Homoclinic Solutions for Fractional Hamiltonian Systems via Variational Method

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Abstract. We study the multiplicity of weak nonzero solutions for fractional Hamiltonian systems of the form

$$ {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) = a(t)\nabla V(t, u(t)), \quad t \in \mathbb{R}, $$

where $\alpha \in (1/2, 1]$, ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and the right Liouville-Weyl fractional derivatives of order α on real line $\mathbb{R}$, $L(t)$ is a positive defined symmetric $n \times n$ matrix and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies specific growth conditions. A result is proved using variational method and the generalized Clark's theorem. Some recent results are extended and improved.

INTRODUCTION

In this paper we are concerned with the existence of infinitely many homoclinic solutions for the fractional Hamiltonian systems

$$ \begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) = a(t)\nabla V(t, u(t)), & t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R}, \mathbb{R}^n), \end{cases} \tag{1} $$

where $\alpha \in (1/2, 1]$, ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and the right Liouville-Weyl fractional derivatives of order α on whole real line $\mathbb{R}$ (see [7], [11], [14]) and $H^\alpha(\mathbb{R}, \mathbb{R}^n)$ is the fractional Sobolev space, defined in Section 2 (see [11], [16]). Next $L(t)$ is a positive defined symmetric $n \times n$ matrix, $n \in \mathbb{N}$, $u(t) \in \mathbb{R}^n$ is a vector valued function, $a : \mathbb{R} \to \mathbb{R}^+$ is a given continuous function, $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies specific growth conditions and $\nabla V(t, u)$ is the gradient of $V(t, u)$ with respect to u. An existence result is proved for the system (1) using variational method and the generalized Clark's theorem due to Liu and Wang [6]. It extends the result of Zhou and Zhang [16, Theorem 1.3], where an other critical point theorem is applied (see [13, Theorem 3.1] and [16, Lemma 2.5]).

Boundary value problems (BVP) for fractional differential equations appear in mathematical models in natural and applied sciences. Recently, variational method and critical point theory are applied to solvability of BVP for fractional differential equations starting from pioneering work of Jiao and Zhou [5]. Several authors showed that Problem (1) has at least one nontrivial solution under different growth assumptions on the functions L, V and a. We refer the reader to the papers [2], [4] [9], [10], [11],[15], [16] and references therein.

We suppose the following conditions on the functions L, V and a. Let $(\cdot.\cdot)$ and $|\cdot|_n$ are the product and norm in $\mathbb{R}^n$.

(L) $L \in C(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$ is a positive defined symmetric matrix and there exists a function $l(t) \in C(\mathbb{R}, (0, +\infty))$, a constant $m > 0$ such that $l(t) \geq m$, $l(t) \to +\infty$ as $|t| \to +\infty$ and $(L(t)x, x) \geq l(t)|x|^2_n$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

(A) $a \in C(\mathbb{R}, \mathbb{R}^+)$ and $a(t) \to 0$ as $|t| \to +\infty$.

(V_1) $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and there are constants q and σ such that $1 < q < 2$ and

$$(\nabla V(t,u), u) \leq qV(t,u), \ \forall u \in \mathbb{R}^n, u \neq 0.$$

(V_2) $\nabla V(t,u) = o(|u|_n)$ as $|u|_n \to 0$

(V_3) $\nabla V(t,u) \leq b_2(t)|u|_n^{q-1}$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$, where $1 < q < 2$, $b_2 : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $b_2 \in L^r(\mathbb{R})$, where $r = \frac{2}{2-q}$. There is an open set $J \subset \mathbb{R}$ and a positive constant b_1 such that $V(t,u) \geq b_1|u|_n^q$ for all $(t,u) \in J \times \mathbb{R}^n$.

(V_4) $V(t,-u) = V(t,u)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$.

Problem (1) has a variational structure and its solutions can be found as critical points of the functional $I : X^\alpha \to \mathbb{R}$

$$I(u) = \int_{\mathbb{R}} \frac{1}{2}(|{}_{-\infty}D_t^\alpha u(t)u(t)|^2 + (L(t)u(t), u(t))) - a(t)V(t, u(t))dt,$$

over an appropriate fractional Sobolev space $X^\alpha \subset H^\alpha(\mathbb{R}, \mathbb{R}^n)$ defined in Section 2. The functional $I \in C^1(X^\alpha, \mathbb{R})$ and

$$< I'(u), v >= \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t)u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), u(t)) - a(t)(\nabla V(t, u(t)), v(t))dt,$$

where $u, v \in X^\alpha$ and $< \cdot, \cdot >$ denotes the duality brackets between the spaces $X^{\alpha*}$ and X^α. The critical points of I, i.e. the points u for which $< I'(u), v >= 0$ for all $v \in X^\alpha$ are the weak solutions of Problem (1). Our main result is as follows:

Theorem 1. *Let the functions a, L and V satisfy assumptions $(A), (L)$, $(V_1) - (V_3)$. Then Problem* (1) *has at least one nontrivial weak solution. If in addition the assumption (V_4) holds, then Problem* (1) *has infinitely many pairs of weak solutions $(u_m, -u_m)$ such that $u_m \neq 0$ and* $\lim_{m\to\infty} \|u_m\|_{C(\mathbb{R},\mathbb{R}^n)} = 0$.

The paper is organized as follows. In Section 2 we give some notes on above mentioned fractional Sobolev spaces, imbedding lemmata, critical point theorems and variational formulation of Problem. In Section 3 we prove Theorem 1 and give an example.

PRELIMINARIES

The left and right Liouville-Weyl fractional integrals of order α, $0 < \alpha < 1$ on $\mathbb{R}$ are defined respectively by

$$\begin{aligned} {}_{-\infty}I_x^\alpha u(x) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} u(\xi)d\xi, \quad x \in \mathbb{R}, \\ {}_xI_{+\infty}^\alpha u(x) &= \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} (\xi - x)^{\alpha-1} u(\xi)d\xi, \quad x \in \mathbb{R}. \end{aligned}$$

The left and right Liouville-Weyl fractional derivatives of order α, $0 < \alpha < 1$ on $\mathbb{R}$ are defined by

$$\begin{aligned} {}_{-\infty}D_x^\alpha u(x) &= \frac{d}{dx} {}_{-\infty}I_x^{1-\alpha} u(x), \quad x \in \mathbb{R}, \\ {}_xD_{+\infty}^\alpha u(x) &= -\frac{d}{dx} {}_xI_{+\infty}^{1-\alpha} u(x), \quad x \in \mathbb{R}. \end{aligned}$$

We introduce the fractional derivative spaces $I^{\alpha}_{-\infty}(\mathbb{R},\mathbb{R}^n)$ and $I^{\alpha}_{+\infty}(\mathbb{R},\mathbb{R}^n)$ as a completion of $C_0^{\infty}(\mathbb{R},\mathbb{R}^n)$ respectively with the norms

$$\|u\|_{I^{\alpha}_{-\infty}} = (\|u\|^2_{L^2} + |u|^2_{I^{\alpha}_{-\infty}})^{1/2}, \ \ \|u\|_{I^{\alpha}_{+\infty}} = (\|u\|^2_{L^2} + |u|^2_{I^{\alpha}_{+\infty}})^{1/2},$$

where

$$|u|_{I^{\alpha}_{-\infty}} = \|_{-\infty}D^{\alpha}_x u\|_{L^2}, \quad |u|_{I^{\alpha}_{+\infty}} = \|_x D^{\alpha}_{+\infty} u\|_{L^2}.$$

Next, the fractional Sobolev space $H^{\alpha}(\mathbb{R},\mathbb{R}^n), 0 < \alpha < 1$ in terms of Fourier transform is defined as a completion of $C_0^{\infty}(\mathbb{R},\mathbb{R}^n)$ with the norm

$$\|u\|_{\alpha} = (\|u\|^2_{L^2} + |u|^2_{\alpha})^{1/2}, \ \ |u|^2_{\alpha} = \||\xi|^{\alpha}\mathcal{F}(u)(\xi)\|_{L^2}$$

where $\mathcal{F}(u)(\xi) = \int\limits_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx$ is the Fourier transform of $u(x)$. Note that $H^{\alpha}(\mathbb{R},\mathbb{R}^n)$, $I^{\alpha}_{-\infty}(\mathbb{R},\mathbb{R}^n)$ and $I^{\alpha}_{+\infty}(\mathbb{R},\mathbb{R}^n)$ have equivalent seminorms and norms (see [4]).

Let $C(\mathbb{R},\mathbb{R}^n)$ denote the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. It is proved in [11, Theorem 2.1],

Lemma 1. *If $1/2 < \alpha < 1$, then $H^{\alpha}(\mathbb{R},\mathbb{R}^n) \subset C(\mathbb{R},\mathbb{R}^n)$ and there exists a constant $C_{\alpha} > 0$ such that*

$$\|u\|_{\infty} := \sup_{t\in\mathbb{R}} |u(t)| \leq C_{\alpha}\|u\|_{\alpha}.$$

Note that if $u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n)$ then $u \in L^p(\mathbb{R},\mathbb{R}^n)$, $p \geq 2$.

We say that a solution $u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n)$ of (1) is homoclinic to 0, if $u(t) \to 0$ as $|t| \to \infty$.

In order to apply the variational metod to solvability of (1) we introduce another fractional Sobolev space $X_{\alpha} \subset H^{\alpha}(\mathbb{R},\mathbb{R}^n)$ as follows

$$X_{\alpha} := \left\{ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n) : \int_{\mathbb{R}} (|_{-\infty}D^{\alpha}_t u(t)|^2 + (L(t)u(t),u(t)))dt < \infty \right\}$$

which is a Hilbert space with scalar product and norm

$$\begin{aligned} (u,v)_{X^{\alpha}} \ &: = \int_{\mathbb{R}} ((_{-\infty}D^{\alpha}_t u(t),_{-\infty} D^{\alpha}_t v(t)) + (L(t)u(t),v(t)))dt, \\ \|u\|_{X^{\alpha}} \ &: = (u,u)^{1/2}_{X^{\alpha}}. \end{aligned}$$

An important continuous embedding lemma is proved in [11, Lemma 2.3].

Lemma 2. *Suppose that L satisfies condition (L). Then X_{α} is continuously embedded in $H^{\alpha}(\mathbb{R},\mathbb{R}^n)$ and there is a constant K_{α} such that $\|u\|_{\alpha} \leq K_{\alpha}\|u\|_{X^{\alpha}}$.*

We present a minimization theorem which will be used in the proof of the main result (see [8],[13, Theorem 2.7]). Recall that the functional J satisfies (PS)-condition if any sequence $u_n \subset E$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ contains a strongly convergent subsequence.

Theorem 2. *(Minimization theorem) Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfying (PS) condition. If J is bounded below, then $c = \inf_E I$ is a critical value of J.*

Rabinowitz [13, p.53] gave a formulation of Clark's theorem as follows

Theorem 3. *(Clark's theorem) Let E be a Banach space, $J \in C^1(E,\mathbb{R})$ with J is even, bounded below and satisfying (PS). Suppose $J(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to $\mathbb{S}^{j-1}$ by an odd map, and $\sup_K J < 0$. Then J possesses at least j distinct pairs of critical points.*

It was recently extended by Z. Liu and Z. Wang [6] as follows:

Theorem 4. *(Generalized Clark's theorem, [6])) Let E be a Banach spa ce, $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the (PS) condition, it is even, bounded from below and $J(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k−dimensional subspace E^k of E and $\rho_k > 0$ such that $\sup_{E^k \cap S_{\rho_k}} J < 0$, where $S_\rho = \{u \in E,\ \|u\|_E = \rho\}$, then at least one of the following conclusions holds:*

1. *There exists a sequence of critical points $\{u_k\}$ satisfying $J(u_k) < 0$ for all k and $\lim_{k\to\infty} \|u_k\|_E = 0$.*
2. *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_E = \alpha$ and $J(u) = 0$.*

Realize that Theorem 4 implies the existence of many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$ such that $J(u_m) \leq 0$, $\lim_{m\to+\infty} J(u_m) = 0$ and $\lim_{m\to+\infty} \|u_m\|_E = 0$.

PROOF OF THE MAIN RESULT

Let $L^p_a(\mathbb{R}, \mathbb{R}^n)$, $p \geq 1$ denote the weighted space of measurable functions $u : \mathbb{R} \to \mathbb{R}^n$ with norm $\|u\|_{p,a} := \left(\int_{\mathbb{R}} a(t)|u(t)|^p_n dt\right)^{1/p}$. Next result is proved in [16, Lemma 3.9]

Lemma 3. *Under assumptions (L) and (A) , the embedding $X_\alpha \subset L^2_a(\mathbb{R}, \mathbb{R}^n)$ is continuous and compact.*

Lemma 4. *Assume that assumptions $(L), (A)$ and (V_2) hold. If $\{u_m\}$ is a sequence in X_α such that $u_m \rightharpoonup u$ weakly in X_α as $m \to \infty$, there exists a subsequence, still denoted by $\{u_m\}$ such that $\nabla V(t, u_m(t)) \to \nabla V(t, u(t))$ in $L^2_a(\mathbb{R}, \mathbb{R}^n)$.*

Proof of Lemma 4 By Banach-Steinhauss theorem there exists a constant $M > 0$ such that

$$\|u_m\|_{X_\alpha} \leq M,\ \|u\|_{X_\alpha} \leq M$$

By (V_2) and Hölder inequality we have for $r = 2/(2-q)$

$$\begin{aligned} I_m &= \int_{\mathbb{R}} a(t)|\nabla V(t, u_m(t)) - \nabla V(t, u(t))|^2_n dt \\ &\leq 2M_a \int_{\mathbb{R}} b_2^2(t)|(|u_m(t)|^{2(q-1)}_n + |u_m(t)|^{2(q-1)}_n)dt \\ &\leq 2M_a\|b_2\|_{L^r(\mathbb{R})}(\|u_m\|^{2(q-1)}_{L^2(\mathbb{R},\mathbb{R}^n)} + \|u\|^{2(q-1)}_{L^2(\mathbb{R},\mathbb{R}^n)}) \\ &\leq 4M_a\|b_2\|_{L^r(\mathbb{R})}M^{2(q-1)}. \end{aligned}$$

By Lemma 3.1 a there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$ such that $u_m \to u$ strongly in $L^2_a(\mathbb{R}, \mathbb{R}^n)$. Then, the assertion follows by the condition (A) and Lebesque's dominated convergence theorem (see [1, Theorem 4.2 and Theorem 4.9]). Since $\{u_m\}$ is bounded in X_α, by (V_2) and Hölder inequality there is a constant $K > 0$ such that

$$\int_{\mathbb{R}} |\nabla V(t, u_m(t))|^2_n dt \leq K, \int_{\mathbb{R}} |\nabla V(t, u(t))|^2_n dt \leq K.$$

Let $\varepsilon > 0$ and choose $R > 0$ by assumption (A) such that $0 < a(t) \leq \frac{\varepsilon}{4K}$ for $|t| \geq R$. Then

$$\int_{|t|\geq R} a(t)|\nabla V(t, u_m(t)) - \nabla V(t, u(t))|^2_n dt \leq \frac{\varepsilon}{2}. \tag{2}$$

Let $0 < a_0 < a(t) \leq M_a$ for $|t| \leq R$ according (A). By $u_m \to u$ strongly in $L^2_a(\mathbb{R}, \mathbb{R}^n)$ it follows that

$$\int_{|t|\leq R} a(t)|u_m(t) - u(t))|^2_n dt \geq a_0 \int_{|t|\leq R} |u_m(t) - u(t)|^2_n dt \to 0.$$

Then $u_m(t) - u(t) \to 0$, a.e. in $|t| \le R$ and by dominated convergence theorem

$$I_{m,R} = \int_{|t|\le R} a(t)|\nabla V(t, u_m(t)) - \nabla V(t, u(t))|_n^2 dt \to 0.$$

Let L is so big, such that $I_{m,R} < \frac{\varepsilon}{2}$ as $m > L$. Then, by (2), $I_m < \varepsilon$ as $m > L$ and the assertion follows. □

Next, we have

Lemma 5. *([16, Lemma 3.11]). Under assumptions* (L),(A), (V_1), (V_2) *and* (V_3) *the functional* $I \in C^1(X_\alpha, \mathbb{R})$ *and for all* $u, v \in X_\alpha$

$$< I'(u), v >= \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t)u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), u(t)) - a(t)(\nabla V(t, u(t)), v(t))dt.$$

Lemma 6. *Under assumptions* (L), (A), (V_1), (V_2) *and* (V_3) *the functional* $I \in C^1(X_\alpha, \mathbb{R})$ *satisfies* $(PS)-$*condition.*

Proof of Lemma 6 Let $\{u_m\}$ be a (PS) sequence for the functional I, i.e. $\{I(u_m)\}$ is bounded and $I'(u_m) \to 0$ in X_α. Then there exists a positive constant $C > 0$, such that

$$\|I(u_m)\|_{X_\alpha} \le C, \quad \|I'(u_m)\|_{X_\alpha^*} \le C.$$

By (V_1) we have

$$\begin{aligned} C + \frac{C}{q}\|u_m\|_{X_\alpha} &\ge \frac{1}{q} < I'(u_m), u_m > -I(u_m) \\ &= \left(\frac{1}{q} - \frac{1}{2}\right)\|u_m\|_{X_\alpha}^2 + \int_{\mathbb{R}} a(t)(V(t, u_m(t)) - \frac{1}{q}(\nabla V(t, u_m(t)), u_m(t))dt \\ &\ge \left(\frac{1}{q} - \frac{1}{2}\right)\|u_m\|_{X_\alpha}^2, \end{aligned}$$

which implies that $\{u_m\}$ is a bounded sequence in X_α. Then, we can assume up to a subsequence, still denoted by $\{u_m\}$, that $u_m \rightharpoonup u$ weakly in X_α and by Lemma 3, $u_m \to u$ in $L_a^2(\mathbb{R}, \mathbb{R}^n)$.Lemma 4 $\nabla V(t, u_m(t)) \to \nabla V(t, u(t))$ in $L_a^2(\mathbb{R}, \mathbb{R}^n)$ and then

$$\begin{aligned} \|u_m - u\|_{X_\alpha}^2 &\le < I'(u_m) - I'(u), u_m - u > \\ +C_a\|\nabla V(t, u_m(t)) - \nabla V(t, u(t))\|_{L_a^2(\mathbb{R},\mathbb{R}^n)}\|u_m - u\|_{X_\alpha} &\to 0, \end{aligned}$$

which implies that $u_m \to u$ in X_α. □

Proof of Theorem 1 We have from (V_2) and Hölder inequality for $r = 2/(2 - q)$

$$\begin{aligned} &\int_{\mathbb{R}} a(t)V(t, u(t))dt \le \frac{1}{q}\int_{\mathbb{R}} a(t)b_2(t)|u(t)|^q dt \\ \le\ & \frac{1}{q}(\int_{\mathbb{R}} a(t)b_2(t)^{\frac{2}{2-q}}dt)^{\frac{2-q}{2}}(\int_{\mathbb{R}} a(t)|u(t)|^2 dt)^{\frac{q}{2}} \\ \le\ & \frac{M_a^{\frac{2-q}{2}}}{q}\|b_2\|_{L^r(\mathbb{R})}\|u\|_{2,a}^q \le \frac{C_{2,a}^q M_a^{\frac{2-q}{2}}}{q}\|b_2\|_{L^r(\mathbb{R})}\|u\|_{X_\alpha}^q, \end{aligned}$$

where according to (A), $0 < a(t) \le M_a, \forall t \in \mathbb{R}$ and $C_{2,a}$ is the embedding constant by Lemma Lemma 3, $X_\alpha \subset L_a^2(\mathbb{R}, \mathbb{R}^n)$, $\|u\|_{2,a} \le C_{2,a}\|u\|_\alpha$.

Let $K_1 = \frac{C^q_{2,a} M_a^{\frac{2-q}{2}}}{q}$. Then we have

$$I(u) \geq \frac{1}{2}\|u\|^2_{X_\alpha} - K_1\|b_2\|_{L^r(\mathbb{R})}\|u\|^q_{X_\alpha}$$

which shows by $1 < q < 2$, that $I(u) \to \infty$ as $\|u\|_{X_\alpha} \to \infty$, i.e. I is a coercive functional bounded from below. The statement follows by Lemma 6 and Minimization theorem.It can be shown that by (V_2) that the minimizer u_0 of I is a nonzero weak solution. Let $v \in X_\alpha$ be such that $v(t) = 0$, $t \in \mathbb{R}\backslash J$ be a nonzero function and $\|v\|_\infty \leq 1$. Then for $\lambda > 0$ by (V_2)

$$\begin{aligned} I(\lambda v) &= \frac{\lambda^2}{2}\|v\|^2_{X_\alpha} - \int_{\mathbb{R}} a(t)V(t, \lambda v(t))dt \\ &= \frac{\lambda^2}{2}\|v\|^2_{X_\alpha} - \int_J a(t)V(t, \lambda v(t))dt \\ &\leq \frac{\lambda^2}{2}\|v\|^2_{X_\alpha} - b_1\lambda^q \int_J a(t)|v(x)|^q dx, \end{aligned}$$

which shows that for sufficiently small $\lambda, I(\lambda v) < 0$ because $1 < q < 2$. Then $I(u_0) = \min\{I(u) : u \in X_\alpha\} < 0$ and u_0 is a nonzero weak solution because $I(0) = 0$. Next, we show that under conditions $(V_1) - (V_4)$ and (A), the functional

I satisfies conditions of generalized Clark's theorem. Let $J = (a, b) \subset \mathbb{R}$ and for $k \in \mathbb{N}$ take k disjoint open intervals $J_j = (x_{j-1}, x_j)$, $j = 1, ..., k$ where $x_j = a + \frac{j}{k}(b - a)$, $j = 0, 1, ..., k$. We have $\cup\{J_j : j = 1, ..., k\} \subset J$. Next, we choose functions $v_j \in C^\infty_0(J_j)$ such that $\|v_j\|_\infty < \infty$ and $\|v_j\|_{X_\alpha} = 1$. Let $X_{k,\alpha}$ be the $k-$dimensional subspace of X_α

$$X_{k,\alpha} := span\{v_1, ..., v_k\}$$

and

$$S_n := \{u \in X_{k,\alpha} : \|u\|_{X_\alpha} = 1\}.$$

For $u = \sum_{j=1}^k \lambda_j v_j \in X_{k,\alpha}$ we have

$$\begin{aligned} \|u\|^2_{X_\alpha} &= \int_{\mathbb{R}} (|_{-\infty}D^\alpha_t u(t)|^2 + (L(t)u(t), u(t)))dt \\ &= \sum_{j=1}^k |\lambda_j|^2 \int_{J_k} (|_{-\infty}D^\alpha_t v_j(t)|^2 + (L(t)v_j(t), v_j(t)))dt \\ &= \sum_{j=1}^k |\lambda_j|^p \end{aligned}$$

and

$$\|u\|^q_{L^q_a} = \sum_{j=1}^k |\lambda_j|^q \int_{J_k} a(t)|v_j(x)|^q dx. \tag{3}$$

Since the norms $\|.\|_{X_\alpha}$ and $\|.\|_{L^q_a}$ are equivalent on the finite dimensional space $X_{k,\alpha}$ there are positive constants d_{1k} and d_{2k} such that for $u \in X_{k,\alpha}$:

$$d_{1n}\|u\|_{X_\alpha} \leq \|u\|_{L^q_a} \leq d_{2n}\|u\|_{X_\alpha}. \tag{4}$$

Then, for $u \in S_k$ by (3) and (4) we have for $\mu > 0$:

$$
\begin{aligned}
I(\mu u) &= \frac{\mu^2}{2}\|u\|^2_{X_\alpha} - \int_{\mathbb{R}} a(t)V(t,\mu u(t))dt \\
&= \frac{\mu^2}{2}\|u\|^2_{X_\alpha} - \sum_{j=1}^{k}\int_{J_k} a(t)V(t,\mu\lambda_j v_j(t))dt \\
&\le \frac{\mu^2}{2}\|u\|^2_{X_\alpha} - b_1\mu^q \sum_{j=1}^{k}|\lambda_j|^q \int_{J_k} a(t)|v_j(t)|^q dt \\
&= \frac{\mu^2}{2}\|u\|^2_{X_\alpha} - b_1\mu^q\|u\|^q_{L^q_a} \\
&\le \frac{\mu^2}{2}\|u\|^2_{X_\alpha} - b_1\mu^q d^q_{1n}\|u\|^q_{X_\alpha}.
\end{aligned}
$$

By the last inequality and $1 < q < 2$ it follows that $I(v) < 0$ for $v \in S^\mu_k := \{\mu u : u \in S_k\}$ with μ sufficiently small. Finally, the functional I satisfies all conditions of Theorem 4 and the Theorem 1 is proved. The assertion $\lim_{m\to\infty}\|u_m\|_{C(\mathbb{R},\mathbb{R}^n)} = 0$ follows from embedding Lemma 1.□

Example. Let $a(t) = a_1(t) := \frac{1}{1+|t|}, t \in \mathbb{R}$ and $q = \frac{3}{2}$. Then $r = \frac{2}{2-q} = 4$. Next choose

$$
b_2(t) = g(t) := \begin{cases} 2 - t^2, & |t| \le 1, \\ \frac{1}{t^2} & |t| \ge 1. \end{cases}
$$

We have $g \in C^1(\mathbb{R}), g(t) > 0, a_1(t) > 0, t \in \mathbb{R}$ and

$$
\int_{\mathbb{R}} |g(t)|^4 dt = \int_{|t|\le 1} (2 - t^2)^4 d + \int_{|t|\ge 1} \frac{1}{t^8} dt = 18.203 + \frac{2}{7}.
$$

Further, let $n \in \mathbb{N}$, $c_1, ..., c_n$ are positive constants and for $u = (u_1, ..., u_n) \in \mathbb{R}^n$, define the even function $V(t,u) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ as

$$
V(t,u) := \frac{2}{3}g(t)(c_1|u_1|^{3/2} + ... + c_n|u_n|^{3/2}).
$$

We have

$$
\nabla V(t,u) := g(t)(c_1|u_1|^{-1/2}u_1, ..., c_n|u_n|^{-1/2}u_n)
$$

and

$$
\begin{aligned}
(\nabla V(t,u), u) &= g(t)(c_1|u_1|^{3/2} + ... + c_n|u_n|^{3/2}), \\
(\nabla V(t,u), u) &= qV(t,u),
\end{aligned}
$$

so condition (V_1) is satisfied. Next let $C_1 = \min\{c_1, ..., c_n\} > 0$ and $C_2 = \max\{c_1, ..., c_n\} > 0$. Let $J = [-1, 1]$. By the equivalence of norms in $|u|_{n,\infty} = \max(|u_1|, ..., |u_n|)$, $|u|_n = (|u_1|^2 + ..., |u_n|^2)^{1/2}$ and $|u|_{n,q} = (|u_1|^q + ..., |u_n|^q)^{1/q}$ we obtain that there are positive constants K_{1n} and K_{1n} such that

$$
\begin{aligned}
|\nabla V(t,u)|_n &\le C_2K_{1n}g(t)|u|^{1/2}_n \quad, t \in \mathbb{R}, \\
V(t,u) &\ge C_1K_{2n}|u|^{3/2}_n \quad, t \in J.
\end{aligned}
$$

So, all assumptions $(V_1) - (V_4)$ are satisfied.

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