: & [0, L(\mathbf{x}^N)] \times [-W_l(s), W_r(s)] & \longmapsto & \mathbb{R}^3 \\ & (s,t) & \longmapsto & \tau_{\mathbf{u}^{N,M}}(s,t) = (\tau_1, \tau_2, \tau_3). \end{array} \tag{2}$$

Finally, it should be noted that from the limits given by $W_l(s)$ and $W_r(s)$, it is usual to acquire an additional w_a-meter strip on both sides of the infrastructure.

2.1.2. Automatic definition of major structures

The previous geometrical model does not take into account the possible need for major structures (tunnels, bridges, overpasses and underpasses). To fix this flaw, we implement a method for automatic identification of these structures and the computation of its corresponding economic costs. In essence, we extend the method introduced by Kim et al. (2007) for tunnels and bridges to overpasses and underpasses, and give a rigorous mathematical formulation of economic costs in terms of the decision vector. This non-discrete formulation is much more suitable for using a gradient type-method in the later optimization process (Section 2.1.4).

Tunnels. A tunnel is considered in cut sections where the vertical alignment is h^c meters lower than the ground elevation and the cut area is greater than a maximum value $A^c_{\max}$ (m^2). For each $s \in [0, L(\mathbf{x}^N)]$, the cut area is given by

$$A^c(s) = \int_{-W_l(s)}^{W_r(s)} (H(\tau_1(s,t), \tau_2(s,t)) - \tau_3(s,t))_+ dt,$$

where $H(x,y)$ denotes the terrain height at point (x,y), and $(x)_+ = \max\{x, 0\}$ is the positive part function. Then, once parameters h^c and $A^c_{\max}$ are set, the need for a tunnel is determined by the following flag function

$$I^t(s) = \begin{cases} 1 & \text{if } \sigma_3(s) + h^c \le H(\sigma_1(s), \sigma_2(s)) \text{ and } A^c(s) \ge A^c_{\max}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

This function is used for modifying the occupation and land acquisition areas in tunnels: if $I^t(s) = 1$ (i.e. tunnel exists), then $W_l(s) = W_r(s) = 0$ and $w_a(s) = 0$. Moreover, it is used for computing the cost of the tunnels, which is given by

$$J_T(\mathbf{u}^{N,M}) = p_t \int_0^{L(\mathbf{x}^N)} I^t(s) ds, \tag{4}$$

where p_t (€/m) is the linear tunnel price.

Bridges. A bridge is considered in two situations:

1. In fill sections where the vertical alignment is h^f meters higher than the ground elevation and the fill area is greater than a maximum value $A^f_{\max}$ (m^2).
2. If the alignment is crossing a river or any *special* area (for example, if the policy of the country is to place the rail line on bridge sections, then the whole domain would be a *special* area).

We denote by $R_1 \subset \mathbb{R}^2$ the set representing rivers and special areas which must be crossed on bridge sections. Moreover, for each section $s \in [0, L(\mathbf{x}^N)]$, we consider the fill area

$$A^f(s) = \int_{-W_l(s)}^{W_r(s)} (\tau_3(s,t) - H(\tau_1(s,t), \tau_2(s,t)))_+ dt,$$

and define the following flag function for bridges

$$I^b(s) = \begin{cases} 1 & \text{if } (\sigma_3(s) - h^c \ge H(\sigma_1(s), \sigma_2(s)) \text{ and } A^f(s) \ge A^f_{\max}) \\ & \text{or} \\ & (\{(\tau_1(s,t), \tau_2(s,t)), t \in [-W_l(s), W_r(s)]\} \cap R_1 \neq \emptyset), \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

In the same way as in tunnels, the occupation area is modified along the bridges: if $I^b(s) = 1$, then $W_l(s) = W_r(s) = W_b/2$, being W_b the width considered for a bridge. Additionally, in this case, the fill area must be redefined, in the sense that, if $I^b(s) = 1$, then $A^f(s) = 0$. Finally, the cost of the bridges is given by

$$J_B(\mathbf{u}^{N,M}) = p_b \int_0^{L(\mathbf{x}^N)} I^b(s) ds, \tag{6}$$

where p_b (€/m) is the linear bridge price.

Over and under passes. To avoid level crossings, where the new bypass intersects an existing infrastructure, a minimum difference in height must be fulfilled between them. In these cases, an overpass or underpass must be built. We denote by $R_2 \subset \mathbb{R}^2$ the set representing the existing infrastructures and compute the cost of overpasses (J_{OP}) and underpasses (J_{UP}) by

$$J_{OP}(\mathbf{u}^{N,M}) = p_{op} \int_0^{L(\mathbf{x}^N)} I^+(s) \left(\int_{-W_l(s)}^{W_r(s)} \chi_{R_2}(\tau_1(s,t), \tau_2(s,t)) dt \right) ds, \tag{7}$$

$$J_{UP}(\mathbf{u}^{N,M}) = p_{up} \int_0^{L(\mathbf{x}^N)} I^-(s) \left(\int_{-W_l(s)}^{W_r(s)} \chi_{R_2}(\tau_1(s,t), \tau_2(s,t)) dt \right) ds, \tag{8}$$

where p_{op} and p_{up} (€/m^2) are, respectively, prices of overpass and underpass per square meter, χ_{R_2} is the characteristic function of R_2

$$\chi_{R_2}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R_2, \\ 0 & \text{otherwise,} \end{cases}$$

and I^+ and I^- are flag functions given by

$$I^+(s) = \begin{cases} 1 & \text{if } H(\sigma_1(s), \sigma_2(s)) > \sigma_3(s), \\ 0 & \text{otherwise,} \end{cases}$$

$$I^-(s) = \begin{cases} 1 & \text{if } H(\sigma_1(s), \sigma_2(s)) < \sigma_3(s), \\ 0 & \text{otherwise.} \end{cases}$$

2.1.3. Optimal design from an economic point of view

As it was previously mentioned, any bypass with N curves and M slope changes is uniquely determined by a vector $\mathbf{u}^{N,M} \in \mathbb{R}^{d_N+3M-2}$. For each of these vectors, acquisition costs (J_A), cleaning and terrain preparation (J_C), railroad construction (J_S), and earthwork (J_{EW}) can be computed as detailed in Vázquez-Méndez et al. (2021a). Therefore, the economic cost of the bypass determined by $\mathbf{u}^{N,M}$ is given by

$$J_E(\mathbf{u}^{N,M}) = J_A(\mathbf{u}^{N,M}) + J_C(\mathbf{u}^{N,M}) + J_S(\mathbf{u}^{N,M}) + J_{EW}(\mathbf{u}^{N,M}) + J_T(\mathbf{u}^{N,M}) + J_B(\mathbf{u}^{N,M}) + J_{OP}(\mathbf{u}^{N,M}) + J_{UP}(\mathbf{u}^{N,M}). \quad (9)$$

On the other hand, the bypass must avoid buildings and forbidden areas (for example urban or special environmental protection areas), and must cross rivers and infrastructures while complying with a minimum difference in height. To deal with these restrictions on passage a penalty technique is used: we consider a penalty function $J_P(\mathbf{u}^{N,M})$ measuring the non fulfillment of these constraints (see Vázquez-Méndez et al., 2021a for a detailed definition), and take the sum of the economic cost (9) and this penalty function as the objective to be minimized

$$J(\mathbf{u}^{N,M}) = J_E(\mathbf{u}^{N,M}) + J_P(\mathbf{u}^{N,M}).$$

Technical constraints are treated as bound (minimum radii, minimum angles of circular curves, maximum gradients, etc.) linear (VIP must be sorted in an increasing order) or nonlinear (minimum length of tangent segments, circular curves and transition curves, etc.) constraints.

Finally, in the design of both roads and railways, the number of curves (N) and the number of slope changes (M) are critical values which determine not only the final result, but also the computational time to obtain it. For horizontal alignment, Sushma and Maji (2020) make a nice analysis of how N is treated in the literature, concluding that it must be chosen in the optimization process instead of being fixed in advance. This conclusion can be also extended to the number of slope changes, and it can be asserted that both N and M must be determined automatically. Therefore, they must be included into the decision variables of the problem, and hence, the optimal design of a railway bypass from an economic point of view consists in solving the following problem

$$\min_{\substack{N,M\in\mathbb{Z}\\ \mathbf{u}^{N,M}\in\mathbb{R}^{d_N+3M-2}}} J(\mathbf{u}^{N,M}) \quad (10)$$

$$\text{subject to}\quad 1 \le N \le N_{\max},\ 1 \le M \le M_{\max}, \quad (11)$$

$$\mathbf{u}_{\min} \le \mathbf{u}^{N,M} \le \mathbf{u}_{\max}, \quad (12)$$

$$\mathbf{A}\mathbf{u}^{N,M} \le \mathbf{b}, \quad (13)$$

$$\mathbf{g}(\mathbf{u}^{N,M}) \le \mathbf{0}, \quad (14)$$

where $N_{\max}$ and $M_{\max}$ are the maximum allowed number of curves and slope changes respectively, $\mathbf{u}_{\min}, \mathbf{u}_{\max} \in \mathbb{R}^{d_N+3M-2}$ collect, respectively, lower and upper bounds for the continuous variable $\mathbf{u}^{N,M}$, $\mathbf{A}$ (square matrix of dimension $d_N + 3M - 2$) and $\mathbf{b} \in \mathbb{R}^{d_N+3M-2}$ define linear constraints, and $\mathbf{g}$ is a vectorial function collecting all nonlinear constraints. Bound (12), linear (13) and non-linear (14) constraints are concerning to technical and safety standards, in order to guarantee, for example, that radii and the length of each alignment (tangent, circular curve and transition curve) are larger than pre-fixed minimum values (see Vázquez-Méndez et al., 2021a for further details).

The problem (10)–(14) is not standard in mixed integer nonlinear programming (MINLP), since the dimension of the continuous variable $\mathbf{u}^{N,M}$ depends on the integer variables N and M. Consequently, the definition of lower and upper bounds ($\mathbf{u}_{\min}$, $\mathbf{u}_{\max}$), linear constraints ($\mathbf{A}$, $\mathbf{b}$), and even nonlinear constraints ($\mathbf{g}$) and objective function (J) depend implicitly on the value of those integer variables N and M.

2.1.4. Automatic generation of multiple (sub)optimal solutions

Before defining the numerical algorithm to solve problem (10)–(14), the following aspects should be taken into account:

- The integer variables are bounded ($1 \le N \le N_{\max}$, $1 \le M \le M_{\max}$). In longer bypasses (particularly in mountainous terrain), the upper bounds $N_{\max}$ and $M_{\max}$ may be large and motion-planning based algorithms (Sushma and Maji, 2020) or deep learning methods (Gao et al., 2022) can be useful. However, in many other practical cases (like the present paper), $N_{\max}$ and $M_{\max}$ are low, and the number of possible combinations on N and M ($N_{\max} \times M_{\max}$) is also low.
- If the values of N and M are fixed in advance, gradient-type techniques have been proved very useful for solving the corresponding NLP-problem in similar situations (Vázquez-Méndez et al., 2018, 2021a). Even so, this NLP-problem is non-convex and a gradient-type method can provide local minima.
- In this particular problem (design of a railway bypass), it is not only interesting for obtaining the global minimum, but other local minima can be also very useful. These local minima provide different alternatives to the engineer, who can and must choose among them, taking into account not only economic criteria, but also other aspects (functional, environmental, social, etc.) that have not been considered in the optimization process.

According to these aspects, we propose an algorithm to solve the problem (10)–(14) based on combining an exhaustive search on integer variables with a random multi-start of the sequential quadratic programming (SQP) method (see Nocedal and Wright, 2006). This combination has been already successfully tested on two academic problems in Vázquez-Méndez et al. (2021b). The number of random multi-starts M_S must be increased with the value $N_{\max}\times M_{\max}$, and they can be also completed with ad-hoc starting points if promising corridors are known or can be obtained in a previous stage. The complete numerical algorithm is schematized in Algorithm 1. The main difficulty of this algorithm is the generation of the vector $\mathbf{x}^N_{\text{inic}}$ determining an admissible horizontal alignment (AHA). If the AHA only has to join two given points (without worrying about azimuth and curvature at these points), this random generation has been studied in detail in Vázquez-Méndez et al. (2021b). The extension to the case where the AHA has to link two given sections requires a previous step, where geometrical properties at both ends must be considered:

- **Case A** (one -or both- given section is a circular curve): we generate a clothoid linking in one of its ends with the circular curve. The other end of the clothoid defines a tangent which is used as new section to be linked in Case B. The clothoid is univocally determined by the angle $\alpha \in [\alpha^{\min}, \alpha^{\max}]$ defining the link point and the clothoid angle $\mu \in (0, \mu^{\max}]$ (see, for instance, Vázquez-Méndez et al., 2021a). Therefore, to generate the clothoid we only take values α and μ in a random way.
- **Case B** (both sections are tangents):
  - Step 1: An AHA with two or three curves linking the two tangents are randomly generated (see Appendix for technical aspects of this problem and a detailed description of the algorithms to do it).
  - Step 2: If the number of curves of the AHA computed in Step 1 is lower than N, then it is modified by including new curves just as described in Vázquez-Méndez et al. (2021b).

2.2. Case study

The proposed method has been applied for obtaining several optimal bypasses to the A Coruña-Lugo railway line at Parga (Spain). The area selected for this study is presented in Fig. 2 which provides information about the railway line crossing this area, topographic data (level curves) and the different elements to be taken into account as passage restrictions: the urban area of Parga, buildings, main roads and Parga river (the river has the same name as the village). It can be observed that the current horizontal alignment was divided into its geometric elements: tangents, circular curves and their corresponding transitional curves (clothoids). This task was carried out by means of a

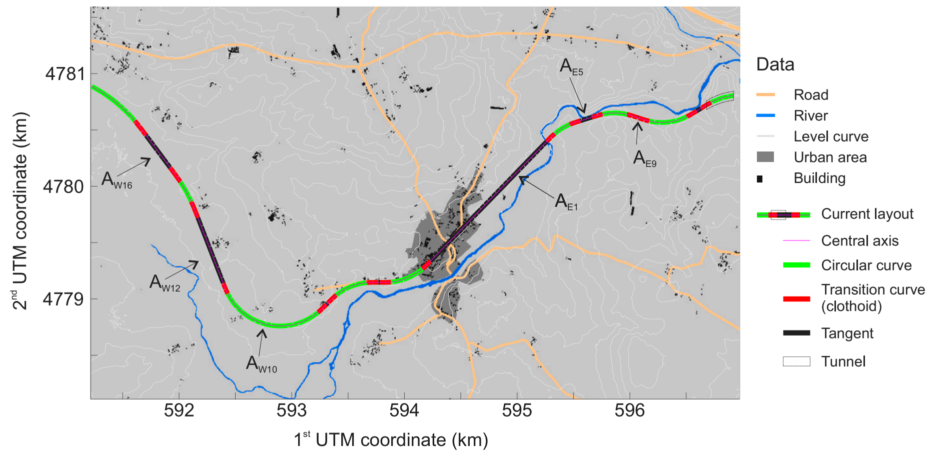

Fig. 2. Case study area including topographic data (level curves), information about the horizontal alignment of the railway line, and the main elements considered as restrictions on passage (roads, rivers, urban area and buildings).

Algorithm 1 Automatically generation of (sub)optimal designs of a railway bypass

Input: Sections of current layout to be linked, prices (p_t, p_b, ...), technical constraints ($R_{\min}$, $m_{\max}$, ...), regions to define passage constraints (R_1, R_2, ...), ground elevation ($H(x,y)$), number of multi-starts (M_S), percentage of multi-starts desired with i-curves ($p_i \in (0,1)$, $i = 1,\dots,N_{\max}$) and with j-slope changes ($q_j \in (0,1)$, $j = 1,\dots,M_{\max}$), maximum project budget ($J_{\max}$)

Output: Multiple (sub)optimal designs of a railway bypass linking A and B

for $m = 1$ to M_S **do**
- Generate $1 \le N \le N_{\max}$, $1 \le M \le M_{\max}$ with probabilities $\{p_i\}$ and $\{q_j\}$
- Compute $\mathbf{u}_{\min}, \mathbf{u}_{\max}$, $\mathbf{A}$, $\mathbf{b}$ and define J and $\mathbf{g}$ for the N and M chosen values
- Generate (randomly) $\mathbf{u}^{N,M}_{\text{inic}} = (\mathbf{x}^N_{\text{inic}}, \mathbf{y}^M_{\text{inic}})$
- Compute $\mathbf{u}^{N,M}$ by solving the NLP-problem associated to (10)–(14) with the SQP method, starting from $\mathbf{u}^{N,M}_{\text{inic}}$

if $J(\mathbf{u}^{N,M}) \le J_{\max}$ **then**
- Save $\mathbf{u}^{N,M}$

end if

end for

Table 1
Connection cases considered in this study and original length (L_0) of the existing horizontal alignment between their sections.

Case	East connection	West connection	L_0 (km)
C1	A_{E5}	A_{W10}	3.330
C2	A_{E5}	A_{W12}	4.728
C3	A_{E1}	A_{W12}	4.265
C4	A_{E5}	A_{W16}	5.595
C5	A_{E9}	A_{W16}	6.013

previous process in which an optimization model (Castro et al., 2023) was used to recreate the original horizontal layout from information about the coordinates of different points of its horizontal axis and data available along the railway line regarding the current geometrical parameters of the tangents, circular curves and clothoids.

This location has been previously considered in Vázquez-Méndez et al. (2021a) as a case study but with a very different approach. In that case, the main goal was finding the shorter optimal bypass, using as much as possible the existing infrastructure. Therefore, the connections of the bypass with the original horizontal alignment were located as close as possible to the urban area of Parga. Taking this fact into account, the main tangent crossing the urban area (A_{E1}) and the nearest sector of the circular curve (A_{W10}) were selected as most suitable sections for those connections. Moreover, only the north region was considered in order to obtain a bypass with a minimum length, avoiding the difficulties presented on the south region. As a consequence, the radii of the circular curves and the vertical slopes were clearly restricted by the goal of obtaining the shortest bypass avoiding the urban area.

In this new approach, the main goal is to show the capabilities of the algorithm to automatically provide a set of initial alternatives for a bypass avoiding the urban area of Parga. In order to do so, at an early stage some sections of the current layout (tangents and circular curves) must be chosen on both sides of Parga. After a visual analysis of the current horizontal alignment, the following sections were considered as candidates for connections: tangents A_{E1}, A_{E5} and A_{E9} on the east side, and circular curve A_{W10} ($R = 520$ m) and tangents A_{W12} and A_{W16} on the west (see Fig. 2). All possible combinations of these sections lead to nine different cases. However, to simplify the subsequent analysis of the solutions, only five of them (collected in Table 1) were selected. Another important difference with the previous work is that, due to the improvements of the new method, it is not necessary to set in advance which region (north/south) is more suitable as passing zone for the bypass. At first glance, the southern region seems a more difficult area, leading to circumvent the urban area of Parga with lengthy alignments, but it is preferable to let the method run freely, and later on analyze all the solutions obtained.

Remark 1. The names A_{W_i} and A_{E_j} stands for all the elements of the alignment placed west (W) and east (E) of the urban area of Parga. All parts of the alignment were labeled. For example, between tangents A_{E_5} and A_{E_9} there exist two transition curves (A_{E_6} and A_{E_8}) and a circular curve (A_{E_7}) in between not named in Fig. 2. For sake of simplicity Fig. 2 only displays the candidate sections where the new bypass will connect with the current layout.

One of the main peculiarities of our approach is that we use a continuous (non-discrete) formulation of the problem in such a way that the solutions, for example the HIPs of the HA, can be located at any point of the study area. The data of the case study area (ground elevation, buildings, urban area, roads, rivers…) was extracted from the databases of the National Geographic Institute (IGN, by its Spanish acronym), and the relevant information was implemented in a Geographic Information System (GIS). The data was obtained on the nodes of a uniform rectangular grid with a resolution of 5×5 m^2, covering a surrounding area around Parga of 20.6 km^2. Finally, the functions used

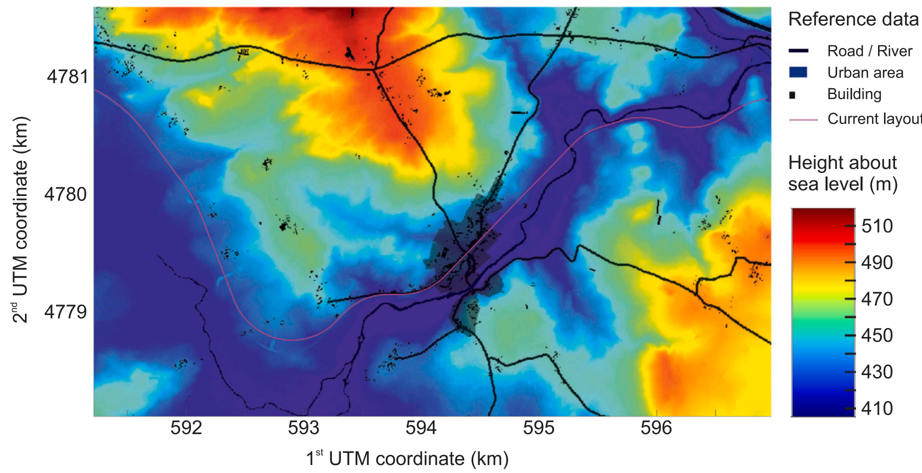

Fig. 3. Ground elevation in the area of interest.

in our continuous formulation were built from this data by numerical interpolation.

For better understanding of the topography of the case study, the continuous function giving the ground elevation of the study area is shown in Fig. 3. It can be observed that the horizontal alignment of the current railway layout was designed along the river Parga basin in order to obtain gentle slopes on its vertical alignment. It is worth to highlight that if the railway horizontal layout moves away from the river area, the ground elevation significantly increases. This effect would be more accentuated on the north region where differences in height are greater than 100 m.

2.2.1. Design criteria and prices

Geometrical constraints considered in this study were selected with the objective of improving the functionality of the existing railway line. Increasing the distance between the bypass connections allows the model to provide solutions with better geometrical parameters. Moreover, the new feature of the model intended to automatically include tunnels and bridges, according to a previously defined design criteria, makes it easier to adapt the new layouts to complex terrains and appropriately crossing sensitive areas, such as the case of rivers.

Concerning technical constraints (inequalities (12)–(14)), they must guarantee an operational speed of 120 km/h. Consequently, according to the Spanish technical standards, the following constraints are assumed: circular curve radii larger than $R_{\min} = 720$ m, transition vertical curve radii larger than $K_{v\min} = 5100$ m, clothoid length larger than $L^{C}_{\min} = 140$ m, tangent and circular curve length larger than $L^{T}_{\min} = L^{CC}_{\min} = 80$ m, and grades (slopes) lower than $m_{max} = 2$ %.

A single-track railway with an Iberian gauge (1668 mm) with a similar cross section to the one used in Vázquez-Méndez et al. (2021a) was also considered. However, some geometrical dimensions were modified taking into account the modernization projects which are currently under construction on nearby sections of the same railway line. The cross section is 11.9 m wide if the profile is in a fill-fill scenario and it has an extra 1.5 m on each side when it is a cut profile. In addition side slopes of 1:1 (45 degrees with the horizontal line) and 2:1 (30 degrees) were selected for cut and fill sides, respectively. Finally, in order to determinate the total land occupation associated with the new layout an expropriation strip $w_a = 8$ m wide on both sides was considered.

The new model automatically determines the need for different types of structures along the new alignment according to the criteria previously defined in Section 2.1.2 based on flag functions. In order to compute those functions the values of some thresholds should be established. In the case of tunnels, a maximum cut area $A^{c}_{\max} = 698$ m^2 corresponding to a vertical distance of $h^c = 20$ m between the new layout and the ground elevation was selected. Similarly, for bridges a fill area threshold ($A^{f}_{\max} = 403.5$ m^2) corresponding to a vertical distance of $h^f = 15$ m was used (in addition, the presence of a river also determines the need of a bridge). Moreover, additional thresholds should be selected in order to assure a minimum vertical distance between the new layout and existing infrastructures. The values adopted in this study were, respectively, 6.5 m and 10 m, for crossing above (underpass) and under (overpass) an existing infrastructure, respectively.

Table 2
Prices used to compute the economic cost given by the formula (9).

Concept	Price
Land acquisition	2.00 €/m^2
Ground preparation	0.75 €/m^2
Cutting	8.20 €/m^3
Filling with re-used material	2.37 €/m^3
Filling with borrowed material	6.92 €/m^3
Ground waste management	1.00 €/m^3
Tunnel	15000 €/m
Bridge	10000 €/m
Underpass	800 €/m^2
Overpass	800 €/m^2
Railway track	1370 €/m
Railway platform	8.05 €/m^2

Finally, the prices used for computing the economic cost given by the formula (9) are collected in Table 2. At this point, it is convenient to highlight that the economic model include realistic acquisition costs, where the land acquisition price is a known function $p_a(x, y)$ depending on the current land purpose, which can be obtained from a geographic information system. In this case study we assume the same purpose for all the land in the study area and, consequently, we take a constant function ($p_a(x, y) = 2$ €/m^2).

3. Results and discussion

We have made many numerical experiments. Particularly, we have been increasing the values for upper bounds $N_{\max}$ and $M_{\max}$, and also for the number of multi-stars M_S, until the results stopped improving. In this section we present and discuss some alignment alternatives obtained for the connection cases included on Table 1, corresponding to considering $N_{\max} = 4$, $M_{\max} = 4$ and $M_S = 100$. Numerical results were obtained with a MATLAB (version 9.4 (R2018a)) code of Algorithm 1, which ran on a cluster with 8 PowerEdge R840 with Intel(R) Xeon(R) Gold 6126 processors, 12 cores per processor and 384 GB of RAM. MATLAB proprietary parallelization was used with a number of 24 threads. The computation (wall clock) times for the connection cases

Table 3
Main characteristics of the solutions.

Solution	S1	S2a	S2b	S3	S4	S5
Passing zone	north	north	south	south	north	north
Total cost (M€)	19.032	30.125	19.122	17.673	23.842	26.652
Total length (km)	3.322	4.734	5.557	5.504	4.157	4.624
Cost/Length (M€/km)	5.760	6.364	3.441	3.211	5.735	5.764
Length reduction (m)	36	5.3	−828.2	−1238.6	1438.4	1389.5
Tunnel length (km)	0.552	0.832	0.000	0.000	0.798	0.771
Bridge length (km)	0.067	0.744	0.654	0.193	0.145	0.305
Underpasses (#)	1	0	0	1	0	0
Overpasses (#)	1	1	1	2	1	1

Table 4
Economic costs (M€) of each solution disaggregated by concepts.

Solution	S1	S2a	S2b	S3	S4	S5
Land acquisition	0.282	0.413	0.628	0.710	0.447	0.487
Ground preparation	0.070	0.103	0.157	0.177	0.112	0.122
Cutting	2.997	1.831	2.436	4.144	2.568	3.052
Filling	0.392	0.476	0.629	0.889	0.615	0.761
Waste management	0.240	0.040	0.053	0.156	0.064	0.067
Tunnels	8.286	12.483	0.000	0.000	11.963	11.562
Bridges	0.667	7.440	6.538	1.926	1.454	3.051
Underpasses	0.947	0.000	0.000	0.340	0.000	0.000
Overpasses	0.237	0.346	0.472	1.196	0.478	0.718
Railway track	4.551	6.486	7.613	7.541	5.695	6.335
Railway platform	0.362	0.508	0.595	0.593	0.446	0.497
TOTAL	19.032	30.125	19.122	17.673	23.842	26.652

Table 5
Geometrical characteristics of the solutions.

Solution	S1	S2a	S2b	S3	S4	S5
N	1	2	4	3	2	2
L^T_{total}(km)	1,100	1.947	1.220	0.854	2.802	2.776
L^{CC}_{total}(km)	0,080	1.989	2.742	3.011	0.714	1.132
L^C_{total}(km)	2.141	0.798	1.595	1.639	0.641	0.716
R_l(m)	1578	746	799	743	729	817
R_L(m)	1578	1150	996	1525	1767	870
R_{av}(m)	1578	853	894	1086	861	858
M	3	3	4	4	4	4
m_L(%)	1.28	1.50	1.72	1.52	2.00	1.96
L^{m_L}(km)	0.048	1.082	1.085	0.441	1.707	1.017
m_{av}(%)	0.57	0.71	0.79	0.43	1.83	1.63
K_{v1}(km)	9.867	9.290	12.108	18.337	5.100	5.100

Horizontal alignment: number of circular curves (N), total length of tangents (L^T_{total}), circular curves (L^{CC}_{total}) and clothoids (L^C_{total}), and respect to the radius of circular curves, lowest (R_l), largest (R_L) and weighted average (R_{av}). Vertical alignment: number of slope changes (M), largest slope (m_L), length with the largest slope (L^{m_L}), weighted average of the absolute values of slopes (m_{av}) and lowest value for parameter K_v (K_{v1}).

included on Table 1 were the following: C1 = 3979 s, C2 = 2711 s, C3 = 2542 s, C4 = 2202 s, C5 = 2836 s.

For simplicity, we only present the global optimal solution for each case, except for the C2, in which two different alternatives (north vs. south) are analyzed to show the good performance of the model. A summary of the main characteristics of these solutions is presented in Table 3, where a number corresponding to the connection case was assigned to each of them. It provides information about the total cost, total length, cost/length ratio and length reduction associated to each solution, including also relevant data about the structures defined automatically by the algorithm (length of tunnels and bridges as well as the number of underpasses and overpasses). In addition, the economic costs disaggregated by the different concepts considered by the model are shown in Table 4.

The horizontal and vertical alignments of each solution are presented in Figs. 4–9. Moreover, in order to assess the technical characteristics of each solution the most relevant geometrical parameters related to the horizontal and vertical alignments are included in Table 5.

All of these layouts are local optima in terms of economic cost but each of them has its pros and cons depending on what the engineer seeks. The results of Table 3 show that the most economic solution in terms of total cost is S3 (17.673 M€) with also the best cost/length ratio (3.211 M€/km) but this alternative will increase significantly the initial length of the railway line needing 1238.6 m more. On the other hand, despite the fact that the solution S4 is more expensive than others (23.842 M€), it is also the best one considering the length reduction obtained with respect to the original layout (1438.4 m) being this solution of great interest if the travel time is the key factor under consideration.

As expected, solution S1 is the shortest (3.322 km) due to the proximity of its connection sections (Fig. 4). This horizontal alignment is similar to the solution proposed in Vázquez-Méndez et al. (2021a) but with a different starting point on the east side (A_{E_1} in the reference therein). Here connection section A_{E_5} is chosen which is located farther from the urban zone of Parga. This fact allows to obtain a better value of horizontal radius (only one circular curve with $R = 1578$ m) and a vertical alignment (Fig. 4) with lower slope values, thanks also to the automatic consideration of a tunnel by the optimization algorithm. It can be appreciated that the model performs well avoiding the urban area of Parga and other buildings located on the surrounding region, as well as the automatic definition of the appropriate structures (tunnel, bridge, underpass and overpass) according to the criteria defined by the designer.

The next solution (S2a) presents a different connection section (A_{W12}) on the west side but the same connection section (A_{E5}) on the east side (Fig. 5). In this case we can appreciate how the algorithm modifies the layout with respect to solution S1 in order to get an appropriate geometrical connection taking into account the azimuth of section (A_{E5}). This solution is clearly worse in various technical parameters with respect to S1, such as horizontal radii and slopes (Fig. 5). In addition, it is the most expensive solution (30.125 M€) and it will cause a highly significant environmental and landscape impact due to the bridge crossing the river Parga.

As mentioned before, solution S2b (Fig. 6) was included in order to show how the model provides alternatives not only passing through the north region but also crossing the south region even when these solutions could be considered in advance inappropriate due to the current railway layout. It shows the capability of the algorithm to provide solutions also in this southern area fulfilling all constraints (including the proper structure if necessary). Despite its moderate economic cost (19.122 M€) this solution increases the length of the railway line with an additional unnecessary 828.2 m which can be considered as a powerful reason to discard this solution.

Solution S3 (Fig. 7) is similar to S2b but starting closer to the urban zone using the connection section A_{E1} on the east side instead of A_{E5}. In general, both solutions present the same advantage: low economic cost, but a great disadvantage due to the fact that circumventing the urban zone of Parga by the south region yields to a considerable longer horizontal alignment.

Finally, solutions S4 and S5 are clearly the most interesting in terms of travel time due to the fact of an important length reduction (1438.4 and 1389.5 m, respectively). Both solutions connect with the former layout on the west side in section A_{W16} whose position (further to the north than A_{W12}) and its azimuth make easier the connection between layouts. The horizontal alignments of these solutions (Figs. 8 and 9) are similar on their west side (right hand circular curve for connecting with the existing layout) and the center zone (a long tangent with a tunnel of similar length in both cases) but differ on the east side due to the connection section used in each case (A_{E5} for S4 and A_{E9} for S5). The solution S5 is more expensive (26.652 M€) vs. S4 (23.842 M€) by reason of the difference in total length and the need of crossing the river Parga twice instead of once. Taking this fact into account and the obtained length reduction solution S4 seems preferable rather than S5. In terms of technical parameters the main differences between these solutions are the values of the maximum slope and its corresponding length (Figs. 8 and 9). Both vertical alignments present values equal

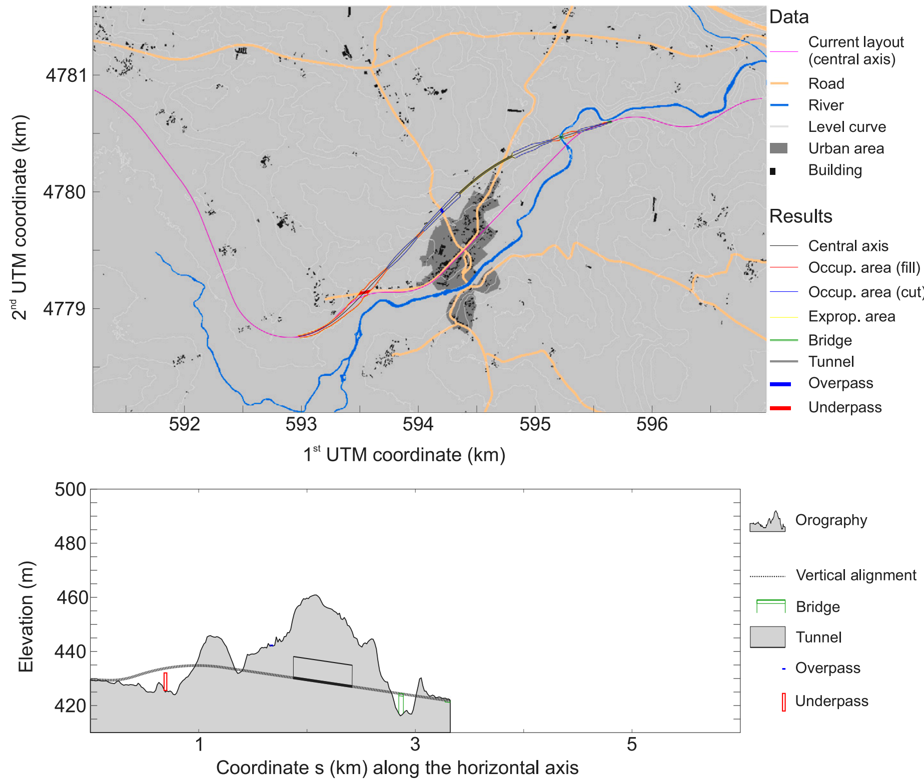

Fig. 4. Solution S1: horizontal (up) and vertical (down) alignments.

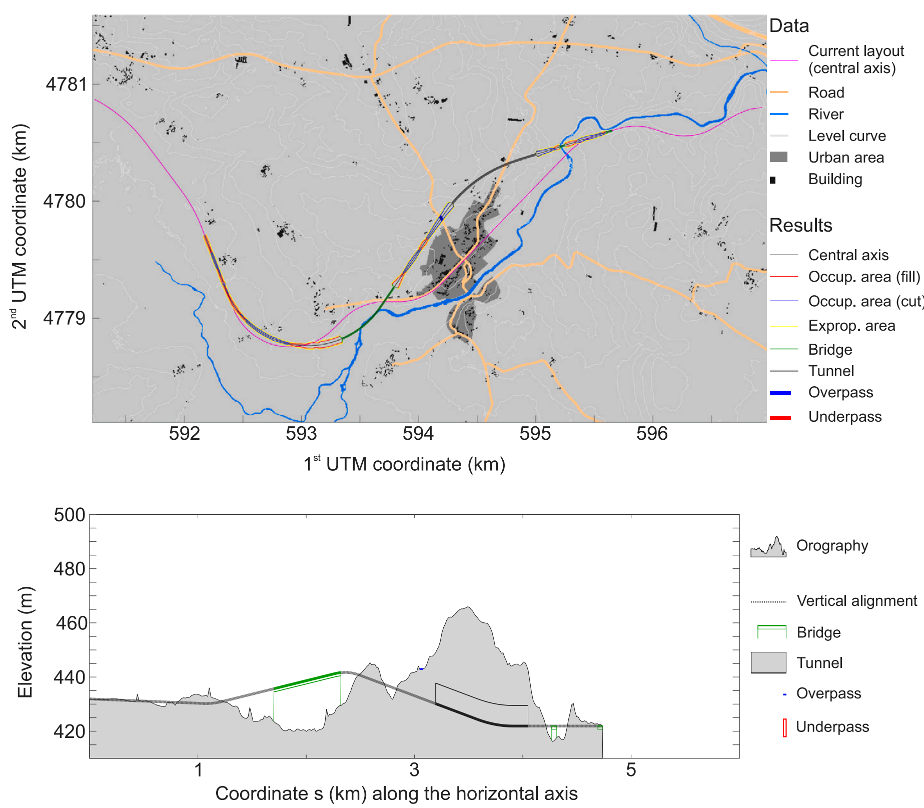

Fig. 5. Solution S2a: horizontal (up) and vertical (down) alignments.

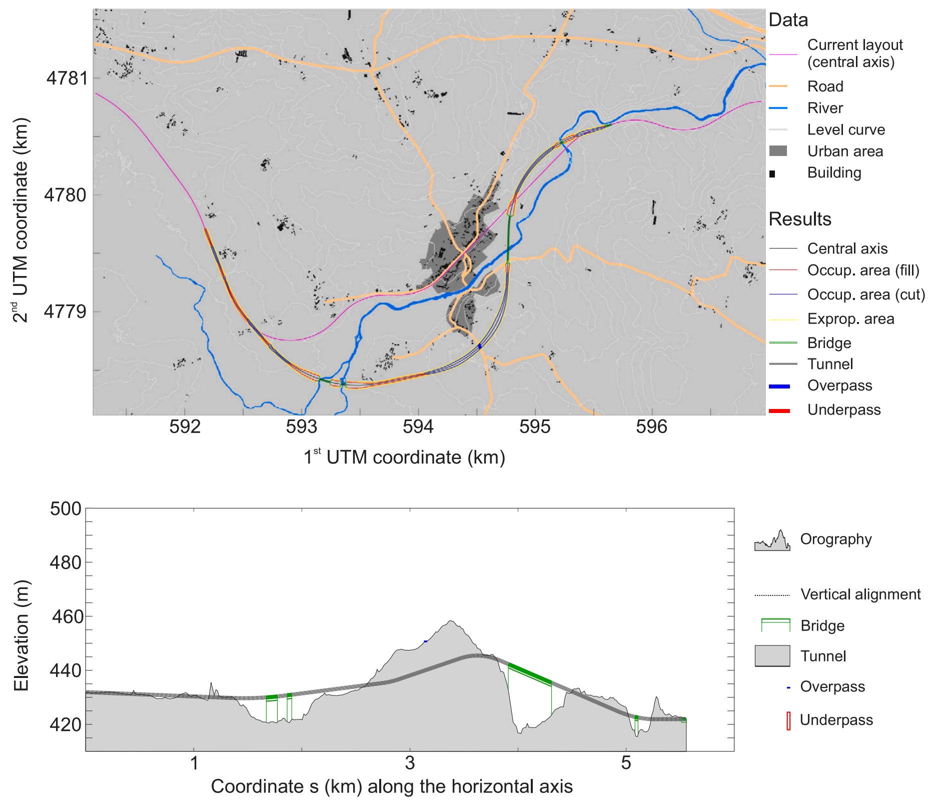

Fig. 6. Solution S2b: horizontal (up) and vertical (down) alignments.

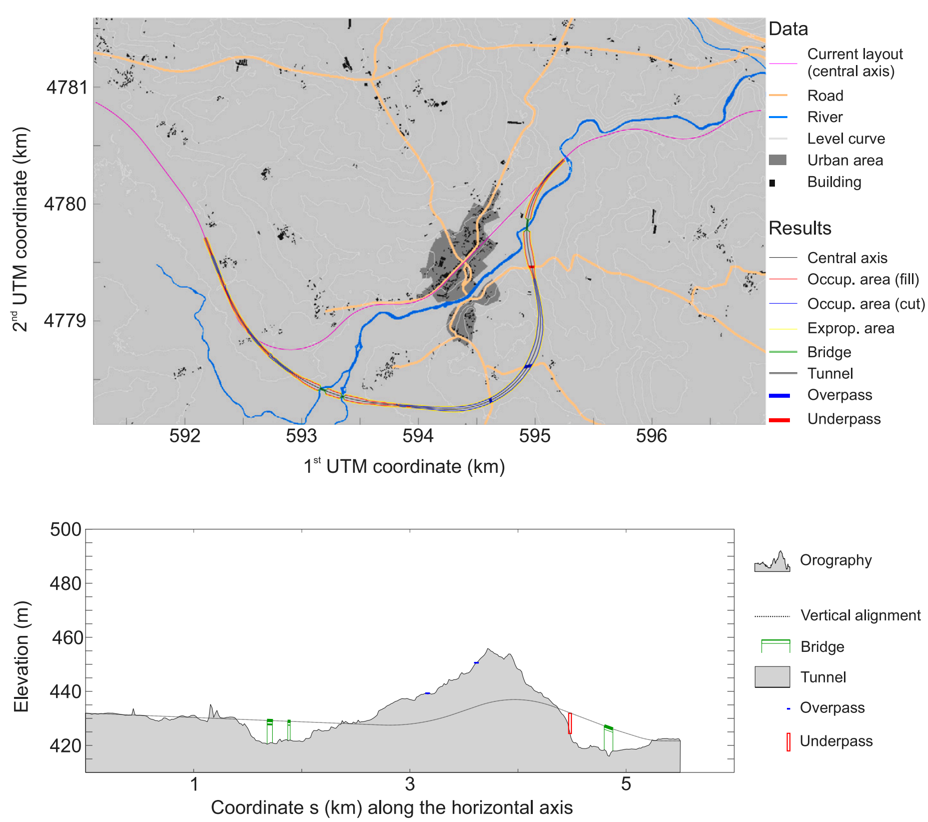

Fig. 7. Solution S3: horizontal (up) and vertical (down) alignments.

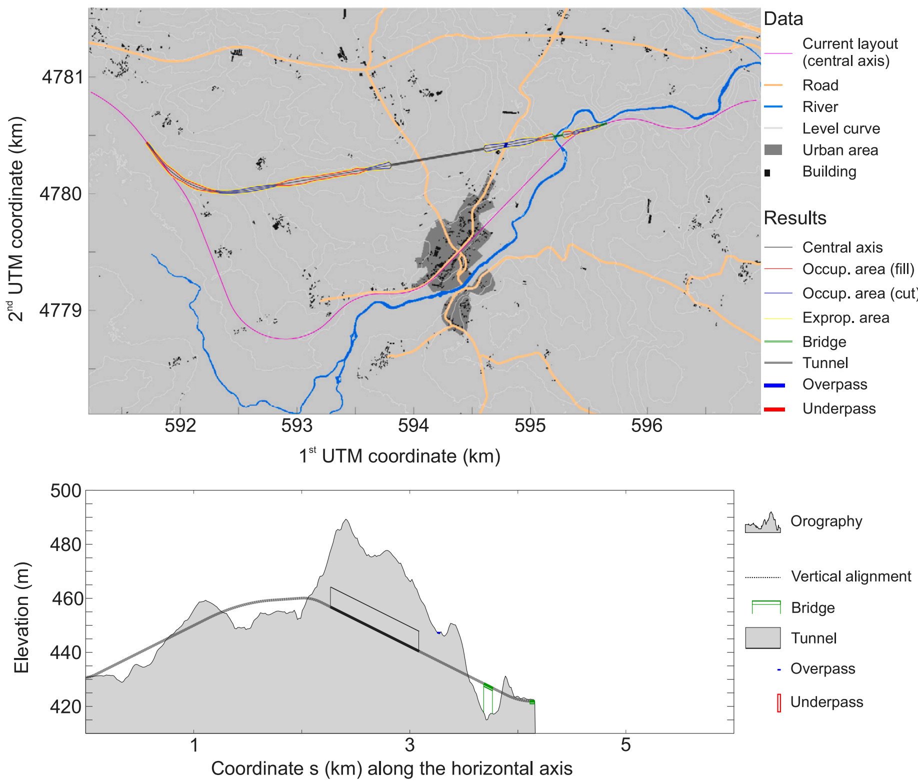

Fig. 8. Solution S4: horizontal (up) and vertical (down) alignments.

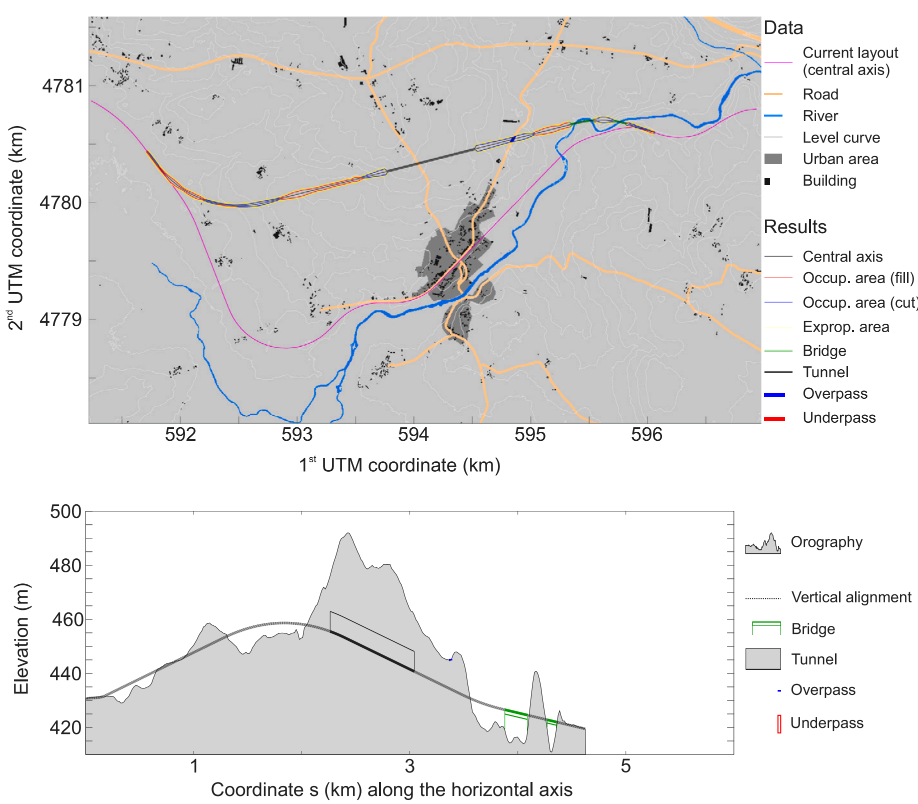

Fig. 9. Solution S5: horizontal (up) and vertical (down) alignments.

or very close to the maximum allowed slope (2.00% for S4 and 1.96% for S5) but the length under maximum slope is significantly worse in the case of S4 (1.707 km) in comparison to S5 with (1.017 km). As regards to the horizontal alignment both solutions have similar values of weighted average radius (861 and 858 m, respectively).

4. Conclusions

In this paper a mathematical model for designing railway bypasses, previously introduced in Vázquez-Méndez et al. (2021a), has been improved. These improvements have been tackled at four different levels:

1. Geometrical model and economic costs: we extend the method introduced by Kim et al. (2007) for tunnels and bridges to overpasses and underpasses, in such a way that the current model automatically identifies the need of major structures. The infrastructure costs considered in Vázquez-Méndez et al. (2021a) are modified to take into account their existence and, of course, their corresponding economic costs.
2. Mathematical formulation of the optimization problem: just as it is recommended in Sushma and Maji (2020), in this work we considered that the number of curves (N) and the number of slope changes (M) are design variables, and therefore the optimization problem is formulated in a more general framework, specifically as a *non standard* Mixed Integer Non Linear Problem (MINLP).
3. Numerical resolution: just as announced in the previous work, we propose a new numerical method for efficiently solving the formulated MINLP if the upper bounds of the integer variables are low. This method is capable of supplying some remarkable solutions (local minima from an economic point of view).
4. Practical application: The proposed method is not just used for obtaining only one solution for the railway bypass design problem, but rather it is used to provide a set of alignment alternatives (all of them sub-optimal in economic terms), which can be subsequently analyzed by the engineers to choose the most appropriate for each particular case, taking into account other functional, social or environmental aspects.

According to the results obtained in this work, it is worth highlighting the following aspects:

- The proposed method can be considered as a very interesting support tool for engineers in order to accomplish the complex and time-consuming task to generate a set of initial alternatives for the design of a railway bypass.
- A thorough analysis of the whole set of solutions carried out by the authors confirm that the method performs successfully according to the expected behavior. All of the solutions provided by the method avoid forbidden areas as well as buildings and cross both the river Parga and other infrastructures assuring a minimum difference in height. In addition, geometrical parameters of both horizontal and vertical alignments satisfy all the technical constraints considered. Finally, the model includes automatically the structures needed along the bypass layout in accordance with the established criteria for each of them.
- Some geometrical parameters of the solutions provided by the method are clearly influenced by the optimization objective (minimizing the economic cost). For example, it can be observed that solutions without or with short tunnels are preferred even though the resulting slopes of the vertical alignment are steeper. This effect could be modulated by considering functional aspects and environmental impacts. Completing the model by including these factors and others already studied in the scientific literature (operating speed trajectories, user costs to passengers, accident costs, noise, geological hazards...) is a pending task. This leads to formulate the problem in the framework of multi-objective optimization and it will be addressed in future works.

Data availability

Data will be made available on request.

Acknowledgments

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Appendix. Randomly generation of an admissible horizontal alignment (AHA) with two (or three) curves linking two given tangent segments

We consider two tangent segments, one starting at point $\mathbf{a} \in \mathbb{R}^2$ with direction (and sense) given by unit vector $\mathbf{u}_a \in \mathbb{R}^2$, and one ending at $\mathbf{b} \in \mathbb{R}^2$ with direction (and sense) given by $\mathbf{u}_b \in \mathbb{R}^2$. We denote by γ_a and γ_b the oriented angles from $\mathbf{b}-\mathbf{a}$ to $\mathbf{u}_a$ and from $\mathbf{b}-\mathbf{a}$ to $\mathbf{u}_b$, respectively (see Fig. 10).

Any horizontal alignment (HA) with symmetric curves joining $\mathbf{a}$ with $\mathbf{b}$ is given by the horizontal intersection points (HIP) $\mathbf{v}_i = (x_i, y_i) \in \mathbb{R}^2$, and the radii $R_i > 0$ and angles $\omega_i \geq 0$ of the circular curves (see Casal et al., 2017). Additionally, if the HA must link with both segments, then $\mathbf{v}_1$ and $\mathbf{v}_N$ must verify $\mathbf{v}_1 = \mathbf{a} + d_a\mathbf{u}_a$, $\mathbf{v}_N = \mathbf{b} - d_b\mathbf{u}_b$, for values $d_a, d_b > 0$. Consequently, an HA with two curves linking both segments (see Fig. 10) is univocally determined by the vector

$$\mathbf{x}^2 = (d_a, R_1, \omega_1, d_b, R_2, \omega_2) \in \mathbb{R}^6.$$

The problem is how to randomly generate this vector guaranteeing that the corresponding HA is admissible (it is an AHA), i.e., that the following constraints are satisfied (Vázquez-Méndez et al., 2021b):

$$R_j \geq R_{\min}, \quad j = 1, 2, \tag{15}$$

$$\frac{L^C_{\min}}{R_j} + \omega_j \leq \theta_j \leq \theta_{\max}, \quad j = 1, 2, \tag{16}$$

$$d_a \geq d_1(\theta_{\max}) + L^T_{\min}, \tag{17}$$

$$\|v_2 - v_1\| \geq d_1(\theta_{\max}) + L^T_{\min} + d_2(\theta_{\max}), \tag{18}$$

$$d_b \geq d_2(\theta_{\max}) + L^T_{\min}, \tag{19}$$

where $R_{\min} > 0$ is the minimum allowable radius of the layout design, θ_j is the deflection angle at the jth curve (see Fig. 10), $\theta_{\max} \in (0, \pi)$ is the maximum allowable deflection angle, $L^C_{\min} > 0$ and $L^T_{\min} > 0$ are, respectively, the minimum length of each clothoid arc and each tangent section, and $d(\theta)$ is the necessary distance to embed a curve of radius R and angle w between two main tangents which deflection angle is θ ($d_j(\theta)$ corresponds with radius R_j and angle w_j).

Below we establish sufficient conditions to guarantee that there exists an AHA with $N = 2$ curves and detail how to generate it in a random way. First, for a given $R > 0$ and $\omega \geq 0$, we denote

$$\theta_{\min} = \frac{L^C_{\min}}{R} + \omega \tag{20}$$

and highlight that $d(\theta)$ is an increasing function, and the minimum length to guarantee that the corresponding curve can be embed between two main tangents is $d(\theta_{\max})$. Then, we define

$$d_{\min} = d(\theta_{\max}) + L^T_{\min}, \tag{21}$$

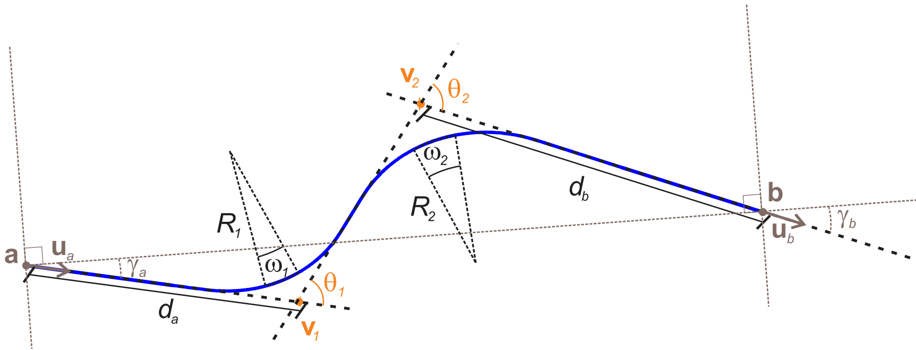

Fig. 10. Example of an HA (in blue) with two curves linking two given tangent segments: decision variables (black), given data (gray) and important elements determined from the decision variables (orange).

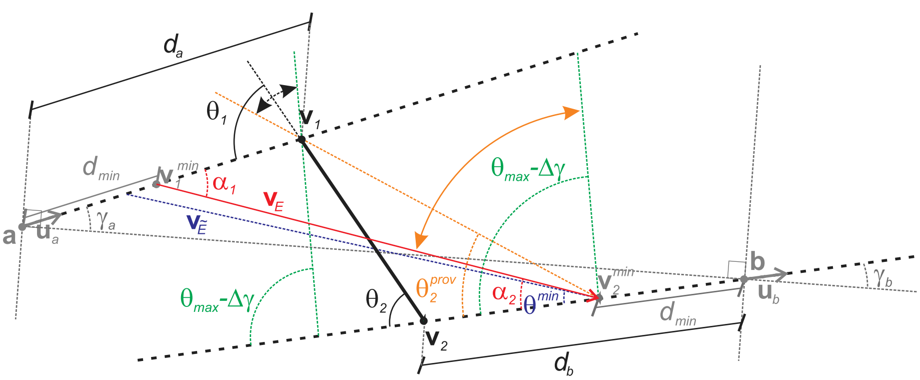

Fig. 11. Diagram of the generation of an AHA in Case 1, with $\alpha_1 > \alpha_2 > 0$: main elements and notation.

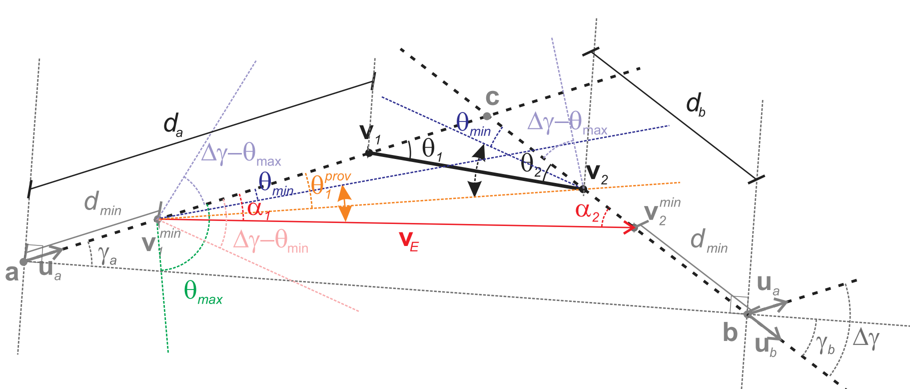

Fig. 12. Diagram of the generation of an AHA in Case 2 ($\alpha_1\alpha_2 < 0$): main elements and notation.

$$L_{\min} = 2d(\theta_{\max}) + L^T_{\min}, \tag{22}$$

consider $\mathbf{v}_1^{\min} = \mathbf{a} + d_{\min}\mathbf{u}_a$, $\mathbf{v}_2^{\min} = \mathbf{b} - d_{\min}\mathbf{u}_b$, $\mathbf{v}_E = \mathbf{v}_2^{\min} - \mathbf{v}_1^{\min}$ and take α_1 and α_2 the oriented angles from $\mathbf{v}_E$ to $\mathbf{u}_a$ and from $\mathbf{v}_E$ to $\mathbf{u}_b$, respectively (see Fig. 11 or Fig. 12). We distinguish two cases:

- Case 1: $\alpha_1\alpha_2 \geq 0$, with $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$
  We assume without loss of generality that $|\alpha_1| \geq |\alpha_2|$ (if $|\alpha_1| < |\alpha_2|$ only subscripts must be exchanged). In this case (see Fig. 11), we must assume the following hypotheses:

Hypothesis 1. Deflection angles large enough must be allowed, specifically,

$$\theta_{\max} \geq \max\{\theta_{\min}, |\alpha_2|\} + \Delta\gamma, \tag{23}$$

where $\Delta\gamma = |\gamma_b - \gamma_a|$.

Hypothesis 2. Terminals $\mathbf{a}$ and $\mathbf{b}$ must be far enough apart, specifically,

$$\min\{\|\mathbf{v}_E\|, \|\mathbf{v}_{\tilde{E}}\|\} \geq L_{\min}, \tag{24}$$

where vector $\mathbf{v}_{\tilde{E}}$ is the vector defined from $\mathbf{v}_2^{\min}$ and $\theta_{\min}$ as is shown in Fig. 11.

Under these hypotheses we can prove the following result. Its constructive proof leads to Algorithm 2, where the method for the random generation of an AHA in this case is detailed.

Theorem 1. *Let be $R \geq R_{\min}$ and $\omega \geq 0$ such that Hypotheses 1 and 2 are verified. There exits θ_2^{prov} and θ_1 such that, taking*

$$d_a = d_{\min} + \frac{\sin\left(\theta_2^{\text{prov}} - |\alpha_2|\right)}{\sin\left(\theta_2^{\text{prov}} + \Delta\gamma\right)} \|\mathbf{v}_E\|,$$

$$\mathbf{v}_1 = \mathbf{a} + d_a \mathbf{u}_a,$$

$$d_b = d_{\min} + \frac{\sin\left(\theta_1 - (\theta_2^{\text{prov}} + \Delta\gamma)\right)}{\sin\left(\theta_1 - \Delta\gamma\right)} \left\|\mathbf{v}_1 - \mathbf{v}_2^{\min}\right\|,$$

the HA given by the vector $\mathbf{x}^2 = (d_a, R, \omega, d_b, R, \omega)$ is an AHA.

Proof. Taking $\theta_{\min}^{\text{aux}} = \max\left\{\theta_{\min}, |\alpha_2|\right\}$, Hypothesis 1 guarantees that the set $[\theta_{\min}^{\text{aux}}, \theta_{\max} - \Delta\gamma]$ is not empty. Taking $d_b = d_{\min}$ (that is, $\mathbf{v}_2 = \mathbf{v}_2^{\min}$), each angle $\theta_2^{\text{prov}} \in [\theta_{\min}^{\text{aux}}, \theta_{\max} - \Delta\gamma]$ defines (see Fig. 11) a vertex $\mathbf{v}_1$ corresponding with a value $d_a \geq d_{\min}$, and a deflection angle $\theta_1 = \theta_2^{\text{prov}} + \Delta\gamma$. Consequently, the corresponding HA verifies (15), (16), (17) and (19). Additionally, Hypothesis 2 guarantees that (18) is also verified if $\theta_2^{\text{prov}} = \theta_{\min}^{\text{aux}}$, and the result is already proven. In fact, if

$$\sin(|\alpha_1|) \|\mathbf{v}_E\| \geq L_{\min}, \tag{25}$$

(18) is satisfied for any $\theta_2^{\text{prov}} \in [\theta_{\min}^{\text{aux}}, \theta_{\max} - \Delta\gamma]$, while if the inequality (25) is not verified, constraint (18) is only satisfied if $\theta_2^{\text{prov}} \in [\theta_{\min}^{\text{aux}}, \theta_{\max}^{\text{aux}} - \Delta\gamma]$, where

$$\theta_{\max}^{\text{aux}} = \min\left\{\arcsin\left(\frac{\sin(|\alpha_1|)\|\mathbf{v}_E\|}{L_{\min}}\right), \theta_{\max}\right\}.$$

Additionally, from each θ_2^{prov} defining the previous AHA, we can achieve more AHAs, by modifying (increasing) the value of d_b. Effectively, any angle $\theta_1 \in \left[\theta_2^{\text{prov}} + \Delta\gamma, \theta_{\max}\right]$ defines (see Fig. 11) a vertex $\mathbf{v}_2$ corresponding with a value $d_b \geq d_{\min}$, and a deflection angle $\theta_2 = \theta_1 - \Delta\gamma$, in such a way that constraints (15), (16), (17) and (19) are satisfied. Constraint (18) is clearly verified if $\theta_1 = \theta_2^{\text{prov}} + \Delta\gamma$ (that is, $\theta_2 = \theta_2^{\text{prov}}$) and also for any $\theta_1 \in \left[\theta_2^{\text{prov}} + \Delta\gamma, \theta_{\max}\right]$ if

$$\sin(\theta_2^{\text{prov}}) \left\|\mathbf{v}_1 - \mathbf{v}_2^{\min}\right\| \geq L_{\min}.$$

If this inequality is not verified, constraint (18) is still satisfied for any $\theta_1 \in \left[\theta_2^{\text{prov}} + \Delta\gamma, \theta_{\max}^{\text{aux}}\right]$, with

$$\theta_{\max}^{\text{aux}} = \min\left\{\arcsin\left(\frac{\sin(\theta_2^{\text{prov}})\|\mathbf{v}_1 - \mathbf{v}_2^{\min}|}{L_{\min}}\right) + \Delta\gamma, \theta_{\max}\right\}. \quad \square$$

Remark 2. Following Vázquez-Méndez et al. (2021b), the choice taking place at step 1 of Algorithm 2 can be random ($R \in [R^{min}, R^{max}]$, $\omega \in [0, \omega^{max}]$, for R^{max} and ω^{max} given values), but can also be supervised to obtain values of $\theta_{\min}$ and $L_{\min}$ helping to verify Hypotheses 1 and 2.

Algorithm 2 Random generation of an AHA joining two given tangent segments. Case 1: $\alpha_1\alpha_2 \geq 0$, with $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ and $|\alpha_1| \geq |\alpha_2|$.

Input: Data of tangent sections to be linked (terminals $\mathbf{a}$, $\mathbf{b}$ and directions $\mathbf{u}_a$, $\mathbf{u}_b$) and constraints ($R_{\min}$, $L_{\min}^C$, $L_{\min}^T$ and $\theta_{\max}$).
Output: Vector $\mathbf{x}^2 = (d_a, R, \omega, d_b, R, \omega)$ determining the AHA.

- Choose $R \geq R_{\min}$, $\omega \geq 0$ such that (23) and (24) are verified
- Compute $\theta_{\min}$, $d_{\min}$ and $L_{\min}$ given by (20)–(22)
- Compute $\mathbf{v}_1^{\min}$, $\mathbf{v}_2^{\min}$, $\mathbf{v}_E$, α_1, α_2 and $\Delta\gamma$
- Compute $\theta_{\min}^{\text{aux}} = \max\left\{\theta_{\min}, |\alpha_2|\right\}$, $\theta_{\max}^{\text{aux}} = \theta_{\max}$

if $\sin(|\alpha_1|)\|\mathbf{v}_E\| < L_{\min}$ **then**

- Compute $\theta_{\max}^{\text{aux}} = \min\left\{\arcsin\left(\frac{\sin(|\alpha_1|)\|\mathbf{v}_E\|}{L_{\min}}\right), \theta_{\max}\right\}$

end if

- Randomly choose $\theta_2^{\text{prov}} \in \left[\theta_{\min}^{\text{aux}}, \theta_{\max}^{\text{aux}} - \Delta\gamma\right]$
- Compute $d_a = d_{\min} + \frac{\sin\left(\theta_2^{\text{prov}} - |\alpha_2|\right)}{\sin\left(\theta_2^{\text{prov}} + \Delta\gamma\right)} \|\mathbf{v}_E\|$
- Compute $\mathbf{v}_1 = \mathbf{a} + d_a \mathbf{u}_a$
- Compute $\theta_{\max}^{\text{aux}} = \theta_{\max}$

if $\sin(\theta_2^{\text{prov}}) \left\|\mathbf{v}_1 - \mathbf{v}_2^{\min}\right\| < L_{\min}$ **then**

- Compute $\theta_{\max}^{\text{aux}} = \min\left\{\arcsin\left(\frac{\sin(\theta_2^{\text{prov}})\left\|\mathbf{v}_1 - \mathbf{v}_2^{\min}\right\|}{L_{\min}}\right) + \Delta\gamma, \theta_{\max}\right\}$

end if

- Randomly choose $\theta_1 \in \left[\theta_2^{\text{prov}} + \Delta\gamma, \theta_{\max}^{\text{aux}}\right]$
- Compute $d_b = d_{\min} + \frac{\sin\left(\theta_1 - (\theta_2^{\text{prov}} + \Delta\gamma)\right)}{\sin(\theta_1 - \Delta\gamma)} \left\|\mathbf{v}_1 - \mathbf{v}_2^{\min}\right\|$

- Case 2: $\alpha_1\alpha_2 < 0$
  Following the same method as in Case 1, under the new Hypotheses 3 and 4 (see below), the existence of an AHA with two curves linking the two given tangent sections can still be guaranteed. In this case, using Fig. 12 and proceeding as in the proof of Theorem 1, the method for random generation of AHAs detailed in Algorithm 3 is obtained.

  Hypothesis 3. Initial angles α_1 and α_2 must verify the established bounds for deflection angles, that is,

  $$\theta_{\min} \leq |\alpha_1|, |\alpha_2| \leq \theta_{\max}.$$

  Remark 3. In this case ($\alpha_1\alpha_2 < 0$), $\Delta\gamma = |\gamma_1| + |\gamma_2| = |\alpha_1| + |\alpha_2|$. Then, Hypothesis 3 is equivalent to require the following inequalities

  $$\theta_{\max} \geq \frac{\Delta\gamma}{2}, \tag{26}$$

  $$\theta_{\max} \geq \Delta\gamma - |\alpha_1| \tag{27}$$

  $$|\alpha_1| \geq \theta_{\min} \tag{28}$$

  $$\Delta\gamma \geq 2\theta_{\min} \tag{29}$$

  Hypothesis 4. Terminals $\mathbf{a}$ and $\mathbf{b}$ must be far enough apart. Specifically, in this case it is only necessary to verify that

  $$\|\mathbf{v}_E\| \geq L_{\min}. \tag{30}$$

Finally, if we are not in any of the two previous cases (if $\alpha_1 = \alpha_2 = 0$), or if Hypothesis 1 (Case 1) or Hypothesis 3 (Case 2) is not verified, but the terminals $\mathbf{a}$ and $\mathbf{b}$ are far enough apart, we cannot guarantee the existence of an AHA with two curves, but one with three curves can be obtained in almost all situations. It can be randomly computed

Algorithm 3 Random generation of an AHA joining two given tangent segments. Case 2: $\alpha_1\alpha_2 < 0$, with $|\alpha_1| \geq |\alpha_2|$.

Input: Data of tangent sections to be linked (terminals **a**, **b** and directions $\mathbf{u}_a$, $\mathbf{u}_b$) and constraints ($R_{\min}$, $L^C_{\min}$, $L^T_{\min}$ and $\theta_{\max}$).
Output: Vector $\mathbf{x}^2 = (d_a, R, \omega, d_b, R, \omega)$ determining the AHA.

- Choose $R \geq R_{\min}$, $\omega \geq 0$ such that (27)–(30) are verified
- Compute $\theta_{\min}$, $d_{\min}$ and $L_{\min}$ given by (20)–(22)
- Compute $\mathbf{v}_1^{\min}$, $\mathbf{v}_2^{\min}$, $\mathbf{v}_E$, α_1, α_2, and $\Delta\gamma$
- Compute $\theta^{\text{aux}}_{\min} = \max\left\{\theta_{\min}, \Delta\gamma - \theta_{\max}\right\}$
- Compute $\theta^{\text{aux}}_{\max} = \min\left\{\theta_{\max}, |\alpha_1|, \Delta\gamma - \theta_{\min}\right\}$

if $\sin(|\alpha_2|)\,\|\mathbf{v}_E\| < L_{\min}$ **then**

- Compute $\theta^{\text{aux}}_{\min} = \max\left\{\Delta\gamma - \arcsin\left(\frac{\sin(|\alpha_2|)\|\mathbf{v}_E\|}{L_{\min}}\right), \theta^{\text{aux}}_{\min}\right\}$

end if

- Randomly choose $\theta_1^{\text{prov}} \in \left[\theta^{\text{aux}}_{\min}, \theta^{\text{aux}}_{\max}\right]$
- Compute $d_b = d_{\min} + \frac{\sin\left(|\alpha_1| - \theta_1^{\text{prov}}\right)}{\sin\left(\Delta\gamma - \theta_1^{\text{prov}}\right)}\|\mathbf{v}_E\|$
- Compute $\mathbf{v}_2 = \mathbf{b} - d_b\mathbf{u}_b$

if $\sin(\theta_1^{\text{prov}})\left\|\mathbf{v}_2 - \mathbf{v}_1^{\min}\right\| < L_{\min}$ **then**

- Compute $\theta^{\text{aux}}_{\min} = \max\left\{\Delta\gamma - \arcsin\left(\frac{\sin(\theta_1^{\text{prov}})\left\|\mathbf{v}_2 - \mathbf{v}_1^{\min}\right\|}{L_{\min}}\right), \theta^{\text{aux}}_{\min}\right\}$

end if

- Randomly choose $\theta_2 \in \left[\theta^{\min}_{\text{aux}}, \Delta\gamma - \theta_1^{\text{prov}}\right]$
- Compute $d_a = d_{\min} + \frac{\sin\left(\Delta\gamma - \theta_1^{\text{prov}} - \theta_2\right)}{\sin(\Delta\gamma - \theta_2)}\left\|\mathbf{v}_2 - \mathbf{v}_1^{\min}\right\|$

by choosing $R \geq R_{\min}$, $w \geq 0$, considering $d_a = d_b = d_{\min}$ ($\mathbf{v}_1 = \mathbf{v}_1^{\min}$, $\mathbf{v}_3 = \mathbf{v}_2^{\min}$), and generating the vertex $\mathbf{v}_2$ (the intermediate curve) with a slight modification of the method proposed in Vázquez-Méndez et al. (2021b) to include a new curve between $\mathbf{v}_1$ and $\mathbf{v}_3$ (the modification is only needed because the initial HA with 2 curves does not verify the constraint (16)).

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