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Three-point bending tests. Part I: Mathematical study and asymptotic analysis

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The goal of this work is to study the static behaviour of a three-dimensional elastic beam when is subjected to a three-point bending test. In the first part, under suitable compatibility conditions on the applied forces and on the geometry of the beam, we will prove the existence of a unique solution for the associated contact elastic problem; these conditions of compatibility on the data come from the absence of a Dirichlet condition on the beam boundary. In the second part, we will study the asymptotic behaviour of this problem; in particular, we will deduce the one-dimensional models associated to the displacement components, and we will give the existence and uniqueness of solution for them. Moreover, we will give an expression for the normal axial stress in the beam which is related to the modulus of rupture of brittle materials. In the final part of the work, we will deal with the regularity of the solution for the bending problem and we will prove some properties of the coincidence set.

Keywords: Modulus of rupture; contact conditions; asymptotic analysis.

1. Introduction

An increasing number of new challenges are constantly arising for engineers, comprising new applications, new materials or possibly both. Very often, the behaviour of a certain material under new conditions cannot be correctly predicted, and many trial tests are needed in order to determine whether it can be used or not. Therefore, an enormous effort is actually being made with the purpose of developing new prediction techniques that allow saving huge amounts of money and time in testing the materials. Strangely enough, there are very few studies about how to predict the behaviour of the different materials under simple conditions. Several factors, such as slight differences in the composition or in the microstructure, can dramatically affect the result of the considered tests, making useless previous tests to use on similar materials. This is particularly critical for the ceramic materials.

In the common applications, materials are subjected to different forces, with complicated components of bending, torsion, compression and usually combinations of all of them. To predict the behaviour of a certain material, without performing expensive destructive tests, it is necessary to understand the mechanisms that affect its resistance to select the suitable failure criteria. The properties of the material are carefully measured in laboratory tests on a sample of material; these tests determine the value of their mechanical properties by measuring the applied forces and the corresponding relative changes of length (see [1]).

One of the most common bending experiments is called *three-point bending test* (see [2]). In this test a sample of a brittle material is placed between three cylinders, without additional support, while with the upper cylinder applies an increasing gradually force until the beam breaks (see Fig. 1). From this test a property known as *modulus of rupture* (MOR) is obtained, which represents the maximum surface stress of the bent beam at the instant of failure. In the classical theory of beams, this value is given by an explicit expression which involves the value of the load of failure, H, the distance, $2l$, between the two lower cylinders, and the second moment of inertia, I_1, of the transversal section of the beam (see [3]). However, some difficulties have arisen when using the three-point bending test to a number of samples of a brittle material. It is quite common that several samples of the same material might lead to different values for the MOR, depending on factors such as the presence of defects, the distance between the two lower cylinders or the ratio between this distance and the length of the beam; but, until now, there

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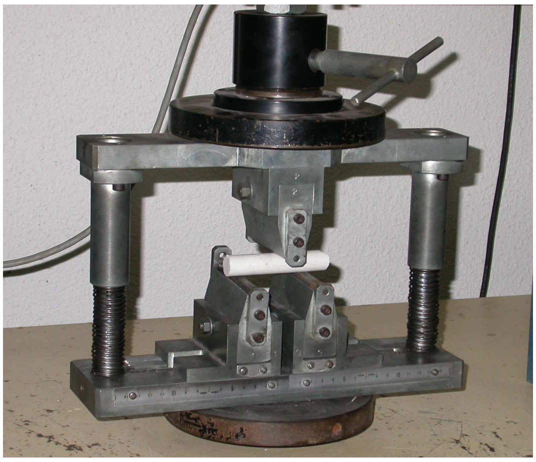

Figure 1. Three-point bending test.

are not many studies about this behaviour. Our aim is to study the mathematical problem associated to three-point bending tests in order to increase the knowledge on the response for brittle materials. Next, in [4] we will obtain an effective method to compute the MOR from different materials using three-point bending tests together with numerical simulations.

In particular, in this paper, we will prove the existence of a unique solution to the elastic problem with frictionless unilateral contact that arises from the mechanical model of the three-point bending test. Furthermore, in order to justify the classic formula for the MOR, we are interested in finding its limit model by using asymptotic analysis techniques when the thickness of the sample beam goes to zero (see [5, 6, 7, 8, 9]). By using this method, classical theories like Bernouilli-Navier, Saint Venant, Timoshenko and Vlassov theories were studied and justified (see [6, 8]). Also beams with a unilateral contact with a rigid foundation have already been studied in [10, 11] but only when the beam is clamped at the ends. Despite the absence of Dirichlet conditions for the three-dimensional mechanical problem associated with the three-point bending test, the asymptotic method will allow us to obtain the limit model and to identify the relationship between the failure load and the corresponding axial normal tension.

The outline of this paper is as follows. In Section 2, we will show the mathematical model associated with the three-point bending test: an elastic problem with a unilateral contact condition, without friction, formulated in displacements. In Section 3, we will propose a weak formulation of the problem in the appropriate functional spaces. Despite not impose Dirichlet conditions, assuming a suitable condition of compatibility on the applied forces, we will prove the existence of a solution in Section 4. Assuming additional conditions of symmetry in the geometry of the beam and in the applied forces, in Section 5 we will demonstrate the uniqueness of the solution. The application of the asymptotic expansion method will be described in Sections 6 and 7, giving a characterization of the terms of first order of the expansion through one-dimensional variational problems. In addition, we will give results of existence and uniqueness of solution for the corresponding axial and bending displacements. Section 8 will be devoted to studying the strong convergences of displacements and stresses. In Section 9, we will rewrite the limit problem on the original domain and, finally, in Section 10, we will present the differential formulations associated with the one-dimensional models and we will give some properties of the coincidence set as well as some regularity results for the bending problem.

2. Mathematical model

As usual in solid mechanics, Latin subscripts are understood to range over the integers $\{1, 2, 3\}$ and Greek subscripts (other than ϵ) over the integers $\{1, 2\}$. Moreover, Einstein summation over repeated subscripts is implied.

Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, with components (u_i), (v_i), respectively, $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ represents their scalar product. We denote by $\mathbb{R}^9_s$ the space of second order symmetric tensors over $\mathbb{R}^3$, endowed with the usual scalar product

$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad \sigma, \tau \in \mathbb{R}^9_s.$$

Let ω be a bounded, open, connected Lipschitz subset of $\mathbb{R}^2$ with area A. Given $0 < \epsilon \leq 1$, we define $\omega^\epsilon = \epsilon\,\omega$ and we consider the domain $\Omega^\epsilon = \omega^\epsilon \times (-L, L)$, $L > 0$, which corresponds to the reference configuration of the beam. Consequently, the beam length is $2L$ and the area of the cross section is $A^\epsilon = \epsilon^2 A$. We refer the motion of the beam to a fixed Cartesian system $0x_1^\epsilon x_2^\epsilon x_3^\epsilon$ whose origin is situated at the center of gravity of the beam (see Figure 2).

The physical problem consists of determining the displacement vector field, $\mathbf{u}^\epsilon$, and the stress tensor field, σ^ϵ, that the beam $\bar{\Omega}^\epsilon$ suffers when it is subjected to a three-point bending test.

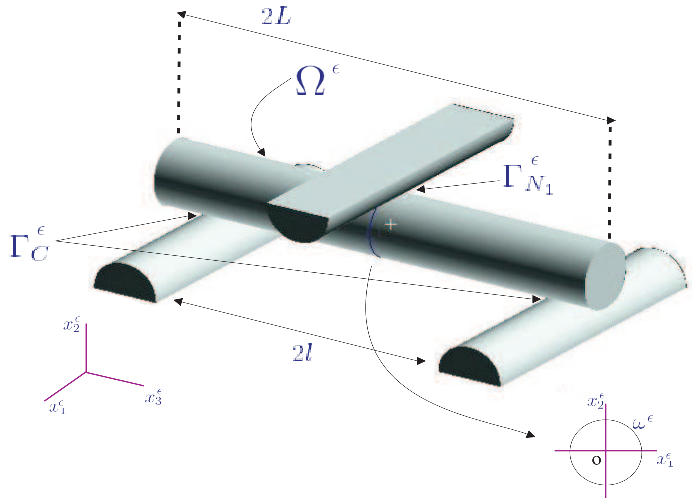

Figure 2. Sketch of the three-point bending test.

Let Γ^ϵ be the boundary of the beam which is the union of the lateral boundary $\Gamma_l^\epsilon = \gamma^\epsilon \times [-L, L]$, where $\gamma^\epsilon = \partial\omega^\epsilon$, and the ends of the beam $\Gamma_\pm^\epsilon = \omega^\epsilon \times \{\pm L\}$; Γ^ϵ is assumed to be Lipschitz-continuous.

An arbitrary point of $\bar{\Omega}^\epsilon$ is denoted by $\mathbf{x}^\epsilon = (x_1^\epsilon, x_2^\epsilon, x_3^\epsilon)$ and $\mathbf{n}^\epsilon$ denotes the outward unit normal vector to Γ^ϵ. Notice that $\mathbf{n}^\epsilon(\mathbf{x}^\epsilon) = (n_1^\epsilon, n_2^\epsilon, 0)$ on the lateral boundary Γ_l^ϵ, where $(n_1^\epsilon(x_1^\epsilon, x_2^\epsilon), n_2^\epsilon(x_1^\epsilon, x_2^\epsilon))$ is the outward unit normal vector to ω^ϵ in $(x_1^\epsilon, x_2^\epsilon) \in \gamma^\epsilon$. Moreover, $\mathbf{n}^\epsilon = (0, 0, \pm 1)$ on $\Gamma_\pm^\epsilon$.

Lateral boundary Γ_l^ϵ is partitioned into three non-empty, open and disjointed parts Γ_C^ϵ, $\Gamma_{N_1}^\epsilon$ and $\Gamma_{N_2}^\epsilon$ satisfying $\bar{\Gamma}_l^\epsilon = \bar{\Gamma}_C^\epsilon \cup \bar{\Gamma}_{N_1}^\epsilon \cup \bar{\Gamma}_{N_2}^\epsilon$, where:

- $\bar{\Gamma}_C^\epsilon$ is the region of the beam where, due to the presence of the two lower cylinders, the normal component of the displacement cannot be positive. To impose this constraint and take into account the reaction force exerted by the two lower cylinders when the contact is effective, we consider a Signorini condition (see [12]):
$$\sigma_\tau^\epsilon = \mathbf{0},\ \sigma_n^\epsilon \leq 0,\ u_n^\epsilon \leq 0,\ u_n^\epsilon \sigma_n^\epsilon = 0,\ \text{ on } \Gamma_C^\epsilon \tag{1}$$
where u_n^ϵ and σ_n^ϵ are the normal displacement and the normal stress at each point of the boundary, respectively, and $\mathbf{u}_\tau^\epsilon$ and σ_τ^ϵ are the tangential displacement and the tangential component of $\sigma^\epsilon \mathbf{n}^\epsilon$, respectively. We suppose that the two-dimensional measure of Γ_C^ϵ is strictly positive.
- $\Gamma_{N_1}^\epsilon$ corresponds to the region of the boundary in which the upper cylinder exerts a compression force with density $\mathbf{h}^\epsilon$ known.
- $\Gamma_{N_2}^\epsilon$ represents the remaining lateral boundary of the beam that is free of forces.

The ends of the beam are also assumed to be free of forces.

Under the assumption of small displacements, the behaviour of the beam is governed by the following problem:

Problem (P^ϵ):

Find the displacement vector field $\mathbf{u}^\epsilon(\mathbf{x}^\epsilon)$ and the stress tensor field $\sigma^\epsilon(\mathbf{x}^\epsilon)$, at each point $\mathbf{x}^\epsilon \in \Omega^\epsilon$, satisfying:

$$-\mathrm{div}^\epsilon(\sigma^\epsilon) = \mathbf{f}^\epsilon \text{ in } \Omega^\epsilon, \tag{2}$$
$$\sigma^\epsilon \mathbf{n}^\epsilon = \mathbf{h}^\epsilon \text{ on } \Gamma_{N_1}^\epsilon, \tag{3}$$
$$\sigma^\epsilon \mathbf{n}^\epsilon = \mathbf{0} \text{ on } \Gamma_{N_2}^\epsilon \cup \Gamma_\pm^\epsilon, \tag{4}$$
$$\sigma_\tau^\epsilon = \mathbf{0},\ \sigma_n^\epsilon \leq 0,\ u_n^\epsilon \leq 0,\ \sigma_n^\epsilon u_n^\epsilon = 0 \text{ on } \Gamma_C^\epsilon, \tag{5}$$
$$\sigma^\epsilon = \mathbf{\Lambda}^\epsilon \varepsilon^\epsilon(\mathbf{u}^\epsilon) = \lambda^\epsilon \mathrm{tr}(\varepsilon^\epsilon(\mathbf{u}^\epsilon))\mathbf{I} + 2\mu^\epsilon \varepsilon^\epsilon(\mathbf{u}^\epsilon) \text{ in } \Omega^\epsilon, \tag{6}$$

where the divergence operator and the linearized strain tensor are given by expressions

$$(\mathrm{div}^\epsilon(\tau^\epsilon))_i = \frac{\partial \tau_{ij}^\epsilon}{\partial x_j^\epsilon}, \quad \varepsilon_{ij}^\epsilon(\mathbf{u}^\epsilon) = \frac{1}{2}\left(\frac{\partial u_i^\epsilon}{\partial x_j^\epsilon} + \frac{\partial u_j^\epsilon}{\partial x_i^\epsilon}\right),$$

and λ^ϵ and μ^ϵ are the Lamé parameters of the material.

3. Weak formulation

To carry out the mathematical analysis of Problem (P^ϵ), we introduce a weak formulation in a similar manner to those used in [13, 14, 12].

We consider the space of displacement vector fields $\mathbf{H}^1(\Omega^\epsilon) = [H^1(\Omega^\epsilon)]^3$ which is a Hilbert space with the norm

$$\|\mathbf{v}^\epsilon\|_{1,\Omega^\epsilon} = \left(\sum_{i=1}^{3} \|v_i^\epsilon\|_{0,\Omega^\epsilon}^2 + \sum_{i,j=1}^{3} \|\varepsilon_{ij}^\epsilon(\mathbf{v}^\epsilon)\|_{0,\Omega^\epsilon}^2 \right)^{1/2},$$

where $\|\cdot\|_{0,\Omega^\epsilon}$ denotes the usual norm in $L^2(\Omega^\epsilon)$. The subset of kinematically admissible displacements is the convex set (see [12])

$$\mathbf{K}(\Omega^\epsilon) = \{\mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon);\, v_n^\epsilon \le 0 \text{ on } \Gamma_C^\epsilon\}.$$

Let us define the stress tensor space $\mathbf{X}(\Omega^\epsilon) = [L^2(\Omega^\epsilon)]_s^9$ where

$$[L^2(\Omega^\epsilon)]_s^9 = \{\tau^\epsilon = (\tau_{ij}^\epsilon) \in [L^2(\Omega^\epsilon)]^9;\, \tau_{ij}^\epsilon = \tau_{ji}^\epsilon\}.$$

$\mathbf{X}(\Omega^\epsilon)$ is a real Hilbert space endowed with the usual norm in $[L^2(\Omega^\epsilon)]^9$. Its subspace

$$\mathbf{H}(\mathrm{div}, \Omega^\epsilon) = \{\tau^\epsilon \in \mathbf{X}(\Omega^\epsilon);\, \mathrm{div}^\epsilon(\tau^\epsilon) \in \mathbf{L}^2(\Omega^\epsilon)\},$$

is also a Hilbert space with the norm

$$\|\tau^\epsilon\|_{\mathrm{H}(\mathrm{div},\Omega^\epsilon)} = \left(\|\tau^\epsilon\|_{0,\Omega^\epsilon}^2 + \|\mathrm{div}^\epsilon(\tau^\epsilon)\|_{0,\Omega^\epsilon}^2\right)^{1/2}.$$

We denote $\mathbf{H}^{1/2}(\Gamma^\epsilon) = [H^{1/2}(\Gamma^\epsilon)]^3$ as the image set of the trace application from $\mathbf{H}^1(\Omega^\epsilon)$. The following Green's formula holds:

$$\int_{\Omega^\epsilon} \tau^\epsilon : \varepsilon^\epsilon(\mathbf{v}^\epsilon) dx^\epsilon + \int_{\Omega^\epsilon} \mathrm{div}^\epsilon(\tau^\epsilon)\cdot \mathbf{v}^\epsilon dx^\epsilon = \langle \tau^\epsilon \mathbf{n}^\epsilon, \mathbf{v}^\epsilon\rangle_{\Gamma^\epsilon},$$

for all $\tau^\epsilon \in \mathbf{H}(\mathrm{div}, \Omega^\epsilon)$ and $\mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon)$, where $\langle\cdot,\cdot\rangle_{\Gamma^\epsilon}$ represents the duality between $\mathbf{H}^{-1/2}(\Gamma^\epsilon)$ and $\mathbf{H}^{1/2}(\Gamma^\epsilon)$.

3.1. Assumptions

From now on we consider the following hypotheses.

(H1$^\epsilon$) The volume forces satisfy $\mathbf{f}^\epsilon \in \mathbf{L}^2(\Omega^\epsilon)$ and the surface forces $\mathbf{h}^\epsilon \in \mathbf{L}^2(\Gamma_{N_1}^\epsilon)$.

(H2$^\epsilon$) The material is homogeneous and isotropic; so, the elasticity tensor $\mathbf{\Lambda}^\epsilon$ defined in (6) is such that $\lambda^\epsilon, \mu^\epsilon \in \mathbb{R}$, $\lambda^\epsilon, \mu^\epsilon > 0$.

(H3$^\epsilon$) The applied forces verify the following compatibility condition

$$\int_{\Omega^\epsilon} \mathbf{f}^\epsilon \cdot \mathbf{w}^\epsilon dx^\epsilon + \int_{\Gamma_{N_1}^\epsilon} \mathbf{h}^\epsilon \cdot \mathbf{w}^\epsilon d\Gamma^\epsilon \le 0,$$

for all $\mathbf{w}^\epsilon \in \mathbf{K}(\Omega^\epsilon) \cap \mathbf{R}(\Omega^\epsilon)$, where

$$\mathbf{R}(\Omega^\epsilon) = \{\mathbf{w}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon) : \varepsilon^\epsilon(\mathbf{w}^\epsilon) = \mathbf{0} \text{ in } [L^2(\Omega^\epsilon)]^9\},$$

is the set of rigid displacements. Moreover,

$$\int_{\Omega^\epsilon} \mathbf{f}^\epsilon \cdot \mathbf{w}^\epsilon dx^\epsilon + \int_{\Gamma_{N_1}^\epsilon} \mathbf{h}^\epsilon \cdot \mathbf{w}^\epsilon d\Gamma^\epsilon = 0,$$

if and only if $\mathbf{w}^\epsilon \in \mathbf{R}_n(\Omega^\epsilon)$, where

$$\mathbf{R}_n(\Omega^\epsilon) = \{\mathbf{w}^\epsilon \in \mathbf{R}(\Omega^\epsilon);\, w_n^\epsilon = 0 \text{ } a.e. \text{ on } \Gamma_C^\epsilon\}.$$

(H4$^\epsilon$) The set $(\mathbf{K}(\Omega^\epsilon) \cap \mathbf{R}(\Omega^\epsilon)) \setminus \mathbf{R}_n(\Omega^\epsilon)$ is non-empty.

3.2. Weak formulation of the problem

To simplify the writing, the following notations are useful:

- The external virtual work for the applied forces over the virtual displacement $\mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon)$ is defined as:

$$F^\epsilon(\mathbf{v}^\epsilon) = \int_{\Omega^\epsilon} \mathbf{f}^\epsilon \cdot \mathbf{v}^\epsilon dx^\epsilon + \int_{\Gamma_{N_1}^\epsilon} \mathbf{h}^\epsilon \cdot \mathbf{v}^\epsilon d\Gamma^\epsilon. \tag{7}$$

- The internal virtual work is related to the bilinear form a^ϵ defined by

$$a^\epsilon(\mathbf{v}^\epsilon, \mathbf{w}^\epsilon) = \int_{\Omega^\epsilon} \Lambda^\epsilon \varepsilon^\epsilon(\mathbf{v}^\epsilon) : \varepsilon^\epsilon(\mathbf{w}^\epsilon) dx^\epsilon, \tag{8}$$

for all $\mathbf{v}^\epsilon, \mathbf{w}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon)$.

With the above notation, we propose the following weak formulation:
Problem (VP^ϵ):
Find $(\mathbf{u}^\epsilon, \sigma^\epsilon) \in \mathbf{K}(\Omega^\epsilon) \times \mathbf{X}(\Omega^\epsilon)$ such that

$$\begin{cases} a^\epsilon(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon - \mathbf{u}^\epsilon) \geq F^\epsilon(\mathbf{v}^\epsilon - \mathbf{u}^\epsilon), \ \forall \mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon), \\ \sigma^\epsilon = \mathbf{\Lambda}^\epsilon \varepsilon^\epsilon(\mathbf{u}^\epsilon). \end{cases} \tag{9}$$

It is easy to prove that every solution of the classic Problem (P^ϵ) is a solution of the weak Problem (VP^ϵ) and the reciprocal is also true (see [12]).

4. Existence of a solution of Problem (VP^ϵ)

Results of existence and uniqueness for Problem (VP^ϵ) when there exists a Dirichlet condition on a part of the boundary can be found in [15] or [16]. If there does not exist a Dirichlet condition, a result of existence was given by [13], rewritten Problem (VP^ϵ) as a minimization problem under the assumption of compatibility **(H3$^\epsilon$)**.

It is a classic result (see [17]) that Problem (VP^ϵ) admits the equivalent formulation:
Problem (MP^ϵ):

$$\begin{cases} \text{Find } \mathbf{u}^\epsilon \in \mathbf{K}(\Omega^\epsilon) \text{ such that:} \\ \qquad J^\epsilon(\mathbf{u}^\epsilon) \leq J^\epsilon(\mathbf{v}^\epsilon), \qquad \forall \mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon), \end{cases}$$

where the minimization functional J^ϵ is defined by

$$J^\epsilon(\mathbf{v}^\epsilon) = \frac{1}{2} a^\epsilon(\mathbf{v}^\epsilon, \mathbf{v}^\epsilon) - F^\epsilon(\mathbf{v}^\epsilon), \quad \mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon). \tag{10}$$

Consequently, we have the following result of existence:

Theorem 1. *Under assumptions* **(H1$^\epsilon$)**-**(H4$^\epsilon$)**, *there exists a solution* $\mathbf{u}^\epsilon$ *of Problem* (VP^ϵ).

Proof. The regularity properties of the domain Ω^ϵ, the assumptions **(H3$^\epsilon$) and (H4$^\epsilon$)** and that the bilinear form a^ϵ verifies

$$a^\epsilon(\mathbf{v}^\epsilon, \mathbf{v}^\epsilon) \geq 2\mu^\epsilon \|\varepsilon^\epsilon(\mathbf{v}^\epsilon)\|^2_{0,\Omega^\epsilon}, \quad \forall \mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon),$$

let us guarantee the hypotheses of theorem XVII of [13] so we can get the existence of a minimum of Problem (MP^ϵ) and consequently the existence of a solution of Problem (VP^ϵ). □

4.1. Properties of a solution of Problem (VP^ϵ)

In the following Lemma we are going to prove that, if there exists a gap between the beam and the lower cylinders, then the movement of the beam is free there.

Lemma 1. *Let* $(\mathbf{u}^\epsilon, \sigma^\epsilon) \in \mathbf{K}(\Omega^\epsilon) \times \mathbf{X}(\Omega^\epsilon)$ *a solution for Problem* (VP^ϵ). *Let* $A \subset \Gamma^\epsilon_C$, $\text{meas}(A) > 0$, *such that* $u^\epsilon_n < 0$ *on* A. *Then* $\sigma^\epsilon_n = 0$ *on* A.

Proof. Since $(\mathbf{u}^\epsilon, \sigma^\epsilon)$ is a solution of Problem (P^ϵ) in the sense of the distributions, we obtain that

$$\langle \sigma^\epsilon \mathbf{n}^\epsilon, \mathbf{v}^\epsilon - \mathbf{u}^\epsilon \rangle_{\Gamma^\epsilon} \geq \int_{\Gamma^\epsilon_{N_1}} \mathbf{h}^\epsilon \cdot (\mathbf{v}^\epsilon - \mathbf{u}^\epsilon) d\Gamma^\epsilon,$$

for every $\mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon)$. By choosing $\mathbf{v}^\epsilon = 2\mathbf{u}^\epsilon$ and $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon/2$, we have that

$$\langle \sigma^\epsilon \mathbf{n}^\epsilon, \mathbf{u}^\epsilon \rangle_{\Gamma^\epsilon} = \int_{\Gamma^\epsilon_{N_1}} \mathbf{h}^\epsilon \cdot \mathbf{u}^\epsilon d\Gamma^\epsilon. \tag{11}$$

Let us assume that $\sigma^\epsilon_n < 0$ on $B \subset A$, with $\text{meas}(B) > 0$. Since $\sigma^\epsilon \mathbf{n}^\epsilon = \mathbf{h}^\epsilon$ on $\Gamma^\epsilon_{N_1}$, $\sigma^\epsilon \mathbf{n}^\epsilon = \mathbf{0}$ on $\Gamma^\epsilon_{N_2}$, $\sigma^\epsilon_\tau = \mathbf{0}$ on Γ^ϵ_C and $\sigma^\epsilon_n \leq 0$ on Γ^ϵ_C, we can deduce that

$$\langle \sigma^\epsilon_n, v^\epsilon_n \rangle_{\Gamma^\epsilon} > \int_{\Gamma^\epsilon_{N_1}} \mathbf{h}^\epsilon \cdot \mathbf{v}^\epsilon d\Gamma^\epsilon,$$

for every $\mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon)$ such that $v^\epsilon_n < 0$ on B. This is a contradiction to (11) if we consider $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon$. □

Lemma 2. *Under assumptions* **(H1$^\epsilon$)**-**(H4$^\epsilon$)**, *there exists a subset* $\Gamma_D^\epsilon \subset \Gamma_C^\epsilon$, $\mathrm{meas}(\Gamma_D^\epsilon) > 0$, *such that the solution* $\mathbf{u}^\epsilon$ *of Problem* (VP^ϵ) *satisfies:*

$$u_n^\epsilon = 0 \text{ on } \Gamma_D^\epsilon.$$

Proof. Let us assume that there does not exist Γ_D^ϵ in those conditions. Then $u_n^\epsilon < 0$ on Γ_C^ϵ and, from Lemma 1, $\sigma^\epsilon \mathbf{n}^\epsilon = \mathbf{0}$ on Γ_C^ϵ. Therefore, the tensor $\sigma^\epsilon \mathbf{n}^\epsilon \in [L^2(\Gamma^\epsilon)]^3$ verifies

$$\sigma^\epsilon \mathbf{n}^\epsilon = \begin{cases} \mathbf{h}^\epsilon & \text{on } \Gamma_{N_1}^\epsilon, \\ \mathbf{0} & \text{on } \Gamma_{N_2}^\epsilon \cup \Gamma_\pm^\epsilon \cup \Gamma_C^\epsilon. \end{cases} \tag{12}$$

Given that $\mathbf{u}^\epsilon$ is a solution of Problem (P^ϵ) in the sense of the distributions, we can replace boundary conditions (3)-(5) by (12) to conclude that

$$a^\epsilon(\mathbf{u}^\epsilon, \mathbf{w}^\epsilon) = F^\epsilon(\mathbf{w}^\epsilon),\ \forall \mathbf{w}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon).$$

By choosing $\mathbf{w}^\epsilon \in (\mathbf{K}(\Omega^\epsilon) \cap \mathbf{R}(\Omega^\epsilon)) \setminus \mathbf{R}_n(\Omega^\epsilon)$, we find

$$0 = F^\epsilon(\mathbf{w}^\epsilon) = \int_{\Omega^\epsilon} \mathbf{f}^\epsilon \cdot \mathbf{w}^\epsilon dx^\epsilon + \int_{\Gamma_{N_1}^\epsilon} \mathbf{h}^\epsilon \cdot \mathbf{w}^\epsilon d\Gamma^\epsilon,$$

which is a contradiction with the assumption **(H3$^\epsilon$)**. □

Lemma 3. *Under assumptions* **(H1$^\epsilon$)**-**(H4$^\epsilon$)**, *if* $\mathbf{u}^\epsilon$ *is a solution of Problem* (VP^ϵ), *any other solution is given by* $\mathbf{u}^\epsilon + \mathbf{w}^\epsilon$, *where* $\mathbf{w}^\epsilon \in \mathbf{R}_n(\Omega^\epsilon)$.

Proof. Thanks to assumption **(H3$^\epsilon$)**, if $\mathbf{u}^\epsilon$ is a solution of Problem (VP^ϵ), then $\mathbf{u}^\epsilon + \mathbf{w}^\epsilon$ is also a solution for each $\mathbf{w}^\epsilon \in \mathbf{R}(\Omega^\epsilon)$ such that $w_n^\epsilon = 0$ on Γ_C^ϵ. Let us see the reciprocal.

Let us assume that $\mathbf{u}^\epsilon$ is a solution of Problem (VP^ϵ). By choosing as test functions $\mathbf{v}^\epsilon + \mathbf{u}^\epsilon$, with $\mathbf{v}^\epsilon = 2\mathbf{u}^\epsilon$ and $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon/2$, we obtain

$$a^\epsilon(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon) - F^\epsilon(\mathbf{v}^\epsilon) \geq 0, \quad \forall \mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon), \tag{13}$$

$$a^\epsilon(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon) - F^\epsilon(\mathbf{u}^\epsilon) = 0. \tag{14}$$

Expressions (13) and (14) allow us to prove the following inequality:

$$\begin{aligned} J^\epsilon(\mathbf{v}^\epsilon) - J^\epsilon(\mathbf{u}^\epsilon) &= \frac{1}{2} a^\epsilon(\mathbf{v}^\epsilon, \mathbf{v}^\epsilon) - F(\mathbf{v}^\epsilon) - \frac{1}{2} a^\epsilon(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon) + F^\epsilon(\mathbf{u}^\epsilon) \\ &\geq \frac{1}{2} a^\epsilon(\mathbf{u}^\epsilon - \mathbf{v}^\epsilon, \mathbf{u}^\epsilon - \mathbf{v}^\epsilon) \geq 0,\ \forall \mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon). \end{aligned}$$

If $\tilde{\mathbf{u}}^\epsilon \in \mathbf{K}(\Omega^\epsilon)$ is another solution of Problem (VP^ϵ) and equivalently of Problem (MP^ϵ), then

$$0 = J^\epsilon(\tilde{\mathbf{u}}^\epsilon) - J^\epsilon(\mathbf{u}^\epsilon) \geq \frac{1}{2} a^\epsilon(\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^\epsilon, \mathbf{u}^\epsilon - \tilde{\mathbf{u}}^\epsilon) \geq 0,$$

that is, $\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^\epsilon \in \mathbf{R}(\Omega^\epsilon)$. Therefore, we have

$$\tilde{\mathbf{u}}^\epsilon = \mathbf{u}^\epsilon + \mathbf{w}^\epsilon, \quad \mathbf{w}^\epsilon \in \mathbf{R}(\Omega^\epsilon).$$

In addition, given that $\mathbf{u}^\epsilon$ is also a solution of Problem (VP^ϵ), if we take $\mathbf{v}^\epsilon = \tilde{\mathbf{u}}^\epsilon$ in (9), we obtain

$$0 \geq F^\epsilon(\mathbf{w}^\epsilon) = \int_{\Omega^\epsilon} \mathbf{f}^\epsilon \cdot \mathbf{w}^\epsilon dx^\epsilon + \int_{\Gamma_{N_1}^\epsilon} \mathbf{h}^\epsilon \cdot \mathbf{w}^\epsilon d\Gamma^\epsilon.$$

Now, exchanging $\mathbf{u}^\epsilon$ and $\tilde{\mathbf{u}}^\epsilon$ in the above reasoning, we can conclude that $F^\epsilon(\mathbf{w}^\epsilon) = 0$. Therefore, from assumption **(H3$^\epsilon$)**, $w_n^\epsilon = 0$ on Γ_C^ϵ. This allows us to conclude the proof. □

5. Existence of a unique solution

In this section we are going to prove the uniqueness of solution of Problem (VP^ϵ) in the convex set $\mathbf{K}(\Omega^\epsilon)$. To do so, we assume the following assumptions of symmetry in the domain and in the applied forces.

(H5$^\epsilon$) • Domain ω^ϵ is symmetrical with respect to the lines $x_1^\epsilon = 0$ and $x_2^\epsilon = 0$.

- Boundaries $\Gamma^\epsilon_{N_1}$ and Γ^ϵ_C are symmetrical with respect to the planes $x^\epsilon_1 = 0$ and $x^\epsilon_3 = 0$. Furthermore,

$$\Gamma^\epsilon_{N_1} = \{\mathbf{x}^\epsilon \in \Gamma^\epsilon_l; |x^\epsilon_3| < \delta, \, (x^\epsilon_1, x^\epsilon_2) \in \gamma^\epsilon_{N_1}(x^\epsilon_3)\},$$
$$\Gamma^\epsilon_C = \{\mathbf{x}^\epsilon \in \Gamma^\epsilon_l; 0 < |x^\epsilon_3| - l < \hat{\delta}, \, (x^\epsilon_1, x^\epsilon_2) \in \gamma^\epsilon_C(x^\epsilon_3)\},$$

  where $2l$ is the distance between the two lower cylinders, $\gamma^\epsilon_{N_1}(x^\epsilon_3), \gamma^\epsilon_C(x^\epsilon_3) \subset \gamma^\epsilon$ and δ, $\hat{\delta}$ being small parameters, verifying $\delta < l - \hat{\delta}$.
- The second component of the outward normal vector to Γ^ϵ is not null on Γ^ϵ_C and with constant sign. Moreover, it is even with respect to x^ϵ_1.

(H6$^\epsilon$) Volume and surface forces verify that

- f^ϵ_1 and h^ϵ_1 are odd with respect to x^ϵ_1 and even with respect to x^ϵ_3,
- f^ϵ_2 and h^ϵ_2 are even with respect to x^ϵ_1 and x^ϵ_3,
- f^ϵ_3 and h^ϵ_3 are even with respect to x^ϵ_1 and odd with respect to x^ϵ_3.

We consider the following subspace of $\mathbf{H}^1(\Omega^\epsilon)$:

$$\mathbf{H}^1_s(\Omega^\epsilon) = \left\{ \mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon); \begin{array}{l} v^\epsilon_1 \text{ is odd in } x^\epsilon_1 \text{ and even in } x^\epsilon_3 \\ v^\epsilon_2 \text{ is even in } x^\epsilon_1 \text{ and } x^\epsilon_3 \\ v^\epsilon_3 \text{ is even in } x^\epsilon_1 \text{ and is odd in } x^\epsilon_3 \end{array} \right\}.$$

Theorem 2. *Under assumptions* **(H1$^\epsilon$)**-**(H6$^\epsilon$)**, *there exists a unique solution* $\mathbf{u}^\epsilon \in \mathbf{H}^1_s(\Omega^\epsilon)$ *of Problem* (VP^ϵ)

Proof. Suppose that $\mathbf{u}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon) \cap \mathbf{K}(\Omega^\epsilon)$ is a solution of Problem (VP^ϵ) and let $\hat{\mathbf{u}}^\epsilon$ be the vector field defined by:

$$\hat{\mathbf{u}}^\epsilon(x^\epsilon_1, x^\epsilon_2, x^\epsilon_3) = (-u^\epsilon_1(-x^\epsilon_1, x^\epsilon_2, x^\epsilon_3), u^\epsilon_2(-x^\epsilon_1, x^\epsilon_2, x^\epsilon_3), u^\epsilon_3(-x^\epsilon_1, x^\epsilon_2, x^\epsilon_3)).$$

Taking into account assumptions **(H5$^\epsilon$)** and **(H6$^\epsilon$)**, we can conclude that $\hat{\mathbf{u}}^\epsilon$ is also a solution of Problem (VP^ϵ). Moreover, it is easy to prove that $\hat{\hat{\mathbf{u}}}^\epsilon = 1/2(\mathbf{u}^\epsilon + \hat{\mathbf{u}}^\epsilon)$ is a solution of Problem (VP^ϵ) such that $\hat{\hat{u}}^\epsilon_1$ is odd with respect to x^ϵ_1 and $\hat{\hat{u}}^\epsilon_2$ and $\hat{\hat{u}}^\epsilon_3$ are even with respect to x^ϵ_1.

Analogously, if $\tilde{\mathbf{u}}^\epsilon$ is the vector field

$$\tilde{\mathbf{u}}^\epsilon(x^\epsilon_1, x^\epsilon_2, x^\epsilon_3) = (\hat{\hat{u}}^\epsilon_1(x^\epsilon_1, x^\epsilon_2, -x^\epsilon_3), \hat{\hat{u}}^\epsilon_2(x_1, x_2, -x_3), -\hat{\hat{u}}^\epsilon_3(x^\epsilon_1, x^\epsilon_2, -x^\epsilon_3)),$$

both $\tilde{\mathbf{u}}^\epsilon$ and $\tilde{\tilde{\mathbf{u}}}^\epsilon = 1/2(\tilde{\mathbf{u}}^\epsilon + \hat{\hat{\mathbf{u}}}^\epsilon)$ are solutions of Problem (VP^ϵ); moreover, $\tilde{\tilde{u}}^\epsilon_3$ is odd with respect to x^ϵ_3 and $\tilde{\tilde{u}}^\epsilon_1$ and $\tilde{\tilde{u}}^\epsilon_2$ are even with respect to x^ϵ_1, that is, $\tilde{\tilde{\mathbf{u}}}^\epsilon \in \mathbf{H}^1_s(\Omega^\epsilon)$.

Finally, let $\bar{\mathbf{u}}^\epsilon \in \mathbf{H}^1_s(\Omega^\epsilon)$ be any solution of Problem (VP^ϵ). From Lemma 3, there exists $\mathbf{w}^\epsilon \in \mathbf{R}_n(\Omega^\epsilon)$ such that

$$\tilde{\tilde{\mathbf{u}}}^\epsilon = \bar{\mathbf{u}}^\epsilon + \mathbf{w}^\epsilon.$$

In addition, $\mathbf{w}^\epsilon$ must belong to $\mathbf{H}^1_s(\Omega^\epsilon)$, so we can conclude $\mathbf{w}^\epsilon = \mathbf{0}$ thanks to assumption **(H6$^\epsilon$)**. □

Remark 1. *Condition* **(H5$^\epsilon$)** *implies that the coordinates system* $0x^\epsilon_1x^\epsilon_2$ *is a principal system of inertia associated with the domain* ω^ϵ, *that is,*

$$\int_{\omega^\epsilon} x^\epsilon_\alpha dx^\epsilon_1 dx^\epsilon_2 = \int_{\omega^\epsilon} x^\epsilon_1 x^\epsilon_2 dx^\epsilon_1 dx^\epsilon_2 = 0. \tag{15}$$

Then, the 2×2 *geometrical inertia tensor of the cross section* w^ϵ *is just a diagonal matrix whose components are:*

$$I^\epsilon_1 = \int_{\omega^\epsilon} (x^\epsilon_2)^2 dx^\epsilon_1 dx^\epsilon_2, \; I^\epsilon_2 = \int_{\omega^\epsilon} (x^\epsilon_1)^2 dx^\epsilon_1 dx^\epsilon_2. \tag{16}$$

6. Rescaling functions and changing variables.

Our next objective is to study the behaviour of the solution $(\mathbf{u}^\epsilon, \sigma^\epsilon)$ of Problem (VP^ϵ) when ϵ tends to be smaller, that is, we try to reduce our three-dimensional problem to a one-dimensional one. To do this, we will use the asymptotic expansion method (see [8]).

Following [8] and in order to simplify the calculations, we are going to introduce the mixed variational formulation of Problem (P^ϵ).

Problem (MVP^ϵ):

Find $\mathbf{u}^\epsilon \in \mathbf{K}(\Omega^\epsilon)$ and $\sigma^\epsilon \in \mathbf{X}(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} [\frac{1+\nu^\epsilon}{E^\epsilon}\sigma^\epsilon - \frac{\nu^\epsilon}{E^\epsilon}\mathrm{tr}(\sigma^\epsilon)\mathbf{I}] : \tau^\epsilon dx^\epsilon = \int_{\Omega^\epsilon} \varepsilon^\epsilon(\mathbf{u}^\epsilon) : \tau^\epsilon dx^\epsilon, \ \forall \tau^\epsilon \in \mathbf{X}(\Omega^\epsilon), \tag{17}$$

$$\int_{\Omega^\epsilon} \sigma^\epsilon : \varepsilon^\epsilon(\mathbf{v}^\epsilon - \mathbf{u}^\epsilon) dx^\epsilon \geq F^\epsilon(\mathbf{v}^\epsilon - \mathbf{u}^\epsilon), \ \forall \mathbf{v}^\epsilon \in \mathbf{K}(\Omega^\epsilon), \tag{18}$$

where E^ϵ and ν^ϵ are the Young's modulus and the Poisson's ratio of the material, respectively, related to the Lamé parameters by the usual relations.

Under assumptions **(H1$^\epsilon$)**-**(H6$^\epsilon$)**, there exists a unique solution of Problem (MVP^ϵ) in $(\mathbf{H}^1_s(\Omega^\epsilon) \cap \mathbf{K}(\Omega^\epsilon)) \times \mathbf{X}(\Omega^\epsilon)$ thanks to Theorem 2.

Given that the dependence of the solution $(\mathbf{u}^\epsilon, \sigma^\epsilon)$ of Problem (MVP^ϵ) with respect to ϵ is very complex, we define an equivalent problem for which this dependence is explicit. To do so, we use a change of variable to a fixed domain and the subsequent rescaling of the displacements and stresses. The reference domain, independent of ϵ, is

$$\Omega = \omega \times (-L, L), \ \Omega = \Omega^1,$$
$$\gamma = \partial\omega = \gamma^1, \ \Gamma = \partial\Omega = \Gamma^1.$$

We associate each point $\mathbf{x} \in \overline{\Omega}$ with a point $\mathbf{x}^\epsilon \in \overline{\Omega}^\epsilon$ through the transformation (see [6])

$$\Pi^\epsilon : \overline{\Omega} \to \overline{\Omega}^\epsilon$$
$$\mathbf{x} = (x_1, x_2, x_3) \rightsquigarrow \Pi^\epsilon(\mathbf{x}) = \mathbf{x}^\epsilon = (\epsilon x_1, \epsilon x_2, x_3), \tag{19}$$

in such a way that

$$\Gamma^\epsilon = \Pi^\epsilon(\Gamma), \ \mathbf{n}^\epsilon(\mathbf{x}^\epsilon) = \mathbf{n}(\mathbf{x}),$$

where $\mathbf{n}$ is the outward unit normal vector to the surface Γ.

We associate the functions $\mathbf{v}^\epsilon \in \mathbf{H}^1(\Omega^\epsilon)$ with the scaled functions $\mathbf{v}(\epsilon) \in \mathbf{H}^1(\Omega)$ through

$$v_\alpha(\epsilon)(\mathbf{x}) = \epsilon v^\epsilon_\alpha(\mathbf{x}^\epsilon), \ v_3(\epsilon)(\mathbf{x}) = v^\epsilon_3(\mathbf{x}^\epsilon), \tag{20}$$

and the functions $\tau^\epsilon \in \mathbf{X}(\Omega^\epsilon)$ with the scaled functions $\tau(\epsilon) \in \mathbf{X}(\Omega)$ through

$$\tau_{\alpha\beta}(\epsilon)(\mathbf{x}) = \epsilon^{-2}\tau^\epsilon_{\alpha\beta}(\mathbf{x}^\epsilon), \ \tau_{3\beta}(\epsilon)(\mathbf{x}) = \epsilon^{-1}\tau^\epsilon_{3\beta}(\mathbf{x}^\epsilon), \ \tau_{33}(\epsilon)(\mathbf{x}) = \tau^\epsilon_{33}(\mathbf{x}^\epsilon). \tag{21}$$

Notice that (20)-(21) are not *a priori* assumptions on the order of magnitude of the displacements or stresses. It will be a result of the forthcoming analysis that the rescaled unknowns are actually bounded.

In what follows, we consider the following additional assumptions on the data:

(H7$^\epsilon$) The Lamé parameters are independent of ϵ: $\lambda^\epsilon = \lambda$, $\mu^\epsilon = \mu$. Consequently, Young's modulus and Poisson's coefficient are independent of ϵ.

(H8$^\epsilon$) There exist functions $\mathbf{f} \in [L^2(\Omega)]^3$ and $\mathbf{h} \in [L^2(\Gamma_{N_1})]^3$ independent of ϵ and such that

$$f^\epsilon_\alpha(\mathbf{x}^\epsilon) = \epsilon f_\alpha(\mathbf{x}), \ f^\epsilon_3(\mathbf{x}^\epsilon) = f_3(\mathbf{x}), \ \mathbf{x} \in \Omega,$$
$$h^\epsilon_\alpha(\mathbf{x}^\epsilon) = \epsilon^2 h_\alpha(\mathbf{x}), \ h^\epsilon_3(\mathbf{x}^\epsilon) = \epsilon h_3(\mathbf{x}), \ \mathbf{x} \in \Gamma_{N_1}.$$

In addition, $\mathbf{f}$ and $\mathbf{h}$ verify:

$$f_\alpha \in L^2(\Omega), \quad f_3 \in H^1((-L, L); L^2(\omega)),$$
$$h_\alpha \in L^2(\Gamma_{N_1}), \quad h_3 \in H^1((-\delta, \delta); L^2(\gamma_{N_1})).$$

Under the previous assumptions we obtain the following result:

Theorem 3. *Under assumptions* **(H1$^\epsilon$)**-**(H8$^\epsilon$)**, *the scaled displacement and stress fields* $(\mathbf{u}(\epsilon), \sigma(\epsilon))$ *of the solution* $(\mathbf{u}^\epsilon, \sigma^\epsilon) \in (\mathbf{H}^1_s(\Omega^\epsilon) \cap \mathbf{K}(\Omega^\epsilon)) \times \mathbf{X}(\Omega^\epsilon)$ *of Problem* (MVP^ϵ) *is the unique solution of the problem:*

Problem $(MVP(\epsilon))$*:*

$$\mathbf{u}(\epsilon) \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega), \quad \sigma(\epsilon) \in \mathbf{X}(\Omega), \tag{22}$$

$$a_0(\sigma(\epsilon), \tau) + \epsilon^2 a_2(\sigma(\epsilon), \tau) + \epsilon^4 a_4(\sigma(\epsilon), \tau) - b(\tau, \mathbf{u}(\epsilon)) = 0, \quad \forall \tau \in \mathbf{X}(\Omega), \tag{23}$$

$$b(\sigma(\epsilon), \mathbf{v} - \mathbf{u}(\epsilon)) \geq F(\mathbf{v} - \mathbf{u}(\epsilon)), \quad \forall \mathbf{v} \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega), \tag{24}$$

where

$$a_0(\sigma,\tau) = \int_\Omega \frac{1}{E}\sigma_{33}\tau_{33}\,dx, \tag{25}$$

$$a_2(\sigma,\tau) = \int_\Omega \big[\frac{2(1+\nu)}{E}\sigma_{3\beta}\tau_{3\beta} - \frac{\nu}{E}(\sigma_{33}\tau_{\mu\mu} + \sigma_{\mu\mu}\tau_{33})\big]\,dx, \tag{26}$$

$$a_4(\sigma,\tau) = \int_\Omega \big[\frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\mu\mu}\delta_{\alpha\beta}\big]\tau_{\alpha\beta}\,dx, \tag{27}$$

$$b(\tau,\mathbf{v}) = \int_\Omega \tau_{ij}\varepsilon_{ij}(\mathbf{v})\,dx, \tag{28}$$

$$F(\mathbf{v}) = \int_\Omega f_i v_i\,dx + \int_{\Gamma_{N_1}} h_i v_i\,d\Gamma. \tag{29}$$

Proof. Taking into account expression (20), we obtain the following relation between the linearized strain tensor $\varepsilon^\epsilon(\mathbf{v}^\epsilon)$ and the scaled linearized strain tensor $\varepsilon(\mathbf{v})$:

$$\varepsilon_{\alpha\beta}(\mathbf{v}(\epsilon)) = \epsilon^2\varepsilon^\epsilon_{\alpha\beta}(\mathbf{v}^\epsilon),\ \varepsilon_{3\beta}(\mathbf{v}(\epsilon)) = \epsilon\,\varepsilon^\epsilon_{3\beta}(\mathbf{v}^\epsilon),\ \varepsilon_{33}(\mathbf{v}(\epsilon)) = \varepsilon^\epsilon_{33}(\mathbf{v}^\epsilon).$$

So, we have that

$$\int_{\Omega^\epsilon} \mathbf{f}^\epsilon\cdot\mathbf{v}^\epsilon\,dx^\epsilon + \int_{\Gamma^\epsilon_{N_1}} \mathbf{h}^\epsilon\cdot\mathbf{v}^\epsilon\,d\Gamma^\epsilon = \epsilon^2\left(\int_\Omega \mathbf{f}\cdot\mathbf{v}(\epsilon)\,dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot\mathbf{v}(\epsilon)\,d\Gamma\right),$$
$$\int_{\Omega^\epsilon} \sigma^\epsilon : \varepsilon^\epsilon(\mathbf{v}^\epsilon)\,dx^\epsilon = \epsilon^2\left(\int_\Omega \sigma(\epsilon) : \varepsilon(\mathbf{v}(\epsilon))\,dx\right),$$

thanks to (19)-(21). Consequently,

$$b(\sigma(\epsilon), \mathbf{v} - \mathbf{u}(\epsilon)) \geq F(\mathbf{v} - \mathbf{u}(\epsilon)),\ \forall \mathbf{v} \in \mathbf{H}^1_s(\Omega)\cap\mathbf{K}(\Omega).$$

Analogously, substituting expressions (20), (21) and (25)-(29) in Problem (MVP^ϵ) and applying the change of variable theorem, we obtain expression (23). ☐

7. Asymptotic expansion method

From now on, unless otherwise stated, we assume hypotheses **(H1$^\epsilon$)-(H8$^\epsilon$)**. We suppose that the solution $(\mathbf{u}(\epsilon), \sigma(\epsilon)) \in (\mathbf{H}^1_s(\Omega)\cap\mathbf{K}(\Omega))\times\mathbf{X}(\Omega)$ of Problem $(MVP(\epsilon))$ can be written as a formal expansion

$$(\mathbf{u}(\epsilon), \sigma(\epsilon)) = (\mathbf{u}^0, \sigma^0) + \epsilon^2(\mathbf{u}^2, \sigma^2) + \epsilon^4(\mathbf{u}^4, \sigma^4) + o(\epsilon^4), \tag{30}$$

where $\mathbf{u}^i \in \mathbf{H}^1_s(\Omega)\cap\mathbf{K}(\Omega)$ and $\sigma^i \in \mathbf{X}(\Omega)$ are independent of ϵ. Substituting expression (30) in Problem $(MVP(\epsilon))$ and identifying the coefficients of the same powers of ϵ, we obtain formally that the terms of order zero must be a solution of the problem:

Problem (MVP^0):

$$\begin{aligned} &\mathbf{u}^0 \in \mathbf{H}^1_s(\Omega)\cap\mathbf{K}(\Omega), \quad \sigma^0 \in \mathbf{X}(\Omega),\\ &a_0(\sigma^0, \tau) - b(\tau, \mathbf{u}^0) = 0, \quad \forall \tau \in \mathbf{X}(\Omega),\\ &b(\sigma^0, \mathbf{v} - \mathbf{u}^0) \geq F(\mathbf{v} - \mathbf{u}^0), \quad \forall \mathbf{v} \in \mathbf{H}^1_s(\Omega)\cap\mathbf{K}(\Omega). \end{aligned} \tag{31}$$

In order to characterize the element $(\mathbf{u}^0, \sigma^0)$, solution of Problem (MVP^0), we consider the decomposition

$$\mathbf{H}^1_s(\Omega) = \mathbf{W}_{s2}(\Omega)\times H^1_s(\Omega),$$

where

$$\mathbf{W}_{s2}(\Omega) = \{(v_1, v_2) \in [H^1(\Omega)]^2; (v_1, v_2, 0) \in \mathbf{H}^1_s(\Omega)\},$$
$$H^1_s(\Omega) = \{v \in H^1(\Omega); (0, 0, v) \in \mathbf{H}^1_s(\Omega)\}.$$

Taking into account that in the contact boundary Γ_C the outward normal vector is given by $(n_1, n_2, 0)$, we have that

$$\mathbf{K}(\Omega) = \mathbf{K}_2(\Omega)\times H^1(\Omega),$$

where

$$\mathbf{K}_2(\Omega) = \{(v_1, v_2) \in [H^1(\Omega)]^2; \, v_\alpha n_\alpha \leq 0 \text{ on } \Gamma_C\}.$$

Consequently, Problem (MVP^0) can be rewritten in the equivalent form:

$$(u_1^0, u_2^0) \in \mathbf{W}_{s2}(\Omega) \cap \mathbf{K}_2(\Omega), \; u_3^0 \in H_s^1(\Omega), \sigma^0 \in \mathbf{X}(\Omega), \tag{32}$$

$$\int_\Omega \frac{1}{E}\sigma_{33}^0 \tau_{33} dx - \int_\Omega \partial_3 u_3^0 \tau_{33} dx = 0, \forall \tau_{33} \in L^2(\Omega), \tag{33}$$

$$\int_\Omega \varepsilon_{3\beta}(\mathbf{u}^0)\tau_{3\beta} dx = 0, \forall (\tau_{3\beta}) \in [L^2(\Omega)]^2, \tag{34}$$

$$\int_\Omega \varepsilon_{\alpha\beta}(\mathbf{u}^0)\tau_{\alpha\beta} dx = 0, \forall (\tau_{\alpha\beta}) \in [L^2(\Omega)]_s^4, \tag{35}$$

$$\int_\Omega (\sigma_{33}^0 \partial_3 v_3 + \sigma_{3\beta}^0 \partial_\beta v_3) dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_{N_1}} h_3 v_3 d\Gamma, \forall v_3 \in H_s^1(\Omega), \tag{36}$$

$$\begin{gathered}\int_\Omega [\sigma_{\alpha\beta}^0 \partial_\alpha (v_\beta - u_\beta^0) + \sigma_{3\beta}^0 \partial_3 (v_\beta - u_\beta^0)] dx \geq \\ \int_\Omega f_\beta (v_\beta - u_\beta^0) dx + \int_{\Gamma_{N_1}} h_\beta (v_\beta - u_\beta^0) d\Gamma, \\ \forall (v_\beta) \in \mathbf{W}_{s2}(\Omega) \cap \mathbf{K}_2(\Omega).\end{gathered} \tag{37}$$

To characterize the solutions of Problem (32)-(37), we introduce the following notations:

- The total force per unit of length is

$$q_i(x_3) = \int_\omega f_i(x_1, x_2, x_3) dx_1 dx_2 + \chi_{[-\delta,\delta]}(x_3) \int_{\gamma_{N_1}(x_3)} h_i(x_1, x_2, x_3) d\gamma. \tag{38}$$

- The total moment about the origin of the cross-section per unit of length is defined as

$$\begin{aligned}\mathbf{M}(x_3) = &\int_\omega (x_2 f_3, -x_1 f_3, x_1 f_2 - x_2 f_1) dx_1 dx_2 + \\ &\chi_{[-\delta,\delta]}(x_3) \int_{\gamma_{N_1}(x_3)} (x_2 h_3, -x_1 h_3, x_1 h_2 - x_2 h_1) d\gamma.\end{aligned} \tag{39}$$

We also introduce the subspace of $\mathbf{H}^1(\Omega)$ corresponding to Bernouilli-Navier displacements (see [8]):

$$\mathbf{V}_{BN}(\Omega) = \{\mathbf{v} \in \mathbf{H}_s^1(\Omega); \, v_\alpha \in H^2(-L, L), \, v_3 = \underline{v}_3 - x_\alpha \partial_3 v_\alpha, \, \underline{v}_3 \in H^1(-L, L)\}.$$

Moreover, to take into account the hypotheses of symmetry, we consider the subspace of $H^m(-L, L)$, $H_e^m(-L, L)$ and $H_o^m(-L, L)$, $m = 1, 2$, corresponding to its even and odd functions, respectively.

The following theorem characterizes the solutions of Problem (32)-(37):

Theorem 4. *Every solution of Problem (32)-(37) verifies:*

i. *The displacement field* $\mathbf{u}^0 \in \mathbf{V}_{BN}(\Omega)$; u_3^0 *being given by*

$$u_3^0(x_1, x_2, x_3) = \underline{u}_3^0(x_3) - x_2 \partial_3 u_2^0(x_3), \; \forall (x_1, x_2, x_3) \in \Omega, \tag{40}$$

with $\underline{u}_3^0 \in H_o^1(-L, L)$. *Moreover,* u_1^0 *is null and* $u_2^0 \in H_e^2(-L, L)$.

ii. *The axial stress* σ_{33}^0 *is given by*

$$\sigma_{33}^0 = E\partial_3 u_3^0 = E(\partial_3 \underline{u}_3^0 - x_2 \partial_{33} u_2^0), \tag{41}$$

and the corresponding component of the bending moment is

$$m_2^0 = \int_\omega x_2 \sigma_{33}^0 dx_1 dx_2 = -EI_1 \partial_{33} u_2^0.$$

iii. *The stretch component* $\underline{u}_3^0$ *is a solution of the axial problem:*
Problem $(\mathbf{P}_3^0)$: *Find* $\underline{u}_3^0 \in H_o^1(-L, L)$ *such that*

$$\int_{-L}^{L} EA\partial_3 \underline{u}_3^0 \partial_3 v_3^0 dx_3 = \int_{-L}^{L} q_3 v_3^0 dx_3, \tag{42}$$

for all $v_3^0 \in H_o^1(-L, L)$.

iv. The displacement u_2^0 is a solution of the bending problem:
Problem ($\mathbf{P}_2^0$)*: Find $u_2^0 \in H_e^2(-L, L)$, $(0, u_2^0) \in \mathbf{K}_2(\Omega)$, such that*

$$\int_{-L}^{L} EI_1 \partial_{33} u_2^0 \partial_{33}(v_2^0 - u_2^0) dx_3 \geq \int_{-L}^{L} q_2(v_2^0 - u_2^0) dx_3 - \int_{-L}^{L} M_1 \partial_3 (v_2^0 - u_2^0) dx_3, \tag{43}$$

for all $v_2^0 \in H_e^2(-L, L)$, $(0, v_2^0) \in \mathbf{K}_2(\Omega)$.

Proof. *(i)* By expression (35) we obtain that

$$\partial_1 u_1^0 = \partial_2 u_2^0 = 0, \quad \partial_1 u_2^0 + \partial_2 u_1^0 = 0.$$

Then, we deduce

$$u_1^0(x_1, x_2, x_3) = \phi_1^0(x_2, x_3),\ u_2^0(x_1, x_2, x_3) = \phi_2^0(x_1, x_3),$$
$$\partial_2 \phi_1^0(x_2, x_3) + \partial_1 \phi_2^0(x_1, x_3) = 0.$$

Given that the variables x_1, x_2 and x_3 are independent, we find that

$$\partial_2 \phi_1^0(x_2, x_3) = -\partial_1 \phi_2^0(x_1, x_3) = z^0(x_3)$$

and consequently,

$$u_1^0(x_1, x_2, x_3) = z_1^0(x_3) + x_2 z^0(x_3),$$
$$u_2^0(x_1, x_2, x_3) = z_2^0(x_3) - x_1 z^0(x_3).$$

In addition, u_1^0 must be odd with respect to x_1, so $z_1^0 = z^0 \equiv 0$. Therefore, $u_1^0 = 0$ and $u_2^0 = z_2^0(x_3)$. Moreover, $(u_1^0, u_2^0) \in \mathbf{W}_{s2}(\Omega)$ and consequently, $z_2^0 \in H_e^1(-L, L)$.
Furthermore, from (34), $\varepsilon_{3\beta}(\mathbf{u}^0) = 0$, thus

$$\partial_1 u_3^0 = -\partial_3 u_1^0 = 0,\ \partial_2 u_3^0 = -\partial_3 u_2^0 = -\partial_3 z_2^0.$$

Integrating, we obtain that

$$u_3^0(x_1, x_2, x_3) = \underline{u}_3^0(x_3) - x_2 \partial_3 z_2^0(x_3) = \underline{u}_3^0(x_3) - x_2 \partial_3 u_2^0(x_3). \tag{44}$$

Summing up, we have proved that:

$$\mathbf{u}^0(x_1, x_2, x_3) = (0, u_2^0(x_3), \underline{u}_3^0(x_3) - x_2 \partial_3 u_2^0(x_3)) \in \mathbf{H}_s^1(\Omega) \cap \mathbf{K}(\Omega). \tag{45}$$

In particular, $\partial_3 u_3^0 = \partial_3 \underline{u}_3^0 - x_2 \partial_{33} u_2^0 \in L^2(\Omega)$, from which we infer that:

$$\int_\Omega (\partial_3 u_3^0)^2 dx = A \int_{-L}^{L} (\partial_3 \underline{u}_3^0(x_3))^2 dx_3 + I_1 \int_{-L}^{L} (\partial_{33} u_2^0(x_3))^2 dx_3 < \infty;$$

so $u_2^0 \in H_e^2(-L, L)$ and $\underline{u}_3^0 \in H_o^1(-L, L)$.

(ii) It is easy to prove (41) substituting expression (44) in equation (33).

(iii) Substituting expression (41) in (36), we deduce that

$$\int_\Omega [E(\partial_3 \underline{u}_3^0 - x_2 \partial_{33} u_2^0) \partial_3 v_3 + \sigma_{3\beta}^0 \partial_\beta v_3] dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_{N_1}} h_3 v_3 d\Gamma,\ \forall v_3 \in H_s^1(\Omega).$$

Choosing $v_3(x_1, x_2, x_3) = v_3^0(x_3) \in H_o^1(-L, L)$ as a test function, we have that

$$\int_\Omega E(\partial_3 \underline{u}_3^0 - x_2 \partial_{33} u_2^0) \partial_3 v_3^0 dx = \int_\Omega f_3 v_3^0 dx + \int_{\Gamma_{N_1}} h_3 v_3^0 d\Gamma.$$

Taking into account (15), we conclude that

$$\int_{-L}^{L} EA \partial_3 \underline{u}_3^0 \partial_3 v_3^0 dx_3 = \int_{-L}^{L} \left(\int_\omega f_3 dx_1 dx_2 \right) v_3^0 dx_3 + \int_{-\delta}^{\delta} \left(\int_{\gamma_{N_1}(x_3)} h_3 d\gamma \right) v_3^0 dx_3,$$

for all $v_3^0 \in H_o^1(-L, L)$. Therefore, from expression (38), $\underline{u}_3^0$ is a solution of Problem ($\mathbf{P}_3^0$).

(iv) By choosing $(v_1, v_2)(x_1, x_2, x_3) = (0, v_2^0(x_3)) \in \mathbf{W}_{s2}(\Omega) \cap \mathbf{K}_2(\Omega)$, with $v_2^0 \in H_e^2(-L, L)$, as a test function in inequality (37), we obtain that

$$\int_\Omega \sigma_{32}^0 \partial_3(v_2^0 - u_2^0)dx \geq \int_\Omega f_2(v_2^0 - u_2^0)dx + \int_{\Gamma_{N_1}} h_2(v_2^0 - u_2^0)d\Gamma. \tag{46}$$

Moreover, choosing $v_3(x_1, x_2, x_3) = x_2\partial_3(v_2^0 - u_2^0)(x_3) \in H_s^1(\Omega)$ in expression (36), we deduce that

$$\begin{aligned}&\int_\Omega [\sigma_{33}^0 \partial_3(x_2\partial_3(v_2^0 - u_2^0)) + \sigma_{32}^0\partial_2(x_2\partial_3(v_2^0 - u_2^0))]dx \\ &= \int_\Omega f_3 x_2 \partial_3(v_2^0 - u_2^0)dx + \int_{\Gamma_{N_1}} h_3 x_2\partial_3(v_2^0 - u_2^0)d\Gamma,\end{aligned}$$

and, consequently,

$$\begin{aligned}&\int_\Omega [\sigma_{33}^0 x_2 \partial_{33}(v_2^0 - u_2^0) + \sigma_{32}^0\partial_3(v_2^0 - u_2^0)]dx \\ &= \int_\Omega f_3 x_2 \partial_3(v_2^0 - u_2^0)dx + \int_{\Gamma_{N_1}} h_3 x_2\partial_3(v_2^0 - u_2^0)d\Gamma.\end{aligned} \tag{47}$$

Now, taking into account expression (41), we have that

$$\begin{aligned}&\int_\Omega \sigma_{33}^0 x_2\partial_{33}(v_2^0 - u_2^0)dx = \\ &\int_{-L}^{L} \partial_{33}(v_2^0 - u_2^0)\left(\int_\omega x_2 E(\partial_3 \underline{u}_3^0 - x_2\partial_{33}u_2^0)dx_1dx_2\right)dx_3 = \\ &-\int_{-L}^{L} EI_1\partial_{33}u_2^0\partial_{33}(v_2^0 - u_2^0)dx_3.\end{aligned} \tag{48}$$

From (46)-(48) we infer that:

$$\begin{aligned}\int_\Omega \sigma_{32}^0\partial_3(v_2^0 - u_2^0)dx &= \int_{-L}^{L} EI_1\partial_{33}u_2^0\partial_{33}(v_2^0 - u_2^0)dx_3 + \\ &\int_\Omega x_2 f_3\partial_3(v_2^0 - u_2^0)dx + \int_{\Gamma_{N_1}} x_2 h_3\partial_3(v_2^0 - u_2^0)d\Gamma, \\ \int_\Omega \sigma_{32}^0\partial_3(v_2^0 - u_2^0)dx &\geq \int_\Omega f_2(v_2^0 - u_2^0)dx + \int_{\Gamma_{N_1}} h_2(v_2^0 - u_2^0)d\Gamma.\end{aligned}$$

Finally, from expressions (38) and (39), we deduce inequality (43) and then (iv).

□

Theorem 5. *Under assumptions* **(H1$^\epsilon$), (H2$^\epsilon$), (H7$^\epsilon$)** *and* **(H8$^\epsilon$)***, there exists a unique solution of Problem* ($\mathbf{P}_3^0$).

Proof. Since $\underline{u}_3^0$ must be an odd function, the bilinear form associated to Problem ($\mathbf{P}_3^0$) is coercive on $H_o^1(-L, L) \subset \mathcal{C}([-L, L])$ and the uniqueness is proved. □

Theorem 6. *Under assumptions* **(H1$^\epsilon$)-(H3$^\epsilon$)**, **(H7$^\epsilon$)** *and* **(H8$^\epsilon$)***, there exists a unique solution of Problem* ($\mathbf{P}_2^0$).

Proof. It can be proved by applying a well-known result of Lions and Stampacchia [16]. Then, we consider $V = H_e^2(-L, L)$ which is a dense subspace of $H_e^1(-L, L) \subset C^0([-L, L])$.

We denote by p_0 the usual norm in $H^1(-L, L)$ which gives a pre-Hilbert structure to V and by p_1 the seminorm in V:

$$p_1(v) = \left\| \frac{d^2 v}{dx_3^2} \right\|_{0,(-L,L)}.$$

The subspace $Y = \{v \in V; p_1(v) = 0\}$ is finite-dimensional since

$$Y = \{v \in V; v(x_3) = c, c \in \mathbb{R}\}.$$

Moreover, due to the compact injection $V \hookrightarrow H_e^1(-L, L)$, we have that (see [18])

$$\inf_{y\in Y} p_0(v - y) \leq c_1 p_1(v), \forall v \in V. \tag{49}$$

If we denote by a the bilinear form in (43), we obtain that a is continuous on V and semicoercive, i.e.,

$$a(v,v) \geq c_2 p_1^2(v),\ \forall v \in V, \quad c_2 > 0. \tag{50}$$

In addition, the right-hand side of equation (43) can be identified as an element of V'

$$(F^0, v) = \int_{-L}^{L} q_2 v\, dx_3 - \int_{-L}^{L} M_1 \partial_3 v\, dx_3,$$

and, thanks to hypothesis (**H3**ffl) for $\varepsilon = 1$, it is easy to prove that

$$(F^0, y) < 0, \quad \forall y \in Y,\ y \neq 0,\ (0, y) \in \mathbf{K}_2(\Omega). \tag{51}$$

Relations (49)-(51) allow us to apply Theorem 5.1 of Lions and Stampacchia [16] and obtain the existence of, at least, one solution of Problem $(\mathbf{P}_2^0)$.

Now, let us assume that u and $\tilde{u}$ are two different solutions of Problem $(\mathbf{P}_2^0)$. Then, we can prove that there exists $y \in Y$such that $\tilde{u} = u + y$. Indeed, given that u and $\tilde{u}$ are solutions, we can write

$$a(u, \tilde{u} - u) \geq (F^0, \tilde{u} - u) = (F^0, y),$$
$$a(\tilde{u}, \tilde{u} - u) \leq (F^0, \tilde{u} - u) = (F^0, y),$$

that is,

$$a(\tilde{u} - u, \tilde{u} - u) \leq 0.$$

Consequently, from (50), $y \in Y$ and $(F^0, y) = 0$; so, from (51), $y = 0$. In conclusion, there exists a unique solution of Problem $(\mathbf{P}_2^0)$. □

8. Convergence of scaled displacements and stresses

In order to complete the asymptotic analysis of our problem, we are going to carry out a convergence analysis of the solution of Problem $(MVP(\varepsilon))$, $(\mathbf{u}(\epsilon), \sigma(\epsilon))_{\varepsilon>0}$, when ϵ goes to 0. For all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, we denote

$$|\mathbf{v}|_{1,\Omega} = \|\mathbf{v}\|_{0,\Omega} + \|\varepsilon(\mathbf{v})\|_{0,\Omega},$$

which is an equivalent norm to $\|\cdot\|_{1,\Omega}$ in $\mathbf{H}^1(\Omega)$; the first addend gives a pre-Hilbert structure to $\mathbf{H}^1(\Omega)$ and the second addend is a seminorm in $\mathbf{H}^1(\Omega)$.

Definition 1. *We define the subspace* $\mathbf{Y}(\Omega) = \mathbf{H}_s^1(\Omega) \cap \mathbf{R}(\Omega)$

$$\mathbf{Y}(\Omega) = \{\mathbf{w} \in \mathbf{H}_s^1(\Omega); \|\varepsilon(\mathbf{w})\|_{0,\Omega} = 0\} = \{\mathbf{w} = (0, c, 0); c \in \mathbb{R}\}.$$

Lemma 4. *Let* P *be the projection of* $\mathbf{H}_s^1(\Omega)$ *onto* $\mathbf{Y}(\Omega)$*, with respect to the pre-Hilbert structure on* $\mathbf{H}_s^1(\Omega)$ *defined by* $\|\cdot\|_{0,\Omega}$*. Then,*

$$\mathrm{P}\mathbf{v} = (0, \mathcal{M}_\Omega(v_2), 0),\ \forall \mathbf{v} \in \mathbf{H}_s^1(\Omega),$$
$$\mathcal{M}_\Omega(v_2) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega v_2\, dx.$$

Furthermore ,

$$\inf_{\mathbf{w} \in \mathbf{Y}(\Omega)} \|\mathbf{v} - \mathbf{w}\|_{0,\Omega} = \|\mathbf{v} - \mathrm{P}\mathbf{v}\|_{0,\Omega} \leq \tilde{C} \|\varepsilon(\mathbf{v})\|_{0,\Omega},\ \forall \mathbf{v} \in \mathbf{H}_s^1(\Omega),\ \tilde{C} > 0, \tag{52}$$

$\tilde{C}$ *depending only on* Ω*.*

Proof. The projection $\mathbf{P}$ is well defined because $\mathbf{Y}(\Omega)$ is a finite-dimensional space. In addition, given that $\mathbf{Y}(\Omega)$ is a closed subspace, we have for each $\mathbf{v} \in \mathbf{H}_s^1(\Omega)$ (see [19])

$$(\mathbf{v} - \mathrm{P}\mathbf{v}, \mathbf{w}) = 0,\ \forall \mathbf{w} \in \mathbf{Y},$$

that is,

$$0 = (\mathbf{v} - \mathrm{P}\mathbf{v}, (0, c, 0)) = c \int_\Omega (v_2 - (\mathrm{P}\mathbf{v})_2)\, dx,\ \forall c \in \mathbb{R},\ c \neq 0.$$

Taking into account that $(Pv)_2$ is a constant, we can deduce

$$(\mathrm{P}\mathbf{v})_2 = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega v_2\, dx = \mathcal{M}_\Omega(v_2).$$

Finally, $\mathcal{M}_\Omega(v_1) = \mathcal{M}_\Omega(v_3) = 0$ since $\mathbf{v} \in \mathbf{H}_s^1(\Omega)$ and from Poincaré-Wirtinger inequality, we obtain (52). □

From now on, we will use an auxiliary stress tensor $\tilde{\sigma}(\epsilon)$ defined by

$$\tilde{\sigma}_{33}(\epsilon) = \sigma_{33}(\epsilon), \quad \tilde{\sigma}_{3\beta}(\epsilon) = \epsilon\sigma_{3\beta}(\epsilon), \quad \tilde{\sigma}_{\alpha\beta}(\epsilon) = \epsilon^2\sigma_{\alpha\beta}(\epsilon). \tag{53}$$

Lemma 5. *The following estimates hold:*

$$\|\tilde{\sigma}(\epsilon)\|_{0,\Omega}^2 \leq C_1 \left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right) \|\mathbf{u}(\epsilon)\|_{1,\Omega}, \tag{54}$$

$$C_2\|\varepsilon(\mathbf{u}(\epsilon))\|_{0,\Omega}^2 \leq F(\mathbf{u}(\epsilon)), \tag{55}$$

$$\|\varepsilon(\mathbf{u}(\epsilon))\|_{0,\Omega}^2 \leq C_3 \left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right) \|\mathbf{u}(\epsilon)\|_{1,\Omega}. \tag{56}$$

Proof. If we consider $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}(\epsilon)$ in (24) and $\tau = \sigma(\epsilon)$ in (23) we obtain that

$$\begin{aligned} a_0(\sigma(\epsilon), \sigma(\epsilon)) + \epsilon^2 a_2(\sigma(\epsilon), \sigma(\epsilon)) + \epsilon^4 a_4(\sigma(\epsilon), \sigma(\epsilon)) = \\ = b(\sigma(\epsilon), \mathbf{u}(\epsilon)) = F(\mathbf{u}(\epsilon)). \end{aligned} \tag{57}$$

From (25)-(29) and (57) and taking into account expressions (53), it is easy to prove that

$$\int_\Omega \left(\frac{1+\nu}{E}\tilde{\sigma}_{ij}(\epsilon) - \frac{\nu}{E}\tilde{\sigma}_{pp}(\epsilon)\delta_{ij}\right)\tilde{\sigma}_{ij}(\epsilon)dx = \int_\Omega f_i u_i(\epsilon)dx + \int_{\Gamma_{N_1}} h_i u_i(\epsilon)d\Gamma.$$

Thanks to the properties of the Lamé coefficients and the assumptions on the volume and surface forces, we have that

$$C_4\|\tilde{\sigma}(\epsilon)\|_{0,\Omega}^2 \leq \int_\Omega \left(\frac{1+\nu}{E}\tilde{\sigma}_{ij}(\epsilon) - \frac{\nu}{E}\tilde{\sigma}_{pp}(\epsilon)\delta_{ij}\right)\tilde{\sigma}_{ij}(\epsilon)dx =$$
$$\int_\Omega f_i u_i(\epsilon)dx + \int_{\Gamma_{N_1}} h_i u_i(\epsilon)d\Gamma \leq C_5\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right)\|\mathbf{u}(\epsilon)\|_{1,\Omega},$$

and consequently,

$$C_4\|\tilde{\sigma}(\epsilon)\|_{0,\Omega}^2 \leq \int_\Omega \mathbf{f}\cdot\mathbf{u}(\epsilon)dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot\mathbf{u}(\epsilon)d\Gamma = F(\mathbf{u}(\epsilon)), \tag{58}$$
$$\|\tilde{\sigma}(\epsilon)\|_{0,\Omega}^2 \leq C_1\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right)\|\mathbf{u}(\epsilon)\|_{1,\Omega},$$

where C_1 and C_4 are positive constants depending on Ω, E and ν. So, we have proved estimate (54).

From (23) and taking into account the definition of bilinear forms a_0, a_2 and a_4, we obtain that

$$\left|\int_\Omega \tau : \varepsilon(\mathbf{u}(\epsilon))dx\right| \leq C_6\|\tilde{\sigma}(\epsilon)\|_{0,\Omega}\|\tau\|_{0,\Omega},\ \forall\tau \in \mathbf{X}(\Omega),\ C_6 > 0,$$

since $0 < \epsilon \leq 1$. By choosing $\tau = \varepsilon(\mathbf{u}(\mathbf{ffl}))$, we deduce that

$$\|\varepsilon(\mathbf{u}(\epsilon))\|_{0,\Omega} \leq C_6\|\tilde{\sigma}(\epsilon)\|_{0,\Omega}. \tag{59}$$

From inequalities (58) and (59), we have that

$$C_2\|\varepsilon(\mathbf{u}(\epsilon))\|_{0,\Omega}^2 \leq \int_\Omega \mathbf{f}\cdot\mathbf{u}(\epsilon)dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot\mathbf{u}(\epsilon)d\Gamma = F(\mathbf{u}(\epsilon)),\ C_2 > 0,$$

therefore,

$$\|\varepsilon(\mathbf{u}(\epsilon))\|_{0,\Omega}^2 \leq C_3\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right)\|\mathbf{u}(\epsilon)\|_{1,\Omega},$$

and hence estimates (55) and (56) are verified. ☐

Theorem 7. *There exist constants C_7 and C_8, independent of ϵ, such that for all $0 < \epsilon \leq 1$ the solution $(\mathbf{u}(\epsilon), \sigma(\epsilon)) \in (\mathbf{H}_s^1(\Omega) \cap \mathbf{K}(\Omega)) \times \mathbf{X}(\Omega)$ of Problem $(MVP(\varepsilon))$ verifies*

$$\|\mathbf{u}(\epsilon)\|_{1,\Omega} \leq C_7, \tag{60}$$

$$\|\sigma_{33}(\epsilon)\|_{0,\Omega} \leq C_8, \quad \epsilon\|\sigma_{3\beta}(\epsilon)\|_{0,\Omega} \leq C_8, \quad \epsilon^2\|\sigma_{\alpha\beta}(\epsilon)\|_{0,\Omega} \leq C_8. \tag{61}$$

Proof. Let us assume that (60) is false, so there exists a sequence $R_n \to +\infty$ as $n \to +\infty$ such that for all $n > 0$ there exists ϵ_n satisfying

$$\|\mathbf{u}(\epsilon_n)\|_{1,\Omega} = R_n. \tag{62}$$

We define the sequence

$$\mathbf{w}_n = \frac{1}{R_n}\mathbf{u}(\epsilon_n), \tag{63}$$

that verifies $\mathbf{w}_n \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega)$ and obviously $\|\mathbf{w}_n\|_{1,\Omega} = 1$. Therefore, there exists a subsequence, still denoted by $(\mathbf{w}_n)_{n\geq 0}$, such that $\mathbf{w}_n \rightharpoonup \mathbf{w}$ in $\mathbf{H}^1(\Omega)$ as $n \to +\infty$ and $\mathbf{w} \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega)$.

From inequality (56), it is verified

$$\|\varepsilon(\mathbf{u}(\epsilon_n))\|^2_{0,\Omega} \leq C_3 \left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right) R_n, \tag{64}$$

and consequently,

$$\|\varepsilon(\mathbf{w}_n)\|_{0,\Omega} = O(1/\sqrt{R_n}), \tag{65}$$

thanks to (63); so

$$\|\varepsilon(\mathbf{w})\|_{0,\Omega} = 0, \tag{66}$$

that is, $\mathbf{w} \in \mathbf{K}(\Omega) \cap \mathbf{Y}(\Omega)$.

Let P be the projection of $\mathbf{H}^1_s(\Omega)$ onto $\mathbf{Y}(\Omega)$ given by Lemma 1. Then,

$$\|\mathrm{P}\mathbf{w}_n\|_{0,\Omega} \geq C_9 > 0, \text{ as } n \to \infty. \tag{67}$$

Indeed, if the subsequence $(\mathbf{w}_n)_{n\geq 0}$ verifies

$$\|\mathrm{P}\mathbf{w}_n\|_{0,\Omega} \to 0,$$

then, from (52) and (65),

$$\|\mathbf{w}_n\|_{0,\Omega} \leq \|\mathbf{w}_n - \mathrm{P}\mathbf{w}_n\|_{0,\Omega} + \|\mathrm{P}\mathbf{w}_n\|_{0,\Omega} \leq \tilde{C}\|\varepsilon(\mathbf{w}_n)\|_{0,\Omega} + \|\mathrm{P}\mathbf{w}_n\|_{0,\Omega} \to 0,$$

and consequently,

$$\|\mathbf{w}_n\|_{1,\Omega} = O(\|\varepsilon(\mathbf{w}_n)\|_{0,\Omega} + \|\mathrm{P}\mathbf{w}_n\|_{0,\Omega}) \to 0,$$

while $\|\mathbf{w}_n\|_{1,\Omega} = 1$ by construction; so (67) is true.

Since $\dim\mathbf{Y}(\Omega) < +\infty$, the projection of $\mathbf{H}^1_s(\Omega)$ onto $\mathbf{Y}(\Omega)$ is compact (see [20]), so $\mathrm{P}\mathbf{w}_n \to \mathrm{P}\mathbf{w}$ and hence, from (67),

$$\|\mathrm{P}\mathbf{w}\|_{0,\Omega} \geq C_9 > 0.$$

Moreover, from (66), $\mathrm{P}\mathbf{w} = \mathbf{w}$, so we can deduce that $\mathbf{w} \in \mathbf{Y}(\Omega) \cap \mathbf{K}(\Omega)$ is not null.

If $\mathbf{Y}(\Omega) \cap \mathbf{K}(\Omega) = \{\mathbf{0}\}$, we obtain a contradiction. If $\mathbf{Y}(\Omega) \cap \mathbf{K}(\Omega) \neq \{\mathbf{0}\}$, from assumptions (**H3**ffl) and (**H5**ffl) for $\epsilon = 1$, and since $w_n = cn_2$, $c \neq 0$, we have that

$$\int_\Omega \mathbf{f}\cdot\mathbf{w}dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot\mathbf{w}d\Gamma = F(\mathbf{w}) = -r^* < 0.$$

So, for each r fixed, $r \in (r^*/2, 3r^*/2)$, there exists n_r such that

$$F(\mathrm{P}\mathbf{w}_n) = \int_\Omega \mathbf{f}\cdot\mathrm{P}\mathbf{w}_n dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot\mathrm{P}\mathbf{w}_n d\Gamma \leq -r < 0, \forall n \geq n_r. \tag{68}$$

From inequality (55),

$$C_2\|\varepsilon(\mathbf{u}(\epsilon_n))\|^2_{0,\Omega} \leq F(\mathbf{u}(\epsilon_n)) = F(\mathbf{u}(\epsilon_n) - \mathrm{P}\mathbf{u}(\epsilon_n)) + F(\mathrm{P}\mathbf{u}(\epsilon_n)),\ C_2 > 0,$$

and we obtain that

$$C_2\|\varepsilon(\mathbf{u}(\epsilon_n))\|^2_{0,\Omega} - R_n F(\mathrm{P}\mathbf{w}(\epsilon_n)) \leq \left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right)\|\mathbf{u}(\epsilon_n) - \mathrm{P}\mathbf{u}(\epsilon_n)\|_{0,\Omega} \leq$$
$$\tilde{C}\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Gamma_{N_1}}\right)\|\varepsilon(\mathbf{u}(\epsilon_n))\|_{0,\Omega} \leq C_{10}\|\varepsilon(\mathbf{u}(\epsilon_n))\|_{0,\Omega},$$

thanks to (52) with $C_{10} > 0$. Finally, from (64),

$$-R_n F(\mathrm{P}\mathbf{w}(\epsilon_n)) \leq C_{10}\sqrt{R_n},$$

and taking into account expression (68), we conclude

$$rR_n \leq C_{10}\sqrt{R_n},\ n > n_r,$$

which is obviously absurd since $R_n \to +\infty$. So, expression (62) cannot be true and, consequently, estimate (60) is verified. Therefore, from Lemma (5), bounds (61) are obtained and the proof of Theorem 7 is complete. ☐

Corollary 1. *The auxiliary stress tensor* $\tilde{\sigma}(\epsilon)$ *defined by (53) and the auxiliary strain tensor* $\tilde{\varepsilon}(\epsilon)$ *given by*

$$\tilde{\varepsilon}_{33}(\epsilon) = \varepsilon_{33}(\mathbf{u}(\epsilon)),\ \tilde{\varepsilon}_{3\beta}(\epsilon) = \epsilon^{-1}\varepsilon_{3\beta}(\mathbf{u}(\epsilon)),\ \tilde{\varepsilon}_{\alpha\beta}(\epsilon) = \epsilon^{-2}\varepsilon_{\alpha\beta}(\mathbf{u}(\epsilon)), \tag{69}$$

verifies that

$$\tilde{\sigma}_{ij}(\epsilon) = \frac{E}{1+\nu}\tilde{\varepsilon}_{ij}(\epsilon) + \frac{\nu E}{(1+\nu)(1-2\nu)}\tilde{\varepsilon}_{pp}(\epsilon)\delta_{ij} = \Lambda_{ijkl}\tilde{\varepsilon}_{kl}(\epsilon),$$
$$\tilde{\varepsilon}_{ij}(\epsilon) = \frac{1+\nu}{E}\tilde{\sigma}_{ij}(\epsilon) - \frac{\nu}{E}\tilde{\sigma}_{pp}(\epsilon)\delta_{ij} = \Lambda^{-1}_{ijkl}\tilde{\sigma}_{kl}(\epsilon).$$

Moreover, there exists $C_{11} > 0$ *such that for all* ϵ, $0 < \epsilon \leq 1$,

$$\|\tilde{\varepsilon}_{33}(\epsilon)\|_{0,\Omega} \leq C_{11}, \quad \|\tilde{\varepsilon}_{3\beta}(\epsilon)\|_{0,\Omega} \leq C_{11}, \quad \|\tilde{\varepsilon}_{\alpha\beta}(\epsilon)\|_{0,\Omega} \leq C_{11}. \tag{70}$$

Theorem 8. *There exists a subsequence of* $(\mathbf{u}(\epsilon), \sigma(\epsilon))$, *also denoted with* ϵ, *and there exists an element* $\Psi = (\Psi_{ij}) \in \mathbf{X}(\Omega)$ *such that*

$$\mathbf{u}(\epsilon) \rightharpoonup \tilde{\mathbf{u}}, \ \text{ in } \mathbf{H}^1_s(\Omega), \tag{71}$$
$$\sigma_{33}(\epsilon) \rightharpoonup \Psi_{33},\ \epsilon\sigma_{3\beta}(\epsilon) \rightharpoonup \Psi_{3\beta},\ \epsilon^2\sigma_{\alpha\beta}(\epsilon) \rightharpoonup \Psi_{\alpha\beta}, \ \text{ in } L^2(\Omega), \tag{72}$$

when ϵ *goes to zero. In addition, the following symmetry properties hold:*

- Ψ_{ii}, $i = 1, 2, 3$, *are even with respect to* x_1 *and* x_3,
- Ψ_{12} *is odd with respect to* x_1 *and even with respect to* x_3,
- Ψ_{13} *is odd with respect to* x_1 *and* x_3,
- Ψ_{23} *is even with respect to* x_1 *and odd with respect to* x_3.

Moreover,

$$\int_\Omega \Psi_{3\beta}\partial_\beta v\, dx = 0,\ \forall v \in H^1_s(\Omega), \tag{73}$$
$$\int_\Omega \Psi_{\alpha\beta}\partial_\alpha v_\beta\, dx \geq 0,\ \forall (v_1, v_2) \in \mathbf{W}_{s2}(\Omega) \cap \mathbf{K}_2(\Omega). \tag{74}$$

Proof. From Theorem 7 and Corollary 1, there exists a subsequence of $(\mathbf{u}(\epsilon), \sigma(\epsilon))_{\epsilon>0}$ such that $(\mathbf{u}(\epsilon), \tilde{\sigma}(\epsilon))$ converges weakly in $\mathbf{H}^1_s(\Omega) \times \mathbf{X}(\Omega)$ when ϵ goes to zero, i.e.:

$$\mathbf{u}(\epsilon) \rightharpoonup \tilde{\mathbf{u}}, \text{ in } \mathbf{H}^1_s(\Omega),$$
$$\sigma_{33}(\epsilon) \rightharpoonup \Psi_{33},\ \epsilon\sigma_{3\beta}(\epsilon) \rightharpoonup \Psi_{3\beta},\ \epsilon^2\sigma_{\alpha\beta}(\epsilon) \rightharpoonup \Psi_{\alpha\beta} \text{ in } L^2(\Omega).$$

The symmetry properties are a direct consequence from $\mathbf{u}(\epsilon) \in \mathbf{H}^1_s(\Omega)$.

By choosing $(v_1 + u_1(\epsilon), v_2 + u_2(\epsilon), u_3(\epsilon))$, with $\mathbf{v} = (v_1, v_2, 0) \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega)$ as a test function in (24) and multiplying by ϵ^2, we have that

$$\epsilon^2 \int_\Omega \sigma_{\alpha\beta}(\epsilon)\varepsilon_{\alpha\beta}(\mathbf{v})\, dx + 2\epsilon^2 \int_\Omega \sigma_{3\beta}(\epsilon)\varepsilon_{3\beta}(\mathbf{v})\, dx \geq \epsilon^2 \left[\int_\Omega f_\beta v_\beta\, dx + \int_{\Gamma_{N_1}} h_\beta v_\beta\, d\Gamma \right].$$

Taking limits when ϵ goes to zero, we obtain

$$\int_\Omega \Psi_{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v})\, dx \geq 0, \quad \forall\, \mathbf{v} = (v_1, v_2, 0) \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega),$$

and thanks to the symmetry properties of $\Psi_{\alpha\beta}$ we deduce that

$$\int_\Omega \Psi_{\alpha\beta}\partial_\alpha v_\beta\, dx \geq 0, \quad \forall\, (v_1, v_2) \in \mathbf{W}_{s2}(\Omega) \cap \mathbf{K}_2(\Omega).$$

With the same procedure, by choosing $\mathbf{u}(\epsilon) \pm (0, 0, v)$, with $v \in H^1_s(\Omega)$, as test functions in (24) and multiplying by ϵ, we have that

$$\epsilon \int_\Omega \sigma_{3\beta}(\epsilon)\partial_\beta v\, dx + \epsilon \int_\Omega \sigma_{33}(\epsilon)\partial_3 v\, dx = \epsilon \left[\int_\Omega f_3 v\, dx + \int_{\Gamma_{N_1}} h_3 v\, d\Gamma \right].$$

Taking again limits when ϵ goes to zero, we find that

$$\int_\Omega \Psi_{3\beta}\partial_\beta v\, dx = 0, \quad \forall\, v \in H^1_s(\Omega).$$

□

Corollary 2. *The limit stress components $\Psi_{\alpha\beta}$ verify*

$$\int_\Omega \Psi_{\alpha\beta}\partial_\alpha v_\beta dx = 0, \forall (v_1, v_2) \in \mathbf{W}_{s2}(\Omega). \tag{75}$$

Moreover,

$$\int_\omega \Psi_{\alpha\alpha} dx_1 dx_2 = \int_\omega x_2 \Psi_{\alpha\alpha} dx_1 dx_2 = 0. \tag{76}$$

Proof. The three-dimensional stress tensor

$$\tilde{\Psi} = \begin{pmatrix} \Psi_{11} & \Psi_{12} & 0 \\ \Psi_{21} & \Psi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

verifies variational problem (74), which can be written in differential form as

$$\begin{aligned} -\mathrm{div}\tilde{\Psi} = -\partial_\alpha \Psi_{\alpha\beta} &= \mathbf{0} \text{ in } \Omega, \\ \tilde{\Psi}\mathbf{n} &= \mathbf{0} \text{ on } \partial\Omega\backslash\Gamma_C, \\ \tilde{\Psi}_\tau &= \mathbf{0} \text{ on } \Gamma_C, \\ \tilde{\Psi}_n &\leq 0 \text{ on } \Gamma_C, \end{aligned}$$

where the boundary conditions must be understood as described in [21]. Consequently,

$$\mathbf{0} = -\int_\Omega \mathrm{div}\tilde{\Psi} dx = \int_{\partial\Omega} \tilde{\Psi}\mathbf{n} d\Gamma = \int_{\Gamma_C} \tilde{\Psi}_n \mathbf{n} d\Gamma. \tag{77}$$

Given that $\tilde{\Psi}_n$ has constant sign on Γ_C and thanks to hypothesis ($\mathbf{H5}^{\text{ffl}}$) concerning to n_2, we deduce that $\tilde{\Psi}_n = 0$ on Γ_C. Therefore, components $\Psi_{\alpha\beta}$ are a weak solution of variational problem (75).

Choosing $(v_1, v_2) = (x_1 v^0, x_2 v^0)$, with $v^0 \in H^1_e(-L, L)$ as a test function in (75), we have that

$$\int_{-L}^{L} v^0 \int_\omega \Psi_{\alpha\alpha} dx_1 dx_2 dx_3 = 0, \forall v^0 \in H^1_e(-L, L),$$

so, since $\Psi_{\alpha\alpha}$ is also even with respect to x_3, we conclude

$$\int_\omega \Psi_{\alpha\alpha} dx_1 dx_2 = 0.$$

Analogously, choosing as test functions in (75) $(v_1, v_2) = (0, \frac{1}{2}x_1^2 v^0)$ and $(v_1, v_2) = (x_1 x_2 v^0, \frac{1}{2}x_2^2 v^0)$, with $v^0 \in H^1_e(-L, L)$, we obtain

$$\begin{gathered} \int_{-L}^{L} v^0 \int_\omega x_1 \Psi_{12} dx_1 dx_2 dx_3 = 0, \\ \int_{-L}^{L} v^0 \int_\omega (x_2 \Psi_{\alpha\alpha} dx_1 dx_2 + x_1 \Psi_{12} dx_1 dx_2) dx_3 = 0, \end{gathered}$$

for all $v^0 \in H^1_e(-L, L)$; then,

$$\int_\omega x_2 \Psi_{\alpha\alpha} dx_1 dx_2 = 0.$$

So, components $\Psi_{\alpha\alpha}$ verify expression (76). □

Corollary 3. *There exists a subsequence of $(\sigma(\epsilon))_{\epsilon>0}$ such that*

$$\sigma_{33}(\epsilon) \rightharpoonup \Psi_{33}, \quad \epsilon\sigma_{3\beta}(\epsilon) \rightharpoonup \Psi_{3\beta}, \quad \epsilon^2 \sigma_{\alpha\beta}(\epsilon) \rightharpoonup \Psi_{\alpha\beta},$$

in $L^2(\Omega)$ and there exist a subsequence of $(\mathbf{u}(\epsilon))_{\epsilon>0}$ and $\varsigma \in \mathbf{X}(\Omega)$ such that

$$\varepsilon_{33}(\mathbf{u}(\epsilon)) \rightharpoonup \varsigma_{33}, \quad \epsilon^{-1}\varepsilon_{3\beta}(\mathbf{u}(\epsilon)) \rightharpoonup \varsigma_{3\beta}, \quad \epsilon^{-2}\varepsilon_{\alpha\beta}(\mathbf{u}(\epsilon)) \rightharpoonup \varsigma_{\alpha\beta},$$

in $L^2(\Omega)$. Moreover,

$$\begin{aligned} \Psi_{ij} &= \frac{E}{1+\nu}\varsigma_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varsigma_{pp}\delta_{ij} = \Lambda_{ijkl}\varsigma_{kl}, \\ \varsigma_{ij} &= \frac{1+\nu}{E}\Psi_{ij} - \frac{\nu}{E}\Psi_{pp}\delta_{ij} = \Lambda^{-1}_{ijkl}\Psi_{kl}. \end{aligned}$$

Proof. The result is a direct consequence of (69), (70) and (72). □

Theorem 9. *The sequence* $(\mathbf{u}(\epsilon))_{\epsilon>0}$ *verifies*

$$\mathbf{u}(\epsilon) \rightharpoonup \mathbf{u}^0 \ \text{ in } \mathbf{H}^1(\Omega), \tag{78}$$

where $\mathbf{u}^0 \in \mathbf{V}_{BN}(\Omega)$ *is the first term in the asymptotic expansion* (30).

Proof. There exists a subsequence, still denoted by $(\mathbf{u}(\epsilon))_{\epsilon>0}$, in $\mathbf{H}^1_s(\Omega)$ that converges weakly in $\mathbf{H}^1(\Omega)$ thanks to Theorem 8; let $\tilde{\mathbf{u}} \in \mathbf{H}^1_s(\Omega)$ be its limit. We will prove that $\tilde{\mathbf{u}} = \mathbf{u}^0$.

Taking limits when ϵ goes to zero in (23) and taking into account expressions (25)-(29), we have that

$$\frac{1}{E}\int_\Omega \Psi_{33}\tau_{33}\,dx - \frac{\nu}{E}\int_\Omega \Psi_{\mu\mu}\tau_{33}\,dx = \int_\Omega \varepsilon_{ij}(\tilde{\mathbf{u}})\tau_{ij}\,dx, \quad \forall \tau \in \mathbf{X}(\Omega),$$

thanks to (72). From this relation we obtain:

$$\begin{aligned} &\varepsilon_{\alpha\beta}(\tilde{\mathbf{u}}) = \varepsilon_{3\beta}(\tilde{\mathbf{u}}) = 0, \\ &\Psi_{33} = E\varepsilon_{33}(\tilde{\mathbf{u}}) + \nu\Psi_{\mu\mu}. \end{aligned} \tag{79}$$

So, a similar methodology to that used in proof of Item (i) in Theorem 4, allow us to deduce

$$\left.\begin{aligned} &\tilde{\mathbf{u}} \in \mathbf{V}_{BN}(\Omega), \\ &\tilde{u}_1 = 0, \\ &\tilde{u}_2 \in H^2_e(-L, L),\ (0, \tilde{u}_2) \in \mathbf{K}_2(\Omega), \\ &\tilde{u}_3 = \underline{\tilde{u}}_3 - x_2\partial_3\tilde{u}_2,\ \underline{\tilde{u}}_3 \in H^1_o(-L, L). \end{aligned}\right\} \tag{80}$$

Consequently, replacing these results on (79), we find that

$$\Psi_{33} = E(\partial_3\underline{\tilde{u}}_3 - x_2\partial_{33}\tilde{u}_2) + \nu\Psi_{\mu\mu}. \tag{81}$$

If we consider $\mathbf{v} \in \mathbf{V}_{BN}(\Omega) \cap \mathbf{K}(\Omega)$ as a test function in (24), we obtain

$$\begin{aligned} &\int_\Omega \sigma_{33}(\epsilon)\varepsilon_{33}(\mathbf{v})\,dx - \int_\Omega \sigma_{ij}(\epsilon)\varepsilon_{ij}(\mathbf{u}(\epsilon))\,dx \geq \\ &\int_\Omega f_i(v_i - u_i(\epsilon))\,dx + \int_{\Gamma_{N_1}} h_i(v_i - u_i(\epsilon))\,d\Gamma,\ \forall \mathbf{v} \in \mathbf{V}_{BN}(\Omega) \cap \mathbf{K}(\Omega). \end{aligned} \tag{82}$$

Moreover, taking into account expression (53) and Corollary 1, we have

$$\int_\Omega \sigma_{ij}(\epsilon)\varepsilon_{ij}(\mathbf{u}(\epsilon))\,dx = \int_\Omega \tilde{\sigma}(\epsilon){:}\tilde{\varepsilon}(\epsilon)\,dx = \int_\Omega \Lambda\tilde{\varepsilon}(\epsilon){:}\tilde{\varepsilon}(\epsilon)\,dx,$$

and, from Corollary 3,

$$\liminf_{\epsilon\to 0}\int_\Omega \sigma_{ij}(\epsilon)\varepsilon_{ij}(\mathbf{u}(\epsilon))\,dx \geq \int_\Omega \Lambda\varsigma{:}\varsigma\,dx = \int_\Omega \Psi{:}\varsigma\,dx = \int_\Omega \Psi_{ij}\varsigma_{ij}\,dx, \tag{83}$$

since Λ is a weakly lower-semicontinuous function (see [12]). So, taking lower limits when ϵ goes to zero in (82), we have for each $\mathbf{v} \in \mathbf{V}_{BN}(\Omega) \cap \mathbf{K}(\Omega)$,

$$\begin{aligned} &\int_\Omega \Psi_{33}\varepsilon_{33}(\mathbf{v})\,dx - \int_\Omega \Psi_{ij}\varsigma_{ij}\,dx \geq \\ &\liminf_{\epsilon\to 0}\int_\Omega \sigma_{33}(\epsilon)\varepsilon_{33}(\mathbf{v})\,dx - \liminf_{\epsilon\to 0}\int_\Omega \sigma_{ij}(\epsilon)\varepsilon_{ij}(\mathbf{u}(\epsilon))\,dx \geq \\ &\liminf_{\epsilon\to 0}\left(\int_\Omega \sigma_{33}(\epsilon)\varepsilon_{33}(\mathbf{v})\,dx - \int_\Omega \sigma_{ij}(\epsilon)\varepsilon_{ij}(\mathbf{u}(\epsilon))\,dx\right) \geq \\ &\liminf_{\epsilon\to 0}\left(\int_\Omega f_i(v_i - u_i(\epsilon))\,dx + \int_{\Gamma_{N_1}} h_i(v_i - u_i(\epsilon))\,d\Gamma\right) = \\ &\int_\Omega f_i(v_i - \tilde{u}_i)\,dx + \int_{\Gamma_{N_1}} h_i(v_i - \tilde{u}_i)\,d\Gamma. \end{aligned}$$

Therefore,

$$\int_\Omega \Psi_{33}\varepsilon_{33}(\mathbf{v}-\tilde{\mathbf{u}})dx \geq \int_\Omega f_i(v_i-\tilde{u}_i)dx + \int_{\Gamma_{N_1}} h_i(v_i-\tilde{u}_i)d\Gamma +$$
$$\int_\Omega \Psi_{3\beta}\varsigma_{3\beta}dx + \int_\Omega \Psi_{\alpha\beta}\varsigma_{\alpha\beta}dx,\ \forall\mathbf{v}\in\mathbf{V}_{BN}(\Omega)\cap\mathbf{K}(\Omega). \tag{84}$$

Moreover, choosing $v = u_3(\epsilon)\in H^1_s(\Omega)$ in expression (73), we find that

$$\int_\Omega \Psi_{3\beta}\partial_\beta u_3(\epsilon)dx = 0.$$

Multiplying by ϵ^{-1} and taking limits when $\epsilon\to 0$, we deduce

$$\int_\Omega \Psi_{3\beta}\varsigma_{3\beta}dx = 0. \tag{85}$$

Furthermore, choosing $(v_1, v_2) = (u_1(\epsilon), u_2(\epsilon))\in\mathbf{W}_{s2}(\Omega)\cap\mathbf{K}_2(\Omega)$ in expression (75), we have that

$$\int_\Omega \Psi_{\alpha\beta}\partial_\alpha u_\beta(\epsilon)dx = 0.$$

Multiplying by ϵ^{-2} and taking limits, we see that

$$\int_\Omega \Psi_{\alpha\beta}\varsigma_{\alpha\beta}dx = 0. \tag{86}$$

Therefore, from (84)-(86) we obtain

$$\int_\Omega \Psi_{33}\varepsilon_{33}(\mathbf{v}-\tilde{\mathbf{u}})dx \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\tilde{\mathbf{u}})dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot(\mathbf{v}-\tilde{\mathbf{u}})d\Gamma,$$
$$\forall\mathbf{v}\in\mathbf{V}_{BN}(\Omega)\cap\mathbf{K}(\Omega).$$

Taking into account the expression for Ψ_{33} obtained in (81), we can write:

$$\int_\Omega [E(\partial_3\underline{\tilde{u}}_3 - x_2\partial_{33}\tilde{u}_2)+\nu\Psi_{\mu\mu}]\varepsilon_{33}(\mathbf{v}-\tilde{\mathbf{u}})dx \geq$$
$$\int_\Omega \mathbf{f}\cdot(\mathbf{v}-\tilde{\mathbf{u}})dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot(\mathbf{v}-\tilde{\mathbf{u}})d\Gamma,\ \forall\mathbf{v}\in\mathbf{V}_{BN}(\Omega)\cap\mathbf{K}(\Omega).$$

Then,

$$\int_\Omega E\partial_3\underline{\tilde{u}}_3\varepsilon_{33}(\mathbf{v}-\tilde{\mathbf{u}})dx - \int_\Omega Ex_2\partial_{33}\tilde{u}_2\varepsilon_{33}(\mathbf{v}-\tilde{\mathbf{u}})dx + \int_\Omega \nu\Psi_{\mu\mu}\varepsilon_{33}(\mathbf{v}-\tilde{\mathbf{u}})dx \geq$$
$$\int_\Omega \mathbf{f}\cdot(\mathbf{v}-\tilde{\mathbf{u}})dx + \int_{\Gamma_{N_1}} \mathbf{h}\cdot(\mathbf{v}-\tilde{\mathbf{u}})d\Gamma,\ \forall\mathbf{v}\in\mathbf{V}_{BN}(\Omega)\cap\mathbf{K}(\Omega). \tag{87}$$

In particular, considering in equation (87) $\mathbf{v}\in\mathbf{V}_{BN}(\Omega)\cap\mathbf{K}(\Omega)$ such that $v_1 = 0$, v_2 depending only on x_3 and even with $(0, v_2)\in\mathbf{K}_2(\Omega)$ and $v_3 = \underline{\tilde{u}}_3 - x_2\partial_3 v_2$, we obtain

$$\int_\Omega Ex_2^2\partial_{33}\tilde{u}_2\partial_{33}(v_2-\tilde{u}_2)dx \geq \int_\Omega f_2(v_2-\tilde{u}_2)dx + \int_{\Gamma_{N_1}} h_2(v_2-\tilde{u}_2)d\Gamma -$$
$$\int_\Omega x_2 f_3\partial_3(v_2-\tilde{u}_2)dx - \int_{\Gamma_{N_1}} x_2 h_3\partial_3(v_2-\tilde{u}_2)d\Gamma + \int_\Omega \nu x_2\Psi_{\mu\mu}\partial_{33}(v_2-\tilde{u}_2)dx,$$

thanks to (80). Taking into account (76), we conclude that

$$\int_{-L}^{L} EI_1\partial_{33}\tilde{u}_2\partial_{33}(v_2-\tilde{u}_2)dx_3 \geq \int_{-L}^{L} q_2(v_2-\tilde{u}_2)dx_3 - \int_{-L}^{L} M_1\partial_3(v_2-\tilde{u}_2)dx_3,$$

for all $v_2\in H^2_e(-L, L)$, $(0, v_2)\in\mathbf{K}_2(\Omega)$; so, $\tilde{u}_2 = u_2^0$ thanks to Theorem 6.

Similarly, choosing $\mathbf{v}\in\mathbf{V}_{BN}(\Omega)\cap\mathbf{K}(\Omega)$ as a test function in (87), such that $v_\alpha = \tilde{u}_\alpha$ and $v_3 = \tilde{u}_3 \pm \underline{v}_3$, with $\underline{v}_3\in H^1_o(-L, L)$, we find that

$$\int_{-L}^{L} EA\partial_3\underline{\tilde{u}}_3\partial_3\underline{v}_3 dx_3 + \int_\Omega \nu\Psi_{\mu\mu}\partial_3\underline{v}_3 dx = \int_{-L}^{L} q_3\underline{v}_3 dx_3,$$

and from (76), $\underline{\tilde{u}}_3 = \underline{u}_3^0$ thanks to Theorem 5, which proves (78) for the subsequence $\mathbf{u}(\epsilon)$. We can repeat this procedure for each subsequence of $(\mathbf{u}(\epsilon))_{\epsilon>0}$ converging weakly in $\mathbf{H}^1_s(\Omega)$; then, the uniqueness of the limit implies that the entire sequence $(\mathbf{u}(\epsilon))_{\epsilon>0}$ converges weakly to $\mathbf{u}^0$. □

Remark 2. *From previous Theorem and Corollary 3,*

$$\varepsilon_{\alpha\beta}(\mathbf{u}(\epsilon)) \rightharpoonup \varepsilon_{\alpha\beta}(\mathbf{u}^0) = 0, \quad \frac{1}{\epsilon^2}\varepsilon_{\alpha\beta}(\mathbf{u}(\epsilon)) \rightharpoonup \varsigma_{\alpha\beta}, \tag{88}$$

$$\varepsilon_{3\beta}(\mathbf{u}(\epsilon)) \rightharpoonup \varepsilon_{3\beta}(\mathbf{u}^0) = 0, \quad \frac{1}{\epsilon}\varepsilon_{3\beta}(\mathbf{u}(\epsilon)) \rightharpoonup \varsigma_{3\beta}, \tag{89}$$

$$\varepsilon_{33}(\mathbf{u}(\epsilon)) \rightharpoonup \varepsilon_{33}(\mathbf{u}^0) = \varsigma_{33}, \tag{90}$$

weakly in $L^2(\Omega)$.

Theorem 10. *For each* $0 < \epsilon \leq 1$*, let* $(\mathbf{u}(\epsilon), \sigma(\epsilon))_{\epsilon>0} \in (\mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega)) \times \mathbf{X}(\Omega)$ *be the solution of problem (23)-(24). Let* $\mathbf{u}^0 \in \mathbf{H}^1_s(\Omega) \cap \mathbf{K}(\Omega)$*,* $\sigma^0_{33} \in L^2(\Omega)$ *be the first term of the asymptotic expansion (30). Then, when* $\epsilon \to 0$*, the following convergences are obtained:*

$$\|\mathbf{u}(\epsilon) - \mathbf{u}^0\|_{1,\Omega} \to 0, \tag{91}$$

$$\|\sigma_{33}(\epsilon) - \sigma^0_{33}\|_{0,\Omega} \to 0, \tag{92}$$

$$\epsilon\|\sigma_{3\beta}\|_{0,\Omega} \to 0, \tag{93}$$

$$\epsilon^2\|\sigma_{\alpha\beta}\|_{0,\Omega} \to 0. \tag{94}$$

Proof. Convergences (92)-(94) are obtained by applying the same methodology given in Proof of Theorem 10.4 in [8].

In order to prove (91), since $\mathbf{u}(\epsilon)$ converges weakly in $\mathbf{H}^1(\Omega)$, there exists a subsequence, still denoted $\mathbf{u}(\epsilon)$, such that converges strongly in $[L^2(\Omega)]^3$,

$$\|\mathbf{u}(\epsilon) - \mathbf{u}^0\|_{0,\Omega} \to 0. \tag{95}$$

From (23) and (31), we obtain the following identity:

$$b(\tau, \mathbf{u}(\epsilon) - \mathbf{u}^0) = a_0(\sigma(\epsilon) - \sigma^0, \tau) + \epsilon^2 a_2(\sigma(\epsilon), \tau) + \epsilon^4 a_4(\sigma(\epsilon), \tau). \tag{96}$$

Moreover, for each $\tau \in \mathbf{X}(\Omega)$

$$\begin{aligned} &|a_0(\sigma(\epsilon) - \sigma^0, \tau) + \epsilon^2 a_2(\sigma(\epsilon), \tau) + \epsilon^4 a_4(\sigma(\epsilon), \tau)| \leq \\ &C_{12}\left[\|\sigma_{33}(\epsilon) - \sigma^0_{33}\|_{0,\Omega} + \epsilon\|\sigma_{3\beta}(\epsilon)\|_{0,\Omega} + \epsilon^2\|\sigma_{\alpha\beta}(\epsilon)\|_{0,\Omega}\right]\|\tau\|_{0,\Omega}, \end{aligned} \tag{97}$$

taking into account that $0 \leq \epsilon < 1$ and then ϵ^2, ϵ^4 can be bounded by ϵ and ϵ^2, respectively, when necessary.

Choosing $\tau = \varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)$ in relations (96) and (97), we have that

$$\begin{aligned} C_{17}\|\varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)\|^2_{0,\Omega} &\leq a_0(\sigma(\epsilon) - \sigma^0, \varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)) + \\ &\epsilon^2 a_2(\sigma(\epsilon), \varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)) + \epsilon^4 a_4(\sigma(\epsilon), \varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)) \leq \\ C_{12}\left(\|\sigma_{33}(\epsilon) - \sigma^0_{33}\|_{0,\Omega} + \epsilon\|\sigma_{3\beta}(\epsilon)\|_{0,\Omega} + \epsilon^2\|\sigma_{\alpha\beta}(\epsilon)\|_{0,\Omega}\right)&\|\varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)\|_{0,\Omega}, \end{aligned}$$

therefore,

$$\|\varepsilon(\mathbf{u}(\epsilon) - \mathbf{u}^0)\|_{0,\Omega} \to 0,$$

and taking into account (95) we conclude (91) for the subsequence $\mathbf{u}(\epsilon)$. This procedure may be applied to any weakly convergent subsequence of $(\mathbf{u}(\epsilon))_{\epsilon>0}$ in $\mathbf{H}^1(\Omega)$; so, the whole sequence converges strongly. □

9. Returning to domain Ω^ϵ

In previous section, we have proved that the first term of the asymptotic expansion $(\mathbf{u}^0, \sigma^0)$ is a good approximation of the scaled displacements and stresses $(\mathbf{u}(\epsilon), \sigma(\epsilon))$ when the cross-section parameter ϵ is small. Then, by applying the inverse transformations of (19)-(21), we obtain an approximation of $(\mathbf{u}^\epsilon, \sigma^\epsilon)$, denoted by $(\mathbf{u}^{0\epsilon}, \sigma^{0\epsilon})$, such that

$$u^{0\epsilon}_\alpha(\mathbf{x}^\epsilon) = \frac{1}{\epsilon}u^0_\alpha(\mathbf{x}),\ u^{0\epsilon}_3(\mathbf{x}^\epsilon) = u^0_3(\mathbf{x}),$$

$$\sigma^{0\epsilon}_{\alpha\beta}(\mathbf{x}^\epsilon) = \epsilon^2\sigma^0_{\alpha\beta}(\mathbf{x}),\ \sigma^{0\epsilon}_{3\beta}(\mathbf{x}^\epsilon) = \epsilon\sigma^0_{3\beta}(\mathbf{x}),\ \sigma^{0\epsilon}_{33}(\mathbf{x}^\epsilon) = \sigma^0_{33}(\mathbf{x}),$$

The following theorem gives a characterization of this approximation:

Theorem 11. *The terms $\mathbf{u}^{0\epsilon}$, $\sigma_{33}^{0\epsilon}$ are determined as follows:*

(i) The displacement field $\mathbf{u}^{0\epsilon}$ belongs to the subspace $\mathbf{V}_{BN}(\Omega^\epsilon)$, where $u_1^{0\epsilon}$ is null, $u_2^{0\epsilon} \in H_e^2(-L, L)$ and

$$u_3^{0\epsilon} = \underline{u}_3^{0\epsilon} - x_2^\epsilon \partial_3 u_2^{0\epsilon},\ \underline{u}_3^{0\epsilon} \in H_o^1(-L, L). \tag{98}$$

(ii) The stress component $\sigma_{33}^{0\epsilon}$ is given by

$$\sigma_{33}^{0\epsilon} = E(\partial_3 \underline{u}_3^{0\epsilon} - x_2^\epsilon \partial_{33} u_2^{0\epsilon}). \tag{99}$$

(iii) The axial displacement $\underline{u}_3^{0\epsilon}$ is the unique solution of the problem:
Problem $(\mathbf{P}_3^{0\text{ffl}})$*:*
Find $\underline{u}_3^{0\epsilon} \in H_o^1(-L, L)$ such that

$$\int_{-L}^{L} EA^\epsilon \partial_3 \underline{u}_3^{0\epsilon}\, \partial_3 v_3^0 dx_3 = \int_{-L}^{L} q_3^\epsilon v_3^0 dx_3,\ \forall v_3^0 \in H_o^1(-L, L), \tag{100}$$

(iv) The displacement $u_2^{0\epsilon}$ is the unique solution of the bending problem:
Problem $(\mathbf{P}_2^{0\text{ffl}})$*:*
Find $u_2^{0\epsilon} \in H_e^2(-L, L)$, $(0, u_2^{0\epsilon}) \in \mathbf{K}_2(\Omega^\epsilon)$, such that

$$\int_{-L}^{L} EI_1^\epsilon \partial_{33} u_2^{0\epsilon} \partial_{33}(v_2^0 - u_2^{0\epsilon}) dx_3 \geq \int_{-L}^{L} q_2^\epsilon (v_2^0 - u_2^{0\epsilon}) dx_3 - \int_{-L}^{L} M_1^\epsilon \partial_3 (v_2^0 - u_2^{0\epsilon}) dx_3, \tag{101}$$

for all $v_2^0 \in H_e^2(-L, L)$, $(0, v_2^0) \in \mathbf{K}_2(\Omega^\epsilon)$.

10. Some properties of the limit models

In the axis of the beam, we distinguish:

- $\Gamma_{N_1}^0 = (-\delta, \delta)$, the part of the axis corresponding to the beam slice in which some surface forces are applied.
- $\Gamma_C^0 = (-l - \hat{\delta}, -l) \cup (l, l + \hat{\delta})$, the parts of the beam axis under sections where their boundaries intersect with Γ_C^ϵ.

From now on and without loss of generality, we consider that the second component of the normal vector is negative, $n_2^\epsilon < 0$, on the contact boundary Γ_C^ϵ.

Theorem 12. *The unique solution $\underline{u}_3^{0\epsilon} \in H_o^1(-L, L)$ of Problem $(\mathbf{P}_3^{0\text{ffl}})$ verifies the following differential formulation:*

$$EA^\epsilon \Delta \underline{u}_3^{0\epsilon} = q_3^\epsilon, \tag{102}$$

$$EA^\epsilon \partial_3 \underline{u}_3^{0\epsilon}(\pm L) = 0. \tag{103}$$

In addition, $\underline{u}_3^{0\epsilon} \in H_o^2(-L, L)$.

Proof. Firstly, choosing an odd function $\phi \in \mathcal{D}(-L, L) \subset H^1(-L, L)$, as test function in (100), and applying integration by parts, we obtain

$$-\int_{-L}^{L} EA^\epsilon \Delta \underline{u}_3^{0\epsilon} \phi dx_3 = \int_{-L}^{L} q_3^\epsilon \phi dx_3.$$

So, expression (102) is obtained in the sense of distributions. Moreover, since q_3^ϵ is smooth,

$$\underline{u}_3^{0\epsilon} \in H_o^2(-L, L).$$

Multiplying (102) by $v \in H_o^1(L, L)$ and applying again Green's theorem, we have

$$\int_{-L}^{L} q_3^\epsilon v dx_3 = \int_{-L}^{L} EA^\epsilon \partial_3 \underline{u}_3^{0\epsilon} \partial_3 v dx_3 - EA^\epsilon \partial_3 \underline{u}_3^{0\epsilon}(L) v(L) + EA^\epsilon \partial_3 \underline{u}_3^{0\epsilon}(-L) v(-L) =$$
$$\int_{-L}^{L} q_3^\epsilon v dx_3 - 2EA^\epsilon \partial_3 \underline{u}_3^{0\epsilon}(L) v(L),$$

taking into account expression (100), so we deduce (103). □

Theorem 13. *Let $u_2^{0\epsilon} \in H_e^2(-L, L)$ be the unique solution of Problem $(\mathbf{P}_2^{0\text{ffl}})$ and let denote the coincidence set $I = \{x_3 \in \bar{\Gamma}_C^0; u_2^{0\epsilon}(x_3) = 0\}$. Then there exists a nonnegative Radon measure μ such that*

$$EI_1^\epsilon \Delta^2 u_2^{0\epsilon} = q_2^\epsilon + \partial_3 M_1^\epsilon + \mu \text{ in } (-L, L), \tag{104}$$

$$\partial_3 \Delta u_2^{0\epsilon}(\pm L) = \Delta u_2^{0\epsilon}(\pm L) = 0, \tag{105}$$

with $\text{supp}(\mu) \subset I$.

Proof. Firstly, let $\phi \in \mathcal{D}(\mathcal{J}) \subset H^2(-L,L)$ be an even function, with $\mathcal{J} = (-L,L)\backslash\bar{\Gamma}_C^0$; taking $v_2^0 = u_2^{0\epsilon} \pm \phi$, as test function in (101), and applying integration by parts, we find that

$$\int_{\mathcal{J}} EI_1^\epsilon \Delta u_2^{0\epsilon} \Delta\phi dx_3 = \int_{\mathcal{J}} q_2^\epsilon \phi dx_3 + \int_{\mathcal{J}} \partial_3 M_1^\epsilon \phi dx_3.$$

So, we obtain in the sense of distributions

$$EI_1^\epsilon \Delta^2 u_2^{0\epsilon} = q_2^\epsilon + \partial_3 M_1^\epsilon \text{ in } \mathcal{D}'(\mathcal{J}). \tag{106}$$

Therefore, multiplying (106) by a test function v, taking into account that $H_e^2(L,L) \subset \mathcal{C}^1(-L,L)$ and since $q_2^\epsilon + \partial_3 M_1 \in L^2(-L,L)$, applying again integration by parts, we deduce that

$$\begin{aligned}\int_{\mathcal{J}} EI_1^\epsilon \Delta^2 u_2^{0\epsilon} v dx_3 &= \int_{\mathcal{J}} EI_1^\epsilon \Delta u_2^{0\epsilon} \Delta v dx_3 - EI_1^\epsilon \partial_3 \Delta u_2^{0\epsilon}(-L) v(-L) + \\ EI_1^\epsilon \partial_3 \Delta u_2^{0\epsilon}(L) v(L) &+ EI_1^\epsilon \Delta u_2^{0\epsilon}(-L) \partial_3 v(-L) - EI_1^\epsilon \Delta u_2^{0\epsilon}(L) \partial_3 v(L) = \\ &= \int_{\mathcal{J}} q_2^\epsilon v dx_3 + \int_{\mathcal{J}} \partial_3 M_1^\epsilon v dx_3,\end{aligned}$$

for all $v \in H_e^2(-L,L)$ such that $v = 0$ in $\bar{\Gamma}_C^0$ and hence, using variational formulation (101), we deduce (105).

Next, let $x_3^0 \in \Gamma_C^0 \backslash I$, $x_3^0 > 0$, be arbitrary and $B_r(\pm x_3^0) = B_r(x_3^0) \cup B_r(-x_3^0)$. Because $(0, u_2^{0\epsilon}) \in \mathbf{K}_2(\Omega^\epsilon)$, there exist $r > 0$ and $\phi \in \mathcal{D}(B_r(\pm x_3^0))$ such that

$$\phi(x) > 0 \text{ in } B_r(\pm x_3^0),$$
$$u_2^{0\epsilon} \geq \phi \text{ in } H^2(B_r(\pm x_3^0)).$$

Then, for all $\xi \in \mathcal{D}(B_{r/2}(\pm x_3^0))$, there exists $\nu > 0$ such that

$$u_2^{0\epsilon} + \nu\xi \geq \frac{\phi}{2} \text{ in } H^2(B_{r/2}(\pm x_3^0)).$$

Choosing $v_2^0 = u_2^{0\epsilon} + \nu\xi$ in (101) and dividing by ν, we have that

$$\int_{B_{r/2}(\pm x_3^0)} EI_1^\epsilon \Delta u_2^{0\epsilon} \Delta\xi dx_3 \geq \int_{B_{r/2}(\pm x_3^0)} q_2^\epsilon \xi dx_3 + \int_{B_{r/2}(\pm x_3^0)} M_1^\epsilon \partial_3 \xi dx_3.$$

Analogously, applying the same procedure to $-\xi$, we obtain the equality

$$\int_{B_{r/2}(\pm x_3^0)} EI_1^\epsilon \Delta u_2^{0\epsilon} \Delta\xi dx_3 = \int_{B_{r/2}(\pm x_3^0)} q_2^\epsilon \xi dx_3 + \int_{B_{r/2}(\pm x_3^0)} M_1^\epsilon \partial_3 \xi dx_3,$$

for all $\xi \in \mathcal{D}(B_{r/2}(\pm x_3^0))$, and then,

$$EI_1^\epsilon \Delta^2 u_2^{0\epsilon} = q_2^\epsilon + \partial_3 M_1^\epsilon \text{ in } \mathcal{D}'(\Gamma_C^0 \backslash I). \tag{107}$$

Now, we notice that from (101),

$$\int_{-L}^{L} EI_1^\epsilon \Delta u_2^{0\epsilon} \Delta v dx_3 \geq \int_{-L}^{L} q_2^\epsilon v dx_3 - \int_{-L}^{L} M_1^\epsilon \partial_3 v dx_3,$$

for all $v \in H_e^2(-L,L)$ with $v \geq 0$ in Γ_C^0; in particular,

$$a(u_2^{0\epsilon}, v) - (f_0^\epsilon, v) \geq 0, \forall v \in H_e^2(-L,L) \cap H_0^1(-L,L),\ v \geq 0 \text{ in } \Gamma_C^0,$$

where a is the bilinear form

$$a(u,v) = \int_{-L}^{L} EI_1^\epsilon \Delta u \Delta v dx_3,\ \forall v \in H_e^2(-L,L),$$

and $f_0^\epsilon \in [H_e^2(-L,L)]'$ is given by

$$(f_0^\epsilon, v) = \int_{-L}^{L} q_2^\epsilon v dx_3 + \int_{-L}^{L} \partial_3 M_1^\epsilon v dx_3,\ \forall v \in H_e^2(-L,L) \cap H_0^1(-L,L).$$

Therefore, by Riesz-Schwartz theorem (see [22]), there exists a positive Radon measure μ in $(-L,L)$ such that

$$a(u_2^{0\epsilon}, v_2^0) - (f_0^\epsilon, v_2^0) = \int_{-L}^{L} v_2^0 d\mu, \forall v_2^0 \in H_e^2(-L,L) \cap H_0^1(-L,L),\ v_2^0 \geq 0 \text{ in } \Gamma_C^0.$$

Finally, from (106) and (107), $supp(\mu) \subset I$ and we deduce (104). $\square$

Remark 3. *The problem defined by equations (104) and (105) constitutes a generalization of the Euler-Bernouilli theory for the case of beams in contact with a rigid foundation, in which becomes into a singularity in the resultant of the applied loads. Other generalizations of Bernouilli-Euler models with singularities are shown in [23] and [8].*

Proposition 1. *The limit vertical displacement satisfies the following regularity properties:*

$$u_2^{0\epsilon} \in \mathcal{C}^2(-L, L) \cap \mathcal{C}^3((-L, L)\backslash I).$$

Moreover, the lateral limits $\partial_3 \Delta u_2^{0\epsilon}(x_3^{\pm})$ exist for all $x_3 \in I$ and

$$\partial_3 \Delta u_2^{0\epsilon}(x_3^+) \geq \partial_3 \Delta u_2^{0\epsilon}(x_3^-), \forall x_3 \in I. \tag{108}$$

Proof. From previous Theorem it is clear that $\Delta^2 u_2^{0\epsilon} \in L^2((-L, L)\backslash I)$. Since I is a closed interval, $u_2^{0\epsilon} \in H^4((-L, L)\backslash I)$, and then, $u_2^{0\epsilon} \in \mathcal{C}^3((-L, L)\backslash I)$.

Moreover, from (104) and given that μ is a positive Radon measure, and hence it can be considered as the derivative of an increasing function φ (see [22]), we can write

$$E I_1^\epsilon \partial_3 \Delta u_2^{0\epsilon} = G^\epsilon + \varphi \text{ in } (-L, L), \tag{109}$$

thanks to (105), where G^ϵ is the continuous function in $(-L, L)$ given by

$$G^\epsilon(x_3) = \int_{-L}^{x_3} (q_2^\epsilon(s) + \partial_3 M_1^\epsilon(s)) ds, \tag{110}$$

and φ is defined by:

$$\varphi(x_3) = \int_{-L}^{x_3} d\mu. \tag{111}$$

By integrating (109) and thanks again to (105), we obtain

$$E I_1^\epsilon \Delta u_2^{0\epsilon}(x_3) = \int_{-L}^{x_3} (G^\epsilon + \varphi) ds, \ x_3 \in (-L, L), \tag{112}$$

where the right-side term is a continuous function in $(-L, L)$; therefore, $\Delta u_2^{0\epsilon}$ is continuous in $(-L, L)$ and, consequently, $u_2^{0\epsilon} \in \mathcal{C}^2(-L, L)$.

Finally, we know that φ defined by (111) is a function of bounded variation in all closed interval; hence the discontinuities of φ are only jump discontinuities (see [24]) and hence, from (109), the lateral limits $\partial_3 \Delta u_2^{0\epsilon}(x_3^{\pm})$ exist and verify (108) thanks to φ is growing. ☐

Proposition 2. *i) If the axial forces F_3 and H_3 are null, the axial stress component $\sigma_{33}^{0\epsilon}$ is null on* int(I).

ii) If $I \subset (-L, L)$, the measure μ verifies:

$$\mu(I) = -\int_{-L}^{L} q_2^\epsilon dx_3, \tag{113}$$

$$\mu = -q_2^\epsilon - \partial_3 M_1^\epsilon \ in \ \mathcal{D}'(\text{int}(I)). \tag{114}$$

iii) The jump discontinuities of $\partial_3 \Delta u_2^{0\epsilon}$ lie in ∂I.

Proof. i) It is easy to prove that $\underline{u}_3^{0\epsilon} = 0$.

Then, let $x_3 \in \text{int}(I)$; there exists $\alpha > 0$ such that

$$u_2^{0\epsilon}(y) = 0, \ \forall y \in B_\alpha(x_3) \subset \text{int}(I),$$

which implies that

$$u_2^{0\epsilon}(y) = \partial_3 u_2^{0\epsilon}(y) = \partial_{33} u_2^{0\epsilon}(y) = \partial_{333} u_2^{0\epsilon}(y) = 0, \ \forall y \in B_\alpha(x_3). \tag{115}$$

So, thanks to (99), $\sigma_{33}^{0\epsilon}$ is null in $B_\alpha(x_3)$, and therefore, $\sigma_{33}^{0\epsilon}$ is null in int(I).

ii) Multiplying expression (109) by $\partial_3 v$, with $v \in H_e^2(-L, L)$, such that $v \geq 0$ in Γ_C^0 and integrating in $(-L, L)$, we find that:

$$\int_{-L}^{L} E I_1^\epsilon \partial_3 \Delta u_2^{0\epsilon} \partial_3 v dx_3 = \int_{-L}^{L} G^\epsilon \partial_3 v dx_3 + \int_{-L}^{L} \varphi \partial_3 v dx_3.$$

By applying Green's identity and the even character of v, we can write

$$-\int_{-L}^{L} E I_1^\epsilon \Delta u_2^{0\epsilon} \Delta v dx_3 = -\int_{-L}^{L} q_2^\epsilon v dx_3 + \left(\int_{-L}^{L} q_2^\epsilon dx_3\right) v(L) + \int_{-L}^{L} M_1^\epsilon \partial_3 v dx_3 + \int_{-L}^{L} \varphi \partial_3 v dx_3. \tag{116}$$

Besides, from the variational formulation (101) we obtain that

$$\int_{-L}^{L} E I_1^{\epsilon} \Delta u_2^{0\epsilon} \Delta v \, dx_3 \geq \int_{-L}^{L} q_2^{\epsilon} v \, dx_3 - \int_{-L}^{L} M_1^{\epsilon} \partial_3 v \, dx_3, \tag{117}$$

By term-by-term addition of (116) and (117), we have that

$$0 \geq \left(\int_{-L}^{L} q_2^{\epsilon} dx_3 \right) v(L) + \int_{-L}^{L} \varphi \partial_3 v \, dx_3, \; \forall v \in H_e^2(-L, L); \, v \geq 0 \text{ in } \Gamma_C^0. \tag{118}$$

Let us choose v such that $\operatorname{supp}(v) \cap \operatorname{supp}(\mu) = \emptyset$ and let us consider $\bar{v} = v - v(L) \in H_e^2(-L, L) \cap H_0^1(-L, L)$; by using the definition of derivative in the sense of distributions,

$$\int_{-L}^{L} \varphi \partial_3 v \, dx_3 = \int_{-L}^{L} \varphi \partial_3 \bar{v} \, dx_3 = -\int_{-L}^{L} \bar{v} \, d\mu =$$
$$- \int_{-L}^{L} v \, d\mu + v(L)\mu(I) = v(L)\mu(I). \tag{119}$$

So, substituting (119) in expression (118), we deduce that

$$v(L) \left(\int_{-L}^{L} q_2^{\epsilon} dx_3 + \mu(I) \right) \leq 0, \forall v \in H_e^2(-L, L); \, v \geq 0 \text{ in } \Gamma_C^0.$$

Then, we conclude (113) considering test functions v such that $v(L) = \pm 1$.
Finally, if $v \in \mathcal{D}(\operatorname{int}(I))$ is an even function, from (116) we obtain that

$$0 = \int_I E I_1^{\epsilon} \Delta u_2^{0\epsilon} \Delta v \, dx_3 = \int_I q_2^{\epsilon} v \, dx_3 + \int_I \partial_3 M_1^{\epsilon} v \, dx_3 + \int_I v \, d\mu,$$

and so (114).

iii) From the demonstration of Proposition 1, we know that $\partial_3 \Delta u_2^{0\epsilon}$ has only jump discontinuities that are in the coincidence set I. Due to (115), $\partial_3 \Delta u_2^{0\epsilon}$ is continuous in $\operatorname{int}(I)$ (identically zero) and, therefore, the discontinuities must appear on ∂I. □

Proposition 3. *Let $u_2^{0\epsilon}$ be a smooth solution of the differential problem (104)-(105). If $u_2^{0\epsilon} \geq 0$ in Γ_C^0, then $u_2^{0\epsilon}$ is the solution of Problem ($\mathbf{P}_2^{0\mathrm{ffl}}$).*

Proof. Following the same reasoning that in Proposition 2, we can write:

$$\int_{-L}^{L} E I_1^{\epsilon} \partial_3 \Delta u_2^{0\epsilon} \partial_3 v \, dx_3 = -\int_{-L}^{L} q_2^{\epsilon} v \, dx_3 + v(L) \int_{-L}^{L} q_2^{\epsilon} dx_3 + \int_{-L}^{L} M_1^{\epsilon} \partial_3 v \, dx_3 -$$
$$- \int_{-L}^{L} v \, d\mu - v(L) \int_{-L}^{L} q_2^{\epsilon} dx_3 =$$
$$- \int_{-L}^{L} q_2^{\epsilon} v \, dx_3 + \int_{-L}^{L} M_1^{\epsilon} \partial_3 v \, dx_3 - \int_{-L}^{L} v \, d\mu,$$

for all $v \in H_e^2(-L, L)$ such that $v \geq 0$ in Γ_C^0.

Besides, by applying integration by parts to the first term in the previous expression and taking into account (105), we have that

$$\int_{-L}^{L} E I_1^{\epsilon} \Delta u_2^{0\epsilon} \Delta v \, dx_3 = \int_{-L}^{L} q_2^{\epsilon} v \, dx_3 - \int_{-L}^{L} M_1^{\epsilon} \partial_3 v \, dx_3 + \int_{-L}^{L} v \, d\mu \geq$$
$$\int_{-L}^{L} q_2^{\epsilon} v \, dx_3 - \int_{-L}^{L} M_1^{\epsilon} \partial_3 v \, dx_3,$$

since μ is a nonnegative measure and, therefore, the theorem is proved. □

11. Conclusions

In this work we have carried out a complete mathematical study for the three-point bending test. We have proved the existence of a unique solution for the associated three-dimensional elastic model with contact constraints when certain assumptions of compatibility, concerning to applied forces and symmetry properties, are satisfied.

Moreover, we have studied the asymptotic behaviour of the beam, when it is subjected to this test, if the area of its cross-section goes to zero. Using the asymptotic expansion method, we have obtained, from the three-dimensional model of elasticity with a Signorini condition and without displacements prescribed, one-dimensional models associated both with the bending problem and that of axial traction as well as the corresponding results of existence and uniqueness of solution. In particular, we have obtained the axial normal stresses associated to the MOR; its expression is given in terms of the derivatives of the vertical and axial displacements.

The variational formulation corresponding to the limit problem of bending involves an inequality due to the contact condition between the beam and the two lower cylinders. Furthermore, the admissible test functions for the axial and bending problems are even and odd, respectively, which is a consequence of the assumptions needed to obtain the existence of a unique solution for the three-dimensional problem of departure. Finally, we have obtained the differential formulations of the problems associated with the one-dimensional models and we have presented a characterization of the zone of effective contact.

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