ESCOLA DE DOUTORAMENTO
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Tese de doutoramento

Homogeneous hypersurfaces in symmetric spaces

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Homogeneous hypersurfaces in symmetric spaces

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Abstract

A hypersurface of a Riemannian manifold is said to be (extrinsically) homogeneous if it can be obtained as an orbit of an action of a subgroup of the isometry group of the ambient space. In this case, such an action is said to be of cohomogeneity one. The study of homogeneous hypersurfaces only makes sense for ambient spaces with a large enough isometry group. This is the case of Riemannian symmetric spaces, which constitute an important class among Riemannian manifolds, and whose study combines ideas from various areas of mathematics like geometry, topology, algebra, and mathematical analysis.

In this thesis, we tackle the classification problem for homogeneous hypersurfaces in symmetric spaces. The results can be divided into two lines. The first of these consists in the development of a structural result for cohomogeneity one actions on symmetric spaces of noncompact type. This result guarantees that any such action can be constructed by one of five standard methods, easily described in terms of Lie algebras. The second line investigates cohomogeneity one actions on products of symmetric spaces of different types. Under certain hypotheses, one can reduce the study of these actions to each factor. This allowed us to produce a classification of codimension one homogeneous foliations on simply connected symmetric spaces.

Resumo en galego

Nas matemáticas, a noción de simetría pode ser entendida vagamente como a invariancia dalgún obxecto ou propiedade matemática baixo a acción dun grupo de transformacións. Na xeometría riemanniana, as simetrías dun espazo veñen dadas polas accións isométricas, e a miúdo determinan as súas propiedades xeométricas. Reciprocamente, as propiedades xeométricas dun obxecto poden implicar a presencia de certas simetrías nel. Neste sentido, a simetría é un concepto central no estudo da xeometría.

Esta tese está motivada pola investigación de subvariedades de variedades riemannianas baixo hipóteses de simetría. En particular, estamos interesados no estudo e a clasificación das hipersuperficies homoxéneas dos espazos simétricos riemannianos. A grandes trazos, un espazo simétrico M é unha variedade de Riemann que é simétrica ao redor de cada un dos seus puntos. Unha hipersuperficie S de M dise entón (extrinsecamente) homoxénea se é unha órbita dalgunha acción isométrica en M. Isto é, as hipersuperficies homoxéneas non só teñen un alto grao de simetría, senón que a súa simetría vén, dalgún xeito, determinada pola restrición das simetrías do espazo ambiente.

As primeiras clasificacións de hipersuperficies homoxéneas remóntanse aos traballos clásicos de Levi-Civita [67] e Segre [81], quen clasificaron ditas hipersuperficies en espazos euclidianos, e Cartan [26], que resolveu o problema de clasificación para espazos hiperbólicos reais. A obtención de resultados similares para espazos de curvatura positiva tardou máis de 30 anos, cando Hsiang e Lawson [55] e Takagi e Takahashi [88] obtiveron a clasificación de hipersuperficies homoxéneas en esferas.

Recentemente téñense obtido varios resultados de clasificación de hipersuperficies homoxéneas en espazos simétricos riemannianos mediante o uso de técnicas de grupos e álxebras de Lie [8, 10–15, 60, 84, 85]. Estes resultados demostran a utilidade de estudar, no canto da xeometría das hipersuperficies homoxéneas, os grupos dos que son órbitas: neste caso, dise que as accións teñen cohomoxeneidade un. As accións de cohomoxeneidade un son interesantes non só dende este punto da xeometría de subvariedades, senón que tamén aparecen de maneira natural no estudo e construción de métricas con propiedades xeométricas especiais, como son as métricas de

Einstein [18], as variedades con holonomía especial [22, 23, 50], ou as métricas de curvatura positiva [52].

As hipersuperficies homoxéneas presentan propiedades xeométricas moi interesantes. Por exemplo, as curvaturas principais dunha hipersuperficie homoxénea son independentes do punto considerado, polo que as hipersuperficies homoxéneas son exemplos sinxelos de subvariedades con curvatura media constante. As hipersuperficies homoxéneas tamén son isoparamétricas, isto é, as súas hipersuperficies paralelas próximas teñen curvatura media constante. A investigación tanto de hipersuperficies de curvatura media constante coma de hipersuperficies isoparamétricas constitúen áreas de investigación moi activas en xeometría diferencial

Por definición, o estudo de hipersuperficies homoxéneas só ten sentido en espazos cun grupo de isometrías grande, como son os espazos homoxéneos riemannianos. Non obstante, a falta de información sobre a estrutura do grupo de isometrías $I(M)$ dun espazo homoxéneo arbitrario M fai difícil o estudo de accións isométricas en M. Por isto, nesta tese restrinxirémonos a espazos simétricos riemannianos como as variedades ambientes nas que realizar o noso estudo.

Os espazos simétricos son unha clase particularmente elegante de variedades de Riemann, xa que engloban varias familias importantes de espazos riemannianos como os espazos forma, os espazos hiperbólicos, ou as grassmannianas. Tamén aparecen dun xeito natural no estudo de diversos problemas de xeometría riemanniana, como as accións polares [31] ou as variedades de Einstein homoxéneas [19]. A estrutura ríxida dos espazos simétricos fai posible transformar moitos enunciados xeométricos en resultados alxébricos. Isto a miúdo permite o uso da maquinaria de grupos e álxebras de Lie para estudar problemas xeométricos complicados, simplificándoos enormemente. Por todo isto, os espazos simétricos son un terreo de proba ideal onde estudar problemas difíciles en xeometría riemanniana.

Un espazo simétrico M sempre se pode expresar como un cociente G/K, onde $G = I^0(M)$ é a compoñente conexa da identidade do grupo de isometrías de M e K é a isotropía dalgún punto base fixado $o \in M$. Grosso modo, os espazos simétricos poden ser clasificados en tres grandes familias: os espazos euclidianos, os espazos simétricos de tipo compacto, e os espazos simétricos de tipo non compacto. Se M é de tipo compacto, tanto M como G son compactos, mentres que se M é de tipo non compacto, M é unha variedade de Hadamard, G un grupo de Lie semisimple non compacto e K un subgrupo compacto maximal de G.

A clasificación das hipersuperficies homoxéneas nos espazos euclidianos séguese do xa mencionado artigo de Segre [81]. Para espazos simétricos de tipo compacto, Kollross deu en [60] unha clasificación completa de accións de cohomoxeneidade un agás equivalencia de órbitas. Non obstante, a diferente natureza dos espazos simétricos de tipo non compacto fixo que aínda non fose posible obter unha clasificación completa neles ata a actualidade. Mentres que a topoloxía desempeña un papel clave nos espazos simétricos de tipo compacto, calquera espazo simétrico de tipo non compacto é isométrico a un grupo de Lie resoluble simplemente conexo cunha métrica invariante pola esquerda. Esta falta de restricións topolóxicas permite que os espazos simétricos de tipo non compacto teñan unha gran variedade de hipersuperficies homoxéneas. A pesar disto, a cantidade de avances parciais de cara a resolver este problema en $[8, 10–15, 84, 85]$ fainos ser optimistas con respecto a completar a súa clasificación no futuro.

No que segue, resumimos brevemente os contidos de cada capítulo. Os capítulos 1 e 2 presentan as nocións básicas e resultados previos necesarios para realizar o noso estudo de hipersuperficies homoxéneas en espazos simétricos, mentres que as contribucións orixinais desta tese se poden atopar nos capítulos 3 e 4.

Accións isométricas e espazos simétricos

Os dous obxectos centrais desta tese, as hipersuperficies homoxéneas e os espazos simétricos, están intimamente relacionados coas nocións de grupo de Lie e as súas accións.

Na sección 1.1, falamos dalgúns resultados básicos sobre accións isométricas e a xeometría das súas órbitas. Unha acción dun grupo de Lie G nunha variedade de Riemann M dise isométrica se as transformacións que induce son isometrías de M. Neste caso as órbitas de G, dotadas cunha certa estrutura diferenciable natural, son subvariedades inxectivamente inmersas de M. No caso de que a acción de G sexa propia, estas órbitas son ademais subvariedades pechadas, e o espazo de órbitas M/G herda unha estrutura topolóxica coa cal é un espazo métrico.

Unha acción isométrica propia $G \curvearrowright M$ dise de cohomoxeneidade un precisamente se ten hipersuperficies homoxéneas como órbitas. Neste caso, o espazo de órbitas é homeomorfo a $\mathbb{R}$, $\mathbb{S}^1$, $[0, 1]$ ou $[0, \infty)$, e o feito de admitir unha acción de cohomoxeneidade un impón condicións na topoloxía de M. Dúas accións isométricas dinse equivalentes por órbitas se existe unha isometría de M levando as órbitas dunha ás órbitas da outra. Polo tanto, o problema de clasificar hipersuperficies homoxéneas agás congruencia nun espazo ambiente dado é esencialmente o mesmo que clasificar accións de cohomoxeneidade un agás equivalencia por órbitas.

Na sección 1.2 pasamos a analizar a estrutura dos espazos simétricos. Os espazos simétricos servirán como as variedades ambiente nas que imos estudar hipersuperficies homoxéneas nesta tese, xa que a súa ríxida estrutura alxébrica facilita enormemente estudar varios problemas relacionados con eles en termos de álxebra linear. Un espazo simétrico M é unha variedade riemanniana para a que en todo punto $p \in M$ existe unha isometría de M que fixa p e invirte xeodésicas. Isto fai que os espazos simétricos sexan variedades homoxéneas e completas, e que polo tanto se poidan expresar como cocientes de grupos de Lie. Se M é un espazo simétrico, e $G = I^0(M)$ é a compoñente conexa do neutro do seu grupo de isometrías, temos que $M = G/K$, onde K é a isotropía de G nalgún punto base fixado $o \in M$. Isto determina unha descomposición naturalmente redutiva $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, cuxas propiedades alxébricas caracterizan o espazo simétrico de partida.

Finalmente, na sección 1.3, poñemos o foco nos espazos simétricos de tipo non compacto. Para espazos simétricos de tipo non compacto, a descomposición $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ é unha descomposición de Cartan da álxebra de Lie semisimple real $\mathfrak{g}$, e calquera elección dun subespazo abeliano $\mathfrak{a} \subseteq \mathfrak{p}$ da lugar a unha descomposición

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{a}^*} \mathfrak{g}_\lambda,$$

onde $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \colon [H, X] = \lambda(H)X\}$. Se $\lambda \neq 0$ e $\mathfrak{g}_\lambda \neq \{0\}$, dicimos que λ é unha raíz, e $\mathfrak{g}_\lambda$ un espazo de raíz. Un pode entón definir certo criterio de positividade entre as raíces de $\mathfrak{g}$. A súma de tódolos espazos de raíces positivas, $\mathfrak{n}$, é unha subálxebra de Lie nilpotente de $\mathfrak{g}$, e dá lugar á descomposición como suma de espazos vectoriais

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Tanto esta descomposición como o correspondente difeomorfismo no nivel de grupos de Lie $G \cong KAN$ se coñecen como a *descomposición de Iwasawa*. Cocientando por K, isto permítenos considerar calquera espazo simétrico M como o grupo de Lie resoluble AN.

Aquí introducimos tamén o concepto de subálxebras parabólicas dunha álxebra de Lie semisimple real $\mathfrak{g}$, e a súa relación coa descomposición de $\mathfrak{g}$ en espazos de raíces. Estas subálxebras resultaran de especial utilidade no desenvolvemento do capítulo 3.

O problema de clasificación de hipersuperficies homoxéneas en espazos simétricos

O capítulo 2 expón, dun xeito breve, unha serie de resultados sobre a clasificación de accións de cohomoxeneidade un en espazos simétricos.

Na sección 2.1 trátase o caso de espazos simétricos de tipo compacto. Comezamos presentando as clasificacións de hipersuperficies homoxéneas en espazos simétricos simplemente conexos de tipo compacto e rango un (isto é, as esferas $\mathbb{S}^n$, e os espazos proxectivos $\mathbb{C}\mathsf{P}^n$, $\mathbb{H}\mathsf{P}^n$ e $\mathbb{O}\mathsf{P}^n$) para despois centrarnos na clasificación de Kollross [60]. Este resultado garante que unha acción de cohomoxeneidade un nun espazo simétrico irreducible de tipo compacto é ou ben unha acción de Hermann, ou a acción do grupo SU_3 en si mesmo dada por $g \cdot h = gh\overline{g}^{-1}$, ou unha acción inducida pola representación de isotropía dun espazo simétrico de rango dous, ou unha de sete accións excepcionais dadas na táboa 2.1.

Na sección 2.2, expóñense as distintas clasificacións de hipersuperficies homoxéneas para os espazos hiperbólicos $\mathbb{R}\mathsf{H}^n$, $\mathbb{C}\mathsf{H}^n$, $\mathbb{H}\mathsf{H}^n$ e $\mathbb{O}\mathsf{H}^n$. Cabe destacar que aquí se introduce unha primeira versión da técnica denominada como *construción nilpotente* que logo se xeneralizou en [15] para espazos simétricos de tipo non compacto e rango arbitrario.

Finalmente, na sección 2.3 estudamos os diversos resultados sobre accións de cohomoxeneidade un en espazos simétricos de tipo non compacto propostos en [12, 13, 15]. Se $M = G/K$ é un espazo simétrico de tipo non compacto, a parte resoluble da descomposición de Iwasawa de G, AN, actúa simple e transitivamente en M. Polo tanto, os subgrupos de Lie de codimensión un en AN dan lugar a foliacións homoxéneas de codimensión un en M. Usando as propiedades dos espazos de raíces, é sinxelo construír exemplos deste tipo de accións, como por exemplo:

- A acción do subgrupo de AN con álxebra de Lie $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ para algunha liña $\ell \subseteq \mathfrak{a}$, onde $\mathfrak{a} \ominus \ell$ é o complemento ortogonal de ℓ en $\mathfrak{a}$ con respecto do produto interior canónico de $\mathfrak{g}$. Estas accións dan lugar a unha familia de hipersuperficies mutuamente congruentes, que se coñece como foliación de *tipo horosférico*.

- A acción do subgrupo de AN con álxebra de Lie $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ para algunha liña $\ell \subseteq \mathfrak{g}_{\alpha_i}$ nun espazo de raíz simple, onde $\mathfrak{n} \ominus \ell$ é o complemento ortogonal de ℓ en $\mathfrak{n}$ con respecto do produto interior canónico de $\mathfrak{g}$. Estas accións dan lugar a unha familia de hipersuperficies con exactamente unha órbita minimal, que se coñece como foliación de *tipo resoluble*.

Se H é un grupo de Lie que actúa con cohomoxeneidade un e sen órbitas singulares en M, foi probado en [15] (para M irreducible) e [10] (para o caso xeral) que H é equivalente por órbitas a un destes dous tipos de accións.

Se H é un grupo de Lie actuando sobre un espazo simétrico irreducible de tipo non compacto M, e H ten unha órbita singular totalmente xeodésica, Berndt e Tamaru foron capaces en [13] de utilizar técnicas de dualidade xunto co resultado de Kollross [60] e a clasificación de Leung de subvariedades reflectivas de espazos simétricos [66] para determinar cando H ten cohomoxeneidade un. Cabe destacar que esta clasificación só é válida cando M é irreducible.

Finalmente, describimos brevemente dous métodos propostos en [15] para a construción de accións de cohomoxeneidade un en espazos simétricos de tipo non compacto que dan lugar a órbitas singulares que non son totalmente xeodésicas:

- O primeiro deles, coñecido como a *extensión canónica*, permite estender accións de cohomoxeneidade un dende certas subvariedades denominadas *compoñentes borde*. As compoñentes borde son subvariedades totalmente xeodésicas do espazo ambiente que son espazos simétricos de por si e, de certo xeito, compórtanse ben con respecto á descomposición en espazos de raíces.

- O segundo método, coñecido como a *construción nilpotente*, é unha xeneralización da técnica co mesmo nome que xa existía para espazos simétricos de tipo non compacto e rango un. Determinar que accións se poden obter mediante este método para un espazo dado require resolver un problema complicado sobre representacións de álxebras de Lie.

Ámbalas dúas construcións están intimamente relacionadas coas subálxebras parabólicas, e son tratadas en detalle posteriormente no capítulo 3.

Accións de cohomoxeneidade un en espazos simétricos de tipo non compacto

A principal contribución do capítulo 3 é a construción dun novo resultado estrutural sobre accións de cohomoxeneidade un. Para poder demostralo, primeiramente estudamos en detalle nas seccións 3.1 e 3.2 tanto o método da extensión canónica como o da construción nilipotente. En particular, demostramos dúas propiedades moi importantes: a extensión canónica dunha extensión canónica é unha extensión canónica, e a extensión canónica dunha construción nilpotente é unha construción nilpotente.

En §3.3 estudamos certas accións diagonais en espazos simétricos reducibles. Se $M = G/K$ é un espazo simétrico reducible escribimos $M = M_1 \times \cdots \times M_n$, onde $M_i = G_i/K_i$ é irreducible, entón un resultado de Dynkin garantiza que calquera subálxebra propia maximal de $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ é ben da forma

$$\mathfrak{l} = \bigoplus_{\substack{i=1 \\ i \neq j}}^{s} \mathfrak{g}_i \oplus \mathfrak{l}_j$$

para algún $j \in \{1, \ldots, s\}$ e unha subálxebra maximal propia $\mathfrak{l}_j$ de $\mathfrak{g}_j$, ou ben

$$\mathfrak{l} = \bigoplus_{\substack{i=1 \\ i \neq j,k}}^{s} \mathfrak{g}_i \oplus \mathfrak{g}_{j,k,\sigma},$$

para dous índices $j, k \in \{1, \ldots, s\}$, $j \neq k$, onde $\sigma \colon \mathfrak{g}_j \to \mathfrak{g}_k$ é un homomorfismo de álxebras de Lie e $\mathfrak{g}_{j,k,\sigma} = \{X + \sigma X : X \in \mathfrak{g}_j\}$. No último caso, $\mathfrak{g}_{j,k,\sigma}$ e $\mathfrak{l}$

son subálxebras redutivas de $\mathfrak{g}$, e $\mathfrak{g}_{j,k,\sigma}$ é unha subálxebra redutiva maximal de $\mathfrak{g}_j \oplus \mathfrak{g}_k$. Estudando a acción do subgrupo conexo de $G_j \times G_k$ con álxebra de Lie $\mathfrak{g}_{j,k,\sigma}$, vemos que dita acción é hiperpolar, e a súa cohomoxeneidade coincide co rango de M_j. Combinando estes resultados xunto coas ferramentas desenvolvidas por Berndt e Tamaru en [12, 13, 15], somos capaces de probar o seguinte:

Teorema A. *Sexa $M = G/K$ un espazo simétrico de tipo non compacto, e sexa H un subgrupo pechado e conexo de G. Entón, H actúa con cohomoxeneidade un en M se, e soamente se, a acción de H é equivalente por órbitas a unha das seguintes:*

(FH) *Unha acción inducindo unha foliación de tipo horosférico.*

(FS) *Unha acción inducindo unha foliación de tipo resoluble.*

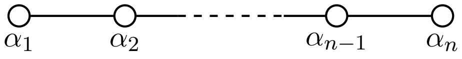
(CEI) *A extensión canónica dunha acción de cohomoxeneidade un cunha órbita singular totalmente xeodésica nunha compoñente borde irreducible.*

(CER) *A extensión canónica dunha acción de cohomoxeneidade un nunha compoñente borde reducible de rango dous cuxos factores son homotéticos.*

(NC) *Unha acción obtida por construción nilpotente.*

As accións producidas polos primeiros catro items no teorema anterior poden ser facilmente determinadas unha vez fixado un espazo M. Isto deixa a construción nilpotente como o principal obstáculo a superar para obter unha clasificación completa das accións de cohomoxeneidade un en espazos simétricos de tipo non compacto. O problema de determinar que accións poden ser obtidas mediante construción nilpotente foi xa resolto para tódolos espazos de rango un [8, 14, 39], e moitos espazos de rango dous [11,15,84]. Non obstante, para espazos de rango superior non se coñecían resultados similares previos á publicación de [37]. Nesta tese somos capaces de resolver dito problema para a familia de espazos $\mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$ con n arbitrario. Isto permítenos dar a primeira clasificación de accións de cohomoxeneidade un nun espazo simétrico de tipo non compacto e rango maior que dous:

Teorema B. *Sexa $M = G/K = \mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$, $n \geq 1$, e sexa $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ un conxunto de raíces simples para $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{R})$ cuxo diagrama de Dynkin é*



Calquera acción de cohomoxeneidade un en M é equivalente a algunha das seguintes accións:

(FH) *A acción do subgrupo conexo de* $\mathsf{SL}_{n+1}(\mathbb{R})$ *con álxebra de Lie* $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$*, para algún subespazo de dimensión un* ℓ *de* $\mathfrak{a}$*.*

(FS) *A acción do subgrupo conexo de* $\mathsf{SL}_{n+1}(\mathbb{R})$ *con álxebra de Lie* $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_j})$*, para algunha raíz simple* $\alpha_j \in \Lambda$*.*

(CE) *A extensión canónica* H_Φ^Λ *da acción do subgrupo conexo* H_Φ *de* G *nunha compoñente borde* B_Φ*, para algún dos casos listados na seguinte táboa:*

$\mathfrak{h}_\Phi$	Φ	B_Φ	$\operatorname{codim}(H_\Phi^\Lambda \cdot o)$	Comentarios
$\mathfrak{k}_{\{\alpha_j\}} \cong \mathfrak{so}_2$	$\{\alpha_j\}$	$\mathbb{R}\mathsf{H}^2$	2	$1 \leq j \leq n$
$\mathfrak{sl}_{k-j+1}(\mathbb{R}) \oplus \mathbb{R}$	$\{\alpha_j, \dots, \alpha_k\}$	$\mathsf{SL}_{k-j+2}(\mathbb{R})/\mathsf{SO}_{k-j+2}$	$k-j+1$	$1 \leq j < k \leq n$
$\mathfrak{sp}_2(\mathbb{R})$	$\{\alpha_j, \alpha_{j+1}, \alpha_{j+2}\}$	$\mathsf{SL}_4(\mathbb{R})/\mathsf{SO}_4$	3	$1 \leq j \leq n-2$
$\mathfrak{s}_{j,k,\sigma} \cong \mathfrak{sl}_2(\mathbb{R})$	$\{\alpha_j, \alpha_k\}$	$\mathbb{R}\mathsf{H}^2 \times \mathbb{R}\mathsf{H}^2$	2	$\lvert k-j \rvert > 1$

Unha diferenza principal entre o noso resultado e os anteriormente existentes é que o noso é aplicable para espazos simétricos de tipo non compacto que non son necesariamente irreducibles. Un estudo coidadoso dos distintos tipos de accións no Teorema A en relación coa descomposición dun espazo simétrico de tipo non compacto nos seus factores irreducibles permítenos afirmar o seguinte:

Teorema C. *Sexa* M *un espazo simétrico de tipo non compacto con descomposición de de Rham* $M = M_1 \times \cdots \times M_s$*, onde* $M_i = G_i/K_i$*,* $i = 1, \dots, s$*, e sexa* $G = \prod_{i=1}^{s} G_i$*. Entón, unha acción de cohomoxeneidade un en* M *é equivalente por órbitas a unha das seguintes accións:*

(Prod) *A acción produto dun subgrupo* $H_j \times \prod_{\substack{i=1 \\ i \neq j}}^{s} G_i$ *de* G*, onde* H_j *é un subgrupo de Lie conexo de* G_j *que actúa con cohomoxeneidade un no factor irreducible* M_j*.*

(FH) *A acción do subgrupo conexo de* G *con álxebra de Lie* $\mathfrak{h} = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$*, para algún subespazo de dimensión un* ℓ *de* $\mathfrak{a}$*.*

(CER) *A extensión canónica dunha acción de cohomoxeneidade un diagonal nunha compoñente borde reducible de rango dous de* M *cuxos factores son homotéticos.*

En particular, este resultado permítenos clasificar accións de cohomoxeneidade un nun produto de espazos simétricos sempre que coñezamos dita clasificación en cada un dos seus factores. Unha aplicación directa deste resultado a produtos arbitrarios de espazos simétricos de rango un (onde as accións de cohomoxeneidade un foron clasificadas en [8, 14, 39]) dá como resultado:

Teorema D. *Sexa $M = M_1 \times \cdots \times M_r$ un produto riemanniano de espazos simétricos de rango un e tipo non compacto $M_i = G_i/K_i = \mathbb{F}_i\mathsf{H}^{n_i}$, onde $\mathbb{F}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $i = 1, \dots, r$, e sexa $G = \prod_{i=1}^{r} G_i$. Entón, unha acción isométrica propia en M é de cohomoxeneidade un se, e soamente se, é equivalente por órbitas á acción dun subgrupo conexo H de G cunha das seguintes álxebras de Lie:*

Type	$\mathfrak{h}$	Comentarios
(FH)	$(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$	$\ell \subseteq \mathfrak{a}$, $\dim \ell = 1$.
(FS)	$\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$	$\ell \subseteq \mathfrak{g}_{\alpha_j}$, $\dim \ell = 1$, $\alpha_j \in \Lambda$.
(CEI)	$\bigoplus_{\substack{i=1 \\ i \neq j}}^{r} \mathfrak{g}_i \oplus \mathfrak{h}_j$	$H_j \subseteq G_j$ *actúa en* M_j *con cohomoxeneidade* 1 *e unha ótbita totalmente xeodésica.*
(CER)	$\bigoplus^{r}{}_{\substack{i=1 \\ i \neq j,k}} \mathfrak{g}_i \oplus \mathfrak{g}_{j,k,\sigma}$	$\mathfrak{g}_{j,k,\sigma} = \{X + \sigma X : X \in \mathfrak{g}_j\}$, $j \neq k$, $\sigma \colon \mathfrak{g}_j \to \mathfrak{g}_k$ *isomorfismo de álxebras de Lie.*
(NC)	$\bigoplus_{\substack{i=1 \\ i \neq j}}^{r} \mathfrak{g}_i \oplus N_{(\mathfrak{k}_j)_0}(\mathfrak{v}) \oplus \mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \mathfrak{v})$	$\mathfrak{v} \subseteq \mathfrak{g}_{\alpha_j}$ *protohomoxéneo,* $\alpha_j \in \Lambda$, $\dim \mathfrak{v} \geq 2$.

Accións de cohomoxeneidade un en produtos

Ata o de agora, as accións de cohomoxeneidade un en espazos simétricos foran estudadas asumindo que os espazos implicados eran dun tipo concreto (é dicir, de tipo compacto, non compacto ou euclidiano). No capítulo 4, empezamos o estudo de accións de cohomoxeneidade un en espazos para os cales esa hipótese previa xa non é certa. En máis detalle, se M é un espazo simétrico simplemente conexo, entón M parte como un produto riemanniano

$$M_+ \times M_0 \times M_-$$

onde M_+ é un espazo simétrico de tipo compacto, M_0 un espazo euclidiano, e M_- un espazo simétrico de tipo non compacto. Para o noso estudo, consideraremos espazos para os cales a anterior descomposición é non trivial, e demostramos que unha acción de cohomoxeneidade está determinada, de certo xeito, por como actúa en cada factor.

Para isto, traballamos no contexto máis xeral dos produtos riemannianos. Dado un produto riemanniano $M_1 \times \cdots \times M_n$, dicimos que unha acción isométrica dun grupo H en $M_1 \times \cdots \times M_n$ descompón se existen accións isométricas $H_i \curvearrowright M_i$ en cada factor tales que a acción de H ten as mesmas órbitas que a acción natural de $H_1 \times \cdots \times H_n$ en $M_1 \times \cdots \times M_n$. No caso dunha acción isométrica propia de cohomoxeneidade un dun grupo H nun produto $M_1 \times \cdots \times M_n$, tense que ou ben a acción descompón ou ben H actúa transitivamente en cada un dos factores. Baixo certas restricións xeométricas, podemos asegurar que este último caso non é posible.

Corolario 4.11. *Sexa M unha variedade de Riemann simplemente conexa, N unha variedade de Hadamard, e sexa $H \subseteq I(M \times N)$ un subgrupo pechado e conexo que actúa en $M \times N$ con cohomoxeneidade un. Entón, a acción de H descompón.*

Séguese de inmediato que se $M_+ \times M_0 \times M_-$ é un espazo simétrico, tomando $M = M_+$ e $N = M_0 \times M_-$ no teorema anterior obtemos:

Teorema E. *Sexa $M = M_+ \times M_0 \times M_-$ un espazo simétrico simplemente conexo, onde M_+ é un espazo simétrico de tipo compacto, $M_0 \cong \mathbb{R}^n$ para algún $n \in \mathbb{N}$, e $M_- = G/K$ é un espazo simétrico de tipo non compacto. Sexa H un subgrupo pechado e conexo de $I(M)$ que actúa en M con cohomoxeneidade un. Entón, ou ben a acción de H en $M_+ \times M_0 \times M_-$ descompón, ou é equivalente por órbitas á acción de $I(M_+) \times H_\Delta$, onde $H_\Delta \subseteq I(M_0) \times I(M_-)$ actúa con cohomoxeneidade un en $M_0 \times M_-$ e transitivamente tanto en M_0 coma en M_-.*

No caso de que a acción de H_Δ no teorema anterior non teña órbitas singulares, somos quen de determinar H_Δ agás equivalencia de órbitas. Se $\mathfrak{r}^n$ denota a álxebra de Lie do grupo de translacións $\mathbb{R}^n$ actuando en $M_0 \cong \mathbb{R}^n$, e $\mathfrak{a} \oplus \mathfrak{n}$ é a parte resoluble da descomposición de Iwasawa de $M_- = G/K$, pódese ver que H_Δ ten as mesmas órbitas có subgrupo conexo de $I(M_0 \times M_-)$ con álxebra de Lie

$$\mathfrak{h}_{E,X} = (\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E + X) \oplus (\mathfrak{a} \ominus X)$$

para certos $E \in \mathfrak{r}^n$ e $X \in \mathfrak{a}$ non nulos. Aquí, $\mathfrak{v} \ominus Y$ denota tomar o complemento ortogonal dun vector Y nun subespazo $\mathfrak{v} \subseteq \mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n} \cong T_o(M_0 \times M_-)$ con respecto do produto interior inducido pola métrica de $M_0 \times M_-$. Isto permítenos clasificar as foliacións homoxéneas de codimensión un en espazos simétricos con gran xeneralidade.

Teorema F. *Sexa $M = M_+ \times M_0 \times M_-$ un espazo simétrico simplemente conexo, onde M_+ é un espazo simétrico de tipo compacto, $M_0 \cong \mathbb{R}^n$ para algún $n \in \mathbb{N}$, e $M_- = G/K$ é un espazo simétrico de tipo non compacto. Sexa $H \subseteq I(M)$ un grupo pechado e conexo que actúa en M con cohomoxeneidade un e sen órbitas singulares. Entón, a acción de H é equivalente por órbitas a unha das seguintes:*

1. *A acción de $I(M_+) \times \mathbb{R}^{n-1} \times G$.*
2. *A acción de $I(M_+) \times \mathbb{R}^n \times H_X$, onde H_X é o subgrupo conexo de G con álxebra de Lie $\mathfrak{h}_X = (\mathfrak{a} \ominus X) \oplus \mathfrak{n}$ para algún vector non nulo $X \in \mathfrak{a}$.*
3. *A acción de $I(M_+) \times \mathbb{R}^n \times H_i$, onde H_i é o subgrupo conexo de G con álxebra de Lie $\mathfrak{h}_i = \mathfrak{a} \oplus (\mathfrak{n} \ominus X)$, para algún vector non nulo $X \in \mathfrak{g}_{\alpha_i}$ dun espazo de raíz simple.*
4. *A acción de $I(M_+) \times H_{E,X}$, onde H é o subgrupo conexo de $\mathbb{R}^n \times G$ con álxebra de Lie $\mathfrak{h}_{E,X} = (\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E + X) \oplus (\mathfrak{a} \ominus X) \oplus \mathfrak{n}$, para algúns vectores non nulos $E \in \mathfrak{r}^n$ e $X \in \mathfrak{a}$.*

En particular, esta descrición alxébrica tamén nos permite calcular o operador forma das órbitas de $H_{E,X}$ en termos de E e X mediante a identificación $M_0 \times M_- \cong \mathbb{R}^n \times AN$, e determinar cando estas son minimais.

Proposición 4.22. *Consideremos o vector normal unitario*

$$\xi = \frac{||X||^2 E - ||E||^2 X}{||E||||X||\sqrt{||E||^2 + ||X||^2}}$$

a $\mathfrak{h}_{E,X}$. *Entón, o operador forma de* $H_{E,X} \subseteq \mathbb{R}^n \times AN$ *con respecto de* ξ *coincide con* $(\mathrm{ad}_\xi)_{|\mathfrak{h}_{E,X}}$, *e a curvatura media de* $H_{E,X}$ *é*

$$\frac{||E||}{||X||\sqrt{||E||^2 + ||X||^2}} \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha)\alpha(X).$$

En particular, as órbitas de $H_{E,X}$ *son minimais se, e soamente se,*

$$\sum_{\alpha \in \Sigma^+} \dim\left(\mathfrak{g}_\alpha\right) \alpha(X) = 0.$$

Introduction

In mathematics, symmetry can be broadly thought of as the invariance of a mathematical object or property under the action of a group of transformations. In Riemannian geometry, the symmetries of a space are given by isometric actions, and they usually shape its geometric properties. Conversely, imposing a geometric condition on an object often implies the presence of symmetry within it. In this sense, symmetry lies in the very core concept of geometry.

This PhD thesis is motivated by the investigation of submanifolds of Riemannian ambient spaces under symmetry assumptions. Namely, we are interested in the study and classification of homogeneous hypersurfaces in Riemannian symmetric spaces. Roughly speaking, a *symmetric space* M is a Riemannian manifold that is symmetric around any of its points. A hypersurface S of M is then said to be (extrinsically) *homogeneous* if it is an orbit of some isometric action on M. That is, homogeneous hypersurfaces not only have a high degree of symmetry, but their symmetry comes, in some sense, from restricting the symmetries of the ambient space itself.

The first classification results of homogeneous hypersurfaces trace back to the works of Levi-Civita [67] and Segre [81], who provided a classification of such hypersurfaces in Euclidean spaces, and Cartan [26], who solved the classification problem in real hyperbolic spaces. Actually, their study, motivated by a problem in geometric optics, dealt with the more general notion of isoparametric hypersurface. Similar results for spaces of positive curvature had to wait for more than thirty years, when Hsiang and Lawson [55] and Takagi and Takahashi [88] provided the classification of homogeneous hypersurfaces in round spheres.

Recently, several Lie-theoretic approaches to the problem have been used successfully in order to obtain classification results in more general symmetric spaces [8, 10–15, 60, 84, 85]. In this fashion, it is often useful to study not the homogeneous hypersurfaces themselves, but rather the actions of which they are orbits: such actions are said to be of cohomogeneity one, or *cohomogeneity one actions*. These actions are not only of interest from this viewpoint of submanifold geometry: they appear naturally in the study and construction of metrics with special geometric properties, such as Einstein metrics [18], manifolds with special holonomy [22, 23, 50], or metrics of positive curvature [52].

Homogeneous hypersurfaces also have some remarkable geometric properties. Namely, the principal curvatures of a homogeneous hypersurface are independent of the choice of point, and so they provide examples of hypersurfaces with constant mean curvature. Homogeneous hypersurfaces are also *isoparametric*, that is, their locally defined, nearby, parallel hypersurfaces have constant mean curvature. Both the invesigation of hypersurfaces with constant mean curvature and isoparametric hypersurfaces are vibrant areas of research in Riemannian geometry.

By definition, the investigation of homogeneous hypersurfaces only makes sense in ambient spaces with a large isometry group. Therefore, Riemannian homogeneous spaces constitute the natural context for their investigation. However, the lack of knowledge about the structure of the isometry group $I(M)$ for an arbitrary homogeneous space M makes it difficult to study isometric actions on M. Thus, in this thesis we restrict ourselves to Riemannian symmetric spaces as the ambient manifolds where to develop our investigation.

Symmetric spaces provide a particularly elegant class of manifolds, as they encompass many important families of Riemannian spaces such as space forms, hyperbolic spaces, compact Lie groups or Grassmannians. They also appear naturally in the study of several problems in Riemannian geometry, such as the topic of polar actions [31] or the investigation of homogeneous Einstein metrics [19]. The rigid structure of symmetric spaces makes it possible to transform many geometric statements into algebraic ones. This often allows for the use of the machinery coming from Lie theory in order to study difficult geometric problems, greatly simplifying them. Therefore, symmetric spaces provide a testing ground where to study harder problems in Riemannian geometry.

A symmetric space M can always be expressed as a quotient G/K, where $G = I^0(M)$ is the identity component of the isometry group of M, and K is the isotropy at some fixed point $o \in M$. Roughly speaking, there are three types of symmetric spaces: Euclidean symmetric spaces, symmetric spaces of compact type, and symmetric spaces of noncompact type. If M is of compact type, then both M and G are compact, whereas for symmetric spaces of noncompact type M is a Hadamard manifold, G a noncompact semisimple Lie group, and K is maximal compact in G.

The classification of homogeneous hypersurfaces in Euclidean spaces follows from the already mentioned classical work of Segre [81]. For irreducible symmetric spaces of compact type, Kollross provided in [60] a complete classification of cohomogeneity one actions, up to orbit equivalence. However, the different nature of symmetric spaces of noncompact type has prevented mathematicians from getting a complete classification up to this day. Namely, whereas topology plays a role in symmetric spaces of compact type, any symmetric space of noncompact type M is isometric to a simply connected solvable Lie group with a left invariant metric. This lack of topological restrictions allows symmetric spaces of noncompact type to have a vast variety of homogeneous hypersurfaces. Nevertheless, the many partial advances of [8, 10–15, 84, 85] provide some hope on eventually completing their classification.

In what follows, we summarize the main contributions of this thesis, which are presented in Chapters 3 and 4.

Cohomogeneity one actions on symmetric spaces of noncompact type

The main contribution in Chapter 3 is the development of a new structural result regarding cohomogeneity one actions on symmetric spaces of noncompact type. This result, which largely builds on the previous efforts [12, 13, 15], states that any cohomogeneity one action on a given symmetric space M of noncompact type can be constructed by one of several procedures:

- The first construction method produces cohomogeneity one actions by considering certain subgroups of the solvable group model of the symmetric space, and give rise to codimension one homogeneous foliations. Depending on the nature of such subgroups, the corresponding foliations are said to be of *horospherical* or *solvable* type.

- The second technique, called the *canonical extension*, allows for extending actions from certain submanifolds, known as boundary components, to the ambient space. Boundary components are symmetric spaces of noncompact type and lower rank which are totally geodesic in M and, in a way, "behave well" with respect to the root space decomposition of the isometry Lie algebra of M. We show that it is enough to consider canonical extensions of two types of actions: cohomogeneity one actions on irreducible boundary components with a totally geodesic singular orbit, or cohomogeneity one actions on products of two homothetic hyperbolic spaces with a maximal diagonal totally geodesic submanifold as a singular orbit.

- The third method, known as the *nilpotent construction*, is a generalization of how cohomogeneity one actions on rank one spaces behave, and involves solving a complicated problem regarding representations of Lie algebras.

Actions that can be constructed by the first two approaches can be easily determined for a particular choice of symmetric space. This leaves the nilpotent construction as the main obstacle for producing classifications of cohomogeneity one actions. The nilpotent construction problem has been solved for all rank one [8, 14, 39] and many rank two [11, 15, 84] spaces, but remains elusive in the higher rank setting. In this thesis, we are able to solve this problem for the family of spaces $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$ for arbitrary n. As an application, we were able to derive the classification of cohomogeneity one actions on such spaces. This provides the first classification result for a symmetric space of noncompact type and rank greater than two.

We should also point out that an important difference between our structural result and the results of [15] is that ours also works for reducible symmetric spaces of noncompact type. Namely, we are able to reduce the classification of cohomogeneity one actions to the irreducible factors, and to two possible types of "diagonal" actions. By means of this, we are also able to produce a classification of cohomogeneity one actions on the product of arbitrarily many hyperbolic spaces.

Cohomogeneity one actions on products

So far, cohomogeneity one actions on symmetric spaces have been studied by assuming the spaces involved to be of a given type. In Chapter 4, we begin the investigation of cohomogeneity one actions without the previous assumption. Namely, if M is a simply connected symmetric space, then it splits as a Riemannian product

$$M_+ \times M_0 \times M_-$$

where M_+ is a symmetric space of compact type, M_0 a Euclidean space, and M_- is a symmetric space of noncompact type. We consider spaces for which the above decomposition is nontrivial, and prove that a cohomogeneity one action, in some sense, can be determined from how it behaves on each of the different factors.

In more detail, we say that an isometric action of a Lie group H on a Riemannian product $M_1 \times \cdots \times M_n$ decomposes if there exist isometric actions $H_i \curvearrowright M_i$, $i = 1, \ldots, n$, such that the H-action has the same orbits as the natural action of $H_1 \times \cdots \times H_n$ on $M_1 \times \cdots \times M_n$. In Section 4.1, we prove that an isometric action on a simply connected symmetric space $M_+ \times M_0 \times M_-$ either decomposes, or it is orbit equivalent to the action of $I(M_+) \times H_\Delta$, where H_Δ acts with cohomogeneity one on $M_0 \times M_-$, but transitively on both M_0 and M_-.

In the particular case that the action has no singular orbits, we are able to prove that any such H_Δ must have the same orbits as the action of the connected subgroup of $I(M_0 \times M_-)$ with Lie algebra

$$(\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E + X) \oplus (\mathfrak{a} \ominus X)$$

for some choice of $E \in \mathfrak{r}^n$ and $X \in \mathfrak{a}$. Here, $\mathfrak{r}^n$ denotes the Lie algebra of the translation group $\mathbb{R}^n$ on $M_0 \cong \mathbb{R}^n$, $\mathfrak{a} \oplus \mathfrak{n}$ is the solvable part of the Iwasawa decomposition of the Lie algebra of $I(M_-)$, and $\mathfrak{v} \ominus X$ denotes the orthogonal complement of a vector X in a subspace $\mathfrak{v} \subseteq \mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$ with respect to the product induced by the metric of $M_0 \times M_-$. This allows us to classify codimension one homogeneous foliations on general symmetric spaces. We are also able to compute the shape operator of the orbits of H_Δ in terms of E and X.

Structure of the thesis

This thesis is organized as follows.

Chapter 1 is devoted to the introduction of the basic notions, concepts and terminology that will be used in this work. More precisely, in Section 1.1 we present some of the properties of isometric actions with a special interest in the geometry of their orbits. In Section 1.2, we introduce the basic notions about Riemannian symmetric spaces, while in Section 1.3 we delve into the topic of symmetric spaces of noncompact type and their relation with semisimple Lie algebras. We also provide a detailed description of parabolic subalgebras of semisimple Lie algebras, since they will be one of the main tools used in Chapter 3.

Chapter 2 provides an exposition about the many previous advances towards classifying homogeneous hypersurfaces in symmetric spaces. This has been divided into three blocks. In Section 2.1 we present some results concerning cohomogeneity one actions on symmetric spaces of compact type, while in Section 2.2 we write about the classification of homogeneous hypersurfaces in hyperbolic spaces. Finally, in Section 2.3 we study symmetric spaces of noncompact type and higher rank. Some of the results in this last section will be of great importance in both of the following chapters.

The original contributions of this thesis can be found in Chapters 3 and 4.

In Chapter 3, we present our structural result regarding cohomogeneity one actions on symmetric spaces of noncompact type. Sections 3.1 and 3.2 provide a detailed description of the two main tools that will be used to build cohomogeneity one actions: the canonical extension and nilpotent construction methods. In Section 3.3, we discuss diagonal cohomogeneity one actions on reducible symmetric spaces. Section 3.4 is entirely devoted to the proof of our main structural result, while in Section 3.5 we deduce from it the classification result for the spaces $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$ and the products of hyperbolic spaces. We also prove the decompostition result for cohomogeneity one actions on reducible symmetric spaces of noncompact type.

Chapter 4 studies cohomogeneity one actions on symmetric spaces that are not of a given particular type. In Section 4.1, we deal with isometric actions on Riemannian products, and prove some decomposability results for such actions. Then, in Section 4.2, we will make use of these results to give our classification of codimension one homogeneous foliations on symmetric spaces. Finally, in Section 4.3, we briefly compute the geometry of the orbits of the new examples of cohomogeneity one actions on $M_0 \times M_-$.

Objectives

The overarching theme of this thesis is the study and ultimate classification of homogeneous hypersurfaces in symmetric spaces. This problem, which is equivalent to the classification of cohomogeneity one actions on symmetric spaces up to orbit equivalence, has constituted a topic of interest in Riemannian geometry over recent years. However, despite having received a remarkable number of contributions, this problem is far from being completely understood.

The classification of cohomogeneity one actions on Euclidean and hyperbolic spaces follows from the classical works [81] and [26] by Segre and Cartan in the 1930's. Similar results for round spheres were obtained in [55, 88] more than 30 years later. Cohomogeneity one actions on irreducible symmetric spaces of compact type were classified by Kollross in [60]. Producing a complete classification for symmetric spaces of noncompact space of higher rank has only been achieved in a handful of low rank symmetric spaces. For instance, in the quaternionic hyperbolic space, this problem was only solved recently [39], more than 80 years after Segre and Cartan's results.

In order to advance towards a solution, this thesis aimed to:

- **O.1** Better understand the structure and behavior of cohomogeneity one actions on symmetric spaces, and develop new tools for their study.
- **O.2** Obtain explicit classifications of homogeneous hypersurfaces for particular symmetric spaces of noncompact type, with a particular interest in spaces of rank greater than two.
- **O.3** Generalize the known classification and structural results for cohomogeneity one actions on irreducible symmetric spaces to the reducible setting.
- **O.4** Generalize the known classification and structural results for spaces of compact, Euclidean, and noncompact type to general symmetric spaces.

Methodology

This thesis has followed the standard approach to research in Mathematics. Namely, this consists in the study of the relevant literature for a given problem, the analysis of the behavioral patterns of the corresponding mathematical structures, the construction of examples and, sometimes, making explicit computations. This allows to use mathematical reasoning to derive general statements about these mathematical structures and properties, which are then proven rigorously via deductive arguments.

In order to carry out these tasks, it was necessary to first assimilate a series of basic ideas and concepts. The particular case of this thesis required the candidate to study some reference texts on submanifold geometry, symmetric spaces, and Lie theory, such as [9,54,58,76,77]. This has been complemented by several courses on differential geometry offered by the doctoral program in Mathematics.

The importance of discussion with other experts in the field of differential geometry also played a key role, not only with the thesis supervisors, but also with other colleagues and established researchers, both through visits of researchers and the attendance to several conferences and workshops. Of particular relevance was the mathematical exchange and collaboration developed in the two research stays I have done during the PhD period. Namely, I spent three months in Turin under the supervision of Professor Anna Fino, and then six weeks in Osaka with Professor Hiroshi Tamaru.

The mathematical methods employed in this thesis fit into the framework of differential geometry, and in particular of Riemannian geometry and Lie groups. More specifically, the contributions of this thesis were achieved by a combination of well-established techniques in the field: isometric actions on Riemannian manifolds, the geometric and algebraic structure of Riemannian symmetric spaces (with particular focus on those of noncompact type), the structure theory of real semisimple Lie algebras and groups, the classification of parabolic subalgebras of real semisimple Lie algebras in terms of root systems, and some classical results on maximal subalgebras of real semisimple Lie algebras.

As is standard practice in all mathematical literature, the concrete application of the techniques mentioned will be carefully detailed in the different chapters of the thesis in order to justify each one of the mathematical arguments and statements included in the text.

Part I

Preliminaries

Chapter 1
Isometric actions and symmetric spaces

This chapter aims to introduce and explain most of the basic (and not so basic) notions and terminology that are used throughout this thesis. This is driven by a desire to make this thesis as self-contained as possible, although most of the proofs are omitted and can be found in the reference texts.

In Section 1.1, we introduce some notions about isometric actions on Riemannian manifolds, paying attention to the geometry of their orbits. We also briefly discuss some topics related to homogeneous hypersurfaces. Section 1.2 provides a quick journey through the basics of symmetric spaces, with a focus on their algebraic properties. Finally, Section 1.3 provides a glimpse of the structure theory of symmetric spaces of noncompact type. Here, we pay special attention to the topic of parabolic subalgebras of real semisimple Lie algebras, since it will be of great importance during de development of Chapter 3.

1.1 Isometric actions and their orbits

The central objects of this thesis, homogeneous hypersurfaces and symmetric spaces, are closely related to the notions of Lie groups and their actions. This section mainly aims to fix some terminology and notation that we use throughout the text. We also discuss briefly some geometric properties of the orbits of isometric actions, as these will be of great relevance in Chapters 3 and 4. For an exhaustive exposition about isometric actions, we refer the reader to [3] or [9, Ch. 2].

An isometric action[1] $G \curvearrowright M$ of a Lie group G on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is a smooth action $G \times M \to M$, $(g, p) \mapsto g \cdot p$, such that the map $p \mapsto g \cdot p$ is an isometry for all $g \in G$. For each point $p \in M$, one can define the *isotropy* or *stabilizer* of the G-action on p as

$$G_p = \{g \in G : \ g \cdot p = p\} \subseteq G,$$

---

[1] Throughout this text, we assume all actions involved to be left actions unless otherwise specified. Although many of the notions which will be defined are still valid if one considers smooth actions, we restrict ourselves to the Riemannian case, since it will be enough for the scope of this thesis.

which is a closed subgroup of G. The *orbit* of the G-action through p is defined to be the set

$$G \cdot p = \{g \cdot p : \; g \in G\} \subseteq M.$$

If $G \cdot p = M$ for some $p \in M$ (equivalently for all $p \in M$), we say that the action is *transitive*, and we call M a *(Riemannian) homogeneous G-space*. The closed subgroup

$$I = \bigcap_{p \in M} G_p$$

is called the *ineffective kernel* of G. An action $G \curvearrowright M$ is said to be *effective* if $I = \{e\}$, and *almost effective* if I is discrete. If $G_p = \{e\}$ for all $p \in M$, the action of G is said to be *free*. An action which is both transitive and free is often called *simply transitive*.

Remark 1.1. Let $I(M)$ denote the isometry group of M. If G is a group acting isometrically on a Riemannian manifold M, the map taking an element $g \in G$ to the transformation $\mu_g \colon p \mapsto g \cdot p$ of M provides a Lie group homomorphism $G \to I(M)$, and its kernel is precisely the kernel of ineffectivity of G. The image of G under this map acts on M with the same orbits as G, and we refer to it as the *effectivization* of the action of G. Because of this, it is common to restrict oneself to the study of actions of subgroups of $I(M)$ for a given space M.

Note that the orbit of a G-action through a point $p \in M$ can be naturally identified with G/G_p. Since isotropies are closed in G, the quotient G/G_p can be given a smooth structure such that the projection map $\rho \colon G \to G/G_p$ determines a G_p-principal bundle

$$G_p \hookrightarrow G \xrightarrow{\rho} G/G_p \cong G \cdot p.$$

From now on, we endow $G \cdot p$ with the smooth structure given by the identification $G \cdot p \cong G/G_p$. Let $\mu^p \colon G \to M$ denote the map $g \mapsto g \cdot p$, and define a map $\tilde{\mu}^p \colon G/G_p \to M$ by requiring the following diagram to commute:

$$\begin{tikzcd}
G \arrow[d, "\rho"'] \arrow[dr, "\mu^p"] & \\
G/G_p \arrow[r, "\tilde{\mu}^p"'] & M
\end{tikzcd}$$

Then, $\tilde{\mu}^p$ is a G-equivariant injective immersion with image $G \cdot p$. Thus, the orbits of an isometric action are injectively immersed submanifolds (and with the induced metric, they become Riemannian homogeneous G-spaces).

Definition 1.2. The *cohomogeneity* of an isometric action $G \curvearrowright M$ is defined to be the least codimension of the orbits of G. An orbit of G is said to be *regular* if it is of maximum codimension, and *singular* otherwise.

Remark 1.3. The family of orbits of an isometric action also determines what is called a *singular Riemannian foliation*. This is a partition $\mathcal{F}$ of M into connected immersed submanifolds, called *leaves*, such that the module $\mathfrak{X}_{\mathcal{F}}$ of smooth vector fields on M that are tangent at each point to the corresponding leaf acts transitively on each leaf, and every geodesic that is perpendicular to a leaf at one point remains perpendicular to every leaf it intersects. In the same way as the orbits of isometric actions, a leaf of $\mathcal{F}$ is called regular if it is of maximal dimension, and singular otherwise. We refer the reader to [4] or the book of Molino [72] for an introduction to singular Riemannian foliations.

A smooth action $G \curvearrowright M$ is said to be *proper* if the map $G \times M \to M \times M$, $(g, p) \mapsto (p, g \cdot p)$ is a proper map (that is, the preimage of any compact set in $M \times M$ is compact in $G \times M$). In this case, the isotropy G_p at every point $p \in M$ is a compact subgroup of G, and the orbits of G are properly embedded closed submanifolds of M. For actions which are both proper and isometric, we have a sort of converse:

Proposition 1.4 [33, Theorems 4 & 5]**.** *Let M be a Riemannian manifold, and let $G \subseteq I(M)$ be a subgroup of the isometry group of M. Then:*

- *The natural action of G on M is proper if and only if G is closed in $I(M)$.*
- *The orbits of G are closed if and only if G has the same orbits as its closure in $I(M)$. In particular, the orbits of an isometric action are closed if and only if they are the orbits of some (possibly different) proper isometric action.*

Every isometric action naturally induces certain orthogonal representations. Let $p \in M$. Then, any element $k \in G_p$ corresponds to an isometry μ_k of M fixing p and leaving $G \cdot p$ invariant. Therefore, the differential of μ_k at p (which, for the sake of convenience, we denote by k_{*p}) is an orthogonal isomorphism of T_pM leaving both the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ to $G \cdot p$ at p invariant. The action

$$G_p \times T_p(G \cdot p) \to T_p(G \cdot p), \quad (k, X) \mapsto k_{*p}X$$

is called the *isotropy representation* of G at p, whereas the action

$$G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p), \quad (k, \xi) \mapsto k_{*p}\xi$$

is known as the *slice representation* of G at p. If $G \subseteq I(M)$ is a closed subgroup, the cohomogeneity of $G \curvearrowright M$ coincides with the cohomogeneity of the slice representation of G at any point $p \in M$.

1.1.1 The orbit space

The orbits of any smooth action $G \curvearrowright M$ define an equivalence relation of M, since any two orbits that have nontrivial intersection must in fact coincide. The quotient

$$M/G = \{G \cdot p \colon\ p \in M\}$$

of M under this relation is known as the *orbit space* of the action. The natural projection $\pi \colon M \to M/G$ is known as the *projection map*, and one usually considers M/G as a topological space with the topology induced by π. Moreover, if the action of G on M is proper, M/G is a Hausdorff topological space. If $G \curvearrowright M$ is a proper isometric action, the orbits of G turn out to be *equidistant*, and hence there is a well-defined *orbital distance* on M/G, turning it into a length metric space.

If $G \curvearrowright M$ is a proper isometric action, one can define a partial ordering on the set of orbit types of G. We say that two orbits $G \cdot p$ and $G \cdot q$ have the same *orbit type* if G_p and G_q are conjugate in G. This defines an equivalence relation in M/G. Let $p, q \in M$, and denote by $[G \cdot p]$ and $[G \cdot q]$ the equivalence classes of the corresponding orbits. We write $[G \cdot p] \leq [G \cdot q]$ if G_q is conjugate in G to some subgroup of G_p. This defines a partial ordering on the set of all orbit types. If M/G is connected, there exists a largest orbit type. Each representative of this equivalence class is called a *principal orbit*, and it must be of maximal dimension.

Definition 1.5. An orbit $G \cdot p$ is *principal* if for every $q \in M$, G_p is, up to conjugation, a subgroup of G_q. A point $p \in M$ is said to be *principal* if the orbit $G \cdot p$ is principal. A non-principal orbit of maximal dimension is called an *exceptional orbit*.

The set of all principal points and the set of all principal orbits of $G \curvearrowright M$ are open and dense subsets of M and M/G, respectively.

The orbit space of cohomogeneity one actions

In the particular case that $G \curvearrowright M$ is a proper cohomogeneity one action on a complete connected Riemannian manifold, the orbit space M/G is known to be homeomorphic to $\mathbb{R}$, $\mathbb{S}^1$, $[0, \infty)$ or $[0, 1]$. Here, nonprincipal orbits correspond to the boundary of such spaces [6]. Thus, the orbits of G are properly embedded hypersurfaces of M, except at most two (the singular orbits). If singular orbits exist, every regular orbit is a tube around any of the singular orbits.

Depending on the geometry and topology of M, some of the possibilities above can be excluded. For instance, if $M/G \cong \mathbb{R}$ or $M/G \cong \mathbb{S}^1$, then all orbits are principal, and M is a fiber bundle over M/G with fiber any principal orbit. In particular, if M is simply connected, M/G cannot be $\mathbb{S}^1$. If $M/G \cong [0, \infty)$, then M is diffeomorphic to a tubular neighborhood of the only nonprincipal G-orbit, say $G \cdot p$, and hence $M \cong (G \cdot p) \times_{G_p} V$ is a Euclidean space bundle over such nonprincipal orbit $G \cdot p$. Lastly, if $M/G \cong [0, 1]$, then there are two nonprincipal orbits, say $G \cdot p_+$ and $G \cdot p_-$, and M admits a decomposition as a union of disk bundles

$$M \cong (G \times_{G_{p_+}} \mathbb{D}_-) \cup_{G/K} (G \times_{G_{p_-}} \mathbb{D}_+),$$

where K is the isotropy at some point in a principal orbit G/K and the union of the disk bundles is made along the principal orbit G/K. If M is a Hadamard manifold, the only possibilities for M/G are $\mathbb{R}$ and $[0, \infty)$.

1.1.2 Homogeneous hypersurfaces and their geometric properties

The ultimate goals of the results in Chapters 3 and 4 is to classify homogeneous hypersurfaces in symmetric spaces. Roughly speaking, these are hypersurfaces that are not only homogeneous as Riemannian manifolds, but whose homogeneous nature comes from the isometries of their ambient space:

Definition 1.6. A submanifold S of a Riemannian manifold M is said to be *(extrinsically) homogeneous* if for every two points $p, q \in S$ there exists an isometry φ of M leaving S invariant and such that $\varphi(p) = q$. If S is a hypersurface of M, we will simply call S a *homogeneous hypersurface* of M.

Note that if $S = M$, we recover the usual (intrinsic) notion of homogeneity discussed at the beginning of this section. By considering the group

$$H_S = \{\varphi \in I(M) \colon \varphi(S) = S\} \subseteq I(M),$$

one can easily see that S is homogeneous if and only if S is an orbit of an isometric action on M. Moreover, S is properly embedded in M if and only if H_S is closed in $I(M)$. In this sense, closed embedded homogeneous hypersurfaces correspond precisely to orbits of proper cohomogeneity one actions on M.

Remark 1.7. From now on in this thesis, and unless otherwise stated, by cohomogeneity one action we will understand a proper isometric action of cohomogeneity one, and homogeneous hypersurfaces will be assumed to be properly embedded.

Although in order to classify homogeneous hypersurfaces we will investigate the groups of which they are orbits, we must not lose track that we are ultimately interested in the orbits themselves and not the action. Thus, when discussing isometric actions, we usually consider actions to be equivalent if their families of orbits are congruent, precisely:

Definition 1.8. Two actions $G \curvearrowright M$ and $H \curvearrowright M$ are said to be *orbit equivalent* if there exists an isometry φ of M such that $\varphi(G \cdot p) = H \cdot \varphi(p)$ for all $p \in M$.

Homogeneous hypersurfaces have some remarkable geometric properties. Since the shape operators (at different points) of a homogeneous hypersurface S of M are conjugate by isometries of M, their eigenvalues are independent of the point, that is, S has constant principal curvatures. Recall that the orbits of an isometric action are locally equidistant. Thus, the nearby equidistant hypersurfaces to a homogeneous hypersurface are also homogeneous, and have constant principal curvatures. This implies that homogeneous hypersurfaces are *isoparametric*: their locally defined, nearby parallel hypersurfaces have constant mean curvature.

Actually, the classification of homogeneous hypersurfaces in Euclidean and real hyperbolic spaces follows from the respective Segre's [81] and Cartan's [26] classifications of isoparametric hypersurfaces in such spaces. It should be noted that, whereas in spaces of constant curvature a hypersurface is isoparametric if and only if it has constant principal curvatures, this is not true in general. Examples of isoparametric hypersurfaces with nonconstant principal curvatures (and hence, inhomogeneous hypersurfaces) have been found in several symmetric spaces, such as complex and quaternionic projective spaces [42, 44], and many symmetric spaces of noncompact type [34, 35, 41, 46]. Conversely, we do not know of any nonisoparametric hypersurface with constant principal curvatures in symmetric spaces, although there do exist examples for some particular conformally flat metrics [79]. There are also important spaces where isoparametric hypersurfaces are known to be homogeneous, such as the homogeneous 3-manifolds with 4-dimensional isometry group [45], or the product of two round 2-spheres [93], besides Euclidean and real hyperbolic spaces. More information about isoparametric hypersurfaces and their interplay with homogeneous hypersurfaces can be found in [29, 30, 40, 83, 91, 92], and the references therein.

1.1.3 Polar and hyperpolar actions

Two important subclasses of isometric actions with special geometric properties are those of *polar* and *hyperpolar actions*. Although they are not the focus of this thesis, they will occasionally make an appearance throughout the text. Because of this, let us briefly define them and state some basic properties. References [9, Ch. 2.3], [71, Ch. VI] and [51] provide a nice introduction to this topic.

Definition 1.9. Let G be a Lie group acting properly by isometries on a complete Riemannian manifold M. A complete connected immersed submanifold $\Sigma \subseteq M$ is called a *section* of the G-action if it intersects all of the G-orbits and every intersection between Σ and a G-orbit is orthogonal. An action $G \curvearrowright M$ is said to be *polar* if it admits a section. A polar action is *hyperpolar* if its sections are flat.

Remark 1.10. In the literature, the action of G is sometimes not required to be proper.

However, its orbits often are automatically closed in many cases, such as when M is simply connected [69]. In this case, the action of G is orbit equivalent to some proper action. Also, there are several results regarding polar actions which assume that the group G is compact. Some authors require the sections Σ to be embedded or properly embedded submanifolds.

Sections of polar actions turn out to be totally geodesic submanifolds of the ambient space M. Also, for every point $p \in M$, there must exist a section Σ of G such that $p \in \Sigma$. Moreover, if p is a principal point for the G-action, the section can be recovered as $\Sigma = \exp_p(\nu_p(G \cdot p))$, where $\exp$ is the Riemannian exponential map.

Polar actions admit a generalization in terms of singular Riemannian foliations. Namely, a singular Riemannian foliation $\mathcal{F}$ which admits a section Σ through every point p of the ambient space M (in the sense that Σ intersects all leaves of $\mathcal{F}$ orthogonally) is called a *singular Riemannian foliation with sections* or *polar foliation*. Polar foliations have some remarkable geometric properties, and have provided an active topic of research in recent years [42, 68, 70].

1.2 Riemannian symmetric spaces

Riemannian symmetric spaces will serve as our ambient spaces where to study hypersurfaces in this thesis. The rigid algebraic structure of symmetric spaces often provides a dictionary that allows to translate problems and properties from the realm of Riemannian geometry to that of Lie theory. This section provides a brief overview of how both notions are intertwined.

The list of texts covering this topic is extensive, but probably the most well-known and complete reference is Helgason's book [54]. For a quicker introduction, we refer the reader to [5, 96]. The books of Besse [17], Kobayashi and Nomizu [59] and O'Neill [75] also include nice chapters on symmetric spaces.

The investigation of Riemannian symmetric spaces can be traced back to the following question of Élie Cartan [24]:

> *For which Riemannian manifolds* (M, g) *is the curvature tensor*
> R *invariant under parallel transport?*

Manifolds satisfying this condition (or equivalently, $\nabla R = 0$) are known as *Riemannian locally symmetric spaces*. Cartan himself proposed in [24] and [25] two different group-theoretic approaches towards the study and classification of such spaces. The first of these methods made use of holonomy groups to (locally) determine the curvature tensor of the space M, while the second one (and the one which is nowadays ubiquitous in the literature) is based on a single but important observation: the invariance of R under parallel transport is equivalent to the condition that the (locally defined) geodesic reflections about every point $p \in M$, $\exp_p(v) \mapsto \exp_p(-v)$,

$v \in T_pM$, be local isometries. Asking these geodesic symmetries to be globally defined motivates the following definition:

Definition 1.11. A *Riemannian globally symmetric space*, or simply *symmetric space*, is a Riemannian manifold (M, g) such that, for every point $p \in M$ there exists an isometry $s_p \in I(M)$ such that $s_p(p) = p$ and $(s_p)_{*p} = -\operatorname{Id}_{T_pM}$. If such s_p exists it must be unique, and is called the *(global) geodesic reflection* or *geodesic symmetry* of M at p.

Remark 1.12. Clearly, a symmetric space is locally symmetric, but the converse is not true. However, if N is a locally symmetric space, for every $p \in N$ there exists a symmetric space M and a neighborhood U of p such that U is isometric to some open subset of M. Moreover, if N is complete and simply connected, then N is a symmetric space. In particular, the universal cover of a complete locally symmetric space is a symmetric space (for more information and detailed proofs, see [54, Ch. IV]).

From the above definition, it is easy to see that symmetric spaces are complete, since geodesics can be extended by means of geodesic reflections. It follows that if M is a symmetric space, any two points $p, q \in M$ are joined by some geodesic segment γ in M. If o is the midpoint of γ, the geodesic reflection about o is an isometry of M taking p to q, and so symmetric spaces are also homogeneous.

1.2.1 Riemannian symmetric pairs

In view of the homogeneity of symmetric space, we may express them as quotients of Lie groups. Let M be a symmetric space, and write $M = G/K$, where $G = I^0(M)$ is the identity component of the isometry group of M and $K = G_o$ is the isotropy at some base point $o \in M$ that we fix from now on. Note that K is compact, since the action of $I^0(M)$ is proper. Under the identification $M = G/K$, it can be seen that the induced metric on G/K is G-invariant.

Let π denote the map $g \in G \mapsto g \cdot o \in M$. Then, its differential at o is a surjective linear map $\pi_{*o} \colon \mathfrak{g} \to T_oM$ whose kernel is precisely $\mathfrak{k}$. The isotropy representation of M is defined to be the isotropy representation of the G-action on M at o, that is, the map $k \in K \mapsto k_{*o} \in \mathsf{GL}(T_oM)$. The symmetric space M is said to be *irreducible* if its isotropy representation is an irreducible representation.

The geodesic reflection about o determines an involution σ of G by

$$\begin{aligned} \sigma \colon G &\longrightarrow G, \\ g &\longmapsto s_o g s_o. \end{aligned}$$

If G_σ denotes the set of fixed points of σ, and $(G_\sigma)^0$ its identity component, we have that $(G_\sigma)^0 \subseteq K \subseteq G_\sigma$. Let θ be the differential of σ at e, and let $\mathfrak{k}$ and $\mathfrak{p}$ denote the 1 and -1-eigenspaces of θ, respectively. Then, $\mathfrak{k}$ coincides with the Lie algebra of the isotropy K, and π_{*o} provides an isomorphism between $\mathfrak{p}$ and the tangent space to M at o. Moreover, we have the bracket relations

$$[\mathfrak{k},\mathfrak{k}]=\mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}]=\mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}]=\mathfrak{k}, \tag{1.1}$$

and so the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$. Under the identification $\mathfrak{p}\cong T_oM$, the isotropy representation of M is equivalent to the adjoint representation of K on $\mathfrak{p}$.

Remark 1.13. The decomposition into the ±1-eigenspaces of an involution of $\mathfrak{g}$ is always a $\mathbb{Z}_2$-grading. Conversely, given a $\mathbb{Z}_2$-grading $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, the map $X+Y\mapsto X-Y$ for $X\in\mathfrak{k}$ and $Y\in\mathfrak{p}$ defines an involutive automorphism of $\mathfrak{g}$. This establishes a one-to-one correspondence between $\mathbb{Z}_2$-gradings and involutive automorphisms of $\mathfrak{g}$.

These algebraic properties exhibited by the pair (G,K) (or the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$) turn out to characterize symmetric spaces. This is really useful, since it is often simpler to express a symmetric space as a quotient G/K where G is not $I^0(M)$ but rather a larger group. Although the description in terms of $G=I^0(M)$ is usually "nicer" to work with in a purely theoretical way, using a different one might be more convenient when one wants to make explicit computations. For example, it is rather common to express the complex projective space $\mathbb{C}P^n$ as the quotient $\mathsf{SU}_{n+1}/\mathsf{S}(\mathsf{U}_n\mathsf{U}_1)$, instead of taking $G=I^0(\mathbb{C}P^n)=\mathsf{SU}_{n+1}/\mathbb{Z}_{n+1}$ and $K=\mathsf{S}(\mathsf{U}_n\mathsf{U}_1)/\mathbb{Z}_{n+1}$. However, one should be careful when writing $M=G/K$, since different choices of G-invariant metrics on G/K could lead to different geometric properties. In order to avoid this unwanted behavior, it is common to take G and K to be a so-called symmetric pair:

Definition 1.14. Let G be a connected Lie group and K a closed subgroup of G. The pair (G,K) is called a Riemannian *symmetric pair* if

(a) K is a *symmetric subgroup* of G, i.e. there exists an involutive automorphism σ of G such that $(G_\sigma)_0\subseteq K\subseteq G_\sigma$, where G_σ is the set of fixed points of σ.

(b) Ad_K is a compact subgroup of the group of inner automorphisms of $\mathfrak{g}$, $\mathrm{Int}(\mathfrak{g})$.

Of course, it follows from the above discussion that, if M is a symmetric space, then $(I^0(M),I^0(M)_o)$ is an effective symmetric pair, and $M=I^0(M)/I^0(M)_o$. Conversely, given a symmetric pair (G,K), one may consider the homogeneous space G/K, with the natural smooth structure making $\pi\colon G\to G/K$ a submersion. Choose an involution σ of G satisfying condition (a) in Definition 1.14, and let $\theta=\sigma_{*e}$ be its differential. Then, the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ of $\mathfrak{g}$ into the

± 1-eigenspaces of θ is a reductive decomposition for G/K, and the differential of π allows us to identify $\mathfrak{p}$ with $T_o(G/K)$, where $o = eK$.

By (1.1), any choice of G-invariant metric $\langle \cdot, \cdot \rangle$ on G/K makes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a naturally reductive decomposition of G/K. Thus, we get the following formula for the curvature tensor at o:

$$R_o(X, Y)Z = -[[X, Y], Z], \tag{1.2}$$

for $X, Y, Z \in \mathfrak{p}$. Moreover, this choice of G-invariant metric turns G/K into a symmetric space.

Remark 1.15. For the sake of convenience, when we write that $M = G/K$ is a symmetric space from now on, we implicitly understand that (G, K) is a symmetric pair and M is isometric to G/K for some choice of G-invariant metric. We also assume that the action of G on G/K is almost effective.

The infinitesimal analog of symmetric pairs is given by the notion of *orthogonal symmetric Lie algebras*. Note that if (G, K) is a symmetric pair, the Lie algebra of Ad_K, $\mathrm{ad}_{\mathfrak{k}}$, is a compact subalgebra of $\mathrm{Int}(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$, and denote by $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ the connected subgroup of $\mathrm{Int}\,\mathfrak{g}$ with Lie algebra $\mathrm{ad}_{\mathfrak{k}}$. Then, $\mathfrak{k}$ is said to be compactly embedded in $\mathfrak{g}$ if $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ is compact. This condition guarantees the existence of an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product in $\mathfrak{g}$.

Definition 1.16. An *orthogonal symmetric Lie algebra* is a pair $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is a Lie algebra, θ is an involutive automorphism of $\mathfrak{g}$, and the set of fixed points of θ, $\mathfrak{k} = \mathrm{Fix}(\theta)$, is a compactly embedded subalgebra of $\mathfrak{g}$. Equivalently, one can think of θ as choosing a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$.

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and K is a connected closed subgroup of G with Lie algebra $\mathfrak{k}$, the pair (G, K) is said to be *a pair associated with* $(\mathfrak{g}, \theta)$. It follows immediately that a symmetric pair (G, K) is associated with its isometry Lie algebra $(\mathfrak{g}, \sigma_*)$ for any choice of σ satisfying condition (a) in Definition 1.14.

If (G, K) is a pair associated with an orthogonal symmetric Lie algebra $(\mathfrak{g}, \sigma_*)$, any $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product in $\mathfrak{g}$ can be extended to a G-invariant inner product on G/K. This turns G/K into a Riemannian locally symmetric space (see [54, Prop. 3.6]). Moreover, if $(\widetilde{G}, \widetilde{K})$ is another pair associated with the same algebra, and $\widetilde{G}$ is simply connected, then $(\widetilde{G}, \widetilde{K})$ is a symmetric pair. For the same choice of $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product in $\mathfrak{g}$, $\widetilde{G}/\widetilde{K}$ is the Riemannian universal cover of G/K.

1.2.2 Totally geodesic submanifolds and rank

Among submanifolds of a symmetric space M, the totally geodesic ones play a specially important role in several results regarding the structure of M. In particular, they allow us to define the rank of a symmetric space, which is perhaps its most important invariant.

Recall that a submanifold $S \subseteq M$ is said to be *totally geodesic* if the geodesics of S are also geodesics of M or, equivalently, its second fundamental form is identically 0. If M is a symmetric space, the geodesic symmetries of M about points of S restrict to geodesic symmetries of S, and so S is itself a symmetric space with the induced metric. After a suitable choice of basepoint (so that $o \in S$), we may identify the tangent space to S at o with a subspace $\mathfrak{s} \subseteq \mathfrak{p} \cong T_oM$. It follows from (1.2) that

$$[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}] \in \mathfrak{s}. \tag{1.3}$$

A subspace $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ satisfying the equation above is said to be a *Lie triple system*. The property of $\mathfrak{s} \cong T_oS$ being a Lie triple system turns out to completely characterize totally geodesic submanifolds of symmetric spaces: namely, if $\mathfrak{s} \subseteq \mathfrak{p}$ is a Lie triple system, $S = \exp_o(\mathfrak{s}) = \mathrm{Exp}(\mathfrak{s}) \cdot o$ is a totally geodesic submanifold of M, where exp and Exp are the Riemannian and Lie exponential maps, respectively. Thus, there is a one-to-one correspondence between complete connected totally geodesic submanifolds $S \subseteq M$ such that $o \in S$ and Lie triple systems $\mathfrak{s} \subseteq \mathfrak{p}$.

In fact, one can always recover S as an orbit of some subgroup of G: it follows immediately that if $\mathfrak{s} \subseteq \mathfrak{p}$ is a Lie triple system, then $\mathfrak{g}' = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ is a subalgebra of $\mathfrak{g}$. If one denotes by G' the connected subgroup of G with Lie algebra $\mathfrak{g}$, and K' the connected subgroup of G' with Lie algebra $[\mathfrak{s}, \mathfrak{s}]$, then $S = G' \cdot o$, and (G', K') is a symmetric pair such that $S = G'/K'$.

Let us now focus on the geodesic submanifolds of M that are flat with the induced metric, often simply referred to as *flats*. Under the previous correspondence, flats correspond to abelian subspaces $\mathfrak{a} \subseteq \mathfrak{p}$. A flat is said to be maximal if it is maximal with respect to inclusion. Any two maximal flats of M are congruent, and thus have the same dimension (the dimension of a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$). Moreover, maximal flats are properly embedded submanifolds of M. This allows us to define the *rank* of a symmetric space:

Definition 1.17. The *rank* of a symmetric space M is defined to be the dimension of a maximal flat, totally geodesic submanifold of M or, equivalently, the dimension of a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$.

1.2.3 Symmetric spaces and type

Symmetric spaces can be divided into three big families by imposing certain conditions on the sign of their curvature: namely, these are the symmetric spaces of compact, Euclidean and noncompact type. Symmetric spaces of a given type, and more specifically irreducible symmetric spaces, serve as building blocks for general symmetric spaces.

Let M be a symmetric space, and write $M = G/K$, where $G = I^0(M)$ and $K = G_o$ for some fixed o. Consider the Killing form $\mathcal{B}$ of $\mathfrak{g}$, given by

$$\mathcal{B}(X,Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$$

for $X, Y \in \mathfrak{g}$. Then, $\mathcal{B}$ is a symmetric bilinear form on $\mathfrak{g}$. Moreover, $\mathcal{B}$ is invariant under automorphisms. This is,

$$\mathcal{B}(\varphi X, \varphi Y) = \mathcal{B}(X,Y)$$

for $X, Y \in \mathfrak{g}$ and any Lie algebra automorphism φ of $\mathfrak{g}$. We have that $\mathcal{B}(\mathfrak{k}, \mathfrak{p}) = 0$, and so $\mathcal{B}$ restricts to an Ad_K-invariant symmetric bilinear form $\mathcal{B}_{|\mathfrak{p}\times\mathfrak{p}}$ on $\mathfrak{p}$. A symmetric space is said to be of *compact*, *Euclidean*, or *noncompact* type if the restriction of $\mathcal{B}$ to $\mathfrak{p}$ is negative definite, zero, or positive definite, respectively.

This type assumption on a symmetric space M imposes strong restrictions on both the curvature of M and its isometry group. Namely, if $G = I^0(M)$, we have the following cases:

- If M is of compact type, then G is a compact semisimple Lie group, and M is compact. Moreover, M has sectional curvature ≥ 0 everywhere.
- If M is of Euclidean type, it is flat.
- If M is of noncompact type, then G is a noncompact semisimple Lie group. Moreover, M has sectional curvature ≤ 0 everywhere.

If $M = G/K$ is irreducible, then both the restriction of the metric $\langle\cdot,\cdot\rangle$ to $\mathfrak{p} \cong T_o(M)$ and $\mathcal{B}_{|\mathfrak{p}\times\mathfrak{p}}$ are Ad_K-invariant inner products on $\mathfrak{p}$. Thus, Schur's Lemma guarantees that $\langle\cdot,\cdot\rangle = \lambda\mathcal{B}_{|\mathfrak{p}\times\mathfrak{p}}$, so M must be exactly of one of the three different types.

Remark 1.18. The notion of *type* can also be defined for an arbitrary orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Roughly speaking, $(\mathfrak{g}, \theta)$ is said to be of compact type if $\mathfrak{g}$ is a compact semisimple Lie algebra, and of noncompact type if $\mathfrak{g}$ is noncompact semisimple and θ is a Cartan involution of $\mathfrak{g}$ (see [54, Ch. V] or [86, §2.1.4] for a detailed discussion).

Suppose that M is a reducible symmetric space, and consider its universal cover $\widetilde{M}$, which is again a symmetric space. By the de Rham decomposition theorem, $\widetilde{M}$ admits a unique (up to isometry and permutation of the factors) decomposition as a product

$$\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k,$$

where $\widetilde{M}_0 \cong \mathbb{R}^m$ for some $m \in \mathbb{N}$ is a maximal Euclidean factor and each $\widetilde{M}_i$ is an irreducible symmetric space for $i \in \{1, \dots, k\}$. By grouping the factors of the same type, we get

Proposition 1.19. *Let M be a Riemannian symmetric space. Then, the universal cover of M splits as a Riemannian product*

$$\widetilde{M} = M_+ \times M_0 \times M_-$$

where M_+ is a symmetric space of compact type, $M_0 \cong \mathbb{R}^m$ for some $m \in \mathbb{N}$, and M_- is a symmetric space of noncompact type.

Remark 1.20. Let $M = M_0 \times M_1 \times \cdots \times M_k$ be a product of irreducible symmetric spaces, where M_0 is a maximal Euclidean factor, and write $M_i = G_i/K_i$ for $i = 0, \dots, k$. Then $(G, K) = (\prod G_i, \prod K_i)$ is a symmetric pair for M, and we have that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = (\mathfrak{k}_0 \oplus \cdots \oplus \mathfrak{k}_k) \oplus (\mathfrak{p}_0 \oplus \cdots \oplus \mathfrak{p}_k),$$

where $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. It follows from our previous discussion that the induced metric on $\mathfrak{p}$ can be expressed as

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{|\mathfrak{p}_0 \times \mathfrak{p}_0} + \lambda_1 \left(\mathcal{B}_1\right)_{|\mathfrak{p}_1 \times \mathfrak{p}_1} + \cdots + \lambda_k \left(\mathcal{B}_k\right)_{|\mathfrak{p}_k \times \mathfrak{p}_k},$$

where $\mathcal{B}_i = \mathcal{B}_{\mathfrak{g}_i}$ is the Killing form of the semisimple Lie algebra $\mathfrak{g}_i$. If M_i is of compact type, we have that $\lambda_i < 0$, whereas if M_i is of noncompact type, $\lambda_i > 0$.

Duality

An important connection between symmetric spaces of compact and noncompact type is given by the notion of duality. Let $M = G/K$ be a symmetric space of noncompact type, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition of its isometry Lie algebra (which is a Cartan decomposition of $\mathfrak{g}$). Consider the complexification of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$. Then, $\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{g}$ is a (noncompact) real form of $\mathfrak{g}$. The subspace $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie algebra, and taking G^* to be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and K^* the connected subgroup of G^* with Lie algebra $\mathfrak{k}$ yields a symmetric pair (G^*, K^*). The corresponding quotient $M^* = G^*/K^*$ is a simply connected symmetric space of compact type when equipped with the metric coming from the Killing form of $\mathfrak{g}^*$, and is known as the dual of M. This gives a one-to-one correspondence between homothethy classes of irreducible simply connected symmetric spaces of compact and of noncompact type.

1.3 Symmetric spaces of noncompact type and noncompact semisimple Lie algebras

In this thesis, the structure of symmetric spaces of noncompact type will be of special relevance both throughout Chapter 3 and Section 4.2. Symmetric spaces of noncompact type turn out to be closely related to the theory of root systems. By means of the Iwasawa decomposition of their isometry group, this will allow us to consider a symmetric space of noncompact type M as a simply connected, solvable Lie group with a left-invariant metric. More information on the structure of symmetric spaces of noncompact type can be found in [54, Ch. VI]. Eberlein's book [49] also includes a chapter on this topic, as well as an in-depth look at parabolic subgroups and subalgebras from a geometric viewpoint. Our main references on the topic of semisimple Lie algebras and their parabolic subalgebras are [58] and [76].

Let $M = G/K$ be a symmetric space of noncompact type and suppose that (G, K) is an almost effective symmetric pair. Then, $\theta = \sigma_{*e}$ is a Cartan involution of the noncompact real semisimple Lie algebra $\mathfrak{g}$, and the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is a Cartan decomposition for $\mathfrak{g}$. In particular, $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$, and K is a maximal compact subgroup of G. Consider the positive definite inner product on $\mathfrak{g}$ given by

$$\mathcal{B}_\theta(X, Y) = -\mathcal{B}(\theta X, Y), \quad \text{for } X, Y \in \mathfrak{g}.$$

Then, $\mathcal{B}_\theta$ is a positive definite inner product on $\mathfrak{g}$. It is easy to check that this inner product satisfies

$$\mathcal{B}_\theta(\operatorname{ad}_X Y, Z) = \mathcal{B}_\theta(Y, \operatorname{ad}_{\theta X} Z), \quad \text{for } X, Y, Z \in \mathfrak{g},$$

so it is $\operatorname{ad}_{\mathfrak{k}}$-invariant. In fact, $\mathcal{B}_\theta$ makes $\mathfrak{k}$ and $\mathfrak{p}$ orthogonal, and the restriction of $\mathcal{B}_\theta$ to $\mathfrak{p}$ (resp. $\mathfrak{k}$) coincides with the Killing form (resp. minus the Killing form).

The isotropy representation of M at o is polar and every maximal abelian subspace of $\mathfrak{p}$ is a section for this action. In particular, any two choices of maximal abelian subspaces $\mathfrak{a}, \mathfrak{a}' \subseteq \mathfrak{p}$ are conjugate under the adjoint action of K. From now, we will fix a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. Since $\mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{a}$ is abelian, the adjoint operators $\operatorname{ad}_H\colon \mathfrak{g} \to \mathfrak{g}$ with $H \in \mathfrak{a}$ form a family of commuting, self-adjoint endomorphisms of $\mathfrak{g}$. Therefore, the elements of $\{\operatorname{ad}_H\colon\ H \in \mathfrak{a}\}$ diagonalize simultaneously. Their common eigenspaces are called the *(restricted) root spaces*, whereas their nonzero eigenvalues (which depend linearly on $H \in \mathfrak{a}$) are called the *(restricted) roots*. In more detail, for every $\lambda \in \mathfrak{a}^*$, define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}\colon\ [H, X] = \lambda(H)X, \text{for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, we say that λ is a root of $\mathfrak{g}$, and $\mathfrak{g}_\lambda$ is its corresponding root space. We call $\dim \mathfrak{g}_\lambda$ the *multiplicity* of the root λ. The set of all roots of $\mathfrak{g}$ will be denoted by Σ. Let us now briefly mention some remarkable properties of root spaces:

- If $\lambda, \mu \in \Sigma \cup \{0\}$, and $\lambda \neq \mu$, then $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to $\mathcal{B}_\theta$.
- For every $\lambda, \mu \in \Sigma \cup \{0\}$, $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.
- The subspace $\mathfrak{g}_0$ splits as a direct sum $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} = N_{\mathfrak{k}}(\mathfrak{a}) = Z_{\mathfrak{k}}(\mathfrak{a})$ is both the normalizer and centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.
- For every $\lambda \in \Sigma \cup \{0\}$, $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$. In particular, λ is a root if and only if $-\lambda$ is.

The orthogonal decomposition of $\mathfrak{g}$ with respect to $\mathcal{B}_\theta$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right)$$

is called the *(restricted) root space decomposition* of $\mathfrak{g}$.

Remark 1.21. Note that the above definitions depend on the choice of Cartan involution θ and maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. However, it can be seen that different choices of θ and $\mathfrak{a}$ give rise to decompositions that are conjugate under the action of G. For simplicity, we will not deal with this nuance but suppose both θ and $\mathfrak{a}$ are fixed and point out whenever these choices become relevant.

Since the restriction of $\mathcal{B}_\theta$ to $\mathfrak{a} \times \mathfrak{a}$ is nondegenerate, it induces an isomorphism $\mathfrak{a} \cong \mathfrak{a}^*$. Namely, if $\phi \in \mathfrak{a}^*$, we denote by $H_\phi \in \mathfrak{a}$ the unique vector such that $\phi(H) = \mathcal{B}_\theta(H_\phi, H)$ for all $H \in \mathfrak{a}$. From now on, we always consider $\mathfrak{a}^*$ to be equipped with the inner product $\langle \cdot, \cdot \rangle$ obtained from $\mathcal{B}_\theta$ via this identification. With this inner product, Σ becomes a (possibly nonreduced) root system in $\mathfrak{a}^*$, that is:

(i) $\mathfrak{a}^*$ is spanned by Σ

(ii) For every $\lambda, \mu \in \Sigma$, the number $n_{\lambda,\mu} = 2\langle \lambda, \mu \rangle / \langle \lambda, \mu \rangle$ is an integer. The numbers $n_{\lambda,\mu}$ are known as the Cartan integers of Σ.

(iii) For every $\lambda, \mu \in \Sigma$, we have $\mu - n_{\lambda,\mu}\lambda \in \Sigma$.

Among the roots of Σ, it is now possible to define a notion of positivity: choose a hyperplane Π in $\mathfrak{a}^*$ such that it does not contain any root, and define those roots lying at one of the two half-spaces determined by Π to be *positive roots*, and the rest of them to be negative. The sets of positive and negative roots will be denoted by Σ^+ and Σ^-, respectively.

A choice of positive roots allows us to relate the Cartan and root space decompositions of $\mathfrak{g}$ in the following manner: write $\mathfrak{k}_\lambda = \pi_{\mathfrak{k}}(\mathfrak{g}_\lambda) = \mathfrak{k} \cap (\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_\lambda)$ and $\mathfrak{p}_\lambda = \pi_{\mathfrak{p}}(\mathfrak{g}_\lambda) = \mathfrak{p} \cap (\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_\lambda)$ where $\pi_{\mathfrak{k}}$ and $\pi_{\mathfrak{p}}$ are the orthogonal projections of $\mathfrak{g}$ onto $\mathfrak{k}$ and $\mathfrak{p}$ with respect to $\mathcal{B}_\theta$. Then

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \right), \quad \text{and} \quad \mathfrak{p} = \mathfrak{a} \oplus \left(\bigoplus_{\lambda \in \Sigma^+} \mathfrak{p}_\lambda \right).$$

Remark 1.22. The positive roots depend again on making a choice: namely, the hyperplane Π. Nevertheless, different choices of positive roots are always conjugate by inner automorphisms of $\mathfrak{g}$.

A positive root is called *simple* if it cannot be written as a sum of two positive roots. The set of simple roots, which will be denoted by Λ, is a basis of $\mathfrak{a}^*$. Thus, the number of simple roots coincides with the rank of M. We usually denote the set of simple roots of $\mathfrak{g}$ by $\{\alpha_1, \dots \alpha_r\}$, where $r = \operatorname{rank}(M)$.

Let $\lambda \in \Sigma$, and write $\lambda = \sum_{\alpha_i \in \Lambda} n_i \alpha_i$. Then, the coefficients n_i are all integers. Moreover, λ is a positive (resp. negative root) if and only if the n_i are all nonnegative (resp. nonpositive) integers. In particular, the sum of two positive (resp. negative) roots is a positive (resp. negative) root.

Remark 1.23. The simple roots, together with the multiplicities of the corresponding root spaces, turn out to completely determine the Lie algebra $\mathfrak{g}$, and thus the symmetric space M. This, together with the use of duality, is in fact how symmetric spaces are usually classified.

This information about simple roots is commonly displayed by means of the Dynkin diagram of Λ. This consists of a graph whose nodes are the simple roots. If the system is nonreduced (i.e. it has different collinear positive roots), two collinear simple roots are usually drawn as concentric circles. Any two roots are then joined by a simple (respectively, double, triple) edge whenever the angle between the two corresponding roots is $2\pi/3$ (respectively, $3\pi/4$, $5\pi/6$), with an arrow pointing to the root of shortest length. If both roots have the same length, the arrow is usually omitted.

From now on, fix a set of simple roots Σ^+, and consider the subspace of $\mathfrak{g}$ given by

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

Since $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ and the sum of positive roots is a positive root, it follows that $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$, and $\mathfrak{a}$ normalizes $\mathfrak{n}$. The following result, commonly known as the *Iwasawa decomposition theorem*, provides a way to express $\mathfrak{g}$ as a direct sum (of vector spaces) of the subalgebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$:

Proposition 1.24. *The Lie algebra* $\mathfrak{g}$ *decomposes as the vector space direct sum*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Note that the Iwasawa decomposition of $\mathfrak{g}$ is neither an orthogonal sum nor a semidirect sum. It is also not unique: as most objects described in this section, it depends on the choices we have made throughout the text (namely, the involution θ, maximal abelian subspace $\mathfrak{a}$, and the choice of positive roots).

The Iwasawa decomposition also admits an analog at the Lie group level. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$, and let A and N denote the connected subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Since $\mathfrak{a}$ normalizes $\mathfrak{n}$, the connected subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ is the semidirect product AN. The global version of the Iwasawa decomposition states that the multiplication map

$$K \times A \times N \to G, \qquad (k, a, n) \mapsto kan,$$

is an analytic diffeomorphism. Moreover, the Lie exponential maps of $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are also diffeomorphisms. In particular, the groups A, N and AN are simply connected.

The Iwasawa decomposition of the Lie group G also allows us to regard a symmetric space of noncompact type as a particular Lie group with a left invariant metric. In more detail, let $M = G/K$ be a symmetric space of noncompact type, and consider the map $\phi\colon G \to M$, $g \mapsto g \cdot o$. Then, the restriction $\phi_{|AN}$ of ϕ to AN is a diffeomorphism. If $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on M, and one considers its pullback to AN, the corresponding metric $\langle \cdot, \cdot \rangle_{AN} = (\phi_{|AN})^* \langle \cdot, \cdot \rangle$ turns out to be left invariant. Thus, we can naturally identify $(M, \langle \cdot, \cdot \rangle)$ with the Lie group $(AN, \langle \cdot, \cdot \rangle_{AN})$ with a left-invariant metric. This provides what is commonly known as the *solvable model* of a symmetric space of noncompact type. In particular, a symmetric space of noncompact type is diffeomorphic to a Euclidean space, and, since any symmetric space of noncompact type must have nonpositive curvature, it must be a Hadamard manifold.

1.3.1 Parabolic subgroups and subalgebras

The lack of topological restrictions allows the isometry group of a symmetric space of noncompact type to have a large lattice of subgroups. Here, we consider an important family of them: the so-called parabolic subgroups. These groups play a crucial role in the construction of cohomogeneity one actions on symmetric spaces of noncompact type, as it will be apparent in Chapters 2 and 3. Although we will mainly use an algebraic description of them, parabolic subgroups can be introduced in a very geometric manner, which we briefly discuss here. For a reader interested in this geometric point of view, we refer to [49]. An extensive discussion about parabolic subalgebras can be found in [58, Ch. VII].

Since a symmetric space of noncompact type M is a Hadamard manifold, we may think of it as an open Euclidean ball endowed with a certain metric. This allows us to consider M as subset of a bigger compact topological space $\overline{M}$ that also encodes the geometry of M "at infinity". For instance, this is common practice when one thinks of the hyperbolic plane $\mathbb{R}\mathsf{H}^2$ as the Poincaré disk model.

Let M be a symmetric space of noncompact type, and let γ_1, γ_2 be unit speed geodesics in M. Then, γ_1 and γ_2 are said to be *asymptotic* if there exists a positive constant C such that $d(\gamma_1(t), \gamma_2(t)) \leq C$ for all $t \in \mathbb{R}$, where d is the Riemannian distance function. The "being asymptotic" relation is an equivalence relation on the unit speed geodesics of M. Each equivalence class for this relation is called a *point at infinity* of M. The set of all points at infinity of M is denoted by $M(\infty)$, and is often called the *ideal boundary* of M. By endowing $\overline{M} = M \sqcup M(\infty)$ with the so-called cone topology, $\overline{M}$ becomes a compact topological space homeomorphic to a closed ball where M corresponds to the interior and $M(\infty)$ to the boundary. Note that isometries of M send asymptotic geodesics to asymptotic geodesics, so any isometric action on M naturally extends to an action on $\overline{M}$.

Definition 1.25. Let $M = G/K$ be a symmetric space of noncompact type. A subgroup $Q \subseteq G$ is called a *parabolic subgroup* of G if $Q = G$ or $Q = G_x$ is the stabilizer in G of some point at infinity $x \in M(\infty)$.

Let us now forget about the geometry of M and the points at infinity and take a look at parabolic subgroups from a purely algebraic point of view. As it is becoming usual, we start at the level of Lie algebras.

Definition 1.26. A subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is called a *Borel subalgebra* if it is maximal among solvable subalgebras of $\mathfrak{g}$. A subalgebra $\mathfrak{q}$ of a real semisimple Lie algebra $\mathfrak{g}$ is said to be *parabolic* if its complexification $\mathfrak{q}_{\mathbb{C}}$ contains some Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$.

This definition, which *a priori* seems detached from the corresponding notion for Lie groups, is justified by the following:

Proposition 1.27. *Let $M = G/K$ be a symmetric space of noncompact type, and let Q be a parabolic subgroup of G. Then, the Lie algebra $\mathfrak{q}$ of Q is a parabolic subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$, its normalizer in G, $N_G(\mathfrak{q})$, is a parabolic subgroup.*

Remark 1.28. In some texts (especially those focusing on the Lie algebra point of view) parabolic subgroups are sometimes defined not as above, but as the connected subgroups of G corresponding to parabolic Lie algebras. This is done because it is usually nicer to work with connected Lie groups, and stabilizers of points at infinity need not be connected. However, it should be noted that for any two points $x, y \in M(\infty)$, their stabilizers G_x and G_y are isomorphic if and only if $(G_x)^0$ and $(G_y)^0$ are. Moreover, G_x always has finitely many components and coincides with its normalizer.

Parabolic subalgebras can be readily built from a choice of simple roots. Indeed, for any subset $\Phi \subseteq \Lambda$ of simple roots, we can consider the root subsystem of Σ generated by Φ, $\Sigma_\Phi = \Sigma \cap \operatorname{span}\{\Phi\}$. Choose a set of positive roots Σ^+, and denote by $\Sigma_\Phi^+ = \Sigma^+ \cap \Sigma_\Phi$ the corresponding positive roots of Σ_Φ. Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{g}_\lambda \right), \qquad \mathfrak{n}_\Phi = \bigoplus_{\lambda \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\lambda,$$

which are reductive and nilpotent, respectively.

Remark 1.29. Throughout this text, when we say that $\mathfrak{l}$ is a reductive subalgebra of a real semisimple Lie algebra $\mathfrak{g}$, we understand that $\mathfrak{l}$ is θ-invariant (where θ is the Cartan involution) or, equivalently, $\mathfrak{l} = (\mathfrak{k} \cap \mathfrak{l}) \oplus (\mathfrak{p} \cap \mathfrak{l})$ for some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This implies that $\mathfrak{l}$ is reductive in the sense that $\operatorname{ad}|_{\mathfrak{h}} \colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ is completely reducible [21, §6] and hence, in particular, $\mathfrak{l}$ is a reductive Lie algebra. If G is a real semisimple Lie group, we say that a Lie subgroup L of G is a reductive subgroup if its Lie algebra $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$. If L is a reductive subgroup of G, the orbit through the base point o that determines the Cartan decomposition is totally geodesic, since $\mathfrak{p} \cap \mathfrak{l}$ is a Lie triple system.

It follows from the properties of root spaces that $[\mathfrak{l}_\Phi, \mathfrak{n}_\Phi] \subseteq \mathfrak{n}_\Phi$. Thus,

$$\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$$

is a subalgebra of $\mathfrak{g}$ which can be seen to be parabolic, since it contains the minimal parabolic subalgebra $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$. The subalgebra $\mathfrak{q}_\Phi$ is called the parabolic subalgebra of $\mathfrak{g}$ associated with the set of simple roots Φ, and the decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is known as the Chevalley decomposition of $\mathfrak{q}_\Phi$.

It turns out that the subalgebras $\mathfrak{q}_\Phi$ constructed in this manner provide representatives for all conjugacy classes of parabolic subalgebras of $\mathfrak{g}$:

Proposition 1.30. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then, every parabolic subalgebra of $\mathfrak{g}$ is conjugate by an inner automorphism to $\mathfrak{q}_\Phi$ for some choice of simple roots $\Phi \subseteq \Lambda$.*

Let $\mathfrak{a}_\Phi = \bigcap_{\alpha\in\Phi} \ker\alpha$ be the split component of $\mathfrak{l}_\Phi$, and let $\mathfrak{a}^\Phi = \mathfrak{a} \ominus \mathfrak{a}_\Phi = \bigoplus_{\alpha\in\Phi} \mathbb{R}H_\alpha$ be its orthogonal complement[2] with respect to $\mathcal{B}_\theta$ (note that the direct sum here is not necessarily orthogonal). Then, $\mathfrak{l}_\Phi$ coincides with both the centralizer and normalizer of $\mathfrak{a}_\Phi$ in $\mathfrak{g}$, and $\mathfrak{a}_\Phi$ normalizes $\mathfrak{n}_\Phi$.

We can define another reductive subalgebra $\mathfrak{m}_\Phi$ of $\mathfrak{g}$ by $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, which normalizes $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. The decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$$

is called the Langlands decomposition of the parabolic subalgebra $\mathfrak{q}_\Phi$.

Remark 1.31. Note that if $\Psi \subseteq \Phi \subseteq \Lambda$ are subsets of simple roots, then $\mathfrak{q}_\Psi \subseteq \mathfrak{q}_\Phi$. In particular, maximal proper parabolic subalgebras of $\mathfrak{g}$ correspond to subsets of simple roots with cardinality $|\Phi| = \dim\mathfrak{a} - 1$.

Recall that for $\lambda \in \Sigma$, we have defined

$$\begin{aligned}\mathfrak{k}_\lambda &= \pi_\mathfrak{k}(\mathfrak{g}_\lambda) = \mathfrak{k} \cap (\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_\lambda),\\ \mathfrak{p}_\lambda &= \pi_\mathfrak{p}(\mathfrak{g}_\lambda) = \mathfrak{p} \cap (\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_\lambda),\end{aligned}$$

where $\pi_\mathfrak{k}$ and $\pi_\mathfrak{p}$ are the orthogonal projection maps onto $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Consider the following subspaces of $\mathfrak{q}_\Phi$

$$\mathfrak{k}_\Phi = \mathfrak{q}_\Phi \cap \mathfrak{k} = \mathfrak{l}_\Phi \cap \mathfrak{k} = \mathfrak{m}_\Phi \cap \mathfrak{k} = \mathfrak{k}_0 \oplus \left(\bigoplus_{\lambda\in\Sigma_\Phi^+} \mathfrak{k}_\lambda \right),$$

$$\mathfrak{b}_\Phi = \mathfrak{m}_\Phi \cap \mathfrak{p} = \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\lambda\in\Sigma_\Phi^+} \mathfrak{p}_\lambda \right).$$

Then, $\mathfrak{k}_\Phi$ is a compact subalgebra of $\mathfrak{q}_\Phi$, and $\mathfrak{b}_\Phi$ is a Lie triple system in $\mathfrak{p}$, and we have that $[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subseteq \mathfrak{m}_\Phi$, $[\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = 0$ and $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subseteq \mathfrak{n}_\Phi$.

Since $\mathfrak{b}_\Phi$ is a Lie triple system, it corresponds to the tangent space at o of some connected totally geodesic submanifold B_Φ of M. Associated with $\mathfrak{b}_\Phi$ one can consider the semisimple Lie algebra $\mathfrak{s}_\Phi = [\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi$, where $[\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \subset \mathfrak{k}_\Phi$. Then,

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[2] Throughout this text, if V is a subspace of an inner product space $(W, \langle\cdot,\cdot\rangle)$, we denote by $W \ominus V$ the orthogonal complement of V in W for the sake of convenience.

$\mathfrak{s}_\Phi = [\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi$ is a Cartan decomposition of $\mathfrak{s}_\Phi$, and $\mathfrak{a}^\Phi$ is a maximal abelian subspace of $\mathfrak{b}_\Phi$. Moreover, the set $\Sigma_\Phi|_{\mathfrak{a}^\Phi} = \{\lambda|_{\mathfrak{a}^\Phi} : \lambda \in \Sigma_\Phi\}$ is a root system for $\mathfrak{s}_\Phi = [\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi$ with respect to the maximal abelian subspace $\mathfrak{a}^\Phi$ of $\mathfrak{b}_\Phi$. Since $\lambda|_{\mathfrak{a}_\Phi} = 0$ for each $\lambda \in \Sigma_\Phi$, the restriction map $\Sigma_\Phi \to \Sigma_\Phi|_{\mathfrak{a}^\Phi}$ is bijective. Thus we can naturally identify Σ_Φ (resp. Φ) with a root system for $\mathfrak{s}_\Phi$ (resp. with a set of simple roots for $\mathfrak{s}_\Phi$) simply by restricting the roots to $\mathfrak{a}^\Phi$. We will implicitly do this identification in what follows. For example, if $\lambda \in \Sigma_\Phi$, the root space $(\mathfrak{s}_\Phi)_\lambda = (\mathfrak{s}_\Phi)_{\lambda|_{\mathfrak{a}^\Phi}}$ of $\mathfrak{s}_\Phi$ coincides with the root space $\mathfrak{g}_\lambda$ of $\mathfrak{g}$, and the root space $(\mathfrak{s}_\Phi)_0$ of $\mathfrak{s}_\Phi$ corresponding to the 0-weight is $(\mathfrak{s}_\Phi)_0 = \mathfrak{s}_\Phi \cap \mathfrak{g}_0 = (\mathfrak{s}_\Phi \cap \mathfrak{k}_0) \oplus \mathfrak{a}^\Phi$. In particular, we have the root space decomposition

$$\mathfrak{s}_\Phi = (\mathfrak{s}_\Phi)_0 \oplus \bigoplus_{\lambda \in \Sigma_\Phi} (\mathfrak{s}_\Phi)_\lambda = (\mathfrak{s}_\Phi \cap \mathfrak{k}_0) \oplus \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{g}_\lambda \right).$$

We can now lift these Lie algebras and decompositions to the Lie group level. First of all, write A_Φ, N_Φ and S_Φ for the connected subgroups of G with Lie algebras $\mathfrak{a}_\Phi$, $\mathfrak{n}_\Phi$ and $\mathfrak{s}_\Phi$, respectively. Then, A_Φ is abelian and N_Φ nilpotent. If we define the reductive group $L_\Phi = Z_G(\mathfrak{a}_\Phi)$ as the centralizer of $\mathfrak{a}_\Phi$ in G, then L_Φ normalizes N_Φ. The group $Q_\Phi = L_\Phi \ltimes N_\Phi$ coincides with the normalizer $N_G(\mathfrak{q}_\Phi)$ of $\mathfrak{q}_\Phi$, and so it is called the parabolic subgroup of G associated with the subset Φ of Λ. Note that both L_Φ and Q_Φ are not necessarily connected.

We also define $K_\Phi = L_\Phi \cap K = Z_K(\mathfrak{a}_\Phi)$ and $M_\Phi = K_\Phi S_\Phi$. Then M_Φ is a (possibly disconnected) closed reductive subgroup of L_Φ, K_Φ is a maximal compact subgroup of M_Φ, and $L_\Phi = M_\Phi \times A_\Phi$. The orbit $S_\Phi \cdot o$ of the S_Φ-action on $M = G/K$ through o is the totally geodesic submanifold B_Φ of M with $T_o B_\Phi \cong \mathfrak{b}_\Phi$. B_Φ is itself a symmetric space of noncompact type whose rank agrees with the cardinality $|\Phi|$ of Φ, and is called the *boundary component* (or boundary symmetric space) of M associated with $\Phi \subseteq \Lambda$ in the context of the maximal Satake compactification of M (see [20]). Moreover,

$$B_\Phi = S_\Phi \cdot o = M_\Phi \cdot o \cong M_\Phi / K_\Phi \cong S_\Phi / (S_\Phi \cap K_\Phi).$$

Since $\mathfrak{s}_\Phi$ is θ-invariant and S_Φ is connected, $(S_\Phi, S_\Phi \cap K) = (S_\Phi, S_\Phi \cap K_\Phi)$ is a symmetric pair, and in particular, $\mathfrak{s}_\Phi$ is the Lie algebra of the isometry group of B_Φ.

The decompositions $Q_\Phi = L_\Phi \ltimes N_\Phi$ and $(M_\Phi \times A_\Phi) \ltimes N_\Phi$ are called the (global) Chevalley and Langlands decompositions of Q_Φ. The Langlands decomposition of Q_Φ induces a diffeomorphism

$$A_\Phi \times N_\Phi \times B_\Phi \to M, \quad (a, n, m \cdot o) \mapsto (anm) \cdot o,$$

known as the horospherical decomposition of the symmetric space M. Indeed, the action of $A_\Phi N_\Phi$ on M turns out to be free, polar with section B_Φ, and with mutually congruent minimal orbits [41,90]. The section B_Φ meets each orbit exactly once (and it does so orthogonally, by polarity).

Let $A^\Phi N^\Phi$ be the connected subgroup of AN with Lie algebra $\mathfrak{a}^\Phi \oplus \mathfrak{n}^\Phi$, where

$$\mathfrak{n}^\Phi = \bigoplus_{\lambda\in\Sigma_\Phi^+} \mathfrak{g}_\lambda.$$

Then $B_\Phi = (A^\Phi N^\Phi)\cdot o$, since the solvable part of the Iwasawa decomposition of the real semisimple Lie algebra $\mathfrak{s}_\Phi$ is precisely $\mathfrak{a}^\Phi \oplus \mathfrak{n}^\Phi$. Similarly, the connected subgroup AN^Φ of AN with Lie algebra $\mathfrak{a}\oplus\mathfrak{n}^\Phi$ acts transitively on the totally geodesic submanifold $F_\Phi = L_\Phi \cdot o \cong (A_\Phi\cdot o)\times B_\Phi$ with Lie triple system $\mathfrak{a}_\Phi \oplus \mathfrak{b}_\Phi$.

Later in Chapter 3, we will need to consider parabolic subalgebras of a boundary component $B_\Phi = S_\Phi/(S_\Phi\cap K_\Phi)$. For this, it will be useful to look at the relation between the parabolic subalgebras of $\mathfrak{s}_\Phi$ and the parabolic subalgebras of $\mathfrak{g}$. If $\Psi \subseteq \Phi \subseteq \Lambda$, we have the following inclusions of boundary components: $B_\Psi \subseteq B_\Phi \subseteq B_\Lambda = M$. By $\mathfrak{q}_{\Psi,\Phi}$ we denote the parabolic subalgebra of $\mathfrak{s}_\Phi$ associated with the subset Ψ of the set Φ of simple roots of $\mathfrak{s}_\Phi$. The corresponding Chevalley and Langlands decompositions can then be written as

$$\mathfrak{q}_{\Psi,\Phi} = \mathfrak{l}_{\Psi,\Phi} \oplus \mathfrak{n}_{\Psi,\Phi} = \mathfrak{m}_{\Psi,\Phi} \oplus \mathfrak{a}_{\Psi,\Phi} \oplus \mathfrak{n}_{\Psi,\Phi},$$

where

$$\begin{aligned}
\mathfrak{l}_{\Psi,\Phi} &= (\mathfrak{s}_\Phi)_0 \oplus \Big(\bigoplus_{\lambda\in\Sigma_\Psi} \mathfrak{g}_\lambda\Big),\\
\mathfrak{n}_{\Psi,\Phi} &= \bigoplus_{\lambda\in\Sigma_\Phi^+\setminus\Sigma_\Psi^+} \mathfrak{g}_\lambda = \mathfrak{n}^\Phi \cap \mathfrak{n}_\Psi,\\
\mathfrak{a}_{\Psi,\Phi} &= \bigcap_{\alpha\in\Psi} \ker \alpha|_{\mathfrak{a}^\Phi} = \mathfrak{a}^\Phi \ominus \Big(\bigoplus_{\alpha\in\Psi}\mathbb{R}H_\alpha\Big) = \mathfrak{a}^\Phi \cap \mathfrak{a}_\Psi,
\end{aligned}$$

and $\mathfrak{m}_{\Psi,\Phi} = \mathfrak{l}_{\Psi,\Phi} \ominus \mathfrak{a}_{\Psi,\Phi}$. In particular, we have $\mathfrak{q}_{\Psi,\Phi} = \mathfrak{q}_\Psi \cap \mathfrak{s}_\Phi$. We also define

$$\mathfrak{k}_{\Psi,\Phi} = \mathfrak{k}\cap \mathfrak{l}_{\Psi,\Phi} = \mathfrak{k}_\Psi \cap \mathfrak{s}_\Phi,$$

and the (possibly disconnected) subgroups $L_{\Psi,\Phi} = Z_{S_\Phi}(\mathfrak{a}_{\Psi,\Phi})$, $K_{\Psi,\Phi} = L_{\Psi,\Phi}\cap K$ and $M_{\Psi,\Phi} = K_{\Psi,\Phi} S_\Psi$ of S_Φ, whose respective Lie algebras are $\mathfrak{l}_{\Psi,\Phi}$, $\mathfrak{k}_{\Psi,\Phi}$, and $\mathfrak{m}_{\Psi,\Phi}$.

Chapter 2

The classification problem for homogeneous hypersurfaces of symmetric spaces

The first classification results for homogeneous hypersurfaces can be attributed to classical work of Levi-Civita [67] and Segre [81] on isoparametric hypersurfaces in Euclidean spaces, and to the work of Cartan [26] for real hyperbolic spaces. Similar results in spaces of positive curvature had to wait until the 1970's, when Hsiang and Lawson [55] and Takagi and Takahashi [88] achieved the classification for round spheres.

Recall that the classification of homogeneous hypersurfaces up to isometric congruence in a given ambient space M and the classification of cohomogeneity one actions on M up to orbit equivalence is, essentially, the same problem. Thus, we will often treat both objects indistinctly, and work with the one which is more convenient at each moment.

For general symmetric spaces, this classification problem is still an ongoing effort. This chapter provides an exposition about the different advances towards solving such problem. This is done, largely, because many of the results regarding cohomogeneity one actions on symmetric spaces of noncompact type presented here will be necessary throughout the different proofs of Chapters 3 and 4. It should be noted that we will not introduce these results in a chronological order for the sake of convenience: for example, the methods described in Section 2.3 for spaces of arbitrary rank largely precedes the classification of homogeneous hypersurfaces in the quaternionic hyperbolic spaces $\mathbb{H}\mathsf{H}^n$ from Section 2.2.

A more detailed treatment of the results displayed in this chapter (as well as the results from Chapter 3) can be found in [38], an expository chapter which was written with the collaboration of José Carlos Díaz Ramos (CITMAga - Universidade de Santiago de Compostela) and Miguel Domínguez Vázquez (CITMAga - Universidade de Santiago de Compostela). This text was composed during the development of this thesis, and also provides a broad look at homogeneous hypersurfaces in symmetric spaces and some related concepts. Chapter 3 in [80] also includes a nice overview of the topic.

In Section 2.1 we present the classification problem of homogeneous hypersurfaces of symmetric spaces of compact type. We start by dealing with the rank one case: namely, the spheres and the projective spaces over the normed division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We then move on to the classification result by Kollross [60] for irreducible symmetric spaces of compact type. Section 2.2 is devoted to report on the classification of homogeneous hypersurfaces of hyperbolic spaces, recently completed in [39]. Finally, Section 2.3 introduces several results regarding cohomogeneity one actions on symmetric spaces of noncompact type, which will serve as the building blocks for the structural results of Chapters 3 and 4.

2.1 Cohomogeneity one actions on symmetric spaces of compact type

Let $M \cong G/K$ be a symmetric space, where $G = I^0(M)$ and K is the isotropy of G at some point $o \in M$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the reductive decomposition of $\mathfrak{g}$ induced by the geodesic reflection about o. Recall that the isotropy representation $k \in K \mapsto k_{*o} \in \mathsf{O}(T_oM)$ can be identified with the adjoint action of K on $\mathfrak{p}$, that is, $K \times \mathfrak{p} \to \mathfrak{p}$, $k \cdot X = \operatorname{Ad}_k X$. Any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ turns out to intersect all the orbits of the isotropy representation, and always perpendicularly (see [9, §2.3.2] for a proof). Actions obtained this way, with G semisimple, are called s-representations, and in a sense provide the general way to obtain polar representations (i.e. linear polar actions), as the following result of Dadok [31] points out:

Theorem 2.1. *Every polar representation is orbit equivalent to an s-representation.*

2.1.1 Homogeneous hypersurfaces in spheres

Since K consists of isometries of M, s-representations restrict to polar actions on the unit sphere $\mathbb{S}^{\dim M-1}$ of $T_oM \cong \mathfrak{p}$ with section $\mathfrak{a} \cap \mathbb{S}^{\dim M-1}$. In particular, if $\operatorname{rank} M = 2$, this procedure provides a way to construct examples of cohomogeneity one actions on spheres. By Hsiang and Lawson's theorem [55], these exhaust all cohomogeneity one actions on $\mathbb{S}^n$, up to orbit equivalence:

Theorem 2.2. *Any homogeneous hypersurface of a round sphere is congruent to a principal orbit of the action obtained by restriction to the unit sphere of the isotropy representation of a symmetric space of rank two.*

The spaces giving rise to these actions are:

$$\begin{array}{ccccc} \mathbb{S}^n \times \mathbb{S}^m, & \mathsf{SU}_3/\mathsf{SO}_3, & \mathsf{SU}_3, & \mathsf{SU}_6/\mathsf{Sp}_3, & \mathsf{E}_6/\mathsf{F}_4, \\ \mathsf{Sp}_2, & \mathsf{SO}_{10}/\mathsf{U}_5, & \mathsf{G}_2^+(\mathbb{R}^{k+2}), & \mathsf{G}_2(\mathbb{C}^{k+2}), & \mathsf{G}_2(\mathbb{H}^{k+2}), \\ & \mathsf{E}_6/\mathsf{U}_1\mathsf{Spin}_{10}, & \mathsf{G}_2/\mathsf{SO}_4, \quad \text{and} & \mathsf{G}_2. & \end{array}$$

The next natural step is to classify homogeneous hypersurfaces in the rest of (simply connected) symmetric spaces of compact type and rank one: these are the projective spaces $\mathbb{C}\mathsf{P}^n = \mathsf{SU}_n/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_{n-1})$, $\mathbb{H}\mathsf{P}^n = \mathsf{Sp}_n/(\mathsf{Sp}_1 \times \mathsf{Sp}_{n-1})$ $(n \geq 2)$, and $\mathbb{O}\mathsf{P}^2 = \mathsf{F}_4/\mathsf{Spin}_9$.

2.1.2 Homogeneous hypersurfaces in complex projective spaces

The classification problem for the complex projective space $\mathbb{C}\mathsf{P}^n$ was solved by Takagi [89]. In a similar way to Theorem 2.2, one gets:

Theorem 2.3. *A hypersurface in $\mathbb{C}\mathsf{P}^n$ is homogeneous if and only if it is congruent to the quotient, via the Hopf fibration, of a principal orbit of the restriction of the isotropy representation of a Hermitian symmetric space G/K of rank two to $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$, where $2n+2 = \dim G/K$.*

The symmetric spaces giving rise to such homogeneous hypersurfaces are:

$$\mathbb{C}\mathsf{P}^{k+1} \times \mathbb{C}\mathsf{P}^{n-k}, \quad \mathsf{G}_2^+(\mathbb{R}^n + 3), \quad \mathsf{G}_2^+(\mathbb{C}^n + 3), \quad \mathsf{SO}_{10}/\mathsf{U}_5 \quad \text{and} \quad \mathsf{E}_6/\mathsf{U}_1\mathsf{Spin}_{10}.$$

Here $\mathsf{G}_k(\mathbb{F}^m)$ denotes the Grassmannian of k-planes in $\mathbb{F}^m$, and $\mathsf{G}_k^+(\mathbb{F}^m)$ the oriented Grassmannian of k-planes in $\mathbb{F}^m$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

If $M \cong G/K$ is a Hermitian symmetric space of rank two (here, Hermitian means that M has a complex structure that is invariant under each geodesic symmetry), the elements of K act as linear holomorphic isometries of $T_{eK}G/K \cong \mathbb{C}^{n+1}$, and so the isotropy representation of M can be restricted to an action on the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. As discussed before, this action on the unit sphere is of cohomogeneity one. Since this action maps complex lines of $\mathbb{C}^{n+1}$ to complex lines of $\mathbb{C}^{n+1}$, it descends to a cohomogeneity one action on the projectivization $\mathsf{P}(\mathbb{C}^{n+1}) \cong \mathbb{C}\mathsf{P}^n$ via the Hopf map $\pi \colon \mathbb{S}^{2n+1} \to \mathbb{C}P^n$.

The isotropy representation of the reducible symmetric space $\mathbb{C}\mathsf{P}^{k+1} \times \mathbb{C}\mathsf{P}^{n-k} = (\mathsf{SU}_{k+2} \times \mathsf{SU}_{n-k+1})/(\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_{k+1}) \times \mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_{n-k}))$ induces the action of $\mathsf{U}_{k+1} \times \mathsf{U}_{n-k}$ on $\mathbb{C}\mathsf{P}^n$, which has totally geodesic $\mathbb{C}\mathsf{P}^{k+1}$ and $\mathbb{C}\mathsf{P}^{n-k}$ as singular orbits. The regular orbits are tubes around them. If $k = 0$, we recover geodesic spheres.

The real oriented two-plane Grassmannian $\mathsf{G}_2^+(\mathbb{R}^{n+3}) = \mathsf{SO}_{n+3}/\mathsf{SO}_2 \times \mathsf{SO}_{n+1}$ induces an action of SO_{n+1} on $\mathbb{C}\mathsf{P}^n$ with two singular orbits: a totally geodesic real projective space $\mathbb{R}\mathsf{P}^n$, and a complex quadric $\mathsf{Q}^{n-1} \cong \mathsf{SO}_{n+1}/\mathsf{SO}_2\mathsf{SO}_{n-1}$. Similarly, the complex two-plane Grassmannian $\mathsf{G}_2(\mathbb{C}^{k+3}) = \mathsf{SU}_{k+3}/\mathsf{S}(\mathsf{U}_2\mathsf{U}_{k+1})$ induces an action on $\mathbb{C}\mathsf{P}^{2k+1}$, one of whose singular orbits is the Segre embedding of $\mathbb{C}\mathsf{P}^1 \times \mathbb{C}\mathsf{P}^k$ in $\mathbb{C}\mathsf{P}^{2k+1}$.

The isotropy representation of $\mathsf{SO}_{10}/\mathsf{U}_5$ induces an action with the Plücker embedding of the complex Grassmannian $\mathsf{G}_2(\mathbb{C}^5)$ into $\mathbb{C}\mathsf{P}^9$ as a singular orbit. Finally, $\mathsf{E}_6/\mathsf{U}_1\mathsf{Spin}_{10}$ induces a cohomogeneity one action on $\mathbb{C}\mathsf{P}^{15}$ one of whose singular orbits is the half spin embedding of the symmetric space $\mathsf{SO}_{10}/\mathsf{U}_5$. We refer the reader to [27, §7] for details about such embeddings.

2.1.3 Homogeneous hypersurfaces in quaternionic projective spaces

The classification problem in quaternionic projective space is attributed to D'Atri [32] and Iwata [56].

Theorem 2.4. *A hypersurface in $\mathbb{H}\mathsf{P}^n$ is homogeneous if and only if it is congruent to the quotient, via the quaternionic Hopf fibration, of a principal orbit of the restriction of the isotropy representation of one of the following symmetric spaces of rank two to $\mathbb{S}^{4n+3} \subseteq \mathbb{H}^{n+1}$:*

$$\mathbb{H}\mathsf{P}^{k+1} \times \mathbb{H}\mathsf{P}^{n-k}, \quad \mathsf{SU}_{n+3}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_{n+1}),$$

where $k = 0, \dots, n-2$.

Remark 2.5. Although this may seem like a quaternionic-Kähler analogue of Theorem 2.3, one should note that the product of quaternionic-Kähler spaces is not necessarily quaternionic-Kähler.

The isotropy action of $\mathbb{H}\mathsf{P}^{k+1} \times \mathbb{H}\mathsf{P}^{n-k}$ induces a cohomogeneity one action of $\mathsf{Sp}_{k+1} \times \mathsf{Sp}_{n-k}$ on $\mathbb{H}\mathsf{H}^n$, which has totally geodesic $\mathbb{H}\mathsf{P}^{k+1}$ and $\mathbb{H}\mathsf{P}^{n-k}$ as singular orbits. If $k = 0$, we retrieve geodesic spheres. The space $\mathsf{SU}_{n+3}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_{n+1})$ induces an action of U_{n+1} on $\mathbb{H}\mathsf{P}^n$, whose principal orbits are tubes around a totally geodesic $\mathbb{C}\mathsf{P}^n$.

2.1.4 Homogeneous hypersurfaces in the Cayley projective plane

We finish our review of homogeneous hypersurfaces in rank one symmetric spaces of compact type recalling the classification result for the Cayley projective plane given by Iwata [57].

Theorem 2.6. *A homogeneous hypersurface in the Cayley projective plane $\mathbb{O}\mathsf{P}^2$ is congruent to a geodesic sphere, or a tube around a totally geodesic $\mathbb{H}\mathsf{P}^2$ in $\mathbb{O}\mathsf{P}^2$.*

A geodesic sphere can be seen as a principal orbit of the isotropy action of Spin_9 on $\mathbb{O}\mathsf{P}^2$. This action has two singular orbits: a fixed point and a totally geodesic $\mathbb{S}^8 = \mathbb{O}\mathsf{P}^1$. The second example in this classification is congruent to a principal orbit of the action of $\mathsf{Sp}_3\mathsf{Sp}_1$, which has two singular orbits: a totally geodesic $\mathbb{H}\mathsf{P}^2$ and a minimal $\mathbb{S}^{11} = \mathsf{Sp}_3\mathsf{Sp}_1/\mathsf{Sp}_2\mathsf{Sp}_1$. As pointed out by Iwata, there are two more groups, up to conjugation, with the same orbits as $\mathsf{Sp}_3\mathsf{Sp}_1$. These are $\mathsf{Sp}_3\mathsf{U}_1$ and Sp_3. Unlike the results presented here, Iwata's classification was obtained up to conjugation by an element of F_4, not up to orbit equivalence.

2.1.5 Homogeneous hypersurfaces in symmetric spaces of compact type and higher rank

The above results were generalized by Kollross [60] to arbitary rank. In order to introduce his main theorem, let us briefly introduce the notion of Hermann actions.

Let G be a compact semisimple Lie group, and H, K be closed symmetric subgroups of G. Then, both (G, H) and (G, K) are symmetric pairs and the corresponding quotients G/H and G/K are symmetric spaces of compact type. A *Hermann action* on G is the action of a product group $H \times K$ given by

$$(h, k) \cdot g = hgk^{-1},$$

for $h \in H$, $k \in K$, and $g \in G$, where H and K are symmetric subgroups. Clearly, this induces an action of H on G/K by taking $h \cdot gK = hgK$. It turns out that the slice representations of both actions have the same cohomogeneity. Thus, classifying cohomogeneity one Hermann H-actions on G/K and classifying cohomogeneity one Hermann $H \times K$-actions on G are equivalent problems. Indeed, there is a correspondence between Hermann actions on symmetric spaces of type II (or group type), that is, compact simple Lie groups, and Hermann actions on symmetric spaces of type III, that is, compact symmetric spaces with simple isometry group, and this correspondence preserves the cohomogeneity.

The main theorem of [60] now reads:

Theorem 2.7. *Let $M = G/K$ be a simply connected, irreducible symmetric space of compact type. A cohomogeneity one action on M is orbit equivalent to one of the following:*

1. *A Hermann action of cohomogeneity one.*
2. *The action of SU_3 on SU_3 given by $g \cdot h = gh\bar{g}^{-1}$, where $\bar{g}$ denotes the complex conjugate of a matrix $g \in \mathsf{SU}_3$.*
3. *An action induced by the isotropy representation of a symmetric space of rank two.*
4. *One of seven exceptions corresponding to the action of $H \times K$ on G, or of the action of H on G/K, where (H, K, G) is a triple of Table 2.1.*

H	G_2	G_2	U_3	Spin_9	$\mathsf{Sp}_1\mathsf{Sp}_n$	SU_3	SU_3
K	$\mathsf{SO}_3 \times \mathsf{SO}_4$	G_2	G_2	$\mathsf{SO}_2 \times \mathsf{SO}_{14}$	$\mathsf{SO}_2 \times \mathsf{SO}_{4n-2}$	SO_4	SU_3
G	SO_7	SO_7	SO_7	SO_{16}	SO_{4n}	G_2	G_2

Table 2.1: The seven exceptional cohomogeneity one actions on symmetric spaces of compact type from Theorem 2.7.

It should be pointed out that this theorem only works if the symmetric space M is irreducible. Indeed, in [63] Kollross studied hyperpolar actions on products of symmetric spaces of compact type. A key idea in order to do this is to try to determine when such actions "split" with respect to the factors of the product (for a more in-depth discussion about this, see Section 4.1). For actions of cohomogeneity greater than one, the problem reduces to studying Hermann actions. However, the classification of cohomogeneity one actions on reducible symmetric spaces of compact type is still open.

Not any Hermann action is of cohomogeneity one, but it is possible to determine explictly which ones are by looking at the classification of symmetric spaces of compact type. Thus, the classification of homogeneous hypersurfaces in an irreducible symmetric space of compact type can be obtained via a case by case study of all these actions in the corresponding space. Obvious examples of cohomogeneity one Hermann actions are the isotropy actions of symmetric spaces G/K of rank one and the corresponding $K \times K$ actions on G. However, there are a few more examples as shown in [60, Theorem B].

2.2 Cohomogeneity one actions on symmetric spaces of noncompact type: the rank one case

The study of homogeneous hypersurfaces in symmetric spaces of noncompact type is of a rather different nature compared to the compact case: the lack of topological restrictions allows symmetric spaces of noncompact type to admit a much richer lattice of subgroups of their isometry group, and thus a rich variety of homogeneous submanifolds. Luckily, the nice properties of noncompact real semisimple Lie algebras often allows to construct and study cohomogeneity one actions by means of the root space and Iwasawa decompositions.

As in the case of symmetric spaces of compact type, we begin our exposition by looking at the rank one symmetric spaces of noncompact type: these are precisely the hyperbolic spaces over the normed real division algebras, namely, $\mathbb{R}\mathsf{H}^n$, $\mathbb{C}\mathsf{H}^n$, $\mathbb{H}\mathsf{H}^n$

$(n \geq 2)$ and $\mathbb{O}\mathsf{H}^2$.

2.2.1 Homogeneous hypersurfaces in real hyperbolic spaces

The classification of homogeneous hypersurfaces on real hyperbolic spaces follows from classical work of Cartan [26] on isoparametric hypersurfaces. Cartan's original aim was to classify isoparametric hypersurfaces in Riemannian manifolds of constant curvature. He succeeded to get such classification in $\mathbb{R}\mathsf{H}^n$, but not in spheres, where the problem remained open for nearly a century. It follows from this classification that an isoparametric hypersurface in $\mathbb{R}\mathsf{H}^n$ is an open part of a homogeneous hypersurface. This implies the classification of homogeneous hypersurfaces in $\mathbb{R}\mathsf{H}^n$:

Theorem 2.8. *A homogeneous hypersurface in* $\mathbb{R}\mathsf{H}^n \cong \mathsf{SO}^0_{1,n}/\mathsf{SO}_n$ *is congruent to:*

(1) *a geodesic sphere, or*

(2) *a tube around a totally geodesic* $\mathbb{R}\mathsf{H}^k$, $k \in \{1, \ldots, n-2\}$, *or*

(3) *a totally geodesic* $\mathbb{R}\mathsf{H}^{n-1}$, *or one of its equidistant hypersurfaces, or*

(4) *a horosphere.*

Geodesic spheres arise as principal orbits of the isotropy SO_n, while the horospheres are the orbits of the nilpotent part N of the Iwasawa decomposition of $\mathsf{SO}^0_{1,n}$. Lastly, the action of $\mathsf{SO}^0_{1,k} \times \mathsf{SO}_{n-k}$ on $\mathbb{R}\mathsf{H}^n$ has a totally geodesic $\mathbb{R}\mathsf{H}^k$ as an orbit, which is singular if $k \neq n-1$. This gives rise to items (3) and (4).

2.2.2 A general approach to homogeneous hypersurfaces in hyperbolic spaces

Consider a hyperbolic space $\mathbb{F}\mathsf{H}^n$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (where $n = 2$ if $\mathbb{F} = \mathbb{O}$), and write $\mathbb{F}\mathsf{H}^n = G/K$ for some symmetric pair (G, K). Then, the root space decomposition of $\mathfrak{g}$ reads

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},$$

where $\mathfrak{g}_{2\alpha} = \mathfrak{g}_{-2\alpha} = 0$ if $\mathbb{F} = \mathbb{R}$. Recall that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, and $\mathfrak{a}$ is 1-dimensional, and let K_0 be the connected subgroup of K whose Lie algebra is $\mathfrak{k}_0$. The possibilities for G, K, and K_0 are summarized in Table 2.2.

The nilpotent part of the Iwasawa decomposition of $\mathfrak{g}$ is simply $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. If $\mathfrak{g}_{2\alpha} = 0$, then $\mathfrak{n}$ is abelian. Otherwise, $\mathfrak{g}_{2\alpha}$ is both the center and the derived subalgebra of $\mathfrak{n}$. In fact, $\mathfrak{g}_{2\alpha}$ can be interpreted as the imaginary part of $\mathbb{F}$: there is a Clifford algebra representation $J\colon \mathrm{Cl}(\mathfrak{g}_{2\alpha}) \to \mathrm{End}(\mathfrak{g}_\alpha)$ which turns $\mathfrak{g}_\alpha$ into a

	$\mathbb{R}\mathsf{H}^n$	$\mathbb{C}\mathsf{H}^n$	$\mathbb{H}\mathsf{H}^n$	$\mathbb{O}\mathsf{H}^2$
G	$\mathsf{SO}^0_{1,n}$	$\mathsf{SU}_{1,n}$	$\mathsf{Sp}_{1,n}$	F_4^{-20}
K	SO_n	$\mathsf{S}(\mathsf{U}_1\mathsf{U}_n)$	$\mathsf{Sp}_1\mathsf{Sp}_n$	Spin_9
K_0	SO_{n-1}	$\mathsf{S}(\mathsf{U}_1\mathsf{U}_{n-1})$	$\mathsf{Sp}_1\mathsf{Sp}_{n-1}$	Spin_7
$\mathfrak{g}_\alpha$	$\mathbb{R}^{n-1}$	$\mathbb{C}^{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{O}$
$\mathfrak{g}_{2\alpha}$	0	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^7$

Table 2.2: Data for each hyperbolic space.

Clifford module (see [16] for further details). The restriction of J to $\mathfrak{g}_{2\alpha}$ gives rise to endomorphisms J_Z of $\mathfrak{g}_\alpha$ that are defined by the relation

$$\langle [U,V], Z\rangle = \langle J_Z U, V\rangle, \qquad U, V \in \mathfrak{g}_\alpha,\, Z \in \mathfrak{g}_{2\alpha}.$$

Moreover, $\mathfrak{g}_\alpha \cong \mathbb{F}^{n-1}$, and the action of K_0 on $\mathfrak{g}_\alpha$ is equivalent to the standard action of K_0 on $\mathbb{F}^{n-1}$.

We can now distinguish three types of cohomogeneity one actions on $\mathbb{F}\mathsf{H}^n$. These results will be generalized later on for spaces of arbitrary rank in Section 2.3.

Actions without singular orbits

If a cohomogeneity one action on $\mathbb{F}\mathsf{H}^n$ has no singular orbits, according to [12] we only have the following two possibilities:

(a) The action is orbit equivalent to the action of N, whose orbits are horospheres.

(b) The action is orbit equivalent to the action of the connected subgroup of AN with Lie algebra $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a codimension one subspace of $\mathfrak{g}_\alpha$.

Since these actions have no singular orbits, their families of orbits are homogeneous regular Riemannian foliations. These homogeneous foliations are known as *horospherical* and *solvable* foliations, respectively.

Actions with a totally geodesic singular orbit

Berndt and Brück classified cohomogeneity one actions on $\mathbb{F}\mathsf{H}^n$ which have a totally geodesic singular orbit [8]. In order to do this, it is enough to determine which totally geodesic submanifolds of $\mathbb{F}\mathsf{H}^n$ have homogeneous tubes, since totally geodesic submanifolds of rank one spaces had previously been classified by Wolf in [94]. Thus, it suffices to compute the stabilizer of each of these submanifolds, as well as its slice representation, in order to determine if such slice representation is of cohomogeneity one.

Actions with a non-totally geodesic singular orbit

Finally, it remains to study cohomogeneity one actions on $\mathbb{F}\mathsf{H}^n$ with a singular orbit which is not totally geodesic. Berndt and Tamaru devised in [14] a procedure to tackle this case. Its generalization to symmetric spaces of arbitrary rank is called the *nilpotent construction*, and will be adressed extensively in Section 3.2.

In brief, the classification of cohomogeneity one actions on $\mathbb{F}\mathsf{H}^n$ with a non-totally geodesic singular orbit reduces to the classification (up to congruence by an element of K_0) of the subspaces $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ such that $N_{K_0}(\mathfrak{w})$, the normalizer of $\mathfrak{w}$ in K_0, acts transitively on the unit sphere of $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$. In this case, the connected subgroup of $K_0AN \subset G$ whose Lie algebra is

$$N_{\mathfrak{k}_0}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$$

acts on $\mathbb{F}\mathsf{H}^n$ with cohomogeneity one. The subspaces $\mathfrak{w} \subset \mathfrak{g}_\alpha$ satisfying this condition have been classified in [14] for $\mathbb{F} \in \{\mathbb{C}, \mathbb{O}\}$, and in [39] for $\mathbb{F} = \mathbb{H}$.

2.2.3 Homogeneous hypersurfaces in complex hyperbolic spaces

The classification of homogeneous hypersurfaces in the complex case was completed by Berndt and Tamaru in [14]. It can be stated as follows:

Theorem 2.9. *A homogeneous hypersurface in* $\mathbb{C}\mathsf{H}^n \cong \mathsf{SU}_{1,n}/\mathsf{S}(\mathsf{U}_1\mathsf{U}_n)$ *is congruent to:*

(1) *a geodesic sphere, or*

(2) *a tube around a totally geodesic* $\mathbb{C}\mathsf{H}^k$, $k \in \{1, \dots, n-1\}$, *or*

(3) *a tube around a totally geodesic* $\mathbb{R}\mathsf{H}^n$, *or*

(4) *a horosphere, or*

(5) *a ruled homogeneous minimal Lohnherr hypersurface* $W^{2n-1}_{\pi/2}$, *or one of its equidistant hypersurfaces, or*

(6) *a tube around a ruled homogeneous minimal Berndt–Brück submanifold* W^{2n-k}_{φ}, *for* $k \in \{2, \dots, n-1\}$, $\varphi \in (0, \pi/2]$, *where* k *is even if* $\varphi \neq \pi/2$.

Tubes around a totally geodesic $\mathbb{C}\mathsf{H}^k$, $k \in \{0,\dots,n-1\}$, are congruent to the principal orbits of the action of $\mathsf{S}(\mathsf{U}_{1,k} \times \mathsf{U}_{n-k})$. The particular case of $k=0$ corresponds to geodesic spheres. The principal orbits of the group $\mathsf{SO}^0_{1,n}$ produce tubes around a totally geodesic real hyperbolic space $\mathbb{R}\mathsf{H}^n$. The group N gives rise to a horosphere foliation, all of whose orbits are mutually congruent and isometric to generalized Heisenberg groups.

Items (5) and (6) correspond to a solvable foliation and a nilpotent construction, respectively, and are closely related to the notion of *Kähler angle*. Let J be the complex structure of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, and $\mathfrak{w} \subseteq \mathfrak{g}_\alpha$ a real subspace. The Kähler angle of a nonzero $v \in \mathfrak{w}$ is the angle between Jv and $\mathfrak{w}$. We say that $\mathfrak{w}$ has constant Kähler angle $\varphi \in [0,\pi/2]$ if the Kähler angle of any nonzero vector of $\mathfrak{w}$ is φ. For example, $\mathfrak{w}$ has constant Kähler angle $\varphi = 0$ precisely if it is a complex subspace, and $\varphi = \pi/2$ if it is totally real.

Two subspaces of $\mathfrak{g}_\alpha$ with the same dimension and the same constant Kähler angle are congruent by an isometry of K_0. It turns out that the normalizer $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$ if and only if $\mathfrak{w}^\perp$ has constant Kähler angle φ. In this case, consider the connected subgroup of $\mathsf{SU}_{1,n}$ with Lie algebra $N_{\mathfrak{k}_0}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and let W^{2n-k}_φ be the orbit through $o \cong eK$, where $k = \dim \mathfrak{w}^\perp$. If $k=1$, then $\mathfrak{w}$ is of codimension one in $\mathfrak{g}_\alpha$, $\varphi = \pi/2$, and the corresponding action does not have any singular orbits. In this case $W^{2n-1}_{\pi/2}$ is a minimal hypersurface of $\mathbb{C}\mathsf{H}^n$ known as the Lohnherr hypersurface, and the rest of the orbits are equidistant hypersurfaces to it. If $k > 1$, then W^{2n-k}_φ is a singular orbit with normal bundle of constant Kähler angle φ known as a Berndt-Brück submanifold, and the rest of the orbits are tubes around it.

2.2.4 Homogeneous hypersurfaces in quaternionic hyperbolic spaces

The classification of cohomogeneity one actions on quaternionic hyperbolic spaces $\mathbb{H}\mathsf{H}^n$ has recently been achieved in [39]. The corresponding classification of homogeneous hypersurfaces follows as:

Theorem 2.10. *A homogeneous hypersurface in* $\mathbb{H}\mathsf{H}^n \cong \mathsf{Sp}_{1,n}/\mathsf{Sp}_1\mathsf{Sp}_n$ *is congruent to:*

(1) *a geodesic sphere, or*

(2) *a tube around a totally geodesic* $\mathbb{H}\mathsf{H}^k$, $k \in \{1,\dots,n-1\}$, *or*

(3) *a tube around a totally geodesic* $\mathbb{C}\mathsf{H}^n$, *or*

(4) *a horosphere, or*

(5) *a homogeneous minimal hypersurface P_1, or one of its equidistant hypersurfaces, or*

(6) *a tube around a homogeneous minimal submanifold $P_{\mathfrak{w}}$ in $\mathbb{H}\mathsf{H}^n$, where $\mathfrak{w}^\perp$ is a protohomogeneous subspace of $\mathfrak{g}_\alpha$.*

Similarly to the complex case, the action of $\mathsf{Sp}_{1,k} \times \mathsf{Sp}_{n-k}$ on $\mathbb{H}\mathsf{H}^n$ provides a cohomogeneity one action with a singular orbit which is a totally geodesic quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^k$. If $k = 0$ we again recover geodesic spheres. The orbits of the action of $\mathsf{SU}_{1,n}$ are a totally geodesic complex hyperbolic space $\mathbb{C}\mathsf{H}^n$ and tubes around it. Although there are more totally geodesic submanifolds of $\mathbb{H}\mathsf{H}^n$, their tubes fail to be homogeneous. The action of N gives rise to a horosphere foliation, and Example (5) corresponds to the leaves of a solvable foliation. This foliation has exactly one minimal leaf, which we have denoted by P_1.

The items in (6) correspond to nilpotent constructions, and similarly to the complex case, are intimately related to the notion of *quaternionic Káhler angle*: the root space $\mathfrak{g}_\alpha \cong \mathbb{H}^{n-1}$ carries a quaternionic structure, that is, a vector subspace $\mathfrak{J} \subseteq \operatorname{End}_{\mathbb{R}}(\mathbb{H}^{n-1})$ which admits a so-called canonical basis $\{J_1, J_2, J_3\}$ satisfying

$$J_i^2 = -\operatorname{Id}, \quad J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i,$$

where the indices are taken modulo 3. For a subspace $\mathfrak{w} \subset \mathfrak{g}_\alpha$, each orthogonal complex structure $J \in \mathfrak{J}$ determines a Kähler angle of a nonzero vector $v \in \mathfrak{w}$. Moreover, we can somehow encode all Kähler angles of v (with respect to all orthogonal complex structures $J \in \mathfrak{J}$) in a certain triple. Indeed, if one considers the symmetric bilinear form

$$L_v \colon \mathfrak{J} \times \mathfrak{J} \to \mathbb{R}, \quad (J_1, J_2) \mapsto \langle \pi_{\mathfrak{w}} J_1 v, \pi_{\mathfrak{w}} J_2 v \rangle,$$

where $\pi_{\mathfrak{w}}$ is the orthogonal projection onto $\mathfrak{w}$, the quaternionic Kähler angle of a nonzero $v \in \mathfrak{w}$ is defined to be the increasingly ordered triple $(\varphi_1(v), \varphi_2(v), \varphi_3(v))$ such that the eigenvalues of L_v are precisely $\cos^2(\varphi_i(v))\langle v, v\rangle$.

It turns out that, if the normalizer $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$ (in which case we say that $\mathfrak{w}^\perp$ is a protohomogeneous subspace of $\mathfrak{g}_\alpha \cong \mathbb{H}^{n-1}$), then $\mathfrak{w}^\perp$ must have constant quaternionic Kähler angle. In [39], this is successfully used to classify all of the possible cohomogeneity actions on $\mathbb{H}\mathsf{P}^n$ which arise via nilpotent construction. The moduli space $\mathcal{M}_{k,n-1}$ of such actions (described in terms of the k-dimensional subspaces $\mathfrak{w}^\perp$of $\mathfrak{g}_\alpha \cong \mathbb{H}^{n-1}$) is described in Table 2.3.

$\mathcal{M}_{k,n}$	$k \leq n$	$n < k \leq \frac{4n}{3}$	$\frac{4n}{3} < k \leq 2n$	$k > 2n$
$k \equiv 0 \,(\mathrm{mod}\ 4)$	$(\mathfrak{R}_4^+ \setminus \mathfrak{R}_4^-) \sqcup (\mathfrak{R}_4^- \times \mathbb{Z}_2)$	$\mathfrak{S}$	$\{(0,\varphi,\varphi)\}_{\varphi\in[0,\frac{\pi}{2}]}$	$\{(0,0,0)\}$
$k \equiv 2 \,(\mathrm{mod}\ 4)$	$\{(\varphi,\frac{\pi}{2},\frac{\pi}{2})\}_{\varphi\in[0,\frac{\pi}{2}]}$	$\{(0,\frac{\pi}{2},\frac{\pi}{2})\}$	$\{(0,\frac{\pi}{2},\frac{\pi}{2})\}$	$\emptyset$
$k \neq 3$ odd	$\{(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})\}$	$\emptyset$	$\emptyset$	$\emptyset$
$k = 3$	$(\mathfrak{R}_3^+ \setminus \mathfrak{R}_3^-) \sqcup (\mathfrak{R}_3^- \times \mathbb{Z}_2)$	$\emptyset$	$\{(\varphi,\varphi,\frac{\pi}{2})\}_{\varphi\in\{0,\frac{\pi}{3}\}}$	$\{(0,0,\frac{\pi}{2})\}$

$$\begin{aligned}
\Lambda &= \{(\varphi_1,\varphi_2,\varphi_3) \in [0,\pi/2]^3 : \varphi_1 \leq \varphi_2 \leq \varphi_3\},\\
\mathfrak{R}_3^+ &= \{(\varphi,\varphi,\pi/2) \in \Lambda : \varphi \in [0,\pi/2]\},\\
\mathfrak{R}_3^- &= \{(\varphi,\varphi,\pi/2) \in \Lambda : \varphi \in [\pi/3,\pi/2)\},\\
\mathfrak{R}_4^+ &= \{(\varphi_1,\varphi_2,\varphi_3) \in \Lambda : \cos(\varphi_1)+\cos(\varphi_2)-\cos(\varphi_3) \leq 1\},\\
\mathfrak{R}_4^- &= \{(\varphi_1,\varphi_2,\varphi_3) \in \Lambda : \cos(\varphi_1)+\cos(\varphi_2)+\cos(\varphi_3) \leq 1,\ \varphi_3 \neq \pi/2\},\\
\mathfrak{S} &= \{(\varphi_1,\varphi_2,\varphi_3) \in \Lambda : \cos(\varphi_1)+\cos(\varphi_2)+\varepsilon\cos(\varphi_3) = 1,\ \text{for } \varepsilon = 1 \text{ or } \varepsilon = -1\}.
\end{aligned}$$

Table 2.3: Moduli space of protohomogeneous subspaces of dimension k in $\mathbb{H}^n$.

This classification includes well-known examples such as totally real subspaces (precisely those with quaternionic Kähler angle $(\pi/2,\pi/2,\pi/2)$), totally complex subspaces (with quaternionic Kähler angle $(0,\pi/2,\pi/2)$) or quaternionic subspaces (with quaternionic Kähler angle $(0,0,0)$). It also includes some other nonclassical examples. See [39] for an explicit construction of these subspaces. While two subspaces with different quaternionic Kähler angles cannot be congruent to each other, a remarkable consequence of this classification implies the existence of noncongruent subspaces of $\mathbb{H}^n$ with the same quaternionic Kähler angles. These correspond precisely to the intersections $\mathfrak{R}_3^+ \cap \mathfrak{R}_3^- = \mathfrak{R}_3^-$ and $\mathfrak{R}_4^+ \cap \mathfrak{R}_4^- = \mathfrak{R}_4^-$ in Table 2.3.

2.2.5 Homogeneous hypersurfaces in the Cayley hyperbolic plane

Finally, we consider the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$, where the classification was completed in [14].

Theorem 2.11. *A homogeneous hypersurface in* $\mathbb{O}\mathsf{H}^2 = \mathsf{F}_4^{-20}/\mathsf{Spin}_9$ *is congruent to:*

(1) *a geodesic sphere, or*

(2) *a tube around a totally geodesic* $\mathbb{O}\mathsf{H}^1$*, or*

(3) *a tube around a totally geodesic* $\mathbb{H}\mathsf{H}^2$*, or*

(4) *a horosphere, or*

(5) *a minimal homogeneous hypersurface* F_1*, or one of its equidistant hypersurfaces, or*

(6) *a tube around the minimal submanifold* F_k *of codimension* $k \in \{2, 3, 6, 7\}$*, or*

(7) *a tube around the minimal submanifold* $F_{4,\varphi}$ *of codimension* 4*, for some* $\varphi \in [0, 1]$.

Geodesic spheres are principal orbits of the isotropy action of Spin_9 on $\mathbb{O}\mathsf{H}^2$, whereas tubes around a totally geodesic $\mathbb{O}\mathsf{H}^1$ are principal orbits of the action of $\mathsf{Spin}^0_{1,8} \subset \mathsf{F}_4^{-20}$, and tubes around a totally geodesic $\mathbb{H}\mathsf{H}^2$ are principal orbits of the action of $\mathsf{Sp}_{1,2}\mathsf{Sp}_1 \subset \mathsf{F}_4^{-20}$.

The group N, which is the nilpotent part of the Iwasawa decompostion of F_4^{-20}, gives rise to the horosphere foliation in $\mathbb{O}\mathsf{H}^2$, and Example (5) of Theorem 2.11 corresponds to the solvable foliation, which is obtained by the action of the subgroup of F_4^{-20} whose Lie algebra is $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a hyperplane in $\mathfrak{g}_\alpha$. This action has a unique minimal orbit which is denoted by F_1.

Examples (6) and (7) correspond to the nilpotent construction. Berndt and Br̈uck classified in [8] all subspaces $\mathfrak{w}$ of $\mathfrak{g}_\alpha \cong \mathbb{O}$ such that $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$. It turns out that any proper subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ with $\dim \mathfrak{w} \neq 3$ satisfies this condition.

The group $K_0 \cong \mathsf{Spin}_7$ acts on $\mathbb{O} \cong \mathbb{R}^8$ by its irreducible 8-dimensional spin representation. This action induces an action on the Grassmannian $\mathsf{G}_k(\mathbb{R}^8)$ of k-planes in $\mathbb{R}^8$. For $k \in \{1, 2, 3, 6, 7\}$, this action is transitive, and any two choices of $\mathfrak{w}$ yield equivalent actions (namely, that of item (6)). If $k = 4$ this action is of cohomogeneity one, and thus there is a one-parameter family of non-orbit equivalent cohomogeneity one actions on $\mathbb{O}\mathsf{H}^2$ with a singular orbit of dimension 4, corresponding to item (7).

2.3 Cohomogeneity one actions on symmetric spaces of noncompact type: some results for higher rank

The jump from hyperbolic spaces to symmetric spaces of noncompact type and higher rank turns out to be a very complicated problem. Unlike the compact case, we still do not have complete classification results for the vast majority of irreducible symmetric spaces of noncompact type. This is partially because the problem of determining which actions can have a non-totally geodesic singular orbit gets increasingly difficult the higher the rank is, as it will become apparent in Chapter 3.

Nevertheless, a series of efforts [10–13, 15, 84, 85] have allowed various partial classifications and structural results. These provide a necessary foundation for the proof of the main theorem of Chapter 3. As in the rank one case, these results can be divided according to whether the orbit has no singular orbits, a totally geodesic singular orbit, or a singular orbit which is not totally geodesic.

2.3.1 Cohomogeneity one actions with no singular orbits

A simple method to produce cohomogeneity one actions on a symmetric space of noncompact type $M = G/K$ is by means of the Iwasawa decomposition. Since the solvable part AN of the Iwasawa decomposition of G acts simply transitively on M, codimension one subgroups of AN give rise to homogeneous codimension one foliations on M. In [12], Berndt and Tamaru proposed two general methods for constructing such subgroups:

- If ℓ is a line in $\mathfrak{a}$, its orthogonal complement in $\mathfrak{a} \oplus \mathfrak{n}$, $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$, is a subalgebra by the properties of root spaces. The corresponding subgroup H_ℓ of G acts on M with cohomogeneity one, and its orbits form a family $\mathcal{F}_\ell$ of mutually congruent hypersurfaces. This family contains horosphere foliations of M as particular choices of ℓ (see [47, Remark 5.4]). We will refer to the $\mathcal{F}_\ell$ as *foliations of horospherical type*.
- If ℓ is a line in some simple root space $\mathfrak{g}_{\alpha_i}$, then $\mathfrak{a}\oplus(\mathfrak{n}\ominus\ell)$ is again a subalgebra. The corresponding subgroup H_i of G acts on M with cohomogeneity one and yields a codimension one foliation $\mathcal{F}_i$ which has a unique minimal leaf. We will refer to the $\mathcal{F}_i$ as *foliations of solvable type*.

It was shown in [12] for irreducible M, and later revisited in [10] for the general case, that any cohomogeneity one action on a symmetric space of noncompact type with no singular orbits is always orbit equivalent to one of the examples described above. Moreover (see [12] and [85] for further details), $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ are congruent precisely whenever there exists an isometry of M which induces a symmetry of the Dynkin diagram of $\mathfrak{g}$ and takes ℓ to ℓ'. Similarly $\mathcal{F}_i$ and $\mathcal{F}_j$ are congruent if and only if there exists an isometry of M inducing a symmetry of the Dynkin diagram of $\mathfrak{g}$ taking α_i to α_j.

Remark 2.12. In fact, it follows from the proof in [10] that the orbit equivalence is obtained by an isometry $g \in G$ (i.e. in the connected component of the identity of the isometry group of M). This will be relevant later in Chapter 3.

2.3.2 Cohomogeneity one actions with a totally geodesic singular orbit

Actions with totally geodesic singular orbits turn out to be related with a particular type of totally geodesic submanifolds, called reflective submanifolds. A *reflective submanifold* F of a symmetric space M is a totally geodesic submanifold of M such that the exponential of its normal space at some (and hence all) point, $F^\perp = \exp(\nu_p F)$, is also totally geodesic in M. Reflective submanifolds of symmetric spaces were classified by Leung in [66]. By means of this, Kollross' classifications and the use of duality, Berndt and Tamaru proved in [13]:

Theorem 2.13. *Let F be a totally geodesic submanifold of an irreducible symmetric space of noncompact type M. Then, F is a singular orbit of a cohomogeneity one action on M if and only if one of the following possibilities holds:*

1. *F is a reflective submanifold such that $F^\perp$ is a symmetric space of rank one (see* [13, Theorem 3.3] *for an explicit list), or*
2. *F is one of five possible nonreflective totally geodesic submanifolds related to the exceptional Lie group G_2 appearing in Table 2.4.*

M	$\mathsf{SO}_{3,7}/\mathsf{SO}_3 \times \mathsf{SO}_7$	$\mathsf{SO}_7(\mathbb{C})/\mathsf{SO}_7$	$\mathsf{G}_2^2/\mathsf{SO}_4$	$\mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2$
F	$\mathsf{G}_2^2/\mathsf{SO}_4$	$\mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2$	$\mathbb{C}\mathsf{H}^2$, $\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$

Table 2.4: Nonreflective totally geodesic submanifolds related to G_2.

It is important to mention that the lists provided in [13] are given up to congruence in M by isometries of the full isometry group $I(M)$. This provides an obstacle to the determination of cohomogeneity one actions, as the canonical extension of actions is only well-behaved with respect to the isometries in the identity component of $I(M)$ (see Remarks 3.2 and 3.9).

2.3.3 Actions with a non-totally geodesic singular orbit

The case of cohomogeneity one actions with a singular orbit which is not totally geodesic turns out to be the most involved of the three. In [15], Berndt and Tamaru proposed a general procedure for their study, which goes like this: if $M \cong G/K$ is an irreducible symmetric space of noncompact type, and $H \subseteq G$ is a group acting on M with cohomogeneity one and a singular orbit, then H must be contained in a maximal proper subgroup Q of G, which, by a result of Mostow [74], is either reductive or parabolic. If Q is reductive (in the sense of Remark 1.29), H and Q have the same orbits, and one of them must be a totally geodesic (proper) submanifold of M. Moreover, if $M \neq \mathbb{R}\mathsf{H}^n$, this must be a totally singular orbit, and so we are in the conditions of Theorem 2.13.

Assume now that Q is a maximal proper parabolic subgroup, say $Q = Q_{\Lambda\setminus\{\alpha_j\}}$ for some α_j in the set Λ of simple roots of $\mathfrak{g}$. There are two known methods for the construction of cohomogeneity one actions with singular orbits. Namely:

- **The canonical extension method:** Consider a cohomogeneity one action of a group $H_{\Lambda\backslash\{\alpha_j\}}$ on the boundary component $B_{\Lambda\backslash\{\alpha_j\}}$. By efectivizing the action, one may suppose that $H_{\Lambda\backslash\{\alpha_j\}} \subseteq M_{\Lambda\backslash\{\alpha_j\}}$. In this case, the group
$$H^{\Lambda}_{\Lambda\backslash\{\alpha_j\}} = H_{\Lambda\backslash\{\alpha_j\}} A_{\Lambda\backslash\{\alpha_j\}} N_{\Lambda\backslash\{\alpha_j\}}$$
acts on M with cohomogeneity one. If $H_{\Lambda\backslash\{\alpha_j\}}$ has a singular orbit, so does $H^{\Lambda}_{\Lambda\backslash\{\alpha_j\}}$.

- **The nilpotent construction method:** This is a generalization of the procedure we introduced before for rank 1 spaces. Essentially, it consists in looking for especial kinds of subspaces $\mathfrak{v} \subseteq \mathfrak{n}_{\Lambda\backslash\{\alpha_j\}}$ such that $N_{K_{\Lambda\backslash\{\alpha_j\}}}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$, and $N_{M_{\Lambda\backslash\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\backslash\{\alpha_j\}} \ominus \mathfrak{v})$ acts transitively on $B_{\Lambda\backslash\{\alpha_j\}}$. In this case, the connected subgroup of G with Lie algebra
$$N_{\mathfrak{m}_{\Lambda\backslash\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\backslash\{\alpha_j\}} \ominus \mathfrak{v}) \oplus \mathfrak{a}_{\Lambda\backslash\{\alpha_j\}} \oplus (\mathfrak{n}_{\Lambda\backslash\{\alpha_j\}} \ominus \mathfrak{v})$$
can be seen to act on M with cohomogeneity one and a singular orbit.

For a detailed description of both of these procedures and their properties, we refer to the discussion on Sections 3.1 and 3.2.

The key result of [15] reads as follows:

Theorem 2.14 [15, Th. 5.8]**.** *Let $M = G/K$ be a symmetric space of noncompact type, and let $H \subseteq G$ be a connected closed subgroup acting on M with cohomogeneity one. If H is contained in a maximal parabolic subgroup of G, then the action of H is orbit equivalent to an action obtained by canonical extension of a cohomogeneity one action on a boundary component, or an action obtained by nilpotent construction.*

Remark 2.15. In the proof of the theorem above, it is implicitly understood that $H \subset gQ_{\Lambda\backslash\{\alpha_j\}}g^{-1}$, for some $g \in G$ and $\alpha_j \in \Lambda$, although it is not explicitly stated. These element $g \in G$ is precisely the one that gives the orbit equivalence between H and the corresponding canonical extension or nilpotent construction, as follows from the proof of [15, Propositions 5.6 & 5.7]. This facts will be important later in Section 3.4.

Since boundary components of a symmetric space of noncompact type are again symmetric spaces of noncompact type, the canonical extension method suggests a rank reduction approach to the classification problem. However, a boundary component of an irreducible symmetric space might be reducible, and this can lead to the approach described above to fail. Namely, one cannot make use of Theorem 2.13 about the classification of actions with a totally geodesic singular orbits, since it only works for irreducible spaces. In Chapter 3, we deal with these inconveniences, and prove a structural result for cohomogeneity one actions on (not necessarily irreducible) symmetric spaces of noncompact type.

Nevertheless, the above approach, along with a case-by-case analysis of the nilpotent construction, allowed for the classification of the cohomogeneity one actions on many irreducible rank 2 spaces in [11, 15, 84], specifically:

$$\begin{gathered}\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3, \quad \mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3, \quad \mathsf{SL}_3(\mathbb{H})/\mathsf{Sp}_3, \quad \mathsf{SO}_5(\mathbb{C})/\mathsf{SO}_5, \\ \mathsf{G}_2^2/\mathsf{SO}_4, \quad \mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2, \quad \mathsf{SO}^0_{2,n}/\mathsf{SO}_2\mathsf{SO}_n, \quad \mathsf{SU}_{2,n}/\mathsf{S}(\mathsf{U}_2\mathsf{U}_n).\end{gathered}$$

Part II

Results

Chapter 3

Cohomogeneity one actions on symmetric spaces of noncompact type

The main goal of this chapter is to present a new structural result regarding cohomogeneity one actions on (not necessarily irreducible) symmetric spaces of noncompact type and arbitrary rank. This result builds on the efforts and techniques of [10–13, 15] which were presented in the previous section, and provides an efficient tool to deal with spaces that are reducible or of rank higher than two. Indeed, for reducible spaces we show that the classification problem can be reduced to the study of the corresponding problem in each of the irreducible factors. On the other hand, we prove the first explicit classification of cohomogeneity one actions on symmetric spaces of noncompact type and rank greater than two, namely in the family of spaces $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$ (for arbitrary n), and on any finite product of rank one spaces. The results presented here were obtained in collaboration with J. C. Díaz Ramos (CITMAga - Universidade de Santiago de Compostela) and M. Domínguez Vázquez (CITMAga - Universidade de Santiago de Compostela), and have been published in [37].

Our new structural result ensures that an isometric action on a symmetric space of noncompact type is always orbit equivalent to an action constructed with the tools of Chapter 2. The subtle but important distinction with Berndt an Tamaru's Theorem 2.14 is that, by means of a rank reduction procedure, we prove that the only cohomogeneity one actions with singular orbits that cannot be obtained by nilpotent construction are canonical extensions of actions of a maximal proper reductive subgroup (or equivalently, with a totally geodesic singular orbit) on a boundary component that is either irreducible or a product of two homothetic hyperbolic spaces:

Theorem A. *Let $M = G/K$ be a symmetric space of noncompact type, and let H be a connected closed subgroup of G. Then H acts on M with cohomogeneity one if and only if the H-action is orbit equivalent to one of the following:*

(FH) *An action inducing a regular codimension one foliation of horospherical type.*

(FS) *An action inducing a regular codimension one foliation of solvable type.*

(CEI) *The canonical extension of a cohomogeneity one action with a totally geodesic singular orbit on an irreducible boundary component.*

(CER) *The canonical extension of a cohomogeneity one diagonal action on a reducible boundary component of rank two whose two factors are homothetic.*

(NC) *An action obtained by nilpotent construction.*

Here, the actions on items (FH) and (FS) correspond to the actions of the subgroups of AN with Lie algebras of the form $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ (where ℓ is a line in $\mathfrak{a}$) and $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ (where ℓ is a line in a simple root space $\mathfrak{g}_{\alpha_i}$), respectively. These are described in §2.3.1.

Actions on item (CER) can roughly be thought of as the extension of the diagonal action of $\Delta I(\mathbb{F}\mathsf{H}^n) \subseteq I(\mathbb{F}\mathsf{H}^n) \times I(\mathbb{F}\mathsf{H}^n)$ on $\mathbb{F}\mathsf{H}^n \times \mathbb{F}\mathsf{H}^n$. In more detail, if $\Phi = \{\alpha, \beta\}$, where α and β are orthogonal simple roots, the corresponding boundary component $B_\Phi = B_{\{\alpha\}} \times B_{\{\beta\}}$ is a product of rank one spaces, which are homothetic precisely if $\mathfrak{s}_{\{\alpha\}}$ and $\mathfrak{s}_{\{\beta\}}$ are isomorphic. In this case, for any isomorphism $\sigma\colon \mathfrak{s}_{\{\alpha\}} \to \mathfrak{s}_{\{\beta\}}$, the connected subgroup H_Φ of S_Φ with Lie algebra $\mathfrak{g}_\Phi = \{X + \sigma X \colon X \in \mathfrak{s}_\alpha\}$ acts on B_Φ with cohomogeneity one and a totally geodesic singular orbit homothetic to $B_{\{\alpha\}} \cong B_{\{\beta\}}$.

Finally, items (CEI) and (NC) correspond to canonical extensions of the actions in §2.3.2 and nilpotent constructions, respectively.

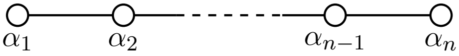
Remark 3.1. Throughout this chapter, we assume that the metric on M is the one induced by the Killing form of $\mathfrak{g}$ (this is also assumed in the proofs of Theorem 2.14 and [10]). Nevertheless, the results hold for an arbitrary symmetric metric on M. Recall from Remark 1.20 that, if $M = M_1 \times \cdots \times M_s$ is reducible (where each $M_i = G_i/K_i$ is irreducible), the induced metric on $\mathfrak{p}$ can be expressed as

$$\langle \cdot, \cdot \rangle = \lambda_1 \left(\mathcal{B}_{\mathfrak{g}_1}\right)_{|\mathfrak{p}_1 \times \mathfrak{p}_1} + \cdots + \lambda_s \left(\mathcal{B}_{\mathfrak{g}_s}\right)_{|\mathfrak{p}_s \times \mathfrak{p}_s},$$

where $\lambda_i > 0$ for $i = 1, \ldots, s$. Thus, given two symmetric metrics g, g' on $M = G/K$, we have $G = I^0(M, g) = I^0(M, g')$. Then, a connected group of isometries of (M, g) acts isometrically with cohomogeneity one on (M, g) if and only if it acts isometrically with cohomogeneity one on (M, g'). Moreover, the orbit equivalences in Theorem 2.14 are achieved by elements of G, as recalled in Remarks 2.12 and 2.15, and hence they hold independently of the symmetric metric on $M = G/K$.

As a first application of the structural result in Theorem A, we derive the explicit classification of cohomogeneity one actions on the spaces $\mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$, $n \geq 1$, whose rank is n. We recall that this family of spaces of noncompact type is universal in the sense that any symmetric space of noncompact type (maybe after rescaling the metric on its irreducible factors) can be isometrically embedded in $\mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$ in an equivariant and totally geodesic manner, for some $n \geq 1$ (see [49, §2.6.5]).

Theorem B. *Let* $M = G/K = \mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$, $n \geq 1$, *and let* $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ *be a set of simple roots for* $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{R})$ *whose Dynkin diagram is*



Any cohomogeneity one action on M *is orbit equivalent to one of the following actions:*

(FH) *The action of the connected subgroup of* $\mathsf{SL}_{n+1}(\mathbb{R})$ *with Lie algebra* $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$, *for some one-dimensional linear subspace* ℓ *of* $\mathfrak{a}$.

(FS) *The action of the connected subgroup of* $\mathsf{SL}_{n+1}(\mathbb{R})$ *with Lie algebra* $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_j})$, *for some simple root* $\alpha_j \in \Lambda$.

(CE) *The canonical extension* H_Φ^Λ *of the action of the connected subgroup* H_Φ *of* G *on a boundary component* B_Φ, *for one of the cases listed on the table below.*

$\mathfrak{h}_\Phi$	Φ	B_Φ	$\operatorname{codim}(H_\Phi^\Lambda \cdot o)$	Comments
$\mathfrak{k}_{\{\alpha_j\}} \cong \mathfrak{so}_2$	$\{\alpha_j\}$	$\mathbb{R}\mathsf{H}^2$	2	$1 \leq j \leq n$
$\mathfrak{sl}_{k-j+1}(\mathbb{R}) \oplus \mathbb{R}$	$\{\alpha_j, \dots, \alpha_k\}$	$\mathsf{SL}_{k-j+2}(\mathbb{R})/\mathsf{SO}_{k-j+2}$	$k - j + 1$	$1 \leq j < k \leq n$
$\mathfrak{sp}_2(\mathbb{R})$	$\{\alpha_j, \alpha_{j+1}, \alpha_{j+2}\}$	$\mathsf{SL}_4(\mathbb{R})/\mathsf{SO}_4$	3	$1 \leq j \leq n-2$
$\mathfrak{s}_{j,k,\sigma} \cong \mathfrak{sl}_2(\mathbb{R})$	$\{\alpha_j, \alpha_k\}$	$\mathbb{R}\mathsf{H}^2 \times \mathbb{R}\mathsf{H}^2$	2	$\lvert k - j\rvert > 1$

In the table we use the notation $\mathfrak{s}_{j,k,\sigma} = \{X + \sigma X : X \in \mathfrak{s}_{\{\alpha_j\}}\}$, for some isomorphism $\sigma\colon \mathfrak{s}_{\{\alpha_j\}} \to \mathfrak{s}_{\{\alpha_k\}}$ between the Lie algebras of the isometry groups of $B_{\{\alpha_j\}}$ and $B_{\{\alpha_k\}}$. Without loss of generality, the only singular orbit of the H_Φ^Λ-action on M is assumed to pass through the base point o. It is important to remark that the group H_Φ does *not* have to be "canonically embedded" or "embedded in the standard way" into the isometry group S_Φ of B_Φ: its Lie algebra is only of the form $\tau(\mathfrak{h}_\Phi^{\text{standard}})$, for some automorphism τ of $\mathfrak{s}_\Phi$, and where $\mathfrak{h}_\Phi^{\text{standard}}$ denotes a standard matrix group embedding. This is important since we have a priori no guarantee that the canonical extensions of orbit equivalent actions (for instance, those corresponding to $\mathfrak{h}_\Phi^{\text{standard}}$ and $\tau(\mathfrak{h}_\Phi^{\text{standard}})$) are orbit equivalent.

Remark 3.2. (*Orbit equivalence of the examples.*) Although the classification results in this chapter are obtained up to orbit equivalence, the explicit determination of the moduli space of cohomogeneity one actions on a given space entails an added difficulty whose solution lies outside the scope of this thesis. The reason is that two orbit equivalent cohomogeneity one actions with a totally geodesic singular orbit on a boundary component B_Φ may (in principle) produce non-orbit equivalent canonical extended actions on M. This can happen if the orbit equivalence in B_Φ

is only obtained via an outer isometry of B_Φ (that is, an isometry not lying in the connected component of the identity of the isometry group of B_Φ), since such outer isometry might not be the restriction of an isometry of M. As it was pointed out in §2.3.2, Berndt and Tamaru's classification of cohomogeneity one actions with a totally geodesic singular orbit on irreducible symmetric spaces [13] is given up to orbit equivalence by a possibly outer isometry. But we do not know if considering the relation of orbit equivalence by an inner isometry (which Solonenko calls "strong orbit equivalence" in [84]) would yield more classes. Addressing this problem would require, in particular, understanding the analogous congruence problem for reflective totally geodesic submanifolds, as [13] rests in part on Leung's classification of such submanifolds [66], where again only congruence by the full group of isometries is considered. In short, this difficulty concerns the actions of type (CEI), as well as of type (CER), where there is a similar problem. The moduli space of actions of foliation types (FH) and (FS) has been determined in [12] in the irreducible case, and in [85] in the reducible setting, whereas the orbit equivalence involving actions obtained by nilpotent construction may in principle require a case-by-case study.

As a second application of our structural result we reduce the classification problem of cohomogeneity one actions (up to orbit equivalence) on a reducible symmetric space of noncompact type to the classification problem on each one of its irreducible factors. The result basically says that if the action is not of (FH) or (CER) type, then it is a product action. It is interesting to point out that there is no known analog of Theorem C below in the compact setting, cf. [63].

Theorem C. *Let M be a symmetric space of noncompact type with De Rham decomposition $M = M_1 \times \cdots \times M_s$, where $M_i = G_i/K_i$, $i = 1, \dots, s$, and let $G = \prod_{i=1}^{s} G_i$. Then, a cohomogeneity one action on M is orbit equivalent to one of the following actions:*

- (Prod) *The product action of a subgroup $H_j \times \prod_{\substack{i=1\\ i\neq j}}^{s} G_i$ of G, where H_j is a connected Lie subgroup of G_j that acts with cohomogeneity one on the irreducible factor M_j.*
- (FH) *The action of the connected subgroup of G with Lie algebra $\mathfrak{h} = (\mathfrak{a}\ominus\ell)\oplus\mathfrak{n}$, for some one-dimensional linear subspace ℓ of $\mathfrak{a}$.*
- (CER) *The canonical extension of a cohomogeneity one diagonal action on a reducible rank two boundary component of M whose two factors are homothetic.*

Since actions of types (FH) and (CER) are well understood, Theorem C easily allows to derive explicit classifications on any product of irreducible spaces $M =$

$M_1 \times \cdots \times M_s$, whenever we know the classification of cohomogeneity one actions up to orbit equivalence on each irreducible factor M_i, $i = 1, \ldots, s$. This is the case, in particular, of the rank one spaces $\mathbb{F}\mathsf{H}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $n \geq 2$. We recall that the real hyperbolic spaces $\mathbb{R}\mathsf{H}^n$ have a root system of type (A_1), whereas the other rank one symmetric spaces $\mathbb{F}\mathsf{H}^n$, $\mathbb{F} \neq \mathbb{R}$, $n \geq 2$, have a root system of type (BC_1). Thus, the set of simple roots associated with a product $M = M_1 \times \cdots \times M_r$, where $M_i = G_i/K_i = \mathbb{F}_i\mathsf{H}^{n_i}$, consists of r mutually orthogonal roots, $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$. We also denote by $\mathfrak{k}_i \oplus \mathfrak{a}_i \oplus \mathfrak{n}_i$ the Iwasawa decomposition of the Lie algebra $\mathfrak{g}_i$ of G_i (in particular, $\mathfrak{n}_i = \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i}$), and put $(\mathfrak{k}_i)_0 = N_{\mathfrak{k}_i}(\mathfrak{a}_i)$. In this context, the application of Theorem C leads to the following classification result

Theorem D. *Let $M = M_1 \times \cdots \times M_r$ be a Riemannian product of rank one symmetric spaces of noncompact type $M_i = G_i/K_i = \mathbb{F}_i\mathsf{H}^{n_i}$, where $\mathbb{F}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $i = 1, \ldots, r$, and let $G = \prod_{i=1}^r G_i$. Then, a proper isometric action on M is of cohomogeneity one if and only if it is orbit equivalent to the action of the connected subgroup H of G with one of the following Lie algebras:*

Type	$\mathfrak{h}$	Comments
(FH)	$(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$	$\ell \subseteq \mathfrak{a}$, $\dim \ell = 1$.
(FS)	$\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$	$\ell \subseteq \mathfrak{g}_{\alpha_j}$, $\dim \ell = 1$, $\alpha_j \in \Lambda$.
(CEI)	$\bigoplus_{\substack{i=1 \\ i \neq j}}^r \mathfrak{g}_i \oplus \mathfrak{h}_j$	*$H_j \subseteq G_j$ acts on M_j with cohom. 1 and a totally geodesic singular orbit.*
(CER)	$\bigoplus_{\substack{i=1 \\ i \neq j,k}}^r \mathfrak{g}_i \oplus \mathfrak{g}_{j,k,\sigma}$	$\mathfrak{g}_{j,k,\sigma} = \{X + \sigma X : X \in \mathfrak{g}_j\}$, $j \neq k$, *$\sigma \colon \mathfrak{g}_j \to \mathfrak{g}_k$ Lie algebra isomorphism.*
(NC)	$\bigoplus_{\substack{i=1 \\ i \neq j}}^r \mathfrak{g}_i \oplus N_{(\mathfrak{k}_j)_0}(\mathfrak{v}) \oplus \mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \mathfrak{v})$	$\mathfrak{v} \subseteq \mathfrak{g}_{\alpha_j}$ *protohomogeneous,* $\alpha_j \in \Lambda$, $\dim \mathfrak{v} \geq 2$.

Here, by a protohomogeneous subspace $\mathfrak{v}$ we mean that $N_{(K_j)_0}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$ (that is, those giving rise to a nilpotent construction on M_j). As discussed in Section 2.2, the cohomogeneity one actions on rank one symmetric spaces of noncompact type are well known (up to orbit equivalence), and their corresponding orbits are listed in Theorems 2.8, 2.9, 2.10 and 2.11.

It is important to remark that, unlike the result for $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$, in the case considered in Theorem D one can easily determine when two given actions are orbit equivalent. Indeed, on the one hand, two orbit equivalent actions must be of the same type in Theorem D, except when $\mathfrak{v} \cong \mathbb{F}_j^l \subset \mathbb{F}_j^{n_j-1}$, $l \in \{0, \ldots, n_j - 2\}$ in type (NC) (which also yields an action of type (CEI)). On the other hand, the moduli space of cohomogeneity one actions on rank one spaces up to orbit equivalence has been completely determined [14], [39]. This immediately gives the moduli space of actions of types (FS), (CEI) and (NC) in Theorem D, since all these fit into type (Prod) of Theorem C. The orbit equivalence of actions of (FH) type has been studied in [12, 85]. Finally, two actions of type (CER) with the same pair (j, k) are orbit equivalent (independently of the isomorphism σ), see Proposition 3.23. We illustrate how this determination of the moduli space can be carried out by considering the case of the product of two real hyperbolic spaces.

Example 3.3. (*Cohomogeneity one actions on* $M = \mathbb{R}\mathsf{H}^n \times \mathbb{R}\mathsf{H}^m$.) Assume first that $m = n$. Then, the moduli space of cohomogeneity one actions up to orbit equivalence is $(I_n \times \Gamma_1) \sqcup \mathbb{R}\mathsf{P}^1/\Gamma_2 \sqcup \{\mathfrak{g}_{1,2,\sigma}\}$, where $I_k = \{0, \dots, k-1\}$, and $(\Gamma_1, \Gamma_2) = (\{0\}, \mathbb{Z}_2)$ if both $\mathbb{R}\mathsf{H}^n$ factors are isometric, or $(\Gamma_1, \Gamma_2) = (\mathbb{Z}_2, \{\mathrm{id}\})$ otherwise. Given $H_1 = \mathsf{SO}^0_{1,k} \times \mathsf{SO}_{n-k}$ (whose action on $\mathbb{R}\mathsf{H}^n$ has a totally geodesic orbit homothetic to $\mathbb{R}\mathsf{H}^k$) and $H_2 = \mathsf{SO}^0_{1,n}$, $(k, 0) \in I_n \times \Gamma_1$ represents the $H_1 \times H_2$-action, and $(k, 1) \in I_n \times \Gamma_1$ the $H_2 \times H_1$-action. Both actions are orbit equivalent if and only if both $\mathbb{R}\mathsf{H}^n$ factors of M are isometric, which motivates the definition of Γ_1. The quotient $\mathbb{R}\mathsf{P}^1/\Gamma_2$ represents the actions of type (FH), where the space of lines ℓ in $\mathfrak{a}$ is represented by $\mathbb{R}\mathsf{P}^1$, and Γ_2 is the group of automorphisms of $\mathfrak{a}$ of the form $\mathrm{Ad}_k|_{\mathfrak{a}}$ and inducing a symmetry of the Dynkin diagram of M, where k is an isometry of M fixing o and such that $\mathrm{Ad}_k\, \mathfrak{a} \subset \mathfrak{a}$. Finally, $\{\mathfrak{g}_{1,2,\sigma}\}$ represents the unique diagonal action of type (CER). If both factors have different dimensions n and m, the moduli space is $I_n \sqcup I_m \sqcup \mathbb{R}\mathsf{P}^1$.

The tools developed in this chapter can be applied to derive explicit classifications in other symmetric spaces of noncompact type. Basically, the only difficulty to do this stems from determining the actions that arise via nilpotent construction. Even in the seemingly simpler case of spaces whose isometry Lie algebra is split real semisimple, this study would entail a long, case-by-case analysis involving various representations of real semisimple Lie algebras. In other cases, the problem seems to get even harder, as illustrated by the solution to the problem for quaternionic hyperbolic spaces [39].

This chapter is organized as follows. In Sections 3.1 and 3.2, we provide a detailed description of the canonical extension and nilpotent construction methods. In Section 3.3, we discuss diagonal cohomogeneity one actions on reducible symmetric spaces. Section 3.4 is entirely devoted to the proof of the structural result stated in Theorem A. Finally, in Section 3.5 we will prove Theorems B, C and D as an application of Theorem A.

3.1 Extending actions from boundary components

The first method for constructing cohomogeneity one actions with singular orbits consists in extending known actions on lower rank symmetric spaces. Recall that, given a subset $\Phi \subseteq \Lambda$ of simple roots, the associated parabolic subgroup Q_Φ of G acts transitively on M. Consider the Langlands decomposition $Q_\Phi = M_\Phi A_\Phi N_\Phi$, and let $B_\Phi \times A_\Phi \times N_\Phi$ be the corresponding horospherical decomposition of M. Recall that, up to a finite quotient, S_Φ is the identity component of the isometry group of B_Φ, and so, any proper isometric action on B_Φ has the same orbits as some closed subgroup H_Φ of S_Φ. Now, $H_\Phi \subseteq S_\Phi \subseteq M_\Phi$, and one can define a new action on M by considering the natural action of $H_\Phi^\Lambda = H_\Phi A_\Phi N_\Phi$, which is a connected closed subgroup of Q_Φ. By construction, one has that $\mathfrak{h}_\Phi^\Lambda \cap \mathfrak{k} = \mathfrak{h}_\Phi \cap \mathfrak{k}$, and the normal space to $H_\Phi \cdot o$ at o in $T_o B_\Phi$ coincides with the normal space to $H_\Phi^\Lambda \cdot o$ at o in $T_o M$. In particular, the slice representations of $H_\Phi \curvearrowright B_\Phi$ and $H_\Phi^\Lambda \curvearrowright M$ coincide, and both actions have the same cohomogeneity. In fact, the orbit of H_Φ^Λ through a point $p \in B_\Phi$ turns out to be the union of the $A_\Phi N_\Phi$-orbits through each of the points of $H_\Phi \cdot p$.

Definition 3.4. The action of H_Φ^Λ on M is known as the *canonical extension* of the action of H_Φ on B_Φ to M.

Remark 3.5. The canonical extension method described above admits a generalization which allows to "enlarge" submanifolds of a given ambient space by means of some free, polar action with minimal orbits (see [41]). This procedure preserves important geometric properties, such as isoparametricity or the constancy of the mean curvature, and has allowed to construct the first examples of inhomogeneous isoparametric submanifolds in symmetric spaces of rank greater than one (cf. [41,46]).

Remark 3.6. The orbits of $A_\Phi N_\Phi$ on M are always minimal, but only totally geodesic if Φ and $\Lambda \setminus \Phi$ are orthogonal subsets (cf. [90]). Thus, the canonical extension of an action $H_\Phi \curvearrowright B_\Phi$ with a minimal orbit always has a minimal orbit. However, the action of H_Φ^Λ will only have a totally geodesic orbit if both Φ and $\Lambda \setminus \Phi$ are orthogonal and there exists a totally geodesic H_Φ-orbit on B_Φ (see [41] for further details).

Let us point out some properties of the canonical extension method. The first of these results, due to Berndt and Tamaru, provides a sufficient condition for two actions obtained by canonical extension from the same boundary component to be orbit equivalent.

Proposition 3.7 [15, Proposition 4.2]**.** *Let M be a Riemannian symmetric space of noncompact type and let B_Φ be a boundary component of M. Let H_Φ^1, H_Φ^2 be two connected closed subgroups of $I(B_\Phi)$ that act on B_Φ with cohomogeneity one. Assume that these two actions are orbit equivalent by an isometry $f \in I^0(B_\Phi)$. Then,* *the two cohomogeneity one actions on M that are obtained by canonical extension of H^1_Φ and H^2_Φ are orbit equivalent.*

Remark 3.8. Although originally formulated for cohomogeneity one actions, the above proposition remains true for the canonical extension of any two isometric actions on B_Φ.

Remark 3.9. The condition that the orbit equivalence in the previous proposition is achieved by an element in $I^0(B_\Phi)$ cannot be weakened. Namely, two actions that are merely orbit equivalent on a boundary component B_Φ can give rise to non-equivalent actions when extended to M (see [15, p. 139]). This is because an isometry $f \in I^0(B_\Phi)$ naturally extends to an isometry on the parabolic subgroup Q_Φ, whereas this cannot be guaranteed for an arbitrary f.

As boundary components of M are symmetric spaces of noncompact type, it makes sense to study what happens if this process is applied inductively. Let B_Φ be a boundary component associated with a subset of simple roots $\Phi \subseteq \Lambda$. Recall that $\Phi_{|\mathfrak{a}^\Phi}$ is a set of simple roots for $\mathfrak{s}_\Phi$. Thus, a boundary component of B_Φ is determined by a subset $\Psi \subseteq \Phi \subseteq \Lambda$, and in fact coincides with the boundary component of M associated with Ψ. One then gets a series of totally geodesic inclusions $B_\Psi \subseteq B_\Phi \subseteq M$. The next lemma states that an iterated canonical extension is a canonical extension itself.

Lemma 3.10. *Let $\Psi \subseteq \Phi \subseteq \Lambda$ be subsets of simple roots, and let H_Ψ be a connected subgroup of S_Ψ. Denote by H^Φ_Ψ the canonical extension of H_Ψ from B_Ψ to B_Φ, by $(H^\Phi_\Psi)^\Lambda$ the canonical extension of H^Φ_Ψ from B_Φ to M, and by H^Λ_Ψ the canonical extension of H_Ψ from B_Ψ to M. Then, $(H^\Phi_\Psi)^\Lambda = H^\Lambda_\Psi$.*

Proof. This is a straightforward computation at the Lie algebra level. First, we have

$$(\mathfrak{h}^\Phi_\Psi)^\Lambda = \mathfrak{h}^\Phi_\Psi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi = (\mathfrak{h}_\Psi \oplus \mathfrak{a}_{\Psi,\Phi} \oplus \mathfrak{n}_{\Psi,\Phi}) \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi.$$

But

$$\mathfrak{a}_\Phi \oplus \mathfrak{a}_{\Psi,\Phi} = (\mathfrak{a} \ominus \mathfrak{a}^\Phi) \oplus (\mathfrak{a}^\Phi \cap \mathfrak{a}_\Psi) = \mathfrak{a}_\Psi$$

and, since $(\mathfrak{s}_\Phi)_\lambda = \mathfrak{g}_\lambda$ for any $\lambda \in \Sigma_\Phi$,

$$\mathfrak{n}_\Phi \oplus \mathfrak{n}_{\Psi,\Phi} = \left(\bigoplus_{\lambda \in \Sigma^+ \setminus \Sigma^+_\Phi} \mathfrak{g}_\lambda \right) \oplus \left(\bigoplus_{\Sigma^+_\Phi \setminus \Sigma^+_\Psi} \mathfrak{g}_\lambda \right) = \mathfrak{n}_\Psi,$$

which shows that $(\mathfrak{h}^\Phi_\Psi)^\Lambda = \mathfrak{h}_\Psi \oplus \mathfrak{a}_\Psi \oplus \mathfrak{n}_\Psi = \mathfrak{h}^\Lambda_\Psi$. □

If one considers a Riemannian product $M = M_1 \times M_2$ of symmetric spaces of noncompact type (where M_1 and M_2 need not be irreducible), any set of simple roots associated with M is a disjoint union $\Lambda = \Lambda_1 \sqcup \Lambda_2$, where Λ_i is a set of simple roots for M_i, and Λ_1, Λ_2 are orthogonal. The boundary component associated with taking $\Phi = \Lambda_i$ as a subset of Λ turns out to be exactly M_i, and the canonical extension behaves nicely with respect to the Riemannian product.

Lemma 3.11. *Let $M_1 = G_1/K_1$ and $M_2 = G_2/K_2$ be symmetric spaces of noncompact type, and let $M = M_1 \times M_2$ be their Riemannian product. Let $\Lambda = \Lambda_1 \sqcup \Lambda_2$ be a set of simple roots for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, where Λ_i is a set of simple roots for $\mathfrak{g}_i$, $i = 1, 2$. Let H_{Λ_1} be a connected closed subgroup of G_1 acting with cohomogeneity one on M_1. Then, the cohomogeneity one action of $H_{\Lambda_1} \times G_2$ on M has the same orbits as the action of the group $H^{\Lambda}_{\Lambda_1}$ obtained by canonical extension of H_{Λ_1} from the boundary component $B_{\Lambda_1} = M_1$ to M.*

Proof. First note that the roots in Λ_1 and Λ_2 are orthogonal to each other. Hence, $\mathfrak{a}_{\Lambda_1} = \mathfrak{a}^{\Lambda_2}$ and $\mathfrak{n}_{\Lambda_1} = \mathfrak{n}^{\Lambda_2}$. Thus $\mathfrak{h}^{\Lambda}_{\Lambda_1} = \mathfrak{h}_{\Lambda_1} \oplus \mathfrak{a}_{\Lambda_1} \oplus \mathfrak{n}_{\Lambda_1} = \mathfrak{h}_{\Lambda_1} \oplus \mathfrak{a}^{\Lambda_2} \oplus \mathfrak{n}^{\Lambda_2} \subseteq \mathfrak{h}_{\Lambda_1} \oplus \mathfrak{g}_2$, and the $H^{\Lambda}_{\Lambda_1}$-orbits are contained in the $H_{\Lambda_1} \times G_2$-orbits. Since $H^{\Lambda}_{\Lambda_1}$ and $H_{\Lambda_1} \times G_2$ act with cohomogeneity one on M and the orbits of both actions are embedded, we conclude that both actions must have the same orbits. ☐

3.2 The nilpotent construction method

Apart from the canonical extension, Berndt and Tamaru proposed in [15] a second approach to constructing cohomogeneity one actions from parabolic subgroups of G: the *nilpotent construction*. This method was later revisited in [11, 84]. The nilpotent construction for spaces of arbitrary rank provides a generalization of the construction method described in §2.2.2 for rank one spaces.

Let $\Phi \subseteq \Lambda$ be a subset of simple roots and consider its corresponding parabolic subalgebra $\mathfrak{q}_\Phi$ with Langlands decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. Let us write $\{H^1, \dots, H^r\} \subseteq \mathfrak{a}$ for the dual basis of $\Lambda = \{\alpha_1, \dots \alpha_r\}$, that is,

$$\alpha_i(H^j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta of i and k. Define

$$H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i,$$

and $m_\Phi = \tilde{\lambda}(H^\Phi)$, where $\tilde{\lambda}$ is the highest root in Σ^+. Note that one has $\lambda(H^\Phi) \in \{0, \dots, m_\Phi\}$ for all $\lambda \in \Sigma^+$, and $\lambda \in \Sigma^+_\Phi$ if and only if $\lambda(H^\Phi) = 0$. Thus, H^Φ

induces an Ad_{K_Φ}-invariant gradation of $\mathfrak{n}_\Phi$,

$$\mathfrak{n}_\Phi = \bigoplus_{\nu=1}^{m_\Phi} \mathfrak{n}_\Phi^\nu,$$

where $\mathfrak{n}_\Phi^\nu = \bigoplus_{\lambda(H^\Phi)=\nu} \mathfrak{g}_\lambda$. Moreover, one has that $[\mathfrak{n}_\Phi^1, \mathfrak{n}_\Phi^\nu] = \mathfrak{n}_\Phi^{\nu+1}$ for every $\nu \in \{1, \dots, m_\Phi - 1\}$.

Suppose that $\mathfrak{v}$ is a subspace of $\mathfrak{n}_\Phi^1$ with dimension $\dim \mathfrak{v} \geq 2$. Then, $\mathfrak{n}_{\Phi,\mathfrak{v}} = \mathfrak{n}_\Phi \ominus \mathfrak{v}$ is a subalgebra of $\mathfrak{n}_\Phi$. Denote by $N_{\Phi,\mathfrak{v}}$ the corresponding connected subgroup of N_Φ, and consider the identity component of the normalizer of $\mathfrak{n}_{\Phi,\mathfrak{v}}$ on M_Φ, $N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N^0_{M_\Phi}(\mathfrak{v})$. Then, the group

$$H_{\Phi,\mathfrak{v}} = N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) A_\Phi N_{\Phi,\mathfrak{v}} = N^0_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) N_{\Phi,\mathfrak{v}}$$

is a closed subgroup of Q_Φ. Hence, it acts properly by isometries on M.

Suppose that $N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts transitively on B_Φ. Then, $B_\Phi \subseteq H_{\Phi,\mathfrak{v}} \cdot o$, and the normal space to $H_{\Phi,\mathfrak{v}} \cdot o$ at o can be identified with $\mathfrak{v}$. Since $H_{\Phi,\mathfrak{v}} \cap K = N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) \cap K = N^0_{K_\Phi}(\mathfrak{v})$, the slice representation of $H_{\Phi,\mathfrak{v}} \curvearrowright M$ coincides with the action of $N^0_{K_\Phi}(\mathfrak{v})$ on $\mathfrak{v}$ by orthogonal transformations. Since an isometric action has the same cohomogeneity as its slice representation, if $N^0_{K_\Phi}(\mathfrak{v})$ is transitive on the spheres of $\mathfrak{v}$, so is the action of $H_{\Phi,\mathfrak{v}}$ on M. This motivates the following definition:

Definition 3.12. If $\mathfrak{v} \subseteq \mathfrak{n}_\Phi^1$ is a subspace of dimension $\dim \mathfrak{v} \geq 2$ satisfying the following conditions:

(NC1) $N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = \Theta N^0_{M_\Phi}(\mathfrak{v})$ acts transitively on $B_\Phi = M_\Phi \cdot o$,

(NC2) $N^0_{K_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N^0_{K_\Phi}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$,

then the cohomogeneity one action of $H_{\Phi,\mathfrak{v}}$ on M is known as the action obtained by *nilpotent construction* from the choices of $\Phi \subseteq \Lambda$ and $\mathfrak{v}$. A subspace $\mathfrak{v}$ satisfying (NC1) is said to be *admissible*, and a subspace $\mathfrak{v}$ satisfying (NC2) is called *protohomogeneous*.

Remark 3.13. It should be noted that in the original formulation of [15], the authors do not require actions obtained by nilpotent construction to satisfy (NC1), but rather the following slightly different condition:

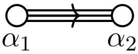
(NC1') $N^0_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = \Theta N^0_{L_\Phi}(\mathfrak{v})$ acts transitively on F_Φ,

where $F_\Phi = L_\Phi \cdot o \cong B_\Phi \times A_\Phi$ is a totally geodesic submanifold. While the latter condition is slightly quicker to introduce, the former turns out to be more manageable in certain situations. It was shown in [11] that both descriptions are equivalent, since $N^0_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) A_\Phi$.

Remark 3.14. For the purposes of our investigation, it will be enough to restrict ourselves to consider subsets of the form $\Phi = \Lambda \setminus \{\alpha_j\}$ in the above construction. In this case, one has that $H^\Phi = H^j$, and $\mathfrak{n}_\Phi^\nu$ is the sum of all root spaces $\mathfrak{g}_\lambda$ such that the coefficient of α_j in the expression of λ as a linear combination of simple roots is precisely ν.

Remark 3.15. If M is a rank one symmetric space of noncompact type, the only proper subset of $\Lambda = \{\alpha\}$ is the empty set, and any boundary component is a point. Thus, condition (NC1) is trivially satisfied, and any subspace $\mathfrak{v} \subseteq \mathfrak{n}_\emptyset^1 = \mathfrak{g}_\alpha$ is admissible. Therefore, for rank one spaces the nilpotent construction amounts to determining the protohomogeneous subspaces. Now, $K_\emptyset^0 = K_0$, from which it is easy to see that the construction described here coincides with the one described in §2.2.2 for symmetric spaces of noncompact type and rank 1.

Example 3.16 [15, pp. 142–143]**.** Consider the space $\mathsf{G}_2^2/\mathsf{SO}_4$, and let $\Lambda = \{\alpha_1, \alpha_2\}$ be a set of simple roots for $\mathfrak{g} = \mathfrak{g}_2^2$ whose Dynkin diagram is



The lie algebra $\mathfrak{g}_2^2$ is a split real form of $\mathfrak{g}_2^{\mathbb{C}}$, which in particular means that $\mathfrak{k}_0 = 0$ and all of the restricted root spaces $\mathfrak{g}_\lambda$ are 1-dimensional. For $\Phi = \{\alpha_1\}$, one gets that

$$\begin{aligned}
\mathfrak{n}_\Phi^1 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2, \\
\mathfrak{n}_\Phi &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_1+3\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+3\alpha_2} \cong \mathbb{R}^5, \\
\mathfrak{a}_\Phi &= \ker \alpha_1 = \operatorname{span}_{\mathbb{R}}\{H^2\} \cong \mathbb{R}, \\
\mathfrak{m}_\Phi &= \mathfrak{g}_{-\alpha_1} \oplus \operatorname{span}_{\mathbb{R}}\{H^1\} \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{sl}_2(\mathbb{R}), \\
\mathfrak{k}_\Phi &= \mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_2.
\end{aligned}$$

Hence, $B_\Phi \cong \mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 = \mathbb{R}\mathsf{H}^2$. Let $\mathfrak{v} = \mathfrak{n}_\Phi^1 \cong \mathbb{R}^2$. Then, $N_{M_\Phi}^0(\mathfrak{n}_{\Phi,\mathfrak{v}}) = M_\Phi \cong \mathsf{SL}_2(\mathbb{R})$ acts transitively on B_Φ, and $N_{K_\Phi}^0 \mathfrak{v} = K_\Phi \cong \mathsf{SO}_2$ acts transitively on the unit sphere of $\mathfrak{v}$. Therefore, $H_{\Phi,\mathfrak{v}}$ acts on $\mathsf{G}_2^2/\mathsf{SO}_4$ with cohomogeneity one. The Lie algebra of $H_{\Phi,\mathfrak{v}}$ is given by

$$\mathfrak{h}_{\Phi,\mathfrak{v}} = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+3\alpha_2}.$$

For the space $\mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2$, an identical choice of $\Phi = \{\alpha_1\}$ and $\mathfrak{v} = \mathfrak{n}_\Phi^1 \cong \mathbb{C}^2$ yields that the connected subgroup of $G_2^{\mathbb{C}}$ with Lie algebra

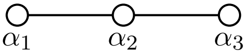
$$\mathfrak{h}_{\Phi,\mathfrak{v}} = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+3\alpha_2}.$$

acts on $\mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2$ with cohomogeneity one and a singular orbit of codimension 4.

The complete determination of all possible subspaces $\mathfrak{v}$ satisfying conditions (NC1)-(NC2) for a given symmetric space is usually a very difficult task. Indeed, this was the main difficulty for classifying cohomogeneity one actions on quaternionic hyperbolic spaces, where condition (NC1) did not play a role. For spaces of arbitrary rank, this becomes even more difficult. In fact, the only symmetric spaces where this problem has been solved are the hyperbolic spaces $\mathbb{F}\mathsf{H}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the rank 2 spaces listed at the end of Section 2.3, and the family of spaces $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$ from Theorem B.

It should be noted, however, that although the nilpotent construction method often produces examples of cohomogeneity one actions, the examples it produces can usually be obtained by extending actions from lower rank. In fact, the only known cases where the nilpotent construction produces actions that cannot be obtained by other methods are the two examples in Example 3.16, and the actions with non-totally geodesic singular orbits in rank one spaces.

Example 3.17. Consider the space $\mathsf{SL}_4(\mathbb{R})/\mathsf{SO}_4$, and let $\Lambda = \{\alpha_1, \alpha_2, \alpha_3\}$ be a set of simple roots for $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{R})$ whose Dynkin diagram is



with all multiplicities equal to 1 and $\mathfrak{k}_0 = 0$. Choose $\Phi = \{\alpha_1, \alpha_2\}$. Then,

$$\begin{aligned}
\mathfrak{n}_\Phi &= \mathfrak{n}_\Phi^1 = \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_3} \cong \mathbb{R}^3, \\
\mathfrak{m}_\Phi &= \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathfrak{sl}_3(\mathbb{R}), \\
\mathfrak{k}_\Phi &= \mathfrak{k}_{\alpha_1} \oplus \mathfrak{k}_{\alpha_2} \oplus \mathfrak{k}_{\alpha_1+\alpha_2} \cong \mathfrak{so}_2.
\end{aligned}$$

Thus, $B_\Phi \cong \mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$. Let $\mathfrak{v} = \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_2+\alpha_3}$. It follows from the properties of root spaces that $\mathfrak{a}^\Phi \oplus \mathfrak{n}^\Phi$ normalizes $\mathfrak{n}_{\Phi,\mathfrak{v}} = \mathfrak{g}_{\alpha_3}$, and so $\mathfrak{v}$ is admissible. We have that $N_{\mathfrak{k}_\Phi}(\mathfrak{v}) = \mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_2$ acts transitively on the unit sphere of $\mathfrak{v}$, so $\mathfrak{v}$ is also protohomogeneous, and $H_{\Phi,\mathfrak{v}}$ acts on $\mathsf{SL}_4(\mathbb{R})/\mathsf{SO}_4$ with cohomogeneity one. However, it can be seen that the action of H_Φ has the same orbits as the canonical extension of an action of $\mathsf{SL}_2 \times \mathbb{R}$ on B_Φ with a totally geodesic singular orbit.

Similarly to the actions obtained via canonical extension, Berndt and Tamaru gave in [15] a sufficient condition for two actions obtained by nilpotent construction to be orbit equivalent. Roughly, their result allows us to reduce our study of subspaces to a representative of each Ad_{K_Φ}-orbit in the Grassmannian $\mathsf{Gr}(\mathfrak{n}_\Phi^1)$:

Proposition 3.18 [15, Proposition 4.3]**.** *Let $\mathfrak{v}_1, \mathfrak{v}_2 \subseteq \mathfrak{n}_\Phi^1$ be two subspaces of dimension ≥ 2, and suppose $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are conjugate by some element in K_Φ. Then, $\mathfrak{v}_1$ is admissible (respectively, protohomogeneous) if and only if $\mathfrak{v}_2$ is. Moreover, if both $\mathfrak{v}_i$*

are admissible and protohomogeneous, the cohomogeneity one actions of $H_{\Phi,\mathfrak{v}_1}$ *and* $H_{\Phi,\mathfrak{v}_2}$ *are orbit equivalent.*

Let us now show that the canonical extension of an action obtained via nilpotent construction on a boundary component B_Φ of M is itself a nilpotent construction on the ambient space. For this, we will restrict ourselves to considering nilpotent constructions coming from maximal subsets of simple roots $\Phi\setminus\{\alpha_j\}$ in Φ, since this will be enough in our proof of Theorem A.

Let $\alpha_j\in\Phi$, and consider the Langlands decomposition of the parabolic subalgebra of $\mathfrak{s}_\Phi$ associated with $\Phi\setminus\{\alpha_j\}\subseteq\Phi$,

$$\mathfrak{q}_{\Phi\setminus\{\alpha_j\},\Phi}=\mathfrak{m}_{\Phi\setminus\{\alpha_j\},\Phi}\oplus\mathfrak{a}_{\Phi\setminus\{\alpha_j\},\Phi}\oplus\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi}.$$

Let $\mathfrak{v}\subseteq\mathfrak{n}^1_{\Phi\setminus\{\alpha_j\},\Phi}$ be a subspace of dimension $\dim\mathfrak{v}\geq 2$, and write $\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}=\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi}\ominus\mathfrak{v}$. Suppose that $\mathfrak{v}$ gives rise to an action via nilpotent construction on B_Φ. Then, $N^0_{M_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}})$ acts transitively on $B_{\Phi\setminus\{\alpha_j\}}$, and $N^0_{K_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$. Denote by

$$H_\Phi=H_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}=N^0_{M_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}})A_{\Phi\setminus\{\alpha_j\},\Phi}N_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}$$

the corresponding Lie group acting on B_Φ with cohomogeneity one. We have:

Lemma 3.19. *Under the previous conditions,* $\mathfrak{v}\subseteq\mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}$*, and* $\mathfrak{v}$ *is admissible and protohomogeneous for* M*. Moreover, the subgroup*

$$H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}=N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})A_{\Lambda\setminus\{\alpha_j\}}N_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$$

of G *obtained by nilpotent construction from the choice of* $\mathfrak{v}$ *has the same orbits as the* H^Λ_Φ*-action obtained by canonical extension of* $H_\Phi\curvearrowright B_\Phi$ *to* M*.*

Proof. First of all, we check that $\mathfrak{v}$ is indeed a subspace of $\mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}$. This is easy, since

$$\mathfrak{v}\subseteq\mathfrak{n}^1_{\Phi\setminus\{\alpha_j\},\Phi}=\bigoplus_{\substack{\lambda\in\Sigma^+_\Phi\\ \lambda(H^j)=1}}\mathfrak{g}_\lambda\subseteq\bigoplus_{\substack{\lambda\in\Sigma^+\\ \lambda(H^j)=1}}\mathfrak{g}_\lambda=\mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}.$$

We now check that this choice of $\mathfrak{v}$ as a subspace of $\mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}$ gives rise to a nilpotent construction, that is, $\mathfrak{v}$ satisfies the conditions (NC1) and (NC2) in 3.12.

We have the inclusion

$$\mathfrak{l}_{\Phi\setminus\{\alpha_j\},\Phi}=(\mathfrak{s}_\Phi)_0\oplus\left(\bigoplus_{\lambda\in\Sigma_{\Phi\setminus\{\alpha_j\}}}\mathfrak{g}_\lambda\right)\subseteq\mathfrak{g}_0\oplus\left(\bigoplus_{\lambda\in\Sigma_{\Lambda\setminus\{\alpha_j\}}}\mathfrak{g}_\lambda\right)=\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}},$$

and hence $\mathfrak{k}_{\Phi\setminus\{\alpha_j\},\Phi} = \mathfrak{l}_{\Phi\setminus\{\alpha_j\},\Phi} \cap \mathfrak{k} \subseteq \mathfrak{l}_{\Lambda\setminus\{\alpha_j\}} \cap \mathfrak{k} = \mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}$. Therefore,

$$\begin{aligned} N_{\mathfrak{l}_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}) &= \theta N_{\mathfrak{l}_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{v}) \subseteq \theta N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}) \\ &= N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}), \end{aligned} \tag{3.1}$$

and $N_{\mathfrak{k}_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{v}) \subseteq N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$.

By hypothesis, $N^0_{K_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$. Since $N^0_{K_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{v}) \subseteq N^0_{K_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$, so does $N^0_{K_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$. Thus $\mathfrak{v}$ is protohomogeneous.

In order to see that $\mathfrak{v}$ is admissible, first note that $N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$ leaves $B_{\Lambda\setminus\{\alpha_j\}} = M_{\Lambda\setminus\{\alpha_j\}} \cdot o$ invariant. Therefore, it is enough to verify the inclusion

$$\mathfrak{b}_{\Lambda\setminus\{\alpha_j\}} \subseteq T_o\big(N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \cdot o\big) \tag{3.2}$$

to see that $N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$ acts transitively on $B_{\Lambda\setminus\{\alpha_j\}}$. For this, decompose $\mathfrak{b}_{\Lambda\setminus\{\alpha_j\}}$ as

$$\begin{aligned} \mathfrak{b}_{\Lambda\setminus\{\alpha_j\}} &= \mathfrak{a}^{\Lambda\setminus\{\alpha_j\}} \oplus \Bigg(\bigoplus_{\lambda\in\Sigma^+_{\Lambda\setminus\{\alpha_j\}}} \mathfrak{p}_\lambda \Bigg) \\ &= \mathfrak{a}_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} \oplus \mathfrak{b}_{\Phi\setminus\{\alpha_j\}} \oplus T_o(N_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} \cdot o), \end{aligned}$$

with

$$\begin{aligned} \mathfrak{a}_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} &= \mathfrak{a}^{\Lambda\setminus\{\alpha_j\}} \cap \mathfrak{a}_{\Phi\setminus\{\alpha_j\}} = \mathfrak{a}^{\Lambda\setminus\{\alpha_j\}} \ominus \mathfrak{a}^{\Phi\setminus\{\alpha_j\}}, \\ \mathfrak{b}_{\Phi\setminus\{\alpha_j\}} &= \mathfrak{a}^{\Phi\setminus\{\alpha_j\}} \oplus \Big(\bigoplus_{\lambda\in\Sigma^+_{\Phi\setminus\{\alpha_j\}}} \mathfrak{p}_\lambda \Big), \end{aligned}$$

and

$$T_o(N_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} \cdot o) \cong \bigoplus_{\lambda\in\Sigma^+_{\Lambda\setminus\{\alpha_j\}}\setminus\Sigma^+_{\Phi\setminus\{\alpha_j\}}} \mathfrak{p}_\lambda.$$

We prove (3.2) by showing that each one of the three addends in the right-hand term of the previous relation for $\mathfrak{b}_{\Lambda\setminus\{\alpha_j\}}$ is contained in $T_o\big(N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \cdot o\big)$.

By assumption, $N^0_{M_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}})$ acts transitively on $B_{\Phi\setminus\{\alpha_j\}}$, so

$$\mathfrak{b}_{\Phi\setminus\{\alpha_j\}} = T_o\big(N^0_{M_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}) \cdot o\big) \subseteq T_o\big(N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \cdot o\big),$$

where in the inclusion we have used (3.1) and $\mathfrak{m}_{\Phi\setminus\{\alpha_j\},\Phi} \subseteq \mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}$.

Let $H \in \mathfrak{a}_{\Phi\setminus\{\alpha_j\}}$ and $X = \sum_{\lambda\in\Sigma_\Phi^+,\, \lambda(H^j)=1} X_\lambda \in \mathfrak{v} \subseteq \mathfrak{n}^1_{\Phi\setminus\{\alpha_j\},\Phi}$, with $X_\lambda \in \mathfrak{g}_\lambda$. Given $\lambda \in \Sigma_\Phi^+$ such that $\lambda(H^j) = 1$, we can write $\lambda = \alpha_j + \sum_{\alpha\in\Phi\setminus\{\alpha_j\}} n_\alpha \alpha$, for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. Then $\lambda(H) = \alpha_j(H)$, since $H \in \mathfrak{a}_{\Phi\setminus\{\alpha_j\}}$. Thus

$$[H, X] = \sum_{\substack{\lambda\in\Sigma_\Phi^+\\ \lambda(H^j)=1}} \lambda(H)X_\lambda = \sum_{\substack{\lambda\in\Sigma_\Phi^+\\ \lambda(H^j)=1}} \alpha_j(H)X_\lambda = \alpha_j(H)X,$$

which means that $\mathfrak{a}_{\Phi\setminus\{\alpha_j\}}$ normalizes $\mathfrak{v}$. Since $\mathfrak{a}_{\Phi\setminus\{\alpha_j\}} \subseteq \mathfrak{a} \subseteq \mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}$, this implies

$$\mathfrak{a}_{\Phi\setminus\{\alpha_j\}} = \theta\mathfrak{a}_{\Phi\setminus\{\alpha_j\}} \subseteq \theta N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}) = N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}). \tag{3.3}$$

Intersecting with $\mathfrak{a}^{\Lambda\setminus\{\alpha_j\}}$, we get $\mathfrak{a}^{\Lambda\setminus\{\alpha_j\}} \cap \mathfrak{a}_{\Phi\setminus\{\alpha_j\}} \subseteq N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$. Thus, $\mathfrak{a}_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} = \mathfrak{a}^{\Lambda\setminus\{\alpha_j\}} \cap \mathfrak{a}_{\Phi\setminus\{\alpha_j\}} \subseteq T_o\big(N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \cdot o\big)$.

We now check that $T_o\big(N_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} \cdot o\big) \subseteq T_o\big(N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \cdot o\big)$. We first prove

$$\mathfrak{n}_\Phi \cap \mathfrak{m}_{\Lambda\setminus\{\alpha_j\}} \subseteq N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}). \tag{3.4}$$

Observe that $\mathfrak{v} \subseteq \mathfrak{n}^1_{\Phi\setminus\{\alpha_j\},\Phi} \subseteq \mathfrak{m}_\Phi \cap \mathfrak{n}_{\Lambda\setminus\{\alpha_j\}}$. Then,

$$\begin{aligned}[\theta(\mathfrak{n}_\Phi \cap \mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}), \mathfrak{v}] &\subseteq [\theta\mathfrak{n}_\Phi, \mathfrak{v}] \cap [\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}, \mathfrak{v}] \subseteq [\theta\mathfrak{n}_\Phi, \mathfrak{m}_\Phi] \cap [\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}, \mathfrak{n}_{\Lambda\setminus\{\alpha_j\}}]\\ &\subseteq \theta[\mathfrak{n}_\Phi, \mathfrak{m}_\Phi] \cap \mathfrak{n}_{\Lambda\setminus\{\alpha_j\}} = \theta\mathfrak{n}_\Phi \cap \mathfrak{n}_{\Lambda\setminus\{\alpha_j\}} \subseteq \theta\mathfrak{n} \cap \mathfrak{n} = 0.\end{aligned}$$

Thus, $\theta(\mathfrak{n}_\Phi \cap \mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}) \subseteq N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}) = \theta N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$, from where (3.4) follows. But then $\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} \subseteq \mathfrak{n}_\Phi \cap \mathfrak{m}_{\Lambda\setminus\{\alpha_j\}} \subseteq N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$, and hence $T_o\big(N_{\Phi\setminus\{\alpha_j\},\Lambda\setminus\{\alpha_j\}} \cdot o\big) \subseteq T_o\big(N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \cdot o\big)$. This concludes the proof of (3.2). Therefore, $N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$ acts transitively on $B_{\Lambda\setminus\{\alpha_j\}}$, and $\mathfrak{v}$ satisfies the condition (NC1) for the nilpotent construction on M. Since, as shown above, (NC2) also holds, we get that $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$ acts on M with cohomogeneity one.

In order to conclude the proof of the lemma, we just have to see that the actions of $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$ and H_Φ^Λ have the same orbits. For this, we show that $\mathfrak{h}_\Phi^\Lambda \subseteq \mathfrak{h}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$. First observe that

$$\mathfrak{h}_\Phi^\Lambda = N_{\mathfrak{l}_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}) \oplus \mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}} \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi,$$

and recall

$$\mathfrak{h}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}} = N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \oplus \mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}.$$

We have seen in (3.1) that $N_{\mathfrak{l}_{\Phi\setminus\{\alpha_j\},\Phi}}(\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}}) \subseteq N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$. Also, $\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi} \subseteq \mathfrak{n}_{\Lambda\setminus\{\alpha_j\}}$, and hence $\mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi,\mathfrak{v}} \subseteq \mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$. By (3.3) we have $\mathfrak{a}_\Phi \subseteq$

$\mathfrak{a}_{\Phi\setminus\{\alpha_j\}} \subseteq N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$. Finally, we show that $\mathfrak{n}_\Phi = \bigoplus_{\lambda\in\Sigma^+\setminus\Sigma^+_\Phi}\mathfrak{g}_\lambda$ is contained in $\mathfrak{h}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$. Let $\lambda \in \Sigma^+ \setminus \Sigma^+_\Phi$. Assume first $\lambda \notin \Sigma^+_{\Lambda\setminus\{\alpha_j\}}$. Then $\mathfrak{g}_\lambda \subseteq \mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$, since $\mathfrak{g}_\lambda \perp \mathfrak{n}_{\Phi\setminus\{\alpha_j\},\Phi} \supset \mathfrak{v}$ as $\lambda \notin \Sigma^+_\Phi$. Now suppose $\lambda \in \Sigma^+_{\Lambda\setminus\{\alpha_j\}}$. Then, by (3.4), $\mathfrak{g}_\lambda \subseteq \mathfrak{n}_\Phi \cap \mathfrak{m}_{\Lambda\setminus\{\alpha_j\}} \subseteq N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}) \subseteq N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$.

Altogether we have $\mathfrak{h}^\Lambda_\Phi \subseteq \mathfrak{h}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$, and by connectedness, $H^\Lambda_\Phi \subseteq H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$. Since both groups act with cohomogeneity one on M, they must have exactly the same orbits. ☐

3.3 Maximal subgroups and diagonal actions

In this section we show that a group acting with cohomogeneity one on a reducible symmetric space of noncompact type is contained in a maximal proper subgroup that either splits nicely with respect to the decomposition into irreducible factors (in which case it is a canonical extension by Lemma 3.11), or it is determined by a diagonal action of a maximal proper reductive subgroup on the product of two rank one irreducible factors.

Let $M = G/K$ be a symmetric space of noncompact type. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the decomposition of the real semisimple Lie algebra $\mathfrak{g}$ into simple ideals, and $M = M_1 \times \cdots \times M_s = G_1/K_1 \times \cdots \times G_s/K_s$ the corresponding decomposition of M into irreducible symmetric spaces of noncompact type. For each $i \in \{1, \dots, s\}$, we have the Cartan decomposition $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$.

Let H be a connected closed Lie subgroup of G. Let $\mathfrak{l}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and with corresponding connected Lie subgroup L of G. Then, it follows from [48, Theorem 15.1, p. 235] (cf. [60, Theorem 2.1]) that either

$$\mathfrak{l} = \bigoplus_{\substack{i=1\\ i\neq j}}^{s} \mathfrak{g}_i \oplus \mathfrak{l}_j$$

for some $j \in \{1, \dots, s\}$ and a maximal proper subalgebra $\mathfrak{l}_j$ of $\mathfrak{g}_j$, or

$$\mathfrak{l} = \bigoplus_{\substack{i=1\\ i\neq j,k}}^{s} \mathfrak{g}_i \oplus \mathfrak{g}_{j,k,\sigma},$$

for two indices $j, k \in \{1, \dots, s\}$, $j \neq k$, where $\sigma\colon \mathfrak{g}_j \to \mathfrak{g}_k$ is a Lie algebra isomorphism and $\mathfrak{g}_{j,k,\sigma} = \{X + \sigma X : X \in \mathfrak{g}_j\}$. In this case, $\mathfrak{g}_{j,k,\sigma}$ and $\mathfrak{l}$ are reductive subalgebras of $\mathfrak{g}$, and $\mathfrak{g}_{j,k,\sigma}$ is a maximal proper reductive subalgebra of $\mathfrak{g}_j \oplus \mathfrak{g}_k$.

Let us focus on the second case, namely, the maximal proper subalgebra $\mathfrak{l}$ has a simple ideal which is diagonal with respect to the decomposition of $\mathfrak{g}$ into simple ideals. Let us recall first that there is a one to one correspondence between homothety classes of irreducible symmetric spaces of noncompact type and noncompact real simple Lie algebras. Hence, since $\mathfrak{g}_j$ and $\mathfrak{g}_k$ are isomorphic, the corresponding irreducible symmetric spaces M_j and M_k are homothetic. Let $G_{j,k,\sigma}$ be the connected closed subgroup of $G_j \times G_k$ with Lie algebra $\mathfrak{g}_{j,k,\sigma}$. Then, according to [15, Proposition 3.1] (it will also follow from Theorem 3.20 below), the action of $G_{j,k,\sigma}$ on $M_j \times M_k$ is not transitive. Hence, since $H \subseteq L$, if H acts with cohomogeneity one on M, the actions of H and L have the same orbits.

It only remains to decide for which real simple Lie algebras $\mathfrak{g}_j \cong \mathfrak{g}_k$ and corresponding isomorphism σ the action of L on M is indeed of cohomogeneity one, and not higher. Equivalently, we have to decide when the action of $G_{j,k,\sigma}$ on $M_j \times M_k$ has cohomogeneity one. The following result answers this question:

Theorem 3.20. *The action of $G_{j,k,\sigma}$ on $M_j \times M_k$ is hyperpolar and its cohomogeneity coincides with the rank of M_j.*

Proof. Without loss of generality, we will assume that $\sigma(\mathfrak{k}_j) = \mathfrak{k}_k$. In other words, the base point o_k we consider in M_k is the one whose isotropy Lie algebra is $\sigma(\mathfrak{k}_j)$. Then the Lie algebra of the isotropy group at (o_j, o_k) is $\mathfrak{k}_{j,k,\sigma} = \mathfrak{g}_{j,k,\sigma} \cap (\mathfrak{k}_j \oplus \mathfrak{k}_k) = \{T + \sigma T : T \in \mathfrak{k}_j\} \cong \mathfrak{k}_j \cong \mathfrak{k}_k$. Moreover, the orbit of $G_{j,k,\sigma}$ through $(o_j, o_k) \in M_j \times M_k$ is singular and of minimum orbit type, according to the proof of [36, Proposition 5.2].

The cohomogeneity of the action of $G_{j,k,\sigma}$ agrees with the cohomogeneity of the slice representation at (o_j, o_k). We calculate this first. We have

$$T_{(o_j,o_k)}(G_{j,k,\sigma} \cdot (o_j, o_k)) \cong \operatorname{pr}_{\mathfrak{p}_j \oplus \mathfrak{p}_k}(\mathfrak{g}_{j,k,\sigma}) = \{X + \sigma X : X \in \mathfrak{p}_j\},$$

where $\operatorname{pr}_{\mathfrak{p}_j \oplus \mathfrak{p}_k} = ((\operatorname{id} - \theta)/2)|_{\mathfrak{g}_j \oplus \mathfrak{g}_k}$ is the projection map onto $\mathfrak{p}_j \oplus \mathfrak{p}_k$. For simplicity we assume that M_j and M_k are isometric, but the proof holds with minor changes if they are only homothetic (cf. Remark 3.22). The normal space to $G_{j,k,\sigma} \cdot (o_j, o_k)$ is

$$\nu_{(o_j,o_k)}(G_{j,k,\sigma} \cdot (o_j, o_k)) \cong \{X - \sigma X : X \in \mathfrak{p}_j\}.$$

Now, the adjoint action of $\mathfrak{k}_{j,k,\sigma}$ on $\nu_{(o_j,o_k)}(G_{j,k,\sigma} \cdot (o_j, o_k))$ is given by

$$\operatorname{ad}_{T+\sigma T}(X - \sigma X) = [T, X] - \sigma[T, X],$$

for $T + \sigma T \in \mathfrak{k}_{j,k,\sigma}$ and $X - \sigma X \in \nu_{(o_j,o_k)}(G_{j,k,\sigma} \cdot (o_j, o_k))$. This representation is clearly equivalent to the adjoint action of $\mathfrak{k}_j$ on $\mathfrak{p}_j$. Therefore, the slice representation at (o_j, o_k) is equivalent to the isotropy representation of the symmetric space M_j, whose cohomogeneity is precisely the rank of M_j.

Let $\mathfrak{a}_{j,k,\sigma} = \{X - \sigma X : X \in \mathfrak{a}_j\}$ and $\Xi = \operatorname{Exp}(\mathfrak{a}_{j,k,\sigma}) \cdot (o_j, o_k) \subseteq M_j \times M_k$, where $\mathfrak{a}_j$ is a maximal abelian subspace of $\mathfrak{p}_j$. As usual we can identify $T_{(o_j,o_k)}\Xi$ with

$\mathfrak{a}_{j,k,\sigma}$, and this is clearly a section for the slice representation of the $G_{j,k,\sigma}$-action on $M_j \times M_k$ at (o_j, o_k). It is also clear that $\langle \mathfrak{g}_{j,k,\sigma}, \mathfrak{a}_{j,k,\sigma} \oplus [\mathfrak{a}_{j,k,\sigma}, \mathfrak{a}_{j,k,\sigma}] \rangle = 0$, since $\mathfrak{a}_{j,k,\sigma} \subseteq \nu_{(o_j,o_k)}(G_{j,k,\sigma} \cdot (o_j, o_k))$ is abelian. Then, [36, Proposition 2.3] guarantees that the $G_{j,k,\sigma}$-action is polar with section Ξ. Since $\mathfrak{a}_{j,k,\sigma}$ is abelian, then Ξ is flat, which shows that the action is hyperpolar. ☐

Remark 3.21. Since $G_{j,k,\sigma}$ is a reductive subgroup of $G_j \times G_k$, its action on $M_j \times M_k$ induces an action on a compact dual symmetric space of $M_j \times M_k$, see [62]. Such dual action turns out to be an indecomposable, hyperpolar, Hermann action in the sense of [63].

Remark 3.22. The singular orbit of $G_{j,k,\sigma}$ through $(o_j, o_k) \in M_j \times M_k$ is a totally geodesic submanifold of $M_j \times M_k$, since its tangent space $T_{(o_j,o_k)}(G_{j,k,\sigma} \cdot (o_j, o_k)) \cong \{X + \sigma X : X \in \mathfrak{p}_j\}$ is a Lie triple system in $\mathfrak{p}_j \oplus \mathfrak{p}_k$. Intrinsically, this singular orbit is homothetic to M_j and to M_k. More specifically, since $\mathfrak{g}_j$ and $\mathfrak{g}_k$ are isomorphic via σ, we can assume that their Killing forms are the same; denote both by $\mathcal{B}$. Then the metrics at o_j and o_k of the irreducible symmetric spaces M_j and M_k can be canonically identified with $\lambda_j \mathcal{B}_{|\mathfrak{p}_j \times \mathfrak{p}_j}$ and $\lambda_k \mathcal{B}_{|\mathfrak{p}_k \times \mathfrak{p}_k}$, for some positive constants λ_j, λ_k. Thus, the metric on $G_{j,k,\sigma} \cdot (o_j, o_k)$ at (o_j, o_k) is given by $(\lambda_j + \lambda_k)\mathcal{B}(\pi_{\mathfrak{p}_j}(\cdot), \pi_{\mathfrak{p}_j}(\cdot))$, where $\pi_{\mathfrak{p}_j} : \mathfrak{p}_j \oplus \mathfrak{p}_k \to \mathfrak{p}_j$ is the projection onto the first factor.

We conclude this section by showing that, up to orbit equivalence in $I(M)$, the role of the automorphism σ is irrelevant. More precisely:

Proposition 3.23. *Let σ, $\tau \colon \mathfrak{g}_j \to \mathfrak{g}_k$ be two Lie algebra isomorphisms. Then the actions of $G_{j,k,\sigma}$ and of $G_{j,k,\tau}$ on $M_j \times M_k$ are orbit equivalent. Moreover, this orbit equivalence is achieved by means of an element of $G_j \times G_k$ if $\sigma\tau^{-1}$ is an inner automorphism of $\mathfrak{g}_k$.*

Proof. As above, we can assume that $\sigma(\mathfrak{k}_j) = \mathfrak{k}_k$ and that there exists $g \in G_k$ such that $\operatorname{Ad}_g \tau(\mathfrak{k}_j) = \mathfrak{k}_k$, since any two maximal compactly embedded subalgebras of a real semisimple Lie algebra are conjugate by an inner automorphism [54, Chapter VI, §2]. Let $\varphi = \sigma\tau^{-1} \operatorname{Ad}_{g^{-1}} \in \operatorname{Aut}(\mathfrak{g}_k)$. Then $\varphi(\mathfrak{k}_k) = \mathfrak{k}_k$, and hence $\varphi(\mathfrak{p}_k) = \mathfrak{p}_k$. Since φ is a Lie algebra automorphism, it preserves the Lie bracket, and then also the curvature tensor of M_k at o_k, and the Killing form of $\mathfrak{g}_k$. Therefore, the map $\varphi_{|\mathfrak{p}_k} : \mathfrak{p}_k \cong T_{o_k} M_k \to \mathfrak{p}_k \cong T_{o_k} M_k$ is a linear isometry that preserves the curvature tensor at o_k. Hence, by a well-known result (see [95, Corollary 2.3.14]), φ is the differential at o_k of an isometry $\psi \in I(M_k)$ that fixes o_k. In other words, $\varphi = \operatorname{Ad}_\psi$, and hence $\sigma = \operatorname{Ad}_{\psi g} \tau$, where Ad is the adjoint representation of the Lie group $I(M_k)$. Then the automorphism $\operatorname{Ad}_{(\mathrm{id},\psi g)}$ of $\mathfrak{g}_j \oplus \mathfrak{g}_k$ satisfies $\operatorname{Ad}_{(\mathrm{id},\psi g)} \mathfrak{g}_{j,k,\tau} = \mathfrak{g}_{j,k,\sigma}$, and therefore the connected Lie groups $G_{j,k,\sigma}$ and $G_{j,k,\tau}$ are conjugate by the isometry $(\mathrm{id}, \psi g) \in I(M_j \times M_k)$. In particular, their actions on $M_j \times M_k$ are orbit equivalent. Finally, if $\sigma\tau^{-1}$ is inner, then φ is also inner, and hence we can assume that $\psi \in G_k$, so $(\mathrm{id}, \psi g) \in G_j \times G_k$. $\square$

3.4 The proof of Theorem A

By construction, an action of any of the five types stated in Theorem A is of cohomogeneity one.

In order to prove the converse, let H be a connected closed subgroup of G acting on $M = G/K$ with cohomogeneity one. If the action of H has no singular orbits, then the results in [12] and [10] (see Remarks 2.12 and 3.1) guarantee that the H-action is orbit equivalent to one of the actions of foliation type, namely (FH) or (FS).

Hence, we assume from now on that the H-action on M has a singular orbit. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the decomposition of the real semisimple Lie algebra $\mathfrak{g}$ into simple ideals, and $M = M_1 \times \cdots \times M_s = G_1/K_1 \times \cdots \times G_s/K_s$ the corresponding decomposition of M into irreducible symmetric spaces of noncompact type. Let $\mathfrak{q}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$ containing the Lie algebra $\mathfrak{h}$ of H, and let Q be the connected subgroup of G with Lie algebra $\mathfrak{q}$. According to the exposition in Section 3.3 we must have $\mathfrak{q} = \bigoplus_{\substack{i=1 \\ i \neq l}}^{s} \mathfrak{g}_i \oplus \mathfrak{q}_l$ for an index $l \in \{1, \dots, s\}$ and a maximal proper subalgebra $\mathfrak{q}_l$ of $\mathfrak{g}_l$, or $\mathfrak{q} = \bigoplus_{\substack{i=1 \\ i \neq j,k}}^{s} \mathfrak{g}_i \oplus \mathfrak{g}_{j,k,\sigma}$, for two indices $j, k \in \{1, \dots, s\}$, $j \neq k$, and an isomorphism $\sigma \colon \mathfrak{g}_j \to \mathfrak{g}_k$, where $\mathfrak{g}_{j,k,\sigma} = \{X + \sigma X : X \in \mathfrak{g}_j\}$ is a maximal proper reductive subalgebra of $\mathfrak{g}_j \oplus \mathfrak{g}_k$.

In the second case, in view of Theorem 3.20, the cohomogeneity of the Q-action on M agrees with the rank of M_j, and so it must be equal to one, since $H \subseteq Q$ acts on M with cohomogeneity one by assumption. Thus, the actions of H and Q have the same orbits. Hence, by Lemma 3.11, the H-action has the same orbits as a canonical extension of a cohomogeneity one diagonal action on the boundary component $B_\Phi = M_j \times M_k$ with $\Phi = \{\beta_j, \beta_k\} \subseteq \Lambda$, $B_{\{\beta_j\}} = M_j$, $B_{\{\beta_k\}} = M_k$, $\mathfrak{g}_j = \mathfrak{s}_{\{\beta_j\}}$, and $\mathfrak{g}_k = \mathfrak{s}_{\{\beta_k\}}$. Thus, the H-action is orbit equivalent to an action of (CER) type.

We consider the first case from now on, i.e. $\mathfrak{h} \subseteq \mathfrak{q} = \bigoplus_{\substack{i=1 \\ i \neq l}}^{s} \mathfrak{g}_i \oplus \mathfrak{q}_l$ for an index $l \in \{1, \dots, s\}$ and a maximal proper subalgebra $\mathfrak{q}_l$ of $\mathfrak{g}_l$. Then, by [15, Theorem 3.2] (which ultimately relies on the work of Mostow [74]), $\mathfrak{q}_l$ is either a maximal proper reductive or a maximal proper parabolic subalgebra of $\mathfrak{g}_l$. If $\mathfrak{q}_l$ is a reductive subalgebra of $\mathfrak{g}_l$, then $\mathfrak{q}$ is a reductive subalgebra of $\mathfrak{g}$, and the H-action and the Q-action have the same orbits by [15, Theorem 3.2]. By the same result and the assumption that H has a singular orbit, we have that such singular orbit is totally geodesic. Using Lemma 3.11, we see that the actions of H and Q have the same orbits as an action obtained by the canonical extension of a cohomogeneity one Q_l-action with a totally geodesic singular orbit on the irreducible boundary component $B_\Phi = M_l$, where Φ is the subset of Λ consisting of all simple roots of $\mathfrak{g}_l$. This corresponds to an action of type (CEI).

Henceforth, we assume that $\mathfrak{q}_l$ is a maximal proper parabolic subalgebra of $\mathfrak{g}_l$, and thus, $\mathfrak{q}$ is a maximal proper parabolic subalgebra of $\mathfrak{g}$. Then, there is an element $g \in G_l \subseteq G$ such that $\mathrm{Ad}_g\,\mathfrak{q}$ is a standard maximal proper parabolic subalgebra $\mathfrak{q}_{\Lambda\setminus\{\alpha_j\}}$, for some simple root $\alpha_j \in \Lambda$ corresponding to $\mathfrak{g}_l \subseteq \mathfrak{g}$. Therefore, we know from Theorem 2.14 (see remarks 2.15 and 3.1) that the H-action on M is orbit equivalent (via $g \in G_l$) to a cohomogeneity one action on M obtained by canonical extension of a cohomogeneity one action on the boundary component $B_{\Lambda\setminus\{\alpha_j\}}$, or to a cohomogeneity one action on M of a group $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$ obtained by nilpotent construction, for some subspace $\mathfrak{v} \subseteq \mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}$. This second case corresponds to an action of type (NC) in the statement of Theorem A.

Hence, we assume that the H-action is orbit equivalent to the canonical extension $H^\Lambda_{\Lambda\setminus\{\alpha_j\}}$ of certain connected closed subgroup $H_{\Lambda\setminus\{\alpha_j\}} \subseteq S_{\Lambda\setminus\{\alpha_j\}}$ acting on $B_{\Lambda\setminus\{\alpha_j\}}$ with cohomogeneity one. We can and will also assume that the $H_{\Lambda\setminus\{\alpha_j\}}$-action on $B_{\Lambda\setminus\{\alpha_j\}}$ has a singular orbit, since otherwise its canonical extension (and hence, the H-action) would yield a homogeneous regular foliation on M, contradicting the assumption that the H-action has a singular orbit.

Let $j_1 = j$. Now we apply all the procedure described so far (for actions with singular orbits) with $B_{\Lambda\setminus\{\alpha_{j_1}\}}$ instead of M and with $H_{\Lambda\setminus\{\alpha_{j_1}\}} \subseteq S_{\Lambda\setminus\{\alpha_{j_1}\}}$ instead of $H \subseteq G$. In the case that the $H_{\Lambda\setminus\{\alpha_{j_1}\}}$-action on $B_{\Lambda\setminus\{\alpha_{j_1}\}}$ is orbit equivalent to a canonical extension of a group $H_{\Lambda\setminus\{\alpha_{j_1},\alpha_{j_2}\}} \subseteq S_{\Lambda\setminus\{\alpha_{j_1},\alpha_{j_2}\}}$ acting on $B_{\Lambda\setminus\{\alpha_{j_1},\alpha_{j_2}\}}$, we continue the procedure. This algorithm ends at some point, since M has finite dimension and the dimensions of successive boundary components $B_{\Lambda\setminus\{\alpha_{j_1}\}} \supset B_{\Lambda\setminus\{\alpha_{j_1},\alpha_{j_2}\}} \supset \dots$ form a strictly decreasing sequence. Say that the sequence of boundary components we get is

$$M = B_{\Phi_0} \supset B_{\Phi_1} \supset B_{\Phi_2} \supset \cdots \supset B_{\Phi_m},$$

where we put $\Phi_0 = \Lambda$ and $\Phi_i = \Lambda \setminus \{\alpha_{j_1}, \dots, \alpha_{j_i}\}$, for $i = 1, \dots, m$, where m must be strictly lower than the rank of M (otherwise B_{Φ_m} is just one point, and there are no cohomogeneity one actions on it). Thus, our recurrence assumption is that we have a finite sequence of groups

$$H = H_{\Phi_0} \subseteq G = S_{\Phi_0}, \quad H_{\Phi_1} \subseteq S_{\Phi_1}, \quad H_{\Phi_2} \subseteq S_{\Phi_2}, \quad \dots, \quad H_{\Phi_m} \subseteq S_{\Phi_m}$$

such that each H_{Φ_i}-action on B_{Φ_i} is orbit equivalent via an element $g_i \in S_{\Phi_i}$ (see Remarks 2.15 and 3.1) to the canonical extension of the $H_{\Phi_{i+1}}$-action on $B_{\Phi_{i+1}}$ to B_{Φ_i}, for each $i = 0, 1, \dots, m-1$, and the H_{Φ_m}-action on B_{Φ_m} is no longer orbit equivalent to a canonical extension from any smaller boundary component of B_{Φ_m}.

Since $g_i H_{\Phi_i} g_i^{-1}$ and $H_{\Phi_{i+1}}^{\Phi_i}$ act on B_{Φ_i} with the same orbits, their canonically extended actions of $(g_i H_{\Phi_i} g_i^{-1})^\Lambda$ and $(H_{\Phi_{i+1}}^{\Phi_i})^\Lambda$ on M have exactly the same orbits, by construction. By Proposition 3.7, the actions of $(g_i H_{\Phi_i} g_i^{-1})^\Lambda$ and $H_{\Phi_i}^\Lambda$ on M are orbit equivalent, because the actions of H_{Φ_i} and $g_i H_{\Phi_i} g_i^{-1}$ on B_{Φ_i} are trivially orbit equivalent by the inner isometry $g_i \in S_{\Phi_i}$ of B_{Φ_i} . Also, by Lemma 3.10, $(H_{\Phi_{i+1}}^{\Phi_i})^\Lambda = H_{\Phi_{i+1}}^\Lambda$. Altogether, we obtain that the actions of $H_{\Phi_i}^\Lambda$ and $H_{\Phi_{i+1}}^\Lambda$ on M are orbit equivalent for each $i = 1, \dots, m-1$. Therefore, the actions of $H_{\Phi_1}^\Lambda$ and $H_{\Phi_m}^\Lambda$ on M are orbit equivalent. Since $g_0 H_{\Phi_0} g_0^{-1}$ and $H_{\Phi_1}^\Lambda$ act on M with the same orbits, we conclude that the action of $H = H_{\Phi_0}$ on M is orbit equivalent to the action of $H_{\Phi_m}^\Lambda$ on M.

Now we apply the procedure described at the beginning of the proof (for actions with singular orbits) to the action of H_{Φ_m} on B_{Φ_m} instead of the action of H on M. Let $\mathfrak{s}_{\Phi_m} = \bigoplus_{i=1}^{s_m} (\mathfrak{s}_{\Phi_m})_i$ be the decomposition of $\mathfrak{s}_{\Phi_m}$ into simple ideals. Since by construction the action of H_{Φ_m} on B_{Φ_m} is not a canonical extension, we have one of the following possibilities:

(i) The H_{Φ_m}-action on B_{Φ_m} has the same orbits as the action of a maximal proper reductive subgroup of S_{Φ_m} with Lie algebra $\bigoplus_{\substack{i=1\\ i\neq j,k}}^{s_m} (\mathfrak{s}_{\Phi_m})_i \oplus (\mathfrak{s}_{\Phi_m})_{j,k,\sigma_m}$, where σ_m is an isomorphism between $(\mathfrak{s}_{\Phi_m})_j$ and $(\mathfrak{s}_{\Phi_m})_k$, $j, k \in \{1, \dots, s_m\}$, $j \neq k$.

(ii) The H_{Φ_m}-action on B_{Φ_m} has the same orbits as the action of a maximal proper reductive subgroup of S_{Φ_m} with Lie algebra $\bigoplus_{\substack{i=1\\ i\neq l}}^{s_m} (\mathfrak{s}_{\Phi_m})_i \oplus \mathfrak{q}_l$, where $\mathfrak{q}_l$ is a maximal proper reductive subalgebra of $(\mathfrak{s}_{\Phi_m})_l$.

(iii) The H_{Φ_m}-action on B_{Φ_m} has the same orbits as $g_m H_{\Phi_m \setminus \{\alpha_k\}, \Phi_m, \mathfrak{v}} g_m^{-1}$, where $g_m \in S_{\Phi_m}$, $\alpha_k \in \Phi_m$, $\mathfrak{v}$ is a subspace of $\mathfrak{n}^1_{\Phi_m \setminus \{\alpha_k\}, \Phi_m}$, and $H_{\Phi_m \setminus \{\alpha_k\}, \Phi_m, \mathfrak{v}} \subseteq S_{\Phi_m}$ is obtained by nilpotent construction.

In case (i) we must have $s_m = 2$, Φ_m has two elements and B_{Φ_m} is the product of two symmetric spaces of rank one, because otherwise (by Lemma 3.11) the H_{Φ_m}-action on B_{Φ_m} would be orbit equivalent to a canonical extension, which would contradict the definition of Φ_m. This situation corresponds to an action of type (CER) in the statement of Theorem A, where the reducible boundary component of rank two is precisely B_{Φ_m}.

Similarly, in case (ii) we have that $\mathfrak{s}_{\Phi_m}$ is a simple Lie algebra for the same reason (and thus $s_m = l = 1$), and this corresponds to an action of type (CEI), where the irreducible boundary component is B_{Φ_m}.

Finally, in case (iii), since $g_m \in S_{\Phi_m}$ is an inner isometry of B_{Φ_m}, we have that the canonical extensions of the actions of H_{Φ_m} and $H_{\Phi_m \setminus \{\alpha_k\}, \Phi_m, \mathfrak{v}}$ on B_{Φ_m} to M

are orbit equivalent. As shown above, the H-action and the $H^{\Lambda}_{\Phi_m}$-action on M are orbit equivalent, so we get that the H-action is orbit equivalent to the $H^{\Lambda}_{\Phi_m\setminus\{\alpha_k\},\Phi_m,\mathfrak{v}}$-action. But then, Lemma 3.19 guarantees that the $H^{\Lambda}_{\Phi_m\setminus\{\alpha_k\},\Phi_m,\mathfrak{v}}$-action has the same orbits as the action of the group $H_{\Lambda\setminus\{\alpha_k\},\mathfrak{v}}$ obtained by nilpotent construction from the choice of $\mathfrak{v}$ as a subset of $\mathfrak{n}^1_{\Lambda\setminus\{\alpha_k\}}$. This corresponds to case (NC) in the statement of Theorem A.

3.5 Applications

The goal of this section is to prove Theorems B, C and D as applications of Theorem A. This will give us explicit descriptions of the cohomogeneity one actions on the symmetric spaces $\mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$, $n \geq 1$ (§3.5.1), on the products of rank one spaces (§3.5.3), and the structure result for cohomogeneity one actions on reducible spaces (§3.5.2).

3.5.1 Cohomogeneity one actions on $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$

For each integer $n \geq 1$, the symmetric space $\mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$ has rank n and its root system is of type (A_n), which in particular means that, for some set of simple roots $\Lambda = \{\alpha_1, \dots, \alpha_n\}$, we have $\Sigma^+ = \{\sum_{i=j}^{k} \alpha_i : 1 \leq j \leq k \leq n\}$. Moreover, $\mathfrak{k}_0 = 0$, $\mathfrak{g}_0 = \mathfrak{a}$, and for each $\lambda \in \Sigma$, the restricted root space $\mathfrak{g}_\lambda$ has dimension one. See [9, Example 13.2.1] for a detailed description.

Note that the case $n = 1$ leads to the real hyperbolic plane $\mathbb{R}\mathsf{H}^2$, in which case the classification is classical, whereas the case $n = 2$ has been studied in [15].

Proof of Theorem B. Let us analyze the different cases arising in Theorem A. First, the foliation types (FH) and (FS) in Theorem A correspond directly to cases (FH) and (FS) of Theorem B.

Let us focus now on case (CEI) of Theorem A, that is, the cohomogeneity one actions on M that arise as canonical extensions from irreducible boundary components.

Any connected subset Φ of simple roots in the Dynkin diagram of Λ is of the form $\Phi = \{\alpha_j, \dots, \alpha_k\}$, for some $j, k \in \{1, \dots, n\}$ with $j \leq k$. In this case, one has that the boundary component B_Φ is isometric to the irreducible symmetric space $\mathsf{SL}_{k-j+2}(\mathbb{R})/\mathsf{SO}_{k-j+2}$. By the description of actions of type (CEI), we have to consider the possible cohomogeneity one actions on B_Φ with a totally geodesic singular orbit. Such actions are induced by maximal proper reductive subgroups of $\mathsf{SL}_{k-j+2}(\mathbb{R})$.

If $j = k$, then $B_\Phi \cong \mathbb{R}\mathsf{H}^2$ admits only one cohomogeneity one action with a totally geodesic orbit, up to orbit equivalence. Such action is the one of the isotropy group at some point of $B_\Phi \cong \mathbb{R}\mathsf{H}^2$, which is given by the action of $K_\Phi^0 \cong \mathsf{SO}_2$ on B_Φ up to orbit equivalence, and has a fixed point as singular orbit. The canonical extension of this action to M leads to the the first row of the table of case (CE) in Theorem B.

If $j < k$, we have to consider the cohomogeneity one actions on the space $B_\Phi \cong \mathsf{SL}_{k-j+2}(\mathbb{R})/\mathsf{SO}_{k-j+2}$ that have a totally geodesic singular orbit. These were classified in [13]. According to the classification, these actions are orbit equivalent to the action of a maximal proper reductive subgroup of $\mathsf{SL}_{k-j+2}(\mathbb{R})$ isomorphic to $\mathsf{SL}_{k-j+1}(\mathbb{R}) \times \mathbb{R}$, or, exceptionally in the case that B_Φ has rank 3, i.e., $k = j + 2$, to the action of a maximal proper reductive subgroup of $\mathsf{SL}_4(\mathbb{R})$ isomorphic to $\mathsf{Sp}_2(\mathbb{R})$. The corresponding totally geodesic singular orbits are isometric to $(\mathsf{SL}_{k-j+1}(\mathbb{R})/\mathsf{SO}_{k-j+1}) \times \mathbb{R}$ or to $\mathsf{Sp}_2(\mathbb{R})/\mathsf{U}_2 \cong \mathsf{SO}^0_{2,3}/\mathsf{SO}_2\mathsf{SO}_3$, respectively. The canonical extensions of such actions from B_Φ to M yield the cohomogeneity one actions described in the second and third rows of the table in Theorem B.

Now, it is straightforward that the actions of type (CER) in Theorem A give rise to the actions of type (CE) described in the fourth row of the table in Theorem B. This is so since any rank two reducible boundary component of $M = \mathsf{SL}_{n+1}(\mathbb{R})/\mathsf{SO}_{n+1}$ is of the form $B_\Phi \cong \mathbb{R}\mathsf{H}^2 \times \mathbb{R}\mathsf{H}^2$, where $\Phi = \{\alpha_j, \alpha_k\}$, $|k-j| \geq 2$, is any disconnected subset of two simple roots in the Dynkin diagram of Λ.

Finally, we have to determine the actions of type (NC) in Theorem A, that is, those obtained via nilpotent construction. For this, fix $j \in \{1, \dots, n\}$. In our context, we have

$$\begin{aligned}
\mathfrak{n}_{\Lambda \setminus \{\alpha_j\}} &= \mathfrak{n}^1_{\Lambda \setminus \{\alpha_j\}} = \bigoplus_{i=1}^{j} \bigoplus_{l=j}^{n} \mathfrak{g}_{\alpha_i + \cdots + \alpha_l} \cong \mathbb{R}^j \otimes (\mathbb{R}^{n-j+1})^*, \\
\mathfrak{m}_{\Lambda \setminus \{\alpha_j\}} &= \mathfrak{a}^{\Lambda \setminus \alpha_j} \oplus \Bigg(\bigoplus_{\alpha \in \Sigma_{\{\alpha_1, \dots, \alpha_{j-1}\}}} \mathfrak{g}_\alpha \Bigg) \oplus \Bigg(\bigoplus_{\alpha \in \Sigma_{\{\alpha_{j+1}, \dots, \alpha_n\}}} \mathfrak{g}_\alpha \Bigg) \\
&\cong \mathfrak{sl}_j(\mathbb{R}) \oplus \mathfrak{sl}_{n-j+1}(\mathbb{R}).
\end{aligned}$$

Moreover, the adjoint representation of $\mathfrak{m}_{\Lambda \setminus \{\alpha_j\}}$ (resp. $\mathfrak{k}_{\Lambda \setminus \{\alpha_j\}}$) on $\mathfrak{n}_{\Lambda \setminus \{\alpha_j\}}$ is equivalent to the exterior tensor product representation of $\mathfrak{sl}_j(\mathbb{R}) \oplus \mathfrak{sl}_{n-j+1}(\mathbb{R})$ (resp. $\mathfrak{so}_j \oplus \mathfrak{so}_{n-j+1}$) on $\mathbb{R}^j \otimes (\mathbb{R}^{n-j+1})^*$. By choosing orthonormal bases $\{e_1, \dots, e_j\}$ of $\mathbb{R}^j$ and $\{f^1, \dots, f^{n-j+1}\}$ of $(\mathbb{R}^{n-j+1})^*$ in such a way that $e_i \otimes f^l$ can be regarded as a generator of $\mathfrak{g}_{\alpha_{j-i+1} + \cdots + \alpha_{j+l-1}}$, the set $\{e_i \otimes f^l : 1 \leq i \leq j,\ 1 \leq l \leq n-j+1\}$ is an orthonormal basis of $\mathfrak{n}_{\Lambda \setminus \{\alpha_j\}} \cong \mathbb{R}^j \otimes (\mathbb{R}^{n-j+1})^*$.

Let us assume, without loss of generality, that $j \leq n-j+1$; the case $j > n-j+1$ is completely analogous due to the symmetry of the Dynkin diagram of Λ. Since the action of $K^0_{\Lambda \setminus \{\alpha_j\}}$ on $\mathfrak{n}_{\Lambda \setminus \{\alpha_j\}}$ is equivalent to the isotropy representation of the symmetric space $\mathsf{SO}^0_{j,n-j+1}/\mathsf{SO}_j\mathsf{SO}_{n-j+1}$, such action is polar with section $\Xi =$

$\operatorname{span}\{e_i\otimes f^i : i=1,\dots,j\}$. Thus, up to conjugation by an element of $K^0_{\Lambda\setminus\{\alpha_j\}}$, we can assume that any nonzero subspace $\mathfrak{v}$ of $\mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}=\mathfrak{n}_{\Lambda\setminus\{\alpha_j\}}$ contains a unit vector $v=\sum_{i=1}^{j} v_i e_i\otimes f^i$, $v_i\in\mathbb{R}$. By the condition (NC2) of the nilpotent construction method, we want $\mathfrak{v}$ to be such that $N^0_{K_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$. Thus, $\mathfrak{v}$ must admit the orthogonal decomposition $\mathfrak{v}=\mathbb{R}v\oplus[N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}),v]$, where the second addend is perpendicular to Ξ by polarity.

Now we take an element in $\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}$, which we identify with some $A+B\in\mathfrak{sl}_j(\mathbb{R})\oplus\mathfrak{sl}_{n-j+1}(\mathbb{R})$, where $A=(a_{il})_{i,l=1}^{j}$ and $B=(b_{il})_{i,l=1}^{n-j+1}$. For the sake of convenience, let us define $v_i=0$ for $i>j$. Then

$$\begin{aligned}[A+B,v]&=\sum_{i=1}^{j}\sum_{l=1}^{n-j+1}(a_{il}v_l-b_{il}v_i)e_i\otimes f^l\\&=\sum_{i=1}^{j}(a_{ii}-b_{ii})v_ie_i\otimes f^i+\sum_{i\neq l}(a_{il}v_l-b_{il}v_i)e_i\otimes f^l.\end{aligned}$$

Note that the first sum after the second equal sign belongs to Ξ, whereas the second sum is perpendicular to Ξ. Thus, if $A+B\in N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$, the first sum must be proportional to v, which implies that there exists $\lambda\in\mathbb{R}$ such that $a_{ii}-b_{ii}=\lambda$ for all $i\in\{1,\dots,j\}$ with $v_i\neq 0$. Hence, if there are at least two indices $i_1,i_2\in\{1,\dots,j\}$ such that $v_{i_1}\neq 0\neq v_{i_2}$, then not every $A+B\in\mathfrak{sl}_j(\mathbb{R})\oplus\mathfrak{sl}_{n-j+1}(\mathbb{R})$, with A and B diagonal, normalizes $\mathfrak{v}$. Under the identification $\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}\cong\mathfrak{sl}_j(\mathbb{R})\oplus\mathfrak{sl}_{n-j+1}(\mathbb{R})$, this means that the orthogonal projection of $N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})=\theta N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$ onto $\mathfrak{p}$ does not contain the whole subspace $\mathfrak{a}^{\{\alpha_1,\dots,\alpha_{j-1}\}}\oplus\mathfrak{a}^{\{\alpha_{j+1},\dots,\alpha_n\}}=\mathfrak{a}^{\Lambda\setminus\{\alpha_j\}}$. In this case, the group $N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$ cannot act transitively on the boundary component $B_{\Lambda\setminus\{\alpha_j\}}\cong(\mathsf{SL}_j(\mathbb{R})/\mathsf{SO}_j)\times(\mathsf{SL}_{n-j-1}(\mathbb{R})/\mathsf{SO}_{n-j-1})$ since $\mathfrak{a}^{\Lambda\setminus\{\alpha_j\}}\subseteq T_oB_{\Lambda\setminus\{\alpha_j\}}$. This means that condition (NC1) does not hold in this case.

Therefore, we must have $v_i=0$ for all except one $i\in\{1,\dots,j\}$. Again, by conjugating by an element of $K^0_{\Lambda\setminus\{\alpha_j\}}$ if necessary, we can assume that $v=e_1\otimes f^1\in\mathfrak{g}_{\alpha_j}$. Now let $S+T\in N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$, where $S\in\mathfrak{k}_{\{\alpha_1,\dots,\alpha_{j-1}\}}\cong\mathfrak{so}_j$ and $T\in\mathfrak{k}_{\{\alpha_{j+1},\dots,\alpha_n\}}\cong\mathfrak{so}_{n-j+1}$. Then the element

$$[S+T,v]=(Se_1)\otimes f^1+e_1\otimes(Tf^1)$$

belongs to $\mathfrak{v}\ominus\mathbb{R}v$. Assume $Se_1\neq 0\neq Tf^1$. Then there exists $P\in\mathsf{SO}_j$ mapping the orthogonal set (e_1,Se_1) to the orthogonal set (e_2,μ_1e_1), for some $\mu_1\neq 0$; and similarly, there exists $Q\in\mathsf{SO}_{n-j+1}$ sending (f^1,Tf^1) to (f^1,μ_2f^2), for some $\mu_2\neq 0$. Thus, the element $g=(P,Q)\in K^0_{\Lambda\setminus\{\alpha_j\}}\cong\mathsf{SO}_j\times\mathsf{SO}_{n-j+1}$ satisfies

$$\operatorname{Ad}_g[S+T,v]=\mu_1e_1\otimes f^1+\mu_2e_2\otimes f^2,\quad\text{with }\mu_1\neq 0\neq\mu_2.\tag{3.5}$$

Therefore, the subspace $\mathrm{Ad}_g\,\mathfrak{v}$ of $\mathfrak{n}_{\Lambda\setminus\{\alpha_j\}}$ intersects Ξ nontrivially. Also, it satisfies conditions (NC1)-(NC2) because $\mathfrak{v}$ does so by assumption. But we have shown in the previous paragraph that no subspace of $\mathfrak{n}_{\Lambda\setminus\{\alpha_j\}}$ satisfying (NC1)-(NC2) and intersecting Ξ contains an element such as the one on the right hand side of (3.5). This yields a contradiction, which implies that, for each $S+T\in N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$, either $Se_1=0$ or $Tf^1=0$. Since $N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$ is a vector space, we actually have either $Se_1=0$ or $Tf^1=0$, for all $S+T\in N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})$. In other words, $[N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}),v]\subseteq\operatorname{span}\{e_1\otimes f^i : i=1,\dots,n-j+1\}$ or $[N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}),v]\subseteq\operatorname{span}\{e_i\otimes f^1 : i=1,\dots,j\}$. Assume that we are in the second case, that is, $\mathfrak{v}=\mathbb{R}v\oplus[N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v}),v]\subseteq\operatorname{span}\{e_i\otimes f^1 : i=1,\dots,j\}$; the first case is completely analogous. Let $k=\dim\mathfrak{v}$. Again, up to conjugation by an element of $K^0_{\{\alpha_1,\dots,\alpha_{j-1}\}}\cong\mathsf{SO}_j$ we can assume

$$\mathfrak{v}=\mathfrak{g}_{\alpha_{j-k+1}+\cdots+\alpha_j}\oplus\mathfrak{g}_{\alpha_{j-k+2}+\cdots+\alpha_j}\oplus\cdots\oplus\mathfrak{g}_{\alpha_{j-1}+\alpha_j}\oplus\mathfrak{g}_{\alpha_j}. \tag{3.6}$$

Let $\Omega=\{\alpha_i : 1\le i\le j-k-1, \text{ or } j-k+1\le i\le j-1, \text{ or } j+2\le i\le n\}$. Then the connected subgroup of $K_{\Lambda\setminus\{\alpha_j\}}$ with Lie algebra

$$N_{\mathfrak{k}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{v})=\bigoplus_{\lambda\in\Sigma^+_\Omega}\mathfrak{k}_\lambda\cong\mathfrak{so}_{j-k}\oplus\mathfrak{so}_k\oplus\mathfrak{so}_{n-j}$$

acts transitively on the unit sphere of $\mathfrak{v}$. Moreover, $N_{\mathfrak{m}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$ contains $\mathfrak{a}^{\Lambda\setminus\{\alpha_j\}}\oplus\mathfrak{n}^{\Lambda\setminus\{\alpha_j\}}$, which is the solvable part of the Iwasawa decomposition of $\mathfrak{s}_{\Lambda\setminus\{\alpha_j\}}$, and hence, $N^0_{M_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})$ acts transitively on $B_{\Lambda\setminus\{\alpha_j\}}$. Thus, the subspace $\mathfrak{v}$ of $\mathfrak{n}^1_{\Lambda\setminus\{\alpha_j\}}$ given in (3.6) satisfies both conditions (NC1)-(NC2) of the nilpotent construction method. However, the corresponding cohomogeneity one action on M is orbit equivalent to a canonical extension. Indeed, on the one hand, the Lie algebra of the resulting group $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$ that acts with cohomogeneity one on M satisfies

$$\mathfrak{h}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}=N_{\mathfrak{l}_{\Lambda\setminus\{\alpha_j\}}}(\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}})\oplus\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}\supset\mathfrak{a}\oplus\mathfrak{n}^{\Lambda\setminus\{\alpha_j\}}\oplus\mathfrak{n}_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}=\mathfrak{a}\oplus(\mathfrak{n}\ominus\mathfrak{v}).$$

By dimension reasons, the singular orbit of the $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$-action is also an orbit of the connected Lie subgroup of AN with Lie algebra $\mathfrak{a}\oplus(\mathfrak{n}\ominus\mathfrak{v})$. On the other hand, let $\Psi=\{\alpha_{j-k+1},\dots,\alpha_{j-1}\}\subseteq\{\alpha_{j-k+1},\dots,\alpha_j\}=\Phi$. Consider the boundary component $B_\Phi\cong\mathsf{SL}_{k+1}(\mathbb{R})/\mathsf{SO}_{k+1}$ and the cohomogeneity one action on B_Φ of the reductive subgroup $L^0_{\Psi,\Phi}\cong\mathsf{SL}_k(\mathbb{R})\times\mathbb{R}$ of $\mathsf{SL}_{k+1}(\mathbb{R})$ with Lie algebra $\mathfrak{l}_{\Psi,\Phi}$ (recall the notation at the end of §1.3.1). Then the Lie algebra $\mathfrak{l}^\Lambda_{\Psi,\Phi}$ of the group obtained by canonical extension of the action of $L^0_{\Psi,\Phi} \cong \mathsf{SL}_k(\mathbb{R}) \times \mathbb{R}$ on B_Φ to M has the following projection onto $\mathfrak{a} \oplus \mathfrak{n}$:

$$\begin{aligned}(\mathfrak{l}^\Lambda_{\Psi,\Phi})_{\mathfrak{a}\oplus\mathfrak{n}} = (\mathfrak{l}_{\Psi,\Phi} \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi)_{\mathfrak{a}\oplus\mathfrak{n}} &= \mathfrak{a}^\Phi \oplus \mathfrak{n}^\Psi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi \\ &= \mathfrak{a}^\Phi \oplus (\mathfrak{n}^\Phi \ominus \mathfrak{v}) \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}),\end{aligned}$$

where we have used $\mathfrak{n}^\Phi \ominus \mathfrak{n}^\Psi = \mathfrak{n}_{\Psi,\Phi} = \mathfrak{g}_{\alpha_{j-k+1}+\cdots+\alpha_j} \oplus \cdots \oplus \mathfrak{g}_{\alpha_j} = \mathfrak{v}$. By dimension reasons, the singular orbits of the cohomogeneity one actions of $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$ and $(L^0_{\Psi,\Phi})^\Lambda$ on M coincide. Hence, both actions have the same orbits. We conclude that the action of $H_{\Lambda\setminus\{\alpha_j\},\mathfrak{v}}$ is orbit equivalent to one of the actions in the second row of the table of item (CE) in Theorem B, namely the canonical extension of the action of $\mathsf{SL}_k(\mathbb{R}) \times \mathbb{R}$ on $B_{\{\alpha_{j-k+1},\dots,\alpha_j\}}$ to M. ☐

3.5.2 Cohomogeneity one actions on products of symmetric spaces of noncompact type

Consider a symmetric space of noncompact type $M = M_1 \times \cdots \times M_s = G/K$, where each $M_i = G_i/K_i$, $i = 1, \dots, s$, is irreducible, $G = \prod_{i=1}^s G_i$, and $K = \prod_{i=1}^s K_i$. Clearly, the root system of $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ splits as the orthogonal disjoint union $\Sigma = \bigsqcup_{i=1}^s \Sigma_i$, where Σ_i is the root system of $\mathfrak{g}_i$. Similarly, a set of simple roots for $\mathfrak{g}$ is given by $\Lambda = \bigsqcup_{i=1}^s \Lambda_s$, where $\Lambda_i \subseteq \Sigma_i^+$ is a set of simple roots for $\mathfrak{g}_i$, $i = 1, \dots, s$. We will denote by $\mathfrak{k}_i \oplus \mathfrak{a}_i \oplus \mathfrak{n}_i$ the associated Iwasawa decomposition of $\mathfrak{g}_i$. Observe that $\mathfrak{a}_i = \mathfrak{a}^{\Lambda_i}$ and $\mathfrak{n}_i = \mathfrak{n}^{\Lambda_i}$, $i = 1, \dots, s$. Of course, we have orthogonal direct sums $\mathfrak{k} = \bigoplus_{i=1}^s \mathfrak{k}_i$, $\mathfrak{a} = \bigoplus_{i=1}^s \mathfrak{a}_i$ and $\mathfrak{n} = \bigoplus_{i=1}^s \mathfrak{n}_i$.

Proof of Theorem C. We have to analyze the different cases arising in Theorem A when applied to a reducible M. First note that cases (FH) and (CER) in Theorem A correspond directly to cases (FH) and (CER) in Theorem D, respectively.

An action of type (FS) in Theorem A is induced by the Lie algebra $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$, where ℓ is a subspace of a simple root space $\mathfrak{g}_\beta$, $\beta \in \Lambda$, with $\dim \ell = 1$. Then $\beta \in \Lambda_j$, for some $j \in \{1, \dots, s\}$, and hence

$$\begin{aligned}\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell) &= \mathfrak{a} \oplus \Bigl(\bigoplus_{\substack{i=1\\ i\neq j}}^{s} \mathfrak{n}_i\Bigr) \oplus (\mathfrak{n}_j \ominus \ell) \\ &= \Bigl(\bigoplus_{\substack{i=1\\ i\neq j}}^{s} (\mathfrak{a}_i \oplus \mathfrak{n}_i)\Bigr) \oplus (\mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \ell)) \\ &= \mathfrak{a}_{\Lambda_j} \oplus \mathfrak{n}_{\Lambda_j} \oplus \mathfrak{h}_j,\end{aligned}$$

where $\mathfrak{h}_j = \mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \ell)$. Then the corresponding action is the canonical extension of the H_j-action on $B_{\Lambda_j} \cong M_j$ to M, where H_j is the connected subgroup of G_j with Lie algebra $\mathfrak{h}_j$. By Lemma 3.11, such action is orbit equivalent to the action of $H_j \times \prod_{\substack{i=1\\ i\neq j}}^{s} G_i$, which corresponds to case (Prod) in the statement of Theorem C.

Case (CEI) of Theorem A concerns canonical extensions of cohomogeneity one actions with a totally geodesic singular orbit on an irreducible boundary component B_Φ, for some connected subset Φ of roots in the Dynkin diagram. The corresponding Lie algebras are of the form $\mathfrak{h}_\Phi^\Lambda = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, for some maximal proper reductive subalgebra $\mathfrak{h}_\Phi$ of $\mathfrak{s}_\Phi$ whose corresponding Lie subgroup of S_Φ acts with cohomogeneity one on B_Φ. In our setting, being B_Φ irreducible implies $\Phi \subseteq \Lambda_j$, for some $j \in \{1, \dots, s\}$. Hence

$$\begin{aligned}\mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi &= \mathfrak{h}_\Phi \oplus (\mathfrak{a}_{\Phi,\Lambda_j} \oplus \mathfrak{a}_{\Lambda_j}) \oplus (\mathfrak{n}_{\Phi,\Lambda_j} \oplus \mathfrak{n}_{\Lambda_j}) \\ &= (\mathfrak{h}_\Phi \oplus \mathfrak{a}_{\Phi,\Lambda_j} \oplus \mathfrak{n}_{\Phi,\Lambda_j}) \oplus \mathfrak{a}_{\Lambda_j} \oplus \mathfrak{n}_{\Lambda_j},\end{aligned}$$

which means that the action is a composition of canonical extensions, firstly from B_Φ to $B_{\Lambda_j} \cong M_j$, and secondly from $B_{\Lambda_j} \cong M_j$ to M. Again by Lemma 3.11 we get that the action of the connected subgroup of G with Lie algebra $\mathfrak{h}_\Phi^\Lambda$ has the same orbits as the action of $H_j \times \prod_{\substack{i=1\\ i\neq j}}^{s} G_i$ on M, where H_j is the connected subgroup of G with Lie algebra $\mathfrak{h}_\Phi \oplus \mathfrak{a}_{\Phi,\Lambda_j} \oplus \mathfrak{n}_{\Phi,\Lambda_j}$. This fits again into case (Prod) in the statement.

Finally, case (NC) of Theorem A describes a nilpotent construction from a subspace $\mathfrak{v}$ of $\mathfrak{n}^1_{\Lambda\setminus\{\beta\}}$ for some $\beta \in \Lambda$, $\dim \mathfrak{v} \geq 2$. Let $j \in \{1, \dots, s\}$ such that $\beta \in \Lambda_j$. Then,

$$\mathfrak{n}^1_{\Lambda\setminus\{\beta\}} = \mathfrak{n}^1_{\Lambda\setminus\{\beta\},\Lambda_j} \subseteq \mathfrak{n}_{\Lambda\setminus\{\beta\}} = \mathfrak{n}_{\Lambda\setminus\{\beta\},\Lambda_j} \subseteq \mathfrak{n}^{\Lambda_j} = \mathfrak{n}_j,$$

since any root not spanned by $\Lambda \setminus \{\beta\}$ must be spanned by roots in Λ_j. Note that $\mathfrak{l}_{\Lambda\setminus\{\beta\}} = \big(\bigoplus_{\substack{i=1\\ i\neq j}}^{s} \mathfrak{g}_i\big) \oplus \mathfrak{l}_{\Lambda_j\setminus\{\beta\},\Lambda_j}$. Hence the Lie algebra of the group $H_{\Lambda_j\setminus\{\beta\},\mathfrak{v}}$ built by nilpotent construction from the choice $\mathfrak{v} \subseteq \mathfrak{n}^1_{\Lambda\setminus\{\beta\}} = \mathfrak{n}^1_{\Lambda\setminus\{\beta\},\Lambda_j}$ is

$$\begin{aligned}N_{\mathfrak{l}_{\Lambda\setminus\{\beta\}}}(\mathfrak{n}_{\Lambda\setminus\{\beta\}} \ominus \mathfrak{v}) \oplus (\mathfrak{n}_{\Lambda\setminus\{\beta\}} \ominus \mathfrak{v}) = \Big(\bigoplus_{\substack{i=1\\ i\neq j}}^{s} \mathfrak{g}_i\Big) &\oplus N_{\mathfrak{l}_{\Lambda_j\setminus\{\beta\},\Lambda_j}}(\mathfrak{n}_{\Lambda_j\setminus\{\beta\},\Lambda_j} \ominus \mathfrak{v}) \\ &\oplus (\mathfrak{n}_{\Lambda_j\setminus\{\beta\},\Lambda_j} \ominus \mathfrak{v}),\end{aligned}$$

where the two last direct addends of the right-hand term constitute a Lie subalgebra of $\mathfrak{g}_j$. (Indeed, it is not difficult to show that the associated connected subgroup of G_j yields the cohomogeneity one action on M_j obtained by nilpotent construction from the choice of $\mathfrak{v}$ as a subspace of $\mathfrak{n}^1_{\Lambda\setminus\{\beta\},\Lambda_j}$). We conclude that the group $H_{\Lambda_j\setminus\{\beta\},\mathfrak{v}}$ splits nicely with respect to the decomposition of G, and hence corresponds again to an action of type (Prod). $\square$

3.5.3 Cohomogeneity one actions on products of hyperbolic spaces

In this subsection, $M = M_1 \times \cdots \times M_r$ is a product of symmetric spaces of non-compact type and rank one, $M_i = G_i/K_i = \mathbb{F}_i\mathsf{H}^{n_i}$, where $\mathbb{F}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $i = 1, \dots, r$. We use the other notations stated at the beginning of §3.5.2. Moreover, we have $\Lambda = \{\alpha_1, \dots, \alpha_r\}$ with $\Lambda_i = \{\alpha_i\}$, $\Sigma^+ = \Lambda \cup \{2\alpha_i : \mathbb{F}_i \neq \mathbb{R}\}$, and $\mathfrak{a}_i \cong \mathbb{R}$, for each $i = 1, \dots, r$.

Proof of Theorem D. We go through the three types of actions in Theorem C. First, assume we have an action of (Prod) type, that is, the action of a subgroup $H = H_j \times \prod_{\substack{i=1\\ i\neq j}}^{r} G_i$ of G, where H_j is a connected subgroup of G_j acting with cohomogeneity one on the rank one space $M_j = B_{\{\alpha_j\}} = \mathbb{F}_j\mathsf{H}^{n_j}$. By the classification of cohomogeneity one actions on rank one spaces [14, 39], we distiguish four cases:

1. H_j produces a foliation of horospherical type. In this case, up to orbit equivalence, we can assume $\mathfrak{h}_j = \mathfrak{n}_j$ (since $\dim \mathfrak{a}_j = 1$), and it is easy to realize that H induces a foliation of horospherical type on M with the same orbits as the action of the connected subgroup of G with Lie algebra $(\mathfrak{a} \ominus \mathfrak{a}_j) \oplus \mathfrak{n}$. This corresponds to item (FH) in the statement.

2. H_j produces a foliation of solvable type. In this case we can assume $\mathfrak{h}_j = \mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \ell)$, for some one-dimensional subspace ℓ of $\mathfrak{g}_{\alpha_j}$. As above, one can see that the H-action is one of the actions described in item (FS) of the statement.

3. H_j acts with cohomogeneity one and a totally geodesic singular orbit on M_j, which translates directly into type (CEI) of the statement.

4. H_j acts with cohomogeneity one and a non-totally geodesic singular orbit on M_j. In this case, $\mathfrak{h}_j$ can be taken of the form $N_{(\mathfrak{k}_j)_0}(\mathfrak{v}) \oplus \mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \mathfrak{v})$, for some protohomogeneous subspace $\mathfrak{v}$ of $\mathfrak{g}_{\alpha_j}$ with $\dim \mathfrak{v} \geq 2$. This yields an action of type (NC) in the statement.

Now, clearly an action of (FH) type in Theorem C corresponds to an action of the same type in Theorem D.

Finally, actions of type (CER) are induced by groups H_Φ^Λ with Lie algebras of the type $\mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, where $\Phi \subseteq \Lambda$ determines a reducible rank two boundary component B_Φ, which, in the current context, is of the form $B_\Phi = M_j \times M_k$, for $\Phi = \{\alpha_j, \alpha_k\}$, $j, k \in \{1, \dots, r\}$, $j \neq k$. Hence, $\mathfrak{s}_{\{\alpha_j\}} = \mathfrak{g}_j$ and $\mathfrak{s}_{\{\alpha_k\}} = \mathfrak{g}_k$, so the Lie algebra of the group acting diagonally on B_Φ is $\mathfrak{h}_\Phi = \{X + \sigma X : X \in \mathfrak{g}_j\} = \mathfrak{g}_{j,k,\sigma}$, for some Lie algebra isomorphism $\sigma \colon \mathfrak{g}_j \to \mathfrak{g}_k$. Since Φ and $\Lambda \setminus \Phi$ are the sets of simple roots associated with $\mathfrak{g}_j \oplus \mathfrak{g}_k$ and $\bigoplus_{\substack{i=1\\ i\neq j,k}} \mathfrak{g}_i$, respectively, we can apply

Lemma 3.11 to conclude that the H_{Φ}^{Λ}-action is orbit equivalent to the action of the connected subgroup of G with Lie algebra $\bigoplus_{\substack{i=1 \\ i\neq j,k}} \mathfrak{g}_i \oplus \mathfrak{g}_{j,k,\sigma}$, as in item (CER) of the statement. $\square$

Chapter 4

Cohomogeneity one actions on products

The purpose of this chapter is the investigation of cohomogeneity one actions on symmetric spaces that are not of a particular given type (namely, compact, noncompact or Euclidean), with the ultimate aim to classify them. As it was pointed out in the exposition in Chapter 2, cohomogeneity one actions on symmetric spaces have been studied in a type by type basis so far. Recall that the universal cover of a symmetric space M splits as a Riemannian product

$$\widetilde{M} = M_+ \times M_0 \times M_-$$

where M_+ is a symmetric space of compact type, M_0 is a Euclidean space, and M_- is of noncompact type. Here, we begin the study of cohomogeneity one actions on symmetric spaces of "mixed type", that is, those for which the above decomposition is nontrivial. The results presented in this chapter were obtained in collaboration with Professor Hiroshi Tamaru of the Osaka Metropolitan University, and are currently under preparation for their publication.

For our investigation, we study the behavior of the action with respect to the aforementioned decomposition: if $\{H_i \curvearrowright M_i\}_{i=1}^n$ is a (finite) collection of isometric actions, one can naturally define an action of $H_1 \times \cdots \times H_n$ on $M_1 \times \cdots \times M_n$, which is called the *product action* of the H_i, and whose cohomogeneity is precisely the sum of the cohomogeneities of the $H_i \curvearrowright M_i$. An action on a product $M_1 \times \cdots \times M_n$ is said to *decompose* if it has the same orbits as some product action. In Section 4.1, we develop some structural results regarding isometric and cohomogeneity one actions on general Riemannian products. Essentially, our aim is to determine the decomposability of the action from its behavior on the different factors of the product. As an application of these results, we obtain the following:

Corollary 4.11. *Let M be a simply connected compact Riemannian manifold, N a Hadamard manifold, and let $H \subseteq I(M \times N)$ be a connected closed subgroup of isometries acting on the Riemannian product $M \times N$ with cohomogeneity one. Then, the H-action decomposes.*

Consider now a Riemannian symmetric space M, and assume that it is simply connected. Let $M = M_+ \times M_0 \times M_-$ be the decomposition of M into its compact, Euclidean, and noncompact factors. Then, since M_+ is compact and $M_0 \times M_-$ is Hadamard, one can apply the above corollary to argue that, with the possible exception of indecomposable actions on $M_0 \times M_-$, it is enough to study cohomogeneity one actions on a type by type basis.

Theorem E. *Let $M = M_+ \times M_0 \times M_-$ be a simply connected symmetric space, and let H be a connected closed subgroup of $I(M)$ acting on M with cohomogeneity one. Then, either the H-action on $M_+ \times M_0 \times M_-$ decomposes, or it is orbit equivalent to the action of $I(M_+) \times H_\Delta$, where $H_\Delta \subseteq I(M_0) \times I(M_-)$ acts with cohomogeneity one on $M_0 \times M_-$ and transtitively on both M_0 and M_-.*

This allows us to produce the main result of this chapter, which provides a complete classification of cohomogeneity one actions without singular orbits (equivalently, homogeneous codimension one foliations) on simply connected symmetric spaces. It is not difficult to show that M_+ does not admit cohomogeneity one actions without singular orbits. For $M_0 \cong \mathbb{R}^n$, the only homogeneous codimension one foliation is, up to congruence, the one given by parallel hyperplanes. For symmetric spaces of noncompact type, actions without singular orbits have been studied extensively in [12, 85], and they correspond to codimension one subgroups of the solvable model of M_-, up to orbit equivalence. By studying actions on $M_0 \times M_-$ that do not decompose, we obtain that every cohomogeneity one action without singular orbits can be constructed in a way similar to the examples given in [12] for symmetric spaces of noncompact type.

In order to introduce the precise statement of this result, let us fix some terminology. Write $M_- = G/K$, where G is the identity component of the isometry group of M_-, and K is the isotropy subgroup at some fixed $o \in M$, and consider the corresponding Cartan decomposition of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Fix a choice of a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ and a set of simple roots Λ, and let Σ^+ be the corresponding positive roots. This determines an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and the connected Lie subgroup AN of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ acts simply transitively on M (cf. Section 1.3 for details). Note that the action of $\mathbb{R}^n$ on itself by translations is also simply transitive. Hence, codimension one subgroups of $\mathbb{R}^n \times AN$ give rise to codimension one homogeneous foliations of $M_0 \times M_-$. Our main result guarantees that every cohomogeneity one action without singular orbits on a simply connected symmetric space can essentially be obtained via this construction.

Theorem F. *Let $M = M_+ \times M_0 \times M_-$ be a simply connected symmetric space, where M_+ is of compact type, $M_0 \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$, and $M_- = G/K$ is a symmetric space of noncompact type. Let $H \subseteq I(M)$ be a connected closed subgroup* *acting on M with cohomogeneity one and no singular orbits. Then, the H-action is orbit equivalent to one of the following:*

1. *The action of $I(M_+) \times \mathbb{R}^{n-1} \times G$.*
2. *The action of $I(M_+) \times \mathbb{R}^n \times H_X$, where H_X is the connected subgroup of G with Lie algebra $\mathfrak{h}_X = (\mathfrak{a} \ominus X) \oplus \mathfrak{n}$, for some nonzero vector $X \in \mathfrak{a}$.*
3. *The action of $I(M_+) \times \mathbb{R}^n \times H_i$, where H_i is the connected subgroup of G with Lie algebra $\mathfrak{h}_i = \mathfrak{a} \oplus (\mathfrak{n} \ominus X)$, for some nonzero vector $X \in \mathfrak{g}_{\alpha_i}$ of a simple root space.*
4. *The action of $I(M_+) \times H_{E,X}$, where H the connected Lie subgroup of $\mathbb{R}^n \times G$ with Lie algebra $\mathfrak{h}_{E,X} = (\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E + X) \oplus (\mathfrak{a} \ominus X) \oplus \mathfrak{n}$, for some nonzero vectors $E \in \mathfrak{r}^n$ and $X \in \mathfrak{a}$.*

Here, $\mathfrak{r}^n$ stands for the abelian Lie algebra of the translation group $\mathbb{R}^n$ on $M_0 \cong \mathbb{R}^n$, and $\mathfrak{v} \ominus Y$ stands for the orthogonal complement of some vector Y in a subspace $\mathfrak{v}$ of $\mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n} \cong T_o(M_0 \times M_-)$ with respect to the inner product induced by the metric of M.

Remark 4.1. Unlike the structural results in Chapters 2 and 3, it should be noted that the results in Theorem F have been stated in terms of the inner product induced by the metric of M, and not the metric coming from the Killing form. This was done mainly for the sake of convenience when computing the geometry of the orbits of $H_{E,X}$ in Section 4.3. Nevertheless, recall (see Remark 1.20) that for any product of symmetric spaces $M_0 \times M_1 \times \cdots \times M_k$ with M_0 maximal Euclidean and $M_i \cong G_i/K_i$ irreducible, we have that the inner product on $\mathfrak{p}$ induced by the metric of M is given by

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{|\mathfrak{p}_0\times\mathfrak{p}_0} + \lambda_1 \left(\mathcal{B}_1\right)_{|\mathfrak{p}_1\times\mathfrak{p}_1} + \cdots + \lambda_k \left(\mathcal{B}_k\right)_{|\mathfrak{p}_k\times\mathfrak{p}_k},$$

for some $\lambda_i \neq 0$, $i = 1, \dots, k$. As already pointed out in Remark 3.1, any two symmetric metrics g, g' yield $I^0(M, g) = I^0(M, g') = I(M)$, and thus a connected subgroup of $I(M)$ acts with cohomogeneity one on (M, g) if and only if it acts with cohomogeneity one on (M, g'). For example, the subalgebras $\mathfrak{h}_X$ and $\mathfrak{h}_i$ described in Theorem F are exactly the ones giving rise to the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_i$ described in §2.3.1. However, here we have described them in terms of the induced metric in $\mathfrak{a} \oplus \mathfrak{n}$ instead of the canonical inner product of $\mathfrak{g}$ induced by its Killing form (so as to avoid fixing the metric of M).

Remark 4.2. In the statement of Theorem F we are implicitly assuming that the factors in decomposition $M_+ \times M_0 \times M_-$ are nontrivial in order to describe the different items. However, this result can easily be adapted by removing items from the list if the corresponding factors are trivial. For example, if M_0 is nontrivial and $M_- = \{*\}$, then H is orbit equivalent to the action described in item (1) with G trivial. If $M_0 = \{*\}$ and M_- is nontrivial, then H is orbit equivalent to one of the actions described in items (2) and (3). Finally, if both M_0 and M_- are trivial factors, none of the actions in items (1)–(4) can be defined, and $M = M_+$ admits no cohomogeneity one actions without singular orbits.

Note that the action in (1) comes essentially from the action of $\mathbb{R}^{n-1}$ on $\mathbb{R}^n$, whose orbits are parallel hyperplanes. Similarly, items (2) and (3) are product actions coming from cohomogeneity one actions on M_-. More precisely, actions in (2) come from actions on M_- whose orbits give rise to a foliation of horospherical type, while the actions (3) come from actions on M_- producing foliations of solvable type. Both foliation types and their properties were described in §2.3.1. The actions (4), corresponding to the new nontrivial examples, do not decompose.

We also study the geometry of orbits in item (4), and show that all orbits are mutually congruent. Moreover, we obtain a minimality condition for the orbits in terms of the roots of M_-. Namely, the orbits of $H_{E,X}$ are minimal if and only if $\sum_{\alpha\in\Sigma^+} \dim(\mathfrak{g}_\alpha)\,\alpha(X) = 0$. This condition turns out to be the same as the one obtained in [12] for the foliations $\mathcal{F}_\ell$ in Section 2.3.1 to be minimal.

Theorems E and F can be applied to derive explicit classifications of homogeneous hypersurfaces on certain ambient manifolds. Recall that homogeneous hypersurfaces provide the simplest examples of hypersurfaces with constant principal curvatures, hypersurfaces with constant mean curvature, or isoparametric hypersurfaces. After the seminal work of Abresch and Rosenberg [1], the study of surfaces with constant mean curvature in products of space forms has been a vibrant area of research. Recently, Chaves and Santos have also studied hypersurfaces with constant principal curvatures in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{R}\mathsf{H}^n \times \mathbb{R}$ in [28], where they give an explicit classification when the number of distinct principal curvatures is ≤ 3. In the recent work of Domínguez-Vázquez and Manzano [45], the authors classify isoparametric hypersurfaces in the family $\mathbb{E}(\kappa,\tau)$ of simply connected homogeneous 3-spaces with a 4-dimensional isometry group. This family contains the products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{R}\mathsf{H}^2 \times \mathbb{R}$. Also, dos Santos and dos Santos [82] have studied isoparametric hypersurfaces in the products of 2-dimensional space forms. It should be noted that all of the examples obtained in [28, 45, 82] are homogeneous hypersurfaces of the corresponding product spaces.

Example 4.3 *(Homogeneous hypersurfaces of $\mathbb{S}^n \times \mathbb{R}^m$ and $\mathbb{S}^n \times \mathbb{R}\mathsf{H}^m$).* Suppose that $P \subseteq \mathbb{S}^n \times \mathbb{R}^m$ is a homogeneous hypersurface. Then, a straightforward application of Theorem E yields that P must split as a product $P_{\mathbb{S}^n} \times \mathbb{R}^m$ or $\mathbb{S}^n \times P_{\mathbb{R}^m}$ where $P_{\mathbb{S}^n}$ and $P_{\mathbb{R}^m}$ are homogeneous hypersurfaces of $\mathbb{S}^n$ and $\mathbb{R}^m$, respectively. It follows from the results of [55] and [81] that $P_{\mathbb{R}^m}$ is a hyperplane, a sphere, or a (generalized) cylinder, and $P_{\mathbb{S}^n}$ is congruent to a principal orbit of the action obtained by restriction to the unit sphere of the isotropy representation of a symmetric space of rank two (for more information, refer to the exposition in §2.1.1).

Similarly, if P is a homogeneous hypersurface of $\mathbb{S}^n \times \mathbb{R}\mathsf{H}^m$, then $P = P_{\mathbb{S}^n} \times \mathbb{R}\mathsf{H}^m$ for some homogeneous $P_{\mathbb{S}^n} \subseteq \mathbb{S}^n$, or $P = \mathbb{S}^n \times P_{\mathbb{R}\mathsf{H}^m}$, where $P_{\mathbb{R}\mathsf{H}^m}$ is either a horosphere, a totally geodesic $\mathbb{R}\mathsf{H}^{n-1}$ or one of its equidistant hypersurfaces, a tube around a totally geodesic $\mathbb{R}\mathsf{H}^k$, or a geodesic sphere in $\mathbb{R}\mathsf{H}^m$ (see Theorem 2.8).

Example 4.4 (*Homogeneous hypersurfaces of* $\mathbb{R} \times \mathbb{R}\mathsf{H}^n$)**.** Suppose that H is a group acting with cohomogeneity one and a singular orbit on $\mathbb{R}\times M_-$, where $M_- = G/K$ is a symmetric space of noncompact type. The isotropy $H \cap (\mathsf{SO}_1 \times K)$ acts transitively in the unit sphere of the normal space $\nu_o(H \cdot o)$, so we must have that $\nu_o(H \cdot o) \subseteq T_o(M_-)$. Since $H \subseteq H' = (\mathsf{SO}_1 \ltimes \mathbb{R}) \times \mathrm{pr}_G(H)$ (where $\mathrm{pr}_G \colon I(\mathbb{R} \times M_-) \to G$ is the projection onto G), the orbits of H are contained in the orbits of H'. Now, H' cannot act transitively on $\mathbb{R} \times M_-$, since $\nu_o(H \cdot o) = \nu_o(H' \cdot o) \subseteq T_o(M_-)$, and it follows that the actions of H and H' have the same orbits. Therefore, the only cohomogeneity one action on $\mathbb{R} \times M_-$ that does not decompose is the one described in item (4) of Theorem F.

In particular, it follows that if $P \subseteq \mathbb{R} \times \mathbb{R}\mathsf{H}^n$ is a homogeneous hypersurface, either $P = \{t\} \times \mathbb{R}\mathsf{H}^n$ or $P = \mathbb{R} \times P_{\mathbb{R}\mathsf{H}^n}$, where $P_{\mathbb{R}\mathsf{H}^n}$ is a geodesic sphere, a horosphere, a totally geodesic $\mathbb{R}\mathsf{H}^{n-1}$ or one of its equidistant hypersurfaces, or a tube around a totally geodesic $\mathbb{R}\mathsf{H}^k$; or P is an orbit of $H_{E,X}$ as described in item (4) of Theorem F. The orbits of $H_{E,X}$ are precisely the only examples of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures obtained in [28, 45, 82] that do not split as products. In [45], the authors refer to them as “parabolic helicoids” for the case $n = 2$.

The structure of this chapter is the following. In Section 4.1, we will deal with isometric actions on Riemannian products, and prove some decomposability results which allow us to prove Theorem E. Then, in Section 4.2, we will restrict ourselves to study actions without singular orbits on simply connected symmetric spaces, culminating in the proof of Theorem F. Finally, in Section 4.3, we briefly compute the geometry of the orbits of $H_{E,X}$.

4.1 Isometric actions on products

Given an isometric action on a Riemannian product $M_1 \times \cdots \times M_n$ it makes sense to ask if one can retrieve information about the action from its behavior on each of the individual factors M_i. In this section, we introduce a notion of decomposability for isometric actions following the approach used by Kollross for products of symmetric spaces of compact type [63]. It should be noted that if one wants to make this kind of study for general Riemannian manifolds, one encounters some technical difficulties. Namely, a proper isometric action on a Riemannian product may not induce actions on its factors and, if it does, this induced actions may fail to be proper. In this section, we deal with these obstacles, and prove some decomposability results for cohomogeneity one actions.

Given a collection of Lie groups $H_1, \dots, H_n$ and a collection of Riemannian manifolds $M_1, \dots, M_n$ such that H_i acts isometrically on M_i for $i = 1, \dots n$, one can define the *product action* of $H_1 \times \cdots \times H_n$ on $M_1 \times \cdots \times M_n$ by

$$(h_1, \dots, h_n) \cdot (p_1, \dots, p_n) = (h_1 \cdot p_1, \dots, h_n \cdot p_n),$$

for $h_i \in H_i$ and $p_i \in M_i$. Note that the orbits of this action are precisely the product of the orbits of the H_i-actions on each M_i. An isometric action of a Lie group H on a Riemannian product $M_1 \times \cdots \times M_n$ is said to *decompose* if there exist Lie groups $H_1, \dots, H_n$ such that each H_i acts isometrically on M_i for $i = 1, \dots, n$ and the H-action has the same orbits as the product action of $H_1 \times \cdots \times H_n$ on $M_1 \times \cdots \times M_n$.

Note that this definition naturally depends on the product expression and thus is not invariant under isometries. In order to solve this, we say that an isometric action of a Lie group H on a Riemannian manifold M is *decomposable* if there exists a decomposition of M as a Riemannian product $M = M_1 \times \cdots \times M_n$ and the H-action on $M_1 \times \cdots \times M_n$ decomposes. Otherwise, the action is said to be *indecomposable*.

Remark 4.5. This notion of decomposability is more restrictive than the one given in [63], where an action of a Lie group H on a product $M_1 \times \cdots \times M_n$ is said to decompose if it is orbit equivalent to a product action. However, this added rigidity allows us to better understand how isometric actions on a Riemannian manifold M behave with respect to a given decomposition of M as a product, and seems not to be bothersome in their study.

Let $M = M_1 \times \cdots \times M_n$ be a Riemannian product, and denote by $I(M_i)$ the isometry group of each factor M_i. Let $H \subset I(M)$ be a group acting isometrically on M, and suppose $H \subseteq I(M_1) \times \cdots \times I(M_n)$ as a subgroup. Then, the action of H on M naturally induces an isometric action on each of the factors by taking

$$(h_1, \dots, h_n) \cdot p_i = h_i \cdot p_i$$

for $(h_1, \dots, h_n) \in H$ and $p_i \in M_i$, known as the *projection action* of H on M_i. The orbits of the projection action of H on M_i are exactly the projections of the H-orbits on M to M_i. If $\mathrm{pr}_{I(M_i)} \colon I(M_1) \times \cdots \times I(M_n) \to I(M_i)$ denotes the natural projection onto $I(M_i)$, the efectivization of the projection action of H on M_i is the natural action of $\mathrm{pr}_{I(M_i)}(H)$ on M_i, so both actions have the same orbits.

Remark 4.6. In general, the isometry group of a Riemannian product $M_1 \times \cdots \times M_n$ is not $I(M_1) \times \cdots \times I(M_n)$, but possibly a larger group. For instance, if $p, q \geq 1$ are positive integers, $I(\mathbb{R}^p) \times I(\mathbb{R}^q) = (\mathsf{O}_p \times \mathsf{O}_q) \ltimes \mathbb{R}^{p+q} \subsetneq \mathsf{O}_{p+q} \ltimes \mathbb{R}^{p+q} = I(\mathbb{R}^p \times \mathbb{R}^q)$. Thus, projection actions are not defined for general products of Riemannian manifolds. However we will see later that under certain assumptions it is possible to guarantee that a proper isometric action on a product $M_1 \times \cdots \times M_n$ is orbit equivalent to the action of some connected closed subgroup $H \subseteq I(M_1) \times \cdots \times I(M_n)$.

If $H \subseteq I(M_1) \times \cdots \times I(M_n)$ is a Lie group acting isometrically on a product $M_1 \times \cdots \times M_n$, the orbit through a point $(p_1, \ldots, p_n)$ is contained in the product of the projection actions of H on each M_i through $p_i \in M_i$, that is, $H \cdot (p_1, \ldots, p_n) \subseteq H \cdot p_1 \times \cdots \times H \cdot p_n$. Thus, if we denote by $\operatorname{cohom}(H \curvearrowright M_1 \times \cdots \times M_n)$ the cohomogeneity of the H-action on $M_1 \times \cdots \times M_n$ and by $\operatorname{cohom}(H \curvearrowright M_i)$ the cohomogeneity of the projection action of H on M_i, one has that

$$\operatorname{cohom}(H \curvearrowright M_1 \times \cdots \times M_n) \geq \sum_{i=1}^{n} \operatorname{cohom}(H \curvearrowright M_i). \tag{4.1}$$

In particular, if $\operatorname{cohom}(H \curvearrowright M_1 \times \cdots \times M_n) = 1$, at most one of the projection actions is of cohomogeneity one and the remaining ones are transitive. Moreover, if the action of H on $M_1 \times \cdots \times M_n$ decomposes, the orbits of the H-action are of the form $H \cdot (p_1, \ldots, p_n) = H_1 \cdot p_1 \times \cdots \times H_n \cdot p_n$ for some Lie groups H_i acting isometrically on each factor. Now, the orbits of the projection action of H on M_i are precisely the projection of the orbits of H to M_i, that is, $H \cdot p_i = H_i \cdot p_i$. Therefore, the H-action decomposes precisely if it has the same orbits as the product of its projection actions. In particular, equality holds in (4.1).

It makes sense to ask if the converse is also true. Namely, if the H-action decomposes when equality holds in (4.1). However, it should be noted that the projection action of H on each factor is not necessarily a proper isometric action, and so its orbits might fail to be embedded. Thus, one could have that $\dim(H \cdot (p_1, \ldots, p_n)) = \sum \dim(H \cdot p_i)$ but $H \cdot (p_1, \ldots, p_n) \subsetneq H \cdot p_1 \times \cdots \times H \cdot p_n$. Nevertheless, if the space is simply connected and H acts with cohomogeneity one, the following proposition ensures that the orbits of the projection actions are closed and embedded and this unwanted behavior does not happen.

Proposition 4.7. *Let $M_1 \times \cdots \times M_n$ be a simply connected complete Riemannian product and $H \subseteq I(M_1) \times \cdots \times I(M_n)$ a connected closed Lie group acting on $M_1 \times \cdots \times M_n$ with cohomogeneity one. Then, the action decomposes if and only if $\sum_{i=1}^{n} \operatorname{cohom}(H \curvearrowright M_i) = 1$.*

Proof. If the action decomposes, we have seen that equality holds in (4.1), so let us prove the converse.

Let $p = (p_1, \dots, p_n) \in M_1 \times \cdots \times M_n$ be a point such that $H \cdot p$ is a regular orbit and let Σ be a normal geodesic to $H \cdot p$ through p. Then, Σ meets all of the H-orbits orthogonally. Since $H \cdot (q_1, \dots, q_n) \subseteq H \cdot q_1 \times \cdots \times H \cdot q_n$ for every $q_1 \in M_1, \dots, q_n \in M_n$, Σ also meets all of the orbits of $H \times \cdots \times H$ on $M_1 \times \cdots \times M_n$. Moreover, since $\dim T_p(H \cdot p) = \sum \dim T_{p_i}(H \cdot p_i)$ by assumption, Σ is orthogonal to $H \cdot p_1 \times \cdots \times H \cdot p_n$. Now, the orbits of $H \times \cdots \times H$ form a singular Riemannian foliation, so Σ remains orthogonal to every orbit it meets, making it a section for the $H \times \cdots \times H$-action. Since $M_1 \times \cdots \times M_n$ is simply connected and complete, it follows from [69, Theorem 1.2] (see also [2, Corollary 1.4]) that the orbits of $H \times \cdots \times H$ are closed embedded submanifolds of M. Thus, the inclusion map $H \cdot p \to H \cdot p_1 \times \cdots \times H \cdot p_n$ is closed and continuous. Now, by dimension reasons $H \cdot p$ is open in $H \cdot p_1 \times \cdots \times H \cdot p_n$, and it follows that H has the same orbits as the product of its projection actions. ☐

Also, if M is a simply connected and complete Riemannian manifold, by the de Rham decomposition theorem, there exists a unique (up to isometry and permutation of the factors) decomposition of M as a Riemannian product

$$M = M_0 \times M_1 \times \cdots \times M_n,$$

where $M_0 \cong \mathbb{R}^m$ for some $m \in \mathbb{N}$ is a maximal Euclidean factor and every other M_i, $i = 1, \dots, n$, is a complete simply connected irreducible Riemannian manifold in the sense that it does not decompose as a Riemannian product. The following result of Solonenko ensures that the isometry group of a simply connected complete Riemannian manifold behaves nicely with respect to this decomposition:

Proposition 4.8 [85, Proposition 2.1]**.** *Let $M = M_0 \times M_1 \times \cdots \times M_n$ be a Riemannian product, where M_0 is a maximal flat factor and all M_i's are connected irreducible Riemannian manifolds for $1 \leq i \leq n$. Then, the isometry group $I(M)$ decomposes as a semidirect product of its subgroups*

$$[I(M_0) \times I(M_1) \times \cdots \times I(M_n)] \rtimes S_{M_1 \times \cdots \times M_n} = I(M)$$

where $S_{M_1 \times \cdots \times M_n}$ stands for the subgroup of the symmetric group S_n consisting of the elements that permute the isometric factors in the above decomposition of M.

It follows immediately that the identity component of the isometry group of a simply connected complete Riemannian manifold M decomposes as a product of the identity components of the isometry groups of its de Rham factors. Thus, if H is a connected group acting by isometries on a complete simply connected Riemannian manifold M, and $M = M_0 \times M_1 \times \cdots \times M_n$ is the de Rham decomposition of M,

$$H \subseteq I^0(M) = I^0(M_0) \times I^0(M_1) \times \cdots \times I^0(M_n) \subseteq I(M_0) \times I(M_1) \times \cdots \times I(M_n),$$

and the projection action of H on the de Rham factors of M is well-defined. In particular, this guarantees that the projection actions on a product $M \times N$ are well-defined as long as either M or N have no Euclidean factors.

Lemma 4.9. *Let M and N be simply connected and complete Riemannian manifolds, and suppose that M has no Euclidean factors. If H is a connected subgroup of $I(M) \times I(N)$, then $H \subseteq I^0(M) \times I^0(N)$. In particular, the projection actions of H on M and on N are well defined.*

Proof. Let $M = M_1 \times \cdots \times M_r$ and $N = N_0 \times N_1 \times \cdots \times N_s$ be the de Rham decompositions of M and N, respectively, where N_0 is the maximal Euclidean factor of N. Then, the de Rham decomposition of $M \times N$ can be written as

$$M \times N = M_1 \times \cdots \times M_r \times N_0 \times N_1 \times \cdots \times N_s.$$

Let H be a connected Lie group acting properly by isometries on $M \times N$. It follows from Proposition 4.8 that $H \subseteq I^0(M \times N) = I^0(M) \times I^0(N)$. ☐

In particular, if $M \times N$ is a simply connected complete Riemannian product, and M is compact, then M does not have a Euclidean factor, and the projection actions of any connected subgroup of $I(M \times N)$ on M and N are well defined. The next result guarantees that, if M is compact and N is Hadamard, only transitive actions on $M \times N$ are transitive on both M and N.

Proposition 4.10. *Let $M \times N$ be a simply connected complete Riemannian product, where M is compact and N Hadamard, and let H be a connected group acting by isometries on $M \times N$. If both the projection action of H on M and the projection action of H on N are transitive, then the action of H on $M \times N$ is also transitive.*

Proof. Although H need not be a compact Lie group, a result of Montgomery [73, Theorem A] ensures us that there exists a compact subgroup K of H such that K acts transitively on M. Now, K is a compact Lie group, so by Cartan's fixed point theorem the projection action of K on N must have a fixed point $o_N \in N$. Consider an arbitrary point $(p, q) \in M \times N$. Fix a point $o_M \in M$. Since the projection action of H on N is transitive, there exists an element $h \in H$ such that $h \cdot q = o_N$. Let $k \in K$ be such that $k \cdot (h \cdot p) = o_M$. We have

$$(kh) \cdot (p, q) = (k \cdot (h \cdot p), k \cdot o_N) = (o_M, o_N).$$

Thus, H acts transitively on $M \times N$. ☐

Suppose now that H acts with cohomogeneity one on $M \times N$. Then, only one of the projection actions of H on M or N can be transitive by the previous result, and the other one is of cohomogeneity one. Now, Proposition 4.7 guarantees that, if the action of H is proper, then it decomposes as the product of its projection actions:

Corollary 4.11. *Let M be a simply connected compact Riemannian manifold, N a Hadamard manifold, and let $H \subseteq I(M \times N)$ be a connected closed subgroup of isometries acting on $M \times N$ with cohomogeneity one. Then, the H-action decomposes.*

If M is a Riemannian symmetric space, the assumption that M is simply connected turns out to be a very natural one. Namely, a cohomogeneity one action on a non-simply connected symmetric space M can always be lifted in the following sense to its universal cover $\widetilde{M}$:

Proposition 4.12. *Let M be a Riemannian symmetric space, and let $\widetilde{M}$ be its universal cover. Let H be a connected Lie group acting properly by isometries on M. Then, there exists a Lie group $\widetilde{H}$ acting properly by isometries on $\widetilde{M}$ such that the orbits of H are the projection of the orbits of $\widetilde{H}$. In particular, the cohomogeneity of the action of H on M and of the action of $\widetilde{H}$ on $\widetilde{M}$ coincide.*

Proof. Write $M = G/K$, and let $\widetilde{G}$ be the universal cover of G, with covering map $\varphi \colon \widetilde{G} \to G$ (which is a homomorphism of Lie groups). Let $\widetilde{K}$ denote the identity component of $\varphi^{-1}(K)$. Then, the universal cover of M can be written as $\widetilde{M} = \widetilde{G}/\widetilde{K}$. The covering map $\psi \colon \widetilde{M} \to M$ is given by $\psi(g\widetilde{K}) = \varphi(g)K$. Without loss of generality, by the properness assumption we can assume that H is a closed connected subgroup of G. Let $\widetilde{H}$ be the identity component of $\varphi^{-1}(H)$. Then, $\varphi_{|\widetilde{H}} \colon \widetilde{H} \to H$ is a covering map, and $\widetilde{H}$ is a connected closed subgroup of $\widetilde{G}$. Therefore, $\widetilde{H}$ acts properly by isometries on $\widetilde{M}$. It now easily follows from the surjectivity of φ and ψ that $\psi(\widetilde{H} \cdot \tilde{p}) = H \cdot p$ for any $p \in M$ and $\tilde{p} \in \psi^{-1}(\{p\})$. ☐

Remark 4.13. Indeed, Proposition 4.12 guarantees that, in order to classify cohomogeneity one actions on a given symmetric space M, we may study the problem in its universal cover $\widetilde{M}$ instead, and then figure out which actions project to M.

The proof of Theorem E follows now easily from Corollary 4.11:

Proof of Theorem E. As discussed in Proposition 1.19, a simply connected symmetric space splits as a product $M = M_+ \times M_0 \times M_-$, where M_+ is a symmetric space of noncompact type, $M_0 \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$, and M_- is a symmetric space of noncompact type. Let H be a connected closed subgroup of $I(M)$ acting with cohomogeneity one. Since M_+ is compact and $N = M_0 \times M_-$ Hadamard, applying the above Corollary 4.11 yields that the action of H decomposes with respect to the decomposition $M = M_+ \times N$.

Suppose now that H is a group acting with cohomogeneity one on $M_0 \times M_-$. Then, by Lemma 4.9, the projection actions of H on M_0 and M_- are well defined. Now, Proposition 4.7 guarantees that if H induces a cohomogeneity one action on one of the factors, then the H-action decomposes. Otherwise, H acts transitively on both M_0 and M_-, which concludes the proof of Theorem E. $\square$

Remark 4.14. The structural results of Chapter 3 allow to describe cohomogeneity one actions on products of symmetric spaces of noncompact type. Namely, Theorem C guarantees that an action on M_- that does not decompose with respect to the irreducible factors of M_- is (up to orbit equivalence) an action in item (FH) producing a foliation of horospherical type (which was described in item (2) of Theorem F in terms of the group H_X), or an action in item (CER) obtained by extension of a diagonal action $I^0(\mathbb{F}H^n) \curvearrowright \mathbb{F}H^n \times \mathbb{F}H^n$.

However, the picture in the compact setting seems to be much more complicated, as there are many examples of indecomposable cohomogeneity one actions on products of symmetric spaces of compact type (see [63, Examples 26–29]). For example, consider the action of Spin_8 on $M_+ = \mathbb{S}^7 \times \mathbb{S}^7 \times \mathbb{S}^7$ given by

$$g \cdot (p_1, p_2, p_3) = (\rho_1(g)p_1, \rho_2(g)p_2, \rho_3(g)p_1),$$

where ρ_1, ρ_2, ρ_3 are the three irreducible 8-dimensional real representations of Spin_8. This action is of cohomogeneity one on M_+, but transitive on each of the factors. Thus, even if one is able to study the indecomposable actions H_Δ of Theorem F, it could be difficult to obtain classification results for spaces with more than one irreducible compact factor.

4.2 Codimension one homogeneous foliations of symmetric spaces

The aim of this section is to prove Theorem F, which describes all of the possible cohomogeneity one actions on symmetric spaces without singular orbits. We will first state two lemmas regarding cohomogeneity one actions without singular orbits on symmetric spaces of compact type and Euclidean spaces, respectively. The first one states that there are no codimension one homogeneous foliations in simply connected symmetric spaces of noncompact type:

Lemma 4.15. *Let H_+ be a connected compact Lie group of isometries acting properly with cohomogeneity one on a simply connected compact Riemannian manifold M_+. Then, the H_+-action has a singular orbit.*

Proof. Since the projection map $\pi \colon M_+ \to M_+/H_+$ is continuous, it follows from the discussion in §1.1.1 that the orbit space M_+/H_+ is either $\mathbb{S}^1$ or $[0,1]$. Now, since M_+ is simply connected, M_+/H_+ cannot be $\mathbb{S}^1$, as this would yield a fiber bundle $M_+ \to \mathbb{S}^1$. Thus, we must have that $M_+/H_+ \cong [0,1]$ and, again since M_+ is simply connected, the nonprincipal orbits of the H_+-action (corresponding to the boundary of $[0,1]$) must be singular (see [3, Prop. 6.39]). ☐

The second lemma will be of good use in the proof of Theorem F. It is a technical lemma which essentially states that if H_0 is a Lie group acting on $\mathbb{R}^n$ with hyperplanes as orbits, then the "rotational part" of H_0 normalizes such hyperplanes.

Lemma 4.16. *Let H_0 be a connected closed solvable subgroup of $I^0(\mathbb{R}^n) = \mathsf{SO}_n \ltimes \mathbb{R}^n$ acting on $\mathbb{R}^n$ with cohomogeneity one and no singular orbits. Then, the projection of H_0 onto SO_n normalizes $T_o(H_0 \cdot o) = H_0 \cdot o$ for every $o \in \mathbb{R}^n$.*

Proof. Let $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denote the inner product on $\mathbb{R}^n$, and define an $\mathrm{ad}_{\mathfrak{so}_n}$-invariant inner product on $\mathfrak{i}(\mathbb{R}^n) = \mathfrak{so}_n \oplus \mathfrak{r}^n$ by $\langle (X,v),(Y,w) \rangle = \mathrm{tr}(XY) + \langle v, w \rangle_{\mathbb{R}^n}$. Let $\mathrm{pr}_{\mathfrak{so}_n} \colon \mathfrak{so}_n \oplus \mathfrak{r}^n \to \mathfrak{so}_n$ denote the orthogonal projection onto the first factor, which is a Lie algebra homomorphism since $\mathfrak{r}^n$ is an ideal of $\mathfrak{so}_n \oplus \mathfrak{r}^n$. Since $\mathfrak{h}_0$ is a solvable algebra by assumption, $\mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{h}_0)$ is a solvable subalgebra of $\mathfrak{so}_n$, so it must be abelian.

Let $\mathrm{pr}_{\mathfrak{r}^n} \colon \mathfrak{so}_n \oplus \mathfrak{r}^n \to \mathfrak{r}^n$ denote the orthogonal projection onto the second factor. We have that $\ker(\mathrm{pr}_{\mathfrak{r}^n})_{|\mathfrak{h}_0} = \mathfrak{h}_0 \cap \mathfrak{so}_n$, and $\mathfrak{h}_0$ decomposes as the sum $\mathfrak{h}_0 = (\mathfrak{h}_0 \cap \mathfrak{so}_n) \oplus (\mathfrak{h}_0 \ominus (\mathfrak{h}_0 \cap \mathfrak{so}_n))$. Moreover, if $X+v, Y+w \in \mathfrak{h}_0$, we have that $[X+v, Y+w] = Xw - Yv \in \mathfrak{h}_0 \cap \mathfrak{r}^{\mathrm{n}} \subseteq \mathfrak{h}_0 \ominus (\mathfrak{h}_0 \cap \mathfrak{so}_n)$, so $\mathfrak{h}_0 \cap \mathfrak{so}_n$ normalizes $\mathrm{pr}_{\mathfrak{r}^n}(\mathfrak{h}_0) \cong T_o(H_0 \cdot o)$ (which can be identified with $H_0 \cdot o$ since the orbits of H_0 must be hyperplanes). Moreover, $\mathfrak{h}_0 \ominus (\mathfrak{h}_0 \cap \mathfrak{so}_n)$ is an ideal in $\mathfrak{h}_0$, and the connected subgroup of $\mathsf{SO}_n \ltimes \mathbb{R}^n$ with Lie algebra $\mathfrak{h}_0 \ominus (\mathfrak{h}_0 \cap \mathfrak{so}_n)$ acts on $\mathbb{R}^n$ with the same orbits as H_0. Thus, from now on we will assume that $\mathfrak{h}_0 \cap \mathfrak{so}_n = \{0\}$.

Write ξ for a nonzero normal vector to $\mathrm{pr}_{\mathfrak{r}^n}(\mathfrak{h}_0)$ in $\mathfrak{r}^n$. Then, the projection $\mathrm{pr}_{\mathfrak{r}^n}$ induces a linear isomorphism $\varphi \colon \mathfrak{h}_0 \to \mathfrak{r}^n \ominus \xi$. Define a linear map $T \colon \mathfrak{r}^n \ominus \xi \to \mathfrak{so}_n$, $v \mapsto T_v$, by requiring T_v to be the unique vector in $\mathfrak{so}_n$ such that $T_v + v \in \mathfrak{h}_0$, that is, $T = \mathrm{pr}_{\mathfrak{so}_n} \circ \varphi^{-1}$. Define an endomorphism $\Phi \colon \mathfrak{r}^n \ominus \xi \to \mathfrak{r}_n \ominus \xi$ by $\Phi(v) = [T_v, \xi]$. This map is well defined because of the $\mathrm{ad}_{\mathfrak{so}_n}$-invariance of the inner product. Moreover, Φ is self-adjoint: given $v, w \in \mathfrak{r}^n \ominus \xi$, we have that $[T_v + v, T_w + w] = [T_v, w] - [T_w, v] \in \mathfrak{h}_0 \cap \mathfrak{r}^n \subseteq \mathfrak{r}^n \ominus \xi$, so

$$0 = \langle [T_v, w] - [T_w, v], \xi \rangle = -\langle w, [T_v, \xi] \rangle + \langle v, [T_w, \xi] \rangle = -\langle \Phi(v), w \rangle + \langle \Phi(w), v \rangle.$$

Suppose that Φ has a nonzero eigenvalue λ, and let u be a corresponding eigenvector (i.e. $[T_u, \xi] = \lambda u$), and $g = \mathrm{Exp}(\frac{1}{\lambda}\xi)$. We have that

$$\mathrm{Ad}_g(T_u + u) = e^{\mathrm{ad}_{\frac{1}{\lambda}\xi}}(T_u + u) = T_u + u + \frac{1}{\lambda}[\xi, T_u] + \frac{1}{\lambda}[\xi, u] = T_u + u - \frac{1}{\lambda}\lambda u = T_u.$$

Since $\dim(\mathrm{Ad}_g \mathfrak{h}_0) = \dim(\mathfrak{h}_0) = n-1$, we deduce that $\dim(\mathrm{pr}_{\mathfrak{r}^n}(\mathrm{Ad}_g \mathfrak{h}_0)) \leq n-2$. Now, $\mathrm{pr}_{\mathfrak{r}^n}(\mathrm{Ad}_g \mathfrak{h}_0)$ is isomorphic to $T_{g^{-1}(o)}(H_0 \cdot g^{-1}(o))$, yielding a contradiction since H has no singular orbits. We conclude that $\Phi = 0$, that is $[T_u, \xi] = 0$ for all $u \in \mathfrak{r}^n \ominus \xi$. Because of the $\mathrm{ad}_{\mathfrak{so}_n}$-invariance of the inner product, $\mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{h}_0)$ normalizes $\mathfrak{r}^n \ominus \xi \cong H_0 \cdot o$. □

4.2.1 The proof of Theorem F

We can now proceed to the proof of the main theorem of this section. For convenience, we have separated the proof in various smaller results which we prove independently.

Let $M = M_+ \times \mathbb{R}^n \times M_-$ be a simply connected symmetric space, where M_+ is of compact type and M_- is of noncompact type, and let H be a connected group acting properly by isometries on M. As mentioned in Section 4.1, we can assume that H is a connected closed subgroup of $I^0(M_+) \times I^0(\mathbb{R}^n) \times I^0(M_-)$. Moreover, if H acts with cohomogeneity one, Theorem E guarantees that either the action of M decomposes or it is orbit equivalent to the action of $I(M_+) \times H_\Delta$, where H_Δ acts with cohomogeneity one on $\mathbb{R}^n \times M_-$ and the projection actions of H_Δ on $\mathbb{R}^n$ and M_- are both transitive. Now, Lemma 4.15 guarantees that M_+ does not admit actions without singular orbits. Also, the only codimension one homogeneous foliation of $\mathbb{R}^n$ is given by affine hyperplanes. This corresponds to item (1) in Theorem F. Recall that cohomogeneity one actions on symmetric spaces of noncompact type have been extensively studied in [10, 12, 85]; see §2.3. Let M_- be a symmetric space of noncompact type, and write $M_- = G/K$ for $G = I^0(M)$ and K the isotropy at some point $o \in M_-$. Then, every cohomogeneity one action on M_- without singular orbits is orbit equivalent to either the action of the connected Lie group H_X of G with Lie algebra

$$\mathfrak{h}_X = (\mathfrak{a} \ominus X) \oplus \mathfrak{n},$$

for some unit vector $X \in \mathfrak{a}$, or to the action of the connected Lie group H_i of G with Lie algebra

$$\mathfrak{h}_i = \mathfrak{a} \oplus (\mathfrak{n} \ominus X)$$

for a unit vector X in a simple root space $\mathfrak{g}_{\alpha_i}$. This corresponds to items (2) and (3) of Theorem F, respectively. Thus, from now on we focus on studying the actions on $\mathbb{R}^n \times M_-$ that do not decompose.

Let H be a connected closed subgroup of $I^0(\mathbb{R}^n) \times I^0(M_-) = (\mathsf{SO}_n \ltimes \mathbb{R}^n) \times G$, and suppose that H acts with cohomogeneity one and no singular orbits on $M_0 \times M_-$. Moreover, suppose that the action of H does not decompose, and so it induces transitive actions on both factors by Proposition 4.7. Our first lemma imposes some extra assumptions on H which will greatly simplify our study.

Proposition 4.17. *There exists a connected solvable subgroup L of H acting freely on $\mathbb{R}^n \times M_-$ such that the orbits of H and L coincide. In this case,* $\mathfrak{l} \cap (\mathfrak{so}_n \oplus \mathfrak{k}) = \{0\}$, *and* $\mathfrak{l} \subseteq (\mathfrak{t}_0 \oplus \mathfrak{r}^n) \oplus (\mathfrak{t}_- \oplus \mathfrak{a} \oplus \mathfrak{n})$*, where* $\mathfrak{t}_0 = \mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{l}) \subset \mathfrak{so}_n$ *and* $\mathfrak{t}_- \subseteq Z_{\mathfrak{k}}(\mathfrak{a})$ *are abelian subalgebras.*

Proof. Since $\mathbb{R}^n \times M_-$ is a Hadamard manifold, according to the discussion at the beginning of [12, Ch. 5] (see also [10, Proposition 2.2]), there exists a connected closed solvable subgroup L of G which acts freely on $\mathbb{R}^n \times M_-$.

Now, let $\mathrm{pr}_{\mathfrak{so}_n}$ and $\mathrm{pr}_{\mathfrak{g}}$ denote the projections of $\mathfrak{so}_n \oplus \mathfrak{r}^n \oplus \mathfrak{g}$ onto $\mathfrak{so}_n$ and $\mathfrak{g}$, respectively. Since $\mathfrak{r}^n \oplus \mathfrak{g}$ and $\mathfrak{so}_n \oplus \mathfrak{r}^n$ are ideals, both $\mathrm{pr}_{\mathfrak{so}_n}$ and $\mathrm{pr}_{\mathfrak{g}}$ are Lie algebra homomorphisms. Thus, $\mathfrak{t}_0 = \mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{l})$ is a solvable subalgebra of $\mathfrak{so}_n$, and so it must be abelian since $\mathfrak{so}_n$ is compact. Similarly, $\mathrm{pr}_{\mathfrak{g}}(\mathfrak{l})$ is an abelian subalgebra of $\mathfrak{g}$, and so it is contained in some maximal solvable subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ (often called a Borel subalgebra of $\mathfrak{g}$). Borel subalgebras are a well-known object in the theory of real semisimple Lie algebras (cf. [74]), and can be parameterized by subsets of a given set of simple roots for $\mathfrak{g}$. Let B be the connected Lie group with Lie algebra $\mathfrak{b}$. Then, the orbit of the projection action of L through a point $p \in M_-$ is contained in the orbit of B through p. By hypothesis, $\mathrm{pr}_{\mathfrak{g}}(\mathfrak{l})$ acts transitively on M_-, so B must also act transitively on M_-. Now, any Borel subalgebra acting transitively on M_- is conjugate to $\mathfrak{t}_- \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{t}_-$ is a maximal abelian subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$, concluding this proof. ☐

Thus, by replacing H with L if necessary, we suppose from now on that H is solvable and acts freely on $\mathbb{R}^n \times M_-$. We then have $\mathfrak{h} \cap (\mathfrak{so}_n \oplus \mathfrak{k}) = \{0\}$ and $\mathfrak{h} \subseteq (\mathfrak{t}_0 \oplus \mathfrak{r}^n) \oplus (\mathfrak{t}_- \oplus \mathfrak{a} \oplus \mathfrak{n})$ for some abelian $\mathfrak{t}_0 \subseteq \mathfrak{so}_n$ and $\mathfrak{t}_- \subseteq \mathfrak{k}_0$. Note that, under these assumptions, the restriction of the projection $\mathfrak{so}_n \oplus \mathfrak{r}^n \oplus \mathfrak{g} \to \mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$ to $\mathfrak{h}$, which is denoted by π, yields an isomorphism between $\mathfrak{h}$ and $\pi(\mathfrak{h}) \cong T_o(H \cdot o)$.

By hypothesis, both projection actions of H on $\mathbb{R}^n$ and H_- are transitive, so we may decompose the tangent space to $H \cdot o$ at o as

$$\pi(\mathfrak{h}) = (\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E + X) \oplus ((\mathfrak{a} \oplus \mathfrak{n}) \ominus X),$$

for some nonzero vectors $E \in \mathfrak{r}^n$ and $X \in \mathfrak{a} \oplus \mathfrak{n}$. We define now three subspaces of $\mathfrak{h}$ by taking the preimage of the summands in the above decomposition. We denote them by $\mathfrak{h}_0 = \pi^{-1}(\mathfrak{r}^n \ominus E)$, $\mathfrak{h}_{mid} = \pi^{-1}(\mathbb{R}(E + X))$ and $\mathfrak{h}_- = \pi^{-1}((\mathfrak{a} \oplus \mathfrak{n}) \ominus X)$. For $V \in \pi(\mathfrak{h})$, we denote by $T_V \in \mathfrak{so}_n \oplus \mathfrak{k}$ the unique (since the action is free) vector such that $V + T_V \in \mathfrak{h}$. Then, the map $V \mapsto V + T_V$ is precisely the inverse of $\pi_{|\mathfrak{h}}$.

Lemma 4.18. $\mathfrak{h}_0$*,* $\mathfrak{h}_{mid}$ *and* $\mathfrak{h}_-$ *are subalgebras of* $\mathfrak{h}$*.*

Proof. Let $V + T_V, W + T_W$ be arbitrary vectors in $\mathfrak{h}_0$, where $V, W \in \mathfrak{r}^n \ominus E$. A straightforward computation gives us

$$[V + T_V, W + T_W] = [V, T_W] + [T_V, W] \in \mathfrak{r}^n \cap \mathfrak{h} \subseteq \mathfrak{h}_0,$$

since $\mathrm{pr}_{\mathfrak{so}_n\oplus\mathfrak{k}}(\mathfrak{h})$ normalizes $\mathfrak{r}^n$. Thus, $\mathfrak{h}_0$ is a subalgebra of $\mathfrak{h}$. Similarly, if $Y+T_Y$, $Z+T_Y$ are arbitrary vectors in $\mathfrak{h}_-$,

$$[Y+T_Y, Z+T_Z] = [Y,T_Z] + [T_Y,Z] + [Y,Z] \in (\mathfrak{a}\oplus\mathfrak{n})\cap\mathfrak{h} \subseteq \mathfrak{h}_-,$$

since $\mathrm{pr}_{\mathfrak{so}_n\oplus\mathfrak{k}}(\mathfrak{h}) \subseteq \mathfrak{t}_0\oplus\mathfrak{t}_- \subseteq \mathfrak{so}_n\oplus\mathfrak{k}_0$, and so normalizes $\mathfrak{a}\oplus\mathfrak{n}$. Thus, $\mathfrak{h}_-$ is a subalgebra of $\mathfrak{h}$. Lastly, $\mathfrak{h}_{mid}$ is one-dimensional, so it is abelian. ☐

In order to determine $\pi(\mathfrak{h})$, the next lemma uses the fact that the connected subgroup of $(\mathsf{SO}_n \ltimes \mathbb{R}^n)\times G$ with Lie algebra $\mathfrak{h}_-$ acts on M_- with cohomogeneity one and no singular orbits (since $\mathfrak{h}_-\cap\mathfrak{k} \subseteq \mathfrak{h}\cap(\mathfrak{so}_n\oplus\mathfrak{k}) = \{0\}$ and $\dim\mathfrak{h}_- = \dim M - 1$).

Lemma 4.19. *With the above notation,* $X\in\mathfrak{a}$, *and* $\mathfrak{n}\subseteq\mathfrak{h}$.

Proof. Let $\mathfrak{s} = \mathrm{pr}_{\mathfrak{g}}(\mathfrak{h}_-)$. Then, $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$, and the connected subgroup S of G with Lie algebra $\mathfrak{s}$ acts on M_- with cohomogeneity one and no singular orbits. It follows from [12, Proposition 5.4] (see also [10]) and the subsequent discussion that $\mathfrak{s}_n = \mathrm{pr}_{\mathfrak{a}\oplus\mathfrak{n}}(\mathfrak{s}) = \mathrm{pr}_{\mathfrak{a}\oplus\mathfrak{n}}(\mathfrak{h}_-) = (\mathfrak{a}\oplus\mathfrak{n})\ominus X$ is a subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$ and (perhaps after conjugation by an element in G) we can suppose that $X\in\mathfrak{a}$ or $X\in\mathfrak{g}_{\alpha_i}$ for some simple root $\alpha_i\in\Lambda$.

Suppose that $X\in\mathfrak{g}_{\alpha_i}$. This implies that $\mathfrak{a}\subseteq\mathfrak{s}_n$, so for every $A\in\mathfrak{a}$ we have that

$$[A+T_A, E+X+T_{E+X}] = \alpha_i(A)X + [T_A,E] + [T_A,X] \in \mathfrak{h}.$$

But $[T_A,E]\perp E$ and $[T_A,X]\perp X$ because of the ad-invariance of the inner product, so $[T_A,E]\in\mathfrak{h}_0$, $[T_A,X]\in\mathfrak{h}_-$ and $\alpha_i(A)X\in\mathfrak{h}$ for all $A\in\mathfrak{a}$. This implies that $X\in\mathfrak{h}$, leading to a contradiction. Therefore, we must have that $X\in\mathfrak{a}$.

Now, let Y_α be a nonzero vector in some positive root space $\mathfrak{g}_\alpha$. Since $\mathfrak{n}\subseteq\pi(\mathfrak{h})$, we have that

$$[E+X+T_{E+X}, Y_\alpha+T_{Y_\alpha}] = \alpha(X)Y_\alpha + [E,T_{Y_\alpha}] + [T_{E+X},Y_\alpha] \in \mathfrak{h}.$$

Using the ad-invariance of the inner product, we get that $[E,T_{Y_\alpha}]\perp E$ and also $[T_{E+X},Y_\alpha]\perp Y_\alpha$, so $\alpha(X)Y_\alpha\in\mathfrak{h}$. Similarly, for any $A\in\mathfrak{a}\ominus X$,

$$[A+T_A, Y_\alpha+T_{Y_\alpha}] = \alpha(A)Y_\alpha + [T_A,Y_\alpha] \in \mathfrak{h}.$$

but $[T_A,Y_\alpha]\perp Y_\alpha$, so $\alpha(A)Y_\alpha\in\mathfrak{h}$. Altogether, we get that $\alpha(A)Y_\alpha\in\mathfrak{h}$ for all $A\in\mathfrak{a}$, and so $\mathfrak{n}\subseteq\mathfrak{h}$. ☐

Note that, under these conditions, the projection $\pi(\mathfrak{h})$ of $\mathfrak{h}$ onto $\mathfrak{r}^n\oplus\mathfrak{a}\oplus\mathfrak{n}$, is precisely the subalgebra $\mathfrak{h}_{E,X}$ described in Therorem F. Next, we prove that the actions of H and the connected group of $(\mathsf{SO}_n\ltimes\mathbb{R}^n)\times G$ with Lie algebra $\mathfrak{h}_{E,X}$ have the same orbits, thus concluding the proof of Theorem F.

Theorem 4.20. *The projection of* $\mathfrak{h}$ *onto* $\mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{h}_{E,V} = \pi(\mathfrak{h}) = (\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E+X) \oplus (\mathfrak{a} \ominus X) \oplus \mathfrak{n}$, *is a subalgebra of* $\mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$*. Moreover, if* $H_{E,V}$ *denotes the connected subgroup of* $(\mathsf{SO}_n \ltimes \mathbb{R}^n) \times G$ *with Lie algebra* $\pi(\mathfrak{h})$*, then* $H_{E,V}$ *acts on* $\mathbb{R}^n \times M_-$ *with the same orbits as* H*.*

Proof. The fact that $\pi(\mathfrak{h})$ is a subalgebra follows immediately from the fact that $X \in \mathfrak{a}$ and the fact that $\mathfrak{a}$ normalizes $\mathfrak{n}$.

Let $\tilde{\mathfrak{h}}_0 = \mathrm{pr}_{\mathfrak{so}_n \oplus \mathfrak{r}^n}(\mathfrak{h}_0)$ be the projection of $\mathfrak{h}_0 = \pi^{-1}(\mathfrak{r}^n \ominus E)$ onto $\mathfrak{so}_n \oplus \mathfrak{r}^n$, and let $\widetilde{H_0}$ be the connected subgroup of $\mathsf{SO}_n \ltimes \mathbb{R}^n$ with Lie algebra $\tilde{\mathfrak{h}}_0$. Then, $\widetilde{H_0}$ is a solvable subgroup of $\mathsf{SO}_n \ltimes \mathbb{R}^n$ acting on $\mathbb{R}^n$ with cohomogeneity one and no singular orbits. By Lemma 4.16, the projection of $\widetilde{H_0}$ onto SO_n normalizes $T_o(\widetilde{H_0} \cdot o) \cong \mathfrak{r}^n \ominus E$. It now follows that $\mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{h}_0) = \mathrm{pr}_{\mathfrak{so}_n}(\tilde{\mathfrak{h}}_0)$ normalizes $\pi(\mathfrak{h})$ from the ad-invariance of the inner product.

If $V \in \mathfrak{r}^n \ominus E$, a straightforward computation yields

$$[E + X + T_{E+X}, V + T_V] = [E, T_V] + [T_{E+X}, V] \in \mathfrak{r}^n \cap \mathfrak{h} \subseteq \mathbb{R}^n \ominus E.$$

Now, $[E, T_V]$ is orthogonal to E, and so $[T_{E+X}, V]$ must also be orthogonal to E. Therefore, $\mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{h}_{mid})$ normalizes $\pi(\mathfrak{h})$. Similarly, if $A \in \mathfrak{a} \ominus X$, one gets

$$[A + T_A, V + T_V] = [T_A, V] \in \mathfrak{r}^n \cap \mathfrak{h} \subseteq \mathbb{R}^n \ominus E,$$

and so $\mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{h}_-)$ normalizes $\pi(\mathfrak{h})$.

Altogether, we get that $\mathfrak{t}_0 = \mathrm{pr}_{\mathfrak{so}_n}(\mathfrak{h})$ normalizes $\pi(\mathfrak{h})$. It follows from the properties of root spaces that $\mathfrak{t}_-$ also normalizes $\pi(\mathfrak{h})$, since $\mathfrak{t}_- \subseteq \mathfrak{k}_0 = Z_{\mathfrak{k}}(\mathfrak{a})$. Define a new subalgebra of $\mathfrak{so}_n \oplus \mathfrak{r}^n \oplus \mathfrak{g}$ as $\mathfrak{h}' = \mathfrak{t}_0 \oplus \mathfrak{t}_- \oplus \pi(\mathfrak{h})$, and denote by H' the corresponding connected subgroup of $(\mathsf{SO}_n \ltimes \mathbb{R}^n) \times G$. It is clear from our construction that $\mathfrak{h} \subseteq \mathfrak{h}'$, and so the orbits of H are contained in the orbits of H'. Now, $T_o(H' \cdot o) \cong \pi(\mathfrak{h}') = \pi(\mathfrak{h}) \cong T_o(H \cdot o)$, so $H \cdot o = H' \cdot o$. Thus, H and H' must have the same orbits. Similarly, since $\pi(\mathfrak{h}) \subseteq \mathfrak{h}'$, the orbits of $H_{E,V}$ are contained in the orbits of H', but $T_o(H' \cdot o) \cong \pi(\mathfrak{h}) \cong T_o(H_{E,V} \cdot o)$, so H, H' and $H_{E,V}$ have the same orbits. ☐

4.3 The geometry of the orbits of $H_{E,X}$

We briefly outline the geometry of the orbits of the action of $H_{E,X}$. Our first step will be to show that we can reduce the investigation to the orbit through a single point $o \in M$. For this, note that the derived subalgebra of $\mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$ is precisely $\mathfrak{n}$. Since $\mathfrak{n} \subseteq \mathfrak{h}_{E,X}$, it follows that $\mathfrak{h}_{E,X}$ is an ideal in $\mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$. The following lemma, due to Kubo and Tamaru, guarantees that all of the orbits of $H_{E,X}$ are mutually congruent.

Lemma 4.21 [64, Lemma 2.1]**.** *Let M be a Riemannian manifold and S be a connected Lie subgroup of $I(M)$ that acts transitively on M. If $\mathfrak{s}'$ is an ideal of $\mathfrak{s}$, then all orbits of S' in M are isometrically congruent to each other, where S' is the connected subgroup of S with Lie algebra $\mathfrak{s}'$.*

Therefore, we can deduce the geometry of the orbits of $H_{E,X}$ by studying only the geometry of the orbit through o. Note that, under the identification $M \cong AN$, the orbit $H_{E,X} \cdot o$ maps precisely to $H_{E,X}$. The vector

$$\xi = \frac{||X||^2 E - ||E||^2 X}{||E||||X||\sqrt{||E||^2 + ||X||^2}}$$

is then a unit normal field to $H_{E,X}$. Denote by S_ξ the shape operator of $H_{E,X} \cdot o$ with respect to ξ, and let ∇ denote the Levi-Civita connection of AN. Note that ad_ξ is a self-adjoint endomorphism of $\mathfrak{r}^n \oplus \mathfrak{a} \oplus \mathfrak{n}$. By means of Koszul's formula and the left-invariance of the metric of AN, we get

$$\begin{aligned} 2\langle S_\xi V, W\rangle &= 2\langle -\nabla_V \xi, W\rangle \\ &= (-\langle [\xi, V], W\rangle + \langle [V, W], \xi\rangle + \langle [\xi, W], V\rangle) = 2\langle \operatorname{ad}_\xi V, W\rangle, \end{aligned}$$

for every $V, W \in \mathfrak{h}_{E,X}$. It is now straightforward to check that ξ centralizes $V_0 = (\mathfrak{r}^n \ominus E) \oplus \mathbb{R}(E + X) \oplus (\mathfrak{a} \ominus X)$ and so V_0 is a principal curvature space of $H_{E,X}$ with principal curvature 0. Similarly, $\operatorname{ad}_\xi Y_\alpha = \frac{-||E||\alpha(X)}{||X||\sqrt{||E||^2+||X||^2}} Y_\alpha$ for any Y_α in a positive root space $\mathfrak{g}_\alpha$. Thus, each $\frac{-||E||\alpha(X)}{||X||\sqrt{||E||^2+||X||^2}}$ with $\alpha \in \Sigma^+$ is a principal curvature with principal curvature space $\mathfrak{g}_\alpha$. To sum up, using the notation above, we have:

Proposition 4.22. *The shape operator of $H_{E,X}$ with respect to ξ coincides with $(\operatorname{ad}_\xi)_{|\mathfrak{h}_{E,X}}$, and the mean curvature of $H_{E,X}$ is*

$$\frac{||E||}{||X||\sqrt{||E||^2 + ||X||^2}} \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha)\alpha(X).$$

In particular, the orbits of $H_{E,X}$ are minimal if and only if $\sum_{\alpha\in\Sigma^+} \dim(\mathfrak{g}_\alpha)\, \alpha(X) = 0$.

Remark 4.23. It should be noted that the geometry of orbits of $H_{E,X}$ is analogous to the behavior exhibited by the orbits of the actions H_X described in item (2) of Theorem F. In [12, Proposition 3.1], it is seen that the orbits of H_X on M_- are mutually congruent orbits with mean curvature $\sum_{\alpha\in\Sigma^+} \dim(\mathfrak{g}_\alpha)\alpha(X)$. In particular, if $M_- = G/K$ is a symmetric space of noncompact type and rank ≥ 2, one can always find a vector $X_0 \in \mathfrak{a}$ such that $\alpha(X_0) = 0$ for every $\alpha \in \Sigma^+$. In this case, the actions given by $I(M_+) \times \mathbb{R}^{n-1} \times G$ and $I(M_+) \times \mathbb{R}^n \times H_{X_0}$ and $I(M_+) \times H_{E,X_0}$ described in Theorem F give rise to noncongruent harmonic foliations on $M_+ \times \mathbb{R}^n \times M_-$. Recall that a foliation is said to be harmonic if all of its orbits are minimal.

Conclusions

The main contributions of this thesis correspond to Theorems A–F presented in Chapters 3 and 4:

C.1 Theorem A provides a new structural result for cohomogeneity one actions on symmetric spaces of noncompact type. Essentially, this result reduces the classification problem of homogeneous hypersurfaces in these ambient spaces to determining which actions arise via the nilpotent construction.

C.2 Theorem B is a classification of cohomogeneity one actions in the family of spaces $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$ for arbitrary n. This is the first classification result for cohomogeneity one actions in a symmetric space of noncompact type and rank greater than two.

C.3 Theorem C proves that, with the exception of the actions of type (FH) and (CER), the classification of cohomogeneity one actions on reducible symmetric spaces of noncompact type reduces to the study of such actions on each of the irreducible factors.

C.4 Theorem D provides a complete classification of cohomogeneity one actions on arbitrary products of hyperbolic spaces.

C.5 Theorem E provides a decomposability result for cohomogeneity one actions on simply connected symmetric spaces that are not of a particular type (compact, noncompact or Euclidean). This ensures that there are no "diagonal" actions between the compact and Hadamard factors of the symmetric space.

C.6 Theorem F is a complete classification of homogeneous codimension one foliations on simply connected symmetric spaces.

This thesis also includes some other results that may be of interest. In Chapter 3, we prove some properties of the canonical extension which could be useful in a more general setting for studying isometric actions that are not of cohomogeneity one. Also, many of the decomposability results in Section 4.1 are stated for the more general setting of proper isometric actions on Riemannian products.

Open problems

There are still several open problems and questions regarding the study of homogeneous hypersurfaces of symmetric spaces, some of which stem from the results presented in this thesis:

P.1 The nilpotent construction problem.

Essentially, solving the nilpotent construction problem would complete the classification of cohomogeneity one actions on symmetric spaces of noncompact type. The combination of the techniques used in the proof of Theorem C and the recent ideas of [84] will be enough to produce classifications for particular families of symmetric spaces (for example, symmetric spaces whose isometry Lie algebra is a split real form).

P.2 Cohomogeneity one actions on symmetric spaces of noncompact type.

Notably, the decomposability results developed in Chapter 4 fail to tackle the products of symmetric spaces of compact type. In view of [63], this seems to be a very difficult topic. Nevertheless, obtaining a description of cohomogeneity one actions on such spaces would generalize the results in [60], and complete the classification of hyperpolar actions on compact symmetric spaces.

P.3 Cohomogeneity one actions on Hadamard symmetric spaces.

Theorem E motivates the study of indecomposable actions on the products $\mathbb{R}^n \times M$, where M is a symmetric space of noncompact type. Obtaining a structural result for this kind of actions seems plausible and would be a great advance towards the classification of cohomogeneity one actions on arbitrary symmetric spaces.

P.4 Minimal homogeneous hypersurfaces in symmetric spaces of noncompact type.

Using the structural result in Theorem A and Jacobi field theory, it seems feasible to study the mean curvature of homogeneous hypersurfaces, and classify those that are minimal.

P.5 Homogeneous hypersurfaces in generalized Heisenberg groups and Damek-Ricci spaces.

Damek-Ricci spaces constitute a generalization of rank one symmetric spaces of noncompact type. They are solvable extensions of certain nilpotent Lie groups called generalized Heisenberg groups [16]. Cohomogeneity one actions on these spaces do not seem to have been studied in the literature. Some partial results on polar actions on these spaces have been obtained by Kollross [61].

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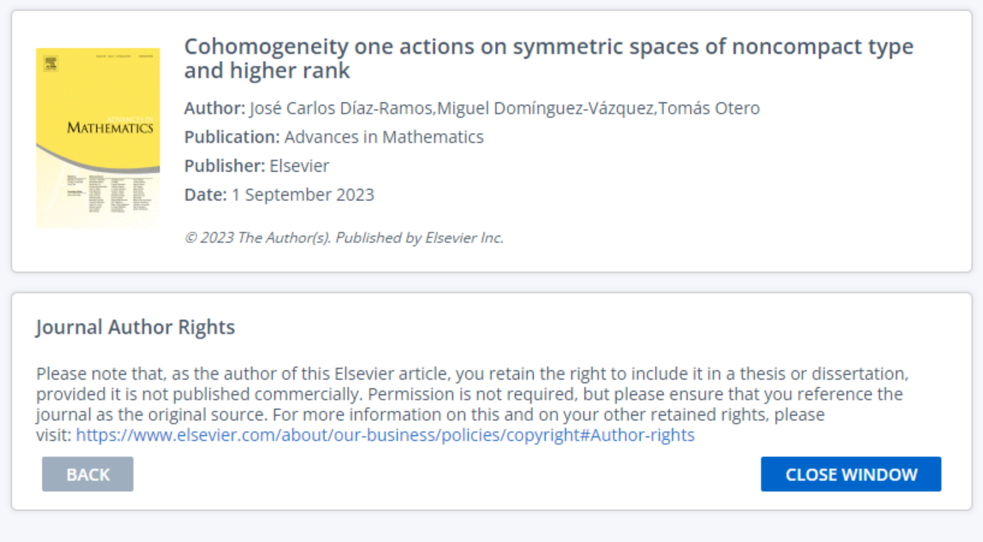

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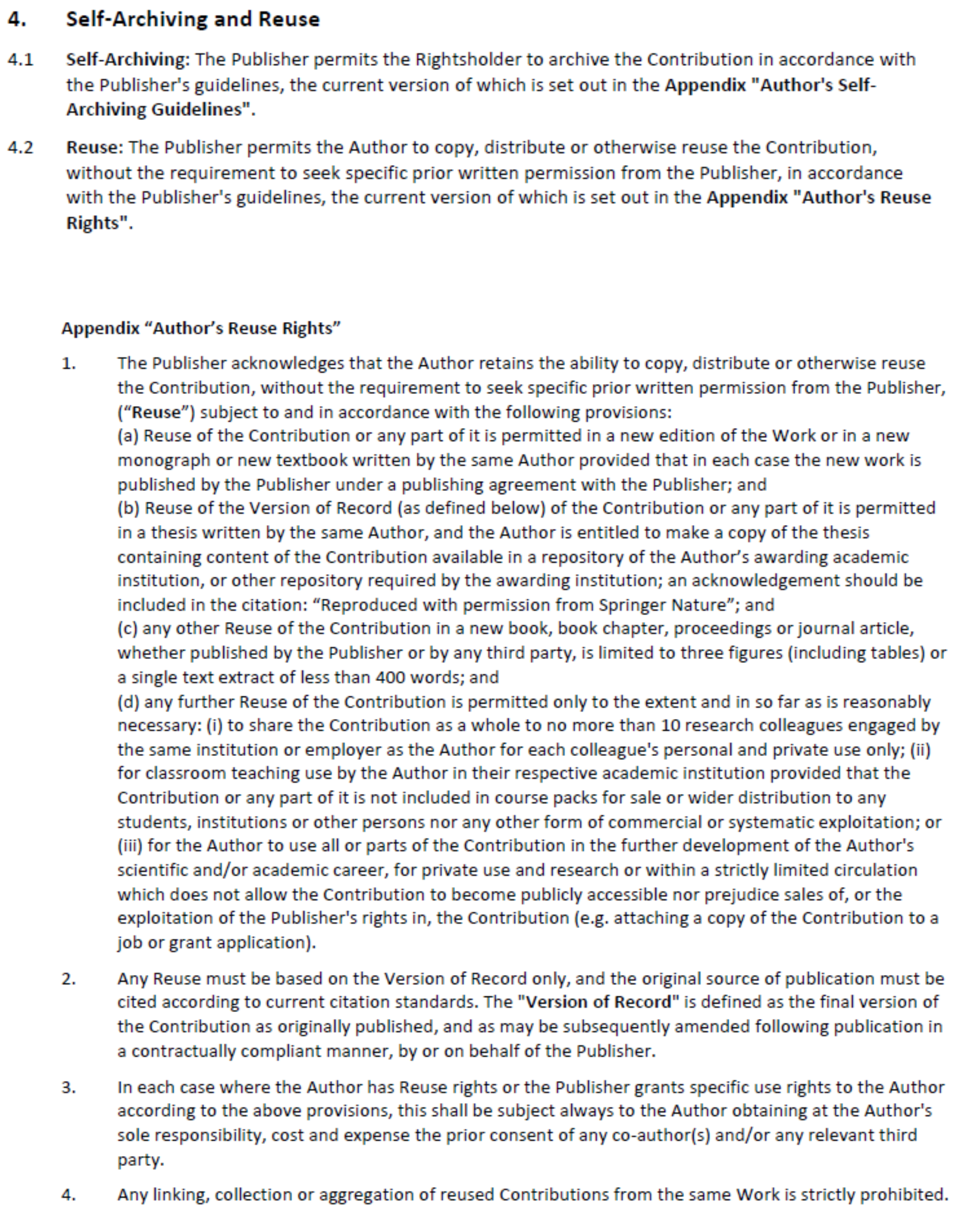

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In this thesis, we tackle the classification problem for homogeneous hypersurfaces in symmetric spaces. The results can be divided into two lines. The first of these consists in the development of a structural result for cohomogeneity one actions on symmetric spaces of noncompact type. This result guarantees that any such action can be constructed by one of five standard methods, easily described in terms of Lie algebras. The second line investigates cohomogeneity one actions on products of symmetric spaces of different types. Under certain hypotheses, one can reduce the study of these actions to each factor. This allowed us to produce a classification of codimension one homogeneous foliations on simply connected symmetric spaces.