

# Modelling and Mathematical Analysis in Thermomechanics

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### **MODELLING AND MATHEMATICAL ANALYSIS IN THERMOMECHANICS**

fue realizada bajo su dirección por Doña María Cristina Naya Riveiro, estimando que la interesada se encuentra en condiciones de optar al grado de Doctor en Ciencias Matemáticas, por lo que solicitan que sea admitida a trámite para su lectura y defensa pública.

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# Preface

Reducing the risk of building collapse during fires, as well as an efficient use of time and cost reduction in material processing has become a priority in civil engineering. Hence, the study of building materials response to high temperatures is a key problem and any contribution on this topic is essential. In addition, the growing capabilities of numerical tools allow the simulation of most coupled thermomechanical problems which model many of the real phenomena in this setting. Consequently, an in-depth knowledge and understanding of modelling and mathematical analysis in thermomechanics is necessary.

Under these premises, the aim of this dissertation thesis is the study of certain nonlinear coupled thermomechanical problems in solid mechanics, arising from real processes subjected to a strong raise in temperature, such as building fires or material processing. Pursuing this goal, this manuscript is divided in three parts with the common topic of modelling and mathematical analysis in thermomechanics. In the first part, the equations of a coupled thermomechanical model for thermoviscoelastic materials with long memory and strongly dependent temperature stresses are derived. These equations can be used to model, for instance, the solidification process during an aluminium casting. The second and third parts follow a similar structure, devoted to the study of existence, uniqueness and regularity of solution in two different submodels. For the first problem, the mechanical submodel with mixed displacement–traction boundary conditions and temperature dependent coefficients is analyzed, assuming that the temperature is known. Mechanical deformations suffered by an alloy structure exposed to fire can be modelled under this scenario. A fully coupled thermoelastic problem with mixed displacement–traction boundary conditions in the mechanical submodel is the second case under study, considering also mixed boundary conditions including a Robin type one for the thermal submodel. This is the adequate setting for studying thermoelastic deformations of a structure exposed to fire.

Specifically, the outline of this dissertation thesis is the following:

## *Part I. Modelling of materials with long memory*

The first part of this thesis, which comprises a single chapter, is devoted to the modelling of thermoviscoelastic materials with long memory, obtaining the corresponding coupled thermomechanical problem. For this purpose, the Eulerian conservation equations are introduced, but not only in terms of the classical thermodynamical variables such as the deformation gradient and the temperature, but also of an internal variable which gives the viscoplastic history of the material. This internal variable is a symmetric second order tensor whose temporal evolution is governed

by an ordinary differential equation. From the Lagrangian formulation of these equations, three linearizations of the equilibrium laws are obtained, assuming small displacements, small temperature variations and small perturbation of the tensorial internal variable, respectively. Finally, Maxwell–Norton materials are presented as an example of this type of materials.

*Part II. Mathematical analysis of a viscoelastic problem with temperature-dependent coefficients*

The second part is divided into two chapters. In Chapter 2, the quasistatic evolution of a thermo-viscoelastic problem with mixed boundary conditions is studied, assuming that the temperature is known. More precisely, for the mechanical problem with Maxwell–Norton type behaviour law and temperature dependent coefficients, the existence and uniqueness of solution is established. Under this model, the deformation rate tensor gathers the elastic, viscoplastic and thermal contributions, namely: a Hooke’s law with temperature dependent coefficients for the elastic term, a Norton–Hoff law for the viscoplastic part and a generalized Arrhenius’s laws for the thermal effects. The main contribution of this chapter lies in the consideration of a complete temperature dependence of the behaviour law which makes the problem fairly difficult. After introducing the appropriate functional framework, the problem is transformed into a homogeneous one, which is discretized in time. Existence and uniqueness of solution of this discretized problem is obtained by using standard theorems on variational inequalities. Some a priori estimates are used in a limit procedure to prove the existence and uniqueness of solution of the continuous problem.

In order to complete this mathematical analysis, local regularity properties of the stress solution are given in Chapter 3. These properties are obtained by means of classical techniques in regularity analysis assuming additional hypotheses on the data. Precisely,  $H^1_{Loc}$  regularity of the stresses is proved, and using a Galgiardo–Nirenberg inequality,  $H^2_{Loc}$  is also achieved.

*Part III. Mathematical analysis of a thermoelastic problem*

This third part is also organized in two chapters. A mathematical analysis of a quasistatic coupled thermoelastic problem is carried out in Chapter 4. Mixed displacement–traction boundary conditions are imposed on the mechanical submodel, whereas mixed boundary conditions, including a Robin boundary condition, are considered for the thermal submodel. The reference temperature, the thermal conductivity and the Lamé’s parameters are assumed to depend on the material point. These considerations allow for the extension of some previous works.

With the aim of studying the existence and uniqueness of solution, an appropriate functional framework is introduced. First, the Galerkin method is applied to derive a sequence approximating the problem for which the existence and uniqueness of solution is shown. Then, based on some a priori estimates, and using a limit procedure, the Galerkin sequence converges and so, the existence of solution for the original problem is obtained. In addition, uniqueness of solution is proved via Gronwall’s lemma.

The mathematical analysis of the thermoelastic problem is concluded in Chapter 5 with the study of the regularity properties of the solution in space and time. Specifically, spatial  $H^2_{Loc}$  regularity of displacements and temperature is obtained, assuming additional hypotheses of reg-

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ularity on some data. The analysis is completed with two novel results concerning the regularity of the solution with respect to time. In the first case, regularity in time is obtained by increasing the smooth properties of the data and the solution at the initial time. In the second result, the regularity analysis is restricted to the corresponding Dirichlet homogeneous problem, by assuming some extra spatial smoothness on the data. In both cases,  $W^{r,\infty}$  ( $r \in \{0\} \cup \mathbb{N}$ ) regularity in time is achieved.

Finally, the last part of the manuscript collects the most relevant conclusions on the different thermomechanical problems analyzed along this dissertation, jointly with a summary of this manuscript in Spanish.



## Part I

# Modelling of materials with long memory.



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# Chapter 1

## Modelling of materials with long memory

### 1.1 Introduction

In this chapter, we deduce different linearized Lagrangian formulations for the motion and energy conservation equations from the conservation principles of continuum thermomechanics for materials with long memory. These results are published in Naya-Riveiro and Quintela [89].

The methodology is a generalization of that used by Carlson [26] for elastic materials and by Bermúdez [19] for Coleman-Noll materials. To define the constitutive laws for thermoviscoelastic materials with long memory we consider not only the classical state variables -deformation gradient and temperature- but also a new specific tensorial internal variable to take into account the memory effect in the mechanical stresses.

The theory of nonlinear materials with specific vectorial internal variables is presented by Coleman and Gurtin in [30], where it is applied to fluids. Also, Simo and Miehe [95] use this methodology to a model of associative coupled thermoplasticity and Tzavaras [97] to a semilinear model problem of stress relaxation. In this paper, we consider this theory to model the thermoviscoelastic materials with long memory and to obtain the associated equilibrium equations; in particular, the energy equation we deduce is analogous to the one given in Tzavaras [97].

The Maxwell-Norton materials with mechanical coefficients strongly dependent on the temperature are a particular example of thermoviscoelastic materials with long memory. These materials are used in the casting processes where there are strong temperature gradients; for their numerical simulation it is very important to include the mechanical dissipation terms in the energy equation and the temperature dependence of all thermomechanical coefficients. That is why we propose different linearizations for the conservation laws associated with thermoviscoelastic materials with long memory and particularly with Maxwell-Norton materials: a first linearization to consider small deformations; a second linearization on the terms of the thermal response mapping which depend linearly on deformation gradient with respect to temperature and a third one to simplify the nonlinearities with respect to tensorial internal variable.

This chapter is organized as follows. Firstly, we will introduce the notation used and we will review some concepts and conservation principles of continuum thermomechanics. In Section 1.3 we will define the thermoviscoelastic materials with long memory, we will deduce the restrictions on their constitutive laws in order to satisfy the second principle of thermodynamics and we will rewrite the associated equilibrium equations in Eulerian coordinates for these materials. Then, in Section 1.4 we will rewrite these equations in the reference configuration; in addition, assuming small displacements, small variations of temperature and small perturbations of the tensorial internal variable, we will present three linearizations of the equilibrium laws. In Section 1.5 we will choose the response functions associated with Maxwell-Norton materials whose mechanical coefficients depend on the temperature and we will rewrite the linearized models for these materials.

## 1.2 Notations and conservation laws

In this section we introduce the notation used along this chapter as in the books of Gurtin [65] or Bermúdez [19].

### 1.2.1 Motion of a body

Let  $\mathcal{E}$  be an affine Euclidean space on a vector space  $\mathcal{V}$ . A body  $\mathcal{B}$  is a regular region of  $\mathcal{E}$ .

**Definition 1.2.1.** *A deformation of  $\mathcal{B}$  is a smooth one-to-one mapping  $f$  which maps  $\mathcal{B}$  onto a closed region in the affine Euclidean space  $\mathcal{E}$ , verifying  $\det \nabla f > 0$ .*

**Definition 1.2.2.** *A motion of  $\mathcal{B}$  is a class  $\mathcal{C}^3$  mapping*

$$X : \mathcal{B} \times \mathbb{R} \longrightarrow \mathcal{E},$$

*$X(\cdot, t)$  being a deformation of  $\mathcal{B}$  for each fixed  $t$ .*

We denote by  $x = X(p, t)$  the place occupied by the material point  $p$  at time  $t$ . We refer to  $\mathcal{B}_t = X(\mathcal{B}, t)$  as the deformed configuration at time  $t$  and we assume that  $\mathcal{B} = \mathcal{B}_0$  is the reference configuration.

**Definition 1.2.3.** *The set  $\mathcal{T} = \{(x, t) : x \in \mathcal{B}_t, t \in \mathbb{R}\}$  is the trajectory of the motion  $X$ .*

Let  $P(\cdot, t) : \mathcal{B}_t \longrightarrow \mathcal{B}$  be the inverse mapping of  $X(\cdot, t)$ , verifying  $X(P(x, t), t) = x$  and  $P(X(p, t), t) = p$ . Then, this mapping is called the reference map of the motion and it indicates the material point  $p$  which occupies the place  $x$  at time  $t$ .

Fields defined in  $\mathcal{T}$  are called spatial fields, while those defined in  $\mathcal{B} \times \mathbb{R}$  are called material fields.

The mappings  $X$  and  $P$  permit to express a spatial field as material field and vice versa:

- the spatial description  $\Phi_s$  of a material field  $\Phi(p, t)$  is

$$\Phi_s(x, t) = \Phi(P(x, t), t),$$

- the material description  $\psi_m$  of a spatial field  $\psi(x, t)$  is

$$\psi_m(p, t) = \psi(X(p, t), t).$$

In fact,

$$(\Phi_s)_m = \Phi, \quad (\psi_m)_s = \psi.$$

**Lemma 1.2.4.** *The reference map  $P$  is of class  $\mathcal{C}^3$ . In consequence, a material field is of class  $\mathcal{C}^N$ ,  $N \leq 3$  if and only if its spatial description is of class  $\mathcal{C}^N$ .*

*Proof.* See Gurtin [65], p. 65. □

Given a material field  $\Phi$ , its material derivative

$$\dot{\Phi}(p, t) = \frac{\partial \Phi}{\partial t}(p, t),$$

is the derivative with respect to time  $t$  holding the material point  $p$  fixed; its material gradient

$$\nabla \Phi(p, t) = \nabla_p \Phi(p, t),$$

is the gradient with respect to  $p$  holding  $t$  fixed.

In particular, we denote by  $\mathbf{F}(p, t) = \nabla X(p, t)$  the material gradient of motion  $X$ . Since,  $X(\cdot, t)$  is a deformation, for each  $t$ , it must verify  $\det \mathbf{F}(p, t) > 0$  for all point  $p \in \mathcal{B}$  and for all time  $t$ . Thus,  $\mathbf{F}$  is called the deformation gradient.

The vector  $\mathbf{u}(p, t) = X(p, t) - p$  represents the displacement of point  $p$  at time  $t$ , and so, it satisfies  $\mathbf{F}(p, t) = (\mathbf{I} + \nabla \mathbf{u})(p, t)$ , where  $\mathbf{I}$  is the identity tensor.

The material derivative with respect to time of  $\mathbf{u}(p, t)$  is denoted by  $\mathbf{v}_m(p, t)$ , this is

$$\mathbf{v}_m(p, t) = \dot{\mathbf{u}}(p, t),$$

and it represents the material velocity of body; its spatial description is  $\mathbf{v}(x, t) = \mathbf{v}_m(P(x, t), t)$ .

Given a spatial field  $\psi$ ,

$$\psi'(x, t) = \frac{\partial \psi}{\partial t}(x, t),$$

denotes the derivative with respect to time  $t$  holding the spatial point  $x$  fixed, and

$$\text{grad } \psi(x, t) = \nabla_x \psi(x, t),$$

represents the gradient with respect to  $x$  holding  $t$  fixed.

In consequence, the material derivative with respect to time of a spatial field  $\psi$ , is

$$\dot{\psi} = ((\psi_m)_s)_s \quad \text{this is, } \dot{\psi}(x, t) = \frac{\partial}{\partial t} \psi(X(p, t), t)|_{p = P(x, t)}.$$

Summing up:

Spatial description $\Phi_s$ of a material field $\Phi(p, t)$	Material description $\psi_m$ of a spatial field $\psi(x, t)$
$\Phi_s(x, t) = \Phi(P(x, t), t)$	$\psi_m(p, t) = \psi(X(p, t), t)$
$(\Phi_s)_m = \Phi, \quad (\psi_m)_s = \psi$	
$\dot{\psi} = ((\psi_m)')_s$	

**Proposition 1.2.5.** *Let  $\varphi$  and  $\mathbf{w}$  be smooth spatial fields with  $\varphi$  a scalar field and  $\mathbf{w}$  a vectorial field. Thus,*

$$\begin{aligned}\dot{\varphi} &= \varphi' + \mathbf{v} \cdot \text{grad } \varphi, \\ \dot{\mathbf{w}} &= \mathbf{w}' + (\text{grad } \mathbf{w})\mathbf{v},\end{aligned}$$

where "." defines the contraction product of 1 subscripts or the scalar product of two vectors.

*Proof.* See Gurtin [65], p. 62. □

The following table summarizes the notation used to different differential operators.

	Material field $\Phi(p, t)$	Spatial field $\psi(x, t)$
Domain	$\mathcal{B} \times \mathbb{R}$	$\mathcal{T}$
Arguments	Material point $p$ and time $t$	Spatial point $x$ and time $t$
Gradient with respect to 1 <sup>st</sup> argument (space)	$\nabla \Phi$	$\text{grad } \psi$
Derivative with respect to 2 <sup>nd</sup> argument (time)	$\dot{\Phi}$	$\psi'$
Divergence	$\text{Div } \Phi$	$\text{div } \psi$

### 1.2.2 Conservation of mass

We assume the existence of a mass distribution defined in the reference configuration by a density  $\rho_0 : \mathcal{B} \rightarrow \mathbb{R}^+$ . The mass conservation law implies that the density in motion  $X$ ,  $\rho(x, t)$ , must satisfy

$$\rho_0(p) = \rho(x, t) \det \mathbf{F}(p, t), \quad \text{with } x = X(p, t). \quad (1.1)$$

**Theorem 1.2.6.** (*Local conservation of mass*) *We have*

$$\dot{\rho} + \rho \text{div } \mathbf{v} = 0, \quad (1.2)$$

$$\rho' + \text{div}(\rho \mathbf{v}) = 0. \quad (1.3)$$

*Proof.* See Gurtin [65], p. 89. □

### 1.2.3 Balance of linear and angular momentum

We represent by

- *Lin* the linear space of endomorphisms from  $\mathcal{V}$ ,

- $Lin^+$  the subset of endomorphisms from  $\mathcal{V}$  whose determinant is positive,
- $Sym$  the subspace of  $Lin$  of all symmetric endomorphisms,
- $Sym_0$  the subspace of  $Lin$  of all symmetric endomorphisms whose trace is null,
- $Skw$  the subspace of  $Lin$  of all skew endomorphisms,
- $Orth$  the subspace of  $Lin$  of all orthogonal endomorphisms,
- $Orth^+$  the subspace of all rotations of  $Orth$ , and
- $\mathcal{N}$  the set of unit vectors of the vector space  $\mathcal{V}$ .

**Definition 1.2.7.** *A system of forces for a body  $\mathcal{B}$  during the motion  $X$  is a pair  $(\mathbf{s}, \mathbf{b})$  of vector fields,*

$$\mathbf{s} : \mathcal{N} \times \mathcal{T} \longrightarrow \mathcal{V}, \quad \mathbf{b} : \mathcal{T} \longrightarrow \mathcal{V},$$

such that

- $\mathbf{s}(\mathbf{n}, \cdot, t)$ , for each  $\mathbf{n} \in \mathcal{N}$  and  $t \in \mathbb{R}$ , is a smooth function of  $x$  in  $\mathcal{B}_t$ ,
- $\mathbf{b}(\cdot, t)$ , for each  $t \in \mathbb{R}$ , is a continuous function of  $x$  in  $\mathcal{B}_t$ .

First component of system of forces,  $\mathbf{s}$ , represents the density of surface force per unit area exerted from positive side on negative side across an oriented surface  $\mathcal{S}$  in  $\mathcal{B}_t$ , when  $\mathbf{n}$  is a unit normal vector in  $x$  (Cauchy's hypothesis). Second component of system forces,  $\mathbf{b}$ , represents the density of body forces per unit volume exerted on the point  $x$  at time  $t$ .

**Theorem 1.2.8. (Cauchy)** *Let  $(\mathbf{s}, \mathbf{b})$  be a system of forces for  $\mathcal{B}$  during a motion  $X$ . A necessary and sufficient condition that principle of conservation of linear and angular momentum be satisfied, is there exists a spatial tensor field  $\mathbf{T}$  (called the Cauchy stress tensor) such that*

1. for each unit vector  $\mathbf{n}$ ,  $s(\mathbf{n})(x, t) := \mathbf{s}(\mathbf{n}, x, t) = \mathbf{T}(x, t)\mathbf{n}$ ,  $\forall (x, t) \in \mathcal{T}$ ,
2.  $\mathbf{T}$  is symmetric,
3.  $\mathbf{T}$  satisfies the equation of motion

$$\rho(x, t)\dot{\mathbf{v}}(x, t) = \operatorname{div} \mathbf{T}(x, t) + \mathbf{b}(x, t), \tag{1.4}$$

being  $\rho$  the density and  $\mathbf{v}$  the velocity of motion.

*Proof.* See Gurtin [65], p. 101. □

The equation (1.4) is known as *the balance of linear and angular momentum*.

### 1.2.4 Balance of energy

We assume Cauchy's hypothesis concerning the surface heat which supposes the existence of a density of surface heat per unit area  $g(\mathbf{n})(x, t) := g(\mathbf{n}, x, t)$ , defined for each unit vector  $\mathbf{n}$  and every  $(x, t)$  in the trajectory of the motion, with the following property:

*If  $\mathcal{S}$  is an oriented surface in  $\mathcal{B}_t$ , with positive unit normal vector  $\mathbf{n}$  at  $x$ , then  $g(\mathbf{n}, x, t)$  is the heat per unit area and time flowing from the negative side of  $\mathcal{S}$  to the positive side of  $\mathcal{S}$  at point  $x$  and time  $t$ .*

We consider a density of body heat per unit volume defined by a scalar field  $f : \mathcal{T} \rightarrow \mathbb{R}$ , where  $f(x, t)$  represents the heat per unit volume supplied by environment at point  $x$  and time  $t$ .

**Definition 1.2.9.** *A system of heat for a body  $\mathcal{B}$  during a motion  $X$ , with trajectory  $\mathcal{T}$ , is a pair of functions  $(g, f)$ ,*

$$g : \mathcal{N} \times \mathcal{T} \rightarrow \mathbb{R} \text{ and } f : \mathcal{T} \rightarrow \mathbb{R},$$

such that

- for each  $\mathbf{n} \in \mathcal{N}$  and  $t \in \mathbb{R}$ ,  $g(\mathbf{n}, \cdot, t)$  is a smooth function of  $x$  on  $\mathcal{B}_t$ ,
- for each  $t \in \mathbb{R}$ ,  $f(\cdot, t)$  is a continuous function of  $x$  on  $\mathcal{B}_t$ .

Let us consider a system of forces  $(\mathbf{s}, \mathbf{b})$  and a system of heat  $(g, f)$  during a motion  $X$  of a body  $\mathcal{B}$ . The *energy conservation law*, which is also called *first principle of thermodynamics*, asserts that there exists a scalar field  $E$ , the specific total energy, such that for every part  $\mathcal{P} \subset \mathcal{B}$  and time  $t$ ,

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho E dV = \int_{\partial \mathcal{P}_t} \mathbf{s}(\mathbf{n}) \cdot \mathbf{v} dA + \int_{\mathcal{P}_t} \mathbf{b} \cdot \mathbf{v} dV - \int_{\partial \mathcal{P}_t} g(\mathbf{n}) dA + \int_{\mathcal{P}_t} f dV,$$

where  $\mathcal{P}_t = X(\mathcal{P}, t)$  and  $\mathbf{n}$  is the outward normal vector to  $\mathcal{P}_t$  at each point.

**Definition 1.2.10.** *We define by specific internal energy per unit mass the scalar field*

$$e = E - \frac{|\mathbf{v}|^2}{2}.$$

**Theorem 1.2.11.** *We suppose that principle of conservation of linear and angular momentum is verified. Thus, a necessary and sufficient condition that first principle of thermodynamics be satisfied, is that there exists a spatial vector field  $\mathbf{q}$  called heat flux per unit area, such that*

1. for each unit vector  $\mathbf{n} \in \mathcal{N}$ ,  $g(\mathbf{n}) := g(\mathbf{n}, x, t) = \mathbf{q}(x, t) \cdot \mathbf{n}$ ,  $\forall (x, t) \in \mathcal{T}$ ,
- 2.

$$\rho \dot{e} = \mathbf{T} : \mathbf{D} - \operatorname{div} \mathbf{q} + f, \tag{1.5}$$

where  $\mathbf{D}$  is the symmetric part of  $\mathbf{L} := \operatorname{grad} \mathbf{v}$  and ":" defines the contraction product of 2 subscripts; in particular  $\mathbf{T} : \mathbf{D}$  is the inner product defined by  $\mathbf{T} : \mathbf{D} = \operatorname{tr}(\mathbf{T}^t \mathbf{D})$ , where  $\operatorname{tr}$  represents the trace and  $\mathbf{T}^t$  the transpose of  $\mathbf{T}$ .

*Proof.* See Bermúdez [19] Theorem 1.4.3, p. 8 and Proposition 1.4.6., p. 9.  $\square$

We will call to equation (1.5) *conservation of energy equation*.

Given a system of heat  $(g, f)$  during a motion  $X$  of a body  $\mathcal{B}$ , *second principle of thermodynamics* asserts the existence of a scalar field  $s$ , the specific entropy per unit mass, and a strictly positive scalar field  $\theta$ , the absolute temperature, such that

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho s \, dV \geq - \int_{\partial \mathcal{P}_t} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} \, dA + \int_{\mathcal{P}_t} \frac{f}{\theta} \, dV, \quad (1.6)$$

for all parts  $\mathcal{P} \subset \mathcal{B}$  and times  $t \in \mathbb{R}$ , being  $\mathcal{P}_t = X(\mathcal{P}, t)$  and  $\mathbf{n}$  the outward unit normal vector to  $\mathcal{P}_t$ .

**Theorem 1.2.12. (*Reynolds's transport*)** *Let  $\Phi$  be a smooth spatial scalar or vector field. Then, for any part  $\mathcal{P}$  and time  $t$ ,*

$$\frac{d}{dt} \int_{\mathcal{P}_t} \Phi \, dV = \int_{\mathcal{P}_t} (\dot{\Phi} + \Phi \operatorname{div} \mathbf{v}) \, dV, \quad (1.7)$$

$$\frac{d}{dt} \int_{\mathcal{P}_t} \Phi \, dV = \int_{\mathcal{P}_t} \Phi' \, dV + \int_{\partial \mathcal{P}_t} \Phi \mathbf{v} \cdot \mathbf{n} \, dA, \quad (1.8)$$

being  $\mathbf{n}$  a outward unit normal vector to  $\mathcal{P}_t$ .

*Proof.* See Gurtin [65], p. 78.  $\square$

**Theorem 1.2.13. (*Localization*)** *Let  $\Phi$  a continuous scalar or vector field on an open set  $\mathcal{R}$  in  $\mathcal{E}$ . Then given any  $x_0 \in \mathcal{R}$ ,*

$$\Phi(x_0) = \lim_{\substack{\operatorname{vol}(R) \rightarrow 0 \\ x_0 \in R}} \frac{1}{\operatorname{vol}(R)} \int_R \Phi \, dV.$$

*Proof.* See Gurtin [65], p. 38.  $\square$

**Corollary 1.2.14.** *If*

$$\int_R \Phi \, dV = 0,$$

for all parts  $R \subset \mathcal{R}$ , then

$$\Phi \equiv 0.$$

*Proof.* See Gurtin [65], p. 38.  $\square$

**Theorem 1.2.15. (Divergence)** Let  $\mathcal{R}$  be a bounded regular region, and let  $\varphi : \mathcal{R} \rightarrow \mathbb{R}$ ,  $\mathbf{w} : \mathcal{R} \rightarrow \mathcal{V}$  and  $\mathbf{H} : \mathcal{R} \rightarrow \text{Lin}$  be smooth fields. Then

$$\begin{aligned}\int_{\mathcal{R}} \nabla \varphi \, dV &= \int_{\partial \mathcal{R}} \varphi \mathbf{n} \, dA, \\ \int_{\mathcal{R}} \operatorname{div} \mathbf{w} \, dV &= \int_{\partial \mathcal{R}} \mathbf{w} \cdot \mathbf{n} \, dA, \\ \int_{\mathcal{R}} \operatorname{div} \mathbf{H} \, dV &= \int_{\partial \mathcal{R}} \mathbf{H} \mathbf{n} \, dA,\end{aligned}$$

where  $\mathbf{n}$  is the outward unit normal vector to  $\mathcal{R}$  on each point from its bordering.

*Proof.* See Gurtin [65], p. 37. □

**Proposition 1.2.16.** Let  $\varphi$  be a smooth scalar field and  $\mathbf{w}$  a smooth vector field. Then

$$\operatorname{div}(\varphi \mathbf{w}) = \varphi \operatorname{div} \mathbf{w} + \mathbf{w} \cdot \operatorname{grad} \varphi.$$

*Proof.* See Gurtin [65], p. 30. □

**Lemma 1.2.17.** The local expression of second principle of thermodynamics is

$$\rho \theta \dot{s} - \rho \dot{e} + \mathbf{T} : \mathbf{D} - \frac{1}{\theta} \mathbf{q} \cdot \operatorname{grad} \theta \geq 0. \quad (1.9)$$

*Proof.* Given inequality (1.6):

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho s \, dV + \int_{\partial \mathcal{P}_t} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} \, dA - \int_{\mathcal{P}_t} \frac{f}{\theta} \, dV \geq 0. \quad (1.10)$$

Taking into account equation (1.7) of Reynold's transport theorem, the first term of inequality (1.10) takes the expression

$$\int_{\mathcal{P}_t} (\dot{\rho s} + \rho s \operatorname{div} \mathbf{v}) \, dV = \int_{\mathcal{P}_t} (s(\dot{\rho} + \rho \operatorname{div} \mathbf{v}) + \rho \dot{s}) \, dV = \int_{\mathcal{P}_t} \rho \dot{s} \, dV, \quad (1.11)$$

where we have used equality (1.2).

Applying Divergence theorem to second term of inequality (1.10), and substituting (1.11) in inequality (1.10), we obtain

$$\int_{\mathcal{P}_t} \rho \dot{s} \, dV + \int_{\mathcal{P}_t} \operatorname{div} \left( \frac{\mathbf{q}}{\theta} \right) \, dV - \int_{\mathcal{P}_t} \frac{f}{\theta} \, dV \geq 0.$$

Tranks to Localization theorem, we deduce

$$\rho \dot{s} + \operatorname{div} \left( \frac{\mathbf{q}}{\theta} \right) - \frac{f}{\theta} \geq 0. \quad (1.12)$$

Multiplying inequality (1.12) by  $\theta$ , and taking into account Proposition 1.2.16, we have

$$\rho \theta \dot{s} + \operatorname{div} \mathbf{q} - \frac{1}{\theta} \mathbf{q} \cdot \operatorname{grad} \theta - f \geq 0. \quad (1.13)$$

Finally, by equation (1.5),

$$f = \rho \dot{e} - \mathbf{T} : \mathbf{D} + \operatorname{div} \mathbf{q},$$

and if we substitute the value of  $f$  in inequality (1.13), we conclude the result.  $\square$

Summing up, under smooth enough assumptions, the following conservation laws and principles must be verified in any deformed configuration:

<b>Conservation laws</b>		
Mass	$\rho_0 = \rho \det \mathbf{F}$	<b>(MC)</b>
Momentum	$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \mathbf{b}$	<b>(FC)</b>
Energy	$\rho \dot{e} = \mathbf{T} : \mathbf{D} - \operatorname{div} \mathbf{q} + f$	<b>(EC)</b>
<b>Thermodynamics principle</b>		
Second	$\rho \theta \dot{s} - \rho \dot{e} + \mathbf{T} : \mathbf{D} - \frac{1}{\theta} \mathbf{q} \cdot \operatorname{grad} \theta \geq 0$	<b>(E1)</b>

### 1.3 Thermoviscoelastic materials with long memory

In this section, we follow the methodology used by Carlson [26] for elastic materials and by Bermúdez [19] for Coleman-Noll materials. Firstly in Subsection 1.3.1, we define the thermoviscoelastic materials with long memory. Then, we deduce the restrictions on their response functions to guarantee the second principle of thermodynamics in Subsection 1.3.2. Finally, we rewrite the equilibrium equations for the thermoviscoelastic materials with long memory in Subsection 1.3.3 and we see under certain assumptions on response mappings that these materials satisfy the principle of material frame indifference and isotropy in Subsection 1.3.4.

The constitutive laws for these materials are introduced considering the theory of internal variables: we suppose the existence of local variables which determine the state of material at each point and time. In particular, the thermoviscoelastic model with long memory considers as observable state variables the deformation tensor,  $\mathbf{F}$ , the absolute temperature,  $\theta$ , and a internal variable  $\mathbf{Z}$  which gives the viscoplastic history of the material. First two observable state variables  $\mathbf{F}$  and  $\theta$  are usual in thermoviscoelastic models, whereas  $\mathbf{Z}$  is introduced to take into account the thermal and mechanical loading histories; as we will see in Section 1.5 these choices will allow us to consider the Maxwell-Norton materials as thermoviscoelastic materials with long memory.

A first study of the thermodynamics of nonlinear materials with constitutive laws defined in terms of internal state variables is given in Coleman and Gurtin [30]; these authors introduce an internal state vector in order to modelize the fluids. Later, Coleman [29] developes a general theory for materials with fading memory but in his work internal variables are not considered. Alber [3] also introduces a vector of internal variables to describe the mechanical behaviour of materials with constitutive equations of monotone type; this theory is used to include particular cases of viscoelastic and viscoplastic materials in Alber and Chelmiński [4], [5] and [6]. Adam and Ponthot [1] model the thermomechanical behaviour of metals submitted to large strains considering also a vector of internal variables.

Other authors like Simo and Miehe [95] and later Serrano *et al.* [93] also include an internal variable to define thermoplastic laws: the scalar corresponding to the part of the entropy dissipating energy. More recently, Tzavaras [97] takes a scalar internal variable to consider the structure of relaxation approximations to conservation laws; and Helm and Haupt [67] describe the material behaviour of shape memory alloy including a scalar internal variable to the fraction of martensite and some stress internal variables to consider the energy storage due to internal stress fields.

Lately, Lattanzio and Tzavaras [77] consider a model of viscoelastic stress-relaxation where the viscoelastic stresses are described by means of tensorial internal variables.

In this chapter we consider as internal variable a symmetric second order tensor  $\mathbf{Z}$ , whose temporal evolution is governed by an ordinary differential equation similar to ones given in the previously mentioned works. The variable  $\mathbf{Z}$ , whose temporal evolution depends on variation of plasticity potential, allows us to check the dissipated energy by history of the viscoplastic stresses from initial time.

### 1.3.1 Constitutive laws

**Definition 1.3.1.** *A thermodynamic process for a body  $\mathcal{B}$ , with a mass distribution  $\rho_0$ , is a set of nine mappings:*

1.  $X : \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{E}$  the motion of  $\mathcal{B}$ ,
2.  $\mathbf{T} \in \mathcal{C}^1(\mathcal{T}; \text{Sym})$  the Cauchy stress tensor,
3.  $\mathbf{b} \in \mathcal{C}^0(\mathcal{T}; \mathcal{V})$  the body forces per unit volume,
4.  $e \in \mathcal{C}^1(\mathcal{T}; \mathbb{R})$  the specific internal energy per unit mass,
5.  $\theta \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^+)$  the absolute temperature,
6.  $\mathbf{q} \in \mathcal{C}^1(\mathcal{T}; \mathcal{V})$  the heat flux per unit area,
7.  $f \in \mathcal{C}^0(\mathcal{T}; \mathbb{R})$  the body heat per unit volume,
8.  $s \in \mathcal{C}^1(\mathcal{T}; \mathbb{R})$  the specific entropy per unit mass,
9.  $\mathbf{Z} \in \mathcal{C}^1(\mathcal{T}; \text{Sym})$  the internal symmetric second order tensor,

verifying Equations (MC), (FC) and (EC) at each point of its trajectory.

**Definition 1.3.2.** *A material body is a triple  $(\mathcal{B}, \rho_0, \mathcal{C})$  consisting of a body  $\mathcal{B}$ , a mass distribution  $\rho_0$  and a family  $\mathcal{C}$  of thermodynamic processes called the constitutive class of the body.*

**Definition 1.3.3.** *A material body  $(\mathcal{B}, \rho_0, \mathcal{C})$  is called thermoviscoelastic with long memory if there*

exist seven smooth enough response mappings  $\widehat{\mathbf{T}}_{elas}$ ,  $\widehat{\mathbf{T}}_{ther}$ ,  $\widehat{\mathbf{T}}_{plas}$ ,  $\widehat{e}$ ,  $\widehat{\mathbf{q}}$ ,  $\widehat{s}$  and  $\mathbf{g}$ ,

$$\begin{aligned}\widehat{\mathbf{T}}_{elas} &: Lin^+ \times \mathbb{R}^+ \times \mathcal{B} \rightarrow Sym, \\ \widehat{\mathbf{T}}_{ther} &: Lin^+ \times \mathbb{R}^+ \times \mathcal{B} \rightarrow Sym, \\ \widehat{\mathbf{T}}_{plas} &: Lin^+ \times \mathbb{R}^+ \times Sym \times \mathcal{B} \rightarrow Sym, \\ \widehat{e} &: Lin^+ \times \mathbb{R}^+ \times Sym \times \mathcal{B} \rightarrow \mathbb{R}, \\ \widehat{\mathbf{q}} &: Lin^+ \times \mathbb{R}^+ \times Sym \times \mathcal{V} \times \mathcal{B} \rightarrow \mathcal{V}, \\ \widehat{s} &: Lin^+ \times \mathbb{R}^+ \times Sym \times \mathcal{B} \rightarrow \mathbb{R}, \\ \mathbf{g} &: Lin^+ \times \mathbb{R}^+ \times Sym \times \mathcal{B} \rightarrow Sym,\end{aligned}$$

such that for every thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  it satisfies

$$\begin{aligned}\mathbf{T}(x, t) &= \widehat{\mathbf{T}}_{elas}(\mathbf{F}(p, t), \theta(x, t), p) + \widehat{\mathbf{T}}_{ther}(\mathbf{F}(p, t), \theta(x, t), p) + \widehat{\mathbf{T}}_{plas}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \\ &= \widehat{\mathbf{T}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p),\end{aligned}\tag{1.14}$$

$$e(x, t) = \widehat{e}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p),\tag{1.15}$$

$$\mathbf{q}(x, t) = \widehat{\mathbf{q}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), \text{grad } \theta(x, t), p),\tag{1.16}$$

$$s(x, t) = \widehat{s}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p),\tag{1.17}$$

$$\dot{\mathbf{Z}}(x, t) = \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p),\tag{1.18}$$

with  $x = \mathbf{X}(p, t)$  and  $\widehat{\mathbf{T}} : Lin^+ \times \mathbb{R}^+ \times Sym \times \mathcal{B} \rightarrow Sym$ , being defined as

$$\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) = \widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta, p) + \widehat{\mathbf{T}}_{ther}(\mathbf{F}, \theta, p) + \widehat{\mathbf{T}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p).\tag{1.19}$$

In practice  $\widehat{\mathbf{T}}_{elas}$  represents the elastic part of Cauchy stress tensor  $\mathbf{T}$ ,  $\widehat{\mathbf{T}}_{ther}$  its thermal part and  $\widehat{\mathbf{T}}_{plas}$  its viscoplastic part which includes the memory of the suffering stresses in  $[0, t]$  through the internal variable  $\mathbf{Z}$ .

**Remark 1.3.4.** *The heat flux response function  $\widehat{\mathbf{q}}$  depends on the temperature gradient as an independent variable to take into account that  $\mathbf{q}$  is strongly dependent on the temperature changes through the material. Consequently, by the Truesdell's principle of equipresence, the variable  $\text{grad } \theta$  should also be present in all constitutive equations. Nevertheless, because of as shown in Coleman [29] or Coleman and Gurtin [30], the presence of  $\text{grad } \theta$  as independent variable in laws (1.14), (1.15), (1.17) or (1.18) contradicts the second principle of thermodynamics (Clausius-Duhem inequality), and in order to simplify our paper we do not consider it.*

### 1.3.2 Verifying the second principle

Throughout this chapter we suppose that the response mappings are smooth enough and verify the following hypotheses:

(H1) *There exists a smooth enough mapping,  $\widehat{\theta} : Lin^+ \times \mathbb{R} \times Sym \times \mathcal{B} \rightarrow \mathbb{R}^+$ , such that if  $\mathbf{F} \in Lin^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in Sym$  and  $p \in \mathcal{B}$  then*

$$s = \widehat{s}(\mathbf{F}, \theta, \mathbf{Z}, p) \text{ if and only if } \theta = \widehat{\theta}(\mathbf{F}, s, \mathbf{Z}, p).$$

*Consequently  $\theta = \widehat{\theta}(\mathbf{F}, s, \mathbf{Z}, p) = \widehat{\theta}(\mathbf{F}, \widehat{s}(\mathbf{F}, \theta, \mathbf{Z}, p), \mathbf{Z}, p)$ .*

(H2) Given a motion  $X$ , a smooth enough function  $\theta : \mathcal{T} \rightarrow \mathbb{R}^+$  and  $\mathbf{Z}_0 \in \text{Sym}$ , there exists a smooth tensorial function  $\mathbf{Z} : \mathcal{T} \rightarrow \text{Sym}$ , unique solution of the system:

$$\begin{cases} \dot{\mathbf{Z}}(x, t) = \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) & \text{in } \mathcal{T}, \quad x = X(p, t), \\ \mathbf{Z}_m(p, 0) = \mathbf{Z}_0. \end{cases}$$

(H3) There exists  $\mathbf{h} : \text{Lin}^+ \times \mathbb{R}^+ \times \text{Sym} \times \mathcal{B} \rightarrow \text{Sym}$  such that,

$$\mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) = \mathbf{h}(\mathbf{F}, \theta, \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p), p), \quad \forall \mathbf{F} \in \text{Lin}^+, \theta \in \mathbb{R}^+, \mathbf{Z} \in \text{Sym} \text{ and } p \in \mathcal{B}.$$

(H4) For all  $\mathbf{F} \in \text{Lin}^+$ , there exists  $\bar{\mathbf{Z}} \in \text{Sym}$  such that the function  $\mathbf{g}$  verifies:

$$\mathbf{g}(\mathbf{F}, \theta, \bar{\mathbf{Z}}, p) = \mathbf{0}, \quad \forall \theta \in \mathbb{R}^+ \text{ and } p \in \mathcal{B}.$$

**Remark 1.3.5.** Hypothesis (H3) relates the temporal evolution of  $\mathbf{Z}$  to the Cauchy stress tensor  $\mathbf{T}$ , and hence,  $\mathbf{Z}$  is connected to the history of suffered stresses by the material,

$$\mathbf{Z}(x, t) = \int_0^t \mathbf{h}(\mathbf{F}(p, s), \theta(X(p, s), s), \mathbf{T}(X(p, s), s), p) ds + \mathbf{Z}_0, \quad \text{where } p = P(x, t).$$

In addition, from a mathematical point of view, this hypothesis allows us to describe  $\dot{\mathbf{Z}}$  in terms of the First Piola-Kirchhoff stress tensor in Lagrangian coordinates and, in consequence, to incorporate the presented linealizations for this tensor in the temporal evolution of  $\mathbf{Z}$ .

**Theorem 1.3.6.** We consider a thermoviscoelastic material with long memory whose constitutive class  $\mathcal{C}$  verifies hypotheses (H1) and (H2).

Then, all elements in  $\mathcal{C}$  satisfy the second principle of thermodynamics if and only if

$$\bullet \quad \frac{\partial \hat{e}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) = \theta \frac{\partial \hat{s}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p), \quad (1.20)$$

$$\bullet \quad \left[ \theta \frac{\partial \hat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{\partial \hat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \geq 0, \quad (1.21)$$

(internal dissipation inequality)

$$\bullet \quad \frac{\partial \hat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) = \theta \frac{\partial \hat{s}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) + \frac{\det \mathbf{F}}{\rho_0} \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^{-t}, \quad (1.22)$$

$$\bullet \quad \left[ \theta \frac{\partial \hat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{\partial \hat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{1}{\rho \theta} \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) \cdot \mathbf{w} \geq 0, \quad (1.23)$$

(general dissipation inequality)

for all  $\mathbf{F} \in \text{Lin}^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in \text{Sym}$ ,  $\mathbf{w} \in \mathcal{V}$  and  $p \in \mathcal{B}$ .

The proof is based on the following results:

**Lemma 1.3.7.** *i) Given a motion  $X$ , we have*

$$\dot{\mathbf{F}} = \mathbf{L}_m \mathbf{F}, \quad (1.24)$$

where  $\mathbf{F}$  is the deformation gradient and  $\mathbf{L}_m$  is the gradient of the velocity in Lagrangian coordinates.

ii) Given  $\bar{\mathbf{F}} \in \text{Lin}^+$  and  $\bar{\mathbf{L}} \in \text{Lin}$ , there exists  $\tilde{\mathbf{F}} : \mathbb{R} \rightarrow \text{Lin}^+$  such that  $\tilde{\mathbf{F}}(0) = \bar{\mathbf{F}}$  and  $\dot{\tilde{\mathbf{F}}}(t)\tilde{\mathbf{F}}(t)^{-1} = \bar{\mathbf{L}}$ . Furthermore  $\det \tilde{\mathbf{F}}(t) = \det \bar{\mathbf{F}} e^{\text{tr}(\bar{\mathbf{L}})t} > 0$ .

iii) Given  $\bar{\mathbf{W}} \in \text{Skw}$  and  $\bar{\mathbf{Q}} \in \text{Orth}^+$ , the solution  $\tilde{\mathbf{Q}}$  of the Cauchy problem:

$$\dot{\tilde{\mathbf{Q}}}(t) = \bar{\mathbf{W}}\tilde{\mathbf{Q}}(t), \quad \tilde{\mathbf{Q}}(0) = \bar{\mathbf{Q}}, \quad (1.25)$$

verifies  $\tilde{\mathbf{Q}}(t) \in \text{Orth}^+$ ,  $\forall t \in \mathbb{R}$ .

*Proof.* i) See Gurtin [65], p. 63; ii) and iii) see Bermúdez [19], p. 20. □

The following Lemma allows us to build up a particular thermodynamic process belonging to each constitutive class  $\mathcal{C}$  of a thermoviscoelastic material with long memory.

**Lemma 1.3.8.** *Let us consider a thermoviscoelastic material with long memory whose constitutive class  $\mathcal{C}$  verifies hypotheses (H1) and (H2). Given  $\mathbf{F}^* \in \text{Lin}^+$  and  $\mathbf{L}^* \in \text{Lin}$ , there exists a motion  $X$  such that  $\mathbf{F}(p, 0) = \mathbf{F}^*$  and  $\text{grad } \mathbf{v}(x, t) = \mathbf{L}^*$ ; let us denote by  $\mathcal{T}$  its trajectory. Furthermore, let  $s^* \in \mathbb{R}$ ,  $\mathbf{Z}^* \in \text{Sym}$ ,  $p^* \in \mathcal{B}$  and  $\theta \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^+)$  satisfying*

$$\theta(x^*, 0) = \theta^*, \quad \text{with } \theta^* := \hat{\theta}(\mathbf{F}^*, s^*, \mathbf{Z}^*, p^*) \quad \text{and } x^* = X(p^*, 0). \quad (1.26)$$

Then there exists a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  such that

$$\mathbf{F}(p, 0) = \mathbf{F}^*, \quad \forall p \in \mathcal{B}; \quad \mathbf{L}(x, t) = \mathbf{L}^*, \quad \forall (x, t) \in \mathcal{T}; \quad s(x^*, 0) = s^*; \quad \mathbf{Z}(x^*, 0) = \mathbf{Z}^*. \quad (1.27)$$

*Proof.* Let  $\tilde{\mathbf{F}}(t)$  be given as in Lemma 1.3.7 for  $\bar{\mathbf{F}} = \mathbf{F}^*$  and  $\bar{\mathbf{L}} = \mathbf{L}^*$ . Let us define  $X$  by  $X(p, t) = o + \tilde{\mathbf{F}}(t)(p - o)$ , where  $o$  is any point in  $\mathcal{E}$ . We have

$$\mathbf{F}(p, t) = \tilde{\mathbf{F}}(t) \quad \text{and} \quad \mathbf{L}(x, t) = \dot{\tilde{\mathbf{F}}}(t)\tilde{\mathbf{F}}(t)^{-1} = \mathbf{L}^*.$$

Let us consider the spatial function  $\theta$  verifying (1.26) and the motion  $X$  previously defined. By hypothesis (H2), for  $\mathbf{Z}_0 = \mathbf{Z}^*$  there exists  $\mathbf{Z}(x, t)$  smooth enough such that

$$\dot{\mathbf{Z}}(x, t) = \mathbf{g} \left( \tilde{\mathbf{F}}(t), \theta(x, t), \mathbf{Z}(x, t), p \right), \quad \mathbf{Z}(x, 0) = \mathbf{Z}^*.$$

Let  $s$  and  $\mathbf{T}$  be defined by

$$\begin{aligned} s(x, t) &= \hat{s}(\tilde{\mathbf{F}}(t), \theta(x, t), \mathbf{Z}(x, t), p), \\ \mathbf{T}(x, t) &= \hat{\mathbf{T}}(\tilde{\mathbf{F}}(t), \theta(x, t), \mathbf{Z}(x, t), p), \quad \text{with } x = X(p, t). \end{aligned}$$

Thus, by hypothesis **(H1)** and using again (1.26), we obtain

$$s(x^*, 0) = \widehat{s}(\mathbf{F}^*, \theta(x^*, 0), \mathbf{Z}(x^*, 0), p^*) = \widehat{s}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) = s^*.$$

Finally, we consider

$$\begin{aligned} e(x, t) &= \widehat{e}(\widetilde{\mathbf{F}}(t), \theta(x, t), \mathbf{Z}(x, t), p), \\ \mathbf{q}(x, t) &= \widehat{\mathbf{q}}(\widetilde{\mathbf{F}}(t), \theta(x, t), \mathbf{Z}(x, t), \text{grad } \theta(x, t), p), \end{aligned}$$

and we define  $\mathbf{b}$  and  $f$  by

$$\begin{aligned} \mathbf{b} &= \rho \dot{\mathbf{v}} - \text{div } \mathbf{T}, \\ f &= \rho \dot{e} - \mathbf{T} : \mathbf{D} + \text{div } \mathbf{q}, \end{aligned}$$

with  $\rho(x, t) = \frac{\rho_0(p)}{\det \widetilde{\mathbf{F}}(t)}$  and  $x = X(p, t)$ .

Therefore, the thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z})$  belongs to the constitutive class of the material and satisfies (1.27).  $\square$

Now we are ready to prove Theorem 1.3.6.

*Proof.* Let us consider the non-conservative expression of the second principle of thermodynamics **(E1)**:

$$\rho \theta \dot{s} - \rho \dot{e} + \mathbf{T} : \mathbf{D} - \frac{1}{\theta} \mathbf{q} \cdot \text{grad } \theta \geq 0. \quad (1.28)$$

By equations (1.15) and (1.17) and taking into account equality (1.18) we deduce

$$\begin{aligned} \dot{e}(x, t) &= \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{F}}(p, t) + \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) \\ &\quad + \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p), \end{aligned} \quad (1.29)$$

$$\begin{aligned} \dot{s}(x, t) &= \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{F}}(p, t) + \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) \\ &\quad + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p), \end{aligned} \quad (1.30)$$

with  $x = X(p, t)$ . Substituting expressions (1.29) and (1.30) in inequality (1.28), we obtain

$$\begin{aligned} &\rho \theta \left[ \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \dot{\mathbf{F}} + \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) \dot{\theta} + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] \\ &- \rho \left[ \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \dot{\mathbf{F}} + \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) \dot{\theta} + \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] \\ &+ \mathbf{T} : \mathbf{L} - \frac{1}{\theta} \mathbf{q} \cdot \text{grad } \theta \geq 0 \text{ in } \mathcal{T}, \end{aligned} \quad (1.31)$$

where  $\mathbf{T} : \mathbf{D}$  was replaced with  $\mathbf{T} : \mathbf{L}$  thanks to the symmetry of the tensor  $\mathbf{T}$ .

- Now we choose a particular thermodynamic process. For this, let  $\mathbf{F}^* \in Lin^+$ ,  $\mathbf{L}^* = \mathbf{0}$ ,  $s^* \in \mathbb{R}$ ,  $\mathbf{Z}^* \in Sym$ ,  $p^* \in \mathcal{B}$  be given and  $\theta \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^+)$  satisfying hypothesis (1.26) and such that  $\text{grad } \theta \equiv \mathbf{0}$  and  $\theta'(x^*, 0) = a$ ,  $a$  being any real number. Lemma 1.3.8 guaranties the existence of a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  verifying equalities (1.27); for this process, inequality (1.31) can be rewritten as follows

$$\begin{aligned} & \rho \theta \left[ \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) \dot{\theta} + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] \\ & - \rho \left[ \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) \dot{\theta} + \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] \geq 0 \quad \text{in } \mathcal{T}, \end{aligned} \quad (1.32)$$

since  $\dot{\mathbf{F}} \equiv \mathbf{0}$ . Taking  $p = p^*$  and  $t = 0$  in inequality (1.32), and given that the density  $\rho^* = \rho(x^*, 0)$  is a positive real number, we deduce

$$\begin{aligned} & \left[ \theta^* \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] a \geq \\ & \left[ \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] : \mathbf{g}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*), \quad \forall a \in \mathbb{R}. \end{aligned}$$

This inequality can be only verified if

$$\left[ \theta^* \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] = 0, \quad (1.33)$$

and, in consequence

$$\left[ \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] : \mathbf{g}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \geq 0. \quad (1.34)$$

Then, we get equality (1.20) from expression (1.33), and inequality (1.21) from expression (1.34).

- In order to prove equality (1.22), we consider again a particular thermodynamic process with the following choices: let  $\mathbf{F}^* \in Lin^+$ ,  $\bar{\mathbf{L}}^* \in Lin$ ,  $s^* \in \mathbb{R}$ ,  $\mathbf{Z}^* \in Sym$ ,  $p^* \in \mathcal{B}$  be given and  $\theta \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^+)$  satisfying hypothesis (1.26) and such that  $\text{grad } \theta \equiv \mathbf{0}$ . We apply Lemma 1.3.8 for  $\mathbf{L}^* = \kappa \bar{\mathbf{L}}^*$ ,  $\kappa$  being any real number, then there exists a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  verifying equalities (1.27).

Thanks to equality (1.20), inequality (1.31) becomes

$$\begin{aligned} & \rho \theta \left[ \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \dot{\mathbf{F}} + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] \\ & - \rho \left[ \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \dot{\mathbf{F}} + \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] + \mathbf{T} : \mathbf{L}^* \geq 0 \quad \text{in } \mathcal{T}. \end{aligned}$$

Taking into account Lemma 1.3.7 and the property

$$\mathbf{O} : (\mathbf{PQ}) = (\mathbf{P}^t \mathbf{O}) : \mathbf{Q} = (\mathbf{OQ}^t) : \mathbf{P}, \quad (1.35)$$

$\mathbf{O}$ ,  $\mathbf{P}$  and  $\mathbf{Q}$  being three second order tensors, the previous inequality can be rewritten as follows

$$\begin{aligned} & \rho \theta \left[ \kappa \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^t : \bar{\mathbf{L}}^* + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] - \rho \kappa \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^t : \bar{\mathbf{L}}^* \\ & - \rho \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) + \kappa \mathbf{T} : \bar{\mathbf{L}}^* \geq 0 \text{ in } \mathcal{T}, \forall \kappa \in \mathbb{R}. \end{aligned} \quad (1.36)$$

Taking  $p = p^*$  and  $t = 0$  in expression (1.36) and given that  $\rho(x^*, 0) = \frac{\rho_0(p^*)}{\det \mathbf{F}^*}$ , we obtain

$$\begin{aligned} & \kappa \left[ \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] \mathbf{F}^{*t} : \bar{\mathbf{L}}^* + \kappa \frac{\det \mathbf{F}^*}{\rho_0} \widehat{\mathbf{T}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) : \bar{\mathbf{L}}^* \\ & \geq \left[ \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] : \mathbf{g}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*), \quad \forall \kappa \in \mathbb{R}, \end{aligned}$$

thanks to equality (1.14). This inequality can only be verified if

$$\left[ \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] \mathbf{F}^{*t} : \bar{\mathbf{L}}^* + \frac{\det \mathbf{F}^*}{\rho_0} \widehat{\mathbf{T}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) : \bar{\mathbf{L}}^* = 0,$$

hence, we conclude equality (1.22).

- Finally, we choose again a particular thermodynamic process: let  $\mathbf{F}^* \in Lin^+$ ,  $\mathbf{L}^* \in Lin$ ,  $s^* \in \mathbb{R}$ ,  $\mathbf{Z}^* \in Sym$ ,  $p^* \in \mathcal{B}$  be given and  $\theta \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^+)$  satisfying hypothesis (1.26) and such that  $\text{grad } \theta(x^*, 0) = \mathbf{w}^*$ , with  $\mathbf{w}^* \in \mathcal{V}$ . Applying Lemma 1.3.8 we obtain a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  verifying equalities (1.27).

Thanks to expressions (1.16), (1.20) and (1.22), inequality (1.31) can be rewritten as follows

$$\begin{aligned} & \rho \left[ \theta \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \\ & - \frac{1}{\theta} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \text{grad } \theta, p) \cdot \text{grad } \theta \geq 0 \text{ in } \mathcal{T}. \end{aligned} \quad (1.37)$$

Taking  $p = p^*$  and  $t = 0$  in inequality (1.37), we deduce

$$\begin{aligned} & \rho^* \left[ \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] : \mathbf{g}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \\ & - \frac{1}{\theta^*} \widehat{\mathbf{q}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, \mathbf{w}^*, p^*) \cdot \mathbf{w}^* \geq 0, \end{aligned}$$

with  $\rho^* = \rho(x^*, 0)$ . Hence, we conclude inequality (1.23).

In order to prove the sufficiency, we notice that if expressions (1.20)-(1.23) are satisfied, the second principle of thermodynamics holds.  $\square$

From now on we suppose that hypotheses **(H1)** and **(H2)** are satisfied, and the second principle of thermodynamics is verified too.

**Corollary 1.3.9.** *If the heat flux response function  $\widehat{\mathbf{q}}$  depends linearly on the temperature gradient, then the heat conduction inequality*

$$\widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) \cdot \mathbf{w} \leq 0, \quad (1.38)$$

holds for all  $\mathbf{F} \in Lin^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in Sym$ ,  $\mathbf{w} \in \mathcal{V}$  and  $p \in \mathcal{B}$ .

*Proof.* We suppose that there exist  $\mathbf{F}^* \in Lin^+$ ,  $\mathbf{L}^* \in Lin$ ,  $s^* \in \mathbb{R}$ ,  $\mathbf{Z}^* \in Sym$ ,  $\mathbf{w}^* \in \mathcal{V}$ ,  $p^* \in \mathcal{B}$  such that

$$\widehat{\mathbf{q}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, \mathbf{w}^*, p^*) \cdot \mathbf{w}^* \geq 0. \quad (1.39)$$

Let  $\theta \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^+)$  be satisfying hypothesis (1.26) and such that  $\text{grad } \theta(x^*, 0) = \gamma \mathbf{w}^*$ , with  $\gamma$  any real number. Applying Lemma 1.3.8 we obtain a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  verifying equalities (1.27) for each  $\gamma \in \mathbb{R}$ .

Using expressions (1.16), (1.20) and (1.22), the second principle of thermodynamics can be written as in inequality (1.37), which for  $p = p^*$  and  $t = 0$  results:

$$\begin{aligned} & \rho^* \left[ \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) - \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \right] : \mathbf{g}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, p^*) \\ & - \frac{\gamma^2}{\theta^*} \widehat{\mathbf{q}}(\mathbf{F}^*, \theta^*, \mathbf{Z}^*, \mathbf{w}^*, p^*) \cdot \mathbf{w}^* \geq 0, \quad \forall \gamma \in \mathbb{R}, \end{aligned} \quad (1.40)$$

with  $\rho^* = \rho(x^*, 0)$ , thanks to the linearity of  $\widehat{\mathbf{q}}$  with respect to its fourth variable. But taking into account inequality (1.21) and (1.39), there exists  $\gamma \in \mathbb{R}$  for which expression (1.40) is negative which is a contradiction.  $\square$

**Corollary 1.3.10.** *If the mapping  $\mathbf{g}$  satisfies (H4) and if the response function of the heat flux  $\widehat{\mathbf{q}}$  is independent of  $\mathbf{Z}$ , then heat conduction inequality (1.38) is verified.*

*Proof.* Let  $\mathbf{F}^* \in Lin^+$ ,  $\mathbf{L}^* \in Lin$ ,  $s^* \in \mathbb{R}$ ,  $\mathbf{w}^* \in \mathcal{V}$  and  $p^* \in \mathcal{B}$  be given,  $\theta \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^+)$  satisfying hypothesis (1.26) and such that  $\text{grad } \theta(x^*, 0) = \mathbf{w}^*$ , and  $\mathbf{Z}^* = \overline{\mathbf{Z}} \in Sym$  verifying hypothesis (H4), such that  $\mathbf{g}(\mathbf{F}^*, \theta^*, \overline{\mathbf{Z}}, p^*) = \mathbf{0}$ . Using Lemma 1.3.8, we deduce the existence of a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  verifying equalities (1.27). Taking into account expressions (1.20) and (1.22), the second principle of thermodynamics written as in inequality (1.37) can be rewritten for  $p = p^*$  and  $t = 0$  as follows:

$$\rho^* \left[ \theta^* \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \overline{\mathbf{Z}}, p^*) - \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}^*, \theta^*, \overline{\mathbf{Z}}, p^*) \right] : \mathbf{g}(\mathbf{F}^*, \theta^*, \overline{\mathbf{Z}}, p^*) - \frac{1}{\theta^*} \widehat{\mathbf{q}}(\mathbf{F}^*, \theta^*, \overline{\mathbf{Z}}, \mathbf{w}^*, p^*) \cdot \mathbf{w}^* \geq 0,$$

with  $\rho^* = \rho(x^*, 0)$ . In consequence, thanks to hypothesis (H4), the positivity of the temperature and the heat flux independence with respect to  $\mathbf{Z}$ , we deduce heat conduction inequality (1.38).  $\square$

### 1.3.3 Writing the conservation laws

Hereafter, we consider the following concepts:

**Definition 1.3.11.** *The specific heat at constant deformation is the scalar field defined by*

$$c_F(x, t) = \widehat{c}_F(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p),$$

with

$$\widehat{c}_F(\mathbf{F}, \theta, \mathbf{Z}, p) = \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p).$$

**Definition 1.3.12.** *The specific Helmholtz free energy is the scalar field  $\psi$  defined by*

$$\psi(x, t) = \widehat{\psi}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p),$$

with

$$\widehat{\psi}(\mathbf{F}, \theta, \mathbf{Z}, p) = \widehat{e}(\mathbf{F}, \theta, \mathbf{Z}, p) - \widehat{s}(\mathbf{F}, \theta, \mathbf{Z}, p)\theta. \quad (1.41)$$

**Proposition 1.3.13.** *We have*

$$\widehat{s}(\mathbf{F}, \theta, \mathbf{Z}, p) = -\frac{\partial \widehat{\psi}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p), \quad (1.42)$$

$$\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) = \frac{\rho_0(p)}{\det \mathbf{F}} \frac{\partial \widehat{\psi}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^t, \quad (1.43)$$

$$\frac{\partial \widehat{\psi}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) + \frac{1}{\rho \theta} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) \cdot \mathbf{w} \leq 0, \quad (1.44)$$

for all  $\mathbf{F} \in \text{Lin}^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in \text{Sym}$ ,  $\mathbf{w} \in \mathcal{V}$  and  $p \in \mathcal{B}$ .

*Proof.* Applying the chain rule to expression (1.41), we have

$$\begin{aligned} \frac{\partial \widehat{\psi}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p)\theta - s, \\ \frac{\partial \widehat{\psi}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \frac{\partial \widehat{e}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p)\theta, \\ \frac{\partial \widehat{\psi}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) - \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p)\theta. \end{aligned}$$

Then, of the first equation thanks to equality (1.20) and taking into account Theorem 1.3.6, we deduce expression (1.42); of the second equation, by equality (1.22), we obtain expression (1.43) and finally, of the third equation, by equality (1.23), we have expression (1.44).  $\square$

**Corollary 1.3.14.** *Given a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$ , we get*

$$\begin{aligned} \dot{s}(x, t) &= -\frac{\det \mathbf{F}(p, t)}{\rho_0(p)} \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{L}(x, t) + \frac{1}{\theta(x, t)} c_F(x, t) \dot{\theta}(x, t) \\ &+ \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p), \quad p \in \mathcal{B} \text{ and } t \in \mathbb{R}. \end{aligned}$$

*Proof.* By equality (1.42) we deduce

$$\begin{aligned} \dot{s}(x, t) &= - \overbrace{\frac{\partial \psi}{\partial \theta}}(x, t) = - \frac{\partial^2 \widehat{\psi}}{\partial \mathbf{F} \partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{F}}(p, t) \\ &\quad - \frac{\partial^2 \widehat{\psi}}{\partial \theta^2}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) - \frac{\partial^2 \widehat{\psi}}{\partial \mathbf{Z} \partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{Z}}(x, t). \end{aligned}$$

Thanks to Proposition 1.3.13, Theorem 1.3.6, Lemma 1.3.7 and property (1.35), this expression can be rewritten as follows:

$$\begin{aligned} \dot{s}(x, t) &= - \frac{\partial}{\partial \theta} \left( \frac{\det \mathbf{F}(p, t)}{\rho_0(p)} \widehat{\mathbf{T}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \mathbf{F}^{-t}(p, t) \right) : \dot{\mathbf{F}}(p, t) \\ &\quad + \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) \\ &\quad + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{Z}}(x, t) \\ &= - \frac{\det \mathbf{F}(p, t)}{\rho_0(p)} \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{L}(p, t) \\ &\quad + \frac{1}{\theta(x, t)} \frac{\partial \widehat{e}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) \\ &\quad + \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p). \end{aligned}$$

Finally, using Definition 1.3.11 we conclude the result.  $\square$

**Theorem 1.3.15.** *For a thermoviscoelastic material with long memory, energy conservation equation (EC) is equivalent to*

$$\rho \theta \dot{s} + \rho \frac{\partial \widehat{\psi}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) = -\operatorname{div} \mathbf{q} + f \text{ in } \mathcal{T}. \quad (1.45)$$

*Proof.* Taking into account expression (1.29) and Theorem 1.3.6,  $\dot{e}$  can be written as follows:

$$\begin{aligned} \dot{e}(x, t) &= \theta(x, t) \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{F}}(p, t) \\ &\quad + \frac{\det \mathbf{F}(p, t)}{\rho_0(p)} \widehat{\mathbf{T}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \mathbf{F}^{-t}(p, t) : \dot{\mathbf{F}}(p, t) \\ &\quad + \theta(x, t) \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) \\ &\quad + \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p). \end{aligned}$$

Using Definition 1.3.12, this expression can be rewritten as follows:

$$\begin{aligned}
\dot{e}(x, t) = & \theta(x, t) \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{F}}(p, t) \\
& + \frac{\det \mathbf{F}(p, t)}{\rho_0(p)} \widehat{\mathbf{T}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \mathbf{F}^{-t}(p, t) : \dot{\mathbf{F}}(p, t) \\
& + \theta(x, t) \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \dot{\theta}(x, t) \\
& + \theta(x, t) \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) \\
& + \frac{\partial \widehat{\psi}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{g}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p).
\end{aligned}$$

Substituting the previous expression in energy conservation equation **(EC)**, using property (1.35), equality (1.24) and mass conservation equation **(MC)**, we have

$$\begin{aligned}
\rho \theta \frac{\partial \widehat{s}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \dot{\mathbf{F}} + \rho \theta \frac{\partial \widehat{s}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) \dot{\theta} + \rho \theta \frac{\partial \widehat{s}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \\
+ \rho \frac{\partial \widehat{\psi}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) = -\operatorname{div} \mathbf{q} + f \text{ in } \mathcal{T},
\end{aligned}$$

since  $\mathbf{T}$  is a symmetric tensor, so  $\mathbf{T} : \mathbf{L} = \mathbf{T} : \mathbf{D}$ .

Finally, by expression (1.30) we conclude the result.  $\square$

**Corollary 1.3.16.** *The energy conservation equation can be rewritten as*

$$\rho_{c_F} \dot{\theta} = \theta \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{L} - \rho \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) - \operatorname{div} \mathbf{q} + f \text{ in } \mathcal{T},$$

where the first two terms on the right-hand side of the equality represent the thermomechanical coupling phenomena.

*Proof.* We obtain the previous equality substituting in expression (1.45) the value of  $\dot{s}$  given in Corollary 1.3.14 and taking into account Definition 1.3.12.  $\square$

Summing up, under hypotheses **(H1)** and **(H2)**, the equilibrium equations for the thermo-viscoelastic materials with long memory verifying the second principle of thermodynamics are summarized in the following problem defined in the deformed configuration:

**Problem (DC)**

Find  $\rho$ ,  $\mathbf{v}$ ,  $\mathbf{T}$ ,  $\theta$  and  $\mathbf{Z}$  verifying:

$$\begin{cases}
\rho_0 = \rho \det \mathbf{F}, & (1.46a) \\
\rho \dot{\mathbf{v}} = \operatorname{div} \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) + \mathbf{b}, & (1.46b) \\
\rho_{c_F} \dot{\theta} = \theta \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{L} - \rho \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) \\
\quad - \operatorname{div} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \operatorname{grad} \theta, p) + f, & (1.46c)
\end{cases}$$

with

$$\dot{\mathbf{Z}} = \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p). \quad (1.47)$$

All these equations of Problem (DC) must be verified on the trajectory  $\mathcal{T}$ , which is an unknown domain.

### 1.3.4 The principle of material frame indifference and isotropy

In order to satisfy the principle of material frame indifference and isotropy, we consider for the associated constitutive laws the same assumptions considered by Bermúdez [19] and Coleman and Gurtin [30]; so, the associated response functions must verify the following properties for all  $\mathbf{F} \in Lin^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in Sym$ ,  $p \in \mathcal{B}$ ,  $\mathbf{Q} \in Orth^+$  and  $\mathbf{w} \in \mathcal{V}$ :

$$\begin{aligned} \widehat{e}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \widehat{e}(\mathbf{QF}, \theta, \mathbf{Z}, p), \\ \widehat{s}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \widehat{s}(\mathbf{QF}, \theta, \mathbf{Z}, p), \\ \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p)\mathbf{Q}^t &= \widehat{\mathbf{T}}(\mathbf{QF}, \theta, \mathbf{Z}, p), \\ \mathbf{Q}\widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) &= \widehat{\mathbf{q}}(\mathbf{QF}, \theta, \mathbf{Z}, \mathbf{Qw}, p), \\ \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \mathbf{g}(\mathbf{QF}, \theta, \mathbf{Z}, p). \end{aligned} \quad (1.48)$$

In these equalities, we suppose that the internal variable  $\mathbf{Z}$  is an ordered  $n \times n$  tuples of scalars ( $n$  being the dimension of space, i.e. a positive integer number), each of which remains invariant under a change in the observer; indeed,  $\mathbf{Z}$  cannot behave as a spatial tensor under a change of frame, since this would imply that

$$\mathbf{Z} \in Sym \text{ is transformed into } \mathbf{QZQ}^t, \forall \mathbf{Q} \in Orth^+, \quad (1.49)$$

but this relation is incompatible with the constitutive equation (1.47). If assertion (1.49) was certain and taking into account behaviour law (1.47) we would obtain:

$$(\mathbf{QZQ}^t)^\cdot = \mathbf{g}(\mathbf{QF}, \theta, \mathbf{QZQ}^t, p), \forall \mathbf{Q} \in Orth^+.$$

In particular, previous equality would be satisfied by  $\tilde{\mathbf{Q}}(t) \in Orth^+$  solution of Cauchy problem (1.25), with  $\overline{\mathbf{Q}} = \mathbf{I}$ ; so

$$\dot{\tilde{\mathbf{Q}}}(t)\mathbf{Z}\tilde{\mathbf{Q}}^t(t) + \tilde{\mathbf{Q}}(t)\dot{\mathbf{Z}}\tilde{\mathbf{Q}}^t(t) + \tilde{\mathbf{Q}}(t)\mathbf{Z}\dot{\tilde{\mathbf{Q}}}(t) = \mathbf{g}(\tilde{\mathbf{Q}}(t)\mathbf{F}, \theta, \tilde{\mathbf{Q}}(t)\mathbf{Z}\tilde{\mathbf{Q}}^t(t), p), \forall t \in \mathbb{R}.$$

Taking the time instant  $t = 0$ , we would get

$$\dot{\tilde{\mathbf{Q}}}(0)\mathbf{Z} + \dot{\mathbf{Z}} + \mathbf{Z}\dot{\tilde{\mathbf{Q}}}(0) = \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p),$$

and using again behaviour law (1.47) and the definition of Cauchy problem (1.25), we would have

$$\overline{\mathbf{W}}\mathbf{Z} - \mathbf{Z}\overline{\mathbf{W}} = \mathbf{0}, \forall \overline{\mathbf{W}} \in Skw.$$

Then, only the variables  $\mathbf{Z}$  could take the form  $\mathbf{Z} = \alpha\mathbf{I}$ ,  $\alpha$  being any scalar field.

In consequence, assertion (1.49) limits the choice of the initial state  $\mathbf{Z}_0$  and the choice of functions  $\mathbf{g}$  for constitutive law (1.47); in particular, it excludes all materials where the deviatoric tensor plays an important role. A similar question was analyzed by Coleman and Gurtin [30] obtaining an analogous conclusion when the specific internal variable is a vector.

## 1.4 Quasistatic thermoviscoelastic problem with linearized thermoviscoelastic response

In this section we will rewrite Problem (DC) in the reference configuration, which is a data of the problem. For this, we will apply the change of variable given by the First Piola-Kirchhoff transformation defined by the own motion  $x = X(p, t)$ . The obtained problem in the reference configuration is nonlinear and it will be linearized using a Taylor's formula around  $\mathbf{F} = \mathbf{I}$ , then around  $\theta = \theta_0$  and finally around  $\mathbf{Z} = \mathbf{Z}_0$ ;  $\mathbf{I}$ ,  $\theta_0$  and  $\mathbf{Z}_0$  being the values at the initial state of the deformation gradient, the temperature and the internal variable  $\mathbf{Z}$  respectively. The linearized equations are obtained by the same methodology used by Bermúdez [19] for Coleman-Noll materials. Nevertheless, there are two significant differences: firstly, we consider the internal variable  $\mathbf{Z}$  to take into account the viscoplastic history of the material. Secondly, in this part the linearization with respect to  $\theta$  is made only on the terms of the first order obtained from the linearization with respect to  $\mathbf{F}$ ; this procedure allows us to write the linearized thermomechanical problem with coefficients depending on temperature.

### 1.4.1 Changing to material or Lagrangian coordinates

In order to apply the change of variable  $x = X(p, t)$ , we need the following results (see Gurtin [65]):

**Theorem 1.4.1.** *Let  $\varphi$ ,  $\mathbf{w}$ ,  $\mathbf{H}$  be a scalar field, a vector field and a tensor field respectively, all them continuous in  $\mathcal{T}$ . Then, given any part  $\mathcal{P}$  of  $\mathcal{B}$ , it results:*

$$\bullet \int_{\mathcal{P}_t} \varphi(x, t) dV_x = \int_{\mathcal{P}} \det \mathbf{F}(p, t) \varphi_m(p, t) dV_p, \quad (1.50)$$

$$\bullet \int_{\partial \mathcal{P}_t} \mathbf{w}(x, t) \cdot \mathbf{m}(x) dA_x = \int_{\partial \mathcal{P}} \det \mathbf{F}(p, t) \mathbf{w}_m(p, t) \cdot \mathbf{F}^{-t}(p, t) \mathbf{n}(p) dA_p, \quad (1.51)$$

$$\bullet \int_{\partial \mathcal{P}_t} \mathbf{H}(x, t) \mathbf{m}(x) dA_x = \int_{\partial \mathcal{P}} \det \mathbf{F}(p, t) \mathbf{H}_m(p, t) \mathbf{F}^{-t}(p, t) \mathbf{n}(p) dA_p, \quad (1.52)$$

where  $\mathcal{P}_t = X(\mathcal{P}, t)$ ;  $\mathbf{m}$  and  $\mathbf{n}$  being the outward unit normal vectors to  $\partial \mathcal{P}_t$  and  $\partial \mathcal{P}$ , respectively.

**Lemma 1.4.2.** *Let  $\Phi$  be a continuous spatial field. Then for any part  $\mathcal{P}$  of  $\mathcal{B}$ , we have*

$$\int_{\mathcal{P}_t} \Phi(x, t) \rho(x, t) dV_x = \int_{\mathcal{P}} \Phi_m(p, t) \rho_0(p) dV_p.$$

**Definition 1.4.3.** *We define the First Piola-Kirchhoff stress tensor in the reference configuration as*

$$\mathbf{S}(p, t) = \det \mathbf{F}(p, t) \mathbf{T}(x, t) \mathbf{F}^{-t}(p, t), \quad x = \mathbf{X}(p, t), \quad \text{with } p \in \mathcal{B} \text{ and } t \in \mathbb{R}. \quad (1.53)$$

*The response function for this tensor is:*

$$\widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) = \widehat{\mathbf{S}}_{elas}(\mathbf{F}, \theta, p) + \widehat{\mathbf{S}}_{ther}(\mathbf{F}, \theta, p) + \widehat{\mathbf{S}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p), \quad (1.54)$$

defined for all  $\mathbf{F} \in \text{Lin}^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in \text{Sym}$  and  $p \in \mathcal{B}$  with

$$\widehat{\mathbf{S}}_{elas}(\mathbf{F}, \theta, p) = \det \mathbf{F} \widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta, p) \mathbf{F}^{-t}, \quad (1.55)$$

$$\widehat{\mathbf{S}}_{ther}(\mathbf{F}, \theta, p) = \det \mathbf{F} \widehat{\mathbf{T}}_{ther}(\mathbf{F}, \theta, p) \mathbf{F}^{-t}, \quad (1.56)$$

$$\widehat{\mathbf{S}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p) = \det \mathbf{F} \widehat{\mathbf{T}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^{-t}. \quad (1.57)$$

Because of the First Piola-Kirchhoff stress tensor  $\mathbf{S}$  is not symmetric, we introduce the following symmetric tensor.

**Definition 1.4.4.** *The Second Piola-Kirchhoff stress tensor is*

$$\boldsymbol{\Sigma}(p, t) = \mathbf{F}^{-1}(p, t) \mathbf{S}(p, t), \quad p \in \mathcal{B}, \quad t \in \mathbb{R}, \quad (1.58)$$

whose response function is defined as

$$\widehat{\boldsymbol{\Sigma}}(\mathbf{F}, \theta, \mathbf{Z}, p) = \widehat{\boldsymbol{\Sigma}}_{elas}(\mathbf{F}, \theta, p) + \widehat{\boldsymbol{\Sigma}}_{ther}(\mathbf{F}, \theta, p) + \widehat{\boldsymbol{\Sigma}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p). \quad (1.59)$$

$\widehat{\boldsymbol{\Sigma}}_{elas}$ ,  $\widehat{\boldsymbol{\Sigma}}_{ther}$  and  $\widehat{\boldsymbol{\Sigma}}_{plas}$  are obtained multiplying equalities (1.55), (1.56) and (1.57) on the left by  $\mathbf{F}^{-1}$ , respectively.

Therefore, thanks to equality (1.53) and using hypothesis **(H3)** we can rewrite  $\dot{\mathbf{Z}}$  in terms of the First Piola-Kirchhoff stress tensor:

$$\begin{aligned} \dot{\mathbf{Z}}(x, t) &= \mathbf{h}(\mathbf{F}(p, t), \theta(x, t), \mathbf{T}(x, t), p) \\ &= \mathbf{h}\left(\mathbf{F}(p, t), \theta_m(p, t), \frac{1}{\det \mathbf{F}(p, t)} \mathbf{S}(p, t) \mathbf{F}^t(p, t), p\right). \end{aligned}$$

Then, defining  $\tilde{\mathbf{h}} : \text{Lin}^+ \times \mathbb{R}^+ \times \text{Lin} \times \mathcal{B} \rightarrow \text{Sym}$  as

$$\tilde{\mathbf{h}}(\mathbf{F}, \theta_m, \mathbf{S}, p) = \mathbf{h}\left(\mathbf{F}, \theta_m, \frac{1}{\det \mathbf{F}} \mathbf{S} \mathbf{F}^t, p\right), \quad (1.60)$$

constitutive law (1.18) can be replaced with

$$\dot{\mathbf{Z}}_m(p, t) = \tilde{\mathbf{h}}(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{S}(p, t), p), \quad \text{with } p \in \mathcal{B} \text{ and } t \in \mathbb{R}.$$

### The motion equation in the reference configuration

Hereinafter, we assume hypotheses **(H1)**-**(H3)** and the second principle of thermodynamics.

**Theorem 1.4.5.** *Motion equation (1.46b) for thermoviscoelastic materials with long memory in the reference configuration is given by*

$$\rho_0 \ddot{\mathbf{u}} = \text{Div} \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) + \mathbf{b}_* \text{ in } \mathcal{B} \times \mathbb{R},$$

where  $\mathbf{b}_*$  is the density of body forces in the reference configuration defined as

$$\mathbf{b}_*(p, t) = \det \mathbf{F}(p, t) \mathbf{b}_m(p, t).$$

*Proof.* Let  $\mathcal{P}$  be any part of  $\mathcal{B}$ . Let us integrate over  $\mathcal{P}_t \subset \mathcal{B}_t$  each term of equation (1.46b), obtaining

$$\int_{\mathcal{P}_t} \rho(x, t) \dot{\mathbf{v}}(x, t) dV_x = \int_{\mathcal{P}_t} \operatorname{div} \mathbf{T}(x, t) dV_x + \int_{\mathcal{P}_t} \mathbf{b}(x, t) dV_x. \quad (1.61)$$

For the member on the left-hand side of the previous equality applying Lemma 1.4.2, we have

$$\int_{\mathcal{P}_t} \rho(x, t) \dot{\mathbf{v}}(x, t) dV_x = \int_{\mathcal{P}} \rho_0(p) \dot{\mathbf{v}}_m(p, t) dV_p = \int_{\mathcal{P}} \rho_0(p) \ddot{\mathbf{u}}(p, t) dV_p,$$

since  $\mathbf{v}_m = \dot{\mathbf{u}}$ .

For the first term on the right-hand side of equality (1.61), using the Divergence Theorem, equation (1.52) and the definition of First Piola-Kirchhoff stress tensor, it results:

$$\begin{aligned} \int_{\mathcal{P}_t} \operatorname{div} \mathbf{T}(x, t) dV_x &= \int_{\partial \mathcal{P}_t} \mathbf{T}(x, t) \mathbf{m}(x) dA_x = \int_{\partial \mathcal{P}} \det \mathbf{F}(p, t) \mathbf{T}_m(p, t) \mathbf{F}^{-t}(p, t) \mathbf{n}(p) dA_p \\ &= \int_{\mathcal{P}} \operatorname{Div} (\det \mathbf{F}(p, t) \mathbf{T}_m(p, t) \mathbf{F}^{-t}(p, t)) dV_p = \int_{\mathcal{P}} \operatorname{Div} \mathbf{S}(p, t) dV_p. \end{aligned}$$

Finally, for the last term of the second member of equality (1.61), we get

$$\int_{\mathcal{P}_t} \mathbf{b}(x, t) dV_x = \int_{\mathcal{P}} \det \mathbf{F}(p, t) \mathbf{b}_m(p, t) dV_p = \int_{\mathcal{P}} \mathbf{b}_*(p, t) dV_p.$$

Summing up, we obtain

$$\int_{\mathcal{P}} \rho_0(p) \ddot{\mathbf{u}}(p, t) dV_p = \int_{\mathcal{P}} \operatorname{Div} \mathbf{S}(p, t) dV_p + \int_{\mathcal{P}} \mathbf{b}_*(p, t) dV_p, \quad \forall \mathcal{P} \subset \mathcal{B}, \quad \forall t \in \mathbb{R},$$

and applying the Localization Theorem we can conclude the result.  $\square$

## The energy equation in the reference configuration

**Theorem 1.4.6.** *Energy equation (1.46c) for thermoviscoelastic materials with long memory in the reference configuration is*

$$\begin{aligned} \rho_0 c_{F_m} \dot{\theta}_m &= \det \mathbf{F} \theta_m \frac{\partial \hat{\mathbf{T}}}{\partial \theta}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \mathbf{L}_m - \rho_0 \frac{\partial \hat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \dot{\mathbf{Z}}_m \\ &\quad - \operatorname{Div} \hat{\mathbf{q}}_*(\mathbf{F}, \theta_m, \mathbf{Z}_m, \nabla \theta_m, p) + f_* \text{ in } \mathcal{B} \times \mathbb{R}, \end{aligned} \quad (1.62)$$

with

$$\begin{aligned} \hat{\mathbf{q}}_*(\mathbf{F}, \theta_m, \mathbf{Z}_m, \mathbf{w}_m, p) &= \det \mathbf{F} \mathbf{F}^{-1} \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{F}^{-t} \mathbf{w}, p), \\ f_*(p, t) &= \det \mathbf{F}(p, t) f_m(p, t). \end{aligned} \quad (1.63)$$

*Proof.* Let  $\mathcal{P}$  be any part of  $\mathcal{B}$ . Let us integrate over  $\mathcal{P}_t \subset \mathcal{B}_t$  each term of equation (1.46c), we have

$$\begin{aligned} \int_{\mathcal{P}_t} \rho(x, t) c_F(x, t) \dot{\theta}(x, t) dV_x &= \int_{\mathcal{P}_t} \theta(x, t) \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{L}(x, t) dV_x \\ &\quad - \int_{\mathcal{P}_t} \rho(x, t) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{Z}}(x, t) dV_x \\ &\quad - \int_{\mathcal{P}_t} \operatorname{div} \mathbf{q}(x, t) dV_x + \int_{\mathcal{P}_t} f(x, t) dV_x. \end{aligned} \quad (1.64)$$

For the member on the left-hand side and for the second term on the right-hand side of the previous expression, applying the variable change theorem given by (1.50) and taking into account mass conservation equation (1.46a), we get:

$$\begin{aligned} \int_{\mathcal{P}_t} \rho(x, t) c_F(x, t) \dot{\theta}(x, t) dV_x &= \int_{\mathcal{P}} \det \mathbf{F}(p, t) \rho_m(p, t) c_{F_m}(p, t) \dot{\theta}_m(p, t) dV_p \\ &= \int_{\mathcal{P}} \rho_0(p) c_{F_m}(p, t) \dot{\theta}_m(p, t) dV_p, \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathcal{P}_t} \rho(x, t) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \dot{\mathbf{Z}}(x, t) dV_x \\ &= \int_{\mathcal{P}} \det \mathbf{F}(p, t) \rho_m(p, t) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{Z}_m(p, t), p) : \dot{\mathbf{Z}}_m(p, t) dV_p \\ &= \int_{\mathcal{P}} \rho_0(p) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{Z}_m(p, t), p) : \dot{\mathbf{Z}}_m(p, t) dV_p. \end{aligned}$$

For the first term and the last term on the right-hand side of expression (1.64), we use again equality (1.50) obtaining:

$$\begin{aligned} &\int_{\mathcal{P}_t} \theta(x, t) \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}(p, t), \theta(x, t), \mathbf{Z}(x, t), p) : \mathbf{L}(x, t) dV_x \\ &= \int_{\mathcal{P}} \det \mathbf{F}(p, t) \theta_m(p, t) \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{Z}_m(p, t), p) : \mathbf{L}_m(p, t) dV_p, \end{aligned}$$

and

$$\int_{\mathcal{P}_t} f(x, t) dV_x = \int_{\mathcal{P}} \det \mathbf{F}(p, t) f_m(p, t) dV_p = \int_{\mathcal{P}} f_*(p, t) dV_p.$$

Finally, for the third term on the right-hand side of equality (1.64), taking into account the

Divergence Theorem and equality (1.51), we deduce

$$\begin{aligned}
\int_{\mathcal{P}_t} \operatorname{div} \mathbf{q}(x, t) dV_x &= \int_{\partial \mathcal{P}_t} \mathbf{q}(x, t) \cdot \mathbf{m}(x) dA_x = \int_{\partial \mathcal{P}} \det \mathbf{F}(p, t) \mathbf{q}_m(p, t) \cdot \mathbf{F}^{-t}(p, t) \mathbf{n}(p) dA_p \\
&= \int_{\partial \mathcal{P}} \det \mathbf{F}(p, t) \mathbf{F}^{-1}(p, t) \mathbf{q}_m(p, t) \cdot \mathbf{n}(p) dA_p \\
&= \int_{\mathcal{P}} \operatorname{Div} (\det \mathbf{F}(p, t) \mathbf{F}^{-1}(p, t) \mathbf{q}_m(p, t)) dV_p \\
&= \int_{\mathcal{P}} \operatorname{Div} \widehat{\mathbf{q}}_*(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{Z}_m(p, t), \nabla \theta_m(p, t), p) dV_p,
\end{aligned}$$

thanks to expression (1.63) and the following equalities

$$\widehat{\mathbf{q}}_*(\mathbf{F}, \theta_m, \mathbf{Z}_m, \nabla \theta_m, p) = \det \mathbf{F} \mathbf{F}^{-1} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{F}^{-t} \nabla \theta_m, p) = \det \mathbf{F} \mathbf{F}^{-1} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \operatorname{grad} \theta, p).$$

Summing up, we get

$$\begin{aligned}
&\int_{\mathcal{P}} \rho_0(p) c_{F_m}(p, t) \dot{\theta}_m(p, t) dV_p \\
&= \int_{\mathcal{P}} \det \mathbf{F}(p, t) \theta_m(p, t) \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{Z}_m(p, t), p) : \mathbf{L}_m(p, t) dV_p \\
&\quad - \int_{\mathcal{P}} \rho_0(p) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}(p, t), \theta_m(p, t), \mathbf{Z}_m(p, t), p) : \dot{\mathbf{Z}}_m(p, t) dV_p \\
&\quad - \int_{\mathcal{P}} \operatorname{Div} \mathbf{q}_*(p, t) dV_p + \int_{\mathcal{P}} f_*(p, t) dV_p, \quad \forall \mathcal{P} \in \mathcal{B}, \forall t \in \mathbb{R}.
\end{aligned}$$

Using the Localization Theorem we conclude expression (1.62).  $\square$

**Corollary 1.4.7.** *The energy conservation equation rewritten in terms of  $\widehat{\mathbf{S}}$  is*

$$\begin{aligned}
\rho_0 c_{F_m} \dot{\theta}_m &= \theta_m \frac{\partial \widehat{\mathbf{S}}}{\partial \theta}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \dot{\mathbf{Z}}_m \\
&\quad - \operatorname{Div} \widehat{\mathbf{q}}_*(\mathbf{F}, \theta_m, \mathbf{Z}_m, \nabla \theta_m, p) + f_* \text{ in } \mathcal{B} \times \mathbb{R}.
\end{aligned}$$

*Proof.* From equation (1.62), taking into account equality (1.24) and using property (1.35), we obtain

$$\begin{aligned}
\rho_0 c_{F_m} \dot{\theta}_m &= \det \mathbf{F} \theta_m \frac{\partial \widehat{\mathbf{T}}}{\partial \theta}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) \mathbf{F}^{-t} : \dot{\mathbf{F}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \dot{\mathbf{Z}}_m \\
&\quad - \operatorname{Div} \widehat{\mathbf{q}}_*(\mathbf{F}, \theta_m, \mathbf{Z}_m, \nabla \theta_m, p) + f_*.
\end{aligned}$$

From definition (1.53) and since  $\dot{\mathbf{F}} = \overline{(\mathbf{I} + \nabla \mathbf{u})} = \nabla \dot{\mathbf{u}}$  we can conclude the result.  $\square$

Summarizing up, the equilibrium equations for the thermoviscoelastic materials with long memory define the following problem in the reference configuration:

**Problem (RC)**

Find  $\mathbf{u}$ ,  $\mathbf{S}$ ,  $\theta_m$  and  $\mathbf{Z}_m$  verifying:

$$\begin{cases} \rho_0 \ddot{\mathbf{u}} = \text{Div } \widehat{\mathbf{S}}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) + \mathbf{b}_*, & (1.65a) \\ \rho_0 c_{F_m} \dot{\theta}_m = \theta_m \frac{\partial \widehat{\mathbf{S}}}{\partial \theta}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta_m, \mathbf{Z}_m, p) : \dot{\mathbf{h}}(\mathbf{F}, \theta_m, \mathbf{S}, p) \\ \quad - \text{Div } \widehat{\mathbf{q}}_*(\mathbf{F}, \theta_m, \mathbf{Z}_m, \nabla \theta_m, p) + f_*, & (1.65b) \end{cases}$$

where  $\dot{\mathbf{Z}}_m = \dot{\mathbf{h}}(\mathbf{F}, \theta_m, \mathbf{S}, p)$ .

**Remark 1.4.8.** Hereafter, we suppose that the forces  $\mathbf{b}_*$  and  $f_*$  are dead forces i.e., they are independent of the deformation.

**Remark 1.4.9.** Equations (1.65a) and (1.65b) are similar to those obtained by Tzavaras [97] and Helm and Haupt [67]. Tzavaras [97] considers a vectorial internal variable whose definition depends on considered models to obtain several relaxation approximations of conservation laws; in his work the heat flux is given by a Fourier law. Helm and Haupt [67] consider a scalar internal variable and some stress internal variables to take into account the stress-induced martensitic phase transitions.

For Coleman-Noll materials the response function for the First Piola-Kirchhoff stress tensor includes a viscous part depending not only on  $\mathbf{F}$ ,  $\theta$ ,  $p$  but also on  $\dot{\mathbf{F}}$  (see Bermúdez [19]) and the dependence with respect to  $\mathbf{Z}$  is not considered. Because of this, the energy equation for Coleman-Noll materials differs from equality (1.65b) in the first two terms of the right-hand side of the equality.

### 1.4.2 Thermoviscoelastic linearizations of the motion and energy equations

It is necessary to take into account that in materials processing, for example in the aluminium casting, the temperature variations are very strong with respect to time and space, then it is convenient to keep without linearizing the energy equation. For this purpose, we present a first linearization in which we only linearize the First Piola-Kirchhoff stress tensor with respect to the deformation gradient around the identity.

In other processes in which it is possible to suppose  $\theta - \theta_0 = O(\varepsilon)$ , or if we want, for example, to use an incremental method in order to make numerical simulations, it is justified the using of a second linearization of the thermal contributions with respect to temperature without eliminating the temperature dependence of the mechanical coefficients.

Finally, in order to simplify the numerical processing, we propose neglecting the nonlinearities with respect to tensorial internal variable but we keep the nonlinear dependence that the own variable has with respect to the First Piola-Kirchhoff stress tensor. This third linealization is numerically used in Barral *et al.* [11].

From now on, in order to simplify the notation, we omit the subscript  $m$  which denotes the material description of the corresponding spatial field.

We choose as reference configuration the initial equilibrium position of the body:  $\mathcal{B}_0 = \mathcal{B}$ , and  $\mathbf{T}_0$ ,  $\mathbf{b}_0$ ,  $\theta_0$ ,  $\mathbf{q}_0$ ,  $f_0$ ,  $\rho_0$  and  $\mathbf{Z}_0$  the initial values for the Cauchy stress tensor, the body forces,

the temperature, the heat flux, the heat source, the density of mass and the internal variable  $\mathbf{Z}$ , respectively. We suppose that they verify the equilibrium equations corresponding to the static problem at the initial time:

$$\mathbf{T}_0(p) = \widehat{\mathbf{T}}(\mathbf{I}, \theta_0(p), \mathbf{Z}_0(p), p), \quad \text{Div } \mathbf{T}_0 + \mathbf{b}_0 = \mathbf{0}, \quad -\text{Div } \mathbf{q}_0 + f_0 = 0 \text{ in } \mathcal{B}_0. \quad (1.66)$$

In this section we consider a thermodynamic process for  $\mathcal{B}$  with small changes with respect to the initial state, i.e. small displacements  $\nabla \mathbf{u} = O(\varepsilon)$ , small temperature changes  $\theta - \theta_0 = O(\varepsilon)$  and small changes of the internal variable  $\mathbf{Z} - \mathbf{Z}_0 = O(\varepsilon)$ ,  $\varepsilon$  being a small parameter. Our aim is to linearize the equilibrium equations with respect to the initial equilibrium position.

### Linearizing the motion equation with respect to $\mathbf{F}$ (1LFM)

We consider a thermodynamic process  $(X, \mathbf{T}, \mathbf{b}, e, \theta, \mathbf{q}, f, s, \mathbf{Z}) \in \mathcal{C}$  such that at the initial time  $X(p, 0) = p$  and  $\mathbf{F}(p, 0) = \mathbf{I}$ . In this first linearization we consider that the deformation gradient is a small perturbation of its initial state, i.e.,  $\nabla \mathbf{u} = O(\varepsilon)$ ,  $\varepsilon$  being a small parameter.

We take into account the following results which are presented in terms of  $\widehat{\mathbf{S}}$  and  $\widehat{\mathbf{T}}$ , but they are also satisfied replacing  $\widehat{\mathbf{S}}$  and  $\widehat{\mathbf{T}}$  with  $\widehat{\mathbf{S}}_{elas}$ ,  $\widehat{\mathbf{T}}_{elas}$  or  $\widehat{\mathbf{S}}_{ther}$ ,  $\widehat{\mathbf{T}}_{ther}$  or  $\widehat{\mathbf{S}}_{plas}$ ,  $\widehat{\mathbf{T}}_{plas}$  respectively, omitting the dependence on  $\mathbf{Z}$  when it is necessary:

**Proposition 1.4.10.** *The response function  $\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p)$ , satisfies*

$$\frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p)(\mathbf{W}\mathbf{F}) = \mathbf{W}\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) + \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p)\mathbf{W}^t,$$

for all  $\mathbf{F} \in Lin^+$ ,  $\mathbf{W} \in Skw$  and  $\theta \in \mathbb{R}^+$ .

*Proof.* Let us consider the ordinary differential equation  $\dot{\mathbf{Q}}(t) = \mathbf{W}\mathbf{Q}(t)$  with initial condition  $\mathbf{Q}(0) = \mathbf{I}$ , where  $\mathbf{W} \in Skw$ . Thanks to item iii) from Lemma 1.3.7, we know that  $\mathbf{Q}(t)$  is a rotation for all  $t$  and  $\dot{\mathbf{Q}}(0) = \mathbf{W}$ .

Considering the derivative with respect to  $t$  of equation (1.48), we have

$$\frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{Q}\mathbf{F}, \theta, \mathbf{Z}, p)(\dot{\mathbf{Q}}\mathbf{F}) = \dot{\mathbf{Q}}(t)\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p)\mathbf{Q}^t(t) + \mathbf{Q}(t)\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p)\dot{\mathbf{Q}}^t(t),$$

and taking  $t = 0$  in the previous equality we conclude the result.  $\square$

**Proposition 1.4.11.** *We have*

$$\begin{aligned} \frac{\partial \widehat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u} = & \text{Div } \mathbf{u} \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) + \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}) \\ & - \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p)\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{W}\widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p), \end{aligned} \quad (1.67)$$

for all  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in Sym$  and  $p \in \mathcal{B}$ , where  $\boldsymbol{\varepsilon}(\mathbf{u})$  is the linearized deformation tensor:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^t),$$

and  $\mathbf{W}$  its skew part:

$$\mathbf{W} = \frac{1}{2} (\nabla \mathbf{u} - \nabla \mathbf{u}^t).$$

*Proof.* Taking into account Definition 1.53, we can deduce

$$\widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^t = \det \mathbf{F} \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p),$$

and applying the product rule of the differential calculus we obtain:

$$\begin{aligned} \frac{\partial \widehat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{U} \mathbf{F}^t + \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{U}^t \\ = \det \mathbf{F} \operatorname{tr}(\mathbf{U} \mathbf{F}^{-1}) \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) + \det \mathbf{F} \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{U}. \end{aligned}$$

If we take  $\mathbf{F} = \mathbf{I}$  and  $\mathbf{U} = \nabla \mathbf{u}$ , we get

$$\begin{aligned} \frac{\partial \widehat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u} = - \widehat{\mathbf{S}}(\mathbf{I}, \theta, \mathbf{Z}, p) \nabla \mathbf{u}^t + \operatorname{tr}(\nabla \mathbf{u}) \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \\ + \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u}. \end{aligned} \quad (1.68)$$

Considering equalities (1.55)-(1.57) we can deduce  $\widehat{\mathbf{S}}(\mathbf{I}, \theta, \mathbf{Z}, p) = \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p)$  and by definition the linearized deformation tensor and its skew part (see Proposition 1.4.11), we obtain  $\nabla \mathbf{u} = \boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{W}$  and  $\nabla \mathbf{u}^t = \boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{W}$ , and we can rewrite expression (1.68) as follows:

$$\begin{aligned} \frac{\partial \widehat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u} = - \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{W}) + \operatorname{Div} \mathbf{u} \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \\ + \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : (\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{W}). \end{aligned}$$

Taking  $\mathbf{F} = \mathbf{I}$  in Proposition 1.4.10 and since the differential is a linear mapping, the previous equality takes the form

$$\begin{aligned} \frac{\partial \widehat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u} = - \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \boldsymbol{\varepsilon}(\mathbf{u}) + \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \mathbf{W} + \operatorname{Div} \mathbf{u} \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \\ + \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{W} \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) + \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \mathbf{W}^t. \end{aligned}$$

As  $\mathbf{W} \in \text{Skew}$ , then  $\mathbf{W} + \mathbf{W}^t = \mathbf{0}$ , therefore we conclude expression (1.67). □

**Corollary 1.4.12.** *We have*

$$\begin{aligned} \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) = (\mathbf{I} + \mathbf{W}) \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) + \operatorname{Div} \mathbf{u} \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) - \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \boldsymbol{\varepsilon}(\mathbf{u}) \\ + \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + o(\nabla \mathbf{u}). \end{aligned} \quad (1.69)$$

*Proof.* The previous equality is deduced applying a Taylor's formula around the point  $(\mathbf{I}, \theta, \mathbf{Z}, p)$  to the response function  $\widehat{\mathbf{S}}$ :

$$\widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) = \widehat{\mathbf{S}}(\mathbf{I}, \theta, \mathbf{Z}, p) + \frac{\partial \widehat{\mathbf{S}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u} + o(\nabla \mathbf{u}), \quad (1.70)$$

taking into account expression (1.53) and thanks to Proposition 1.4.11. □

**Definition 1.4.13.** *The elasticity tensor is defined as*

$$\mathbf{C}(p, t) = \widehat{\mathbf{C}}(\theta(p, t), p), \quad p \in \mathcal{B}, \quad t \in \mathbb{R},$$

where the operator  $\widehat{\mathbf{C}}(\theta, p) \in \mathcal{L}(\text{Lin}, \text{Sym})$  is given by

$$\widehat{\mathbf{C}}(\theta(p, t), p) = \frac{\partial \widehat{\mathbf{T}}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta(p, t), p).$$

Considering equation (1.65a), substituting  $\mathbf{S}$  by equality (1.54), we get

$$\rho_0 \ddot{\mathbf{u}} - \text{Div} (\widehat{\mathbf{S}}_{elas}(\mathbf{F}, \theta, p) + \widehat{\mathbf{S}}_{ther}(\mathbf{F}, \theta, p) + \widehat{\mathbf{S}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p)) = \mathbf{b}_*. \quad (1.71)$$

Thanks to Corollary 1.4.12, Definition 1.4.13, equality (1.19) and neglecting the terms  $o(\nabla \mathbf{u}) = o(\varepsilon)$ , we can approximate motion equation (1.71) by its linearized one with respect to the deformation gradient as follows:

First linearization of the motion equation (1FLM)

$$\rho_0 \ddot{\mathbf{u}} - \text{Div} \mathbf{S}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \mathbf{b}_*,$$

$\mathbf{S}^*$  being the linearized part of the First Piola-Kirchhoff stress tensor (see equality (1.69)),

$$\begin{aligned} \mathbf{S}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) + \text{Div} \mathbf{u} \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) - \widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p) \varepsilon(\mathbf{u}) \\ & + \widehat{\mathbf{C}}(\theta, p) : \varepsilon(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \varepsilon(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \varepsilon(\mathbf{u}), \end{aligned} \quad (1.72)$$

with  $\widehat{\mathbf{T}}(\mathbf{I}, \theta, \mathbf{Z}, p)$  deduced from equality (1.19).

**Corollary 1.4.14.** *The quasistatic motion equation associated with (1FLM) in terms of the Second Piola-Kirchhoff stress tensor is*

$$- \text{Div} \mathbf{S}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \mathbf{b}_*, \quad (1.73)$$

$\mathbf{S}^*$  defined by equality (1.72) written in terms of the Second Piola-Kirchhoff stress tensor,

$$\begin{aligned} \mathbf{S}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) + \text{Div} \mathbf{u} \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) - \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) \varepsilon(\mathbf{u}) \\ & + \widehat{\mathbf{C}}(\theta, p) : \varepsilon(\mathbf{u}) + (\mathbf{I} + \mathbf{W} + \varepsilon(\mathbf{u})) \widehat{\Sigma}_{ther}(\mathbf{I}, \theta, p) + \frac{\partial \widehat{\Sigma}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \varepsilon(\mathbf{u}) \\ & + (\mathbf{I} + \mathbf{W} + \varepsilon(\mathbf{u})) \widehat{\Sigma}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) + \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \varepsilon(\mathbf{u}). \end{aligned} \quad (1.74)$$

*Proof.* In order to simplify the proof, we consider  $\widehat{\mathbf{T}}$  and  $\widehat{\Sigma}$  instead of  $\widehat{\mathbf{T}}_{ther}$  or  $\widehat{\mathbf{T}}_{plas}$  and  $\widehat{\Sigma}_{ther}$  or  $\widehat{\Sigma}_{plas}$  respectively, omitting the dependence on  $\mathbf{Z}$  in  $\widehat{\mathbf{T}}_{ther}$  and  $\widehat{\Sigma}_{ther}$ .

In quasistatic case, inertia term  $\rho_0 \ddot{\mathbf{u}}$  is neglected.

From equalities (1.53) and (1.58), we deduce

$$\widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) \det \mathbf{F} = \mathbf{F} \widehat{\Sigma}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^t, \quad (1.75)$$

and applying the product rule of differential calculus we obtain:

$$\begin{aligned} \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) &: \mathbf{U} \det \mathbf{F} + \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) \det \mathbf{F} \operatorname{tr}(\mathbf{U} \mathbf{F}^{-1}) \\ &= \mathbf{U} \widehat{\boldsymbol{\Sigma}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^t + \mathbf{F} \frac{\partial \widehat{\boldsymbol{\Sigma}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, p) : \mathbf{U} \mathbf{F}^t + \mathbf{F} \widehat{\boldsymbol{\Sigma}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{U}^t. \end{aligned}$$

Taking  $\mathbf{F} = \mathbf{I}$ ,  $\mathbf{U} = \boldsymbol{\varepsilon}(\mathbf{u})$ , and thanks to equality (1.75) we get

$$\begin{aligned} \frac{\partial \widehat{\mathbf{T}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}) &= -\operatorname{Div} \mathbf{u} \widehat{\boldsymbol{\Sigma}}(\mathbf{I}, \theta, \mathbf{Z}, p) + \boldsymbol{\varepsilon}(\mathbf{u}) \widehat{\boldsymbol{\Sigma}}(\mathbf{I}, \theta, \mathbf{Z}, p) \\ &\quad + \widehat{\boldsymbol{\Sigma}}(\mathbf{I}, \theta, \mathbf{Z}, p) \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\boldsymbol{\Sigma}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}). \end{aligned} \quad (1.76)$$

Finally, replacing in first linearization (1FLM)  $\frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \mathbf{F}}$  and  $\frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}$  with its corresponding expressions deduced from equality (1.76), we conclude the result thanks to expression (1.75).  $\square$

### Linearizing the energy equation with respect to $\mathbf{F}$ (1FLE)

**Proposition 1.4.15.** *We have*

$$\begin{aligned} \frac{\partial \mathbf{q}^*}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p) &= -\nabla \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p) + \operatorname{Div} \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p) \\ &\quad + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p)(\nabla \mathbf{u}) - \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p) \nabla \mathbf{u}^t \nabla \theta, \end{aligned}$$

for all  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in \operatorname{Sym}$ ,  $\mathbf{w} \in \mathcal{V}$  and  $p \in \mathcal{B}$ .

*Proof.* Taking into account equality (1.63), we can deduce

$$\mathbf{F} \widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) = \det \mathbf{F} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{F}^{-t} \mathbf{w}, p),$$

and applying the product rule of the differential calculus we obtain:

$$\begin{aligned} \mathbf{U} \widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) + \mathbf{F} \frac{\partial \widehat{\mathbf{q}}_*}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p)(\mathbf{U}) &= \det \mathbf{F} \operatorname{tr}(\mathbf{U} \mathbf{F}^{-1}) \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{F}^{-t} \mathbf{w}, p) \\ + \det \mathbf{F} \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{F}^{-t} \mathbf{w}, p)(\mathbf{U}) + \det \mathbf{F} \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{F}^{-t} \mathbf{w}, p) &(-\mathbf{F}^{-t} \mathbf{U}^t \mathbf{F}^{-t} \mathbf{w}). \end{aligned}$$

If we take  $\mathbf{F} = \mathbf{I}$ ,  $\mathbf{w} = \nabla \theta$  and  $\mathbf{U} = \nabla \mathbf{u}$ , we get

$$\begin{aligned} \frac{\partial \widehat{\mathbf{q}}_*}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p)(\nabla \mathbf{u}) &= -\nabla \mathbf{u} \widehat{\mathbf{q}}_*(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) + \operatorname{tr}(\nabla \mathbf{u}) \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) \\ &\quad + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p)(\nabla \mathbf{u}) - \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) \nabla \mathbf{u}^t \nabla \theta, \end{aligned}$$

and since  $\widehat{\mathbf{q}}_*(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) = \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p)$ , we can conclude the result.  $\square$

**Proposition 1.4.16.** *The approximation of the heat flux is given by*

$$\mathbf{q}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) = \widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \nabla \theta, p),$$

where  $\widehat{\mathbf{q}}_*$  is the linearization of the heat flux around  $\mathbf{F} = \mathbf{I}$  as follows:

$$\begin{aligned} \widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \nabla \theta, p) &= \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) - \nabla \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) \\ &\quad + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p)(\nabla \mathbf{u}) - \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) \nabla \mathbf{u}^t \nabla \theta + o(\nabla \mathbf{u}). \end{aligned} \quad (1.77)$$

*Proof.* It is deduced applying a Taylor's formula around the point  $(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p)$  to the response function  $\widehat{\mathbf{q}}_*$ :

$$\widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) = \widehat{\mathbf{q}}_*(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p) + \frac{\partial \widehat{\mathbf{q}}_*}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p)(\nabla \mathbf{u}) + o(\nabla \mathbf{u}), \quad (1.78)$$

using Proposition 1.4.15, equality (1.63) and taking  $\mathbf{w} = \nabla \theta$ .  $\square$

Considering equation (1.65b), equalities (1.19), (1.54), (1.60), Corollary 1.4.12, expressions (1.72), (1.77) and neglecting the terms  $o(\nabla \mathbf{u}) = o(\varepsilon)$ , we have:

First linearization of the energy equation (1FLE)

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} &= \theta \frac{\partial \mathbf{S}^*}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\ &\quad - \text{Div } \mathbf{q}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*. \end{aligned}$$

**Corollary 1.4.17.** *The energy equation associated with (1FLE) in terms of the Second Piola-Kirchhoff stress tensor is*

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} &= \theta \frac{\partial \mathbf{S}^*}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\ &\quad - \text{Div } \mathbf{q}^*(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*, \end{aligned} \quad (1.79)$$

with  $\mathbf{S}^*$  given by equality (1.74).

*Proof.* It is deduced analogously to the proof of Corollary 1.4.14, using expression (1FLE), equality (1.60), considering expressions (1.75) and (1.76) for  $\widehat{\mathbf{T}}_{ther}$  and  $\widehat{\mathbf{T}}_{plas}$ , taking into account that  $\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) = \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p)$  and Definition 1.4.13.  $\square$

**Obtaining a second linearization with respect to  $\theta$**

This second linearization is done with respect to the temperature only on the thermal contribution of the first order obtained from previous linearization i.e., in the term  $\frac{\partial \widehat{\mathbf{S}}_{ther}}{\partial \mathbf{F}}$ . For this purpose, we consider that the deformation gradient  $\mathbf{F}$  and the temperature  $\theta$  are small perturbations of their initial states, i.e.,  $\nabla \mathbf{u} = O(\varepsilon)$  and  $\theta - \theta_0 = O(\varepsilon)$ ,  $\varepsilon$  being a small parameter.

**Corollary 1.4.18.** *We have*

$$\begin{aligned}
\widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) &= (\mathbf{I} + \mathbf{W})\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u}\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) - \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p)\boldsymbol{\varepsilon}(\mathbf{u}) \\
&\quad + \frac{\partial \widehat{\mathbf{T}}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u}\widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) + \mathbf{W}\widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) \\
&\quad - \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p)\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + (\mathbf{I} + \mathbf{W})\widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) \\
&\quad + \text{Div } \mathbf{u}\widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) - \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p)\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}) \\
&\quad + o(\nabla \mathbf{u}) + o(\theta - \theta_0).
\end{aligned} \tag{1.80}$$

*Proof.* The previous equality is deduced applying a Taylor's formula around the point  $(\mathbf{I}, \theta_0, \mathbf{Z}, p)$  on the thermal part of the first order obtained of the previous linearization (see expression (1.70)) to the response function  $\widehat{\mathbf{S}}$ , in fact:

$$\begin{aligned}
\widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \widehat{\mathbf{S}}_{elas}(\mathbf{I}, \theta, p) + \frac{\partial \widehat{\mathbf{S}}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \nabla \mathbf{u} + \widehat{\mathbf{S}}_{ther}(\mathbf{I}, \theta, p) + \frac{\partial \widehat{\mathbf{S}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \nabla \mathbf{u} \\
&\quad + \widehat{\mathbf{S}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) + \frac{\partial \widehat{\mathbf{S}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \nabla \mathbf{u} + o(\nabla \mathbf{u}) + o(\theta - \theta_0).
\end{aligned} \tag{1.81}$$

Hence, thanks to expression (1.53) and Proposition 1.4.11 we deduce the result.  $\square$

**Definition 1.4.19.** *We define the tensor*

$$\mathbf{Y}(p, t) = \widehat{\mathbf{Y}}(\mathbf{F}(p, t), p),$$

$\widehat{\mathbf{Y}}$  being

$$\widehat{\mathbf{Y}}(\mathbf{F}(p, t), p) = \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \theta}(\mathbf{F}(p, t), \theta_0, p).$$

Considering first linearization (**1FLM**), linearizing with respect to  $\theta$  in the terms  $O(\nabla \mathbf{u})$  for the thermal part and neglecting the terms  $o(\varepsilon)$ , motion equation can be approximated as follows:

Second linearization of the motion equation (**2θLM**)

$$\rho_0 \ddot{\mathbf{u}} - \text{Div } \mathbf{S}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \mathbf{b}_*,$$

where  $\mathbf{S}^{**}$  is the linearized part of the First Piola-Kirchhoff stress tensor (see equality (1.80)),

$$\begin{aligned}
\mathbf{S}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) &= (\mathbf{I} + \mathbf{W})\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u}\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) - \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p)\boldsymbol{\varepsilon}(\mathbf{u}) \\
&\quad + \widehat{\mathbf{C}}(\theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u}\widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) \\
&\quad + \mathbf{W}\widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) - \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p)\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \boldsymbol{\varepsilon}(\mathbf{u}) \\
&\quad + (\mathbf{I} + \mathbf{W})\widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) + \text{Div } \mathbf{u}\widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) - \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p)\boldsymbol{\varepsilon}(\mathbf{u}) \\
&\quad + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}).
\end{aligned} \tag{1.82}$$

**Corollary 1.4.20.** *The quasistatic motion equation associated with (2 $\theta$ LM) in terms of the Second Piola-Kirchhoff stress tensor is*

$$-\text{Div } \mathbf{S}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \mathbf{b}_*, \quad (1.83)$$

with  $\mathbf{S}^{**}$  defined by equality (1.82) written in terms of the Second Piola-Kirchhoff stress tensor,

$$\begin{aligned} \mathbf{S}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) - \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) \boldsymbol{\varepsilon}(\mathbf{u}) \\ & + \widehat{\mathbf{C}}(\theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \widehat{\Sigma}_{ther}(\mathbf{I}, \theta, p) + (\mathbf{W} + \boldsymbol{\varepsilon}(\mathbf{u})) \widehat{\Sigma}_{ther}(\mathbf{I}, \theta_0, p) \\ & + \frac{\partial \widehat{\Sigma}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + (\mathbf{I} + \mathbf{W} + \boldsymbol{\varepsilon}(\mathbf{u})) \widehat{\Sigma}_{plas}(\mathbf{I}, \theta, \mathbf{Z}, p) \\ & + \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \boldsymbol{\varepsilon}(\mathbf{u}). \end{aligned} \quad (1.84)$$

*Proof.* The proof is deduced from second linearization (2 $\theta$ LM) thanks to equalities (1.75), (1.76) for  $\widehat{\mathbf{T}}_{ther}$  and  $\widehat{\mathbf{T}}_{plas}$ , taking into account that  $\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) = \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p)$  and Definitions 1.4.13, 1.4.19.  $\square$

**Proposition 1.4.21.** *The approximation of the heat flux considering this linearization takes the form:*

$$\begin{aligned} \mathbf{q}^{**}(\mathbf{F}, \theta, \mathbf{Z}, \nabla \theta, p) = & \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}, \nabla \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta_0, \mathbf{Z}, \nabla \theta, p) - \nabla \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta_0, \mathbf{Z}, \nabla \theta, p) \\ & + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, \mathbf{Z}, \nabla \theta, p)(\nabla \mathbf{u}) - \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{I}, \theta_0, \mathbf{Z}, \nabla \theta, p)(\nabla \mathbf{u}^t \nabla \theta) \\ & + o(\nabla \mathbf{u}) + o(\theta - \theta_0). \end{aligned} \quad (1.85)$$

*Proof.* It is deduced applying a Taylor's formula around the point  $(\mathbf{I}, \theta_0, \mathbf{Z}, \mathbf{w}, p)$  on the first order term  $\frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}$  of the previous linearization (see expression (1.78)) to response function  $\widehat{\mathbf{q}}_*$ :

$$\widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) = \widehat{\mathbf{q}}_*(\mathbf{I}, \theta, \mathbf{Z}, \mathbf{w}, p) + \frac{\partial \widehat{\mathbf{q}}_*}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, \mathbf{Z}, \mathbf{w}, p)(\nabla \mathbf{u}) + o(\nabla \mathbf{u}) + o(\theta - \theta_0), \quad (1.86)$$

using Proposition 1.4.15, equality (1.63) and taking  $\mathbf{w} = \nabla \theta$ .  $\square$

Considering first linearization (1FLE), linearizing with respect to  $\theta$  in the terms  $O(\nabla \mathbf{u})$  for the thermal stress and the heat flux response functions, neglecting the terms  $o(\varepsilon)$ , energy equation can be approximated as follows:

Second linearization of the energy equation (2 $\theta$ LE)

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} = & \theta \frac{\partial \mathbf{S}^{**}}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\ & - \text{Div } \mathbf{q}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*, \end{aligned}$$

with  $\mathbf{S}^{**}$  given by expression (1.82) and  $\mathbf{q}^{**}$  by equality (1.85).

**Corollary 1.4.22.** *The energy equation associated with (2θLE) in terms of the Second Piola-Kirchhoff stress tensor is*

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} = & \theta \frac{\partial \mathbf{S}^{**}}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\ & - \text{Div } \mathbf{q}^{**}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*, \end{aligned} \quad (1.87)$$

where  $\mathbf{S}^{**}$  and  $\mathbf{q}^{**}$  are given by expressions (1.84) and (1.85) respectively.

*Proof.* This equation is obtained from expression (2θLE), equality (1.60), thanks to expressions (1.75), (1.76) for  $\widehat{\mathbf{T}}_{ther}$  and  $\widehat{\mathbf{T}}_{plas}$ , taking into account that  $\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) = \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p)$ , Definitions 1.4.13 and 1.4.19.  $\square$

### Obtaining a third linearization with respect to $\mathbf{Z}$

This third linearization is done considering that the internal variable  $\mathbf{Z}$  is a small perturbation of its initial state, i.e.,  $\mathbf{Z} - \mathbf{Z}_0 = O(\varepsilon)$ ,  $\varepsilon$  being a small parameter, besides considering that the deformation gradient  $\mathbf{F}$  and the temperature  $\theta$  are small perturbations of their initial states, i.e.,  $\nabla \mathbf{u} = O(\varepsilon)$  and  $\theta - \theta_0 = O(\varepsilon)$  in same way the previous linearization.

**Corollary 1.4.23.** *We have*

$$\begin{aligned} \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) - \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) \boldsymbol{\varepsilon}(\mathbf{u}) \\ & + \frac{\partial \widehat{\mathbf{T}}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) + \mathbf{W} \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) \\ & - \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + (\mathbf{I} + \mathbf{W}) \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) \\ & + \text{Div } \mathbf{u} \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) - \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : \boldsymbol{\varepsilon}(\mathbf{u}) \\ & + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0) + o(\nabla \mathbf{u}) + o(\theta - \theta_0) + o(\mathbf{Z} - \mathbf{Z}_0). \end{aligned} \quad (1.88)$$

*Proof.* The previous equality is deduced applying a Taylor's formula around the point  $(\mathbf{I}, \theta, \mathbf{Z}_0, p)$  on the plastic part of the previous linearization (see expression (1.81)) to the response function  $\widehat{\mathbf{S}}$ , in fact:

$$\begin{aligned} \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{Z}, p) = & \widehat{\mathbf{S}}_{elas}(\mathbf{I}, \theta, p) + \frac{\partial \widehat{\mathbf{S}}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \nabla \mathbf{u} + \widehat{\mathbf{S}}_{ther}(\mathbf{I}, \theta, p) + \frac{\partial \widehat{\mathbf{S}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \nabla \mathbf{u} \\ & + \widehat{\mathbf{S}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) + \frac{\partial \widehat{\mathbf{S}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : \nabla \mathbf{u} + \frac{\partial \widehat{\mathbf{S}}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0) \\ & + o(\nabla \mathbf{u}) + o(\theta - \theta_0) + o(\mathbf{Z} - \mathbf{Z}_0). \end{aligned} \quad (1.89)$$

Hence, thanks to expression (1.53) and Proposition 1.4.11 we deduce the result.  $\square$

Therefore, given the second linearization (**2θLM**), linearizing  $\widehat{\mathbf{T}}_{plas}$  and  $\frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}$  around  $\mathbf{Z} = \mathbf{Z}_0$  and neglecting the terms  $o(\varepsilon)$ , we can approximate the motion equation as follows:

Third linearization of the motion equation (3ZLM)

$$\rho_0 \ddot{\mathbf{u}} - \text{Div } \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \mathbf{b}_*,$$

$\mathbf{S}^{***}$  being the linearized part of the First Piola-Kirchhoff stress tensor (see equality (1.88))

$$\begin{aligned} \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) - \widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) \varepsilon(\mathbf{u}) \\ & + \widehat{\mathbf{C}}(\theta, p) : \varepsilon(\mathbf{u}) + \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) + \mathbf{W} \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) \\ & - \widehat{\mathbf{T}}_{ther}(\mathbf{I}, \theta_0, p) \varepsilon(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \varepsilon(\mathbf{u}) + (\mathbf{I} + \mathbf{W}) \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) \\ & + \text{Div } \mathbf{u} \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) - \widehat{\mathbf{T}}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) \varepsilon(\mathbf{u}) + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : \varepsilon(\mathbf{u}) \\ & + \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0). \end{aligned} \quad (1.90)$$

**Corollary 1.4.24.** *The quasistatic motion equation associated with (3ZLM) in terms of the Second Piola-Kirchhoff stress tensor is*

$$- \text{Div } \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \mathbf{b}_*, \quad (1.91)$$

with  $\mathbf{S}^{***}$  defined by equality (1.90) written in terms of the Second Piola-Kirchhoff stress tensor,

$$\begin{aligned} \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) - \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) \varepsilon(\mathbf{u}) \\ & + \widehat{\mathbf{C}}(\theta, p) : \varepsilon(\mathbf{u}) + \widehat{\Sigma}_{ther}(\mathbf{I}, \theta, p) + (\mathbf{W} + \varepsilon(\mathbf{u})) \widehat{\Sigma}_{ther}(\mathbf{I}, \theta_0, p) \\ & + \frac{\partial \widehat{\Sigma}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, p) : \varepsilon(\mathbf{u}) + (\mathbf{I} + \mathbf{W} + \varepsilon(\mathbf{u})) \widehat{\Sigma}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) \\ & + \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : \varepsilon(\mathbf{u}) + \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0). \end{aligned} \quad (1.92)$$

*Proof.* The proof is deduced from expression (**3ZLM**) thanks to equalities (1.75), (1.76) for  $\widehat{\mathbf{T}}_{ther}$  and  $\widehat{\mathbf{T}}_{plas}$ , taking into account that

$$\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta, p) = \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p), \quad (1.93)$$

$$\frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0) = \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0), \quad (1.94)$$

and Definitions 1.4.13 and 1.4.19. □

**Proposition 1.4.25.** *The approximation of the heat flux considering this linearization takes the form:*

$$\begin{aligned} \mathbf{q}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) = & \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \nabla \theta, p) - \nabla \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \nabla \theta, p) \\ & + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \nabla \theta, p)(\nabla \mathbf{u}) - \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \nabla \theta, p)(\nabla \mathbf{u}^t \nabla \theta) \\ & + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{Z}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \nabla \theta, p)(\mathbf{Z} - \mathbf{Z}_0) + o(\nabla \mathbf{u}) + o(\theta - \theta_0) + o(\mathbf{Z} - \mathbf{Z}_0). \end{aligned} \quad (1.95)$$

*Proof.* It is deduced applying a Taylor's formula around the point  $\mathbf{Z} = \mathbf{Z}_0$  on the previous linearization (see expression (1.86)) to response function  $\widehat{\mathbf{q}}_*$ :

$$\begin{aligned} \widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) = & \widehat{\mathbf{q}}_*(\mathbf{I}, \theta, \mathbf{Z}_0, \mathbf{w}, p) + \frac{\partial \widehat{\mathbf{q}}_*}{\partial \mathbf{F}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \mathbf{w}, p)(\nabla \mathbf{u}) + \frac{\partial \widehat{\mathbf{q}}_*}{\partial \mathbf{Z}}(\mathbf{I}, \theta_0, \mathbf{Z}_0, \mathbf{w}, p)(\mathbf{Z} - \mathbf{Z}_0) \\ & + o(\nabla \mathbf{u}) + o(\theta - \theta_0) + o(\mathbf{Z} - \mathbf{Z}_0), \end{aligned}$$

using Proposition 1.4.15, equality (1.63) and taking  $\mathbf{w} = \nabla \theta$ .  $\square$

Finally, given the second linearization ( $2\theta\mathbf{LE}$ ), linearizing  $\widehat{\mathbf{T}}_{plas}$ ,  $\frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \mathbf{F}}$  and the heat flux  $\widehat{\mathbf{q}}$  around  $\mathbf{Z} = \mathbf{Z}_0$ , and neglecting the terms  $o(\varepsilon)$  we can approximate the energy equation as follows:

Third linearization of the energy equation ( $3\mathbf{ZLE}$ )

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} = & \theta \frac{\partial \mathbf{S}^{***}}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\ & - \text{Div } \mathbf{q}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*, \end{aligned}$$

where  $\mathbf{S}^{***}$  is given by expression (1.90) and  $\mathbf{q}^{***}$  by equality (1.95).

**Corollary 1.4.26.** *The energy equation associated with ( $3\mathbf{ZLE}$ ) in terms of the Second Piola-Kirchhoff stress tensor is*

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} = & \theta \frac{\partial \mathbf{S}^{***}}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\ & - \text{Div } \mathbf{q}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*, \end{aligned} \quad (1.96)$$

$\mathbf{S}^{***}$  and  $\mathbf{q}^{***}$  defined by equalities (1.92) and (1.95), respectively.

*Proof.* This equation is obtained from expression ( $3\mathbf{ZLE}$ ), equality (1.60), thanks to expressions (1.75), (1.76) for  $\widehat{\mathbf{T}}_{ther}$  and  $\widehat{\mathbf{T}}_{plas}$ , taking into account (1.93) and (1.94) and Definitions 1.4.13 and 1.4.19.  $\square$

## 1.5 Example: Maxwell-Norton materials with heat conduction

In this section we present an example of thermoviscoelastic materials with long memory: the Maxwell-Norton materials with heat conduction. The main difficulty is the choice of appropriate response functions for the Cauchy stress tensor associated with these materials in order to guarantee the second principle of thermodynamics, that is verifying hypotheses **(H1)**-**(H4)**.

**Definition 1.5.1.** *A thermoviscoelastic material with long memory is called a Maxwell-Norton*

material with heat conduction if

$$\widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta, p) = \frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}(\theta, p) : \mathbf{E} \mathbf{F}^t, \quad (1.97)$$

$$\widehat{\mathbf{T}}_{ther}(\mathbf{F}, \theta, p) = -\frac{1}{\det \mathbf{F}} \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{\Lambda}^{-1}(\theta, p) : \mathbf{I} \mathbf{B}, \quad (1.98)$$

$$\widehat{\mathbf{T}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p) = -\frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}(\theta, p) : \mathbf{Z} \mathbf{F}^t, \quad (1.99)$$

$$\widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{Z}, \mathbf{w}, p) = -\widehat{k}(\theta, p) \mathbf{w}, \quad (1.100)$$

$$\mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) = \varphi(\theta) \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta, p) : (\mathbf{E} - \mathbf{Z}) \right), \quad (1.101)$$

for all  $\mathbf{F} \in Lin^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{Z} \in Sym$ ,  $\mathbf{w} \in \mathcal{V}$  and  $p \in \mathcal{B}$ , where:

- $\widehat{\Lambda}$  is the elasticity tensorial function defined as

$$(\theta, p) \in \mathbb{R}^+ \times \mathcal{B} \rightarrow \widehat{\Lambda}(\theta, p) : Sym \longrightarrow Sym$$

$$\boldsymbol{\tau} \longrightarrow \widehat{\Lambda}(\theta, p) : \boldsymbol{\tau} = \frac{1 + \widehat{\nu}(\theta, p)}{\widehat{E}(\theta, p)} \boldsymbol{\tau} - \frac{\widehat{\nu}(\theta, p)}{\widehat{E}(\theta, p)} \text{tr}(\boldsymbol{\tau}) \mathbf{I},$$

$\widehat{E}(\theta, p)$ ,  $\widehat{\nu}(\theta, p) \in C^1(\mathbb{R}^+ \times \mathcal{B}; \mathbb{R})$  being the response functions associated with Young's modulus and Poisson's coefficient, respectively. The inverse of  $\widehat{\Lambda}$  is defined by:

$$(\theta, p) \in \mathbb{R}^+ \times \mathcal{B} \rightarrow \widehat{\Lambda}^{-1}(\theta, p) : Sym \longrightarrow Sym$$

$$\boldsymbol{\tau} \longrightarrow \widehat{\Lambda}^{-1}(\theta, p) : \boldsymbol{\tau} = \widehat{\lambda}(\theta, p) \text{tr}(\boldsymbol{\tau}) \mathbf{I} + 2\widehat{\mu}(\theta, p) \boldsymbol{\tau}, \quad (1.102)$$

with  $\widehat{\lambda}(\theta, p)$ ,  $\widehat{\mu}(\theta, p) \in C^1(\mathbb{R}^+ \times \mathcal{B}; \mathbb{R})$  the response functions associated with Lamé's parameters of the material and related to  $\widehat{E}$ ,  $\widehat{\nu}$  by,

$$\widehat{\lambda}(\theta, p) = \frac{\widehat{E}(\theta, p) \widehat{\nu}(\theta, p)}{(1 + \widehat{\nu}(\theta, p))(1 - 2\widehat{\nu}(\theta, p))}, \quad \widehat{\mu}(\theta, p) = \frac{\widehat{E}(\theta, p)}{2(1 + \widehat{\nu}(\theta, p))}.$$

- $\mathbf{E} = \frac{\mathbf{C} - \mathbf{I}}{2}$  is the Green-Saint Venant strain tensor,  $\mathbf{C} = \mathbf{F}^t \mathbf{F}$  and  $\mathbf{B} = \mathbf{F} \mathbf{F}^t$  are the right and left Cauchy-Green strain tensors, respectively.
- $\widehat{\alpha} : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a smooth enough response function associated with coefficient of thermal expansion where  $\theta_0 \in \mathbb{R}^+$  given.
- $\widehat{k} : \mathbb{R}^+ \times \mathcal{B} \mapsto \mathbb{R}$  is the smooth enough response function associated with the thermal conductivity of the material.
- $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a smooth enough function, such that  $\varphi(s) \geq \varphi_{min} > 0$ .
- $\phi_q$  is the plasticity potential defined from  $Sym$  to  $\mathbb{R}$  as

$$\phi_q(\boldsymbol{\tau}) = \frac{1}{q} |\boldsymbol{\tau}^D|^q, \quad (1.103)$$

where  $q \geq 2$  is a strictly positive material parameter and  $\boldsymbol{\tau}^D$  represents its deviatoric tensor  $\boldsymbol{\tau}^D = \boldsymbol{\tau} - \frac{1}{n} \text{tr}(\boldsymbol{\tau}) \mathbf{I}$ ,  $n$  being the dimension of the space, in particular, in our case  $n = 3$ . Its subdifferential  $\partial\phi_q : \text{Sym} \rightarrow \text{Sym}_0$  is

$$\partial\phi_q(\boldsymbol{\tau}) = |\boldsymbol{\tau}^D|^{q-2} \boldsymbol{\tau}^D. \quad (1.104)$$

### 1.5.1 Verifying the hypotheses (H2)-(H4)

In order to prove the following result we introduce the spaces:

$$H = L^2(\mathcal{B}, \text{Sym}) \text{ and } H^q = \{\boldsymbol{\tau} \in H \text{ such that } \boldsymbol{\tau}^D \in L^q(\mathcal{B}; \text{Sym})\} \subset H.$$

**Lemma 1.5.2.** *The Maxwell-Norton materials satisfy hypothesis (H2): given a motion  $X$ , a function  $\theta : \mathcal{T} \rightarrow \mathbb{R}^+$  such that  $\theta_m \in L^\infty(\mathcal{B} \times \mathbb{R}; \mathbb{R}^+)$  and a tensorial function  $\mathbf{Z}_0 : \mathcal{B} \rightarrow \text{Sym}$  such that  $\mathbf{Z}_0 \in H^q$ . There exists a smooth tensorial function  $\mathbf{Z} : \mathcal{T} \rightarrow \text{Sym}$ , unique solution of the system:*

$$\begin{cases} \dot{\mathbf{Z}}(x, t) = \varphi(\theta(x, t)) \partial\phi_q\left(\widehat{\Lambda}^{-1}(\theta(x, t), p) : (\mathbf{E}(p, t) - \mathbf{Z}(x, t))\right) \text{ in } \mathcal{T}, x = X(p, t), \\ \mathbf{Z}(X(p, 0), 0) = \mathbf{Z}_0(p), \end{cases} \quad (1.105)$$

in  $L^2(\mathcal{T}, \text{Sym})$ .

*Proof.* In order to prove this lemma, we use a result given by Attouch and Damlamian [9] and the methodology used by Blanchard and Le Tallec [22]. Let  $X, \theta$  and  $\mathbf{Z}_0$  be given, and  $t_f \in \mathbb{R}^+$  arbitrary. We rewrite problem (1.105) in its material formulation:

$$\begin{cases} \dot{\mathbf{Z}}_m(p, t) = \varphi(\theta_m(p, t)) \partial\phi_q\left(\widehat{\Lambda}^{-1}(\theta_m(p, t), p) : (\mathbf{E}(p, t) - \mathbf{Z}_m(p, t))\right) \text{ in } \mathcal{B} \times (0, t_f), \\ \mathbf{Z}_m(p, 0) = \mathbf{Z}_0(p), \quad p \in \mathcal{B}, \end{cases} \quad (1.106)$$

$\mathbf{Z}_m(p, t)$  belonging to  $L^2(\mathcal{B} \times (0, t_f); \text{Sym})$ .

To simplify the proof, we consider the following notation:

- $\tilde{\varphi}(p, t) = \varphi(\theta_m(p, t))$ ,
- $\tilde{\Lambda}(p, t) : \boldsymbol{\tau} = \widehat{\Lambda}^{-1}(\theta_m(p, t), p) : \boldsymbol{\tau}$ ,
- $\tilde{\mu}(p, t) = \widehat{\mu}(\theta_m(p, t), p)$ ,

at each point  $p \in \mathcal{B}$  and at each time instant  $t \in (0, t_f)$ . If we define

$$\Psi_q(p, t, \boldsymbol{\tau}) = \tilde{\varphi}(p, t) \frac{1}{2\tilde{\mu}(p, t)} \phi_q(\tilde{\Lambda}(p, t) : \boldsymbol{\tau}),$$

and taking into account definitions (1.102)-(1.104), we obtain

$$\Psi_q(p, t, \boldsymbol{\tau}) = \tilde{\varphi}(p, t) (2\tilde{\mu}(p, t))^{q-1} \phi_q(\boldsymbol{\tau}^D). \quad (1.107)$$

Consequently,

$$\partial\Psi_q(p, t, \cdot)(\boldsymbol{\tau}) = \tilde{\varphi}(p, t) [2\tilde{\mu}(p, t)]^{q-1} |\boldsymbol{\tau}^D|^{q-2} \boldsymbol{\tau}^D = \tilde{\varphi}(p, t) \partial\phi_q(\tilde{\Lambda}(p, t) : \boldsymbol{\tau}).$$

Thanks to properties of the subdifferential (see Subsection 2.2.1 or Djaoua and Suquet [35]) the function

$$\Phi_q(t, \boldsymbol{\tau}) = \int_{\mathcal{B}} \Psi_q(p, t, \boldsymbol{\tau}) dp, \quad (1.108)$$

verifies that  $\partial\Phi_q(t, \cdot)(\boldsymbol{\tau}(p)) \equiv \partial\Psi_q(p, t, \cdot)(\boldsymbol{\tau}(p))$ . Therefore, if we consider the change of variable  $\mathbf{U} = \mathbf{E} - \mathbf{Z}_m \in \text{Sym}$ , we have an equivalent formulation to problem (1.106) given by:

$$\begin{cases} \dot{\mathbf{U}} + \partial\Phi_q(t, \mathbf{U}) &= \dot{\mathbf{E}} \text{ in } L^2(\mathcal{B}, \text{Sym}) \text{ and a.e. } t \in (0, t_f), \\ \mathbf{U}(0) &= \mathbf{Z}_0. \end{cases} \quad (1.109)$$

In order to solve Cauchy problem (1.109) using Theorem 1 from Attouch and Damllamian [9], we need the following properties:

- i)  $H$  is a real Hilbert space.
- ii)  $(\Phi_q(t, \cdot))_{t \in (0, t_f)}$  is a family of convex, proper and lower semi-continuous functions defined from  $H \mapsto (-\infty, +\infty]$  (see Subsection 2.2.1 or Barral *et al.* [12]).
- iii) Domain  $((\Phi_q(t, \cdot))_{t \in (0, t_f)}) = H^q$  is independent of time.
- iv)  $\forall \boldsymbol{\tau} \in H^q, \forall t, s \in (0, t_f)$  there exists  $a \in W^{1,1}(0, t_f; \mathbb{R})$  and  $c \geq 0$  such that

$$|\Phi_q(t, \boldsymbol{\tau}) - \Phi_q(s, \boldsymbol{\tau})| \leq |a(t) - a(s)| [\Phi_q(t, \boldsymbol{\tau}) + c]. \quad (1.110)$$

In order to prove this assertion, we take into account that  $\tilde{\varphi}$  and  $\tilde{\mu}$  verify:

- $\tilde{\varphi} \in L^\infty(\mathcal{B} \times (0, t_f); \mathbb{R})$  and  $\tilde{\varphi}(p, t) \geq \varphi_{min} > 0$ , a.e. in  $\mathcal{B} \times (0, t_f)$ ; in consequence

$$\frac{\tilde{\varphi}(p, t)}{\varphi_{min}} \geq 1 \text{ a.e. in } \mathcal{B} \times (0, t_f). \quad (1.111)$$

- $\tilde{\mu} \in L^\infty(\mathcal{B} \times (0, t_f); \mathbb{R})$  and  $\tilde{\mu}(p, t) \geq \mu_{min} > 0$ , a.e. in  $\mathcal{B} \times (0, t_f)$ ; in consequence

$$\frac{\tilde{\mu}(p, t)}{\mu_{min}} \geq 1 \text{ a.e. in } \mathcal{B} \times (0, t_f). \quad (1.112)$$

Then using definitions (1.108), (1.107) and (1.103) we have

$$\begin{aligned} & |\Phi_q(t, \boldsymbol{\tau}) - \Phi_q(s, \boldsymbol{\tau})| \\ &= \left| \int_{\mathcal{B}} \frac{1}{q} \tilde{\varphi}(p, t) (2\tilde{\mu}(p, t))^{q-1} |\boldsymbol{\tau}^D(p)|^q dp - \int_{\mathcal{B}} \frac{1}{q} \tilde{\varphi}(p, s) (2\tilde{\mu}(p, s))^{q-1} |\boldsymbol{\tau}^D(p)|^q dp \right| \\ &\leq \frac{1}{q} \int_{\mathcal{B}} |\tilde{\varphi}(p, t) (2\tilde{\mu}(p, t))^{q-1} - \tilde{\varphi}(p, s) (2\tilde{\mu}(p, s))^{q-1}| |\boldsymbol{\tau}^D(p)|^q dp \\ &\leq \frac{1}{q} \int_{\mathcal{B}} \left[ |(2\tilde{\mu}(p, t))^{q-1} (\tilde{\varphi}(p, t) - \tilde{\varphi}(p, s))| + |\tilde{\varphi}(p, s) ((2\tilde{\mu}(p, t))^{q-1} - (2\tilde{\mu}(p, s))^{q-1})| \right] \\ & \quad |\boldsymbol{\tau}^D(p)|^q dp. \end{aligned}$$

Thanks to properties (1.111) and (1.112), we get

$$\begin{aligned} & |\Phi_q(t, \boldsymbol{\tau}) - \Phi_q(s, \boldsymbol{\tau})| \\ & \leq \left\{ \sup_{p \in \mathcal{B}} \left| (2\tilde{\mu}(p, t))^{q-1} (\tilde{\varphi}(p, t) - \tilde{\varphi}(p, s)) \right| + \sup_{p \in \mathcal{B}} \left| \tilde{\varphi}(p, s) ((2\tilde{\mu}(p, t))^{q-1} - (2\tilde{\mu}(p, s))^{q-1}) \right| \right\} \\ & \quad \frac{1}{q} \int_{\mathcal{B}} \frac{\tilde{\varphi}(p, t) (2\tilde{\mu}(p, t))^{q-1}}{\varphi_{\min} (2\mu_{\min})^{q-1}} |\boldsymbol{\tau}^D(p)|^q dp. \end{aligned}$$

Furthermore, using again definitions (1.103), (1.107) and (1.108), we obtain

$$\begin{aligned} |\Phi_q(t, \boldsymbol{\tau}) - \Phi_q(s, \boldsymbol{\tau})| & \leq \frac{1}{\varphi_{\min} (2\mu_{\min})^{q-1}} \|(2\tilde{\mu}(\cdot, t))^{q-1}\|_{L^\infty(\mathcal{B})} \|\tilde{\varphi}(\cdot, t) - \tilde{\varphi}(\cdot, s)\|_{L^\infty(\mathcal{B})} \Phi_q(t, \boldsymbol{\tau}) \\ & + \frac{1}{\varphi_{\min} (2\mu_{\min})^{q-1}} \|\tilde{\varphi}(\cdot, s)\|_{L^\infty(\mathcal{B})} \|(2\tilde{\mu}(\cdot, t))^{q-1} - (2\tilde{\mu}(\cdot, s))^{q-1}\|_{L^\infty(\mathcal{B})} \Phi_q(t, \boldsymbol{\tau}). \end{aligned}$$

Hence, we consider the functions  $a_1$  and  $a_2$  defined as:

- $\|\tilde{\varphi}(\cdot, t) - \tilde{\varphi}(\cdot, s)\|_{L^\infty(\mathcal{B})} = \sup_{p \in \mathcal{B}} \left| \int_s^t \frac{\partial \tilde{\varphi}(p, u)}{\partial u} du \right| \leq \int_s^t \sup_{p \in \mathcal{B}} \left| \frac{\partial \tilde{\varphi}(p, u)}{\partial u} \right| du = \int_s^t a_1(u) du,$
- $\|(2\tilde{\mu}(\cdot, t))^{q-1} - (2\tilde{\mu}(\cdot, s))^{q-1}\|_{L^\infty(\mathcal{B})} = \sup_{p \in \mathcal{B}} \left| \int_s^t \frac{\partial (2\tilde{\mu}(p, u))^{q-1}}{\partial u} du \right|$   
 $\leq \int_s^t \sup_{p \in \mathcal{B}} \left| \frac{\partial (2\tilde{\mu}(p, u))^{q-1}}{\partial u} \right| du = \int_s^t a_2(u) du.$

Taking into account the previous inequalities, we deduce

$$\begin{aligned} |\Phi_q(t, \boldsymbol{\tau}) - \Phi_q(s, \boldsymbol{\tau})| & \leq \frac{c_1}{\varphi_{\min} (2\mu_{\min})^{q-1}} \left( \int_s^t \sup_i a_i(u) du \right) \Phi_q(t, \boldsymbol{\tau}) \\ & \leq \frac{c_1}{\varphi_{\min} (2\mu_{\min})^{q-1}} \left| \int_0^t \sup_i a_i(u) du - \int_0^s \sup_i a_i(u) du \right| \Phi_q(t, \boldsymbol{\tau}), \end{aligned}$$

where  $c_1$  is a real positive constant.

Finally, we can conclude inequality (1.110), taking

$$a(t) = \frac{c_1}{\varphi_{\min} (2\mu_{\min})^{q-1}} \int_0^t \sup_i a_i(u) du$$

and  $c = 0$ .

Therefore, we can apply Theorem 1 given by Attouch and Damlamian [9] which guarantees the existence and uniqueness of the solution of problem (1.109) and in consequence of problem (1.105), with  $\hat{\mathbf{E}}(\cdot, t) \in L^2(\mathcal{B}, Sym)$ ). Furthermore,

$$\begin{aligned} \mathbf{Z}(X(p, t), t) & = \int_0^t \varphi \left( \theta(X(p, s), s) \right) \partial \phi_q \left( \hat{\Lambda}^{-1}(\theta(X(p, s), s), p) : (\mathbf{E}(p, s) - \mathbf{Z}((X(p, s), s), s)) \right) ds \\ & \quad + \mathbf{Z}_0(p), \quad \forall p \in \mathcal{B} \text{ and } t \in (0, t_f). \end{aligned} \tag{1.113}$$

□

**Lemma 1.5.3.** *The Maxwell-Norton materials satisfy hypothesis (H3): there exists  $\mathbf{h} : \text{Lin}^+ \times \mathbb{R}^+ \times \text{Sym} \times \mathcal{B} \rightarrow \text{Sym}$  such that,*

$$\mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) = \mathbf{h}(\mathbf{F}, \theta, \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p), p), \quad \forall \mathbf{F} \in \text{Lin}^+, \theta \in \mathbb{R}^+, \mathbf{Z} \in \text{Sym} \text{ and } p \in \mathcal{B}.$$

*Proof.* Taking into account that the deviatoric tensor of  $\left(\int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr\right) \widehat{\Lambda}^{-1}(\theta, p) : \mathbf{I}$  is null, expressions (1.97)-(1.99), (1.101) and equality (1.19), we can deduce

$$\begin{aligned} \mathbf{g}(\mathbf{F}, \theta, \mathbf{Z}, p) &= \varphi(\theta) \partial\phi_q \left( \widehat{\Lambda}^{-1}(\theta, p) : (\mathbf{E} - \mathbf{Z}) \right) \\ &= \varphi(\theta) \partial\phi_q \left( \widehat{\Lambda}^{-1}(\theta, p) : \left( \mathbf{E} - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} - \mathbf{Z} \right) \right) \\ &= \varphi(\theta) \partial\phi_q \left( \det \mathbf{F} \mathbf{F}^{-1} \left[ \frac{\mathbf{F}}{\det \mathbf{F}} \left( \widehat{\Lambda}^{-1}(\theta, p) : \left( \mathbf{E} - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} - \mathbf{Z} \right) \right) \mathbf{F}^t \right] \mathbf{F}^{-t} \right) \\ &= \varphi(\theta) \partial\phi_q \left( \det \mathbf{F} \mathbf{F}^{-1} \left[ \widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta, p) + \widehat{\mathbf{T}}_{ther}(\mathbf{F}, \theta, p) + \widehat{\mathbf{T}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}, p) \right] \mathbf{F}^{-t} \right) \\ &= \varphi(\theta) \partial\phi_q \left( \det \mathbf{F} \mathbf{F}^{-1} \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{Z}, p) \mathbf{F}^{-t} \right). \end{aligned}$$

Hence, we can conclude the result defining

$$\mathbf{h}(\mathbf{F}, \theta, \mathbf{T}, p) = \varphi(\theta) \partial\phi_q \left( \det \mathbf{F} \mathbf{F}^{-1} \mathbf{T} \mathbf{F}^{-t} \right), \quad (1.114)$$

for all  $\mathbf{F} \in \text{Lin}^+$ ,  $\theta \in \mathbb{R}^+$ ,  $\mathbf{T} \in \text{Sym}$  and  $p \in \mathcal{B}$ . □

**Lemma 1.5.4.** *The Maxwell-Norton materials satisfy hypothesis (H4): For all  $\mathbf{F} \in \text{Lin}^+$ , there exists  $\overline{\mathbf{Z}} \in \text{Sym}$  such that the function  $\mathbf{g}$  verifies:*

$$\mathbf{g}(\mathbf{F}, \theta, \overline{\mathbf{Z}}, p) = \mathbf{0}, \quad \forall \theta \in \mathbb{R}^+ \text{ and } p \in \mathcal{B}.$$

*Proof.* Evaluating expression (1.101) with  $\mathbf{Z} = \mathbf{E}$ , we obtain the result, i.e. in this case  $\overline{\mathbf{Z}} = \mathbf{E}$ . □

**Remark 1.5.5.** *For the Maxwell-Norton materials with heat conduction Theorem 1.3.6 and Lemma 1.3.8 are satisfied under the assumptions of Lemma 1.5.2.*

## 1.5.2 The motion and energy equations in the deformed configuration

From now on, we omit the dependence on the different operators and functions with respect to the material point  $p$ .

In order to give the expressions of the motion and energy conservation equations, we calculate some derivatives of the response functions associated with the Maxwell-Norton materials with

heat conduction:

$$\bullet \quad \frac{\partial \widehat{\mathbf{T}}_{elas}}{\partial \theta}(\mathbf{F}, \theta) = \frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}'(\theta) : \mathbf{E} \mathbf{F}^t, \quad (1.115)$$

$$\bullet \quad \frac{\partial \widehat{\mathbf{T}}_{ther}}{\partial \theta}(\mathbf{F}, \theta) = -\frac{1}{\det \mathbf{F}} \left[ \widehat{\alpha}(\theta) \widehat{\Lambda}^{-1}(\theta) : \mathbf{I} \mathbf{B} + \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{\Lambda}^{-1}'(\theta) : \mathbf{I} \mathbf{B} \right], \quad (1.116)$$

$$\bullet \quad \frac{\partial \widehat{\mathbf{T}}_{plas}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{Z}) = -\frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}'(\theta) : \mathbf{Z} \mathbf{F}^t, \quad (1.117)$$

for all  $\mathbf{F} \in Lin^+$ ,  $\theta \in \mathbb{R}^+$  and  $\mathbf{Z} \in Sym$ .

**Corollary 1.5.6.** *The momentum and energy conservation equations in the deformed configuration for Maxwell-Norton materials with heat conduction are respectively as follow*

$$\begin{aligned} \rho \dot{\mathbf{v}} &= \operatorname{div} \left[ \frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E}_{elas} \mathbf{F}^t \right] + \mathbf{b} \text{ in } \mathcal{T}, \\ \rho c_F \dot{\theta} &= \theta \frac{\mathbf{F}}{\det \mathbf{F}} \left[ \widehat{\Lambda}^{-1}'(\theta) : \mathbf{E}_{elas} - \widehat{\alpha}(\theta) \widehat{\Lambda}^{-1}(\theta) : \mathbf{I} \right] \mathbf{F}^t : \mathbf{L} \\ &\quad - \rho \varphi(\theta) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}) : \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : (\mathbf{E} - \mathbf{Z}) \right) + \operatorname{div} \left( \widehat{k}(\theta) \operatorname{grad} \theta \right) + f \text{ in } \mathcal{T}, \end{aligned}$$

with

$$\begin{aligned} \rho_0 &= \rho \det \mathbf{F}, \\ \mathbf{E}_{elas} &= \mathbf{E} - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} - \mathbf{Z}, \\ \dot{\mathbf{Z}} &= \varphi(\theta) \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : (\mathbf{E} - \mathbf{Z}) \right) \text{ in } \mathcal{T}. \end{aligned}$$

*Proof.* They are deduced from equations (1.46a)-(1.46c), taking into account equalities (1.97)-(1.101) and (1.115)-(1.117).  $\square$

### 1.5.3 The motion and energy equations in the reference configuration

**Lemma 1.5.7.** *We have*

$$\begin{aligned} \frac{\partial \widehat{\mathbf{T}}_{elas}}{\partial \mathbf{F}}(\mathbf{F}, \theta) : \mathbf{U} &= -\frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \mathbf{F}^t \operatorname{tr}(\mathbf{U} \mathbf{F}^{-1}) + \frac{1}{\det \mathbf{F}} \mathbf{U} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \mathbf{F}^t \\ &\quad + \frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \mathbf{U}^t + \frac{1}{\det \mathbf{F}} \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \left( \frac{\mathbf{U}^t \mathbf{F}}{2} + \frac{\mathbf{F}^t \mathbf{U}}{2} \right) \mathbf{F}^t, \end{aligned}$$

for all  $\mathbf{F} \in Lin^+$ ,  $\theta \in \mathbb{R}^+$  and  $\mathbf{U} \in Lin$ .

*Proof.* Taking equation (1.97) we get

$$\widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta) \det \mathbf{F} = \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \mathbf{F}^t,$$

and applying the product rule of the differential calculus, we deduce:

$$\begin{aligned} \frac{\partial \widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta)}{\partial \mathbf{F}} : \mathbf{U} \det \mathbf{F} + \widehat{\mathbf{T}}_{elas}(\mathbf{F}, \theta) \det \mathbf{F} \operatorname{tr}(\mathbf{U}\mathbf{F}^{-1}) \\ = \mathbf{U} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \mathbf{F}^t + \mathbf{F} \frac{\partial}{\partial \mathbf{F}} \left( \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \right) : \mathbf{U} \mathbf{F}^t + \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \mathbf{U}^t. \end{aligned} \quad (1.118)$$

Thanks to definition of the Green-Saint Venant strain tensor and expression (1.102) we have:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{F}} \left[ \widehat{\Lambda}^{-1}(\theta) : \mathbf{E} \right] : \mathbf{U} &= \frac{\partial}{\partial \mathbf{F}} \left[ \widehat{\lambda}(\theta) \operatorname{tr} \left( \frac{\mathbf{C} - \mathbf{I}}{2} \right) \mathbf{I} + 2\widehat{\mu}(\theta) \frac{\mathbf{C} - \mathbf{I}}{2} \right] : \mathbf{U} \\ &= \frac{\partial}{\partial \mathbf{F}} \left[ \widehat{\lambda}(\theta) \frac{1}{2} (\operatorname{tr}(\mathbf{F}^t \mathbf{F}) - 3) \mathbf{I} + 2\widehat{\mu}(\theta) \frac{\mathbf{F}^t \mathbf{F} - \mathbf{I}}{2} \right] : \mathbf{U} \\ &= \frac{\partial}{\partial \mathbf{F}} \left[ \widehat{\lambda}(\theta) \frac{1}{2} (\mathbf{F} : \mathbf{F} - 3) \mathbf{I} + \widehat{\mu}(\theta) (\mathbf{F}^t \mathbf{F} - \mathbf{I}) \right] : \mathbf{U} \\ &= \widehat{\lambda}(\theta) \frac{1}{2} (2\mathbf{U} : \mathbf{F}) \mathbf{I} + \widehat{\mu}(\theta) (\mathbf{U}^t \mathbf{F} + \mathbf{F}^t \mathbf{U}) = \widehat{\lambda}(\theta) (\mathbf{U} : \mathbf{F}) \mathbf{I} + \widehat{\mu}(\theta) (\mathbf{U}^t \mathbf{F} + \mathbf{F}^t \mathbf{U}). \end{aligned}$$

Thus, taking into this equality in expression (1.118) we can conclude the result.  $\square$

In order to obtain the motion and energy equation in Lagrangian coordinates we calculate the expressions for the First Piola-Kirchhoff stress tensors; thanks to equalities (1.55)-(1.57), (1.97)-(1.99) we get:

$$\widehat{\mathbf{S}}_{elas}(\mathbf{F}, \theta) = \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E}, \quad (1.119)$$

$$\widehat{\mathbf{S}}_{ther}(\mathbf{F}, \theta) = -3 \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{K}(\theta) \mathbf{F}, \quad (1.120)$$

$$\widehat{\mathbf{S}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}) = -\mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{Z}, \quad (1.121)$$

where  $\widehat{K}(\theta) = \frac{1}{3} (3\widehat{\lambda}(\theta) + 2\widehat{\mu}(\theta))$  is the bulk modulus and hence, using (1.58) we deduce the following expressions for the Second Piola-Kirchhoff stress tensors:

$$\widehat{\boldsymbol{\Sigma}}_{elas}(\mathbf{F}, \theta) = \widehat{\Lambda}^{-1}(\theta) : \mathbf{E}, \quad (1.122)$$

$$\widehat{\boldsymbol{\Sigma}}_{ther}(\mathbf{F}, \theta) = -3 \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{K}(\theta) \mathbf{I}, \quad (1.123)$$

$$\widehat{\boldsymbol{\Sigma}}_{plas}(\mathbf{F}, \theta, \mathbf{Z}) = -\widehat{\Lambda}^{-1}(\theta) : \mathbf{Z}. \quad (1.124)$$

**Corollary 1.5.8.** *The motion equation and the energy equation in the reference configuration for Maxwell-Norton materials with heat conduction are*

$$\begin{aligned} \rho_0 \ddot{\mathbf{u}} &= \operatorname{Div} \left( \mathbf{F} \widehat{\Lambda}^{-1}(\theta) : \mathbf{E}_{elas} \right) + \mathbf{b}_*, \\ \rho_0 c_F \dot{\theta} &= \theta \mathbf{F} \left[ \widehat{\Lambda}^{-1'}(\theta) : \mathbf{E}_{elas} - 3\widehat{\alpha}(\theta) \widehat{K}(\theta) \mathbf{I} \right] : \nabla \dot{\mathbf{u}} \\ &\quad - \rho_0 \varphi(\theta) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{F}, \theta, \mathbf{Z}) : \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : \mathbf{E}_{elas} \right) + \operatorname{Div} \left( \widehat{k}(\theta) \det \mathbf{F} (\mathbf{F}^t \mathbf{F})^{-1} \nabla \theta \right) + f_*, \end{aligned}$$

where

$$\dot{\mathbf{Z}} = \varphi(\theta) \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : \mathbf{E}_{elas} \right).$$

*Proof.* The results are deduced from equations (1.65a)-(1.65b) using expressions (1.119)-(1.121) and taking into account equality (1.60) to obtain that

$$\dot{\mathbf{Z}} = \varphi(\theta) \partial \phi_q (\mathbf{F}^{-1} \mathbf{S}).$$

□

#### 1.5.4 Thermoviscoelastic linearizations of the motion and energy equations

We assume as in Subsection 1.4.2 that the reference configuration is the initial equilibrium position of the body:  $\mathcal{B}_0 = \mathcal{B}$ , and  $\mathbf{T}_0$ ,  $\mathbf{b}_0$ ,  $\theta_0$ ,  $\mathbf{q}_0$ ,  $f_0$ ,  $\rho_0$  and  $\mathbf{Z}_0$  are the initial values for the Cauchy stress tensor, the body forces, the temperature, the heat flux, the heat source, the density of mass and the internal variable  $\mathbf{Z}$ , respectively; we assume that they verify equilibrium equations (1.66) with  $\mathbf{Z}_0 = \mathbf{0}$  and  $\mathbf{T}_0 = \mathbf{0}$  in order to have the reference configuration like a natural state.

**Lemma 1.5.9.** *For Maxwell-Norton materials:*

i) the elasticity tensor  $\widehat{\mathbf{C}}(\theta)$  is related with the elasticity tensorial function  $\widehat{\Lambda}^{-1}(\theta)$  by:

$$\widehat{\mathbf{C}}(\theta) : \boldsymbol{\tau} = \widehat{\Lambda}^{-1}(\theta) : \left( \frac{\boldsymbol{\tau}^t + \boldsymbol{\tau}}{2} \right) = \widehat{\lambda}(\theta) \text{tr}(\boldsymbol{\tau}) \mathbf{I} + 2\widehat{\mu}(\theta) \left( \frac{\boldsymbol{\tau}^t + \boldsymbol{\tau}}{2} \right), \quad \forall \boldsymbol{\tau} \in \text{Lin},$$

ii) the tensor  $\widehat{\mathbf{Y}}$  is

$$\widehat{\mathbf{Y}}(\mathbf{F}) = - \frac{1}{\det \mathbf{F}} \widehat{\alpha}(\theta_0) \widehat{\mathbf{C}}(\theta_0) : \mathbf{I} \mathbf{B} = -3 \frac{\widehat{\alpha}(\theta_0)}{\det \mathbf{F}} \widehat{K}(\theta_0) \mathbf{B}.$$

*Proof.* i) Considering Definition 1.4.13 and Lemma 1.5.7, we have for all  $\mathbf{U} \in \text{Lin}$

$$\widehat{\mathbf{C}}(\theta) : \mathbf{U} = \frac{\partial \widehat{\mathbf{T}}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta) : \mathbf{U} = \widehat{\Lambda}^{-1}(\theta) : \left( \frac{\mathbf{U}^t + \mathbf{U}}{2} \right) = \left[ \widehat{\lambda}(\theta) \text{tr}(\mathbf{U}) \mathbf{I} + 2\widehat{\mu}(\theta) \left( \frac{\mathbf{U}^t + \mathbf{U}}{2} \right) \right],$$

since for  $\mathbf{F} = \mathbf{I}$ ,  $\mathbf{E} = \mathbf{0}$ .

ii) Thanks to Definition 1.4.19, equalities (1.116), (1.102) and taking into account the previous item, we conclude the result.

□

Hereafter, we take into account the following properties of the response functions associated with Maxwell-Norton materials with heat conduction:

- $\widehat{\mathbf{T}}_{elas}(\mathbf{I}, \theta) = \widehat{\mathbf{S}}_{elas}(\mathbf{I}, \theta) = \widehat{\boldsymbol{\Sigma}}_{elas}(\mathbf{I}, \theta) = \mathbf{0}$  for all  $\theta \in \mathbb{R}^+$ , since the Green-Saint Venant strain tensor is null as  $\mathbf{F} = \mathbf{I}$ ,

- $\widehat{\mathbf{T}}_{ther}(\mathbf{F}, \theta_0) = \widehat{\mathbf{S}}_{ther}(\mathbf{F}, \theta_0) = \widehat{\boldsymbol{\Sigma}}_{ther}(\mathbf{F}, \theta_0) = \mathbf{0}$ , for all  $\mathbf{F} \in Lin^+$ , and
- $\widehat{\boldsymbol{\Sigma}}_{ther}(\mathbf{F}, \theta)$  and  $\widehat{\boldsymbol{\Sigma}}_{plas}(\mathbf{F}, \theta, \mathbf{Z})$  are independent of  $\mathbf{F}$ .

Thanks to equalities (1.122)-(1.124) we deduce from equations (1.73) and (1.79) associated with the first linearization of motion equation (**1FLM**) and of energy equation (**1FLE**), respectively, taking into account expression (1.74) for Maxwell-Norton materials with heat conduction:

**Problem (1FL-MN)**

$$\left\{ \begin{array}{l} -\text{Div} \left( \widehat{\Lambda}^{-1}(\theta) : \boldsymbol{\varepsilon}(\mathbf{u}) - (\mathbf{I} + \nabla \mathbf{u}) \left[ \widehat{\Lambda}^{-1}(\theta) : (\mathbf{E} - \mathbf{E}_{elas}) \right] \right) = \mathbf{b}_*, \\ \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}) \dot{\theta} = -\theta \left[ (\mathbf{I} + \nabla \mathbf{u}) \left( \widehat{\Lambda}^{-1'}(\theta) : (\mathbf{E} - \mathbf{E}_{elas}) \right) \right] : \nabla \dot{\mathbf{u}} \\ \quad + \theta \left[ \widehat{\Lambda}^{-1'}(\theta) : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\widehat{\alpha}(\theta) \widehat{K}(\theta) (\mathbf{I} + \nabla \mathbf{u}) \right] : \nabla \dot{\mathbf{u}} \\ \quad - \rho_0 \varphi(\theta) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}) : \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : \boldsymbol{\varepsilon}(\mathbf{u}) - (\mathbf{I} + \nabla \mathbf{u}) \left[ \widehat{\Lambda}^{-1}(\theta) : (\mathbf{E} - \mathbf{E}_{elas}) \right] \right) \\ \quad - \text{Div} \left( -\widehat{k}(\theta) (1 + \text{Div} \mathbf{u}) \nabla \theta + 2\widehat{k}(\theta) \boldsymbol{\varepsilon}(\mathbf{u}) \nabla \theta \right) + f_*, \\ \dot{\mathbf{Z}} = \varphi(\theta) \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : \boldsymbol{\varepsilon}(\mathbf{u}) - (\mathbf{I} + \nabla \mathbf{u}) \left[ \widehat{\Lambda}^{-1}(\theta) : (\mathbf{E} - \mathbf{E}_{elas}) \right] \right). \end{array} \right. \quad (1.125a)$$

$$(1.125b)$$

Considering equalities (1.122)-(1.124) again, we deduce from equations (1.83) and (1.87) associated with the second linearization of motion equation (**2θLM**) and of energy equation (**2θLE**), respectively, taking into account expression (1.84) for Maxwell-Norton materials with heat conduction:

**Problem (2θL-MN)**

$$\left\{ \begin{array}{l} -\text{Div} \left( \widehat{\Lambda}^{-1}(\theta) : \left( \boldsymbol{\varepsilon}(\mathbf{u}) - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} \right) - (\mathbf{I} + \nabla \mathbf{u}) \widehat{\Lambda}^{-1}(\theta) : \mathbf{Z} \right) = \mathbf{b}_*, \\ \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}) \dot{\theta} = \theta \widehat{\Lambda}^{-1'}(\theta) : \left( \boldsymbol{\varepsilon}(\mathbf{u}) - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} - \mathbf{Z} \right) : \nabla \dot{\mathbf{u}} \\ \quad - \theta \left[ \nabla \mathbf{u} (3\widehat{\alpha}(\theta_0) \widehat{K}(\theta_0) \mathbf{I} + \widehat{\Lambda}^{-1'}(\theta) : \mathbf{Z}) + 3\widehat{\alpha}(\theta) \widehat{K}(\theta) \mathbf{I} \right] : \nabla \dot{\mathbf{u}} \\ \quad - \rho_0 \varphi(\theta) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}) : \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : \left( \boldsymbol{\varepsilon}(\mathbf{u}) - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} \right) - (\mathbf{I} + \nabla \mathbf{u}) \widehat{\Lambda}^{-1}(\theta) : \mathbf{Z} \right) \\ \quad - \text{Div} \left( -\widehat{k}(\theta) \nabla \theta - \widehat{k}(\theta_0) \text{Div} \mathbf{u} \nabla \theta + \widehat{k}(\theta_0) \nabla \mathbf{u} \nabla \theta + \widehat{k}(\theta_0) \nabla \mathbf{u}^t \nabla \theta \right) + f_*, \\ \dot{\mathbf{Z}} = \varphi(\theta) \partial \phi_q \left( \widehat{\Lambda}^{-1}(\theta) : \left( \boldsymbol{\varepsilon}(\mathbf{u}) - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} \right) - (\mathbf{I} + \nabla \mathbf{u}) \widehat{\Lambda}^{-1}(\theta) : \mathbf{Z} \right). \end{array} \right. \quad (1.126a)$$

$$(1.126b)$$

Taking into account equalities (1.122)-(1.124) again, we deduce from equations (1.91) and (1.96) associated with the third linearization of motion equation (**3ZLM**) and of energy equation (**3ZLE**), respectively, considering expression (1.92) for Maxwell-Norton materials with heat conduction:

**Problem (3ZL-MN)**

$$\left\{ \begin{array}{l} -\text{Div} \widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z}) = \mathbf{b}_*, \\ \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}) \dot{\theta} = \theta \left[ \frac{\partial \widehat{\boldsymbol{\sigma}}}{\partial \theta}(\theta, \mathbf{u}, \mathbf{Z}) - 3\widehat{\alpha}(\theta_0) \widehat{K}(\theta_0) \nabla \mathbf{u} \right] : \nabla \dot{\mathbf{u}} \\ \quad - \rho_0 \varphi(\theta) \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}) : \partial \phi_q(\widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z})) \\ \quad - \text{Div} \left( -\widehat{k}(\theta) \nabla \theta - \widehat{k}(\theta_0) \text{Div} \mathbf{u} \nabla \theta + \widehat{k}(\theta_0) \nabla \mathbf{u} \nabla \theta + \widehat{k}(\theta_0) \nabla \mathbf{u}^t \nabla \theta \right) + f_*, \\ \dot{\mathbf{Z}} = \varphi(\theta) \partial \phi_q(\widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z})), \end{array} \right.$$

being

$$\widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z}) = \widehat{\Lambda}^{-1}(\theta) : \boldsymbol{\varepsilon}_{elas}(\mathbf{u}), \quad (1.128)$$

with

$$\boldsymbol{\varepsilon}_{elas}(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathbf{u}) - \left( \int_{\theta_0}^{\theta} \widehat{\alpha}(r) dr \right) \mathbf{I} - \mathbf{Z}. \quad (1.129)$$

**Remark 1.5.10.** We notice that for Problem (3ZL-MN),  $\widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z}) = \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta)$  is the response function of  $\boldsymbol{\sigma}$  which is the linearized Second Piola-Kirchhoff stress tensor, considering a linearization for  $\widehat{\boldsymbol{\Sigma}}$  as the described one in the proof of Corollary 1.4.24.

**Proposition 1.5.11.** The classical mechanical behaviour law for Maxwell-Norton materials (see Friauf [52], Lemaitre and Chaboche [79] and Barral et al. [12]) is

$$\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(p, t)) = \overline{\widehat{\Lambda}(\theta(p, t)) : \boldsymbol{\sigma}(p, t)} + \widehat{\alpha}(\theta(p, t)) \dot{\theta}(p, t) \mathbf{I} + \varphi(\theta(p, t)) \partial \phi_q(\boldsymbol{\sigma}(p, t)),$$

for all  $p \in \mathcal{B}$  and  $t \in (0, t_f)$ .

*Proof.* Applying  $\widehat{\Lambda}(\theta)$  to expression (1.128), taking into account equality (1.129) and deriving with respect to the time, we obtain

$$\overline{\widehat{\Lambda}(\theta(p, t)) : \boldsymbol{\sigma}(p, t)} = \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(p, t)) - \widehat{\alpha}(\theta(p, t)) \dot{\theta}(p, t) \mathbf{I} - \dot{\mathbf{Z}}(p, t),$$

for all  $p \in \mathcal{B}$  and  $t \in (0, t_f)$ , with  $\dot{\mathbf{Z}}(p, t)$  given by

$$\dot{\mathbf{Z}}(p, t) = \varphi(\theta(p, t)) \partial \phi_q(\boldsymbol{\sigma}(p, t)).$$

Hence, we can deduce the result.  $\square$

**Remark 1.5.12.** The energy equation from Adam and Ponthot [1] is similar to our one of Problem (3ZL-MN): their coefficients also depend on temperature but they do not consider the term  $-3\widehat{\alpha}(\theta_0) \widehat{K}(\theta_0) \nabla \mathbf{u} : \nabla \dot{\mathbf{u}}$  and they only include the term  $-\widehat{k}(\theta) \nabla \theta$  for heat flux. Furthermore these authors consider the same motion equation like Problem (3ZL-MN).

**Remark 1.5.13.** The motion equation from Problem (3ZL-MN) has been considered to simulate the aluminium casting process (see Barral and Quintela [14] and Barral et al. [11]).



## Part II

Mathematical analysis of a viscoelastic  
problem with temperature-dependent  
coefficients.



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## Chapter 2

# Existence and uniqueness of a viscoelastic problem

### 2.1 Introduction

In this chapter, we consider a thermoviscoelastic problem with mixed displacement-traction boundary conditions. The behaviour law is the sum of the thermal and viscoelastic effects modeled by Arrhenius and Maxwell-Norton laws respectively, with coefficients depending on temperature. This problem arises from the modelling of processes in the aluminium industry (see Drezet *et al.* [36]) and it has several applications like extrusion and lamination of metals and alloys as well as the modelling of the deformation of alloy structures exposed to fire, where temperature changes play a fundamental role. Although for the numerical simulations of such deformation it is necessary to consider a fully thermomechanical coupled problem, for simplicity, in this part we only consider the mechanical problem assuming that the temperature has been previously computed. This problem coincides with the linearized motion equation of Problem **(3ZL-MN)** studied in the previous chapter.

The mathematical analysis has been divided into two chapters. In this first one, we prove the existence and uniqueness of solution, which was published in Barral *et al.* [12]; the regularity properties of this solution will be the aim of Chapter 3, whose results were gathered in paper Barral *et al.* [13].

In the literature there exist several existence results for viscoelastic problems of the Maxwell-Norton type: Djaoua and Suquet [35] and Le Tallec [78] consider mixed boundary conditions and Barral and Quintela [15] extend the results for a Signorini contact problem; in all these papers the coefficients are time independent but, as we have remarked, our aim is to model processes with temperature variation. In Blanchard and Le Tallec [22], the proof given in Le Tallec [78] is extended to a Maxwell-Norton law with viscoelastic coefficients depending on time and space; nevertheless, the behaviour law includes neither the thermal strains nor the derivative of the elasticity tensor with respect to time. It is also worth noting that there are several existence results for other viscoelastic and plastic behaviour laws: for example, a thermoviscoelastic law under contact

conditions with Coulomb friction is presented in Eck [46]; however, the plastic part is linear and the elastic and viscoelastic coefficients are temperature independent. In Hlaváček and Whiteman [69] a result of existence and uniqueness for elastoplastic laws with isotropic hardening is given but temperature dependence is not considered.

The main contribution of this part is that we assume a complete temperature dependence of the behaviour law. For that purpose, we follow the techniques used in Barral and Quintela [15] and Djaoua and Suquet [35]. The proof consists of discretizing the problem in time, proving the existence and uniqueness of solution for the discretized problem and passing to the limit. The main difficulties are calculating a priori estimates and deducing the constitutive law for the limit of the discrete solutions; those difficulties come from the temperature dependence of the elasticity tensor.

This chapter is outlined as follows: firstly, we will describe the mathematical model. After introducing in the Section 2.3 the appropriate functional framework, in Section 2.4 we will prove the existence and uniqueness of solution of the problem.

## 2.2 Mathematical Model

Following with the model developed in the previous chapter, we consider the mechanical submodel of Problem **(3ZL-MN)**, corresponding to the third linearization of the quasistatic motion equation **(3ZLM)** for Maxwell-Norton materials with heat conduction (see Subsection 1.5.4, Chapter 1). So, we are going to study the existence and uniqueness of a mechanical problem whose equilibrium equation is given by:

$$-\text{Div} \hat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z}) = \mathbf{b}_*, \quad (2.1)$$

with

$$\dot{\mathbf{Z}} = \partial \phi_q(\hat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z})). \quad (2.2)$$

The linearized Second Piola-Kirchhoff stress tensor  $\hat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z})$  is obtained as follows:

$$\hat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, \mathbf{Z}) = \hat{\Lambda}^{-1}(\theta) : \boldsymbol{\varepsilon}_{elas}(\mathbf{u}), \quad (2.3)$$

where

$$\boldsymbol{\varepsilon}_{elas}(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathbf{u}) - \left( \int_{\theta_0}^{\theta} \hat{\alpha}(r) dr \right) \mathbf{I} - \mathbf{Z}. \quad (2.4)$$

It is recalled that in expressions (2.1)-(2.4) we use the following notation:

- $\theta$  is the absolute temperature.
- $\mathbf{u}$  is the displacement and  $\boldsymbol{\varepsilon}_{elas}(\mathbf{u})$  is the elastic part of  $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$ .
- $\mathbf{Z}$  is the internal symmetric second order tensor and  $\dot{\mathbf{Z}}$  its evolution with respect to time.
- $\mathbf{b}_*$  are the body forces per unit volume at the reference configuration.

- $\phi_q$  is the plasticity potential defined from  $Sym$  to  $\mathbb{R}$  as

$$\phi_q(\boldsymbol{\tau}) = \frac{\Theta(\theta)}{q} |\boldsymbol{\tau}^D|^q, \quad (2.5)$$

where  $\Theta : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a smooth enough function, such that  $\Theta(s) \geq \Theta_{min} > 0$  and  $q \geq 2$  is a strictly positive material parameter. Given any tensor  $\boldsymbol{\tau}$ ,  $\boldsymbol{\tau}^D$  represents the deviatoric tensor  $\boldsymbol{\tau}^D = \boldsymbol{\tau} - \frac{1}{n} \text{tr}(\boldsymbol{\tau})\mathbf{I}$ ,  $n$  being the dimension of the space and  $\mathbf{I}$  being the identity tensor. Its subdifferential  $\partial\phi_q : Sym \rightarrow Sym_0$  is

$$\partial\phi_q(\boldsymbol{\tau}) = \Theta(\theta) |\boldsymbol{\tau}^D|^{q-2} \boldsymbol{\tau}^D. \quad (2.6)$$

Here  $|\cdot|$  denotes the norm induced by the usual scalar product,  $Sym$  is the subspace of  $Lin$  of all symmetric endomorphisms and  $Sym_0$  is the subspace of  $Lin$  of all symmetric endomorphisms whose trace is null.

Notice that equalities (2.1)-(2.2) with (2.5) are equivalent to expression (2.2) introduced in the previous chapter (see Problem **(3ZL-MN)** Subsection 1.5.4). In this part we take this expression of the plasticity potential in order to simplify and improve the notation of the behaviour law.

- $\widehat{\Lambda}$  is the elasticity tensorial function defined as

$$\begin{aligned} (\theta, p) \in \mathbb{R}^+ \times \mathcal{B} &\rightarrow \widehat{\Lambda}(\theta, p) : Sym \longrightarrow Sym \\ \boldsymbol{\tau} &\longrightarrow \widehat{\Lambda}(\theta, p) : \boldsymbol{\tau} = \frac{1 + \widehat{\nu}(\theta, p)}{\widehat{E}(\theta, p)} \boldsymbol{\tau} - \frac{\widehat{\nu}(\theta, p)}{\widehat{E}(\theta, p)} \text{tr}(\boldsymbol{\tau})\mathbf{I}, \end{aligned}$$

$\widehat{E}(\theta, p)$ ,  $\widehat{\nu}(\theta, p)$  being the response functions associated with Young's modulus and Poisson's coefficient, respectively, and  $p$  the material point. The inverse of  $\widehat{\Lambda}$  is defined by:

$$\begin{aligned} (\theta, p) \in \mathbb{R}^+ \times \mathcal{B} &\rightarrow \widehat{\Lambda}^{-1}(\theta, p) : Sym \longrightarrow Sym \\ \boldsymbol{\tau} &\longrightarrow \widehat{\Lambda}^{-1}(\theta, p) : \boldsymbol{\tau} = \widehat{\lambda}(\theta, p) \text{tr}(\boldsymbol{\tau})\mathbf{I} + 2\widehat{\mu}(\theta, p)\boldsymbol{\tau}, \end{aligned}$$

where  $\widehat{\lambda}(\theta, p)$ ,  $\widehat{\mu}(\theta, p)$  are the response functions associated with Lamé's parameters of the material and related to  $\widehat{E}$ ,  $\widehat{\nu}$  by

$$\widehat{\lambda}(\theta, p) = \frac{\widehat{E}(\theta, p)\widehat{\nu}(\theta, p)}{(1 + \widehat{\nu}(\theta, p))(1 - 2\widehat{\nu}(\theta, p))}, \quad \widehat{\mu}(\theta, p) = \frac{\widehat{E}(\theta, p)}{2(1 + \widehat{\nu}(\theta, p))}.$$

A detailed description of the elastic behaviour law can be found in Ciarlet [28] and Gurtin [65].

- $\widehat{\alpha} : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a smooth enough response function associated with coefficient of thermal expansion. In equation (2.4),  $\theta_0 \in \mathbb{R}^+$  is the initial temperature.

In this chapter, we choose as reference configuration the initial equilibrium position of the body:  $B_0 = \mathcal{B}$ . From now on, in order to simplify the notation we will omit the  $\widehat{\cdot}$  of the response functions and the  $\cdot_*$  of the body forces.

From equality (2.4), we can obtain the deformation tensor,  $\boldsymbol{\varepsilon}(\mathbf{u})$ , as a superposition of elastic,  $\boldsymbol{\varepsilon}_{elas}(\mathbf{u})$ , plastic,  $\boldsymbol{\varepsilon}_{plas}(\mathbf{u})$ , and thermal,  $\varepsilon_{ther}\mathbf{I}$ , contributions:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}_{elas}(\mathbf{u}) + \varepsilon_{ther}\mathbf{I} + \boldsymbol{\varepsilon}_{plas}(\mathbf{u}), \quad (2.7)$$

where:

- the elastic deformations are related to the stress tensor  $\boldsymbol{\sigma}$  by Hooke's law with coefficients depending on temperature, given by

$$\boldsymbol{\varepsilon}_{elas}(\mathbf{u}) = \Lambda(\theta) : \boldsymbol{\sigma}. \quad (2.8)$$

- the viscoplastic part is

$$\dot{\boldsymbol{\varepsilon}}_{plas}(\mathbf{u}) = \dot{\mathbf{Z}}.$$

Notice that from equality (2.2), we obtain that

$$\dot{\boldsymbol{\varepsilon}}_{plas}(\mathbf{u}) = \partial\phi_q(\boldsymbol{\sigma}), \quad (2.9)$$

which is the classical Norton-Hoff law (see Friaâ [51, 52]):

- and finally, we consider that the thermal law is a generalization of Arrhenius's law:

$$\varepsilon_{ther} = \int_{\theta_0}^{\theta} \alpha(r) \, dr. \quad (2.10)$$

Furthermore, if we derive equality (2.7) with respect to the time and we replace equations (2.8)-(2.10), we obtain the classical mechanical behaviour law for Maxwell-Norton materials:

$$\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) = \overline{\Lambda(\theta) : \boldsymbol{\sigma}} + \alpha(\theta)\dot{\theta}\mathbf{I} + \partial\phi_q(\boldsymbol{\sigma}). \quad (2.11)$$

In Proposition 1.5.11 of the previous chapter we can check the calculus. This law were studied in the works of Drezet *et al.* [36], Mariaux *et al.* [84] and Barral and Quintela [14, 15].

**Remark 2.2.1.** *An example of an alloy that can be modelled with this kind of law is aluminium (see Lemaitre and Chaboche [79], Drezet and Plata [37] and Drezet et al. [36]) for which*

$$\Theta(\theta) = \Theta_0 e^{\frac{-G}{R(\theta+273)}},$$

where  $\Theta_0$  is a strictly positive material parameter,  $G$  is the activation energy for the process and  $R$  is the universal gas constant.

**Remark 2.2.2.** *In practice, to take into account the volume changes due to possible phase transformations  $\alpha$  is given by*

$$\alpha(\theta) = -\frac{1}{3} \frac{\rho_0^{\frac{1}{3}}}{\rho(\theta)^{\frac{4}{3}}} \frac{d\rho(\theta)}{d\theta},$$

$\rho$  being the density of the material and  $\rho_0 = \rho(\theta_0)$  its density at the initial temperature.

**Remark 2.2.3.** *In this part, as in whole dissertation thesis we consider the case  $n = 3$ , but the results of this part are also true for the case  $n = 2$ . Therefore, we keep the notation with  $n$ .*

### 2.2.1 Plasticity potential's properties

In this part we summarize the main properties of the plasticity potential  $\phi_q$ , which will be needed hereinafter. It verifies:

- $\phi_q$  is a convex, proper and lower semicontinuous (l.s.c.) function on  $S_n$  (see Friaâ [52], p. 8), being  $S_n$  the space of symmetric second order tensors over  $\mathbb{R}^n$ .
- $\phi_q$  is continuous and differentiable on  $S_n$ , so its subdifferential coincides with its derivative (see Friaâ [52], p. 9).
- Since  $\phi_q$  is convex and differentiable, its subdifferential  $\partial\phi_q$  is monotonous on  $S_n$  (see Ekeland and Teman [47]).
- $\partial\phi_q$  is hemicontinuous (see Lions [82], p. 157), this is: for all  $\tau_1, \tau_2, \tau_3 \in S_n$  the function

$$s \in \mathbb{R} \rightarrow \partial\phi_q(\tau_1 + s\tau_2) : \tau_3, \quad (2.12)$$

is continue from  $\mathbb{R}$  into  $\mathbb{R}$ .

- Furthermore, since plasticity potential  $\phi_q$  is convex, its integral over  $\Omega$  is also convex (see Ekeland and Teman [47]) and it verifies (see Djaoua and Suquet [35], p. 197):

$$\int_{\Omega} \phi_q(\tau_1) dp - \int_{\Omega} \phi_q(\tau_2) dp \leq \int_{\Omega} \partial\phi_q(\tau_1) : (\tau_1 - \tau_2) dp, \quad \forall \tau_1, \tau_2 \in S_n. \quad (2.13)$$

### 2.2.2 Problem ( $P_T$ )

Let  $\Omega \subset \mathbb{R}^n$ , be a bounded, open, connected region with a Lipschitz boundary  $\Gamma = \partial\Omega$  and  $\mathbf{n}$  its outward unit normal vector. We assume that  $\Gamma_D$  and  $\Gamma_N$  are open subsets of  $\Gamma$ , verifying

- $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ ,
- $\Gamma_D \cap \Gamma_N = \emptyset$ ,

and  $\text{meas}(\Gamma_D) > 0$ . We represent by  $[0, t_f]$  the time interval of interest. At each time  $t \in [0, t_f]$ , the temperature field  $\theta(p, t)$  at each point  $p \in \Omega$  is assumed to be known. The mechanical problem consists of determining the displacement field  $\mathbf{u}(p, t)$  and the stress tensor  $\boldsymbol{\sigma}(p, t) = \boldsymbol{\sigma}(\theta(p, t), \mathbf{u}(p, t), \mathbf{Z}(p, t))$  at each  $(p, t) \in \Omega \times (0, t_f]$ . Then, the problem we are going to study is the following:

#### **Problem ( $P_T$ )**

Given  $\theta(p, t)$  in  $\Omega \times [0, t_f]$ , find  $\mathbf{u}(p, t)$  and  $\boldsymbol{\sigma}(p, t)$  in  $\Omega \times (0, t_f]$ , verifying:

Equilibrium equation:

$$-\text{Div } \boldsymbol{\sigma} = \mathbf{b} \quad \text{in } \Omega \times (0, t_f]. \quad (2.14)$$

Behaviour law:

$$\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) = \overline{\Lambda(\theta) : \boldsymbol{\sigma}} + \partial\phi_q(\boldsymbol{\sigma}) + \alpha(\theta)\dot{\theta}\mathbf{I} \quad \text{in } \Omega \times (0, t_f]. \quad (2.15)$$

Boundary conditions:

$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N \times (0, t_f], \quad (2.16)$$

$$\mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma_D \times (0, t_f]. \quad (2.17)$$

Initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega. \quad (2.18)$$

Here, we use the following notation:

- $\mathbf{g}$  is the density of surface forces on the Neumann boundary, and
- $\mathbf{u}_D$  is the displacement on the Dirichlet boundary.

In order to complete the mathematical model, we consider the following set of initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega,$$

where  $\mathbf{u}_0$  and  $\boldsymbol{\sigma}_0$  must verify the following compatibility conditions:

$$\left\{ \begin{array}{lll} -\text{Div } \boldsymbol{\sigma}_0 & = & \mathbf{b}(0) \quad \text{in } \Omega, \\ \boldsymbol{\sigma}_0 \mathbf{n} & = & \mathbf{g}(0) \quad \text{on } \Gamma_N, \\ \mathbf{u}_0 & = & \mathbf{u}_D(0) \quad \text{on } \Gamma_D, \\ \boldsymbol{\varepsilon}(\mathbf{u}_0) & = & \Lambda(\theta_0)\boldsymbol{\sigma}_0 \quad \text{in } \Omega. \end{array} \right. \quad (2.19)$$

Then, at the beginning of the process  $\boldsymbol{\varepsilon}^{plas}(\mathbf{u}_0) = \mathbf{0}$  and  $\boldsymbol{\varepsilon}^{ther} = 0$ .

## 2.3 Choice of a Functional Framework

In this section, we first introduce the spaces of vectors and tensors defined in the domain  $\Omega$  which were studied by Geymonat and Suquet [63]. Next, for the time dependence we use the usual functional spaces (see for instance Lions and Magenes [83]).

From now on, we suppose that the parameter  $q$  of the Norton-Hoff law verifies  $q \geq 2$ , so its conjugate  $p$  is such that  $1 < p \leq 2$ .

### 2.3.1 Space of displacement fields

Let us introduce the displacement space

$$\mathbf{U}^p = \{\mathbf{v} \in [W^{1,p}(\Omega)]^n; \text{Div } \mathbf{v} = \text{tr}(\boldsymbol{\varepsilon}(\mathbf{v})) \in L^2(\Omega)\}.$$

$\mathbf{U}^p$  is a Banach space with the norm

$$\|\mathbf{v}\|_{\mathbf{U}^p} = \|\mathbf{v}\|_{[L^p(\Omega)]^n} + \|\varepsilon^D(\mathbf{v})\|_{[L^p(\Omega)]^{n \times n}} + \|\text{Div } \mathbf{v}\|_{L^2(\Omega)}.$$

In order to consider the boundary condition on  $\Gamma_D$  we define the closed subspace of  $\mathbf{U}^p$ :

$$\mathbf{U}_0^p = \{\mathbf{v} \in \mathbf{U}^p; \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}.$$

### 2.3.2 Space of stress fields

Let us define the space of stresses

$$\mathbf{X}^q = \{\boldsymbol{\tau} = (\tau_{ij}); \tau_{ij} = \tau_{ji}, \boldsymbol{\tau}^D \in [L^q(\Omega)]^{n \times n}, \text{tr}(\boldsymbol{\tau}) \in L^2(\Omega)\}.$$

$\mathbf{X}^q$  is a Banach space with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{X}^q} = \|\boldsymbol{\tau}^D\|_{[L^q(\Omega)]^{n \times n}} + \|\text{tr}(\boldsymbol{\tau})\|_{L^2(\Omega)}.$$

We introduce the subspace of  $\mathbf{X}^q$

$$\mathbf{H}^q = \{\boldsymbol{\tau} \in \mathbf{X}^q; \text{Div } \boldsymbol{\tau} \in [L^q(\Omega)]^n\},$$

which is also a Banach space with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{H}^q} = \|\boldsymbol{\tau}\|_{\mathbf{X}^q} + \|\text{Div } \boldsymbol{\tau}\|_{[L^q(\Omega)]^n}.$$

It can be proved that the space of distributions  $[D(\bar{\Omega})]^{n \times n}$  is dense in  $\mathbf{H}^q$  (see Geymonat and Suquet [63]). Moreover, the following properties are verified:

**Lemma 2.3.1.** *The mapping*

$$\boldsymbol{\tau} \in \mathbf{H}^q \rightarrow \boldsymbol{\tau} \mathbf{n} \in [W^{-\frac{1}{q}, q}(\Gamma)]^n,$$

*is linear and continuous. Furthermore, the following Green's formula is verified:*

$$\int_{\Omega} \boldsymbol{\tau} : \varepsilon(\mathbf{v}) \, dp + \int_{\Omega} \text{Div } \boldsymbol{\tau} \cdot \mathbf{v} \, dp = \langle \boldsymbol{\tau} \mathbf{n}, \mathbf{v} \rangle_{\Gamma}, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q, \quad \forall \mathbf{v} \in \mathbf{U}^p,$$

*where  $\langle \cdot, \cdot \rangle_{\Gamma}$  denotes the duality product between  $[W^{-\frac{1}{q}, q}(\Gamma)]^n$  and  $[W^{1-\frac{1}{p}, p}(\Gamma)]^n$ .*

Finally, given  $\mathbf{b} \in [L^q(\Omega)]^n$  and  $\mathbf{g} \in [W^{-\frac{1}{q}, q}(\Gamma)]^n \cap [L^q(\Gamma_N)]^n$ , we introduce the following space of stresses

$$\mathbf{H}^q(\mathbf{b}, \mathbf{g}) = \{\boldsymbol{\tau} \in \mathbf{H}^q; -\text{Div } \boldsymbol{\tau} = \mathbf{b} \text{ in } \Omega, \boldsymbol{\tau} \mathbf{n} = \mathbf{g} \text{ on } \Gamma_N\}.$$

The following results, whose proofs can be found in Geymonat and Suquet [63], give an orthogonality relation between stresses belonging to  $\mathbf{X}^q$  and deformation strain for displacements in  $\mathbf{U}^p$ .

**Lemma 2.3.2.** *For all  $\boldsymbol{\tau} \in \mathbf{X}^q$  and  $\mathbf{v} \in \mathbf{U}^p$  the following relation is satisfied:*

$$\int_{\Omega} \boldsymbol{\tau} : \varepsilon(\mathbf{v}) \, dp = \int_{\Omega} \boldsymbol{\tau}^D : \varepsilon^D(\mathbf{v}) \, dp + \frac{1}{n} \int_{\Omega} \text{tr}(\boldsymbol{\tau}) \text{tr}(\varepsilon(\mathbf{v})) \, dp.$$

**Proposition 2.3.3.** *Let  $\mathbf{e} \in \mathbf{X}^p$ ,  $1 < p \leq 2$ . A necessary and sufficient condition for the existence of  $\mathbf{v} \in \mathbf{U}_0^p$ , such that  $\varepsilon(\mathbf{v}) = \mathbf{e}$ , is that*

$$\int_{\Omega} \boldsymbol{\tau} : \mathbf{e} \, dp = 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}).$$

## 2.4 Existence and Uniqueness of Solution

### 2.4.1 Assumptions

From now on we assume the following hypotheses:

**(H1)** The temperature field  $\theta \in W^{2,\infty}(0, t_f; L^\infty(\Omega))$ . Moreover, it is positive and  $\dot{\theta} \geq 0$  *a.e.* in  $\Omega$ ,  $\forall t \in [0, t_f]$ .

**(H2)** The applied forces verify:

$$\mathbf{b} \in W^{2,\infty}(0, t_f; [L^q(\Omega)]^n), \quad \mathbf{g} \in W^{2,\infty}(0, t_f; [W^{-\frac{1}{q},q}(\Gamma)]^n \cap [L^q(\Gamma_N)]^n).$$

Consequently, there exists  $\boldsymbol{\chi} \in W^{2,\infty}(0, t_f; \mathbf{X}^q)$  such that  $\boldsymbol{\chi}(t) \in \mathbf{H}^q(\mathbf{b}(t), \mathbf{g}(t))$ ,  $\forall t \in [0, t_f]$  (see Djaoua and Suquet [35]).

**(H3)** The tensor of Hooke's law  $\Lambda$  is symmetric and the following hold true:

- $\Lambda \in [W^{2,\infty}(\mathbb{R})]^{n^4}$ , therefore, there exists  $\beta_1 > 0$  such that  $|\Lambda(s)\boldsymbol{\tau}| \leq \beta_1|\boldsymbol{\tau}|$  and there exists  $\beta_2 > 0$  such that  $|\Lambda'(s)\boldsymbol{\tau}| \leq \beta_2|\boldsymbol{\tau}|$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .
- there exists  $\kappa > 0$  such that  $\Lambda(s)\boldsymbol{\tau} : \boldsymbol{\tau} \geq \kappa |\boldsymbol{\tau}|^2$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .
- $\Lambda'(s)\boldsymbol{\tau} : \boldsymbol{\tau} \geq 0$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .

**(H4)** The coefficient of thermal expansion  $\alpha \in W^{1,\infty}(\mathbb{R})$ .

**(H5)** The coefficient of the plasticity potential  $\Theta \in L^\infty(\mathbb{R})$  and  $\Theta(s) \geq \Theta_{min} > 0$ ,  $\forall s \in \mathbb{R}$ .

**(H6)** The initial stress  $\boldsymbol{\sigma}_0$  and the initial displacement field  $\mathbf{u}_0$  verify:

$$\boldsymbol{\sigma}_0 \in \mathbf{H}^q(\mathbf{b}(0), \mathbf{g}(0)) \quad \text{and} \quad \mathbf{u}_0 \in \mathbf{U}^p.$$

Furthermore, they satisfy the compatibility conditions (2.19).

**(H7)** The boundary Dirichlet condition satisfies:

$$\mathbf{u}_D \in W^{2,\infty}(0, t_f; [W^{1-\frac{1}{p},p}(\Gamma)]^n).$$

**(H8)** The exponent  $q$  verifies  $q \geq 2$ , and its conjugate  $p$  is such that  $1 < p \leq 2$ .

**Remark 2.4.1.** Taking into account hypothesis **(H1)**, **(H3)**, **(H4)** and **(H5)**, we have:

- $\Lambda(\theta) \in W^{2,\infty}(0, t_f; [L^\infty(\Omega)]^{n^4})$ ,
- $\alpha(\theta) \in W^{1,\infty}(0, t_f; L^\infty(\Omega))$ , and
- $\Theta(\theta) \in L^\infty(0, t_f; L^\infty(\Omega))$ .

**Remark 2.4.2.** *The hypothesis*

$$(\Lambda'(s) : \boldsymbol{\tau}) : \boldsymbol{\tau} \geq 0, \quad \forall \boldsymbol{\tau} \in S_n, \quad \forall s \in \mathbb{R},$$

is necessary for the treatment of the time derivative of elastic stress (see behaviour law (2.15)). This hypothesis is verified for a thermomechanical process with a temperature increment, as in building fires. It is easy to check this expression for materials which are used generally under these premises.

**Remark 2.4.3.** *Thanks to hypothesis (H1) and (H3), we obtain*

$$\overline{(\Lambda(\theta) : \boldsymbol{\tau})} : \boldsymbol{\tau} \geq 0, \quad \forall \boldsymbol{\tau} \in S_n \text{ a.e. in } \Omega, \quad \forall t \in [0, t_f].$$

**Remark 2.4.4.** *From hypothesis (H7) we can deduce that there exists a lifting of*

$$\mathbf{u}_D \in W^{2,\infty}(0, t_f; [W^{1-\frac{1}{p}, p}(\Gamma)]^n) \text{ on } W^{2,\infty}(0, t_f; \mathbf{U}^p),$$

(see Geymonat and Suquet [63]). From now on we denote this lifting by  $\mathbf{u}_D$ .

**Remark 2.4.5.** *The above assumptions are satisfied in extrusion or lamination processes of aluminium alloys.*

## 2.4.2 Existence and uniqueness of a solution of Problem ( $P_T$ )

**Theorem 2.4.6.** *Under assumptions (H1)-(H8) there exists a unique solution  $(\mathbf{u}, \boldsymbol{\sigma})$  of Problem ( $P_T$ ) such that  $\mathbf{u} \in W^{1,2}(0, t_f; \mathbf{U}^p)$  and  $\boldsymbol{\sigma} \in W^{1,2}(0, t_f; \mathbf{X}^2) \cap L^\infty(0, t_f; \mathbf{X}^q)$ .*

The proof is divided into eight steps with the following scheme:

- i) a change of variable by translation transforms Problem ( $P_T$ ) into a homogeneous one,
- ii) this problem is discretized in time by an implicit scheme obtaining a variational inequality in stresses,
- iii) the weak discretized problem has a unique solution,
- iv) the existence and uniqueness of the displacement field is obtained,
- v) a priori estimates are achieved, that let us pass to the limit,
- vi) the convergence of the sequences is established,
- vii) the limit is a solution of the Problem ( $P_T$ ), which is checked by means of monotony's techniques,
- viii) and finally, the uniqueness of the continuous problem is proved.

### Step I: The homogeneous problem

In order to transform Problem  $(P_T)$  into a homogeneous problem we introduce the unknowns  $\boldsymbol{\eta} = \boldsymbol{\sigma} - \boldsymbol{\chi} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  and  $\underline{\mathbf{u}} = \mathbf{u} - \mathbf{u}_D \in \mathbf{U}_0^p$  with  $\boldsymbol{\chi}$  as in **(H2)**. With respect to these new unknowns, the behaviour law satisfies:

$$\varepsilon(\underline{\dot{\mathbf{u}}}) = \varepsilon(\dot{\mathbf{u}}) - \varepsilon(\dot{\mathbf{u}}_D) = \overline{\Lambda(\theta) : \dot{\boldsymbol{\eta}}} + \overline{\Lambda(\theta) : \dot{\boldsymbol{\chi}}} + \alpha(\theta)\dot{\theta}\mathbf{I} + \partial\phi_q(\boldsymbol{\eta} + \boldsymbol{\chi}) - \varepsilon(\dot{\mathbf{u}}_D) \text{ in } \Omega \times (0, t_f].$$

Problem  $(P_T)$  can be rewritten as follows:

#### **Problem** $(HP_T)$

Given  $\theta$  in  $\Omega \times [0, t_f]$ , find  $\underline{\mathbf{u}}$  and  $\boldsymbol{\eta}$  such that:

$$\begin{cases} \text{Div } \boldsymbol{\eta} = \mathbf{0} & \text{in } \Omega \times (0, t_f], & (2.20a) \\ \boldsymbol{\eta} \mathbf{n} = \mathbf{0} & \text{on } \Gamma_N \times (0, t_f], & (2.20b) \\ \underline{\mathbf{u}} = \mathbf{0} & \text{on } \Gamma_D \times (0, t_f], & (2.20c) \\ \varepsilon(\underline{\dot{\mathbf{u}}}) - \overline{\Lambda(\theta) : \dot{\boldsymbol{\eta}}} + \underline{\mathbf{g}} = \partial\phi_q(\boldsymbol{\eta} + \boldsymbol{\chi}) & \text{in } \Omega \times (0, t_f], & (2.20d) \\ \underline{\mathbf{u}}(0) = \mathbf{0}, \boldsymbol{\eta}(0) = \boldsymbol{\sigma}_0 - \boldsymbol{\chi}(0) & \text{in } \Omega, & (2.20e) \end{cases}$$

where  $\underline{\mathbf{g}} = \varepsilon(\dot{\mathbf{u}}_D) - \overline{\Lambda(\theta) : \dot{\boldsymbol{\chi}}} - \alpha(\theta)\dot{\theta}\mathbf{I}$ .

### Step II: Time discretization

Let us consider a regular partition of the time interval  $[0, t_f]$  into  $I + 1$  points:  $t^i = i\Delta t$ ,  $i = 0, 1, \dots, I$ , with step size  $\Delta t = \frac{t_f}{I}$ . From now on, given any function  $f$  we will denote by  $f^i$  its approximation at time  $t^i$ . Then, using an implicit Euler method, we obtain the following scheme:

Given  $\boldsymbol{\eta}^0 = \boldsymbol{\eta}(0) = \boldsymbol{\sigma}_0 - \boldsymbol{\chi}(0)$ ,  $\underline{\mathbf{u}}^0 = \underline{\mathbf{u}}(0) = \mathbf{0}$ , for  $i > 0$  and  $\underline{\mathbf{u}}^{i-1}$ ,  $\boldsymbol{\eta}^{i-1}$  known,  $\underline{\mathbf{u}}^i \in \mathbf{U}_0^p$ ,  $\boldsymbol{\eta}^i \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  are determined as weak solution of the equation

$$\varepsilon\left(\frac{\underline{\mathbf{u}}^i - \underline{\mathbf{u}}^{i-1}}{\Delta t}\right) - \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} + \underline{\mathbf{g}}^i = \partial\phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i), \quad (2.21)$$

with

$$\underline{\mathbf{g}}^i = (\varepsilon(\dot{\mathbf{u}}_D))^i - \left(\overline{\Lambda(\theta) : \dot{\boldsymbol{\chi}}}\right)^i - \alpha(\theta^i)(\dot{\theta})^i \mathbf{I}, \quad (2.22)$$

and  $\phi_q^i$  given by (2.5) for  $\theta^i = \theta(p, t^i)$ .

For this, let us denote by  $\bar{\mathbf{g}}^i = \underline{\mathbf{g}}^i - \frac{\varepsilon(\underline{\mathbf{u}}^{i-1})}{\Delta t}$ , then (2.21) can be written as

$$\varepsilon\left(\frac{\underline{\mathbf{u}}^i}{\Delta t}\right) - \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} + \bar{\mathbf{g}}^i = \partial\phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i). \quad (2.23)$$

Multiplying this equality by  $\boldsymbol{\tau} - \boldsymbol{\eta}^i$ , integrating over  $\Omega$  and taking into account that  $\phi_q$  is differentiable and convex (see Subsection 2.2.1), we get

$$\begin{aligned} & \int_{\Omega} \varepsilon \left( \frac{\mathbf{u}^i}{\Delta t} \right) : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp - \int_{\Omega} \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp \\ & + \int_{\Omega} \bar{\mathbf{g}}^i : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp \leq \int_{\Omega} \phi_q^i(\boldsymbol{\tau} + \boldsymbol{\chi}^i) dp - \int_{\Omega} \phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) dp, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}). \end{aligned}$$

Using the orthogonality property given by Proposition 2.3.3, we have the following variational stress problem

$$\begin{aligned} & - \int_{\Omega} \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp + \int_{\Omega} \bar{\mathbf{g}}^i : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp \\ & \leq \int_{\Omega} \phi_q^i(\boldsymbol{\tau} + \boldsymbol{\chi}^i) dp - \int_{\Omega} \phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) dp, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}). \end{aligned} \quad (2.24)$$

If we denote by

$$\Psi_q^i(\boldsymbol{\tau}) = \int_{\Omega} \phi_q^i(\boldsymbol{\tau}) dp, \quad (2.25)$$

and

$$\tilde{\mathbf{g}}^i = \bar{\mathbf{g}}^i + \frac{\Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t},$$

the inequality (2.24) can be rewritten as the following discretized problem in stresses:

**Problem**  $(HP_T)^i$

Find  $\boldsymbol{\eta}^i \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  such that for all  $\boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  verifies

$$\int_{\Omega} \left( \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} - \tilde{\mathbf{g}}^i \right) : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp + \Psi_q^i(\boldsymbol{\tau} + \boldsymbol{\chi}^i) - \Psi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \geq 0. \quad (2.26)$$

**Step III: Existence and uniqueness of a solution of the discretized problem**

In order to prove the existence and uniqueness of solution of Problem  $(HP_T)^i$  we use the following result, which can be directly deduced from Theorem 3.1 and Remark 3.4 in Chapter II of Ekeland and Teman [47], p. 40.

**Theorem 2.4.7.** *Let  $\mathbf{V}$  be a reflexive Banach space and  $\mathbf{V}^*$  its dual. Let  $\mathcal{A}$  be an operator from  $\mathbf{V}$  into its dual  $\mathbf{V}^*$  and  $\varphi$  a function from  $\mathbf{V}$  into  $\mathbb{R}$  verifying:*

- $\varphi$  is convex, lower semicontinuous (l.s.c) and proper function,
- $\mathcal{A}$  is weakly continuous on the finite dimension subspaces from  $\mathbf{V}$ ,
- $\mathcal{A}$  is monotonous:  $\langle \mathcal{A}\mathbf{w} - \mathcal{A}\mathbf{v}, \mathbf{w} - \mathbf{v} \rangle \geq 0, \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}$ ,
- there exists  $\mathbf{v}_0$  in the domain of  $\varphi$  such that

$$\frac{\langle \mathcal{A}\mathbf{v}, \mathbf{v} - \mathbf{v}_0 \rangle + \varphi(\mathbf{v})}{\|\mathbf{v}\|} \rightarrow +\infty \text{ if } \|\mathbf{v}\| \rightarrow \infty.$$

Therefore, given  $\mathbf{l} \in \mathbf{V}^*$  there exists  $\mathbf{w} \in \mathbf{V}$  verifying:

$$\langle \mathcal{A}\mathbf{w} - \mathbf{l}, \mathbf{v} - \mathbf{w} \rangle + \varphi(\mathbf{v}) - \varphi(\mathbf{w}) \geq 0, \quad \forall \mathbf{v} \in \mathbf{V}. \quad (2.27)$$

Furthermore, if operator  $\mathcal{A}$  is strictly monotonous, there exists a unique  $\mathbf{w} \in \mathbf{V}$  verifying inequality (2.27).

**Proposition 2.4.8.** *Under assumptions (H1)-(H8) given  $\tilde{\mathbf{g}}^i \in \mathbf{X}^p$  Problem  $(HP_T)^i$  admits a unique solution  $\boldsymbol{\eta}^i \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  for each  $i$ ,  $1 \leq i \leq I$ .*

*Proof.* The proof is based on Theorem 2.4.7 which is a classical result for variational inequalities. We define the operator  $\mathcal{A}^i$  from  $\mathbf{X}^q$  into its dual  $\mathbf{X}^p$  as

$$\langle \mathcal{A}^i \boldsymbol{\xi}, \boldsymbol{\tau} \rangle = \int_{\Omega} \left( \frac{\Lambda(\theta^i)}{\Delta t} : \boldsymbol{\xi} \right) : \boldsymbol{\tau} dp,$$

where  $\langle \cdot, \cdot \rangle$  denotes duality pairing on  $\mathbf{X}^p \times \mathbf{X}^q$ . We consider

$$\varphi(\boldsymbol{\xi}) = \mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\boldsymbol{\xi}) + \Psi_q^i(\boldsymbol{\xi} + \boldsymbol{\chi}^i), \quad \boldsymbol{\xi} \in \mathbf{X}^q, \quad (2.28)$$

where  $\mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}$  is the indicator function of  $\mathbf{H}^q(\mathbf{0}, \mathbf{0})$ ,

$$\mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\boldsymbol{\xi}) = \begin{cases} 0 & \text{if } \boldsymbol{\xi} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}), \\ +\infty & \text{if } \boldsymbol{\xi} \notin \mathbf{H}^q(\mathbf{0}, \mathbf{0}). \end{cases} \quad (2.29)$$

From equalities (2.5) and (2.25), we obtain

$$\Psi_q^i(\boldsymbol{\xi} + \boldsymbol{\chi}^i) = \frac{1}{q} \int_{\Omega} \Theta(\theta^i) |(\boldsymbol{\xi} + \boldsymbol{\chi}^i)^D|^q dp, \quad \text{with } \boldsymbol{\chi}^i \in \mathbf{H}^q(\mathbf{b}^i, \mathbf{g}^i).$$

With this notation the Problem  $(HP_T)^i$  can be rewritten as:

Find  $\boldsymbol{\eta}^i \in \mathbf{X}^q$  such that

$$\langle \mathcal{A}\boldsymbol{\eta}^i - \tilde{\mathbf{g}}^i, \boldsymbol{\tau} - \boldsymbol{\eta}^i \rangle + \varphi(\boldsymbol{\tau}) - \varphi(\boldsymbol{\eta}^i) \geq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{X}^q. \quad (2.30)$$

This problem has the following properties:

- The functional  $\varphi$  is convex, proper and l.s.c. Indeed, these properties can be deduced taking into account that  $\mathbf{H}^q(\mathbf{0}, \mathbf{0})$  is a convex and closed, not empty, subset of  $\mathbf{X}^q$  then its indicator function  $\mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}$  is convex, l.s.c and  $\mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})} \not\equiv +\infty$  (see Brezis [25], p. 13). Besides, the potential  $\phi_q^i$  is convex, proper and continuous (see Subsection 2.2.1), therefore  $\Phi_q^i$  is convex, proper and l.s.c. since it is the integral of a convex, proper and continuous function. Hence,  $\varphi$  is a convex, proper and l.s.c. function because it is the addition of two convex, proper and l.s.c. functions.
- Thanks to hypothesis (H3) and applying Hölder's inequality, the operator  $\mathcal{A}$  is continuous:

$$|\langle \mathcal{A}\boldsymbol{\xi}, \boldsymbol{\tau} \rangle| \leq \int_{\Omega} \left| \frac{\Lambda(\theta^i)}{\Delta t} : \boldsymbol{\xi} \right| : \boldsymbol{\tau} dp \leq \frac{\beta_1}{\Delta t} \|\boldsymbol{\xi}\|_{\mathbf{X}^p} \|\boldsymbol{\tau}\|_{\mathbf{X}^q} \leq \tilde{c} \frac{\beta_1}{\Delta t} \|\boldsymbol{\xi}\|_{\mathbf{X}^q} \|\boldsymbol{\tau}\|_{\mathbf{X}^q}, \quad \forall \boldsymbol{\xi}, \boldsymbol{\tau} \in \mathbf{X}^q,$$

where  $\check{c}$  is a strictly positive constant. In consequence, it is weakly continuous on the finite dimension subspaces from  $\mathbf{X}^q$ . Furthermore, taking into account again hypothesis **(H3)**,  $\mathcal{A}$  is strictly monotonous:

$$\langle \mathcal{A}\xi - \mathcal{A}\tau, \xi - \tau \rangle \geq \frac{\kappa}{\Delta t} \|\xi - \tau\|_{\mathbf{X}^2}^2 > 0, \forall \xi, \tau \in \mathbf{X}^q, \xi \neq \tau,$$

since  $\langle \mathcal{A}\xi - \mathcal{A}\tau, \xi - \tau \rangle = 0 \Leftrightarrow \xi = \tau$ .

By definition of operator  $\mathcal{A}$ , we get for  $\tau \in \mathbf{X}^q$

$$\frac{\langle \mathcal{A}\xi, \xi - \tau \rangle + \varphi(\xi)}{\|\xi\|_{\mathbf{X}^q}} = \frac{1}{\|\xi\|_{\mathbf{X}^q}} \left[ \int_{\Omega} \frac{\Lambda(\theta^i) : \xi}{\Delta t} : (\xi - \tau) dp + \chi_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\xi) + \int_{\Omega} \phi_q(\xi + \chi^i) dp \right].$$

Let us take  $\tau = \mathbf{0} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}) \subset \mathbf{X}^q$  and from hypotheses **(H1)**, **(H3)**, **(H5)** and **(H8)** we can deduce,

$$\begin{aligned} \frac{\langle \mathcal{A}\xi, \xi \rangle + \varphi(\xi)}{\|\xi\|_{\mathbf{X}^q}} &= \frac{1}{\|\xi\|_{\mathbf{X}^q}} \left[ \int_{\Omega} \frac{\Lambda(\theta^i) : \xi}{\Delta t} : \xi dp + \chi_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\xi) + \int_{\Omega} \frac{\Theta(\theta^i)}{q} |(\xi + \chi^i)^D|^q dp \right] \\ &\geq \frac{1}{\|\xi\|_{\mathbf{X}^q}} \left( \frac{\kappa}{\Delta t} \|\xi\|_{\mathbf{X}^2}^2 + \chi_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\xi) + \frac{\Theta_{min}}{q} \|(\xi + \chi^i)^D\|_{[L^q(\Omega)]^{n \times n}}^q \right) \\ &\geq \frac{c}{\|\xi\|_{\mathbf{X}^q}} \left( \|\xi\|_{\mathbf{X}^2}^2 + \chi_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\xi) + \|(\xi + \chi^i)^D\|_{[L^q(\Omega)]^{n \times n}}^q \right), \end{aligned}$$

where  $c$  is a positive constant. Therefore,

$$\lim_{\|\xi\|_{\mathbf{X}^q} \rightarrow +\infty} \frac{\langle \mathcal{A}\xi, \xi \rangle + \varphi(\xi)}{\|\xi\|_{\mathbf{X}^q}} = +\infty. \quad (2.31)$$

Indeed, since

$$\|\xi\|_{\mathbf{X}^q} = \|\xi^D\|_{[L^q(\Omega)]^{n \times n}} + \|\text{tr } \xi\|_{L^2(\Omega)},$$

then, as  $\|\xi\|_{\mathbf{X}^q} \rightarrow +\infty$  we have three possibilities:

– If  $\|\xi^D\|_{[L^q(\Omega)]^{n \times n}} \rightarrow +\infty$ :

$$\begin{aligned} \frac{\langle \mathcal{A}\xi, \xi \rangle + \varphi(\xi)}{\|\xi\|_{\mathbf{X}^q}} &\geq \frac{k\Theta_{min}}{q\Delta t \|\xi\|_{\mathbf{X}^q}} \left( \|\xi\|_{\mathbf{X}^2}^2 + \chi_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\xi) + \|(\xi + \chi^i)^D\|_{[L^q(\Omega)]^{n \times n}}^q \right) \\ &\geq \frac{2k\Theta_{min}}{q\Delta t \|\xi\|_{\mathbf{X}^q}} \left( \|\xi^D\|_{[L^q(\Omega)]^{n \times n}}^q \right), \end{aligned}$$

and therefore, in this case (2.31) is true.

– If  $\|\text{tr } \xi\|_{L^2(\Omega)} \rightarrow +\infty$ :

$$\begin{aligned} \frac{\langle \mathcal{A}\xi, \xi \rangle + \varphi(\xi)}{\|\xi\|_{\mathbf{X}^q}} &\geq \frac{k\Theta_{min}}{q\Delta t \|\xi\|_{\mathbf{X}^q}} \left( \|\xi\|_{\mathbf{X}^2}^2 + \chi_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\xi) + \|(\xi + \chi^i)^D\|_{[L^q(\Omega)]^{n \times n}}^q \right) \\ &\geq \frac{2k\Theta_{min}}{q\Delta t \|\xi\|_{\mathbf{X}^q}} \left( \|\text{tr } \xi\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

and therefore, in this case (2.31) is also true.

– Finally, if  $\|\boldsymbol{\xi}^D\|_{[L^q(\Omega)]^{n \times n}} \rightarrow +\infty$  and  $\|\text{tr } \boldsymbol{\xi}\|_{L^2(\Omega)} \rightarrow +\infty$

$$\begin{aligned} \frac{\langle \mathcal{A}\boldsymbol{\xi}, \boldsymbol{\xi} \rangle + \varphi(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_{\mathbf{X}^q}} &\geq \frac{k\Theta_{\min}}{q\Delta t \|\boldsymbol{\xi}\|_{\mathbf{X}^q}} \left( \|\boldsymbol{\xi}\|_{\mathbf{X}^2}^2 + \mathcal{X}_{\mathbf{H}^q(\mathbf{0},\mathbf{0})}(\boldsymbol{\xi}) + \|(\boldsymbol{\xi} + \boldsymbol{\chi}^i)^D\|_{[L^q(\Omega)]^{n \times n}}^q \right) \\ &\geq \frac{k\Theta_{\min}}{q\Delta t \|\boldsymbol{\xi}\|_{\mathbf{X}^q}} \left( \|\text{tr } \boldsymbol{\xi}\|_{L^2(\Omega)}^2 + \|\boldsymbol{\xi}^D\|_{[L^q(\Omega)]^{n \times n}}^q \right), \end{aligned}$$

and (2.31) holds true.

We can deduce the existence and uniqueness of a solution for problem (2.30) directly from Theorem 2.4.7.  $\square$

#### Step IV: Construction of the corresponding displacement field

Our aim in this step is to prove the existence and uniqueness of displacements field  $\underline{\mathbf{u}}^i$  which is solution of equation (2.21).

**Proposition 2.4.9.** *Under assumptions (H1)-(H8), given  $\boldsymbol{\eta}^i$  the solution of Problem  $(HP_T)^i$ , there exists a unique displacement field  $\underline{\mathbf{u}}^i \in \mathbf{U}_0^p$  solution of equation (2.21).*

*Proof.* Let us consider  $\mathbf{e}^i = \Lambda(\theta^i) : \boldsymbol{\eta}^i + \partial\phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i)\Delta t - \tilde{\mathbf{g}}^i\Delta t$ . Since the tensor  $\boldsymbol{\eta}^i$  is the solution of the inequality (2.26), it verifies

$$\begin{aligned} \int_{\Omega} \left( \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} - \tilde{\mathbf{g}}^i \right) : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp + \mathcal{X}_{\mathbf{H}^q(\mathbf{0},\mathbf{0})}(\boldsymbol{\tau}) + \Psi_q^i(\boldsymbol{\tau} + \boldsymbol{\chi}^i) \\ - \mathcal{X}_{\mathbf{H}^q(\mathbf{0},\mathbf{0})}(\boldsymbol{\eta}^i) - \Psi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \geq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{X}^q, \end{aligned}$$

and it can be rewritten as follows:

$$\int_{\Omega} \left( \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} - \tilde{\mathbf{g}}^i \right) : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp + \varphi(\boldsymbol{\tau}) - \varphi(\boldsymbol{\eta}^i) \geq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{X}^q.$$

By definition of subdifferential, we get

$$-\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i \in \partial\varphi(\boldsymbol{\eta}^i),$$

and thanks to the properties of  $\varphi$  (convex and l.s.c.), we obtain

$$-\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i \in \partial\mathcal{X}_{\mathbf{H}^q(\mathbf{0},\mathbf{0})}(\boldsymbol{\eta}^i) + \partial\Psi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i).$$

Furthermore, since  $\partial\Psi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) = \partial\phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i)$  we have

$$-\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i - \partial\phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \in \partial\mathcal{X}_{\mathbf{H}^q(\mathbf{0},\mathbf{0})}(\boldsymbol{\eta}^i),$$

this is,

$$\int_{\Omega} \left( -\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i - \partial\phi_q(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \right) : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp + \mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\boldsymbol{\eta}^i) - \mathcal{X}_{\mathbf{H}^q(\mathbf{0}, \mathbf{0})}(\boldsymbol{\tau}) \leq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{X}^q,$$

and, in consequence

$$\int_{\Omega} \left( -\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i - \partial\phi_q(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \right) : (\boldsymbol{\tau} - \boldsymbol{\eta}^i) dp \leq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}).$$

In particular, if we consider  $\boldsymbol{\tau} + \boldsymbol{\eta}^i \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  we get

$$\int_{\Omega} \left( -\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i - \partial\phi_q(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \right) : \boldsymbol{\tau} dp \leq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}),$$

and if we take  $-\boldsymbol{\tau} + \boldsymbol{\eta}^i \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$ , we have

$$\int_{\Omega} \left( -\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i}{\Delta t} + \tilde{\mathbf{g}}^i - \partial\phi_q(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) \right) : \boldsymbol{\tau} dp \geq 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}).$$

Then, we can conclude that

$$\int_{\Omega} \mathbf{e}^i : \boldsymbol{\tau} dp = 0, \quad \forall \boldsymbol{\tau} \in \mathbf{H}^q(\mathbf{0}, \mathbf{0}).$$

Thus, from Proposition 2.3.3, we deduce the existence of a displacement field  $\mathbf{u}^i \in \mathbf{U}_0^p$ , such that  $\mathbf{e}^i = \boldsymbol{\varepsilon}(\mathbf{u}^i)$ . Furthermore, this displacement field is unique, since if there exist two solutions  $\mathbf{u}_1^i$  and  $\mathbf{u}_2^i$  such that

$$\mathbf{u}_1^i = \mathbf{u}_2^i = \mathbf{0} \text{ on } \Gamma_D \text{ and } \boldsymbol{\varepsilon}(\mathbf{u}_1^i) = \boldsymbol{\varepsilon}(\mathbf{u}_2^i),$$

then  $\mathbf{u}_1^i = \mathbf{u}_2^i$  thanks to Korn's inequality in  $\mathbf{U}_0^p$ .  $\square$

### Step V: A priori estimates

We define  $\boldsymbol{\sigma}^i = \boldsymbol{\eta}^i + \boldsymbol{\chi}^i \in \mathbf{H}^q(\mathbf{b}^i, \mathbf{g}^i)$ ,  $0 \leq i \leq I$ , and we consider the following functions:

- the linear interpolant

$$\boldsymbol{\sigma}^I(t) = (\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^{i-1}) \frac{t - t^{i-1}}{t^i - t^{i-1}} + \boldsymbol{\sigma}^{i-1}, \quad \forall t \in [t^{i-1}, t^i], \quad (2.32)$$

- the step functions

$$\boldsymbol{\sigma}^{I*}(t) = \boldsymbol{\sigma}^i, \quad \forall t \in [t^{i-1}, t^i], \quad (2.33)$$

$$\boldsymbol{\sigma}_*^I(t) = \boldsymbol{\sigma}^{i-1}, \quad \forall t \in [t^{i-1}, t^i], \quad i = 1, \dots, I. \quad (2.34)$$

Analogously, we define  $\boldsymbol{\eta}^I$ ,  $\boldsymbol{\eta}^{I*}$ ,  $\boldsymbol{\eta}_*^I$ ,  $\mathbf{u}^I$ ,  $\mathbf{u}^{I*}$  and  $\mathbf{u}_*^I$ .

Our aim is to obtain the limit of these sequences as  $I \rightarrow +\infty$ . For that purpose, it is necessary to deduce some a priori estimates independent of  $I$ . The main difficulty arises from the dependence of the elasticity tensor on temperature which can be overcome with the following lemma:

**Lemma 2.4.10.** *Under hypothesis **(H3)** we have the following inequality:*

$$\begin{aligned} (\Lambda(s_1) : \boldsymbol{\tau}_1 - \Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_1 &\geq \frac{1}{2}(\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - \frac{1}{2}(\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 \\ &+ \frac{1}{2}((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1, \quad \forall s_1, s_2 \in \mathbb{R}, \quad \forall \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in S_n. \end{aligned} \quad (2.35)$$

*Proof.* Let us consider the first two terms of the second member of inequality (2.35) without considering the factor  $\frac{1}{2}$ . Adding and subtracting  $(\Lambda(s_2) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1$  and  $(\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_1$  we have

$$\begin{aligned} (\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - (\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 &= ((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 \\ &+ (\Lambda(s_2) : \boldsymbol{\tau}_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) + (\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : \boldsymbol{\tau}_1. \end{aligned}$$

Thanks to symmetry of  $\Lambda(s)$ , it results

$$\begin{aligned} (\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - (\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 &= ((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 : (\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) \\ &+ (\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : \boldsymbol{\tau}_1, \end{aligned}$$

and adding and subtracting the term  $(\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : \boldsymbol{\tau}_1$ , using the symmetry of  $\Lambda(s)$  again, we obtain

$$\begin{aligned} (\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - (\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 &= 2(\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : \boldsymbol{\tau}_1 - (\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \\ &+ ((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1. \end{aligned}$$

Thanks to the hypothesis **(H3)**, we have

$$(\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - (\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 \leq 2(\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : \boldsymbol{\tau}_1 + ((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1;$$

hence, we obtain the inequality

$$\frac{1}{2}(\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - \frac{1}{2}(\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 \leq (\Lambda(s_2) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)) : \boldsymbol{\tau}_1 + \frac{1}{2}((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1.$$

Finally, adding and subtracting the term  $(\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1$  to the second member of this inequality we get

$$\begin{aligned} \frac{1}{2}(\Lambda(s_1) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - \frac{1}{2}(\Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_2 &\leq (\Lambda(s_1) : \boldsymbol{\tau}_1 - \Lambda(s_2) : \boldsymbol{\tau}_2) : \boldsymbol{\tau}_1 \\ &+ \frac{1}{2}((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1 - ((\Lambda(s_1) - \Lambda(s_2)) : \boldsymbol{\tau}_1) : \boldsymbol{\tau}_1, \end{aligned}$$

and hence we can deduce inequality (2.35).  $\square$

**Lemma 2.4.11.** *Given  $I \in \mathbb{N}$ , under assumptions **(H1)**-**(H8)** we have the following two estimates:*

*i) For any constant  $c_1 > 0$ , there exists  $c_2 > 0$  such that*

$$c_1 \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} \leq \Psi_q^i(\boldsymbol{\sigma}^i) + c_2, \quad 1 \leq i \leq I; \quad (2.36)$$

ii)

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 \leq c_9 + c_{10} \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2, \quad (2.37)$$

where  $\hat{I} \leq I$ ; from now on,  $c_l, l \geq 1$ , will denote a positive constant independent of  $I$ .

*Proof.* i) Taking into account definitions (2.5) and (2.25), we get

$$\begin{aligned} \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} &= \|(\boldsymbol{\sigma}^i)^D\|_{[\mathbf{L}^q(\Omega)]^{n \times n}} = \left( \int_{\Omega} |(\boldsymbol{\sigma}^i)^D|^q dp \right)^{\frac{1}{q}} \\ &= \left( \frac{q}{\Theta_{min}} \right)^{\frac{1}{q}} \left( \int_{\Omega} \frac{\Theta_{min}}{q} |(\boldsymbol{\sigma}^i)^D|^q dp \right)^{\frac{1}{q}} \leq \left( \frac{q}{\Theta_{min}} \right)^{\frac{1}{q}} \left( \int_{\Omega} \frac{\Theta(\theta)}{q} |(\boldsymbol{\sigma}^i)^D|^q dp \right)^{\frac{1}{q}} \\ &= \left( \frac{q}{\Theta_{min}} \right)^{\frac{1}{q}} (\Psi_q(\boldsymbol{\sigma}^i))^{\frac{1}{q}}. \end{aligned}$$

Using the inequality

$$s^{\frac{1}{r}} \leq \frac{s}{\delta} + \delta^{\frac{1}{r-1}}, \quad \forall \delta > 0, \quad \forall r > 1, \quad \forall s \geq 0, \quad (2.38)$$

for  $r = q$ , we obtain

$$\|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} \leq \left( \frac{q}{\Theta_{min}} \right)^{\frac{1}{q}} \Psi_q^i(\boldsymbol{\sigma}^i)^{\frac{1}{q}} \leq \left( \frac{q}{\Theta_{min}} \right)^{\frac{1}{q}} \left( \frac{\Psi_q^i(\boldsymbol{\sigma}^i)}{\delta} + \delta^{\frac{1}{q-1}} \right), \quad \forall \delta > 0, \quad (2.39)$$

where  $\Theta_{min}$  is the lower bound of the function  $\Theta(\theta)$ . Then, taking  $\delta = c_1 \left( \frac{q}{\Theta_{min}} \right)^{\frac{1}{q}}$ , we deduce estimate (2.36).

ii) Multiplying both sides of the discretized behaviour law (2.21) by  $\Delta t \boldsymbol{\eta}^i$ , integrating over  $\Omega$  and considering the sum for  $1 \leq i \leq \hat{I}$ , we obtain

$$\begin{aligned} &\sum_{i=1}^{\hat{I}} \int_{\Omega} (\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}) : \boldsymbol{\eta}^i dp + \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \partial \phi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) : \boldsymbol{\eta}^i dp \\ &= \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \underline{\mathbf{g}}^i : \boldsymbol{\eta}^i dp, \end{aligned}$$

thanks to the orthogonality relation given in Proposition 2.3.2. Hence, taking into account inequality (2.35) and the convexity of the function  $\Psi_q^i$ , we get

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} (\Lambda(\theta^{\hat{I}}) : \boldsymbol{\eta}^{\hat{I}}) : \boldsymbol{\eta}^{\hat{I}} dp - \frac{1}{2} \int_{\Omega} (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0 dp \\ &+ \frac{1}{2} \sum_{i=1}^{\hat{I}} \int_{\Omega} ((\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp \\ &+ \Delta t \sum_{i=1}^{\hat{I}} (\Psi_q^i(\boldsymbol{\eta}^i + \boldsymbol{\chi}^i) - \Psi_q^i(\boldsymbol{\chi}^i)) \leq \Delta t \sum_{i=1}^{\hat{I}} \int_{\Omega} \underline{\mathbf{g}}^i : \boldsymbol{\eta}^i dp. \end{aligned}$$

From hypothesis **(H3)** and Hölder's inequality, we have

$$\begin{aligned} & \frac{\kappa}{2} \|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 + \frac{1}{2} \sum_{i=1}^{\hat{I}} \int_{\Omega} ((\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp + \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i) \\ & \leq \frac{\beta_1}{2} \|\boldsymbol{\eta}^0\|_{\mathbf{X}^2}^2 + \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\chi}^i) + \Delta t \sum_{i=1}^{\hat{I}} \|\underline{\mathbf{g}}^i\|_{\mathbf{X}^p} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^q}. \end{aligned} \quad (2.40)$$

Taking into account that assumptions **(H1)**-**(H8)** guarantee that  $\underline{\mathbf{g}} \in W^{1,\infty}(0, t_f; \mathbf{X}^p)$ , we deduce

$$\|\underline{\mathbf{g}}^i\|_{\mathbf{X}^p} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^q} \leq c_3 \|\boldsymbol{\eta}^i\|_{\mathbf{X}^q} \leq c_3 \|\boldsymbol{\sigma}^i\|_{\mathbf{X}^q} + c_3 \|\boldsymbol{\chi}^i\|_{\mathbf{X}^q}. \quad (2.41)$$

On other hand, since

$$\|\boldsymbol{\sigma}^i\|_{\mathbf{X}^q} \leq \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} + \|\boldsymbol{\sigma}^i\|_{\mathbf{X}^2}, \quad (2.42)$$

inequality (2.41), applying estimate (2.36) and given that  $\boldsymbol{\chi}^i \in \mathbf{X}^q$ , we can bound the last term of equation (2.40) as follows:

$$\Delta t \sum_{i=1}^{\hat{I}} \|\underline{\mathbf{g}}^i\|_{\mathbf{X}^p} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^q} \leq c_3 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^q} \leq \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i) + c_3 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\sigma}^i\|_{\mathbf{X}^2} + c_4.$$

Using the previous expression in inequality (2.40), we get

$$\begin{aligned} & \frac{\kappa}{2} \|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 + \frac{1}{2} \sum_{i=1}^{\hat{I}} \int_{\Omega} ((\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp + \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i) \\ & \leq \frac{\beta_1}{2} \|\boldsymbol{\eta}^0\|_{\mathbf{X}^2}^2 + \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\chi}^i) + \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i) + c_3 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\sigma}^i\|_{\mathbf{X}^2} + c_4, \end{aligned}$$

hence, since  $\boldsymbol{\sigma}^i = \boldsymbol{\eta}^i + \boldsymbol{\chi}^i$ , we can deduce

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 \leq c_5 + c_6 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} + c_7 \sum_{i=1}^{\hat{I}} \int_{\Omega} ((\Lambda(\theta^{i-1}) - \Lambda(\theta^i)) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp.$$

Applying hypotheses **(H1)** and **(H3)**, we obtain

$$\begin{aligned} \|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 & \leq c_5 + c_6 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} + c_7 \Delta t \|\widehat{\Lambda(\theta)}\|_{[L^\infty(0, t_f, L^\infty(\Omega))]}^{n^4} \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2 \\ & \leq c_5 + c_6 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} + c_8 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2. \end{aligned} \quad (2.43)$$

Finally, from this inequality we deduce (2.37). In effect:

- If  $\|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} \leq 1$  for all  $i = 1, \dots, I$ , we have

$$\sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} \leq \hat{I}.$$

Since  $\hat{I} \leq I$  and  $\Delta t = \frac{t_f}{I}$ , then

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 \leq c_5 + c_6 \Delta t \hat{I} + c_8 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2. \quad (2.44)$$

- If  $\|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} \geq 1$  for all  $i = 1, \dots, I$ , we get

$$\sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2} \leq \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2,$$

then, from expression (2.43) we deduce

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 \leq c_5 + (c_6 + c_8) \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2. \quad (2.45)$$

- Let us suppose that there exists a set  $\{i_1, \dots, i_k\}$  with  $k < \hat{I}$ , such that  $\|\boldsymbol{\eta}^{i_j}\|_{\mathbf{X}^2} < 1$  with  $j = 1, \dots, k$  and furthermore,  $\|\boldsymbol{\eta}^{i_j}\|_{\mathbf{X}^2} \geq 1$  with  $i_j \notin \{i_1, \dots, i_k\}$ . Thus,

$$\begin{aligned} \|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 &\leq c_5 + c_6 \Delta t \sum_{i_j \in \{i_1, \dots, i_k\}} \|\boldsymbol{\eta}^{i_j}\|_{\mathbf{X}^2} + c_6 \Delta t \sum_{i_j \notin \{i_1, \dots, i_k\}} \|\boldsymbol{\eta}^{i_j}\|_{\mathbf{X}^2} + c_8 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2 \\ &\leq c_5 + c_6 \Delta t k + c_6 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2 + c_8 \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2 \\ &\leq (c_5 + c_6 \Delta t k) + (c_6 + c_8) \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^i\|_{\mathbf{X}^2}^2. \end{aligned} \quad (2.46)$$

Taking  $c_9 = c_5 + c_6 \Delta t \hat{I}$  and  $c_{10} = c_6 + c_8$ , from inequalities (2.44), (2.45) and (2.46) we conclude the result.  $\square$

From these estimates, using Gronwall's discrete lemma (see Lions [82]) and proceeding analogously to Barral and Quintela [15] we obtain the following result:

**Corollary 2.4.12.** *Under assumptions (H1)-(H8),  $\boldsymbol{\eta}^I$ ,  $\boldsymbol{\eta}^{I*}$ ,  $\boldsymbol{\sigma}^I$  and  $\boldsymbol{\sigma}^{I*}$  are bounded in  $L^\infty(0, t_f; \mathbf{X}^2)$  for  $I \rightarrow +\infty$ .*

*Proof.* We have to prove that  $\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}$ ,  $\hat{I} \leq I$ , is bounded independent of  $\hat{I}$  for  $\Delta t$  enough small. For this purpose, we apply Gronwall's discrete lemma (see Lions [82]) to expression (2.37) and we get,

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 \leq \frac{c_9}{1 - c_{10}\Delta t} e^{\frac{c_{10}t_f}{1 - c_{10}\Delta t}},$$

then  $\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2$  remain in a bounded set of  $\mathbf{X}^2$  independent of  $\hat{I}$  for  $\Delta t$  enough small. Therefore, we deduce

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2}^2 \leq c_{11}, \quad \forall \hat{I} \leq I, \quad \text{as } I \rightarrow +\infty,$$

and we conclude

$$\|\boldsymbol{\eta}^{\hat{I}}\|_{\mathbf{X}^2} \leq c_{12}, \quad \forall \hat{I} \leq I, \quad \text{as } I \rightarrow +\infty,$$

being  $c_{11}$  and  $c_{12}$  positive constants independent of  $\hat{I}$ . In consequence, we obtain

$$\boldsymbol{\eta}^I \text{ and } \boldsymbol{\eta}^{I*} \text{ remain in a bounded set of } L^\infty(0, t_f; \mathbf{X}^2) \text{ as } I \rightarrow +\infty.$$

Furthermore, thanks to hypothesis **(H2)** we have also that  $\boldsymbol{\sigma}^I$  and  $\boldsymbol{\sigma}^{I*}$  remain in a bounded set of  $L^\infty(0, t_f; \mathbf{X}^2)$  as  $I \rightarrow +\infty$ .  $\square$

In order to obtain an analogous result for the time derivatives and to generalize the previous bound to  $L^\infty(0, t_f; \mathbf{X}^q)$  we need the following estimate.

**Lemma 2.4.13.** *Under assumptions (H1)-(H8) we have*

$$\frac{\kappa}{2} \sum_{i=1}^{\hat{I}} \Delta t \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 + \frac{1}{2} \Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) \leq c_{25} + c_{26} \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i), \quad (2.47)$$

where  $\hat{I} \leq I$ .

*Proof.* Let us consider again the discretized behaviour law (2.21). If we multiply this relation by  $\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^{i-1}$  and we proceed analogously to the previous proof, we arrive at

$$\begin{aligned} & \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} dp + \sum_{i=1}^{\hat{I}} \int_{\Omega} \partial \phi_q^i(\boldsymbol{\sigma}^i) : (\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^{i-1}) dp \\ &= \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^i) - \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{i-1})}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \\ & - \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \\ & + \sum_{i=1}^{\hat{I}} \int_{\Omega} \underline{\mathbf{g}}^i : (\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^{i-1}) dp. \end{aligned} \quad (2.48)$$

In order to bound the first addend, we write its integrand as follows:

$$\begin{aligned} \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} &= \frac{\Lambda(\theta^i) : (\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1})}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \\ &+ \frac{(\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t}. \end{aligned}$$

Thanks to **(H3)** and the convexity of  $\Psi_q^i$ , from expression (2.48) we obtain

$$\begin{aligned} &\kappa \sum_{i=1}^{\hat{I}} \Delta t \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 + \Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) - \Psi_q^0(\boldsymbol{\sigma}^0) \\ &\leq \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{(\Lambda(\theta^{i-1}) - \Lambda(\theta^i)) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} dp \\ &\quad + \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\boldsymbol{\varepsilon}(\mathbf{u}^i) - \boldsymbol{\varepsilon}(\mathbf{u}^{i-1})}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \\ &\quad - \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \\ &\quad + \sum_{i=1}^{\hat{I}} \int_{\Omega} \underline{\mathbf{g}}^i : (\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^{i-1}) dp. \end{aligned} \tag{2.49}$$

Let us bound the first term of the second member under hypotheses **(H1)** and **(H3)**, applying Hölder's inequality, the mean value theorem and Corollary 2.4.12:

$$\begin{aligned} &\left| \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{(\Lambda(\theta^{i-1}) - \Lambda(\theta^i)) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} dp \right| \\ &\leq \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \left| \frac{(\Lambda(\theta^{i-1}) - \Lambda(\theta^i)) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right| dp \\ &\leq \Delta t \sum_{i=1}^{\hat{I}} \|\widehat{\Lambda(\theta)}\|_{L^\infty(0,t_f,L^\infty(\Omega))} \|\boldsymbol{\eta}^{i-1}\|_{\mathbf{X}^2} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} \\ &\leq \|\widehat{\Lambda(\theta)}\|_{L^\infty(0,t_f,L^\infty(\Omega))} \Delta t \sum_{i=1}^{\hat{I}} \|\boldsymbol{\eta}^{i-1}\|_{\mathbf{X}^2} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} \leq c_{13} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}. \end{aligned} \tag{2.50}$$

For the second term on the right hand-side of equation (2.49), applying Hölder's inequality, the mean value theorem again and hypothesis **(H2)**, we get

$$\begin{aligned} \left| \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\boldsymbol{\varepsilon}(\mathbf{u}^i) - \boldsymbol{\varepsilon}(\mathbf{u}^{i-1})}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \right| &\leq \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \left| \frac{\boldsymbol{\varepsilon}(\mathbf{u}^i) - \boldsymbol{\varepsilon}(\mathbf{u}^{i-1})}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} \right| dp \\ &\leq \|\dot{\boldsymbol{\chi}}\|_{L^\infty(0,t_f;\mathbf{X}^q)} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\varepsilon}(\mathbf{u}^i) - \boldsymbol{\varepsilon}(\mathbf{u}^{i-1})}{\Delta t} \right\|_{\mathbf{X}^p}. \end{aligned}$$

In addition, from behaviour law (2.21), adding and subtracting the term  $\Lambda(\theta^i) : \boldsymbol{\eta}^{i-1}$  and using the same reasoning, we deduce

$$\begin{aligned} \left\| \frac{\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^i) - \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{i-1})}{\Delta t} \right\|_{\mathbf{X}^p} &\leq \left\| \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^p} + \|\underline{\mathbf{g}}^i\|_{\mathbf{X}^p} + \|\partial\phi_q(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^p} \\ &\leq \left\| \frac{\Lambda(\theta^i) : (\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1})}{\Delta t} \right\|_{\mathbf{X}^p} + \left\| \frac{(\Lambda(\theta^i) - \Lambda(\theta^{i-1}))\boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^p} \\ &\quad + \|\underline{\mathbf{g}}^i\|_{\mathbf{X}^p} + \|\partial\phi_q^i(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^p}; \end{aligned}$$

hence, using hypotheses **(H1)**-**(H8)** and Corollary 2.4.12

$$\begin{aligned} \left\| \frac{\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^i) - \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{i-1})}{\Delta t} \right\|_{\mathbf{X}^p} &\leq \beta_1 \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} + \|\widehat{\Lambda(\theta)}\|_{L^\infty(0,t_f;L^\infty(\Omega))} \|\boldsymbol{\eta}^{i-1}\|_{\mathbf{X}^2} + \|\underline{\mathbf{g}}^i\|_{\mathbf{X}^p} \\ &\quad + \|\partial\phi_q(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^p} \leq \beta_1 \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} + \|\partial\phi_q^i(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^p} + c_{14}. \end{aligned} \quad (2.51)$$

The term  $\|\partial\phi_q^i(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^p}$  is bounded using definitions (2.5), (2.25) and inequality (2.38) for  $\delta = 1$ ,  $s = \frac{q}{\Theta_{\min}}$   $\|\Psi_q^i(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^q}$  and  $r = \frac{q}{q-1}$ , obtaining

$$\begin{aligned} \|\partial\phi_q^i(\boldsymbol{\sigma}^i)\|_{\mathbf{X}^p} &\leq \Theta_S \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^p}^{q-1} \leq \Theta_S \left( \frac{q}{\Theta_{\min}} \right)^{\frac{q-1}{q}} \Psi_q(\boldsymbol{\sigma}^i)^{\frac{q-1}{q}} \\ &\leq \Theta_S \left( \frac{q}{\Theta_{\min}\delta} \Psi_q(\boldsymbol{\sigma}^i) + \delta^{q-1} \right) = q \frac{\Theta_S}{\Theta_{\min}} \Psi_q^i(\boldsymbol{\sigma}^i) + \Theta_S, \end{aligned} \quad (2.52)$$

where  $\Theta_S$  is an upper bound for  $\Theta(\theta)$ . Summing up,

$$\begin{aligned} \left| \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^i) - \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{i-1})}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \right| &\leq c_{15} + c_{16} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} \\ &\quad + c_{17} \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i). \end{aligned} \quad (2.53)$$

Next, we search for an upper bound for the third addend of the second member of (2.49). For that

purpose we add and subtract the term  $\frac{\Lambda(\theta^i) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t}$ , deducing

$$\begin{aligned}
& \left| \sum_{i=1}^{\hat{I}} \int_{\Omega} \Delta t \frac{\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} dp \right| \\
& \leq \sum_{i=1}^{\hat{I}} \left[ \int_{\Omega} \Delta t \left| \frac{\Lambda(\theta^i) : (\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1})}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} \right| dp \right. \\
& \quad \left. + \int_{\Omega} \Delta t \left| \frac{(\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^{i-1}}{\Delta t} : \frac{\boldsymbol{\chi}^i - \boldsymbol{\chi}^{i-1}}{\Delta t} \right| dp \right] \\
& \leq \beta_1 \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} \|\dot{\boldsymbol{\chi}}\|_{L^\infty(0,t_f;\mathbf{X}^q)} + \Delta t \sum_{i=1}^{\hat{I}} \|\widehat{\Lambda(\theta)}\|_{L^\infty(0,t_f;L^\infty(\Omega))} \|\boldsymbol{\eta}^{i-1}\|_{\mathbf{X}^2} \|\dot{\boldsymbol{\chi}}\|_{L^\infty(0,t_f;\mathbf{X}^q)} \\
& \leq c_{18} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} + c_{19}, \tag{2.54}
\end{aligned}$$

where we have taken into account Hölder's inequality, the mean value theorem, hypotheses **(H1)**-**(H3)** and Corollary 2.4.12. The last term of the second member of expression (2.49) is bounded similarly as in Djaoua and Suquet [35], where they prove that

$$\begin{aligned}
& \left| \sum_{i=1}^{\hat{I}} \int_{\Omega} \underline{\mathbf{g}}^i : (\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^{i-1}) dp \right| = \left| \sum_{i=1}^{\hat{I}-1} \int_{\Omega} (\underline{\mathbf{g}}^i - \underline{\mathbf{g}}^{i+1}) : \boldsymbol{\sigma}^i dp + \int_{\Omega} [(\underline{\mathbf{g}}^{\hat{I}} : \boldsymbol{\sigma}^{\hat{I}}) - (\underline{\mathbf{g}}^1 : \boldsymbol{\sigma}^0)] dp \right| \\
& \leq \|\underline{\mathbf{g}}\|_{L^\infty(0,t_f;\mathbf{X}^p)} \sum_{i=1}^{\hat{I}-1} \Delta t \|\boldsymbol{\sigma}^i\|_{\mathbf{X}^q} + \|\underline{\mathbf{g}}\|_{L^\infty(0,t_f;\mathbf{X}^p)} (\|\boldsymbol{\sigma}^{\hat{I}}\|_{\mathbf{X}^q} + \|\boldsymbol{\sigma}^0\|_{\mathbf{X}^q}) \\
& \leq c_{20} \sum_{i=1}^{\hat{I}-1} \Delta t \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} + c_{21} \|(\boldsymbol{\sigma}^{\hat{I}})^D\|_{\mathbf{X}^q} + c_{22}. \tag{2.55}
\end{aligned}$$

To sum up, from inequalities (2.49)-(2.55) we obtain

$$\begin{aligned}
& \kappa \sum_{i=1}^{\hat{I}} \Delta t \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 + \Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) \leq c_{23} + \Psi_q^0(\boldsymbol{\sigma}^0) + c_{17} \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i) \\
& + c_{24} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} + c_{20} \sum_{i=1}^{\hat{I}-1} \Delta t \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} + c_{21} \|(\boldsymbol{\sigma}^{\hat{I}})^D\|_{\mathbf{X}^q}. \tag{2.56}
\end{aligned}$$

In addition, if we get  $\delta = \frac{2c_{24}}{\kappa}$ ,  $r = 2$  and  $s = \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2$  in expression (2.38), we deduce that

$$c_{24} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} \leq \frac{\kappa}{2} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 + \frac{2(c_{24})^2}{\kappa} t_f. \tag{2.57}$$

Furthermore, from inequality (2.39), with  $\delta = 2c_{20} \left( \frac{q}{\Theta_{\min}} \right)^{\frac{1}{q}}$  for the fifth term on the right-hand side of inequality (2.56) and with  $\delta = 2c_{21} \left( \frac{q}{\Theta_{\min}} \right)^{\frac{1}{q}}$  for the sixth term on the right-hand side of inequality (2.56), we get

$$c_{20} \|(\boldsymbol{\sigma}^i)^D\|_{\mathbf{X}^q} \leq \frac{1}{2} \Psi_q^i(\boldsymbol{\sigma}^i) + \left( 2c_{20} \left( \frac{q}{\Theta_{\min}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q-1}}, \quad (2.58)$$

and

$$c_{21} \|(\boldsymbol{\sigma}^{\hat{I}})^D\|_{\mathbf{X}^q} \leq \frac{1}{2} \Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) + \left( 2c_{21} \left( \frac{q}{\Theta_{\min}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q-1}}. \quad (2.59)$$

Finally, if we add expressions (2.57), (2.58) and (2.59) to the inequality (2.56),

$$\begin{aligned} \kappa \sum_{i=1}^{\hat{I}} \Delta t \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 + \Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) &\leq c_{23} + \Psi_q^0(\boldsymbol{\sigma}^0) + c_{17} \Delta t \sum_{i=1}^{\hat{I}} \Psi_q^i(\boldsymbol{\sigma}^i) + \frac{\kappa}{2} \Delta t \sum_{i=1}^{\hat{I}} \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 \\ &+ \frac{2(c_{24})^2}{\kappa} t_f + \sum_{i=1}^{\hat{I}-1} \Delta t \left( \frac{1}{2} \Psi_q^i(\boldsymbol{\sigma}^i) + \left( 2c_{20} \left( \frac{q}{\Theta_{\min}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q-1}} \right) + \frac{1}{2} \Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) + \left( 2c_{21} \left( \frac{q}{\Theta_{\min}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q-1}}, \end{aligned}$$

we deduce result (2.47).  $\square$

From these estimates, we deduce the following corollaries:

**Corollary 2.4.14.** *Under assumptions (H1)-(H8),*

- $\boldsymbol{\sigma}^I, \boldsymbol{\sigma}^{I*}, \boldsymbol{\eta}^I, \boldsymbol{\eta}^{I*}$  remain in a bounded set of  $L^\infty(0, t_f; \mathbf{X}^q)$  as  $I \rightarrow +\infty$ .
- $\dot{\boldsymbol{\eta}}^I$  and  $\dot{\boldsymbol{\sigma}}^I$  remain in a bounded set of  $L^2(0, t_f; \mathbf{X}^2)$  as  $I \rightarrow +\infty$ .

*Proof.* Using Gronwall's discrete lemma (see Lions[82]) we deduce from inequality (2.47) that

$$\Psi_q^{\hat{I}}(\boldsymbol{\sigma}^{\hat{I}}) \leq c_{27}, \quad \forall \hat{I} \leq I \text{ as } I \rightarrow +\infty. \quad (2.60)$$

Then, from expression (2.36) and Corollary 2.4.12 we can conclude that

$$\|\boldsymbol{\sigma}^{\hat{I}}\|_{\mathbf{X}^q} \leq c_{28}, \quad \forall \hat{I} \leq I \text{ as } I \rightarrow +\infty,$$

so that  $\boldsymbol{\sigma}^I, \boldsymbol{\sigma}^{I*}, \boldsymbol{\eta}^I, \boldsymbol{\eta}^{I*}$  remain in a bounded set of  $L^\infty(0, t_f; \mathbf{X}^q)$  as  $I \rightarrow +\infty$ . Furthermore, applying inequality (2.60) in expression (2.47) we deduce

$$\sum_{i=1}^{\hat{I}} \Delta t \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 \leq c_{29},$$

then  $\dot{\boldsymbol{\eta}}^I$ , so  $\dot{\boldsymbol{\sigma}}^I$ , remain in a bounded set of  $L^2(0, t_f; \mathbf{X}^2)$  as  $I \rightarrow +\infty$ .  $\square$

The inequality (2.52) is combined with (2.60) to obtain the following property of  $\partial\phi_q(\boldsymbol{\sigma}^{I*})$ .

**Corollary 2.4.15.** *Under assumptions (H1)-(H8),  $\partial\phi_q^{I*}(\boldsymbol{\sigma}^{I*})$  remains in a bounded set of  $L^\infty(0, t_f; \mathbf{X}^p)$  as  $I \rightarrow +\infty$ .*

*Proof.* It is sufficient to consider inequality (2.52) and estimate (2.60).  $\square$

Finally, applying these results to behaviour law (2.21) we can bound the field of displacements.

**Corollary 2.4.16.** *Under assumptions (H1)-(H8), it follows that*

- $\underline{\mathbf{u}}^{I*}$  and  $\underline{\mathbf{u}}^I$  remain in a bounded set of  $L^\infty(0, t_f; \mathbf{U}_0^p)$  as  $I \rightarrow +\infty$ .
- $\dot{\underline{\mathbf{u}}}^I$  remains in a bounded set of  $L^2(0, t_f; \mathbf{U}_0^p)$  as  $I \rightarrow +\infty$ .
- $\underline{\mathbf{u}}_*^I$  and  $\underline{\mathbf{u}}^I$  remain in a bounded set of  $L^\infty(0, t_f; \mathbf{U}^p)$  as  $I \rightarrow +\infty$ .
- $\dot{\underline{\mathbf{u}}}^I$  remains in a bounded set of  $L^2(0, t_f; \mathbf{U}^p)$  as  $I \rightarrow +\infty$ .

*Proof.* Considering the sum for  $1 \leq i \leq \hat{I}$  of equation (2.21),

$$\sum_{i=1}^{\hat{I}} (\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^i) - \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{i-1})) = \sum_{i=1}^{\hat{I}} (\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}) - \sum_{i=1}^{\hat{I}} \Delta t \underline{\mathbf{g}}^i + \sum_{i=1}^{\hat{I}} \Delta t \partial\phi_q(\boldsymbol{\sigma}^i),$$

for  $\hat{I} \leq I$ , we deduce

$$\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{\hat{I}}) = \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^0) + \Lambda(\theta^{\hat{I}}) : \boldsymbol{\eta}^{\hat{I}} - \Lambda(\theta^0) : \boldsymbol{\eta}^0 - \sum_{i=1}^{\hat{I}} \Delta t \underline{\mathbf{g}}^i + \sum_{i=1}^{\hat{I}} \Delta t \partial\phi_q(\boldsymbol{\sigma}^i).$$

Using Corollaries 2.4.14 and 2.4.15, we can conclude that  $\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^{\hat{I}})$  remains in a bounded set of  $\mathbf{X}^p$ , independent of  $\hat{I}$ , for  $\Delta t$  small enough. Thus, thanks to the Dirichlet condition on  $\Gamma_D$ , we deduce that  $\underline{\mathbf{u}}^{\hat{I}}$  remains in a bounded set of  $\mathbf{U}_0^p$  for all  $\hat{I} \leq I$ , as  $I \rightarrow +\infty$ . Consequently, also for  $\underline{\mathbf{u}}^{I*}$ ,  $\underline{\mathbf{u}}^I$ ,  $\underline{\mathbf{u}}_*^I$  and  $\underline{\mathbf{u}}^I$  taking into account Remark 2.4.4.

From expression (2.51) and applying Corollaries 2.4.14 and 2.4.15 once again, with the same reasoning, we obtain the result for the velocity fields.  $\square$

### Step VI: Passage to the limit as $I \rightarrow +\infty$

As we have announced, the previous estimates let us obtain the following convergences.

**Corollary 2.4.17.** *Under assumptions (H1)-(H8), there exist*

$$\begin{aligned} \boldsymbol{\eta}, \boldsymbol{\sigma} &\in W^{1,2}(0, t_f; \mathbf{X}^2) \cap L^\infty(0, t_f; \mathbf{X}^q), \quad \boldsymbol{\eta}^*, \boldsymbol{\sigma}^* \in L^\infty(0, t_f; \mathbf{X}^q), \\ \underline{\mathbf{u}} &\in W^{1,2}(0, t_f; \mathbf{U}_0^p), \quad \underline{\mathbf{u}}^* \in L^\infty(0, t_f; \mathbf{U}_0^p), \\ \underline{\mathbf{u}} &\in W^{1,2}(0, t_f; \mathbf{U}^p), \quad \underline{\mathbf{u}}^* \in L^\infty(0, t_f; \mathbf{U}^p), \end{aligned}$$

and subsequences (again indexed with  $I$ ) such that

$$\boldsymbol{\eta}^I \rightharpoonup \boldsymbol{\eta} \text{ in } L^\infty(0, t_f; \mathbf{X}^q) \text{ weak-star,} \quad (2.61)$$

$$\boldsymbol{\sigma}^I \rightharpoonup \boldsymbol{\sigma} \text{ in } L^\infty(0, t_f; \mathbf{X}^q) \text{ weak-star,} \quad (2.62)$$

$$\dot{\boldsymbol{\eta}}^I \rightharpoonup \dot{\boldsymbol{\eta}} \text{ in } L^2(0, t_f; \mathbf{X}^2) \text{ weakly,} \quad (2.63)$$

$$\dot{\boldsymbol{\sigma}}^I \rightharpoonup \dot{\boldsymbol{\sigma}} \text{ in } L^2(0, t_f; \mathbf{X}^2) \text{ weakly,} \quad (2.64)$$

$$\boldsymbol{\eta}^{I*} \rightharpoonup \boldsymbol{\eta}^* \text{ in } L^\infty(0, t_f; \mathbf{X}^q) \text{ weak-star,} \quad (2.65)$$

$$\boldsymbol{\sigma}^{I*} \rightharpoonup \boldsymbol{\sigma}^* \text{ in } L^\infty(0, t_f; \mathbf{X}^q) \text{ weak-star,} \quad (2.66)$$

$$\underline{\mathbf{u}}^I \rightharpoonup \underline{\mathbf{u}} \text{ in } L^\infty(0, t_f; \mathbf{U}_0^p) \text{ weak-star,} \quad (2.67)$$

$$\mathbf{u}^I \rightharpoonup \mathbf{u} \text{ in } L^\infty(0, t_f; \mathbf{U}^p) \text{ weak-star,} \quad (2.68)$$

$$\dot{\underline{\mathbf{u}}}^I \rightharpoonup \dot{\underline{\mathbf{u}}} \text{ in } L^2(0, t_f; \mathbf{U}_0^p) \text{ weakly,} \quad (2.69)$$

$$\dot{\mathbf{u}}^I \rightharpoonup \dot{\mathbf{u}} \text{ in } L^2(0, t_f; \mathbf{U}^p) \text{ weakly,} \quad (2.70)$$

$$\underline{\mathbf{u}}^{I*} \rightharpoonup \underline{\mathbf{u}}^* \text{ in } L^\infty(0, t_f; \mathbf{U}_0^p) \text{ weak-star,} \quad (2.71)$$

$$\mathbf{u}^{I*} \rightharpoonup \mathbf{u}^* \text{ in } L^\infty(0, t_f; \mathbf{U}^p) \text{ weak-star.} \quad (2.72)$$

Furthermore, from definitions (2.32)-(2.34), it is easy to prove some strong convergences, which are summarized in the following lemmas.

**Lemma 2.4.18.** *Under assumptions (H1)-(H8),*

$$\lim_{I \rightarrow +\infty} (\boldsymbol{\eta}^I - \boldsymbol{\eta}^{I*}) = \mathbf{0} \text{ in } L^\infty(0, t_f; \mathbf{X}^2) \text{ strong,} \quad (2.73)$$

$$\lim_{I \rightarrow +\infty} (\boldsymbol{\sigma}^I - \boldsymbol{\sigma}^{I*}) = \mathbf{0} \text{ in } L^\infty(0, t_f; \mathbf{X}^2) \text{ strong,} \quad (2.74)$$

$$\lim_{I \rightarrow +\infty} (\underline{\mathbf{u}}^I - \underline{\mathbf{u}}^{I*}) = \mathbf{0} \text{ in } L^\infty(0, t_f; \mathbf{U}_0^p) \text{ strong,} \quad (2.75)$$

$$\lim_{I \rightarrow +\infty} (\mathbf{u}^I - \mathbf{u}^{I*}) = \mathbf{0} \text{ in } L^\infty(0, t_f; \mathbf{U}^p) \text{ strong.} \quad (2.76)$$

*Proof.* By definition of  $\boldsymbol{\eta}^I$  and  $\boldsymbol{\eta}^{I*}$  (see expressions (2.32) and (2.33)), we have

$$\begin{aligned} \|\boldsymbol{\eta}^I(t) - \boldsymbol{\eta}^{I*}(t)\|_{\mathbf{X}^2} &\leq \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2} |t - t^{i-1}| + \|\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}\|_{\mathbf{X}^2} \\ &\leq 2\|\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}\|_{\mathbf{X}^2} \leq 2 \left( \sum_{i=1}^{\hat{I}} \Delta t \left\| \frac{\boldsymbol{\eta}^i - \boldsymbol{\eta}^{i-1}}{\Delta t} \right\|_{\mathbf{X}^2}^2 \right)^{\frac{1}{2}} \left( \frac{t_f}{I} \right)^{\frac{1}{2}}, \end{aligned}$$

for  $t \in [t^{i-1}, t^i)$  and from Corollary 2.4.14 we conclude result (2.73).

Convergences (2.74)-(2.76) are proved in the same way. The proof of the remaining results can be proved with a similar methodology, thus for the second case thanks to

□

From Corollary 2.4.17 and Lemma 2.4.18 we deduce the following result.

**Corollary 2.4.19.** *Under assumptions (H1)-(H8),  $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ ,  $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}$ ,  $\underline{\mathbf{u}}^* = \underline{\mathbf{u}}$  and  $\mathbf{u}^* = \mathbf{u}$ .*

*Proof.* Thanks to convergences (2.61) and (2.65) and since  $L^\infty(0, t_f; \mathbf{X}^q)$  is the dual space of  $L^1(0, t_f; \mathbf{X}^p)$ , we can deduce

$$\boldsymbol{\eta}^I \rightharpoonup \boldsymbol{\eta} \text{ in } L^\infty(0, t_f; \mathbf{X}^2) \text{ weakly}$$

and

$$\boldsymbol{\eta}^{I*} \rightharpoonup \boldsymbol{\eta}^* \text{ in } L^\infty(0, t_f; \mathbf{X}^2) \text{ weakly.}$$

Hence, we obtain

$$\boldsymbol{\eta}^I - \boldsymbol{\eta}^{I*} \rightharpoonup \boldsymbol{\eta} - \boldsymbol{\eta}^* \text{ in } L^\infty(0, t_f; \mathbf{X}^2) \text{ weakly.}$$

Then, taking into account convergence (2.73) and the uniqueness of limit we have  $\boldsymbol{\eta} = \boldsymbol{\eta}^*$  a.e. in  $\Omega \times (0, t_f)$ . With the same reasoning, we achieve the other results.  $\square$

**Lemma 2.4.20.** *Under assumptions (H1)-(H8), we have*

$$\mathbf{u}_D^I \rightarrow \mathbf{u}_D \text{ in } W^{1,\infty}(0, t_f; \mathbf{U}^p) \text{ strong,} \quad (2.77)$$

$$\mathbf{u}_D^{I*} \rightarrow \mathbf{u}_D \text{ in } L^\infty(0, t_f; \mathbf{U}^p) \text{ strong,} \quad (2.78)$$

$$\boldsymbol{\chi}^I \rightarrow \boldsymbol{\chi} \text{ in } W^{1,\infty}(0, t_f; \mathbf{X}^q) \text{ strong,} \quad (2.79)$$

$$\boldsymbol{\chi}^{I*} \rightarrow \boldsymbol{\chi} \text{ in } L^\infty(0, t_f; \mathbf{X}^q) \text{ strong,} \quad (2.80)$$

$$\theta^I \rightarrow \theta \text{ in } W^{1,\infty}(0, t_f; L^\infty(\Omega)) \text{ strong,} \quad (2.81)$$

$$\theta^{I*} \rightarrow \theta \text{ in } L^\infty(0, t_f; L^\infty(\Omega)) \text{ strong.} \quad (2.82)$$

*Proof.* It is sufficient to consider hypotheses (H1), (H2) and (H7).  $\square$

Taking into account the definition of  $\mathbf{g}^{I*}$  given in (2.22), we deduce from Lemma 2.4.20 the strong convergence given in the following result.

**Corollary 2.4.21.** *Under assumptions (H1)-(H8) the following convergence holds:*

$$\underline{\mathbf{g}}^{I*} \rightarrow \underline{\mathbf{g}} \text{ in } L^\infty(0, t_f; \mathbf{X}^p) \text{ strong.} \quad (2.83)$$

Corollary 2.4.15 implies the following convergence.

**Lemma 2.4.22.** *Under assumptions (H1)-(H8) there exists a function  $\boldsymbol{\Upsilon}$  such that*

$$\partial \phi_q^{I*}(\boldsymbol{\sigma}^{I*}) \rightharpoonup \boldsymbol{\Upsilon} \text{ in } L^\infty(0, t_f; \mathbf{X}^p) \text{ weak-star.}$$

*Proof.* It is obtained taking into account Corollary 2.4.15 and applying a result from Brezis [25].  $\square$

Therefore, it has been proved that

$$\boldsymbol{\sigma} = \boldsymbol{\eta} + \boldsymbol{\chi} \text{ and } \mathbf{u} = \underline{\mathbf{u}} + \mathbf{u}_D \text{ a.e. in } \Omega \times (0, t_f).$$

**Step VII: Verifying that  $(\mathbf{u}, \boldsymbol{\sigma})$  is a solution of Problem  $(P_T)$** 

In order to complete the proof of the existence of a solution it is necessary to prove that  $(\mathbf{u}, \boldsymbol{\sigma})$  verifies Problem  $(P_T)$ . The proof is similar to that given in Barral and Quintela [15] except for the behaviour law.

**Initial conditions.** Since  $\mathbf{u}$  and  $\mathbf{u}^I$  belong to  $W^{1,2}(0, t_f; \mathbf{U}^p)$ , they are continuous functions from  $[0, t_f]$  into  $\mathbf{U}^p$ , except on a null subset of  $(0, t_f)$  (see Lions [81]). Then, taking into account weak convergences (2.68) and (2.70) we can assert that  $\mathbf{u}^I(0) \rightharpoonup \mathbf{u}(0)$  in  $\mathbf{U}^p$ , and since  $\mathbf{u}^I(0) = \mathbf{u}_0$ ,  $\forall I$ , we conclude  $\mathbf{u}(0) = \mathbf{u}_0$ . The same reasoning holds true to prove that  $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ .

**Equilibrium equations and boundary conditions.** Since  $\boldsymbol{\eta}^I(t)$  belongs to  $\mathbf{H}^q(\mathbf{0}, \mathbf{0})$ ,  $\boldsymbol{\eta}^I(t) \rightharpoonup \boldsymbol{\eta}(t)$  in  $\mathbf{X}^q$  a.e. in  $(0, t_f)$ , and  $\mathbf{H}^q(\mathbf{0}, \mathbf{0})$  is a closed convex subset of  $\mathbf{X}^q$ , we conclude that  $\boldsymbol{\eta}(t) \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$ , and therefore

$$\boldsymbol{\sigma}(t) \in \mathbf{H}^q(\mathbf{b}, \mathbf{g}) \text{ a.e. in } (0, t_f).$$

Furthermore since  $\mathbf{u} = \underline{\mathbf{u}} + \mathbf{u}_D$  and  $\underline{\mathbf{u}} \in \mathbf{U}_0^p$ , then  $\mathbf{u}$  verifies Dirichlet's condition on  $\Gamma_D$ .

**Behaviour law.** From equation (2.21) and using definitions (2.32) and (2.33), we can write the behaviour law as follows

$$\boldsymbol{\varepsilon}(\underline{\mathbf{u}}^I) - \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} + \underline{\mathbf{g}}^{I*} = \partial\phi_q^{I*}(\boldsymbol{\sigma}^{I*}) \text{ a.e. in } \Omega \times (0, t_f). \quad (2.84)$$

Let us choose  $\boldsymbol{\tau} \in \mathcal{D}(\Omega)^{n \times n}$  and  $\vartheta \in \mathcal{D}((0, t_f))$ ; multiplying the previous equality by  $\boldsymbol{\tau}(p)\vartheta(t)$  and integrating over  $\Omega \times (0, t_f)$ , we obtain

$$\begin{aligned} & \int_0^{t_f} \int_{\Omega} \boldsymbol{\tau}(p)\vartheta(t) : \overline{(\Lambda(\theta(p, t)) : \boldsymbol{\eta}(p, t))^I} dp dt - \int_0^{t_f} \int_{\Omega} \boldsymbol{\tau}(p)\vartheta(t) : \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^I(p, t)) dp dt \\ & - \int_0^{t_f} \int_{\Omega} \boldsymbol{\tau}(p)\vartheta(t) : (\underline{\mathbf{g}}^{I*}(p, t) - \partial\phi_q^{I*}(\boldsymbol{\sigma}^{I*}(p, t))) dp dt = 0. \end{aligned} \quad (2.85)$$

In order to prove the convergence of the first term of equality (2.85), we rewrite the term  $\overline{(\Lambda(\theta) : \boldsymbol{\eta})^I}$ , as follows

$$\overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} = \overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^{I*} + (\Lambda(\theta))_*^I : \dot{\boldsymbol{\eta}}^I \text{ a.e. in } \Omega \times (0, t_f).$$

Since  $\Lambda(\theta) \in W^{2,\infty}(0, t_f; [L^\infty(\Omega)]^{n^4})$ , and thanks to convergences (2.63) and (2.65) we get the weak convergence

$$\overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^{I*} + (\Lambda(\theta))_*^I : \dot{\boldsymbol{\eta}}^I \rightharpoonup \overline{\Lambda(\theta) : \boldsymbol{\eta}} \text{ in } L^2(0, t_f; \mathbf{X}^2). \quad (2.86)$$

Hence,

$$\int_0^{t_f} \int_{\Omega} \tau(p) \vartheta(t) : \overline{((\Lambda(\theta) : \boldsymbol{\eta})^I)}(p, t) dp dt \rightarrow \int_0^{t_f} \int_{\Omega} \tau(p) \vartheta(t) : \overline{(\Lambda(\theta) : \boldsymbol{\eta})}(p, t) dp dt.$$

The convergence of the remaining terms of expression (2.85) is deduced thanks to convergences (2.69), (2.83) and Lemma 2.4.22; so, we can pass to the limit as  $I \rightarrow +\infty$  in equality (2.85) and we obtain

$$\int_0^{t_f} \int_{\Omega} \tau(p) \vartheta(t) : \overline{(\Lambda(\theta) : \boldsymbol{\eta})} - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \underline{\mathbf{g}} + \boldsymbol{\Upsilon}(p, t) dp dt = 0.$$

Consequently,

$$\overline{(\Lambda(\theta) : \boldsymbol{\eta})} - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \underline{\mathbf{g}} + \boldsymbol{\Upsilon} = \mathbf{0} \text{ a.e. in } \Omega \times (0, t_f). \quad (2.87)$$

In order to conclude that  $(\mathbf{u}, \boldsymbol{\sigma})$  verifies the behaviour law, left to prove is that

$$\boldsymbol{\Upsilon} = \partial\phi_q(\boldsymbol{\sigma}) \text{ a.e. in } \Omega \times (0, t_f). \quad (2.88)$$

For that purpose, the following lemma is necessary.

**Lemma 2.4.23.** *Under assumptions (H1)-(H8), for all  $\tau \in L^\infty(0, t_f; \mathbf{X}^q)$  the following inequality holds:*

$$\liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : (\boldsymbol{\sigma}^{I^*} - \tau) dp dt \geq \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})} : (\boldsymbol{\sigma} - \tau) dp dt,$$

where  $(\Lambda(\theta) : \boldsymbol{\eta})^I$  denotes the sequence defined by expression (2.32).

*Proof.* Since  $\boldsymbol{\sigma}^{I^*} = \boldsymbol{\eta}^{I^*} + \boldsymbol{\chi}^{I^*}$ , and from convergences (2.80) and (2.86), we get

$$\begin{aligned} \liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : (\boldsymbol{\sigma}^{I^*} - \tau) dp dt &= \liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : (\boldsymbol{\eta}^{I^*} + \boldsymbol{\chi}^{I^*} - \tau) dp dt \\ &= \liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : \boldsymbol{\eta}^{I^*} dp dt + \liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : (\boldsymbol{\chi}^{I^*} - \tau) dp dt \\ &= \liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : \boldsymbol{\eta}^{I^*} dp dt + \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})} : (\boldsymbol{\chi} - \tau) dp dt. \end{aligned}$$

Therefore, we only need to prove that

$$\liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : \boldsymbol{\eta}^{I^*} dp dt \geq \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})} : \boldsymbol{\eta} dp dt. \quad (2.89)$$

Taking into account definitions (2.32) and (2.33) and applying inequality (2.35) of Lemma 2.4.10, we can deduce

$$\begin{aligned}
\int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : \boldsymbol{\eta}^{I*} dp dt &= \sum_{i=1}^I \int_{\Omega} (\Lambda(\theta^i) : \boldsymbol{\eta}^i - \Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}) : \boldsymbol{\eta}^i dp \\
&\geq \sum_{i=1}^I \int_{\Omega} \left[ \frac{1}{2} (\Lambda(\theta^i) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i - \frac{1}{2} (\Lambda(\theta^{i-1}) : \boldsymbol{\eta}^{i-1}) : \boldsymbol{\eta}^{i-1} \right] dp \\
&\quad + \sum_{i=1}^I \int_{\Omega} \left[ \frac{1}{2} ((\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i \right] dp \\
&= \frac{1}{2} \int_{\Omega} (\Lambda(\theta^I) : \boldsymbol{\eta}^I) : \boldsymbol{\eta}^I - (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0 dp \tag{2.90} \\
&\quad + \frac{1}{2} \sum_{i=1}^I \int_{\Omega} ((\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp, \tag{2.91}
\end{aligned}$$

where we notice that here the term  $\Lambda(\theta^I) \boldsymbol{\eta}^I : \boldsymbol{\eta}^I$  represents the value of  $\Lambda(\theta^i) : \boldsymbol{\eta}^i : \boldsymbol{\eta}^i$  for  $i = I$ . If we express

$$\frac{\Lambda(\theta^i) - \Lambda(\theta^{i-1})}{\Delta t} = \overline{(\Lambda(\theta))^I}(\dot{t}), \quad t \in (t^{i-1}, t^i),$$

it results,

$$\Lambda(\theta^{i-1}) = \Lambda(\theta^i) - \overline{(\Lambda(\theta))^I} \Delta t,$$

hence, we can deduce

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^I \int_{\Omega} ((\Lambda(\theta^i) - \Lambda(\theta^{i-1})) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp &= \frac{1}{2} \sum_{i=1}^I \int_{\Omega} ((\Lambda(\theta^i) - \Lambda(\theta^i) + \overline{(\Lambda(\theta))^I} \Delta t) : \boldsymbol{\eta}^i) : \boldsymbol{\eta}^i dp \\
&= \frac{1}{2} \sum_{i=1}^I \int_{\Omega} \Delta t \overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^i : \boldsymbol{\eta}^i dp \\
&= \frac{1}{2} \sum_{i=1}^I \int_{\Omega} \int_{t^{i-1}}^{t^i} \overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^i : \boldsymbol{\eta}^i dt dp \\
&= \frac{1}{2} \int_{\Omega} \int_0^{t_f} \overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^{I*} : \boldsymbol{\eta}^{I*} dt dp.
\end{aligned}$$

Thanks to this expression we can rewrite inequality (2.91) as follows:

$$\begin{aligned}
&\liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : \boldsymbol{\eta}^{I*} dp dt \\
&\geq \liminf_{I \rightarrow +\infty} \left\{ \frac{1}{2} \int_{\Omega} [(\Lambda(\theta^I) : \boldsymbol{\eta}^I) : \boldsymbol{\eta}^I - (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0] dp + \frac{1}{2} \int_{\Omega} \int_0^{t_f} \overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^{I*} : \boldsymbol{\eta}^{I*} dt dp \right\}. \tag{2.92}
\end{aligned}$$

Using convergences (2.61) and (2.63), applying Lemma 1.2 in Lions [81] and taking into account that  $\Lambda(\theta^I)$  is the value of  $\Lambda(\theta)$  for  $t = t_f$ , we can bound the first addend on the right-hand side of expression (2.92) as

$$\liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} [(\Lambda(\theta^I) : \boldsymbol{\eta}^I) : \boldsymbol{\eta}^I - (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0] dp \geq \frac{1}{2} \int_0^{t_f} \int_{\Omega} \frac{d}{dt} [(\Lambda(\theta) : \boldsymbol{\eta}) : \boldsymbol{\eta}] dp dt. \quad (2.93)$$

In effect,  $\boldsymbol{\eta}^I \in L^\infty(0, t_f; \mathbf{X}^q)$ ,  $q \geq 2$  and since  $\mathbf{X}^q \subset \mathbf{X}^2$ , we get

$$\boldsymbol{\eta}^I \in L^\infty(0, t_f; \mathbf{X}^2) \subset L^2(0, t_f; \mathbf{X}^2).$$

Furthermore,  $\dot{\boldsymbol{\eta}}^I \in L^2(0, t_f; \mathbf{X}^2)$ , and applying the result from Lions [81] (see Lemma 1.2, p. 7), we obtain

$$\begin{aligned} \boldsymbol{\eta}^I : [0, t_f] &\longrightarrow \mathbf{X}^2 \\ t_* &\longrightarrow \boldsymbol{\eta}^I(t_*) \text{ is continuous except on a null subset.} \end{aligned}$$

The same reasoning holds for  $\boldsymbol{\eta}$ .

Hence, we can write  $\boldsymbol{\eta}^I(t_f)$  and  $\boldsymbol{\eta}(t_f)$ , and in order to obtain the result, left to prove is that

$$\boldsymbol{\eta}^I(t_f) \rightharpoonup \boldsymbol{\eta}(t_f) \text{ in } \mathbf{X}^2 \text{ weak star.}$$

In fact, thanks to weak-star convergence (2.61), we deduce the convergence

$$\boldsymbol{\eta}^I \rightharpoonup \boldsymbol{\eta} \text{ in } L^\infty(0, t_f; \mathbf{X}^2) \text{ weak star.} \quad (2.94)$$

Taking into account Corollary 2.4.14, there exists a positive constant  $c_{30}$ , such that  $\|\boldsymbol{\eta}^I\|_{L^\infty(0, t_f; \mathbf{X}^2)} \leq c_{30}$ , hence we can deduce  $\|\boldsymbol{\eta}^I(t)\|_{\mathbf{X}^2} \leq c_{31}$  for all  $t \in (0, t_f)$ . Therefore, there exists  $\hat{\boldsymbol{\eta}}(t) \in \mathbf{X}^2$  such that

$$\boldsymbol{\eta}^I(t) \rightharpoonup \hat{\boldsymbol{\eta}}(t) \text{ in } \mathbf{X}^2 \text{ weak star.} \quad (2.95)$$

From the previous convergence, we have

$$\|\hat{\boldsymbol{\eta}}(t)\|_{\mathbf{X}^2} \leq \liminf_{I \rightarrow +\infty} \|\boldsymbol{\eta}^I(t)\|_{\mathbf{X}^2} \leq c_{32} \text{ a.e. } t \in (0, t_f).$$

In consequence  $\hat{\boldsymbol{\eta}} \in L^\infty(0, t_f; \mathbf{X}^2)$ . Thus, from convergence (2.95), we get the strong convergence

$$\int_{\Omega} \boldsymbol{\eta}^I(t) : \boldsymbol{\xi} dp \rightarrow \int_{\Omega} \hat{\boldsymbol{\eta}}(t) : \boldsymbol{\xi} dp, \quad \forall \boldsymbol{\xi} \in \mathbf{X}^2.$$

And applying Dominated Convergence Theorem from Brezis [25] (see p. 54), taking  $\boldsymbol{\tau} \in L^\infty(0, t_f; \mathbf{X}^2)$ , we obtain

$$\int_{\Omega} \hat{\boldsymbol{\eta}}(t) : \boldsymbol{\tau}(t) dp \in L^1(0, t_f),$$

since

$$\int_{\Omega} \boldsymbol{\eta}^I(t) : \boldsymbol{\tau}(t) dp \in L^1(0, t_f),$$

$$\int_{\Omega} \boldsymbol{\eta}^I(t) : \boldsymbol{\tau}(t) dp \longrightarrow \int_{\Omega} \hat{\boldsymbol{\eta}}(t) : \boldsymbol{\tau}(t) dp,$$

and

$$\left| \int_{\Omega} \boldsymbol{\eta}^I(t) : \boldsymbol{\tau}(t) dp \right| \leq \|\boldsymbol{\eta}^I(t)\|_{\mathbf{X}^2}.$$

Hence, we can deduce

$$\int_0^{t_f} \int_{\Omega} \boldsymbol{\eta}^I(t) : \boldsymbol{\tau}(t) dp dt \longrightarrow \int_0^{t_f} \int_{\Omega} \hat{\boldsymbol{\eta}}(t) : \boldsymbol{\tau}(t) dp dt,$$

this is  $\boldsymbol{\eta}^I(t) \rightharpoonup \hat{\boldsymbol{\eta}}(t)$  in  $L^\infty(0, t_f; \mathbf{X}^2)$  weak star.

Then, taking into account convergence (2.94), and since if there exists the limit, it must be unique, we get  $\boldsymbol{\eta} = \hat{\boldsymbol{\eta}}$ , and we obtain the convergence  $\boldsymbol{\eta}^I(t) \rightharpoonup \boldsymbol{\eta}(t)$  in  $\mathbf{X}^2$  weak star. Therefore,

$$\begin{aligned} & \liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} [(\Lambda(\theta^I(t_f)) : \boldsymbol{\eta}^I(t_f)) : \boldsymbol{\eta}^I(t_f) - (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0] dp \\ &= \liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} [((\Lambda(\theta^I(t_f)) - \Lambda(\theta(t_f)) + \Lambda(\theta(t_f))) : \boldsymbol{\eta}^I(t_f)) : \boldsymbol{\eta}^I(t_f) - (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0] dp \\ &\geq \frac{1}{2} \int_{\Omega} [(\Lambda(\theta(t_f)) : \boldsymbol{\eta}(t_f)) : \boldsymbol{\eta}(t_f) - (\Lambda(\theta^0) : \boldsymbol{\eta}^0) : \boldsymbol{\eta}^0] dp, \end{aligned}$$

and hence we conclude (2.93).

For the second addend on the right-hand side of (2.92), we consider

$$\begin{aligned} & \liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \int_0^{t_f} (\overline{(\Lambda(\theta))^I} : \boldsymbol{\eta}^{I*}) : \boldsymbol{\eta}^{I*} dt dp \\ &\geq \liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \int_0^{t_f} ([\overline{(\Lambda(\theta))^I} - \overline{(\Lambda(\theta))}] : \boldsymbol{\eta}^{I*}) : \boldsymbol{\eta}^{I*} dt dp \\ &+ \liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \int_0^{t_f} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}^{I*}) : \boldsymbol{\eta}^{I*} dt dp. \end{aligned} \tag{2.96}$$

Firstly, let us prove that

$$\liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \int_0^{t_f} ([\overline{(\Lambda(\theta))^I} - \overline{(\Lambda(\theta))}] : \boldsymbol{\eta}^{I*}) : \boldsymbol{\eta}^{I*} dt dp = 0. \tag{2.97}$$

Applying Hölder's inequality, we have

$$\begin{aligned} & \left| \frac{1}{2} \int_{\Omega} \int_0^{t_f} ([\overline{(\Lambda(\theta))^I} - \overline{(\Lambda(\theta))}] : \boldsymbol{\eta}^{I*}) : \boldsymbol{\eta}^{I*} dt dp \right| \\ &\leq \frac{1}{2} \left\| [\overline{(\Lambda(\theta))^I} - \overline{(\Lambda(\theta))}] : \boldsymbol{\eta}^{I*} \right\|_{L^\infty(0, t_f; \mathbf{X}^q)} \|\boldsymbol{\eta}^{I*}\|_{L^1(0, t_f; \mathbf{X}^p)} \\ &\leq \frac{1}{2} \|\overline{(\Lambda(\theta))^I} - \overline{(\Lambda(\theta))}\|_{L^\infty(0, t_f; [L^\infty(\Omega)]^{n^4})} \|\boldsymbol{\eta}^{I*}\|_{L^\infty(0, t_f; \mathbf{X}^q)} \|\boldsymbol{\eta}^{I*}\|_{L^1(0, t_f; \mathbf{X}^p)} \\ &\leq c_{33} \|\overline{(\Lambda(\theta))^I} - \overline{(\Lambda(\theta))}\|_{L^\infty(0, t_f; [L^\infty(\Omega)]^{n^4})} \|\boldsymbol{\eta}^{I*}\|_{L^\infty(0, t_f; \mathbf{X}^q)}^2, \end{aligned}$$

where  $c_{33}$  is a positive constant. So, using the weak convergence (2.65) and the strong convergence

$$\overline{(\Lambda(\theta))^I} \rightarrow \overline{\Lambda(\theta)} \text{ in } W^{1,\infty}(0, t_f; L^\infty(\Omega)),$$

we obtain equality (2.97).

In order to bound the second term of the second member of (2.96) we use a classical result of convex analysis<sup>1</sup> (see Ciarlet [28], p. 122). For that purpose, let us define the function

$$\begin{aligned} g : \Omega \times (0, t_f) \times \mathbb{R}^{n \times n} &\rightarrow [0, +\infty] \\ ((p, t), \boldsymbol{\tau}) &\rightarrow (\overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau}) : \boldsymbol{\tau}, \end{aligned}$$

which verifies

- $g((p, t), \cdot)$  is convex for almost every  $(p, t) \in \Omega \times (0, t_f)$ .  
In effect, let us consider  $\underline{\lambda} \in (0, 1)$ ,  $\boldsymbol{\tau}, \boldsymbol{\xi} \in \mathbb{R}^{n \times n}$ , then

$$\begin{aligned} g((p, t), \underline{\lambda}\boldsymbol{\tau} + (1 - \underline{\lambda})\boldsymbol{\xi}) &= \overline{(\Lambda(\theta(p, t)))} : (\underline{\lambda}\boldsymbol{\tau} + (1 - \underline{\lambda})\boldsymbol{\xi}) : (\underline{\lambda}\boldsymbol{\tau} + (1 - \underline{\lambda})\boldsymbol{\xi}) \\ &= \underline{\lambda}^2 \overline{(\Lambda(\theta(p, t)))} \boldsymbol{\tau} : \boldsymbol{\tau} + (1 - \underline{\lambda})^2 \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\xi} : \boldsymbol{\xi} + 2\underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau} : \boldsymbol{\xi}. \end{aligned}$$

Taking into account the equality

$$\begin{aligned} 2\underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau} : \boldsymbol{\xi} &= \underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau} : \boldsymbol{\tau} \\ &+ \underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\xi} : \boldsymbol{\xi} - \underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : (\boldsymbol{\tau} - \boldsymbol{\xi}) : (\boldsymbol{\tau} - \boldsymbol{\xi}), \end{aligned}$$

and since  $\underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : (\boldsymbol{\tau} - \boldsymbol{\xi}) : (\boldsymbol{\tau} - \boldsymbol{\xi}) \geq 0$  thanks to hypotheses **(H1)** and **(H3)**, we deduce

$$\begin{aligned} &g((p, t), \underline{\lambda}\boldsymbol{\tau} + (1 - \underline{\lambda})\boldsymbol{\xi}) \\ &= \underline{\lambda}^2 \overline{(\Lambda(\theta(p, t)))} \boldsymbol{\tau} : \boldsymbol{\tau} + (1 - \underline{\lambda})^2 \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\xi} : \boldsymbol{\xi} + 2\underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} \boldsymbol{\tau} : \boldsymbol{\xi} \\ &= \underline{\lambda}^2 \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau} : \boldsymbol{\tau} + (1 - \underline{\lambda})^2 \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\xi} : \boldsymbol{\xi} + \underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau} : \boldsymbol{\tau} \\ &\quad + \underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\xi} : \boldsymbol{\xi} - \underline{\lambda}(1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : (\boldsymbol{\tau} - \boldsymbol{\xi}) : (\boldsymbol{\tau} - \boldsymbol{\xi}) \\ &< \underline{\lambda} \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\tau} : \boldsymbol{\tau} + (1 - \underline{\lambda}) \overline{(\Lambda(\theta(p, t)))} : \boldsymbol{\xi} : \boldsymbol{\xi} = \underline{\lambda}g((p, t), \boldsymbol{\tau}) + (1 - \underline{\lambda})g((p, t), \boldsymbol{\xi}). \end{aligned}$$

<sup>1</sup>Result from Ciarlet [28] p. 122:

Theorem 2.5.1. Let  $m$  and  $\nu$  be two integers  $\geq 1$  and  $a \in \mathbb{R}$ . Let  $\Omega$  be a open set from  $\mathbb{R}^m$  and  $g : \Omega \times \mathbb{R}^\nu \rightarrow [a, +\infty]$  with  $a = 0$  if  $\text{meas } \Omega = +\infty$ , such that  $g(p, \cdot) : q \in \mathbb{R}^\nu \rightarrow g(p, q) \in [a, +\infty]$  is convex and continuous *a.e.*  $p \in \Omega$  and  $g(\cdot, q) : p \in \Omega \rightarrow g(p, q) \in [a, +\infty]$  is a measurable function for all  $q \in \mathbb{R}^\nu$ . Then,

$$q_n \rightharpoonup q \text{ in } L^1(\Omega) \Rightarrow \int_{\Omega} g(p, q(p)) dp \leq \liminf_{n \rightarrow \infty} \int_{\Omega} g(p, q_n(p)) dp.$$

- $g((p, t), \cdot)$  is continue for almost every  $(p, t) \in \Omega \times (0, t_f)$ , since thanks to hypothesis **(H3)** it can be rewrite as the composition of two continuous functions.
- $g(\cdot, \boldsymbol{\tau})$  is a measurable function for all  $\boldsymbol{\tau} \in \mathbb{R}^{n \times n}$ .

Furthermore, from convergence (2.65) we deduce the weak convergence  $\boldsymbol{\eta}^{I^*} \rightharpoonup \boldsymbol{\eta}$  in  $L^1(0, t_f; \mathbf{X}^q)$ ; therefore, from Theorem 2.5.1 in Ciarlet [28], we obtain

$$\liminf_{I \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \int_0^{t_f} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}^{I^*}) : \boldsymbol{\eta}^{I^*} dt dp \geq \frac{1}{2} \int_{\Omega} \int_0^{t_f} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dt dp. \quad (2.98)$$

Summing up, from expressions (2.92), (2.96) and (2.98) we arrive at

$$\begin{aligned} & \liminf_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta) : \boldsymbol{\eta})^I} : \boldsymbol{\eta}^{I^*}) dp dt \\ & \geq \frac{1}{2} \int_{\Omega} [(\Lambda(\theta(p, t_f)) : \boldsymbol{\eta}(p, t_f)) : \boldsymbol{\eta}(p, t_f) - (\Lambda(\theta(p, 0)) : \boldsymbol{\eta}(p, 0)) : \boldsymbol{\eta}(p, 0)] dp \\ & + \frac{1}{2} \int_{\Omega} \int_0^{t_f} (\overline{(\Lambda(\theta(p, t)) : \boldsymbol{\eta}(p, t))} : \boldsymbol{\eta}(p, t)) dt dp \\ & \geq \frac{1}{2} \int_0^{t_f} \int_{\Omega} \frac{d}{dt} ((\Lambda(\theta) : \boldsymbol{\eta}) : \boldsymbol{\eta}) dp dt + \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt. \end{aligned} \quad (2.99)$$

But, if we apply the chain rule to the second member of this inequality we deduce

$$\begin{aligned} & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \frac{d}{dt} ((\Lambda(\theta) : \boldsymbol{\eta}) : \boldsymbol{\eta}) dp dt + \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt \\ & = \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta) : \boldsymbol{\eta})} : \boldsymbol{\eta}) dp dt + \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\Lambda(\theta) : \boldsymbol{\eta}) : \dot{\boldsymbol{\eta}} dp dt + \frac{1}{2} \int_{\Omega} \int_0^{t_f} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt \\ & = \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt + \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\Lambda(\theta) : \boldsymbol{\eta}) : \dot{\boldsymbol{\eta}} dp dt + \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\Lambda(\theta) : \dot{\boldsymbol{\eta}}) : \boldsymbol{\eta} dp dt \\ & + \frac{1}{2} \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt \\ & = \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt + \int_0^{t_f} \int_{\Omega} (\Lambda(\theta) : \dot{\boldsymbol{\eta}}) : \boldsymbol{\eta} dp dt \\ & = \int_0^{t_f} \int_{\Omega} (\overline{(\Lambda(\theta))} : \boldsymbol{\eta}) : \boldsymbol{\eta} dp dt. \end{aligned} \quad (2.100)$$

Hence, we obtain inequality (2.89).  $\square$

**Corollary 2.4.24.** *Under assumptions **(H1)**-**(H8)***

$$\mathbf{Y} = \partial\phi_q(\boldsymbol{\sigma}) \text{ a.e. in } \Omega \times (0, t_f).$$

*Proof.* Let  $\boldsymbol{\tau} \in L^\infty(0, t_f; \mathbf{X}^q)$ , since the subdifferential is a monotone operator, we have

$$\int_0^{t_f} \int_{\Omega} (\partial\phi_q^{I^*}(\boldsymbol{\sigma}^{I^*}) - \partial\phi_q^{I^*}(\boldsymbol{\tau})) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) dp dt \geq 0. \quad (2.101)$$

From the discrete behaviour law (2.84), we obtain

$$\begin{aligned} & \int_0^{t_f} \int_{\Omega} \partial \phi_q^{I^*}(\boldsymbol{\sigma}^{I^*}) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt = \int_0^{t_f} \int_{\Omega} \underline{\mathbf{g}}^{I^*} : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt \\ & + \int_0^{t_f} \int_{\Omega} \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^I) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt - \int_0^{t_f} \int_{\Omega} \overline{((\Lambda(\theta) : \boldsymbol{\eta})^I)} : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt. \end{aligned} \quad (2.102)$$

The convergence of the first term of the second member of this expression is deduced from weak convergence (2.66) and strong convergence (2.83):

$$\lim_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \underline{\mathbf{g}}^{I^*} : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt = \int_0^{t_f} \int_{\Omega} \underline{\mathbf{g}} : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt.$$

For the next term, if we use that  $\boldsymbol{\sigma}^{I^*} = \boldsymbol{\eta}^{I^*} + \boldsymbol{\chi}^{I^*}$  and we apply the orthogonality property given in Proposition 2.3.3, we have

$$\int_0^{t_f} \int_{\Omega} \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^I) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt = \int_0^{t_f} \int_{\Omega} \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^I) : (\boldsymbol{\chi}^{I^*} - \boldsymbol{\tau}) \, dp \, dt.$$

Therefore, thanks to convergences (2.70) and (2.80) and applying again Proposition 2.3.3, we deduce

$$\lim_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \boldsymbol{\varepsilon}(\underline{\mathbf{u}}^I) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt = \int_0^{t_f} \int_{\Omega} \boldsymbol{\varepsilon}(\underline{\mathbf{u}}) : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt.$$

Then, if we pass to the upper limit in expression (2.102), we can apply Lemma 2.4.23, obtaining

$$\begin{aligned} & \limsup_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \partial \phi_q^{I^*}(\boldsymbol{\sigma}^{I^*}) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt \leq \int_0^{t_f} \int_{\Omega} \underline{\mathbf{g}} : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt \\ & + \int_0^{t_f} \int_{\Omega} \boldsymbol{\varepsilon}(\underline{\mathbf{u}}) : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt - \int_0^{t_f} \int_{\Omega} \overline{((\Lambda(\theta) : \boldsymbol{\eta})^I)} : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt. \end{aligned}$$

So, taking into account (2.87), we get

$$\limsup_{I \rightarrow +\infty} \int_0^{t_f} \int_{\Omega} \partial \phi_q^{I^*}(\boldsymbol{\sigma}^{I^*}) : (\boldsymbol{\sigma}^{I^*} - \boldsymbol{\tau}) \, dp \, dt \leq \int_0^{t_f} \int_{\Omega} \boldsymbol{\Upsilon} : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt.$$

Consequently, using (2.101) we have

$$\int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial \phi_q(\boldsymbol{\tau})) : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \, dp \, dt \geq 0, \quad \forall \boldsymbol{\tau} \in L^\infty(0, t_f; \mathbf{X}^q).$$

Finally, we can conclude the result applying Minty trick (see Fučík and Kufner [54], p. 146). In effect, the previous expression is valid for all  $\boldsymbol{\tau} \in L^\infty(0, t_f; \mathbf{X}^q)$ , then, taking  $\boldsymbol{\tau} = \boldsymbol{\sigma} + s\tilde{\boldsymbol{\tau}}$ , with  $s > 0$  and  $\tilde{\boldsymbol{\tau}} \in L^\infty(0, t_f; \mathbf{X}^q)$ , it results

$$\begin{aligned} 0 & \leq \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial \phi_q(\boldsymbol{\sigma} + s\tilde{\boldsymbol{\tau}})) : (\boldsymbol{\sigma} - (\boldsymbol{\sigma} + s\tilde{\boldsymbol{\tau}})) \, dp \, dt \\ & = -s \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial \phi_q(\boldsymbol{\sigma} + s\tilde{\boldsymbol{\tau}})) : \tilde{\boldsymbol{\tau}} \, dp \, dt, \end{aligned}$$

hence,

$$0 \geq \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial\phi_q(\boldsymbol{\sigma} + s\tilde{\boldsymbol{\tau}})) : \tilde{\boldsymbol{\tau}} \, dp \, dt, \quad \forall \tilde{\boldsymbol{\tau}} \in L^\infty(0, t_f; \mathbf{X}^q).$$

Passing to limit as  $s \rightarrow 0$  on the right and applying the hemicontinuity of  $\partial\phi_q$ , (see Subsection 2.2.1, definition (2.12)), we get

$$0 \geq \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial\phi_q(\boldsymbol{\sigma})) : \tilde{\boldsymbol{\tau}} \, dp \, dt, \quad \forall \tilde{\boldsymbol{\tau}} \in L^\infty(0, t_f; \mathbf{X}^q). \quad (2.103)$$

If we replace stress  $\tilde{\boldsymbol{\tau}}$  in the previous expression with  $-\tilde{\boldsymbol{\tau}}$ , we obtain

$$0 \geq \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial\phi_q(\boldsymbol{\sigma})) : (-\tilde{\boldsymbol{\tau}}) \, dp \, dt = - \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial\phi_q(\boldsymbol{\sigma})) : \tilde{\boldsymbol{\tau}} \, dp \, dt,$$

and we can deduce

$$0 \leq \int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial\phi_q(\boldsymbol{\sigma})) : \tilde{\boldsymbol{\tau}} \, dp \, dt, \quad \forall \tilde{\boldsymbol{\tau}} \in L^\infty(0, t_f; \mathbf{X}^q). \quad (2.104)$$

Then, from inequalities (2.103) and (2.104), we get

$$\int_0^{t_f} \int_{\Omega} (\boldsymbol{\Upsilon} - \partial\phi_q(\boldsymbol{\sigma})) : \tilde{\boldsymbol{\tau}} \, dp \, dt = 0, \quad \forall \tilde{\boldsymbol{\tau}} \in L^\infty(0, t_f; \mathbf{X}^q),$$

and we conclude the result.  $\square$

### Step VIII: Uniqueness of solution of Problem $(P_T)$

Let  $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ ,  $(\mathbf{u}_2, \boldsymbol{\sigma}_2)$  be two solutions of Problem  $(P_T)$  and let us denote by  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \in \mathbf{H}^q(\mathbf{0}, \mathbf{0})$  and  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2 \in \mathbf{U}_0^p$ . They verify

$$\begin{aligned} \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) &= \overline{\Lambda(\theta) : \boldsymbol{\sigma}_1} - \overline{\Lambda(\theta) : \boldsymbol{\sigma}_2} + \partial\phi_q(\boldsymbol{\sigma}_1) - \partial\phi_q(\boldsymbol{\sigma}_2) \quad a.e. \text{ in } \Omega \times (0, t_f], \\ \boldsymbol{\sigma}(0) &= \mathbf{0} \text{ and } \mathbf{u}(0) = \mathbf{0} \quad a.e. \text{ in } \Omega. \end{aligned} \quad (2.105)$$

Thanks to the orthogonality between  $\mathbf{H}^q(\mathbf{0}, \mathbf{0})$  and  $\mathbf{U}_0^p$ , multiplying (2.105) by  $\boldsymbol{\sigma}$  and integrating in time and space, we obtain for all  $t \in (0, t_f]$

$$\int_0^t \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\sigma})} : \boldsymbol{\sigma} \, dp \, ds + \int_0^t \int_{\Omega} (\partial\phi_q(\boldsymbol{\sigma}_1) - \partial\phi_q(\boldsymbol{\sigma}_2)) : (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \, dp \, ds = 0. \quad (2.106)$$

The second term of expression (2.106) is positive due to the monotony of the subdifferential  $\partial\phi_q$ , therefore

$$\int_0^t \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\sigma})} : \boldsymbol{\sigma} \, dp \, ds \leq 0.$$

If we use the equality

$$\overline{(\Lambda(\theta) : \boldsymbol{\tau})} : \boldsymbol{\tau} = \frac{1}{2} \overline{(\Lambda(\theta) : \boldsymbol{\tau})} : \boldsymbol{\tau} + \frac{1}{2} \overline{(\Lambda(\theta) : \boldsymbol{\tau})} : \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in S_n,$$

we have

$$\frac{1}{2} \int_0^t \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\sigma})} : \boldsymbol{\sigma} dp dt + \frac{1}{2} \int_0^t \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\sigma})} : \boldsymbol{\sigma} dp ds \leq 0.$$

Furthermore, from hypotheses **(H1)** and **(H3)** we deduce that the term  $\overline{(\Lambda(\theta) : \boldsymbol{\tau})} : \boldsymbol{\tau}$  is positive. Thus

$$\int_0^t \int_{\Omega} \overline{(\Lambda(\theta) : \boldsymbol{\sigma})} : \boldsymbol{\sigma} dp ds \leq 0.$$

Then, if we integrate in time, and since  $\boldsymbol{\sigma}(p, 0) = \mathbf{0}$ , we obtain

$$\int_{\Omega} (\Lambda(\theta(p, t)) : \boldsymbol{\sigma}(p, t)) : \boldsymbol{\sigma}(p, t) dp \leq \int_{\Omega} (\Lambda(\theta(p, 0)) : \boldsymbol{\sigma}(p, 0)) : \boldsymbol{\sigma}(p, 0) dp = 0;$$

hence  $\boldsymbol{\sigma} = \mathbf{0}$ , so,  $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$ . Now, from equation (2.105) we obtain  $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}$ . In consequence, since  $\mathbf{u}(p, 0) = \mathbf{0}$ ,  $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}$ . Thus, from Korn's inequality in  $\mathbf{U}_0^p$  we conclude that  $\mathbf{u}_1 = \mathbf{u}_2$ .  $\square$

**Remark 2.4.25.** *The uniqueness of solution guarantees the convergence for the whole sequence  $(\mathbf{u}^I, \boldsymbol{\sigma}^I)$  to  $(\mathbf{u}, \boldsymbol{\sigma})$  and not only for a certain subsequence.*



## Chapter 3

# Regularity of a viscoelastic problem

### 3.1 Introduction

This chapter deals with the question of regularity for the stress solution of the Maxwell-Norton model with temperature dependent coefficients studied in the previous chapter. We recall that the mathematical model arises from the modelling of the deformation of aluminium structures exposed to fire where temperature changes play a fundamental role. The existence and uniqueness of solution for this problem has been proved in the previous chapter and their results are published in Barral *et al.* [12].

Bensoussan and Frehse [16] proved that the stress solution of a Maxwell-Norton model belongs to  $[H_{\text{Loc}}^1(\Omega)]^{n^2}$  for the static case. Afterwards, they obtained a similar result for the time dependent case (see [17]). They also proved that, when the deviatoric stress tensor is bounded, the  $H_{\text{Loc}}^2$  regularity is reached. In this chapter by assuming that all coefficients of the behaviour law depend on temperature, we will prove the regularity properties for the behaviour law studied in Chapter 2, which is a combination of two parts: an Arrhenius thermal part and a Maxwell-Norton viscoelastic part.

The outline of this chapter is as follows: after giving some preliminary notations in Section 3.2, we will recall the mathematical model and the result of existence and uniqueness of solution proved in Chapter 2 in Section 2.4. In Section 3.4 we will prove the  $H_{\text{Loc}}^1$  regularity of stresses after introducing the appropriate hypotheses. Finally, in Section 3.5 we will obtain the  $H_{\text{Loc}}^2$  regularity.

### 3.2 Preliminary notation

We introduce some useful notation about differential calculus and tensorial products.

Let  $g(p, t)$  be a scalar function defined in  $\mathbb{R}^n \times \mathbb{R}$ . We represent by  $g(t)$  the function  $p \rightarrow g(p, t)$ ,  $g_t$  its partial derivative with respect to  $t$  and  $\partial_i g$  or  $g_{,i}$  its partial derivative with respect to the variable  $p_i$ ,  $1 \leq i \leq n$ ,  $n = 3$ .  $\nabla g$ ,  $D^2 g$ ,  $D^3 g$  and  $\Delta g$  denote its gradient, Hessian, differential of

third order and Laplacian respectively, defined by

$$\begin{aligned} (\nabla g)_i &= \partial_i g = g_{,i}, \quad (D^2 g)_{ij} = \partial_{ij} g = g_{,ij}, \quad (D^3 g)_{ijk} = \partial_{ijk} g = g_{,ijk}, \quad 1 \leq i, j, k \leq n, \\ \Delta g &= \partial_{ii} g, \end{aligned}$$

where the sum over repeated subscripts is implied.

If  $\mathbf{u}$  is a vectorial function in  $\mathbb{R}^n$ , we denote by  $\nabla \mathbf{u}$ ,  $D^2 \mathbf{u}$ ,  $\Delta \mathbf{u}$  and  $\text{Div } \mathbf{u}$  its gradient, Hessian, Laplacian and divergence respectively, determined by

$$\begin{aligned} (\nabla \mathbf{u})_{ij} &= \partial_i u_j = u_{j,i}, \quad (D^2 \mathbf{u})_{ijk} = \partial_{ij} u_k = u_{k,ij}, \quad (\Delta \mathbf{u})_i = \Delta u_i, \quad 1 \leq i, j, k \leq n, \\ \text{Div } \mathbf{u} &= \partial_i u_i. \end{aligned}$$

For  $\boldsymbol{\tau} \in S_n$ , we denote by  $D\boldsymbol{\tau}$ ,  $D^2\boldsymbol{\tau}$  and  $\text{Div } \boldsymbol{\tau}$  the differential operator of first and second order and the divergence of the tensor valued function  $\boldsymbol{\tau}$  respectively, defined by

$$\begin{aligned} (D\boldsymbol{\tau})_{kij} &= \partial_k \tau_{ij} = \tau_{ij,k}, \quad (D^2\boldsymbol{\tau})_{kl ij} = \partial_{kl} \tau_{ij} = \tau_{ij,kl}, \quad 1 \leq i, j, k, l \leq n, \\ (\text{Div } \boldsymbol{\tau})_j &= \partial_i \tau_{ij} = \tau_{ij,i}, \quad 1 \leq j \leq n. \end{aligned}$$

If  $\mathbf{V}$  is a fourth order tensor valued function, we denote by  $D\mathbf{V}$  and  $D^2\mathbf{V}$  its differential operator of first and second order respectively, with components

$$\begin{aligned} (D\mathbf{V})_{kijlh} &= \partial_k V_{ijlh} = V_{ijlh,k}, \quad 1 \leq i, j, l, h, k \leq n, \\ (D^2\mathbf{V})_{kmi jlh} &= \partial_{km} V_{ijlh} = V_{ijlh,km}, \quad 1 \leq i, j, l, h, k, m \leq n. \end{aligned}$$

Let  $L^r(\Omega)$ , for  $1 \leq r < \infty$ , be the usual space of  $r$ -integrable functions on  $\Omega$ . We denote by  $[L^r(\Omega)]^n$  the space of vectors whose components are in  $L^r(\Omega)$ , dotted with the norm

$$\|\mathbf{u}\|_{[L^r(\Omega)]^n} = \left[ \sum_{i=1}^n \int_{\Omega} |u_i|^r dp \right]^{\frac{1}{r}}.$$

Analogously, we can define the corresponding spaces of tensors provided with the usual norm. So, given  $\boldsymbol{\tau} \in [L^r(\Omega)]^{n^2}$  we consider the following norms:

$$\begin{aligned} \|\boldsymbol{\tau}\|_{[L^r(\Omega)]^{n^2}} &= \left[ \sum_{i,j=1}^n \int_{\Omega} |\tau_{ij}|^r dp \right]^{\frac{1}{r}}, \\ \|D\boldsymbol{\tau}\|_{[L^r(\Omega)]^{n^3}} &= \left[ \sum_{k=1}^n \|\partial_k \boldsymbol{\tau}\|_{[L^r(\Omega)]^{n^2}}^r \right]^{\frac{1}{r}}, \end{aligned}$$

and

$$\|D^2\boldsymbol{\tau}\|_{[L^r(\Omega)]^{n^4}} = \left[ \sum_{k,l=1}^n \|\partial_{kl} \boldsymbol{\tau}\|_{[L^r(\Omega)]^{n^2}}^r \right]^{\frac{1}{r}}.$$

**Definition 3.2.1.** Let  $\mathbf{A}$  be an  $n$ -th order tensor and  $\mathbf{B}$  be an  $m$ -th order tensor. Then the tensor product  $\mathbf{AB}$  is an  $(n+m)$ -th order tensor defined by

$$(\mathbf{AB})_{i_1, \dots, i_{n+m}} = A_{i_1 \dots i_n} B_{i_{n+1} \dots i_{n+m}}.$$

**Definition 3.2.2.** Let  $\mathbf{A}$  be an  $n$ -th order tensor and  $\mathbf{B}$  be an  $m$ -th order tensor. We define the contraction product of  $l$  subscripts, with  $l \leq n, m$ , as the following  $(n-l) + (m-l)$ -th order tensor:

$$\begin{aligned} (\mathbf{A} \cdot_l \mathbf{B})_{i_1, \dots, i_{n-l} j_1, \dots, j_{m-l}} &= A_{i_1 \dots i_{n-l} k_1 \dots k_l} B_{k_1 \dots k_l j_1 \dots j_{m-l}}, & \text{if } l < n, m, \\ (\mathbf{A} \cdot_l \mathbf{B})_{j_1, \dots, j_{m-l}} &= A_{k_1 \dots k_l} B_{k_1 \dots k_l j_1 \dots j_{m-l}}, & \text{if } l = n < m, \\ \mathbf{A} \cdot_l \mathbf{B} &= A_{k_1 \dots k_l} B_{k_1 \dots k_l}, & \text{if } l = n = m. \end{aligned}$$

We notice that if  $\mathbf{u}, \mathbf{v}$  are vectors in  $\mathbb{R}^n$ , with components  $u_i$  and  $v_i$ , their scalar product can be understood as a contraction product of one subscript:  $\mathbf{u} \cdot_1 \mathbf{v} = u_i v_i$ . Analogously, for  $\boldsymbol{\tau}, \boldsymbol{\xi} \in S_n$ , their scalar product can be represented by  $\boldsymbol{\tau} \cdot_2 \boldsymbol{\xi} = \tau_{ij} \xi_{ij}$ . Hereinafter,  $|\boldsymbol{\tau}|$  denotes the norm induced by this scalar product.

Examples: Using the previous notation and taking into account Definitions 3.2.1 and 3.2.2, we have

$$\begin{aligned} (\nabla g \mathbf{u})_{ij} &= (\nabla g)_i u_j = g_{,i} u_j, \\ \nabla g \cdot_1 \mathbf{u} &= (\nabla g)_i u_i = g_{,i} u_i, \\ \boldsymbol{\tau} \cdot_2 \nabla \mathbf{u} &= \tau_{ij} (\nabla \mathbf{u})_{ij} = \tau_{ij} u_{j,i}, \\ (\mathbf{V} \cdot_2 \boldsymbol{\tau}) \cdot_2 \boldsymbol{\tau} &= (V_{ijlm} \tau_{lm}) \tau_{ij}, \\ (\mathbf{V} \cdot_3 D\boldsymbol{\tau})_i &= V_{ijlm} (D\boldsymbol{\tau})_{jlm} = V_{ijlm} \tau_{lm,j}, \\ \mathbf{V} \cdot_4 D^2 \boldsymbol{\tau} &= V_{kl ij} (D^2 \boldsymbol{\tau})_{kl ij} = V_{kl ij} \tau_{ij,kl}. \end{aligned}$$

In order to familiarize readers with this notation see Segel [92].

### 3.3 Mathematical Model

We recall the mathematical model studied in the previous chapter in order to introduce the new notation.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded, open and connected domain with a Lipschitz boundary  $\Gamma$ ; we assume

$$\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset \text{ and } \text{meas}(\Gamma_D) > 0,$$

where  $\Gamma_D$  and  $\Gamma_N$  are  $n-1$  dimensional open subsets of  $\Gamma$ .

Let  $[0, t_f]$  be the time interval of interest. The field of temperatures  $\theta(p, t)$  is assumed to be known at each  $(p, t) \in \Omega \times (0, t_f]$ . We denote by  $\mathbf{u}(p, t)$  the displacement field and by  $\boldsymbol{\sigma}(p, t)$  the stress field at each  $(p, t) \in \Omega \times (0, t_f]$ .

We study the regularity properties of the stress tensor which is the solution of the following problem:

**Problem** ( $P_T$ )

Given  $\theta(p, t)$  in  $\Omega \times [0, t_f]$ , find  $\mathbf{u}(p, t)$  and  $\boldsymbol{\sigma}(p, t)$  in  $\Omega \times (0, t_f]$ , such that:

$$\begin{cases} -\text{Div } \boldsymbol{\sigma} = \mathbf{b} & \text{in } \Omega \times (0, t_f], & (3.1a) \\ \boldsymbol{\sigma} \cdot_1 \mathbf{n} = \mathbf{g} & \text{on } \Gamma_N \times (0, t_f], & (3.1b) \\ \mathbf{u} = \mathbf{u}_D & \text{on } \Gamma_D \times (0, t_f], & (3.1c) \\ \boldsymbol{\varepsilon}(\mathbf{u}_t) = (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma}) + \alpha(\theta)\theta_t \mathbf{I} & \text{in } \Omega \times (0, t_f], & (3.1d) \\ \mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 & \text{in } \Omega. & (3.1e) \end{cases}$$

Here

- $\mathbf{b}$  are the applied body forces,
- $\mathbf{n}$  denotes the outward unit normal to  $\Omega$ ,
- $\mathbf{g}$  is the surface force,
- $\mathbf{u}_D$  is the boundary Dirichlet condition,
- $\boldsymbol{\varepsilon}(\mathbf{u}_t)$  is the linearized deformation rate tensor,
- $\Lambda$  represents the tensor of Hooke's law,
- $\alpha$  is the coefficient of thermal expansion and
- $D\phi_q$  is the differential of the plasticity potential given by

$$D\phi_q(\boldsymbol{\sigma}) = \Theta(\theta)|\boldsymbol{\sigma}^D|^{q-2}\boldsymbol{\sigma}^D, \quad (3.2)$$

where  $q$  is a strictly positive material parameter and  $\Theta$  is a function depending on temperature.

Furthermore, the initial conditions  $\mathbf{u}_0$  and  $\boldsymbol{\sigma}_0$  must verify the usual compatibility conditions:

$$\begin{cases} -\text{Div } \boldsymbol{\sigma}_0 = \mathbf{b}(0) & \text{in } \Omega, \\ \boldsymbol{\sigma}_0 \cdot_1 \mathbf{n} = \mathbf{g}(0) & \text{on } \Gamma_N, \\ \mathbf{u}_0 = \mathbf{u}_D(0) & \text{on } \Gamma_D, \\ \boldsymbol{\varepsilon}(\mathbf{u}_0) = \Lambda(\theta_0) \cdot_2 \boldsymbol{\sigma}_0 & \text{in } \Omega, \end{cases} \quad (3.3)$$

with  $\theta_0 = \theta(0)$ . For more details of the model see Chapter 2 or Barral *et al.* [12].

**Remark 3.3.1.** *In behaviour law (3.1d) the coefficients only depend on temperature. Nevertheless, the methodology used in this chapter could be extended to behaviour laws that include a coefficient of material damage.*

### 3.3.1 Existence and uniqueness of solution

Let us remember the existence and uniqueness result proved in the previous chapter. In order to simplify the writing of regularity results we rearrange the hypotheses of that theorem in the following way:

(H1) The temperature field is positive and  $\theta_t \geq 0$  a. e. in  $\Omega$ ,  $\forall t \in [0, t_f]$ .

(H2) The applied surface force verifies

$$\mathbf{g} \in W^{2,\infty}(0, t_f; [W^{-\frac{1}{q},q}(\Gamma)]^n \cap [L^q(\Gamma_N)]^n).$$

(H3) The exponent  $q$  verifies  $q \geq 2$ , thus its conjugate  $p$  is such that  $1 < p \leq 2$ .

(H4) The initial displacement field verifies

$$\mathbf{u}_0 \in \mathbf{U}^p = \{\mathbf{v} \in [W^{1,p}(\Omega)]^n; \text{Div } \mathbf{v} = \text{tr}(\boldsymbol{\varepsilon}(\mathbf{v})) \in L^2(\Omega)\}.$$

(H5) The Dirichlet function verifies

$$\mathbf{u}_D \in W^{2,\infty}(0, t_f; [W^{1-\frac{1}{p},p}(\Gamma)]^n).$$

(H6) The tensor of Hooke's law  $\Lambda$  is symmetric and the following hold true:

- $\Lambda \in [W^{2,\infty}(\mathbb{R})]^{n^4}$ .
- there exists  $\kappa > 0$  such that  $(\Lambda(s) \cdot_2 \boldsymbol{\tau}) \cdot_2 \boldsymbol{\tau} \geq \kappa |\boldsymbol{\tau}|^2$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .
- $(\Lambda'(s) \cdot_2 \boldsymbol{\tau}) \cdot_2 \boldsymbol{\tau} \geq 0$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .

(H7) The coefficient of thermal expansion verifies  $\alpha \in W^{1,\infty}(\mathbb{R})$ .

(H8) The coefficient of the plasticity potential verifies

$$\Theta \in L^\infty(\mathbb{R}) \text{ and } \Theta(s) \geq \Theta_{min} > 0, \forall s \in \mathbb{R}.$$

(H9) The temperature field verifies  $\theta \in W^{2,\infty}(0, t_f; L^\infty(\Omega))$ .

(H10) The body force verifies  $\mathbf{b} \in W^{2,\infty}(0, t_f; [L^q(\Omega)]^n)$ .

(H11) The initial stress verifies

$$\boldsymbol{\sigma}_0 \in \mathbf{X}^q = \{\boldsymbol{\tau} = (\tau_{ij}); \tau_{ij} = \tau_{ji}, \boldsymbol{\tau}^D \in [L^q(\Omega)]^{n^2}, \text{tr}(\boldsymbol{\tau}) \in L^2(\Omega)\}.$$

The functional spaces  $\mathbf{U}^p$  and  $\mathbf{X}^q$  are studied in-depth in Geymonat & Suquet [63] and they are defined in Subsection 2.3.

In the previous chapter the following result was proved:

**Theorem 3.3.2.** *Under assumptions (H1)-(H11) and assuming also that  $\mathbf{u}_0, \boldsymbol{\sigma}_0$  verify compatibility conditions (3.3), there exists a unique solution  $(\mathbf{u}, \boldsymbol{\sigma})$  of problem  $(P_T)$  such that*

$$\mathbf{u} \in W^{1,2}(0, t_f; \mathbf{U}^p) \text{ and } \boldsymbol{\sigma} \in W^{1,2}(0, t_f; \mathbf{X}^2) \cap L^\infty(0, t_f; \mathbf{X}^q). \quad (3.4)$$

### 3.4 $H_{\text{Loc}}^1$ Regularity

The aim of this section is to prove the  $H_{\text{Loc}}^1$  regularity of stresses. To do so, throughout this section we replace **(H8)**-**(H11)** with the following hypotheses:

$$\overline{\text{(H8)}} \quad \Theta \in W^{1,\infty}(\mathbb{R}) \text{ and } \Theta(s) \geq \Theta_{\min} > 0, \forall s \in \mathbb{R}.$$

$$\overline{\text{(H9)}} \quad \theta \in W^{2,\infty}(0, t_f; W^{1,\infty}(\Omega)).$$

$$\overline{\text{(H10)}} \quad \mathbf{b} \in W^{2,\infty}(0, t_f; [H^q(\Omega)]^n) \text{ and } \Delta \mathbf{b} \in L^\infty(0, t_f; [L^q(\Omega)]^n).$$

$$\overline{\text{(H11)}} \quad \boldsymbol{\sigma}_0 \in \mathbf{X}^q \cap [H_{\text{Loc}}^1(\Omega)]^{n^2}.$$

$$\overline{\text{(H12)}} \quad \text{The trace of the stress tensor verifies } \text{tr}(\boldsymbol{\sigma}) \in L^\infty(0, t_f; L^q(\Omega)).$$

Theorem 3.3.2 establishes that the stress tensor  $\boldsymbol{\sigma}(t)$ , solution of problem  $(P_T)$ , belongs to  $\mathbf{X}^2$  and hypothesis  $\overline{\text{(H12)}}$  assures that  $\boldsymbol{\sigma}(t) \in [L^q(\Omega)]^{n^2}$  a. e. in  $(0, t_f)$ . Because  $\boldsymbol{\sigma}(t)$  cannot be derived in  $[L^q(\Omega)]^{n^2}$  with respect to any spacial variable, in the forthcoming proofs its derivatives should be replaced with the corresponding difference quotients which have the same regularity as  $\boldsymbol{\sigma}(t)$ ; nevertheless, since this is a classical technique in regularity analysis, we present for the sake of simplicity a formal proof using the derivatives of  $\boldsymbol{\sigma}(t)$ ; for details, see Bensoussan and Frehse [17] or Frehse and Málek [50]. In this context, given  $\Omega' \subset\subset \Omega$  and  $\boldsymbol{\tau}$  a second order tensor valued function in  $[L^r(\Omega)]^{n^2}$ , with  $1 \leq r < \infty$ , we consider the following norm for its difference quotients:

$$\|D\boldsymbol{\tau}\|_{[L^r(\Omega')]^{n^3}} = \left[ \sum_{k=1}^n \|\partial_k \boldsymbol{\tau}\|_{[L^r(\Omega')]^{n^2}}^r \right]^{\frac{1}{r}},$$

where

$$\partial_k \boldsymbol{\tau}(p) = \frac{\boldsymbol{\tau}(p + h\mathbf{e}^k) - \boldsymbol{\tau}(p)}{h}, \quad p \in \Omega', \quad \mathbf{e}^k = (e_1^k, \dots, e_n^k) \text{ with } e_j^k = \delta_{kj},$$

$\delta_{kj}$  being the Kronecker delta and  $|h| \leq \text{dist}(\Omega', \partial\Omega)$ .

**Theorem 3.4.1.** *Under assumptions **(H1)**-**(H7)** and  $\overline{\text{(H8)}}$ - $\overline{\text{(H12)}}$ , the stress solution of problem  $(P_T)$  satisfies  $\boldsymbol{\sigma}(t) \in [H_{\text{Loc}}^1(\Omega)]^{n^2}$ ,  $\forall t \in [0, t_f]$ .*

The proof is based on the following lemmas.

**Lemma 3.4.2.** *Let  $\varphi$  be a time independent smooth function with compact support in  $\Omega$ . Then,*

under the assumptions of Theorem 3.4.1, the following equality holds:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varphi (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma} dp + \frac{1}{2} \int_{\Omega} \varphi (D\boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dp \\
& + \int_{\Omega} \varphi (D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_3 D\boldsymbol{\sigma} dp + \int_{\Omega} \varphi (D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_3 D\boldsymbol{\sigma} dp \\
& + \frac{1}{2} \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} \nabla(\Theta(\theta)) \cdot_1 \nabla |\boldsymbol{\sigma}^D|^2 dp + \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp \\
& + \frac{q-2}{4} \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} |\nabla |\boldsymbol{\sigma}^D|^2|^2 dp + \int_{\Omega} \varphi \nabla(\alpha(\theta)\theta_t) \cdot_1 \nabla(\text{tr}(\boldsymbol{\sigma})) dp \\
& - \int_{\Omega} \mathbf{u}_t \cdot_1 [D^3\varphi \cdot_2 \boldsymbol{\sigma} - D^2\varphi \cdot_1 \mathbf{b}] dp + \int_{\Omega} \mathbf{u}_t \cdot_1 (\nabla\varphi \cdot_1 \nabla\mathbf{b} + \varphi\Delta\mathbf{b}) dp \\
& - \int_{\Omega} \text{Div} \mathbf{u}_t (D^2\varphi \cdot_2 \boldsymbol{\sigma} - \nabla\varphi \cdot_1 \mathbf{b}) dp + 2 \int_{\Omega} \{[(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla\varphi\} \cdot_3 D\boldsymbol{\sigma} dp \\
& - 2 \int_{\Omega} \alpha(\theta)\theta_t \nabla\varphi \cdot_1 \mathbf{b} dp = 0 \text{ in } L^\infty(0, t_f). \tag{3.5}
\end{aligned}$$

*Proof.* Let us consider a time independent smooth function  $\varphi$  with compact support in  $\Omega$ . Following the techniques used in Bensoussan and Frehse [16, 17], the proof consists of multiplying constitutive law (3.1d) by the test tensor function  $-\partial_k(\varphi\partial_k\boldsymbol{\sigma})$ , integrating over  $\Omega$  and summing over  $k$ ,  $k = 1, \dots, n$ :

$$\begin{aligned}
& - \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_k(\varphi\partial_k\boldsymbol{\sigma}) dp = - \int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_k(\varphi\partial_k\boldsymbol{\sigma}) dp \\
& - \int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_k(\varphi\partial_k\boldsymbol{\sigma}) dp - \int_{\Omega} \alpha(\theta)\theta_t \mathbf{I} \cdot_2 \partial_k(\varphi\partial_k\boldsymbol{\sigma}) dp. \tag{3.6}
\end{aligned}$$

For the first member, using the definition of the strain tensor and integrating twice by parts, we have

$$\begin{aligned}
- \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_k(\varphi\partial_k\boldsymbol{\sigma}) dp &= \int_{\Omega} \partial_k \varepsilon_{ij}(u_t) \varphi \partial_k \sigma_{ij} dp = \int_{\Omega} \partial_k \left[ \frac{1}{2} (\partial_i(u_t)_j + \partial_j(u_t)_i) \right] \varphi \partial_k \sigma_{ij} dp \\
&= \frac{1}{2} \int_{\Omega} \partial_k \partial_i (u_t)_j \varphi \partial_k \sigma_{ij} dp + \frac{1}{2} \int_{\Omega} \partial_k \partial_j (u_t)_i \varphi \partial_k \sigma_{ij} dp \\
&= -\frac{1}{2} \int_{\Omega} \partial_k (u_t)_j \partial_i (\varphi \partial_k \sigma_{ij}) dp - \frac{1}{2} \int_{\Omega} \partial_k (u_t)_i \partial_j (\varphi \partial_k \sigma_{ij}) dp \\
&= -\frac{1}{2} \int_{\Omega} \partial_k (u_t)_j \partial_i \varphi \partial_k \sigma_{ij} dp - \frac{1}{2} \int_{\Omega} \partial_k (u_t)_j \varphi \partial_{ik} \sigma_{ij} dp \\
&\quad - \frac{1}{2} \int_{\Omega} \partial_k (u_t)_i \partial_j \varphi \partial_k \sigma_{ij} dp - \frac{1}{2} \int_{\Omega} \partial_k (u_t)_i \varphi \partial_{jk} \sigma_{ij} dp.
\end{aligned}$$

Taking into account the equilibrium equation and the symmetry of the tensor  $\boldsymbol{\sigma}$ , it follows that

$$\begin{aligned} -\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) \, dp &= -\frac{1}{2} \int_{\Omega} \partial_k(u_t)_j \partial_i \varphi \partial_k \sigma_{ij} \, dp + \frac{1}{2} \int_{\Omega} \partial_k(u_t)_j \varphi \partial_k b_j \, dp \\ &\quad - \frac{1}{2} \int_{\Omega} \partial_k(u_t)_i \partial_j \varphi \partial_k \sigma_{ji} \, dp + \frac{1}{2} \int_{\Omega} \partial_k(u_t)_i \varphi \partial_k b_i \, dp \\ &= -\int_{\Omega} \partial_k(u_t)_j \partial_i \varphi \partial_k \sigma_{ij} \, dp + \int_{\Omega} \partial_k(u_t)_j \varphi \partial_k b_j \, dp. \end{aligned} \quad (3.7)$$

The second term on the right-hand side of equation (3.7) can be rewritten as

$$\int_{\Omega} \partial_k(u_t)_j \varphi \partial_k b_j \, dp = -\int_{\Omega} (u_t)_j [\partial_k \varphi \partial_k b_j + \varphi \partial_{kk} b_j] \, dp. \quad (3.8)$$

In addition, the first term of the second member of equation (3.7) takes the expression

$$-\int_{\Omega} \partial_k(u_t)_j \partial_i \varphi \partial_k \sigma_{ij} \, dp = -\int_{\Omega} 2\varepsilon_{jk}(\mathbf{u}_t) \partial_i \varphi \partial_k \sigma_{ij} \, dp + \int_{\Omega} \partial_j(u_t)_k \partial_i \varphi \partial_k \sigma_{ij} \, dp. \quad (3.9)$$

From behaviour law (3.1d), we convert the first term on the right-hand side of this equality into

$$\begin{aligned} &-\int_{\Omega} 2\varepsilon_{jk}(\mathbf{u}_t) \partial_i \varphi \partial_k \sigma_{ij} \, dp \\ &= -2 \int_{\Omega} [(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})]_{jk} \partial_i \varphi \partial_k \sigma_{ij} \, dp - 2 \int_{\Omega} \alpha(\theta) \theta_t \delta_{jk} \partial_i \varphi \partial_k \sigma_{ij} \, dp. \end{aligned}$$

Thanks to equilibrium equation (3.1a), we obtain

$$\begin{aligned} &-\int_{\Omega} 2\varepsilon_{jk}(\mathbf{u}_t) \partial_i \varphi \partial_k \sigma_{ij} \, dp \\ &= -2 \int_{\Omega} [(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})]_{jk} \partial_i \varphi \partial_k \sigma_{ij} \, dp + 2 \int_{\Omega} \alpha(\theta) \theta_t \partial_i \varphi b_i \, dp. \end{aligned} \quad (3.10)$$

Let us consider the second term on the right-hand side of equation (3.9); integrating by parts and using the equilibrium equation, we deduce

$$\begin{aligned} \int_{\Omega} \partial_j(u_t)_k \partial_i \varphi \partial_k \sigma_{ij} \, dp &= -\int_{\Omega} (u_t)_k \partial_{ji} \varphi \partial_k \sigma_{ij} \, dp - \int_{\Omega} (u_t)_k \partial_i \varphi \partial_{jk} \sigma_{ij} \, dp \\ &= -\int_{\Omega} (u_t)_k \partial_{ji} \varphi \partial_k \sigma_{ij} \, dp + \int_{\Omega} (u_t)_k \partial_i \varphi \partial_k b_i \, dp, \end{aligned}$$

and, integrating by parts again, we obtain

$$\begin{aligned} &\int_{\Omega} \partial_j(u_t)_k \partial_i \varphi \partial_k \sigma_{ij} \, dp \\ &= \int_{\Omega} \partial_k(u_t)_k [\partial_{ji} \varphi \sigma_{ij} - \partial_i \varphi b_i] \, dp + \int_{\Omega} (u_t)_k [\partial_{kji} \varphi \sigma_{ij} - \partial_{ki} \varphi b_i] \, dp. \end{aligned} \quad (3.11)$$

Therefore, from equalities (3.7)-(3.11) we have the following expression for the first member of equation (3.6)

$$\begin{aligned}
-\int_{\Omega} \varepsilon(\mathbf{u}_t) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) dp &= -\int_{\Omega} 2\{[(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla \varphi\} \cdot_3 D\boldsymbol{\sigma} dp + 2\int_{\Omega} \alpha(\theta) \theta_t \nabla \varphi \cdot_1 \mathbf{b} dp \\
&+ \int_{\Omega} \text{Div } \mathbf{u}_t (D^2 \varphi \cdot_2 \boldsymbol{\sigma} - \nabla \varphi \cdot_1 \mathbf{b}) dp \\
&+ \int_{\Omega} \mathbf{u}_t \cdot_1 [D^3 \varphi \cdot_2 \boldsymbol{\sigma} - D^2 \varphi \cdot_1 \mathbf{b}] dp \\
&- \int_{\Omega} \mathbf{u}_t \cdot_1 (\nabla \varphi \cdot_1 \nabla \mathbf{b} + \varphi \Delta \mathbf{b}) dp. \tag{3.12}
\end{aligned}$$

Integrating by parts the first term of the second member of equation (3.6) we obtain the expression

$$\begin{aligned}
-\int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) dp &= \int_{\Omega} \varphi \partial_k (\Lambda(\theta)_{ijlm} \boldsymbol{\sigma}_{lm})_t \cdot_2 \partial_k \boldsymbol{\sigma}_{ij} dp \\
&= \int_{\Omega} \varphi \partial_k ((\Lambda(\theta)_{ijlm})_t \boldsymbol{\sigma}_{lm}) \cdot_2 \partial_k \boldsymbol{\sigma}_{ij} dp \\
&+ \int_{\Omega} \varphi \partial_k (\Lambda(\theta)_{ijlm} ((\boldsymbol{\sigma}_{lm})_t)) \cdot_2 \partial_k \boldsymbol{\sigma}_{ij} dp \\
&= \int_{\Omega} \varphi \partial_k (\Lambda(\theta)_{ijlm})_t \boldsymbol{\sigma}_{lm} \partial_k \boldsymbol{\sigma}_{ij} dp + \int_{\Omega} \varphi (\Lambda(\theta)_{ijlm})_t \partial_k \boldsymbol{\sigma}_{lm} \partial_k \boldsymbol{\sigma}_{ij} dp \\
&+ \int_{\Omega} \varphi \partial_k (\Lambda(\theta)_{ijlm}) (\boldsymbol{\sigma}_{lm})_t \partial_k \boldsymbol{\sigma}_{ij} dp \\
&+ \int_{\Omega} \varphi \Lambda(\theta)_{ijlm} \partial_k (\boldsymbol{\sigma}_{lm})_t \partial_k \boldsymbol{\sigma}_{ij} dp. \tag{3.13}
\end{aligned}$$

From the chain rule, we have

$$\begin{aligned}
((D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma})_t &= (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta))_t \cdot_3 D\boldsymbol{\sigma} + (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma}_t \\
&= (D\boldsymbol{\sigma}_t \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma} + (D\boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} + (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma}_t,
\end{aligned}$$

and we can deduce the following equality:

$$(D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma}_t = \frac{1}{2} ((D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma})_t - \frac{1}{2} (D\boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma},$$

so expression (3.13) can be rewritten as

$$\begin{aligned}
-\int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) dp &= \int_{\Omega} \varphi (D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_3 D\boldsymbol{\sigma} dp + \frac{1}{2} \int_{\Omega} \varphi (D\boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dp \\
&+ \int_{\Omega} \varphi (D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_3 D\boldsymbol{\sigma} dp \\
&+ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varphi (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma} dp. \tag{3.14}
\end{aligned}$$

For the second term on the right-hand side of equation (3.6), and taking into account expression (3.2) and integrating by parts, we obtain

$$\begin{aligned} - \int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) dp &= \int_{\Omega} \partial_k(\partial\phi_q(\boldsymbol{\sigma})) \cdot_2 \varphi \partial_k \boldsymbol{\sigma} dp \\ &= \int_{\Omega} \partial_k(\Theta(\theta)) |\boldsymbol{\sigma}^D|^{q-2} \sigma_{ij}^D \varphi \partial_k \sigma_{ij} dp + \int_{\Omega} \Theta(\theta) \partial_k |\boldsymbol{\sigma}^D|^{q-2} \sigma_{ij}^D \varphi \partial_k \sigma_{ij} dp \\ &\quad + \int_{\Omega} \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} \partial_k \sigma_{ij}^D \varphi \partial_k \sigma_{ij} dp. \end{aligned}$$

Since

$$\begin{aligned} \partial_k((\boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D)^{\frac{q-2}{2}}) &= \frac{q-2}{2} (\boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D)^{\frac{q-2}{2}-1} 2 \partial_k \boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D \\ &= (q-2) (\boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D)^{\frac{q-4}{2}} \partial_k \boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D = \frac{q-2}{2} (\boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D)^{\frac{q-4}{2}} \partial_k |\boldsymbol{\sigma}^D|^2, \end{aligned}$$

and

$$\boldsymbol{\sigma}^D \cdot_2 \partial_k \boldsymbol{\sigma}^D = \frac{\partial_k (|\boldsymbol{\sigma}^D|^2)}{2},$$

we obtain,

$$\partial_k |\boldsymbol{\sigma}^D|^{q-2} = (q-2) |\boldsymbol{\sigma}^D|^{q-4} \frac{\partial_k |\boldsymbol{\sigma}^D|^2}{2}. \quad (3.15)$$

Then we deduce

$$\begin{aligned} - \int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) dp &= \frac{1}{2} \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} \nabla(\Theta(\theta)) \cdot_1 \nabla |\boldsymbol{\sigma}^D|^2 dp \\ &\quad + \frac{q-2}{4} \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} |\nabla |\boldsymbol{\sigma}^D|^2|^2 dp \\ &\quad + \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp. \end{aligned} \quad (3.16)$$

Finally, for the last term of the second member of equation (3.6), we obtain

$$\begin{aligned} - \int_{\Omega} \alpha(\theta) \theta_t \mathbf{I} \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}) dp &= \int_{\Omega} \partial_k(\alpha(\theta)) \theta_t \delta_{ij} \varphi \partial_k \sigma_{ij} dp + \int_{\Omega} \alpha(\theta) \partial_k \theta_t \delta_{ij} \varphi \partial_k \sigma_{ij} dp \\ &= \int_{\Omega} \varphi \theta_t \nabla(\alpha(\theta)) \cdot_1 \nabla(\text{tr}(\boldsymbol{\sigma})) dp + \int_{\Omega} \varphi \alpha(\theta) \nabla \theta_t \cdot_1 \nabla(\text{tr}(\boldsymbol{\sigma})) dp \\ &= \int_{\Omega} \varphi \nabla(\alpha(\theta) \theta_t) \cdot_1 \nabla(\text{tr}(\boldsymbol{\sigma})) dp. \end{aligned} \quad (3.17)$$

Therefore, from equalities (3.6), (3.12), (3.14), (3.16) and (3.17) we conclude result (3.5).  $\square$

**Lemma 3.4.3.** *Let  $\varphi$  be a time independent smooth function with compact support in  $\Omega$ . Under assumptions (H1)-(H7) and (H8)-(H12), the following inequality holds:*

$$\begin{aligned} \frac{\kappa}{2} \|\varphi D\boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^3}}^2 &+ \frac{\Theta_{\min}}{2} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp ds + \frac{\Theta_{\min}}{8} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-4} |\nabla |\boldsymbol{\sigma}^D|^2|^2 dp ds \\ &\leq c_{29}(\varphi) + c_{30}(\varphi) \int_0^t \|\varphi D\boldsymbol{\sigma}(s)\|_{[L^2(\Omega)]^{n^3}}^2 ds, \quad \forall t \in (0, t_f), \end{aligned} \quad (3.18)$$

where  $\Theta_C$  is a lower bound of  $\Theta(\theta)$  and  $c_l(\varphi)$ ,  $l \geq 0$ , denotes a positive constant depending at most on  $\varphi$ .

*Proof.* Let us replace  $\varphi$  with  $\varphi^2$  in expression (3.5) and integrate over  $(0, t)$ ,

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma} dp + \frac{1}{2} \int_0^t \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dpds \\
& + \int_0^t \int_{\Omega} \varphi^2 \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dpds + \frac{q-2}{4} \int_0^t \int_{\Omega} \varphi^2 \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} |\nabla |\boldsymbol{\sigma}^D|^2|^2 dpds \\
& = \frac{1}{2} \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma}_0 \cdot_2 \Lambda(\theta_0)) \cdot_3 D\boldsymbol{\sigma}_0 dp + \int_0^t \int_{\Omega} \mathbf{u}_t \cdot_1 [D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma} - D^2 \varphi^2 \cdot_1 \mathbf{b}] dpds \\
& - \int_0^t \int_{\Omega} \mathbf{u}_t \cdot_1 (\nabla \varphi^2 \cdot_1 \nabla \mathbf{b} + \varphi^2 \Delta \mathbf{b}) dpds + \int_0^t \int_{\Omega} \text{Div } \mathbf{u}_t (D^2 \varphi^2 \cdot_2 \boldsymbol{\sigma} - \nabla \varphi^2 \cdot_1 \mathbf{b}) dpds \\
& + 2 \int_0^t \int_{\Omega} \alpha(\theta) \theta_t \nabla \varphi^2 \cdot_1 \mathbf{b} dpds - 2 \int_0^t \int_{\Omega} (\Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} \boldsymbol{\sigma}^D \nabla \varphi^2) \cdot_3 D\boldsymbol{\sigma} dpds \\
& - 2 \int_0^t \int_{\Omega} ((\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \nabla \varphi^2) \cdot_3 D\boldsymbol{\sigma} dpds - \int_0^t \int_{\Omega} \varphi^2 (D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_3 D\boldsymbol{\sigma} dpds \\
& - \int_0^t \int_{\Omega} \varphi^2 (D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_3 D\boldsymbol{\sigma} dpds - \frac{1}{2} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} \nabla(\Theta(\theta)) \cdot_1 \nabla |\boldsymbol{\sigma}^D|^2 dpds \\
& - \int_0^t \int_{\Omega} \varphi^2 \nabla(\alpha(\theta) \theta_t) \cdot_1 \nabla(\text{tr}(\boldsymbol{\sigma})) dpds. \tag{3.19}
\end{aligned}$$

Using hypotheses **(H6)** and  $(\overline{\mathbf{H9}})$ , we have

$$\frac{1}{2} \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma} dp \geq \frac{\kappa}{2} \|\varphi D\boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^3}}^2.$$

Thanks to hypotheses **(H1)**, **(H6)**, we get

$$\frac{1}{2} \int_0^t \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dpds \geq 0.$$

Applying hypotheses **(H6)** and  $(\overline{\mathbf{H9}})$ , we obtain

$$\frac{1}{2} \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma}_0 \cdot_2 \Lambda(\theta_0)) \cdot_3 D\boldsymbol{\sigma}_0 dp \leq \frac{\|\Lambda(\theta_0)\|_{[L^\infty(\Omega)]^{n^4}}}{2} \|\varphi D\boldsymbol{\sigma}_0(t)\|_{[L^2(\Omega)]^{n^3}}^2.$$

Then, from equality (3.19) we deduce

$$\begin{aligned}
& \frac{\kappa}{2} \|\varphi D\boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^3}}^2 + \Theta_{\min} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dpds \\
& + \frac{(q-2)\Theta_{\min}}{4} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-4} |\nabla |\boldsymbol{\sigma}^D|^2|^2 dpds \\
& \leq \frac{\|\Lambda(\theta_0)\|_{[L^\infty(\Omega)]^{n^4}}}{2} \|\varphi D\boldsymbol{\sigma}_0(t)\|_{[L^2(\Omega)]^{n^3}}^2 + \sum_{l=1}^{10} |I_l|, \tag{3.20}
\end{aligned}$$

where to simplify the notation,  $I_l$  denotes the  $(l + 1)$ -th term of the second member of equality (3.19).

The first term of the second member of equation (3.20) is bounded by hypothesis  $(\overline{\mathbf{H11}})$ .

Since  $\mathbf{u}_t \in L^2(0, t_f; \mathbf{U}^p)$ ,  $\boldsymbol{\sigma} \in L^\infty(0, t_f; \mathbf{X}^q)$  and taking into account assumptions  $(\mathbf{H7})$ ,  $(\overline{\mathbf{H9}})$ ,  $(\overline{\mathbf{H10}})$  and  $(\mathbf{H12})$ , terms  $I_l$ ,  $l = 1, 2, 3, 4$  are bounded. Indeed:

- We obtain an estimation for  $I_1$  thanks to  $\mathbf{u}_t \in L^2(0, t_f; \mathbf{U}^p)$ ,  $\boldsymbol{\sigma} \in L^\infty(0, t_f; \mathbf{X}^q)$ , hypotheses  $(\overline{\mathbf{H10}})$ ,  $(\mathbf{H12})$  and applying Hölder's inequality on  $L^2(0, t_f; [L^p(\Omega)]^n) \times L^2(0, t_f; [L^q(\Omega)]^n)$  (see Ženíšek [101] p. 40 and 299):

$$\begin{aligned} |I_1| &\leq \int_0^t \int_\Omega |\mathbf{u}_t \cdot_1 (D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma})| \, dp ds + \int_0^t \int_\Omega |\mathbf{u}_t \cdot_1 (D^2 \varphi^2 \cdot_1 \mathbf{b})| \, dp ds \\ &\leq \|\mathbf{u}_t\|_{L^2(0, t_f; [L^p(\Omega)]^n)} \left[ \|D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} + \|D^2 \varphi^2 \cdot_1 \mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} \right] \\ &\leq \|\mathbf{u}_t\|_{L^2(0, t_f; [L^p(\Omega)]^n)} \left[ c_1(\varphi) \|\boldsymbol{\sigma}\|_{L^2(0, t_f; [L^q(\Omega)]^{n^2})} + c_2(\varphi) \|\mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} \right]. \end{aligned}$$

Since

$$\begin{aligned} \|D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} &= \left[ \int_0^t \|D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{[L^q(\Omega)]^n}^2 ds \right]^{\frac{1}{2}} \\ &= \left[ \int_0^t \left( \int_\Omega |D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \leq \left[ \int_0^t \left( \int_\Omega |D^3 \varphi^2|^q |\boldsymbol{\sigma}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \\ &\leq c_1(\varphi) \left[ \int_0^t \left( \int_\Omega |\boldsymbol{\sigma}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} = c_1(\varphi) \|\boldsymbol{\sigma}\|_{L^2(0, t_f; [L^q(\Omega)]^{n^2})}, \end{aligned}$$

and

$$\begin{aligned} \|D^2 \varphi^2 \cdot_1 \mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} &= \left[ \int_0^t \|D^2 \varphi^2 \cdot_1 \mathbf{b}\|_{[L^q(\Omega)]^n}^2 ds \right]^{\frac{1}{2}} = \left[ \int_0^t \left( \int_\Omega |D^2 \varphi^2 \cdot_1 \mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \\ &\leq \left[ \int_0^t \left( \int_\Omega |D^2 \varphi^2|^q |\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \leq c_2(\varphi) \left[ \int_0^t \left( \int_\Omega |\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} = c_2(\varphi) \|\mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)}. \end{aligned}$$

- For  $I_2$  if we consider that  $\mathbf{u}_t \in L^2(0, t_f; \mathbf{U}^p)$ , hypothesis  $(\overline{\mathbf{H10}})$  and Hölder's inequality on  $L^2(0, t_f; [L^p(\Omega)]^n) \times L^2(0, t_f; [L^q(\Omega)]^n)$ , we have

$$\begin{aligned} |I_2| &\leq \int_0^t \int_\Omega |\mathbf{u}_t \cdot_1 (\nabla \varphi^2 \cdot_1 \nabla \mathbf{b})| \, dp ds + \int_0^t \int_\Omega |\mathbf{u}_t \cdot_1 (\varphi^2 \Delta \mathbf{b})| \, dp ds \\ &\leq \|\mathbf{u}_t\|_{L^2(0, t_f; [L^p(\Omega)]^n)} \left[ \|\nabla \varphi^2 \cdot_1 \nabla \mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} + \|\varphi^2 \Delta \mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} \right] \\ &\leq \|\mathbf{u}_t\|_{L^2(0, t_f; [L^p(\Omega)]^n)} \left[ c_3(\varphi) \|\nabla \mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^{n^2})} + c_4(\varphi) \|\Delta \mathbf{b}\|_{L^2(0, t_f; [L^q(\Omega)]^n)} \right], \end{aligned}$$

since

$$\begin{aligned} \|\nabla\varphi^2 \cdot_1 \nabla\mathbf{b}\|_{L^2(0,t_f;[L^q(\Omega)]^n)} &= \left[ \int_0^t \|\nabla\varphi^2 \cdot_1 \nabla\mathbf{b}\|_{[L^q(\Omega)]^n}^2 ds \right]^{\frac{1}{2}} = \left[ \int_0^t \left( \int_{\Omega} |\nabla\varphi^2 \cdot_1 \nabla\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \\ &\leq \left[ \int_0^t \left( \int_{\Omega} |\nabla\varphi^2|^q |\nabla\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \leq c_3(\varphi) \left[ \int_0^t \left( \int_{\Omega} |\nabla\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \\ &= c_3(\varphi) \|\nabla\mathbf{b}\|_{L^2(0,t_f;[L^q(\Omega)]^{n^2})}, \end{aligned}$$

and

$$\begin{aligned} \|\varphi^2 \Delta\mathbf{b}\|_{L^2(0,t_f;[L^q(\Omega)]^n)} &= \left[ \int_0^t \|\varphi^2 \Delta\mathbf{b}\|_{[L^q(\Omega)]^n}^2 ds \right]^{\frac{1}{2}} = \left[ \int_0^t \left( \int_{\Omega} |\varphi^2 \Delta\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \\ &\leq \left[ \int_0^t \left( \int_{\Omega} |\varphi^2|^q |\Delta\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \leq c_4(\varphi) \left[ \int_0^t \left( \int_{\Omega} |\Delta\mathbf{b}|^q dp \right)^{\frac{2}{q}} ds \right]^{\frac{1}{2}} \\ &= c_4(\varphi) \|\Delta\mathbf{b}\|_{L^2(0,t_f;[L^q(\Omega)]^n)}. \end{aligned}$$

- Analogously, for  $I_3$  we get

$$\begin{aligned} |I_3| &\leq \int_0^t \int_{\Omega} |\text{Div } \mathbf{u}_t (D^2\varphi^2 \cdot_2 \boldsymbol{\sigma})| dp ds + \int_0^t \int_{\Omega} |\text{Div } \mathbf{u}_t (\nabla\varphi^2 \cdot_1 \mathbf{b})| dp ds \\ &\leq \|\text{Div } \mathbf{u}_t\|_{L^2(0,t_f;L^2(\Omega))} \left[ \|D^2\varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{L^2(0,t_f;L^2(\Omega))} + \|\nabla\varphi^2 \cdot_1 \mathbf{b}\|_{L^2(0,t_f;L^2(\Omega))} \right] \\ &\leq \|\text{Div } \mathbf{u}_t\|_{L^2(0,t_f;L^2(\Omega))} \left[ c_5(\varphi) \|\boldsymbol{\sigma}\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} + c_6(\varphi) \|\mathbf{b}\|_{L^2(0,t_f;[L^2(\Omega)]^n)} \right], \end{aligned}$$

since

$$\begin{aligned} \|D^2\varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{L^2(0,t_f;L^2(\Omega))} &= \left[ \int_0^t \|D^3\varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{L^2(\Omega)}^2 ds \right]^{\frac{1}{2}} = \left[ \int_0^t \left( \int_{\Omega} |D^2\varphi^2 \cdot_2 \boldsymbol{\sigma}|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \\ &\leq \left[ \int_0^t \left( \int_{\Omega} |D^2\varphi^2|^2 |\boldsymbol{\sigma}|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \leq c_5(\varphi) \left[ \int_0^t \left( \int_{\Omega} |\boldsymbol{\sigma}|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \\ &= c_5(\varphi) \|\boldsymbol{\sigma}\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla\varphi^2 \cdot_1 \mathbf{b}\|_{L^2(0,t_f;L^2(\Omega))} &= \left[ \int_0^t \|\nabla\varphi^2 \cdot_1 \mathbf{b}\|_{L^2(\Omega)}^2 ds \right]^{\frac{1}{2}} = \left[ \int_0^t \left( \int_{\Omega} |\nabla\varphi^2 \cdot_1 \mathbf{b}|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \\ &\leq \left[ \int_0^t \left( \int_{\Omega} |\nabla\varphi^2|^2 |\mathbf{b}|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \leq c_6(\varphi) \left[ \int_0^t \left( \int_{\Omega} |\mathbf{b}|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \\ &= c_6(\varphi) \|\mathbf{b}\|_{L^2(0,t_f;[L^2(\Omega)]^n)}. \end{aligned}$$

- For  $I_4$ , using hypotheses **(H7)**, **(H9)** and **(H10)**, we obtain the estimation

$$\begin{aligned} |I_4| &\leq 2 \int_0^t \int_{\Omega} |\alpha(\theta)\theta_t \nabla \varphi^2 \cdot_1 \mathbf{b}| \, dp ds \leq \|\alpha(\theta)\theta_t \nabla \varphi^2\|_{L^2(0,t_f;[L^2(\Omega)]^n)} \|\mathbf{b}\|_{L^2(0,t_f;[L^2(\Omega)]^n)} \\ &\leq c_7(\varphi) \|\alpha(\theta)\theta_t\|_{L^2(0,t_f;L^2(\Omega))} \|\mathbf{b}\|_{L^2(0,t_f;[L^2(\Omega)]^n)}, \end{aligned}$$

since

$$\begin{aligned} \|\alpha(\theta)\theta_t \nabla \varphi^2\|_{L^2(0,t_f;L^2(\Omega))} &= \left[ \int_0^t \|\alpha(\theta)\theta_t \nabla \varphi^2\|_{L^2(\Omega)}^2 ds \right]^{\frac{1}{2}} = \left[ \int_0^t \left( \int_{\Omega} |\alpha(\theta)\theta_t \nabla \varphi^2|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \\ &\leq \left[ \int_0^t \left( \int_{\Omega} |\nabla \varphi^2|^2 |\alpha(\theta)\theta_t|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \leq c_7(\varphi) \left[ \int_0^t \left( \int_{\Omega} |\alpha(\theta)\theta_t|^2 dp \right)^{\frac{2}{2}} ds \right]^{\frac{1}{2}} \\ &= c_7(\varphi) \|\alpha(\theta)\theta_t\|_{L^2(0,t_f;L^2(\Omega))}. \end{aligned}$$

Terms  $I_l$ ,  $5 \leq l \leq 10$  of equation (3.19) are bounded combining Hölder's and Young's<sup>1</sup> inequalities. In effect:

- In order to bound  $I_5$ , we apply Green's formula in space to obtain

$$\begin{aligned} |I_5| &\leq 2 \left| \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} [(\boldsymbol{\sigma}^D \cdot_1 \nabla(\Theta(\theta))) \nabla \varphi^2] \cdot_2 \boldsymbol{\sigma} \, dp ds \right| \\ &\quad + 2 \left| \int_0^t \int_{\Omega} \Theta(\theta) [(\boldsymbol{\sigma}^D \cdot_1 \nabla |\boldsymbol{\sigma}^D|^{q-2}) \nabla \varphi^2] \cdot_2 \boldsymbol{\sigma} \, dp ds \right| \\ &\quad + 2 \left| \int_0^t \int_{\Omega} \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} (\operatorname{div} \boldsymbol{\sigma}^D \nabla \varphi^2) \cdot_2 \boldsymbol{\sigma} \, dp ds \right| \\ &\quad + 2 \left| \int_0^t \int_{\Omega} \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} (\boldsymbol{\sigma}^D \cdot_1 D^2 \varphi^2) \cdot_2 \boldsymbol{\sigma} \, dp ds \right| \\ &= 2 (|I_{5,1}| + |I_{5,2}| + |I_{5,3}| + |I_{5,4}|). \end{aligned} \tag{3.21}$$

From hypotheses **(H8)** and **(H9)**, we get

$$|I_{5,1}| \leq 2 \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} |\boldsymbol{\sigma}^D| |\nabla(\Theta(\theta))| |\nabla \varphi^2| |\boldsymbol{\sigma}| \, dp ds \leq c_8(\varphi) \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-1} |\boldsymbol{\sigma}| \, dp ds.$$

<sup>1</sup>We recall Young's inequality:

$$ab \leq \varepsilon a^s + \frac{1}{\varepsilon^{\frac{1}{s-1}}} b^{s'}, \quad \forall a \geq 0, \quad \forall b \geq 0, \quad \frac{1}{s} + \frac{1}{s'} = 1 \text{ and } \varepsilon > 0.$$

Since  $\sigma(t) \in [L^q(\Omega)]^{n^2}$ , applying Hölder's inequality on  $L^{\frac{q}{q-1}}(\Omega) \times L^q(\Omega)$ ,

$$\begin{aligned} |I_{5,1}| &\leq c_8(\varphi) \int_0^t \left\{ \left( \int_{\Omega} |\sigma^D|^{(q-1)\frac{q}{q-1}} dp \right)^{\frac{q-1}{q}} \left( \int_{\Omega} |\sigma|^q dp \right)^{\frac{1}{q}} \right\} ds \\ &\leq c_8(\varphi) \int_0^t \left\{ \left[ \left( \int_{\Omega} |\sigma^D|^q dp \right)^{\frac{1}{q}} \right]^{q-1} \left( \int_{\Omega} |\sigma|^q dp \right)^{\frac{1}{q}} \right\} ds \\ &= c_8(\varphi) \int_0^t \|\sigma^D\|_{[L^q(\Omega)]^{n^2}}^{q-1} \|\sigma\|_{[L^q(\Omega)]^{n^2}} ds, \end{aligned}$$

hence, we conclude that  $|I_{5,1}| \leq c_9(\varphi)$ .

Let us apply equality (3.15) in the next term  $I_{5,2}$  of expression (3.21), so it can be bounded as follows

$$\begin{aligned} |I_{5,2}| &\leq \frac{q-2}{2} \left| \int_0^t \int_{\Omega} [|\sigma^D|^{q-4} \Theta(\theta)(\sigma^D \cdot \nabla |\sigma^D|^2) \nabla \varphi^2] \cdot \sigma dp ds \right| \\ &\leq q-2 \int_0^t \int_{\Omega} |\Theta(\theta)| |\sigma^D|^{q-4} |\nabla |\sigma^D|^2| |\sigma^D \varphi| |\nabla \varphi| |\sigma| dp ds \\ &\leq c_{10}(\varphi) \int_0^t \int_{\Omega} \varphi |\sigma^D|^{q-3} |\nabla |\sigma^D|^2| |\sigma| dp ds \\ &\leq c_{10}(\varphi) \int_0^t \int_{\Omega} \varphi |\sigma^D|^{\frac{q-4}{2}} |\nabla |\sigma^D|^2| |\sigma^D|^{\frac{q-2}{2}} |\sigma| dp ds, \end{aligned}$$

where we have taken into account that  $\nabla \varphi^2 = 2\varphi \nabla \varphi$ . Then, thanks to hypothesis  $(\overline{\mathbf{H12}})$  we can apply again Hölder's inequality firstly on  $L^2(\Omega) \times L^2(\Omega)$  and secondly on  $L^{\frac{q}{q-2}}(\Omega) \times L^{\frac{q}{2}}(\Omega)$ , obtaining

$$\begin{aligned} |I_{5,2}| &\leq c_{10}(\varphi) \int_0^t \left( \int_{\Omega} \varphi^2 |\sigma^D|^{q-4} |\nabla |\sigma^D|^2|^2 dp \right)^{\frac{1}{2}} \left( \int_{\Omega} |\sigma^D|^{q-2} |\sigma|^2 dp \right)^{\frac{1}{2}} ds \\ &\leq c_{10}(\varphi) \int_0^t \left( \int_{\Omega} \varphi^2 |\sigma^D|^{q-4} |\nabla |\sigma^D|^2|^2 dp \right)^{\frac{1}{2}} \left( \left[ \int_{\Omega} (|\sigma^D|^{q-2})^{\frac{q}{q-2}} dp \right]^{\frac{q-2}{q}} \left[ \int_{\Omega} (|\sigma|^2)^{\frac{q}{2}} dp \right]^{\frac{2}{q}} \right)^{\frac{1}{2}} ds \\ &= c_{10}(\varphi) \int_0^t \left( \int_{\Omega} \varphi^2 |\sigma^D|^{q-4} |\nabla |\sigma^D|^2|^2 dp \right)^{\frac{1}{2}} \|\sigma^D\|_{[L^q(\Omega)]^{n^2}}^{\frac{q-2}{2}} \|\sigma\|_{[L^q(\Omega)]^{n^2}} ds. \end{aligned}$$

Using Young's inequality for  $\varepsilon = \frac{\Theta_{\min}}{16c_{10}(\varphi)}$ , we conclude that

$$\begin{aligned} |I_{5,2}| &\leq \frac{\Theta_{\min}}{16} \int_0^t \int_{\Omega} \varphi^2 |\sigma^D|^{q-4} |\nabla |\sigma^D|^2|^2 dp ds + \frac{c_{10}(\varphi)16}{\Theta_{\min}} \int_0^t \|\sigma^D\|_{[L^q(\Omega)]^{n^2}}^{q-2} \|\sigma\|_{[L^q(\Omega)]^{n^2}}^2 ds \\ &\leq \frac{\Theta_{\min}}{16} \int_0^t \int_{\Omega} \varphi^2 |\sigma^D|^{q-4} |\nabla |\sigma^D|^2|^2 dp ds + c_{11}(\varphi). \end{aligned}$$

Taking into account hypotheses  $(\overline{\mathbf{H8}})$  and  $(\overline{\mathbf{H9}})$ , we have

$$|I_{5,3}| \leq c_{12}(\varphi) \int_0^t \int_{\Omega} |\sigma^D|^{q-2} |\text{Div} \sigma^D| |\sigma| dp ds \leq c_{12}(\varphi) \int_0^t \int_{\Omega} |\sigma^D|^{q-2} |D \sigma^D| |\sigma| dp ds.$$

In a similar way as term  $I_{5,2}$ , under hypothesis  $(\overline{\mathbf{H12}})$ , we apply Hölder's inequality firstly on  $L^2(\Omega) \times L^2(\Omega)$  and secondly on  $L^{\frac{q}{q-2}}(\Omega) \times L^{\frac{q}{2}}(\Omega)$ , obtaining

$$\begin{aligned} |I_{5,3}| &\leq c_{12}(\varphi) \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{\frac{q-2}{2}} |D\boldsymbol{\sigma}^D| |\boldsymbol{\sigma}^D|^{\frac{q-2}{2}} |\boldsymbol{\sigma}| dp ds \\ &\leq c_{12}(\varphi) \int_0^t \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} |\boldsymbol{\sigma}|^2 dp \right)^{\frac{1}{2}} ds \\ &\leq c_{12}(\varphi) \int_0^t \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \|\boldsymbol{\sigma}^D\|_{[L^q(\Omega)]^{n^2}}^{\frac{q-2}{2}} \|\boldsymbol{\sigma}\|_{[L^q(\Omega)]^{n^2}} ds, \end{aligned}$$

and using Young's inequality for  $\varepsilon = \frac{\Theta_{min}}{8c_{12}(\varphi)}$ , we conclude that

$$\begin{aligned} |I_{5,3}| &\leq \frac{\Theta_{min}}{8} \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp ds + \frac{c_{12}(\varphi)8}{\Theta_{min}} \int_0^t \|\boldsymbol{\sigma}^D\|_{[L^q(\Omega)]^{n^2}}^{\frac{q-2}{2}} \|\boldsymbol{\sigma}\|_{[L^q(\Omega)]^{n^2}} ds \\ &\leq \frac{\Theta_{min}}{8} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp ds + c_{13}(\varphi). \end{aligned}$$

Finally, the bound of  $I_{5,4}$  is obtained analogously to term  $I_{5,1}$ :

$$|I_{5,4}| \leq 2 \int_0^t \int_{\Omega} |\Theta(\theta)| |\boldsymbol{\sigma}^D|^{q-2} |\boldsymbol{\sigma}^D| |D^2\varphi^2| |\boldsymbol{\sigma}| dp ds \leq c_{14}(\varphi) \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-1} |\boldsymbol{\sigma}| dp ds,$$

and considering Hölder's inequality on  $L^{\frac{q}{q-1}}(\Omega) \times L^q(\Omega)$ ,

$$\begin{aligned} |I_{5,4}| &\leq c_{14}(\varphi) \int_0^t \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-1} |\boldsymbol{\sigma}| dp ds \\ &\leq c_{14}(\varphi) \int_0^t \left\{ \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^{(q-1)\frac{q}{q-1}} dp \right)^{\frac{q-1}{q}} \left( \int_{\Omega} |\boldsymbol{\sigma}|^q dp \right)^{\frac{1}{q}} \right\} ds \\ &\leq c_{14}(\varphi) \int_0^t \left\{ \left[ \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^q dp \right)^{\frac{1}{q}} \right]^{q-1} \left( \int_{\Omega} |\boldsymbol{\sigma}|^q dp \right)^{\frac{1}{q}} \right\} ds \\ &= c_{14}(\varphi) \int_0^t \|\boldsymbol{\sigma}^D\|_{[L^q(\Omega)]^{n^2}}^{q-1} \|\boldsymbol{\sigma}\|_{[L^q(\Omega)]^{n^2}} ds \leq c_{15}(\varphi). \end{aligned}$$

- Let us apply Hölder's inequality on  $[L^2(\Omega)]^{n^2} \times [L^2(\Omega)]^{n^2}$  to term  $I_6$ :

$$|I_6| \leq c_{16}(\varphi) \left( \int_0^t \|(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}}.$$

Then, taking into account that

$$(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t = (\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma} + \Lambda(\theta) \cdot_2 \boldsymbol{\sigma}_t, \quad (3.22)$$

using hypotheses **(H6)**,  $(\overline{\mathbf{H9}})$  and since  $\boldsymbol{\sigma} \in L^\infty(0, t_f; \mathbf{X}^2)$  and  $\boldsymbol{\sigma}_t \in L^2(0, t_f; \mathbf{X}^2)$  we conclude that

$$\begin{aligned} |I_6| &\leq c_{16}(\varphi) \left( \int_0^t \|(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma} + \Lambda(\theta) \cdot_2 (\boldsymbol{\sigma})_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{16}(\varphi) \left( \int_0^t (\|(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}} + \|\Lambda(\theta) \cdot_2 (\boldsymbol{\sigma})_t\|_{[L^2(\Omega)]^{n^3}})^2 ds \right)^{\frac{1}{2}} \\ &\quad \left( \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{17}(\varphi) \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{18}(\varphi), \end{aligned}$$

by applying Young's inequality with  $\varepsilon = \frac{1}{2}$ .

- Analogously, terms  $I_l$ ,  $l = 7, \dots, 10$  can be bounded from hypothesis  $(\overline{\mathbf{H9}})$ , applying Hölder's and Young's inequalities consecutively for an appropriate  $\varepsilon$

– and using hypothesis **(H6)** for term  $I_7$ :

$$\begin{aligned} |I_7| &\leq c_{19}(\varphi) \left( \int_0^t \|D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}\|_{\mathbf{X}^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{\mathbf{X}^2}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{20}(\varphi) \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{21}(\varphi); \end{aligned}$$

– and thanks to  $\boldsymbol{\sigma}_t \in L^2(0, t_f; \mathbf{X}^2)$  and **(H6)**, for term  $I_8$ :

$$\begin{aligned} |I_8| &\leq c_{22}(\varphi) \left( \int_0^t \|D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t\|_{\mathbf{X}^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{\mathbf{X}^2}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{23}(\varphi) \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{24}(\varphi); \end{aligned}$$

– and hypothesis  $(\overline{\mathbf{H8}})$  for term  $I_9$ :

$$\begin{aligned} |I_9| &\leq c_{25}(\varphi) \int_0^t \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{\frac{q-2}{2}} |D\boldsymbol{\sigma}^D| |\boldsymbol{\sigma}^D|^{\frac{q-2}{2}} |\boldsymbol{\sigma}^D| dp ds \\ &\leq c_{25}(\varphi) \int_0^t \left( \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^{q-2} |\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} ds \\ &\leq c_{25}(\varphi) \int_0^t \left( \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \int_{\Omega} |\boldsymbol{\sigma}^D|^q dp \right)^{\frac{1}{2}} ds \\ &\leq c_{25}(\varphi) \int_0^t \left( \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \|\boldsymbol{\sigma}^D\|_{L^q(\Omega)}^{\frac{q}{2}} ds \\ &\leq \frac{\Theta_{\min}}{4} \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} |D\boldsymbol{\sigma}^D|^2 dp ds + c_{26}(\varphi); \end{aligned}$$

– and hypothesis **(H7)** for term  $I_{10}$ :

$$\begin{aligned}
|I_{10}| &\leq \int_0^t \int_{\Omega} |\varphi \nabla(\alpha(\theta)\theta_t)| |\varphi \nabla(\operatorname{tr}(\boldsymbol{\sigma}))| dp ds \leq \int_0^t \int_{\Omega} |\varphi \nabla(\alpha(\theta)\theta_t)| |\varphi \operatorname{tr}(D\boldsymbol{\sigma})| dp ds \\
&\leq c_{27}(\varphi) \int_0^t \int_{\Omega} |\nabla(\alpha(\theta)\theta_t)| |\varphi D\boldsymbol{\sigma}| dp ds \\
&\leq c_{27}(\varphi) \int_0^t \left( \int_{\Omega} |\nabla(\alpha(\theta)\theta_t)|^2 dp \right)^{\frac{1}{2}} \left( \int_{\Omega} |\varphi D\boldsymbol{\sigma}|^2 dp \right)^{\frac{1}{2}} ds \\
&\leq c_{27}(\varphi) \int_0^t \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{28}(\varphi).
\end{aligned}$$

Summing up, we deduce the result.  $\square$

*Proof of Theorem 3.4.1.* From inequality (3.18), since all terms on the left-hand side are positive we get

$$\|\varphi D\boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^3}}^2 \leq \frac{2c_{29}(\varphi)}{\kappa} + \frac{2c_{30}(\varphi)}{\kappa} \int_0^t \|\varphi D\boldsymbol{\sigma}(s)\|_{[L^2(\Omega)]^{n^3}}^2 ds.$$

and using Gronwall's lemma (see Lions [82], p. VI-8), it follows that

$$\|\varphi D\boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^3}}^2 \leq \frac{2c_{29}(\varphi)}{\kappa} e^{\frac{2c_{30}(\varphi)}{\kappa}t} \leq c_{31}(\varphi).$$

The function  $\varphi$  may now be chosen as an appropriate cut-off function to conclude that  $\boldsymbol{\sigma}(t) \in [H^1(\Omega')]^{n^2}$  for any subdomain  $\Omega' \subset\subset \Omega$  (see for instance Gilbarg and Trudinger [64], p. 155).  $\square$

In the following Lemma, we prove that if  $\boldsymbol{\sigma}^D$  is bounded, then  $\mathbf{u}_t(t)$  belongs to  $[H_{\text{Loc}}^1(\Omega)]^n$ . This result allows us to obtain the  $H_{\text{Loc}}^1$  regularity of stresses replacing hypothesis **(H12)** with the bound of  $\boldsymbol{\sigma}^D$  and relaxing assumption **(H10)** about  $\mathbf{b}$ .

**Lemma 3.4.4.** *Under assumptions **(H1)**-**(H11)** if  $\boldsymbol{\sigma}^D$  verifies*

$$\boldsymbol{\sigma}^D \in [L^\infty((0, t_f) \times \Omega)]^{n^2}, \quad (3.23)$$

then

$$\mathbf{u}_t \in L^2(0, t_f; [H_{\text{Loc}}^1(\Omega)]^n). \quad (3.24)$$

*Proof.* Since  $\boldsymbol{\sigma}^D$  is bounded due to hypothesis (3.23), from Theorem 3.3.2 and behaviour law (3.1d) we deduce that  $\boldsymbol{\varepsilon}(\mathbf{u}_t) \in L^2(0, t_f; [L^2(\Omega)]^{n^2})$ . Therefore, from Corollary 1 of Bolley & Camus [24] we obtain (3.24).  $\square$

**Theorem 3.4.5.** *Under hypotheses **(H1)**-**(H7)**, **(H8)**-**(H9)**, **(H11)**, assuming that*

$$\mathbf{b} \in W^{2,\infty}(0, t_f; [L^q(\Omega)]^n) \cap L^2(0, t_f; [H^1(\Omega)]^n), \quad \Delta \mathbf{b} \in L^2(0, t_f; [L^2(\Omega)]^n) \quad (3.25)$$

and  $\boldsymbol{\sigma}^D$  verifying property (3.23), then the stress solution of problem  $(P_T)$  satisfies

$$\boldsymbol{\sigma}(t) \in [H_{\text{Loc}}^1(\Omega)]^{n^2}, \quad \forall t \in [0, t_f].$$

*Proof.* The proof is analogous to Theorem 3.4.1, except for the bound of terms  $I_l$  with  $l = 1, \dots, 5$  of Lemma 3.4.3.

Applying Theorem 3.3.2 and Lemma 3.4.4 we obtain that  $\boldsymbol{\sigma} \in W^{1,2}(0, t_f; [L^2(\Omega)]^{n^2})$  and  $\mathbf{u}_t \in L^2(0, t_f; [H_{\text{Loc}}^1(\Omega)]^n)$ ; therefore hypothesis (3.25) about  $\mathbf{b}$  is sufficient in order that terms  $I_l$  with  $l = 1, \dots, 4$  are well defined and bounded. In effect, denoting  $c_l(\varphi)$ ,  $l > 0$  a positive constant depending at most on  $\varphi$ , we get:

- For  $I_1$ ,

$$\begin{aligned} |I_1| &\leq \int_0^t \int_{\Omega} |\mathbf{u}_t \cdot_1 (D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma})| \, dpds + \int_0^t \int_{\Omega} |\mathbf{u}_t \cdot_1 (D^2 \varphi^2 \cdot_1 \mathbf{b})| \, dpds \\ &\leq \|\mathbf{u}_t\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \left[ c_1(\varphi) \|\boldsymbol{\sigma}\|_{L^2(0, t_f; [L^2(\Omega)]^{n^2})} + c_2(\varphi) \|\mathbf{b}\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \right] \leq c_3(\varphi), \end{aligned}$$

where we have used  $\mathbf{u}_t \in L^2(0, t_f; [H_{\text{Loc}}^1(\Omega)]^n)$ ,  $\boldsymbol{\sigma} \in W^{1,2}(0, t_f; [L^2(\Omega)]^{n^2})$ , assumption (3.25) and Hölder's inequality on  $L^2(0, t_f; [L^2(\Omega)]^n) \times L^2(0, t_f; [L^2(\Omega)]^n)$ .

- Terms  $I_2$  and  $I_3$  are bounded considering again  $\mathbf{u}_t \in L^2(0, t_f; [H_{\text{Loc}}^1(\Omega)]^n)$ , assumption (3.25) and Hölder's inequality on  $L^2(0, t_f; [L^2(\Omega)]^n) \times L^2(0, t_f; [L^2(\Omega)]^n)$ :

$$\begin{aligned} |I_2| &\leq \int_0^t \int_{\Omega} |\mathbf{u}_t \cdot_1 (\nabla \varphi^2 \cdot_1 \nabla \mathbf{b})| \, dpds + \int_0^t \int_{\Omega} |\mathbf{u}_t \cdot_1 (\varphi^2 \Delta \mathbf{b})| \, dpds \\ &\leq \|\mathbf{u}_t\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \left[ c_4(\varphi) \|\nabla \mathbf{b}\|_{L^2(0, t_f; [L^2(\Omega)]^{n^2})} + c_5(\varphi) \|\Delta \mathbf{b}\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \right] \leq c_6(\varphi), \end{aligned}$$

and

$$\begin{aligned} |I_3| &\leq \int_0^t \int_{\Omega} |\text{Div } \mathbf{u}_t (D^2 \varphi^2 \cdot_2 \boldsymbol{\sigma})| \, dpds + \int_0^t \int_{\Omega} |\text{Div } \mathbf{u}_t (\nabla \varphi^2 \cdot_1 \mathbf{b})| \, dpds \\ &\leq \|\text{Div } \mathbf{u}_t\|_{L^2(0, t_f; L^2(\Omega))} \left[ \|D^2 \varphi^2 \cdot_2 \boldsymbol{\sigma}\|_{L^2(0, t_f; L^2(\Omega))} + \|\nabla \varphi^2 \cdot_1 \mathbf{b}\|_{L^2(0, t_f; L^2(\Omega))} \right] \\ &\leq \|\text{Div } \mathbf{u}_t\|_{L^2(0, t_f; L^2(\Omega))} \left[ c_7(\varphi) \|\boldsymbol{\sigma}\|_{L^2(0, t_f; [L^2(\Omega)]^{n^2})} + c_8(\varphi) \|\mathbf{b}\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \right] \leq c_9(\varphi). \end{aligned}$$

- For  $I_4$ , using hypotheses **(H7)**, **(H9)**, (3.25) and Hölder's inequality on  $L^2(0, t_f; [L^2(\Omega)]^n) \times L^2(0, t_f; [L^2(\Omega)]^n)$ , we obtain the estimate

$$\begin{aligned} |I_4| &\leq 2 \int_0^t \int_{\Omega} |\alpha(\theta) \theta_t \nabla \varphi^2 \cdot_1 \mathbf{b}| \, dpds \leq \|\alpha(\theta) \theta_t \nabla \varphi^2\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \|\mathbf{b}\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \\ &\leq c_{10}(\varphi) \|\alpha(\theta) \theta_t\|_{L^2(0, t_f; L^2(\Omega))} \|\mathbf{b}\|_{L^2(0, t_f; [L^2(\Omega)]^n)} \leq c_{11}(\varphi). \end{aligned}$$

Finally, term  $I_5$  is bounded taking into account that  $\boldsymbol{\sigma}^D \in [L^\infty((0, t_f) \times \Omega)]^{n^2}$ , hypotheses **(H8)**, **(H9)** and Hölder's and Young's inequalities:

$$\begin{aligned} |I_5| &\leq 4 \int_0^t \int_{\Omega} |\Theta(\theta) \nabla \varphi| |\boldsymbol{\sigma}^D|^{q-1} |\varphi D \boldsymbol{\sigma}| \, dpds \\ &\leq 4 \int_0^t \|\boldsymbol{\sigma}^D\|_{[L^\infty(\Omega)]^{n^2}}^{q-1} \left( \int_{\Omega} |\Theta(\theta) \nabla \varphi|^2 \, dp \right)^{\frac{1}{2}} \left( \int_{\Omega} |\varphi D \boldsymbol{\sigma}|^2 \, dp \right)^{\frac{1}{2}} \, ds \\ &\leq c_{12}(\varphi) \int_0^t \|\varphi D \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 \, ds + c_{13}(\varphi). \end{aligned}$$

□

In the following section, we extend the previous theorem to prove the  $H_{\text{Loc}}^2$  regularity of stresses.

### 3.5 $H_{\text{Loc}}^2$ Regularity

In this section we replace **(H6)**, **(H7)** and **(H8)**-**(H12)** with the following seven hypotheses:

**(H6)** The tensor of Hooke's law  $\Lambda$  is symmetric and the following hold true:

- $\Lambda \in [W^{3,\infty}(\mathbb{R})]^{n^4}$ .
- there exists  $\kappa > 0$  such that  $(\Lambda(s) \cdot_2 \boldsymbol{\tau}) \cdot_2 \boldsymbol{\tau} \geq \kappa |\boldsymbol{\tau}|^2$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .
- $(\Lambda'(s) \cdot_2 \boldsymbol{\tau}) \cdot_2 \boldsymbol{\tau} \geq 0$ ,  $\forall \boldsymbol{\tau} \in S_n$ ,  $\forall s \in \mathbb{R}$ .

**(H7)**  $\alpha \in W^{2,\infty}(\mathbb{R})$ .

**(H8)**  $\Theta \in W^{2,\infty}(\mathbb{R})$  and  $\Theta(s) \geq \Theta_{\min} > 0$ ,  $\forall s \in \mathbb{R}$ .

**(H9)**  $\theta \in W^{2,\infty}((0, t_f) \times \Omega)$ .

**(H10)**  $\mathbf{b} \in W^{2,\infty}(0, t_f; [W^{2,q}(\Omega)]^n)$  and  $\Delta \mathbf{b} \in L^\infty(0, t_f; [H^1(\Omega)]^n)$ .

**(H11)**  $\boldsymbol{\sigma}_0 \in \mathbf{X}^q \cap [H_{\text{Loc}}^2(\Omega)]^{n^2}$ .

**(H12)** The deviatoric tensor verifies  $\boldsymbol{\sigma}^D \in [L^\infty((0, t_f) \times \Omega)]^{n^2}$ .

**Theorem 3.5.1.** *Under assumptions **(H1)**-**(H5)** and **(H6)**-**(H12)** the stress solution of problem  $(P_T)$  satisfies  $\boldsymbol{\sigma}(t) \in [H_{\text{Loc}}^2(\Omega)]^{n^2}$ ,  $\forall t \in [0, t_f]$ .*

To prove this result, we need the following lemmas:

**Lemma 3.5.2.** *Under assumptions **(H1)**-**(H7)**, **(H8)**-**(H11)** and **(H12)** there exists  $\tilde{c} > 0$  such that*

$$\int_0^t \|\boldsymbol{\sigma}_t(s)\|_{[H_{\text{Loc}}^1(\Omega)]^{n^2}}^2 ds \leq \tilde{c}, \quad t \in [0, t_f]. \quad (3.26)$$

*Proof.* Let  $\varphi$  be a time independent smooth function with compact support in  $\Omega$ . Following the techniques used in the proof of Lemma 3.4.2, we multiply constitutive law (3.1d) by the test tensor function  $-\partial_k(\varphi \partial_k \boldsymbol{\sigma}_t)$ , we integrate over  $\Omega$  and sum over  $k$ ,  $k = 1, \dots, n$  to obtain:

$$\begin{aligned} - \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp &= - \int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp \\ &\quad - \int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp - \int_{\Omega} \alpha(\theta) \theta_t \mathbf{I} \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp. \end{aligned} \quad (3.27)$$

For the first member, we deduce, as in equation (3.12), that

$$\begin{aligned}
-\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp &= -2 \int_{\Omega} ([(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla \varphi) \cdot_3 D\boldsymbol{\sigma}_t dp \\
&+ 2 \int_{\Omega} \alpha(\theta) \theta_t \nabla \varphi \cdot_1 \mathbf{b}_t dp + \int_{\Omega} \operatorname{div} \mathbf{u}_t [D^2 \varphi \cdot_2 \boldsymbol{\sigma}_t - \nabla \varphi \cdot_1 \mathbf{b}_t] dp \\
&+ \int_{\Omega} \mathbf{u}_t \cdot_1 [D^3 \varphi \cdot_2 \boldsymbol{\sigma}_t - D^2 \varphi \cdot_1 \mathbf{b}_t] dp + \int_{\Omega} \varphi \nabla \mathbf{u}_t \cdot_2 \nabla \mathbf{b}_t dp, \quad (3.28)
\end{aligned}$$

where the last term is obtained without considering the change of expression (3.8).

For the first term on the right-hand side of equation (3.27), taking into account (3.22) and integrating by parts, we get

$$\begin{aligned}
-\int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp &= \int_{\Omega} \varphi (D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_3 D\boldsymbol{\sigma}_t dp + \int_{\Omega} \varphi (D\boldsymbol{\sigma}_t \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dp \\
&+ \int_{\Omega} \varphi (D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_3 D\boldsymbol{\sigma}_t dp \\
&+ \int_{\Omega} \varphi (D\boldsymbol{\sigma}_t \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma}_t dp. \quad (3.29)
\end{aligned}$$

For the next term on the right-hand side of equation (3.27), integrating by parts and taking into account expressions (3.2) and (3.15) we obtain

$$\begin{aligned}
-\int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp &= \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} (\nabla(\Theta(\theta)) \boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}_t dp \\
&+ \frac{(q-2)}{2} \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} (\nabla |\boldsymbol{\sigma}^D|^2 \boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}_t dp \\
&+ \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} D\boldsymbol{\sigma}^D \cdot_3 D\boldsymbol{\sigma}_t dp. \quad (3.30)
\end{aligned}$$

Finally, for the last term on the right-hand side of equation (3.27),

$$-\int_{\Omega} \alpha(\theta) \theta_t \mathbf{I} \cdot_2 \partial_k(\varphi \partial_k \boldsymbol{\sigma}_t) dp = \int_{\Omega} \varphi \nabla(\alpha(\theta) \theta_t) \cdot_1 \nabla(\operatorname{tr}(\boldsymbol{\sigma}_t)) dp. \quad (3.31)$$

Therefore, from equalities (3.27)-(3.31), we obtain

$$\begin{aligned}
& -2 \int_{\Omega} [(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla \varphi \cdot_3 D\boldsymbol{\sigma}_t dp + 2 \int_{\Omega} \alpha(\theta) \theta_t \nabla \varphi \cdot_1 \mathbf{b}_t dp \\
& + \int_{\Omega} \operatorname{div} \mathbf{u}_t [D^2 \varphi \cdot_2 \boldsymbol{\sigma}_t - \nabla \varphi \cdot_1 \mathbf{b}_t] dp + \int_{\Omega} \mathbf{u}_t \cdot_1 [D^3 \varphi \cdot_2 \boldsymbol{\sigma}_t - D^2 \varphi \cdot_1 \mathbf{b}_t] dp \\
& + \int_{\Omega} \varphi \nabla \mathbf{u}_t \cdot_2 \nabla \mathbf{b}_t dp = \int_{\Omega} \varphi (D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_3 D\boldsymbol{\sigma}_t dp + \int_{\Omega} \varphi (D\boldsymbol{\sigma}_t \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dp \\
& + \int_{\Omega} \varphi (D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_3 D\boldsymbol{\sigma}_t dp + \int_{\Omega} \varphi (D\boldsymbol{\sigma}_t \cdot_2 \Lambda(\theta)) \cdot_3 D\boldsymbol{\sigma}_t dp \\
& + \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} (\nabla(\Theta(\theta)) \boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}_t dp \\
& + \frac{(q-2)}{2} \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} (\nabla |\boldsymbol{\sigma}^D|^2 \boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}_t dp \\
& + \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} D\boldsymbol{\sigma}^D \cdot_3 D\boldsymbol{\sigma}_t dp + \int_{\Omega} \varphi \nabla(\alpha(\theta) \theta_t) \cdot_1 \nabla(\operatorname{tr}(\boldsymbol{\sigma}_t)) dp,
\end{aligned}$$

rearranging the terms, integrating over  $(0, t)$ , replacing  $\varphi$  with  $\varphi^2$  and using hypothesis **(H6)** and **(H9)**, we have

$$\begin{aligned}
& \kappa \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \leq \left| \int_0^t \int_{\Omega} \mathbf{u}_t \cdot_1 [D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma}_t - D^2 \varphi^2 \cdot_1 \mathbf{b}_t] dp ds \right| \\
& + \left| \int_0^t \int_{\Omega} \varphi^2 \nabla \mathbf{u}_t \cdot_2 \nabla \mathbf{b}_t dp ds \right| + \left| \int_0^t \int_{\Omega} \operatorname{div} \mathbf{u}_t [D^2 \varphi^2 \cdot_2 \boldsymbol{\sigma}_t - \nabla \varphi^2 \cdot_1 \mathbf{b}_t] dp ds \right| \\
& + \left| 2 \int_0^t \int_{\Omega} \alpha(\theta) \theta_t \nabla \varphi^2 \cdot_1 \mathbf{b}_t dp ds \right| + \left| 2 \int_0^t \int_{\Omega} (D\phi_q(\boldsymbol{\sigma}) \nabla \varphi^2) \cdot_3 D\boldsymbol{\sigma}_t dp ds \right| \\
& + \left| 2 \int_0^t \int_{\Omega} ((\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \nabla \varphi^2) \cdot_3 D\boldsymbol{\sigma}_t dp ds \right| + \left| \int_0^t \int_{\Omega} \varphi^2 (D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_3 D\boldsymbol{\sigma}_t dp ds \right| \\
& + \left| \int_0^t \int_{\Omega} \varphi^2 (D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_3 D\boldsymbol{\sigma}_t dp ds \right| \\
& + \left| \int_0^t \int_{\Omega} \varphi^2 |\boldsymbol{\sigma}^D|^{q-2} (\nabla(\Theta(\theta)) \boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}_t dp ds \right| \\
& + \left| \int_0^t \int_{\Omega} \varphi^2 \nabla(\alpha(\theta) \theta_t) \cdot_1 \nabla(\operatorname{tr}(\boldsymbol{\sigma}_t)) dp ds \right| + \left| \int_0^t \int_{\Omega} \varphi^2 (D\boldsymbol{\sigma}_t \cdot_2 (\Lambda(\theta))_t) \cdot_3 D\boldsymbol{\sigma} dp ds \right| \\
& + \left| \frac{(q-2)}{2} \int_0^t \int_{\Omega} \varphi^2 \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} (\nabla |\boldsymbol{\sigma}^D|^2 \boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}_t dp ds \right| \\
& + \left| \int_0^t \int_{\Omega} \varphi^2 \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} D\boldsymbol{\sigma}^D \cdot_3 D\boldsymbol{\sigma}_t dp ds \right|. \tag{3.32}
\end{aligned}$$

We denote by  $I_l$  the  $l$ -th term of the second member of this inequality.

Following the methodology used in Lemma 3.4.3, we can bound terms  $I_l$ ,  $l = 1, 2, 3, 4$  thanks to Theorem 3.3.2, Theorem 3.4.5 and hypotheses **(H7)**, **(H9)**, **(H10)**, obtaining:

- For  $I_1$ ,

$$|I_1| = \left( \int_0^t \|\mathbf{u}_t\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|D^3 \varphi^2 \cdot_2 \boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^n} ds \right)^{\frac{1}{2}} \\ + \left( \int_0^t \|\mathbf{u}_t\|_{[L^p(\Omega)]^n}^p ds \right)^{\frac{1}{p}} \left( \int_0^t \|D^2 \varphi^2 \cdot_1 \mathbf{b}_t\|_{[L^q(\Omega)]^n}^q ds \right)^{\frac{1}{q}} \leq c_1(\varphi), \quad (3.33)$$

thanks to Theorem 3.3.2, hypothesis  $(\overline{\mathbf{H10}})$  and Hölder's inequality.

- $I_2$  is bounded as follows,

$$|I_2| \leq \left( \int_0^t \|\varphi D \mathbf{u}_t\|_{[L^2(\Omega)]^{n^2}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D \mathbf{b}_t\|_{[L^2(\Omega)]^{n^2}}^2 ds \right)^{\frac{1}{2}} \leq c_2(\varphi), \quad (3.34)$$

considering Theorem 3.4.5, hypothesis  $(\overline{\mathbf{H10}})$  and Hölder's inequality.

- For  $I_3$ ,

$$|I_3| \leq \left( \int_0^t \|\operatorname{div} \mathbf{u}_t\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|D^2 \varphi^2 \cdot_2 \boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^n} ds \right)^{\frac{1}{2}} \\ + \left( \int_0^t \|\operatorname{div} \mathbf{u}_t\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \varphi^2 \cdot_1 \mathbf{b}_t\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \leq c_3(\varphi), \quad (3.35)$$

taking into account Theorem 3.3.2, hypothesis  $(\overline{\mathbf{H10}})$  and Hölder's inequality.

- Finally,  $I_4$  is bounded thanks to hypotheses  $(\mathbf{H7})$ ,  $(\overline{\mathbf{H9}})$  and Hölder's inequality:

$$|I_4| \leq 2 \left( \int_0^t \|\alpha(\theta) \theta_t \nabla \varphi^2\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\mathbf{b}_t\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \leq c_4(\varphi). \quad (3.36)$$

Terms  $I_l$ ,  $l = 5, \dots, 13$  can be bounded taking into account hypotheses  $(\mathbf{H6})$ ,  $(\mathbf{H7})$ ,  $(\overline{\mathbf{H8}})$ ,  $(\overline{\mathbf{H9}})$ ,  $(\overline{\mathbf{H12}})$ , applying Hölder's and Young's inequalities consecutively for an appropriate  $\varepsilon$  and using  $H_{\text{Loc}}^1$  regularity of  $\boldsymbol{\sigma}(t)$  proved in the previous section, in the form:

$$|I_l| \leq \frac{\kappa}{20} \int_0^t \|\varphi D \boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_l(\varphi), \quad l = 5, \dots, 13. \quad (3.37)$$

Indeed:

- $I_5$  is bounded considering hypotheses  $(\overline{\mathbf{H8}})$ ,  $(\overline{\mathbf{H9}})$ ,  $(\overline{\mathbf{H12}})$  and applying Hölder's and Young's inequalities consecutively for  $\varepsilon = \frac{80 \|\Theta(\theta)\|_{L^\infty(\Omega)}}{\kappa}$

$$|I_5| \leq 4 \|\Theta(\theta)\|_{L^\infty(\Omega)} \int_0^t \left( \int_\Omega |\nabla \varphi|^2 |\boldsymbol{\sigma}^D|^{2(q-2)} |\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \int_\Omega |\varphi D \boldsymbol{\sigma}_t|^2 dp \right)^{\frac{1}{2}} ds \\ \leq \frac{\kappa}{20} \int_0^t \|\varphi D \boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_5(\varphi).$$

- For  $I_6$  we have,

$$\begin{aligned} |I_6| &\leq 4 \left( \int_0^t \|\nabla\varphi(\Lambda(\theta)) \cdot 2\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_6(\varphi), \end{aligned}$$

thanks to Theorem 3.3.2, ( $H_{\text{Loc}}^1$  regularity of  $\boldsymbol{\sigma}(t)$ ), hypotheses **(H3)**, **(H9)**, and Hölder's and Young's inequalities for  $\varepsilon = \frac{80}{\kappa}$ .

- $I_7$  and  $I_8$  can be bounded taking into account hypotheses **(H3)**, **(H9)**, Hölder's and Young's inequalities for  $\varepsilon = \frac{20}{\kappa}$ :

$$\begin{aligned} |I_7| &\leq \left( \int_0^t \|\varphi D(\Lambda(\theta))_t \cdot 2\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_7(\varphi), \end{aligned}$$

$$\begin{aligned} |I_8| &\leq \left( \int_0^t \|\varphi D(\Lambda(\theta)) \cdot 2\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_8(\varphi). \end{aligned}$$

- For  $I_9$  we obtain

$$\begin{aligned} |I_9| &\leq \|\nabla\Theta(\theta)\|_{L^\infty(\Omega)} \int_0^t \left( \int_\Omega \varphi |\boldsymbol{\sigma}^D|^{2(q-2)} |\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 \right)^{\frac{1}{2}} ds \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_9(\varphi), \end{aligned}$$

considering hypotheses **(H8)**, **(H9)**, **(H12)** and Hölder's and Young's inequalities for  $\varepsilon = \frac{20\|\nabla\Theta(\theta)\|_{L^\infty(\Omega)}}{\kappa}$ .

- Applying hypotheses **(H7)**, **(H9)** and Hölder's and Young's inequalities for  $\varepsilon = \frac{20}{\kappa}$ , we get the following bound for  $I_{10}$

$$\begin{aligned} |I_{10}| &\leq \int_0^t \int_\Omega |\varphi \nabla(\alpha(\theta)\theta_t)| |\varphi \nabla(\text{tr}(\boldsymbol{\sigma}_t))| dp ds \leq \int_0^t \int_\Omega |\varphi \nabla(\alpha(\theta)\theta_t)| |\varphi(\text{tr}(D\boldsymbol{\sigma}_t))| dp ds \\ &\leq \int_0^t \int_\Omega |\varphi \nabla(\alpha(\theta)\theta_t)| |\varphi D\boldsymbol{\sigma}_t| dp ds \\ &\leq \left( \int_0^t \|\varphi \nabla(\alpha(\theta)\theta_t)\|_{[L^2(\Omega)]^n}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{10}(\varphi). \end{aligned}$$

- Term  $I_{11}$  is bounded thanks to hypotheses **(H3)**, **(H9)**, Theorem 3.4.1 and Hölder's and Young's inequalities for  $\varepsilon = \frac{\kappa}{20\|(\Lambda(\theta))_t\|_{L^\infty(\Omega)}}$ ,

$$\begin{aligned} |I_{11}| &\leq \left( \int_0^t \|\varphi D\sigma_t \cdot_2 (\Lambda(\theta))_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\sigma\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \|(\Lambda(\theta))_t\|_{L^\infty(\Omega)} \left( \int_0^t \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D\sigma\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{11}(\varphi). \end{aligned}$$

- For  $I_{12}$  we deduce the estimation

$$\begin{aligned} |I_{12}| &\leq \frac{(q-2)}{2} \|\Theta(\theta)\|_{L^\infty(\Omega)} \int_0^t \left( \int_\Omega |\sigma^D|^{2(q-2)} |\varphi D\sigma^D|^2 dp \right)^{\frac{1}{2}} \left( \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 \right)^{\frac{1}{2}} ds \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{12}(\varphi), \end{aligned}$$

using hypotheses **(H8)**, **(H9)**, **(H12)**, Theorem 3.4.1 and Hölder's and Young's inequalities for  $\varepsilon = \frac{20\frac{(q-2)}{2}\|\Theta(\theta)\|_{L^\infty(\Omega)}}{\kappa}$ .

- If  $\|\sigma^D\|_{[L^\infty(\Omega)]^{n^2}}$  is no null -another case is trivial- and  $\varepsilon = \frac{\kappa}{20\Theta_{\min}\|\sigma^D\|_{[L^\infty(\Omega)]^{n^2}}^{q-2}}$  we have the following bounded for  $I_{13}$ :

$$\begin{aligned} |I_{13}| &= \left| \int_0^t \int_\Omega \varphi^2 \Theta(\theta) |\sigma^D|^{q-2} D\sigma^D \cdot_3 D\sigma_t dp ds \right| \\ &\leq \Theta_{\min} \|\sigma^D\|_{[L^\infty(\Omega)]^{n^2}}^{q-2} \int_0^t \left( \|\varphi D\sigma^D\|_{[L^2(\Omega)]^{n^3}}^2 \right)^{\frac{1}{2}} \left( \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 \right)^{\frac{1}{2}} ds \\ &\leq \frac{\kappa}{20} \int_0^t \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds + c_{13}(\varphi). \end{aligned}$$

Summing up, from (3.32)-(3.37) we get

$$\kappa \int_0^t \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \leq \sum_{l=1}^{13} c_l(\varphi) + \frac{\kappa}{2} \int_0^t \|\varphi D\sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds,$$

hence, we conclude the result.  $\square$

**Lemma 3.5.3.** *Let  $\varphi$  be a time independent smooth function with compact support in  $\Omega$ . Then,*

under assumptions **(H1)**-**(H5)** and **(H6)**-**(H12)** the following equality holds:

$$\begin{aligned}
& - \int_{\Omega} 2(D[(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla \varphi) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} 2\nabla(\alpha(\theta)\theta_t) \cdot_1 (\nabla \mathbf{b} \cdot_1 \nabla \varphi) dp \\
& - \int_{\Omega} \nabla \mathbf{u}_t \cdot_2 (D^2 \boldsymbol{\sigma} \cdot_2 D^2 \varphi - D^2 \mathbf{b} \cdot_1 \nabla \varphi) dp - \int_{\Omega} \nabla \mathbf{u}_t \cdot_2 (\nabla \varphi \cdot_1 D^2 \mathbf{b} + \nabla(\Delta \mathbf{b})\varphi) dp \\
& = \int_{\Omega} \varphi(D^2(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_4 D^2 \boldsymbol{\sigma} dp + 2 \int_{\Omega} \varphi(D^2 \boldsymbol{\sigma} \cdot_3 D(\Lambda(\theta))_t) \cdot_3 D \boldsymbol{\sigma} dp \\
& + \frac{1}{2} \int_{\Omega} \varphi(D^2 \boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} \varphi(D^2(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
& + 2 \int_{\Omega} \varphi(D^2 \boldsymbol{\sigma} \cdot_3 D(\Lambda(\theta))) \cdot_3 D \boldsymbol{\sigma}_t dp + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varphi(D^2 \boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
& + \int_{\Omega} \varphi|\boldsymbol{\sigma}^D|^{q-2}(D^2(\Theta(\theta))\boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + 2 \int_{\Omega} \varphi(\nabla|\boldsymbol{\sigma}^D|^{q-2}\nabla(\Theta(\theta))\boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
& + 2 \int_{\Omega} \varphi|\boldsymbol{\sigma}^D|^{q-2}(\nabla(\Theta(\theta))D\boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} \varphi\Theta(\theta)(D^2|\boldsymbol{\sigma}^D|^{q-2}\boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
& + 2 \int_{\Omega} \varphi\Theta(\theta)(\nabla|\boldsymbol{\sigma}^D|^{q-2}D\boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} \varphi\Theta(\theta)|\boldsymbol{\sigma}^D|^{q-2}D^2\boldsymbol{\sigma}^D \cdot_4 D^2 \boldsymbol{\sigma} dp \\
& + \int_{\Omega} \varphi D^2(\alpha(\theta)\theta_t) \cdot_2 D^2(\text{tr}(\boldsymbol{\sigma})) dp. \tag{3.38}
\end{aligned}$$

*Proof.* Following the techniques used in Lemma 3.4.2, the proof consists of multiplying constitutive law (3.1d) by the test tensor function  $\partial_{kl}(\varphi\partial_{kl}\boldsymbol{\sigma})$ , integrating over  $\Omega$  and summing over  $k, l$ , with  $k, l = 1, \dots, n$ :

$$\begin{aligned}
\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_{kl}(\varphi\partial_{kl}\boldsymbol{\sigma}) dp &= \int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_{kl}(\varphi\partial_{kl}\boldsymbol{\sigma}) dp \\
&+ \int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_{kl}(\varphi\partial_{kl}\boldsymbol{\sigma}) dp + \int_{\Omega} \alpha(\theta)\theta_t \mathbf{I} \cdot_2 \partial_{kl}(\varphi\partial_{kl}\boldsymbol{\sigma}) dp. \tag{3.39}
\end{aligned}$$

Applying integration by parts twice, taking into account equilibrium equation (3.1a), the symmetry

of the tensor  $\boldsymbol{\sigma}$  and behaviour law (3.1d), the first member of equation (3.39) can be written as

$$\begin{aligned}
\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot_2 \partial_{kl}(\varphi \partial_{kl} \boldsymbol{\sigma}) dp &= - \int_{\Omega} 2 \partial_k [(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})]_{lj} \partial_i \varphi \partial_{kl} \sigma_{ij} dp \\
&\quad - \int_{\Omega} 2 \partial_k (\alpha(\theta) \theta_t) \delta_{lj} \partial_i \varphi \partial_{kl} \sigma_{ij} dp - \int_{\Omega} \partial_k (u_t)_l [\partial_{ji} \varphi \partial_{kl} \sigma_{ij} - \partial_i \varphi \partial_{kl} b_i] dp \\
&\quad - \int_{\Omega} \partial_k (u_t)_j [\partial_l \varphi \partial_{kl} f_j + \varphi \partial_k \Delta f_j] dp \\
&= - \int_{\Omega} 2 (D [(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla \varphi) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
&\quad + \int_{\Omega} 2 \nabla (\alpha(\theta) \theta_t) \cdot_1 (\nabla \mathbf{b} \cdot_1 \nabla \varphi) dp \\
&\quad - \int_{\Omega} \nabla \mathbf{u}_t \cdot_2 (D^2 \boldsymbol{\sigma} \cdot_2 D^2 \varphi - D^2 \mathbf{b} \cdot_1 \nabla \varphi) dp \\
&\quad - \int_{\Omega} \nabla \mathbf{u}_t \cdot_2 (\nabla \varphi \cdot_1 D^2 \mathbf{b} + \nabla (\Delta \mathbf{b}) \varphi) dp. \tag{3.40}
\end{aligned}$$

Due to the equality

$$(D^2 \boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_4 D^2 \boldsymbol{\sigma}_t = \frac{1}{2} ((D^2 \boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_4 D^2 \boldsymbol{\sigma})_t - \frac{1}{2} (D^2 \boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_4 D^2 \boldsymbol{\sigma},$$

and using integration by parts, we deduce for the first term on the right-hand side of equation (3.39) the expression

$$\begin{aligned}
&\int_{\Omega} (\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t \cdot_2 \partial_{kl}(\varphi \partial_{kl} \boldsymbol{\sigma}) dp \\
&= \int_{\Omega} \varphi \partial_{kl} (\Lambda(\theta)_{ijmn})_t \sigma_{mn} \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \varphi \partial_k (\Lambda(\theta)_{ijmn})_t \partial_l \sigma_{mn} \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \varphi \partial_l (\Lambda(\theta)_{ijmn})_t \partial_k \sigma_{mn} \partial_{kl} \sigma_{ij} dp + \frac{1}{2} \int_{\Omega} \varphi (\Lambda(\theta)_{ijmn})_t \partial_{kl} \sigma_{mn} \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \varphi \partial_{kl} (\Lambda(\theta)_{ijmn}) (\sigma_{mn})_t \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \varphi \partial_k \Lambda(\theta)_{ijmn} \partial_l (\sigma_{mn})_t \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \varphi \partial_l \Lambda(\theta)_{ijmn} \partial_k (\sigma_{mn})_t \partial_{kl} \sigma_{ij} dp + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varphi \Lambda(\theta)_{ijmn} \partial_{kl} \sigma_{mn} \partial_{kl} \sigma_{ij} dp \\
&= \int_{\Omega} \varphi (D^2 (\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_4 D^2 \boldsymbol{\sigma} dp + 2 \int_{\Omega} \varphi (D^2 \boldsymbol{\sigma} \cdot_3 D (\Lambda(\theta))_t) \cdot_3 D \boldsymbol{\sigma} dp \\
&\quad + \frac{1}{2} \int_{\Omega} \varphi (D^2 \boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} \varphi (D^2 (\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
&\quad + 2 \int_{\Omega} \varphi (D^2 \boldsymbol{\sigma} \cdot_3 D (\Lambda(\theta))) \cdot_3 D \boldsymbol{\sigma}_t dp + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varphi (D^2 \boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_4 D^2 \boldsymbol{\sigma} dp. \tag{3.41}
\end{aligned}$$

For the second term on the right-hand side of equation (3.39), taking into account expression (3.2)

we obtain:

$$\begin{aligned}
& \int_{\Omega} D\phi_q(\boldsymbol{\sigma}) \cdot_2 \partial_{kl}(\varphi \partial_{kl} \boldsymbol{\sigma}) dp \\
&= \int_{\Omega} \partial_{kl}(\Theta(\theta)) |\boldsymbol{\sigma}^D|^{q-2} \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \partial_l(\Theta(\theta)) \partial_k(|\boldsymbol{\sigma}^D|^{q-2}) \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \partial_l(\Theta(\theta)) |\boldsymbol{\sigma}^D|^{q-2} \partial_k \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \partial_k(\Theta(\theta)) \partial_l(|\boldsymbol{\sigma}^D|^{q-2}) \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \Theta(\theta) \partial_{kl}(|\boldsymbol{\sigma}^D|^{q-2}) \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \Theta(\theta) \partial_l(|\boldsymbol{\sigma}^D|^{q-2}) \partial_k \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \partial_k(\Theta(\theta)) |\boldsymbol{\sigma}^D|^{q-2} \partial_l \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \Theta(\theta) \partial_k(|\boldsymbol{\sigma}^D|^{q-2}) \partial_l \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} \partial_{kl} \sigma_{ij}^D \varphi \partial_{kl} \sigma_{ij} dp \\
&= \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} (D^2(\Theta(\theta)) \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + 2 \int_{\Omega} \varphi (\nabla |\boldsymbol{\sigma}^D|^{q-2} \nabla(\Theta(\theta)) \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
&\quad + 2 \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} (\nabla(\Theta(\theta)) D \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} \varphi \Theta(\theta) (D^2 |\boldsymbol{\sigma}^D|^{q-2} \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
&\quad + 2 \int_{\Omega} \varphi \Theta(\theta) (\nabla |\boldsymbol{\sigma}^D|^{q-2} D \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + \int_{\Omega} \varphi \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-2} D^2 \boldsymbol{\sigma}^D \cdot_4 D^2 \boldsymbol{\sigma} dp \\
&= \int_{\Omega} \varphi |\boldsymbol{\sigma}^D|^{q-2} D^2(\Theta(\theta) \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + 2 \int_{\Omega} \varphi (\nabla |\boldsymbol{\sigma}^D|^{q-2} \nabla(\Theta(\theta)) \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp \\
&\quad + \int_{\Omega} \varphi \Theta(\theta) (D^2 |\boldsymbol{\sigma}^D|^{q-2} \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp + 2 \int_{\Omega} \varphi \Theta(\theta) (\nabla |\boldsymbol{\sigma}^D|^{q-2} D \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp. \tag{3.42}
\end{aligned}$$

Finally, for the last term of the second member of equation (3.39), we have

$$\begin{aligned}
\int_{\Omega} \alpha(\theta) \theta_t \mathbf{I} \cdot_2 \partial_{kl}(\varphi \partial_{kl} \boldsymbol{\sigma}) dp &= \int_{\Omega} \partial_{kl}(\alpha(\theta)) \theta_t \delta_{ij} \varphi \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \partial_k(\alpha(\theta)) \partial_l \theta_t \delta_{ij} \varphi \partial_{kl} \sigma_{ij} dp \\
&\quad + \int_{\Omega} \partial_l(\alpha(\theta)) \partial_k \theta_t \delta_{ij} \varphi \partial_{kl} \sigma_{ij} dp + \int_{\Omega} \alpha(\theta) \partial_{kl} \theta_t \delta_{ij} \varphi \partial_{kl} \sigma_{ij} dp \\
&= \int_{\Omega} \varphi \theta_t D^2(\alpha(\theta)) \cdot_2 D^2(\text{tr}(\boldsymbol{\sigma})) dp + 2 \int_{\Omega} \varphi (\nabla(\alpha(\theta)) \nabla \theta_t) \cdot_2 D^2(\text{tr}(\boldsymbol{\sigma})) dp \\
&\quad + \int_{\Omega} \varphi \alpha(\theta) D^2 \theta_t \cdot_2 D^2(\text{tr}(\boldsymbol{\sigma})) dp \\
&= \int_{\Omega} \varphi D^2(\alpha(\theta) \theta_t) \cdot_2 D^2(\text{tr}(\boldsymbol{\sigma})) dp. \tag{3.43}
\end{aligned}$$

Therefore, from (3.39)-(3.43) we conclude (3.38).  $\square$

The main inequality of this section will be proved in Lemma 3.5.5. For that purpose we need a Gagliardo-Nirenberg inequality involving difference quotients, which is given in the following Lemma.

**Lemma 3.5.4.** *Let  $\boldsymbol{\tau}$  be any tensor valued function of second order in  $[H^1(\Omega)]^{n^2} \cap [L^\infty(\Omega)]^{n^2}$  and  $\varphi$  a smooth function with compact support in  $\Omega$ . Then,*

$$\|D(\varphi\boldsymbol{\tau})\|_{[L^4(\Omega)]^{n^3}}^4 \leq c(\varphi, \|\boldsymbol{\tau}\|_{[L^\infty(\Omega)]^{n^2}}) \left\{ \|D(\varphi\boldsymbol{\tau})\|_{[L^2(\Omega)]^{n^3}}^2 + \|D^2(\varphi\boldsymbol{\tau})\|_{[L^2(\Omega)]^{n^4}}^2 \right\}, \quad (3.44)$$

where  $c(\varphi, \|\boldsymbol{\tau}\|_{[L^\infty(\Omega)]^{n^2}})$ , is a positive constant depending on  $\varphi$  and  $\|\boldsymbol{\tau}\|_{[L^\infty(\Omega)]^{n^2}}$ .

*Proof.* Since  $\boldsymbol{\tau} \in [H^1(\Omega)]^{n^2} \cap [L^\infty(\Omega)]^{n^2}$ , we obtain the result applying Theorem 10.1 of Friedman [53] (see p. 27) to difference quotient  $\partial_k(\varphi\boldsymbol{\tau})$  for the choice

- $a = 0, p = q = 4, j = 0, m = 1$  and  $r = 2$  if  $n = 2$ ,
- $a = \frac{1}{2}, p = 4, q = 3, j = 0, m = 1$  and  $r = 2$  if  $n = 3$

and taking into account Lemma 15.1 of Friedman [53] (see p. 46). □

**Lemma 3.5.5.** *Let  $\varphi$  be a time-independent smooth function with compact support in  $\Omega$ . Under assumptions **(H1)**-**(H5)** and **(H6)**-**(H12)** the following inequality holds:*

$$\frac{\kappa}{2} c_1(\varphi) \|\varphi D^2 \boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^4}}^2 \leq c_{19}(\varphi) + c_{20}(\varphi) \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds, \quad (3.45)$$

where  $c_l(\varphi)$ ,  $l > 0$ , denotes a positive constant depending at most on  $\varphi$ .

*Proof.* Integrating in time over  $(0, t)$  and replacing  $\varphi$  by  $\varphi^4$  in (3.38), we deduce that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varphi^4 (D^2 \boldsymbol{\sigma} \cdot_2 \Lambda(\theta)) \cdot_4 D^2 \boldsymbol{\sigma} dp + \frac{1}{2} \int_0^t \int_{\Omega} \varphi^4 (D^2 \boldsymbol{\sigma} \cdot_2 (\Lambda(\theta))_t) \cdot_4 D^2 \boldsymbol{\sigma} dp ds \\ &= \frac{1}{2} \int_{\Omega} \varphi^4 (D^2 \boldsymbol{\sigma}_0 \cdot_2 \Lambda(\theta_0)) \cdot_4 D^2 \boldsymbol{\sigma}_0 dp + \int_0^t \int_{\Omega} 2 \nabla(\alpha(\theta)\theta_t) \cdot_1 (\nabla \mathbf{b} \cdot_1 \nabla \varphi^4) dp ds \\ & \quad - 2 \int_0^t \int_{\Omega} (D[(\Lambda(\theta) \cdot_2 \boldsymbol{\sigma})_t + D\phi_q(\boldsymbol{\sigma})] \nabla \varphi^4) \cdot_4 D^2 \boldsymbol{\sigma} dp ds \\ & \quad - \int_0^t \int_{\Omega} \nabla \mathbf{u}_t \cdot_2 (D^2 \boldsymbol{\sigma} \cdot_2 D^2 \varphi^4 - D^2 \mathbf{b} \cdot_1 \nabla \varphi^4 + \nabla \varphi^4 \cdot_1 D^2 \mathbf{b} + \nabla(\Delta \mathbf{b}) \varphi^4) dp ds \\ & \quad - \int_0^t \int_{\Omega} \varphi^4 (D^2(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}) \cdot_4 D^2 \boldsymbol{\sigma} dp ds - \int_0^t \int_{\Omega} \varphi^4 (D^2(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t) \cdot_4 D^2 \boldsymbol{\sigma} dp ds \\ & \quad - 2 \int_0^t \int_{\Omega} \varphi^4 (D^2 \boldsymbol{\sigma} \cdot_3 D(\Lambda(\theta))_t) \cdot_3 D \boldsymbol{\sigma} dp ds - 2 \int_0^t \int_{\Omega} \varphi^4 (D^2 \boldsymbol{\sigma} \cdot_3 D(\Lambda(\theta))) \cdot_3 D \boldsymbol{\sigma}_t dp ds \\ & \quad - \int_0^t \int_{\Omega} \varphi^4 [|\boldsymbol{\sigma}^D|^{q-2} D^2(\Theta(\theta)\boldsymbol{\sigma}^D) + 2 \nabla |\boldsymbol{\sigma}^D|^{q-2} \nabla(\Theta(\theta)\boldsymbol{\sigma}^D)] \cdot_4 D^2 \boldsymbol{\sigma} dp ds \\ & \quad - \int_0^t \int_{\Omega} \varphi^4 D^2(\alpha(\theta)\theta_t) \cdot_2 D^2(\text{tr}(\boldsymbol{\sigma})) dp ds - \int_0^t \int_{\Omega} \varphi^4 \Theta(\theta) (D^2 |\boldsymbol{\sigma}^D|^{q-2} \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp ds \\ & \quad - 2 \int_0^t \int_{\Omega} \varphi^4 \Theta(\theta) (\nabla |\boldsymbol{\sigma}^D|^{q-2} D \boldsymbol{\sigma}^D) \cdot_4 D^2 \boldsymbol{\sigma} dp ds. \end{aligned}$$

From **(H6)**, **(H9)** and **(H11)**, we get

$$\frac{\kappa}{2}c_1(\varphi)\|\varphi D^2\boldsymbol{\sigma}(t)\|_{[L^2(\Omega)]^{n^4}}^2 \leq c_2(\varphi) + \sum_{l=1}^{11}|I_l|. \quad (3.46)$$

Taking into account the regularity given in Theorem 3.4.5 and Lemma 3.5.2, terms  $I_l$ ,  $l = 1, \dots, 9$  can be bounded using the same techniques as in the previous lemmas. In effect,

- We bound term  $I_1$  considering hypotheses **(H7)**, **(H9)**, **(H10)** and using Hölder's inequality:

$$|I_1| \leq 2\|\nabla(\alpha(\theta)\theta_t)\|_{L^2(0,t_f;[L^2(\Omega)]^n)}\|\nabla\mathbf{b} \cdot \nabla\varphi^4\|_{L^2(0,t_f;[L^2(\Omega)]^n)} \leq c_3(\varphi).$$

- For  $I_2$ , taking into account hypotheses **(H6)**, **(H8)**, **(H9)**, **(H12)**, Theorem 3.4.5, Lemma 3.5.2 and applying Hölder's and Young's inequalities we obtain the following estimation:

$$\begin{aligned} |I_2| &\leq 8 \left( \int_0^t \|\varphi^2 \nabla \varphi D(\Lambda(\theta))_t \cdot_2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &+ 8 \left( \int_0^t \|\varphi^2 \nabla \varphi (\Lambda(\theta))_t \cdot_2 D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &+ 8 \left( \int_0^t \|\varphi^2 \nabla \varphi D(\Lambda(\theta)) \cdot_2 \boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &+ 8 \left( \int_0^t \|\varphi^2 \nabla \varphi \Lambda(\theta) \cdot_2 D\boldsymbol{\sigma}_t\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &+ 8\|\nabla\Theta(\theta)\|_{L^\infty(\Omega)} \int_0^t \left( \int_\Omega \varphi^2 |\nabla\varphi| |\boldsymbol{\sigma}^D|^{2(q-1)} dp \right)^{\frac{1}{2}} \left( \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 \right)^{\frac{1}{2}} ds \\ &+ 8\|\Theta(\theta)\|_{L^\infty(\Omega)} \int_0^t \left( \int_\Omega \varphi^2 |\nabla\varphi| |\boldsymbol{\sigma}^D|^{2(q-4)} |\boldsymbol{\sigma}^D|^4 |\varphi D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 \right)^{\frac{1}{2}} ds \\ &+ 8\|\Theta(\theta)\|_{L^\infty(\Omega)} \int_0^t \left( \int_\Omega \varphi^2 |\nabla\varphi| |\boldsymbol{\sigma}^D|^{2(q-2)} |\varphi D\boldsymbol{\sigma}^D|^2 dp \right)^{\frac{1}{2}} \left( \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 \right)^{\frac{1}{2}} ds \\ &\leq c_4(\varphi) + c_5(\varphi) \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds. \end{aligned}$$

- Under assumptions **(H1)**-**(H5)** and **(H6)**-**(H12)** we can also obtain the bound for  $\mathbf{u}_t$  from Lemma 3.4.4, hence term  $I_3$  is bounded using Hölder's and Young's inequalities and hypothesis **(H10)**:

$$\begin{aligned} |I_3| &\leq 4 \left( \int_0^t |\varphi^2 D^2 \varphi|^2 |\nabla \mathbf{u}_t|^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &+ \|\nabla \mathbf{u}_t\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} \|D^2 \mathbf{b} \cdot \nabla \varphi^4\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} \\ &+ \|\nabla \mathbf{u}_t\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} \|\nabla \varphi^4 \cdot \nabla D^2 \mathbf{b}\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} \\ &+ \|\nabla \mathbf{u}_t\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} \|\nabla(\Delta \mathbf{b})\varphi^4\|_{L^2(0,t_f;[L^2(\Omega)]^{n^2})} \\ &\leq c_6(\varphi) + c_7(\varphi) \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds. \end{aligned}$$

- Considering hypotheses **(H6)**, **(H9)**, Theorem 3.4.5 and Hölder's and Young's inequalities for terms  $I_4$ ,  $I_5$ ,  $I_6$  and  $I_7$  we get:

$$\begin{aligned} |I_4| &\leq \left( \int_0^t \|\varphi^3 D^2(\Lambda(\theta))_t \cdot_2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_8(\varphi) + c_9(\varphi) \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds, \end{aligned}$$

$$\begin{aligned} |I_5| &\leq \left( \int_0^t \|\varphi^3 (D^2(\Lambda(\theta)) \cdot_2 \sigma_t)\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{10}(\varphi) + c_{11}(\varphi) \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds, \end{aligned}$$

$$\begin{aligned} |I_6| &\leq 2 \left( \int_0^t \|\varphi (D^2 \sigma \cdot_3 D(\Lambda(\theta))_t)\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi^3 D \sigma\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq 2 \|D(\Lambda(\theta))_t\|_{[L^\infty(\Omega)]^{n^5}} \left( \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi^3 D \sigma\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{12}(\varphi) + c_{13}(\varphi) \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds, \end{aligned}$$

and

$$\begin{aligned} |I_7| &\leq 2 \left( \int_0^t \|\varphi (D^2 \sigma \cdot_3 D(\Lambda(\theta)))\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi^3 D \sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq 2 \|D(\Lambda(\theta))\|_{[L^\infty(\Omega)]^{n^5}} \left( \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi^3 D \sigma_t\|_{[L^2(\Omega)]^{n^3}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{14}(\varphi) + c_{15}(\varphi) \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds. \end{aligned}$$

- We bound term  $I_8$  thanks to hypotheses **(H8)**, **(H9)**, **(H12)**, Theorem 3.4.5 and Hölder's and Young's inequalities:

$$\begin{aligned} |I_8| &\leq \int_0^t \int_{\Omega} \varphi^4 [|\sigma^D|^{q-2} D^2(\Theta(\theta) \sigma^D) + 2 \nabla |\sigma^D|^{q-2} \nabla(\Theta(\theta)) \sigma^D] \cdot_4 D^2 \sigma dp ds \\ &\leq \left( \int_0^t \|\varphi^3 |\sigma^D|^{q-2} D^2(\Theta(\theta) \sigma^D)\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &\quad + 2 \left( \int_0^t \|\varphi^3 \nabla |\sigma^D|^{q-2} \nabla(\Theta(\theta)) \sigma^D\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\ &\leq c_{16}(\varphi) + c_{17}(\varphi) \int_0^t \|\varphi D^2 \sigma\|_{[L^2(\Omega)]^{n^4}}^2 ds. \end{aligned}$$

- Finally for term  $I_9$ , applying hypotheses **(H7)**, **(H9)** and Hölder's and Young's inequalities, we have:

$$\begin{aligned}
|I_9| &\leq \left( \int_0^t \|\varphi^3 D^2(\alpha(\theta)\theta_t)\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2(\text{tr}(\boldsymbol{\sigma}))\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\
&\leq \left( \int_0^t \|\varphi^3 D^2(\alpha(\theta)\theta_t)\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds \right)^{\frac{1}{2}} \\
&\leq c_{18}(\varphi) + c_{19}(\varphi) \int_0^t \|\varphi D^2 \boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^4}}^2 ds.
\end{aligned}$$

The main difficulty is to bound the last two terms, since those techniques lead to an expression like  $\int_0^t \int_{\Omega} \varphi^4 |D\boldsymbol{\sigma}^D|^2 |D^2 \boldsymbol{\sigma}^D| dp ds$ . Indeed, let us begin with  $I_{10}$ ; taking into account (3.15), we obtain

$$\begin{aligned}
|I_{10}| &\leq \left| \int_0^t \int_{\Omega} (q-2)\varphi^4 \Theta(\theta) |\nabla |\boldsymbol{\sigma}^D|^{q-4} (D\boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D)] \cdot_2 (D^2 \boldsymbol{\sigma} \cdot_2 \boldsymbol{\sigma}^D) dp ds \right| \\
&\quad + \left| \int_0^t \int_{\Omega} (q-2)\varphi^4 \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} ((D^2 \boldsymbol{\sigma} \cdot_2 \boldsymbol{\sigma}^D) \cdot_1 D\boldsymbol{\sigma}^D) \cdot_3 D\boldsymbol{\sigma}^D dp ds \right| \\
&\quad + \left| \int_0^t \int_{\Omega} (q-2)\varphi^4 \Theta(\theta) |\boldsymbol{\sigma}^D|^{q-4} (D^2 \boldsymbol{\sigma}^D \cdot_2 \boldsymbol{\sigma}^D) \cdot_2 (D^2 \boldsymbol{\sigma} \cdot_2 \boldsymbol{\sigma}^D) dp ds \right|.
\end{aligned}$$

Then, by hypotheses **(H8)**, **(H9)** and **(H12)**, we can bound the first addend as follows:

$$|I_{10,1}| \leq c_{20} \int_0^t \int_{\Omega} \varphi^4 |D\boldsymbol{\sigma}^D|^2 |D^2 \boldsymbol{\sigma}^D| dp ds, \quad (3.47)$$

and, applying Young's inequality for  $\varepsilon = 2$ , we have

$$\begin{aligned}
|I_{10,1}| &\leq 2c_{20} \int_0^t \int_{\Omega} \varphi^4 |D\boldsymbol{\sigma}^D|^4 dp ds + \frac{1}{2} c_{21}(\varphi) \int_0^t \int_{\Omega} |\varphi D^2 \boldsymbol{\sigma}^D|^2 dp ds \\
&\leq c_{22} \int_0^t \|\varphi D\boldsymbol{\sigma}^D\|_{[L^4(\Omega)]^{n^3}}^4 ds + \frac{1}{2} c_{21}(\varphi) \int_0^t \|\varphi D^2 \boldsymbol{\sigma}^D\|_{[L^2(\Omega)]^{n^4}}^2 ds.
\end{aligned} \quad (3.48)$$

In order to bound the term  $\|\varphi D\boldsymbol{\sigma}^D\|_{[L^4(\Omega)]^{n^3}}^4$ , we use the Gagliardo-Nirenberg inequality given in Lemma 3.5.4. Indeed, from hypothesis **(H12)** we deduce

$$\|D(\varphi \boldsymbol{\sigma}^D)\|_{[L^4(\Omega)]^{n^3}}^4 \leq c_{23}(\varphi) \left[ \|D(\varphi \boldsymbol{\sigma}^D)\|_{[L^2(\Omega)]^{n^3}}^2 + \|D^2(\varphi \boldsymbol{\sigma}^D)\|_{[L^2(\Omega)]^{n^4}}^2 \right]. \quad (3.49)$$

Since  $\boldsymbol{\sigma}(t) \in [H_{\text{loc}}^1(\Omega)]^{n^2}$  and using hypothesis **(H12)** again together with the product derivation rule, we obtain from (3.49) that

$$\begin{aligned}
\|D(\varphi \boldsymbol{\sigma}^D)\|_{[L^4(\Omega)]^{n^3}}^4 &\leq c_{23}(\varphi) \left[ \|\nabla \varphi \boldsymbol{\sigma}^D\|_{[L^2(\Omega)]^{n^3}}^2 + \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}}^2 + 2\|\nabla \varphi \boldsymbol{\sigma}^D\|_{[L^2(\Omega)]^{n^3}} \|\varphi D\boldsymbol{\sigma}\|_{[L^2(\Omega)]^{n^3}} \right. \\
&\quad \left. + \|D^2(\varphi \boldsymbol{\sigma}^D)\|_{[L^2(\Omega)]^{n^4}}^2 \right] \\
&\leq c_{24}(\varphi) + c_{23}(\varphi) \|D^2(\varphi \boldsymbol{\sigma}^D)\|_{[L^2(\Omega)]^{n^4}}^2,
\end{aligned}$$

so, with an analogous reasoning for  $\|D^2(\varphi\sigma^D)\|_{[L^2(\Omega)]^{n^4}}^2$ , we get

$$\begin{aligned} \|D(\varphi\sigma^D)\|_{[L^4(\Omega)]^{n^3}}^4 &\leq c_{24}(\varphi) + c_{23}(\varphi) \left[ \|D(\nabla\varphi)\sigma^D\|_{[L^2(\Omega)]^{n^4}} + 2\|\nabla\varphi D\sigma^D\|_{[L^2(\Omega)]^{n^4}} \right. \\ &\quad \left. + \|\varphi D^2\sigma^D\|_{[L^2(\Omega)]^{n^4}} \right]^2 \end{aligned} \quad (3.50)$$

$$\leq c_{25}(\varphi) + c_{26}(\varphi)\|\varphi D^2\sigma^D\|_{[L^2(\Omega)]^{n^4}}^2. \quad (3.51)$$

Besides, since  $\sigma^D$  is bounded, we have

$$\|\varphi D\sigma^D\|_{[L^4(\Omega)]^{n^3}}^4 \leq [\|D(\varphi\sigma^D)\|_{[L^4(\Omega)]^{n^3}} + c_{27}(\varphi)]^4,$$

and, using (3.51)

$$\|\varphi D\sigma^D\|_{[L^4(\Omega)]^{n^3}}^4 \leq c_{28}(\varphi) + c_{29}(\varphi)\|\varphi D^2\sigma^D\|_{[L^2(\Omega)]^{n^4}}^2. \quad (3.52)$$

Summing up, from (3.48) and (3.52), we obtain

$$|I_{10,1}| \leq c_{30}(\varphi) + c_{31}(\varphi) \int_0^t \|\varphi D^2\sigma^D\|_{[L^2(\Omega)]^{n^4}}^2 ds.$$

Term  $I_{10,2}$  can be bounded with the same technique, and for  $I_{10,3}$  we have

$$\begin{aligned} |I_{10,3}| &= \left| \int_0^t \int_{\Omega} (q-2)\varphi^4 \Theta(\theta) |\sigma^D|^{q-4} (D^2\sigma^D \cdot_2 \sigma^D) \cdot_2 (D^2\sigma \cdot_2 \sigma^D) dp ds \right| \\ &\leq c_{32}(\varphi) \int_0^t \int_{\Omega} |\varphi D^2\sigma^D|^2 dp ds = c_{32}(\varphi) \int_0^t \|\varphi D^2\sigma^D\|_{[L^2(\Omega)]^{n^4}}^2 ds, \end{aligned}$$

using hypotheses **(H6)**, **(H9)** and **(H12)**. Therefore

$$|I_{10}| \leq c_{33}(\varphi) + c_{34}(\varphi) \int_0^t \|\varphi D^2\sigma^D\|_{[L^2(\Omega)]^{n^4}}^2 ds.$$

For term  $I_{11}$ , using (3.15) and hypotheses **(H8)** and **(H9)** we deduce

$$|I_{11}| \leq c_{35} \int_0^t \int_{\Omega} \varphi^4 |D\sigma^D|^2 |D^2\sigma^D| dp ds,$$

which is analogous to (3.47).

Substituting all bounds in (3.46), we deduce the result.  $\square$

*Proof of Theorem 3.5.1.* From inequality (3.45) and using Gronwall's lemma, it follows that

$$\|\varphi D^2\sigma(t)\|_{[L^2(\Omega)]^{n^4}} \leq c_{21}(\varphi).$$

The function  $\varphi$  may now be chosen as an appropriate cut-off function to conclude that  $\sigma(t) \in [H^2(\Omega')]^{n^2}$  for any subdomain  $\Omega' \subset\subset \Omega$ .  $\square$

**Remark 3.5.6.** *The regularity properties of stresses proved in this chapter can be generalized to the contact problem studied in Barral and Quintela [15], given that they are regularity results in the interior.*

**Remark 3.5.7.** *Since the regularity results proved in this chapter are local in Section 3.4 and 3.5, it would be enough to consider local regularity hypotheses on both temperature and the applied body force.*



## Part III

# Mathematical analysis of a thermoelastic problem.



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## Chapter 4

# Existence and uniqueness of a thermoelastic problem

### 4.1 Introduction

Once we have studied the modelling of Maxwell-Norton materials with heat conduction and the mathematical analysis of a quasistatic thermoviscoelastic problem of the Maxwell-Norton type, in this chapter, our purpose is to carry out a mathematical analysis of the coupling between the motion and energy conservation equations for thermoelastic materials. This problem is deduced from equations obtained in Chapter 1 of this manuscript, and its mechanical part is a simplification of the mechanical problem studied in Chapters 2 and 3.

Next, we present a bibliographical review of some results of existence and uniqueness of thermoelastic problems arranged chronologically.

In literature, there exists several existence results for thermoelastic problems, although not many deal with quasistatic problems with a Robin boundary condition. The equations of the coupled thermoelasticity are considered for the first time by Duhamel [38] in 1837. A step forward in the study of this problem is carried out by Biot [20] and Lessen [80], and for the first time, in 1960 Boley and Weiner [23] face the quasistatic theory for thermoelastic problems. Other authors, like Hetnarski [68] presents a solution for dimensionless equations of a coupled dynamic thermoelastic problem. Dafermos [33] is concerned about the dynamic initial-boundary value problem of linear thermoelasticity theory in which the boundary conditions are homogeneous and the initial conditions and source terms (body forces and heat supply function) are non-zero. The problem is formulated for an inhomogeneous, anisotropic material occupying a bounded region in space and subject to small disturbances over a natural reference configuration. It is proved, in particular, that the solution is kinetically stable in the sense of exhibiting continuous dependence on the initial data and the source terms. Later, Duvaut and Lions [40, 41, 42, 43] consider the dynamic thermoelastic problem with the particular boundary conditions of contact and semi-permeable wall, and they give two theorems of existence and uniqueness.

Gawinecki in [55] proves the existence of the solution of a generalized dynamic thermoelastic

problem with null Dirichlet boundary conditions using the Faedo-Galerkin method, later in [56] he studies its uniqueness and regularity. Taking the same problem of these two previous articles but unconsidering the dynamic term to the energy equation, Gawinecki obtains its existence and uniqueness in [57], and a year later he presents its regularity properties in [58]. In the same year, Bermúdez and Viaño [18] give a result of existence for a dynamic thermoelastic problem, where they considerer null Dirichlet and Neumann boundary conditions, with constant reference temperature. Gawinecki *et al.* [62] analyze also the existence and uniqueness of a solution of the dynamic initial value problem, with mixed boundary conditions, in linear thermoelasticity, whose energy equation presents the mechanic dissipated term. In Gawinecki [61] the existence, uniqueness and regularity of several problems of the dynamic thermoelasticity with different boundary conditions are studied applying the Faedo-Galerkin method. In addition, Day [34] presents a quasistatic thermoelastic problem with the body clamped at the boundary, whose static case with unilateral Signorini boundary condition together with heat flux has been investigated by Duvaut [39].

Afterwards, Shi and Shillor [94] present the  $n$ -dimensional quasistatic thermoelastic problem of frictionless contact of a thermoelastic body with a rigid foundation and they give the same results which guarantee the existence of a solution. Later, Ames and Payne [7] obtain the uniqueness and the continuous dependence on the initial temperature for this problem.

Next, Figueiredo and Trabucho in [48, 49] give a summary of some existence results for dynamical contact problems with friction in the framework of thermoelasticity and thermoviscoelasticity theories. These works extend the previous results of Duvaut and Lions [43, 45], Martins and Oden [85] and Rabier *et al.* [91]. Also, Muñoz Rivera and Racke in [88] consider dynamical thermoelastic problems with a contact condition of Signorini's type without source terms and with homogeneous boundary conditions, and they study the quasi-static case. They obtain a result of existence of solution weaker than our one without obtaining the uniqueness by applying a penalty method.

On the other hand, the uniqueness of mathematical models of thermoelastic problems has been considered in many papers (some already named previously), amongst which, let us mention: Weiner [99] proves a uniqueness theorem for the coupled dynamic thermoelastic problem without source terms for an isotropic elastic solid, which is generalized later by Ionescu-Cazimir [73, 74]. Ieşan [70] gives the form to the solutions of the coupled dynamic thermoelastic mixed problem for non homogeneous and anisotropic materials. Also in [71] he studies the same problem but for a homogeneous and isotropic solid with two temperatures establishing some general uniqueness theorems. In [72], Ieşan derives some uniqueness theorems for these problems. Knops and Payne [76] examine the continuous dependence on the initial data of solutions of the linear anisotropic thermoelastic initial boundary value problem. Carlson [26] gives a uniqueness theorem for the dynamic mixed problem of the theory of thermoelasticity.

Lately, Chrzęszczuk [27] formulates results about uniqueness, regularity and continuous dependence on the data for generalized solutions of some coupled problems in the nonlinear theory of thermoelastic shells. Blanchard and Francfort [21] present a study of a dynamic behaviour of a three-dimensional linear thermoelastic flat plate.

Nowadays, many of these results are used for example, for the thermoelastic thermistor problem given by Wu and Xu [100] and Badii [10], in planar thermoelasticity considered by Copetti [31] and in a contact problem in thermoelasticity with second sound studied by Sprenger [96]; although new

existence and uniqueness results for quasistatic thermoelasticity problems have not been found.

Therefore, our purpose in this chapter is the studying of the existence and uniqueness of a quasistatic thermoelastic problem with mixed displacement-traction boundary conditions for the mechanical submodel and mixed boundary conditions including a Robin boundary condition for the thermal one, considering non homogeneous materials. Besides local regularity properties of the solution are studied in the next chapter. A similar problem has been studied by Viaño in [98] for both anisotropic and isotropic elastic materials in contact with another elastic body, and by Figueiredo and Trabucho in [48, 49] for three different types of materials and considering the dynamic motion equation. We use the same methodology, the Galerkin's method, in order to prove the existence of a solution of the problem. Our main contributions are:

- the problem is quasistatic,
- a convection heat transfer boundary condition is considered, and
- the reference temperature, the thermal conductivity and the Lamé's parameters depend on the material point.

These contributions cause some difficulties to calculate *a priori* estimates, since different and new terms appear in the weak formulation of the problem. Furthermore, in order to prove the uniqueness their techniques can not be applied to our case. To overcome this difficulty, we apply a result used by Gawinecki [56, 58, 59, 60] and Gawinecki *et al.* [62].

The outline of this Chapter 4 is as follows: firstly, we will describe the mathematical model. After introducing in Section 4.3 the appropriate functional framework, in Section 4.4 we will prove the existence and uniqueness of a solution of the problem.

## 4.2 Mathematical model

Taking into account the model developed in Chapter 1, we consider a thermodynamic process with small changes with respect to a reference state, i.e. small displacements  $\nabla \mathbf{u} = O(\varepsilon)$ , small velocities  $\nabla \dot{\mathbf{u}} = O(\varepsilon)$ , small temperature changes with respect to a reference temperature  $\theta_r$ ,  $\theta - \theta_r = O(\varepsilon)$ , and small changes of the internal symmetric second order tensor with respect to initial internal symmetric second order tensor,  $\mathbf{Z} - \mathbf{Z}_0 = O(\varepsilon)$ ,  $\varepsilon$  being a small parameter (see Section 1.4, Chapter 1). So, neglecting the terms  $o(\varepsilon)$ , the equilibrium equations for quasistatic thermoviscoelastic materials with long memory can be expressed as follows:

$$\begin{aligned}
 -\text{Div } \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) &= \mathbf{b}_*, & (4.1) \\
 \rho_0 \widehat{c}_F(\mathbf{I}, \theta, \mathbf{Z}, p) \dot{\theta} &= \theta \frac{\partial \mathbf{S}^{***}}{\partial \theta}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) : \nabla \dot{\mathbf{u}} - \rho_0 \frac{\partial \widehat{e}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}, p) : \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p) \\
 &\quad - \text{Div } \mathbf{q}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) + f_*, & (4.2)
 \end{aligned}$$

where  $\mathbf{S}^{***}$  is the linearized part of the First Piola-Kirchhoff stress tensor considering a **(3ZLM)** linearization written in terms of the Second Piola-Kirchhoff stress tensor (see Subsection 1.4.2,

Chapter 1), which is given by the expression

$$\begin{aligned}
\mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = & (\mathbf{I} + \mathbf{W}) \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) + \text{Div } \mathbf{u} \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) - \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) \boldsymbol{\varepsilon}(\mathbf{u}) \\
& + \widehat{\mathbf{C}}(\theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \widehat{\Sigma}_{ther}(\mathbf{I}, \theta, p) + (\mathbf{W} + \boldsymbol{\varepsilon}(\mathbf{u})) \widehat{\Sigma}_{ther}(\mathbf{I}, \theta, p) \\
& + \frac{\partial \widehat{\Sigma}_{ther}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + (\mathbf{I} + \mathbf{W} + \boldsymbol{\varepsilon}(\mathbf{u})) \widehat{\Sigma}_{plas}(\mathbf{I}, \theta, \mathbf{Z}_0, p) \\
& + \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\partial \widehat{\Sigma}_{plas}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, p) : (\mathbf{Z} - \mathbf{Z}_0), \quad (4.3)
\end{aligned}$$

and  $\mathbf{q}^{***}$  is a (3ZLM) linearization of the heat flux as follows:

$$\begin{aligned}
\mathbf{q}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, \nabla \theta, p) = & \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p) + \text{Div } \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p) - \nabla \mathbf{u} \widehat{\mathbf{q}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p) \\
& + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{F}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p)(\nabla \mathbf{u}) - \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{w}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p)(\nabla \mathbf{u}^t \nabla \theta) \\
& + \frac{\partial \widehat{\mathbf{q}}}{\partial \mathbf{Z}}(\mathbf{I}, \theta, \mathbf{Z}_0, \nabla \theta, p)(\mathbf{Z} - \mathbf{Z}_0). \quad (4.4)
\end{aligned}$$

It is recalled that in expressions (4.1)-(4.4) we use the following notation:

- $\mathbf{u}$  is the displacement and  $\nabla \mathbf{u}$  its gradient,
- $\theta$  is the absolute temperature and  $\nabla \theta$  its gradient,
- $\mathbf{Z}$  is the internal symmetric second order tensor,
- $p$  is a material point,
- $\mathbf{b}_*$  are the body forces per unit volume at the reference configuration,
- $\rho_0$  is the reference density,
- $\widehat{c}_F$  is the specific heat at constant deformation,
- $\widehat{e}$  is the specific internal energy per unit mass,
- $\widetilde{\mathbf{h}}$  is the function which defines the evolution of internal variable  $\mathbf{Z}$ , verifying

$$\dot{\mathbf{Z}} = \widetilde{\mathbf{h}}(\mathbf{I}, \theta, \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p), p),$$

- $\widehat{\mathbf{q}}$  is the heat flux per unit area,
- $f_*$  is the body heat per unit volume at the reference configuration,
- $\mathbf{I}$  is the identity tensor,  $\mathbf{W} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^t)$ ,  $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$ ,
- $\widehat{\Sigma}_{elas}$ ,  $\widehat{\Sigma}_{ther}$ ,  $\widehat{\Sigma}_{plas}$ , are the elastic, thermal and plastic part of the Second Piola-Kirchhoff stress tensor,

- and finally,  $\widehat{\mathbf{C}}$  is the elasticity tensor which is defined by

$$\widehat{\mathbf{C}}(\theta, p) : \mathbf{U} = \mathbf{U} \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) + \frac{\widehat{\Sigma}_{elas}}{\partial \mathbf{F}}(\mathbf{I}, \theta, p) : \mathbf{U} + \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) \mathbf{U}^t - \text{tr} \mathbf{U} \widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p), \quad (4.5)$$

for all  $\mathbf{U} \in Lin$ , where  $Lin$  is the linear space of endomorphisms from a vector space  $\mathcal{V}$ . It can be deduced of Definition 1.4.13, Subsection 1.4.2 of Chapter 1.

Furthermore, given any function  $s$ ,  $\widehat{s}$  indicates the response function of  $s$  and  $s_0(p) = s(p, 0)$  its value at the time  $t = 0$ .

In this chapter, we choose  $\mathcal{B}$  as the reference configuration, we neglect the plastic part of the behaviour law and so, we consider thermoelastic materials whose response functions in Lagrangian coordinates are given by:

$$\widehat{\Sigma}_{elas}(\mathbf{F}, \theta, p) = \widehat{\Lambda}^{-1}(\theta, p) : \mathbf{E}, \quad (4.6)$$

$$\widehat{\Sigma}_{ther}(\mathbf{F}, \theta, p) = -3 \left( \int_{\theta_r}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{K}(\theta, p) \mathbf{I}, \quad (4.7)$$

$$\widehat{\mathbf{q}}_*(\mathbf{F}, \theta, \mathbf{w}, p) = \det \mathbf{F} \mathbf{F}^{-1} \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{F}^{-t} \mathbf{w}, p) = -\widehat{k}(\theta, p) \det \mathbf{F} (\mathbf{F}^t \mathbf{F})^{-1} \mathbf{w}, \quad (4.8)$$

where:

- $\widehat{\Lambda}$  is the elasticity tensorial function defined as

$$\begin{aligned} (\theta, p) \in \mathbb{R}^+ \times \mathcal{B} &\rightarrow \widehat{\Lambda}(\theta, p) : Sym \longrightarrow Sym \\ \boldsymbol{\tau} &\longrightarrow \widehat{\Lambda}(\theta, p) : \boldsymbol{\tau} = \frac{1 + \widehat{\nu}(\theta, p)}{\widehat{E}(\theta, p)} \boldsymbol{\tau} - \frac{\widehat{\nu}(\theta, p)}{\widehat{E}(\theta, p)} \text{tr}(\boldsymbol{\tau}) \mathbf{I}, \end{aligned}$$

$\widehat{E}(\theta, p)$ ,  $\widehat{\nu}(\theta, p)$  being the response functions associated with Young's modulus and Poisson's coefficient, respectively, and  $Sym$  being the subspace of  $Lin$  of all symmetric endomorphisms. The inverse of  $\widehat{\Lambda}$  is defined by:

$$\begin{aligned} (\theta, p) \in \mathbb{R}^+ \times \mathcal{B} &\rightarrow \widehat{\Lambda}^{-1}(\theta, p) : Sym \longrightarrow Sym \\ \boldsymbol{\tau} &\longrightarrow \widehat{\Lambda}^{-1}(\theta, p) : \boldsymbol{\tau} = \widehat{\lambda}(\theta, p) \text{tr}(\boldsymbol{\tau}) \mathbf{I} + 2\widehat{\mu}(\theta, p) \boldsymbol{\tau}, \end{aligned} \quad (4.9)$$

where  $\widehat{\lambda}(\theta, p)$ ,  $\widehat{\mu}(\theta, p)$  are the response functions associated with Lamé's parameters of the material and related to  $\widehat{E}$ ,  $\widehat{\nu}$  by

$$\widehat{\lambda}(\theta, p) = \frac{\widehat{E}(\theta, p) \widehat{\nu}(\theta, p)}{(1 + \widehat{\nu}(\theta, p))(1 - 2\widehat{\nu}(\theta, p))}, \quad \widehat{\mu}(\theta, p) = \frac{\widehat{E}(\theta, p)}{2(1 + \widehat{\nu}(\theta, p))}.$$

Considering elastic response function (4.6) and taking into account the expression of the elasticity tensor (4.5), it is easy to deduce that

$$\widehat{\mathbf{C}}(\theta, p) : \mathbf{U} = \widehat{\Lambda}^{-1}(\theta, p) : \left( \frac{\mathbf{U}^t + \mathbf{U}}{2} \right),$$

for all  $\mathbf{U} \in Lin$ . The proof of this assertion can be seen in Lemma 1.5.9, Subsection 1.5.4 of Chapter 1.

- $\mathbf{E} = \frac{\mathbf{C} - \mathbf{I}}{2}$  is the Green-Saint Venant strain tensor, where  $\mathbf{C} = \mathbf{F}'\mathbf{F}$  is the right Cauchy-Green strain tensor.
- $\widehat{K}$  is the bulk modulus of the material:  $\widehat{K}(\theta, p) = \frac{1}{3} \left( 3\widehat{\lambda}(\theta, p) + 2\widehat{\mu}(\theta, p) \right)$ .
- $\widehat{\alpha} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a smooth enough response function associated with the coefficient of thermal expansion.  
In equation (4.7),  $\theta_r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a smooth enough given function. In practice,  $\theta_r$  will be the reference temperature or the initial temperature.
- $\widehat{\mathbf{q}}$  is the heat flux which is defined by the Fourier's law in the deformation configuration as follows:

$$\widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{w}, p) = -\widehat{k}(\theta, p)\mathbf{w}, \quad (4.10)$$

for all  $\mathbf{w} \in \mathcal{V}$ .

- $\widehat{k} : \mathbb{R}^+ \times \mathcal{B} \mapsto \mathbb{R}$  is the smooth enough response function associated with the thermal conductivity of the material.

Under these assumptions, thanks to response functions (4.6)-(4.7) and since  $\widehat{\Sigma}_{plas} \equiv \mathbf{0}$  and  $\widehat{\Sigma}_{elas}(\mathbf{I}, \theta, p) \equiv \mathbf{0}$ ,  $\mathbf{S}^{***}$  takes the following form:

$$\mathbf{S}^{***}(\nabla \mathbf{u}, \theta, p) = \widehat{\Lambda}^{-1}(\theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) - 3 \left( \int_{\theta_r}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{K}(\theta, p)\mathbf{I}. \quad (4.11)$$

Furthermore, from expressions (4.4) and (4.10), we obtain  $\mathbf{q}^{***}$  as follows:

$$\mathbf{q}^{***}(\nabla \mathbf{u}, \theta, \nabla \theta, p) = -\widehat{k}(\theta, p)\nabla \theta - \widehat{k}(\theta_r, p)\text{Div } \mathbf{u}\nabla \theta + 2\widehat{k}(\theta_r, p)\boldsymbol{\varepsilon}(\mathbf{u})\nabla \theta. \quad (4.12)$$

If we replace equalities (4.11) and (4.12) in equilibrium equations (4.1) and (4.2), we get:

$$-\text{Div} \left( \widehat{\Lambda}^{-1}(\theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) - 3 \left( \int_{\theta_r}^{\theta} \widehat{\alpha}(r) dr \right) \widehat{K}(\theta, p)\mathbf{I} \right) = \mathbf{b}_*, \quad (4.13)$$

$$\begin{aligned} \rho_0 \widehat{c}_F(\mathbf{I}, \theta, p)\dot{\theta} &= \theta \left( \frac{\partial \widehat{\Lambda}^{-1}}{\partial \theta}(\theta, p) : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\widehat{\alpha}(\theta)\widehat{K}(\theta, p)\mathbf{I} \right) : \nabla \dot{\mathbf{u}} \\ &\quad - 3\theta \left( \int_{\theta_r}^{\theta} \widehat{\alpha}(r) dr \right) \frac{\partial \widehat{K}}{\partial \theta}(\theta, p)\mathbf{I} : \nabla \dot{\mathbf{u}} - \text{Div} \left( (-\widehat{k}(\theta, p) - \widehat{k}(\theta_r, p))\text{Div } \mathbf{u}\nabla \theta \right) \\ &\quad - \text{Div} \left( 2\widehat{k}(\theta_r, p)\boldsymbol{\varepsilon}(\mathbf{u})\nabla \theta \right) + f_*. \end{aligned} \quad (4.14)$$

In order to study the mathematical analysis of a thermoelastic submodel, as a first approximation, we simplify the previous equations, considering the following hypotheses:

- (H1) The mechanic coefficients  $E$  and  $\nu$  are independent of temperature and consequently,  $\widehat{\Lambda}$  and  $\widehat{K}$  are also independent of temperature.

( $\bar{\text{H}}2$ ) The thermal coefficients  $\widehat{k}$ ,  $\widehat{c}_F$  are also independent of temperature and the coefficient of thermal expansion  $\widehat{\alpha}$  is a constant.

Furthermore, to avoid the nonlinearities in equation (4.14), we assume:

1.  $\nabla \mathbf{u} = O(\varepsilon)$ ,  $\nabla \dot{\mathbf{u}} = O(\varepsilon)$  and  $\theta - \theta_r = O(\varepsilon)$ ,  $\varepsilon$  being a small parameter.

Therefore, taking the temperature as  $\theta = \theta_r + (\theta - \theta_r) = \theta_r + O(\varepsilon)$ , neglecting the terms  $o(\varepsilon)$  and considering that the coefficients are independent of temperature, equations (4.13) and (4.14) can be rewritten as:

$$-\text{Div } \widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, p) = \mathbf{b}_*, \quad (4.15)$$

$$\rho_0 \widehat{c}_F(\mathbf{I}, p) \dot{\theta} = -\theta_r \left( 3\widehat{\alpha} \widehat{K}(p) \mathbf{I} \right) : \nabla \dot{\mathbf{u}} + \text{Div}(\widehat{k}(p) \nabla \theta) + f_*, \quad (4.16)$$

where we denote by

$$\widehat{\boldsymbol{\sigma}}(\theta, \mathbf{u}, p) = \mathbf{S}^{***}(\nabla \mathbf{u}, \theta, \mathbf{Z}, p) = \widehat{\Lambda}^{-1}(p) : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\widehat{\alpha}(\theta - \theta_r) \widehat{K}(p) \mathbf{I},$$

(see Section 1.5.4, Chapter 1).

From now on, in order to simplify the notation we will omit the dependence of the different operators and functions with respect to the material point  $p$ , the  $\widehat{\cdot}$  of the response functions, and  $\cdot_*$  of the body forces and the body heat. So, we rewrite the equilibrium equations as follows:

$$-\text{Div } \boldsymbol{\sigma}(\theta, \mathbf{u}) = \mathbf{b}, \quad (4.17)$$

$$\rho_0 c_F \dot{\theta} = -\theta_r \alpha (3\lambda + 2\mu) \text{Div } \dot{\mathbf{u}} + \text{Div}(k \nabla \theta) + f, \quad (4.18)$$

with

$$\boldsymbol{\sigma}(\theta, \mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\alpha(\theta - \theta_r) K \mathbf{I}.$$

#### 4.2.1 Problem (P)

Let  $\Omega \subset \mathbb{R}^3$  be an open and bounded set with smooth boundary  $\Gamma$  and  $\mathbf{n}$  its outward unit normal vector. We assume that  $\Gamma_{\mathbf{u},D}$ ,  $\Gamma_{\mathbf{u},N}$ ,  $\Gamma_{\theta,D}$ ,  $\Gamma_{\theta,N}$  and  $\Gamma_{\theta,R}$  are open subsets of  $\Gamma$ , such that

- $\Gamma = \bar{\Gamma}_{\mathbf{u},D} \cup \bar{\Gamma}_{\mathbf{u},N} = \bar{\Gamma}_{\theta,D} \cup \bar{\Gamma}_{\theta,N} \cup \bar{\Gamma}_{\theta,R}$ ,
- $\Gamma_{\mathbf{u},D} \cap \Gamma_{\mathbf{u},N} = \emptyset$ ,  $\Gamma_{\theta,D} \cap \Gamma_{\theta,N} = \emptyset$ ,  $\Gamma_{\theta,D} \cap \Gamma_{\theta,R} = \emptyset$ ,  $\Gamma_{\theta,R} \cap \Gamma_{\theta,N} = \emptyset$ ,

$\text{meas}(\Gamma_{\mathbf{u},D}) > 0$  and  $\text{meas}(\Gamma_{\theta,D} \cup \Gamma_{\theta,R}) > 0$ . Let  $[0, t_f]$  be the time interval of interest. Then, the problem we are going to study is the following:

**Problem (P)**

Find  $\mathbf{u}(p, t)$  and  $\theta(p, t)$  in  $\Omega \times (0, t_f]$ , verifying:

Equilibrium equations:

$$-\text{Div } \boldsymbol{\sigma}(\theta, \mathbf{u}) = \mathbf{b} \quad \text{in } \Omega \times (0, t_f], \quad (4.19)$$

$$\rho_0 c_F \dot{\theta} = -\theta_r \alpha (3\lambda + 2\mu) \text{Div } \dot{\mathbf{u}} + \text{Div } (k \nabla \theta) + f \quad \text{in } \Omega \times (0, t_f]. \quad (4.20)$$

Behaviour law:

$$\boldsymbol{\sigma}(\theta, \mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - \alpha(\theta - \theta_r)(3\lambda + 2\mu)\mathbf{I} \quad \text{in } \Omega \times (0, t_f]. \quad (4.21)$$

Boundary conditions:

$$\mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma_{\mathbf{u},D} \times (0, t_f], \quad (4.22)$$

$$\boldsymbol{\sigma}(\theta, \mathbf{u}) \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathbf{u},N} \times (0, t_f], \quad (4.23)$$

$$k \nabla \theta \cdot \mathbf{n} = \alpha_c (\theta^e - \theta) \quad \text{on } \Gamma_{\theta,R} \times (0, t_f], \quad (4.24)$$

$$k \nabla \theta \cdot \mathbf{n} = h \quad \text{on } \Gamma_{\theta,N} \times (0, t_f], \quad (4.25)$$

$$\theta = \theta_D \quad \text{on } \Gamma_{\theta,D} \times (0, t_f]. \quad (4.26)$$

Initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \quad (4.27)$$

Here, we use the following notation:

- $\mathbf{u}_D$  is the displacement on the Dirichlet boundary,
- $\mathbf{g}$  is the density of surface forces of the mechanical problem,
- $\alpha_c$  is the coefficient of convective heat transfer,
- $\theta^e$  is the external convection temperature,
- $h$  is the heat flux on the Neumann thermal boundary, and
- $\theta_D$  is the temperature on the Dirichlet thermal boundary.

In order to complete the model,  $\mathbf{u}_0$  and  $\theta_0$  must verify the following compatibility conditions:

$$\left\{ \begin{array}{ll} \boldsymbol{\sigma}(\theta_0, \mathbf{u}_0) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}_0) - \alpha(\theta_0 - \theta_r)(3\lambda + 2\mu)\mathbf{I} & \text{in } \Omega, \\ -\text{Div } \boldsymbol{\sigma}(\theta_0, \mathbf{u}_0) = \mathbf{b}(0) & \text{in } \Omega, \\ \mathbf{u}_0 = \mathbf{u}_D(0) & \text{on } \Gamma_{\mathbf{u},D}, \\ \boldsymbol{\sigma}(\theta_0, \mathbf{u}_0) \mathbf{n} = \mathbf{g}(0) & \text{on } \Gamma_{\mathbf{u},N}, \\ \theta_0 = \theta_D(0) & \text{on } \Gamma_{\theta,D}. \end{array} \right. \quad (4.28)$$

### 4.3 Functional framework

In this section, we introduce the spaces of admissible displacements and temperatures. We consider  $\mathbf{D}(\Omega) = [\mathcal{D}(\Omega)]^3$ ,  $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^3$ ,  $\mathbf{L}^\infty(\Omega) = [L^\infty(\Omega)]^3$  and  $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^3$  with their usual norms.

- The admissible displacements space is

$$\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : v_i|_{\Gamma_{\mathbf{u},D}} = 0, 1 \leq i \leq 3\},$$

which is a Hilbert space with the usual norm in  $\mathbf{H}^1(\Omega)$ .

- The admissible temperature space is

$$H_{0,\Gamma_{\theta,D}}^1(\Omega) = \{\phi \in H^1(\Omega) : \phi|_{\Gamma_{\theta,D}} = 0\},$$

which is a Hilbert space with the usual norm of  $H^1(\Omega)$ .

Furthermore, we have the classical continuous, dense and compact embeddings:

$$\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega) \subset \mathbf{L}^2(\Omega) \subset \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega) \quad \text{and} \quad H_{0,\Gamma_{\theta,D}}^1(\Omega) \subset L^2(\Omega) \subset H_{0,\Gamma_{\theta,D}}^{1'}(\Omega),$$

where  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)$  and  $H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)$  are the dual spaces of  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)$  and  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$ , respectively.

And finally, we denote by:

- $\langle \cdot, \cdot \rangle_{\mathbf{u}}$  the duality between  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)$  and  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)$ , and
- $\langle \cdot, \cdot \rangle_{\theta}$  the duality between  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$  and  $H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)$ .

### 4.4 Existence and uniqueness of solution

#### 4.4.1 A weak variational formulation

Let us assume that  $(\mathbf{u}, \theta)$  is a smooth enough solution of Problem (P). Then, taking into account equation (4.19), we have

$$- \int_{\Omega} \text{Div } \boldsymbol{\sigma}(\theta, \mathbf{u}) \cdot \mathbf{v} \, dp = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, dp,$$

for all  $\mathbf{v} \in \mathbf{D}(\bar{\Omega})$  with  $\mathbf{v} = \mathbf{0}$  on  $\Gamma_{\mathbf{u},D}$  and  $t \in (0, t_f]$ .

Applying Green's formula, we obtain

$$\int_{\Omega} \boldsymbol{\sigma}(\theta, \mathbf{u}) : \nabla \mathbf{v} \, dp - \int_{\Gamma} \boldsymbol{\sigma}(\theta, \mathbf{u}) \mathbf{n} \cdot \mathbf{v} \, d\Gamma = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, dp,$$

and using boundary condition (4.23), we deduce

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{\theta}, \mathbf{u}) : \nabla \mathbf{v} \, dp = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g} \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, dp.$$

Besides, thanks to expression (4.21), definition (4.9), and the properties of scalar product, this expression can be rewritten as follows:

$$\begin{aligned} \int_{\Omega} (\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dp - \int_{\Omega} \alpha(\theta - \theta_r)(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v}) \, dp &= \int_{\Gamma_{\mathbf{u},N}} \mathbf{g} \cdot \mathbf{v} \, d\Gamma \\ &+ \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, dp, \end{aligned} \quad (4.29)$$

for all  $\mathbf{v} \in \mathbf{D}(\bar{\Omega})$  with  $\mathbf{v} = \mathbf{0}$  on  $\Gamma_{\mathbf{u},D}$  and  $t \in (0, t_f]$ .

Analogously, considering energy equation (4.20) since  $\theta_r$  is not null in  $\Omega$ , we have

$$\int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \dot{\theta} \phi \, dp = - \int_{\Omega} \alpha(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \phi \, dp + \int_{\Omega} \frac{1}{\theta_r} \text{Div}(k\nabla\theta) \phi \, dp + \int_{\Omega} \frac{f}{\theta_r} \phi \, dp,$$

for all  $\phi \in D(\bar{\Omega})$  with  $\phi = 0$  on  $\Gamma_{\theta,D}$  and  $t \in (0, t_f]$ .

Applying Green's formula, we obtain

$$\begin{aligned} \int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \dot{\theta} \phi \, dp &= - \int_{\Omega} \alpha(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \phi \, dp - \int_{\Omega} k\nabla\theta \cdot \nabla\left(\frac{\phi}{\theta_r}\right) \, dp + \int_{\Gamma} \frac{k}{\theta_r} \nabla\theta \cdot \mathbf{n} \phi \, d\Gamma \\ &+ \int_{\Omega} \frac{f}{\theta_r} \phi \, dp. \end{aligned}$$

Using boundary conditions (4.24)-(4.25), we get

$$\begin{aligned} \int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \dot{\theta} \phi \, dp &= - \int_{\Omega} \alpha(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \phi \, dp - \int_{\Omega} k\nabla\theta \cdot \nabla\left(\frac{\phi}{\theta_r}\right) \, dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e}{\theta_r} \phi \, d\Gamma \\ &- \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta}{\theta_r} \phi \, d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h}{\theta_r} \phi \, d\Gamma + \int_{\Omega} \frac{f}{\theta_r} \phi \, dp, \end{aligned} \quad (4.30)$$

for all  $\phi \in D(\bar{\Omega})$  with  $\phi = 0$  on  $\Gamma_{\theta,D}$  and  $t \in (0, t_f]$ .

Then, we propose the following weak variational formulation for Problem (P) as follows:

Find  $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$  such that  $\mathbf{u}(t) = \mathbf{u}_D(t)$  on  $\Gamma_{\mathbf{u},D}$  and  $\theta(t) = \theta_D(t)$  on  $\Gamma_{\theta,D}$  a.e.  $t \in (0, t_f)$ , and verifying:

$$\left\{ \begin{array}{l} \int_{\Omega} (\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dp - \int_{\Omega} \alpha(\theta - \theta_r)(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v}) \, dp \\ = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g} \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{D}(\bar{\Omega}) \text{ with } \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\mathbf{u},D} \text{ and } t \in (0, t_f], \\ \int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \dot{\theta} \phi \, dp = - \int_{\Omega} \alpha(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \phi \, dp - \int_{\Omega} k\nabla\theta \cdot \nabla\left(\frac{\phi}{\theta_r}\right) \, dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e}{\theta_r} \phi \, d\Gamma \\ - \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta}{\theta_r} \phi \, d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h}{\theta_r} \phi \, d\Gamma + \int_{\Omega} \frac{f}{\theta_r} \phi \, dp, \quad \forall \phi \in D(\bar{\Omega}) \text{ with } \phi = 0 \text{ on } \Gamma_{\theta,D} \text{ and } t \in (0, t_f]. \end{array} \right.$$

In addition, let us introduce:

- the bilinear form  $a(\cdot, \cdot)$  defined on  $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$  by:

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) dp. \quad (4.31)$$

- the bilinear form  $\kappa(\cdot, \cdot)$  on  $H^1(\Omega) \times H^1(\Omega)$  such that:

$$\kappa(\phi, \psi) = \int_{\Omega} k \nabla \phi \cdot \nabla \left( \frac{\psi}{\theta_r} \right) dp. \quad (4.32)$$

For simplicity of notation,  $\kappa$  will be considered as the sum of bilinear forms  $\kappa_1$  and  $\kappa_2$ , this is:

$$\kappa(\phi, \psi) = \kappa_1(\phi, \psi) + \kappa_2(\phi, \psi) = \int_{\Omega} \frac{k}{\theta_r} \nabla \phi \cdot \nabla \psi dp - \int_{\Omega} k \nabla \phi \cdot \frac{\nabla \theta_r}{\theta_r^2} \psi dp, \quad (4.33)$$

relations deduced applying the product rule to definition (4.32) of  $\kappa$ .

- Finally, the bilinear form  $m(\cdot, \cdot)$  defined on  $L^2(\Omega) \times \mathbf{H}^1(\Omega)$  by:

$$m(\phi, \mathbf{v}) = \int_{\Omega} \phi \alpha (3\lambda + 2\mu) \mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v}) dp = \int_{\Omega} \phi \alpha (3\lambda + 2\mu) \mathbf{I} : \nabla \mathbf{v} dp. \quad (4.34)$$

Taking into account this notation, we can rewrite the proposed weak variational formulation for Problem (P) as follows:

Find  $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$  such that  $\mathbf{u}(t) = \mathbf{u}_D(t)$  on  $\Gamma_{\mathbf{u},D}$  and  $\theta(t) = \theta_D(t)$  on  $\Gamma_{\theta,D}$  a.e.  $t \in (0, t_f)$ , and verifying:

$$\left\{ \begin{array}{l} a(\mathbf{u}, \mathbf{v}) - m(\theta - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g} \cdot \mathbf{v} d\Gamma + \int_{\Omega} \mathbf{b} \cdot \mathbf{v} dp, \quad \forall \mathbf{v} \in \mathbf{D}(\bar{\Omega}) \text{ with } \mathbf{v} = \mathbf{0} \\ \text{on } \Gamma_{\mathbf{u},D} \text{ and } t \in (0, t_f], \\ \int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \dot{\theta} \phi dp = -m(\phi, \dot{\mathbf{u}}) - \kappa(\theta, \phi) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e}{\theta_r} \phi d\Gamma - \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta}{\theta_r} \phi d\Gamma \\ + \int_{\Gamma_{\theta,N}} \frac{h}{\theta_r} \phi d\Gamma + \int_{\Omega} \frac{f}{\theta_r} \phi dp, \quad \forall \phi \in D(\bar{\Omega}) \text{ with } \phi = 0 \text{ on } \Gamma_{\theta,D} \text{ and } t \in (0, t_f]. \end{array} \right. \quad (4.35a)$$

$$(4.35b)$$

#### 4.4.2 Assumptions

Besides assumptions  $(\bar{\text{H}}1)$ - $(\bar{\text{H}}2)$ , from now on we will also assume the following hypotheses:

(H1) The elasticity tensor  $\Lambda^{-1} \in [\mathbf{L}^\infty(\Omega)]^4$  and there exists  $a_{min} > 0$  such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geq a_{min} |\boldsymbol{\tau}|^2,$$

for all  $\boldsymbol{\tau} \in S_3$ , being  $S_3$  the space of symmetric second-order tensors over  $\mathbb{R}^3$ .

- (H2) The reference temperature  $\theta_r \in W^{1,\infty}(\Omega)$ , and there exists  $\theta_{r,min} > 0$  such that  $\theta_r(p) \geq \theta_{r,min}$  in  $\Omega$ .
- (H3) The reference density  $\rho_0 > 0$ , the specific heat at constant deformation  $c_F > 0$  and the coefficient of thermal expansion  $\alpha > 0$ .
- (H4) The thermal conductivity coefficient  $k \in W^{1,\infty}(\Omega)$ , and there exists  $k_{min} > 0$  such that  $k(p) \geq k_{min}$  in  $\Omega$ .
- (H5) The body forces  $\mathbf{b} \in W^{2,2}(0, t_f; \mathbf{L}^2(\Omega))$ .
- (H6) The body heat  $f \in W^{1,2}(0, t_f; L^2(\Omega))$ .
- (H7)  $\mathbf{u}_D$  is the restriction to  $\Gamma_{\mathbf{u},D} \times (0, t_f)$  of a function called  $\bar{\mathbf{u}}_D$  such that  $\bar{\mathbf{u}}_D \in W^{2,2}(0, t_f; \mathbf{H}^{\frac{1}{2}}(\Gamma))$ .
- (H8)  $\theta_D$  is the restriction to  $\Gamma_{\theta,D} \times (0, t_f)$  of a function called  $\bar{\theta}_D$  such that  $\bar{\theta}_D \in W^{2,2}(0, t_f; H^{\frac{1}{2}}(\Gamma))$ .
- (H9) The surface forces  $\mathbf{g} \in W^{2,2}(0, t_f; \mathbf{L}^2(\Gamma_{\mathbf{u},N}))$  and  $h \in W^{1,2}(0, t_f; L^2(\Gamma_{\theta,N}))$ .
- (H10) The coefficient of convective heat transfer  $\alpha_c \in L^\infty(\Gamma_{\theta,R})$ , and there exists  $\alpha_{c,min} > 0$  verifying  $\alpha_c(p) \geq \alpha_{c,min}$  a.e. on  $\Gamma_{\theta,R}$ .
- (H11) The external convection temperature  $\theta^e \in W^{1,2}(0, t_f; L^2(\Gamma_{\theta,R}))$ .
- (H12) The initial conditions  $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$  and  $\theta_0 \in H^1(\Omega)$ .
- (H13) The initial conditions  $\mathbf{u}_0$  and  $\theta_0$  verify:

$$a(\mathbf{u}_0, \mathbf{v}) - m(\theta_0 - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(0) \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b}(0) \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega),$$

$$\mathbf{u}_0 = \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D},$$

$$\theta_0 = \theta_D(0) \text{ on } \Gamma_{\theta,D}.$$

**Remark 4.4.1.** From hypothesis (H1) we can deduce that expression (4.31) defines a symmetric and surjective continuous form in  $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ . Furthermore, since  $\text{meas}(\Gamma_{\mathbf{u},D}) > 0$ ,

$$\forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega) \quad a(\mathbf{v}, \mathbf{v}) \geq a_{min} \|\mathbf{v}\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2, \quad (4.36)$$

i.e. the bilinear form  $a(\cdot, \cdot)$  is  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)$ -elliptic.

**Remark 4.4.2.** Taking into account hypotheses (H2) and (H4), equality (4.32) defines a continuous form and, thanks to Poincaré's inequality in  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$ , the expression of  $\kappa_1$  is an equivalent norm to the usual one of  $H^1(\Omega)$  (see Brezis [25]); therefore, it verifies:

$$\forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega) \quad \kappa_1(\phi, \phi) \geq \frac{k_{min}}{\|\theta_r\|_{L^\infty(\Omega)}} \|\phi\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2. \quad (4.37)$$

**Remark 4.4.3.** Considering assumption (H1), expression (4.34) defines a continuous form in  $L^2(\Omega) \times \mathbf{H}^1(\Omega)$ .

**Remark 4.4.4.** Thanks to hypotheses (H2)-(H3), we can consider  $L^2(\Omega)$  with the scalar product

$$(\phi, \psi)_2 = \int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \phi \psi \, dp,$$

which is equivalent to the usual one and we denote its associated norm as  $\|\cdot\|_2$ .

Using the expressions (4.35a)-(4.35b) and taking into account Remark 4.4.4, we obtain a weak formulation of Problem (P) as follows:

**Problem (VP)**

Find  $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$  such that  $\mathbf{u}(t) = \mathbf{u}_D(t)$  on  $\Gamma_{\mathbf{u},D}$  and  $\theta(t) = \theta_D(t)$  on  $\Gamma_{\theta,D}$  a.e.  $t \in (0, t_f)$ , and verifying:

$$\left\{ \begin{array}{l} a(\mathbf{u}(t), \mathbf{v}) - m(\theta(t) - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \quad (4.38a) \\ (\dot{\theta}(t), \phi)_2 + \kappa(\theta(t), \phi) + m(\phi, \dot{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta(t)}{\theta_r} \phi \, d\Gamma = \int_{\Omega} \frac{f(t)}{\theta_r} \phi \, dp \\ + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi \, d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi \, d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \quad (4.38b) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \text{ in } \Omega. \quad (4.38c) \end{array} \right.$$

#### 4.4.3 Existence of a solution of Problem (VP)

**Theorem 4.4.5.** Under assumptions (H1)-(H13), there exists a solution  $(\mathbf{u}, \theta)$  of Problem (VP) such that

$$\mathbf{u} \in L^\infty(0, t_f; \mathbf{H}^1(\Omega)), \quad \dot{\mathbf{u}} \in L^2(0, t_f; \mathbf{H}^1(\Omega)), \quad \text{and} \quad (4.39)$$

$$\theta \in L^\infty(0, t_f; H^1(\Omega)), \quad \dot{\theta} \in L^2(0, t_f; L^2(\Omega)). \quad (4.40)$$

We prove this result applying the Galerkin's method. The proof, which is divided into five steps, follows the following scheme:

- i) we make a change of variable by translation in order to obtain a problem whose Dirichlet boundary conditions are null,
- ii) we define approximations of the solution using the Galerkin's method, and we prove the existence and uniqueness of the approximated solutions,
- iii) we get some estimations for the Galerkin sequence that let us pass to the limit,
- iv) we prove the convergence of the Galerkin sequence,
- v) and finally, we check the limit of the Galerkin sequence is a solution of Problem (VP) verifying regularity properties (4.39)-(4.40).

**Step I: A variable change by translation**

Assumptions (H7)-(H8) imply the existence of  $\underline{\mathbf{u}}$  and  $\underline{\theta}$  verifying (see Duvaut and Lions [44], p. 125):

$$\underline{\mathbf{u}} \in W^{2,2}(0, t_f; \mathbf{H}^1(\Omega)) \text{ and } \underline{\mathbf{u}} = \mathbf{u}_D \text{ on } \Gamma_{\mathbf{u},D} \times (0, t_f], \quad (4.41)$$

$$\underline{\theta} \in W^{2,2}(0, t_f; H^1(\Omega)) \text{ and } \underline{\theta} = \theta_D \text{ on } \Gamma_{\theta,D} \times (0, t_f]. \quad (4.42)$$

Hence, we deduce that  $\underline{\mathbf{u}} \in C^1([0, t_f]; \mathbf{H}^1(\Omega))$  and  $\underline{\theta} \in C^1([0, t_f]; H^1(\Omega))$ .

We take

$$\tilde{\mathbf{u}} = \mathbf{u} - \underline{\mathbf{u}}, \quad \tilde{\mathbf{u}}_0 = \mathbf{u}_0 - \underline{\mathbf{u}}(0), \quad \tilde{\theta} = \theta - \underline{\theta}, \quad \tilde{\theta}_0 = \theta_0 - \underline{\theta}(0). \quad (4.43)$$

Therefore, with respect to these new unknowns, Problem (VP) can be transformed into an equivalent one:

**Problem** ( $\widetilde{VP}$ )

Find  $(\tilde{\mathbf{u}}(t), \tilde{\theta}(t)) \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega) \times H_{0,\Gamma_{\theta,D}}^1(\Omega)$  a.e.  $t \in (0, t_f)$  verifying:

$$\begin{cases} a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) = \langle L_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), & (4.44a) \\ (\dot{\tilde{\theta}}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \dot{\tilde{\mathbf{u}}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \phi \, d\Gamma = \langle L_{\theta}(t), \phi \rangle_{\theta}, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), & (4.44b) \\ \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0, \quad \tilde{\theta}(0) = \tilde{\theta}_0, & (4.44c) \end{cases}$$

where  $L_{\mathbf{u}}(t)$  and  $L_{\theta}(t)$  are the linear forms defined by:

$$\langle L_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}} = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} \, dp - a(\underline{\mathbf{u}}(t), \mathbf{v}) + m(\underline{\theta}(t) - \theta_r, \mathbf{v}), \quad (4.45)$$

with  $\mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)$  and

$$\begin{aligned} \langle L_{\theta}(t), \phi \rangle_{\theta} &= \int_{\Omega} \frac{f(t)}{\theta_r} \phi \, dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi \, d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi \, d\Gamma - (\dot{\underline{\theta}}(t), \phi)_2 - \kappa(\underline{\theta}(t), \phi) \\ &- m(\phi, \dot{\underline{\mathbf{u}}}(t)) - \int_{\Gamma_{\theta,R}} \frac{\alpha_c \underline{\theta}(t)}{\theta_r} \phi \, d\Gamma, \quad \text{with } \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned} \quad (4.46)$$

Furthermore, taking into account that  $a, m, (\cdot, \cdot)_2, k$  are continuous forms and thanks to hypotheses (H2), (H5), (H6), (H9)-(H11),  $L_{\mathbf{u}}(t)$  and  $L_{\theta}(t)$  are also continuous forms for all  $t \in [0, t_f]$ .

Summing up, it is sufficient to prove the existence of a solution of Problem ( $\widetilde{VP}$ ) satisfying

$$\begin{cases} \tilde{\mathbf{u}} \in L^\infty(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)), \quad \dot{\tilde{\mathbf{u}}} \in L^2(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)), \text{ and} \\ \tilde{\theta} \in L^\infty(0, t_f; H_{0,\Gamma_{\theta,D}}^1(\Omega)), \quad \dot{\tilde{\theta}} \in L^2(0, t_f; L^2(\Omega)). \end{cases} \quad (4.47)$$

**Step II: Existence and uniqueness of a solution of the approximated problem**

First of all we remark that thanks to hypothesis (H13) and to change of variable (4.43), we deduce:

$$\tilde{\mathbf{u}}_0 \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \quad \tilde{\theta}_0 \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \quad a(\tilde{\mathbf{u}}_0, \mathbf{v}) - m(\tilde{\theta}_0, \mathbf{v}) = \langle L_{\mathbf{u}}(0), \mathbf{v} \rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega). \quad (4.48)$$

Since the spaces  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)$  and  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$  are separable, there exist numerable bases  $\{w_1^{\mathbf{u}}, w_2^{\mathbf{u}}, \dots, w_m^{\mathbf{u}}, \dots\}$  and  $\{w_1^{\theta}, w_2^{\theta}, \dots, w_m^{\theta}, \dots\}$ , respectively, such that for all  $m \geq 1$ ,  $\{w_1^{\mathbf{u}}, w_2^{\mathbf{u}}, \dots, w_m^{\mathbf{u}}\}$  and  $\{w_1^{\theta}, w_2^{\theta}, \dots, w_m^{\theta}\}$  are linearly independent, and the finite linear combinations of the  $w_j^{\mathbf{u}}$  and  $w_j^{\theta}$ ,  $j \geq 1$  are dense in  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)$  and  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$ , respectively. In addition, we can choose a base in  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$  verifying:

$$(w_k^{\theta}, w_l^{\theta})_2 = \delta_{kl}, \quad l, k \geq 1,$$

and furthermore,

$$w_1^{\mathbf{u}} = \tilde{\mathbf{u}}_0, \quad w_1^{\theta} = \frac{\tilde{\theta}_0}{\|\tilde{\theta}_0\|_2} \text{ if } \|\tilde{\theta}_0\|_2 \neq 0 \text{ and any other function with unitary norm in other case.}$$

We notice that it is possible to do thanks to (4.43).

We denote by  $\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{\mathbf{u}m} = [w_1^{\mathbf{u}}, w_2^{\mathbf{u}}, \dots, w_m^{\mathbf{u}}]$  and  $H_{0,\Gamma_{\theta,D}}^{\theta m} = [w_1^{\theta}, w_2^{\theta}, \dots, w_m^{\theta}]$  the subspaces generated by  $\{w_1^{\mathbf{u}}, w_2^{\mathbf{u}}, \dots, w_m^{\mathbf{u}}\}$  and  $\{w_1^{\theta}, w_2^{\theta}, \dots, w_m^{\theta}\}$ , respectively.

We consider the following approximated problem:

**Problem** ( $\widetilde{VP}_m$ )

Find  $(\tilde{\mathbf{u}}_m(t), \tilde{\theta}_m(t)) \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{\mathbf{u}m} \times H_{0,\Gamma_{\theta,D}}^{\theta m}$  for all  $t \in (0, t_f)$  verifying:

$$\begin{cases} a(\tilde{\mathbf{u}}_m(t), \mathbf{v}) - m(\tilde{\theta}_m(t), \mathbf{v}) = \langle L_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{\mathbf{u}m}, & (4.49a) \\ (\dot{\tilde{\theta}}_m(t), \phi)_2 + \kappa(\tilde{\theta}_m(t), \phi) + m(\phi, \dot{\tilde{\mathbf{u}}}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_m(t)}{\theta_r} \phi d\Gamma = \langle L_{\theta}(t), \phi \rangle_{\theta}, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^{\theta m}, & (4.49b) \\ \tilde{\mathbf{u}}_m(0) = \tilde{\mathbf{u}}_0, \quad \tilde{\theta}_m(0) = \tilde{\theta}_0. & (4.49c) \end{cases}$$

**Lemma 4.4.6.** *Under assumptions (H1)-(H13) there exists a unique solution  $(\tilde{\mathbf{u}}_m, \tilde{\theta}_m)$  of Problem ( $\widetilde{VP}_m$ ) such that*

$$\tilde{\mathbf{u}}_m \in C^1([0, t_f]; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)) \text{ and } \tilde{\theta}_m \in C^1([0, t_f]; H_{0,\Gamma_{\theta,D}}^1(\Omega)). \quad (4.50)$$

*Proof.* The proof is based on transforming Problem ( $\widetilde{VP}_m$ ) into an equivalent one expressed as a differential system, whose existence and uniqueness is easy to prove. For this purpose, we write  $\tilde{\mathbf{u}}_m$  and  $\tilde{\theta}_m$  as:

$$\begin{aligned} \tilde{\mathbf{u}}_m(t) &= \sum_{i=1}^m \tilde{g}_{im}(t) w_i^{\mathbf{u}}, \quad \tilde{\theta}_m(t) = \sum_{i=1}^m \tilde{h}_{im}(t) w_i^{\theta}, \\ \text{and } \tilde{\mathbf{u}}_0 &= \sum_{i=1}^m \tilde{g}_{im}(0) w_i^{\mathbf{u}} = w_1^{\mathbf{u}}, \quad \tilde{\theta}_0 = \sum_{i=1}^m \tilde{h}_{im}(0) w_i^{\theta} = \|\tilde{\theta}_0\|_2 w_1^{\theta}. \end{aligned}$$

Then, Problem  $(\widetilde{VP}_m)$  is equivalent to the following differential system:

$$\begin{cases} [A_m]^t \{\tilde{g}_m(t)\} - [M_m]^t \{\tilde{h}_m(t)\} = \{L_{\mathbf{u}m}(t)\}, \\ \{\dot{\tilde{h}}_m(t)\} + [K_m]^t \{\tilde{h}_m(t)\} + [M_m] \{\dot{\tilde{g}}_m(t)\} + [H_m]^t \{h_m(t)\} = \{L_{\theta m}(t)\}, \end{cases} \quad (4.51)$$

with the initial condition

$$\{\tilde{g}_m(0)\} = \mathbf{e}_1, \quad \{\tilde{h}_m(0)\} = \|\tilde{\theta}_0\|_2 \mathbf{e}_1, \quad (4.52)$$

where  $\mathbf{e}_1$  is the first vector of the canonic base.

In system (4.51), we have used the following notation:

$$[A_m]_{ij} = a(w_i^{\mathbf{u}}, w_j^{\mathbf{u}}), \quad [M_m]_{ij} = m(w_i^{\theta}, w_j^{\mathbf{u}}), \quad [K_m]_{ij} = \kappa(w_i^{\theta}, w_j^{\theta}), \quad \text{and}$$

$$[H_m]_{ij} = \int_{\Gamma_{\theta, R}} \frac{\alpha_c}{\theta_r} w_i^{\theta} w_j^{\theta} d\Gamma \quad \text{with} \quad 1 \leq i, j \leq m.$$

$$\{\tilde{g}_m(t)\} = (\tilde{g}_{1m}(t), \dots, \tilde{g}_{mm}(t))^t \quad \text{and} \quad \{\tilde{h}_m(t)\} = (\tilde{h}_{1m}(t), \dots, \tilde{h}_{mm}(t))^t.$$

$$\{L_{\mathbf{u}m}(t)\} = (\langle L_{\mathbf{u}}(t), w_1^{\mathbf{u}} \rangle, \dots, \langle L_{\mathbf{u}}(t), w_m^{\mathbf{u}} \rangle)^t \quad \text{and} \quad \{L_{\theta m}(t)\} = (\langle L_{\theta}(t), w_1^{\theta} \rangle, \dots, \langle L_{\theta}(t), w_m^{\theta} \rangle)^t.$$

Deriving with respect to time the first equation of system (4.51), and substituting  $\{\dot{\tilde{g}}_m(t)\}$  in the second equation, we obtain the following equivalent problem:

**Problem**  $(\widetilde{VP}_m)_a$

Find  $\{\tilde{g}_m\}, \{\tilde{h}_m\} \in C^1([0, t_f]; \mathbb{R}^m)$  verifying:

$$\begin{cases} [[I_m] + [M_m][A_m]^{-1}[M_m]^t] \{\dot{\tilde{h}}_m(t)\} + [[K_m]^t + [H_m]^t] \{\tilde{h}_m(t)\} = \{L_m(t)\}, & (4.53a) \\ \{\tilde{g}_m(t)\} = [A_m]^{-1} [\{L_{\mathbf{u}m}(t)\} + [M_m]^t \{\tilde{h}_m(t)\}], & (4.53b) \\ \{\tilde{h}_m(0)\} = \|\tilde{\theta}_0\|_2 \mathbf{e}_1, \quad \{\tilde{g}_m(0)\} = \mathbf{e}_1, & (4.53c) \end{cases}$$

with

$$\{L_m(t)\} = \{L_{\theta m}(t)\} - [M_m][A_m]^{-1} \{\dot{L}_{\mathbf{u}m}(t)\}. \quad (4.54)$$

We deduce of hypothesis (H1) that the matrix  $[A_m]$  is symmetric and positive defined thanks to the coercivity of  $a(\cdot, \cdot)$ . Therefore,  $[A_m]^{-1}$  is also symmetric and positive defined:

$$([A_m]^{-1} \mathbf{p}, \mathbf{p})_{\mathbb{R}^m} \geq \gamma \|\mathbf{p}\|^2, \quad \forall \mathbf{p} \in \mathbb{R}^m, \quad (\gamma > 0). \quad (4.55)$$

Taking into account assumptions (H2), (H5), (H6), (H9)-(H11), properties (4.41)-(4.42) and that  $a, \kappa, m$  and  $(\cdot, \cdot)_2$  are continuous forms thanks to (H1), (H2) and (H4), we deduce

$$L_{\mathbf{u}m} \in C^1([0, t_f]; \mathbb{R}^m) \quad \text{and} \quad L_{\theta m} \in C([0, t_f]; \mathbb{R}^m).$$

So, Problem  $(\widetilde{VP}_m)$  is equivalent to Problem  $(\widetilde{VP}_m)_a$ .

Problem (4.53a) with the first initial condition from (4.53c) admits a unique solution, since  $L_m \in C([0, t_f]; \mathbb{R}^m)$  and the matrix  $[S_m] = [I_m] + [M_m][A_m]^{-1}[M_m]^t$  is invertible. Indeed,  $-1$  is

not a proper value of  $[M_m][A_m]^{-1}[M_m]^t$  because if there existed any  $\mathbf{p} \in \mathbb{R}^m$ ,  $\mathbf{p} \neq \mathbf{0}$ ,  $\|\mathbf{p}\| = 1$  such that  $[M_m][A_m]^{-1}[M_m]^t\mathbf{p} = -\mathbf{p}$ , taking this value of  $\mathbf{p}$  into inequality (4.55), we would obtain:

$$-1 = (-\mathbf{p}, \mathbf{p}) = ([M_m][A_m]^{-1}[M_m]^t\mathbf{p}, \mathbf{p}) = ([A_m]^{-1}[M_m]^t\mathbf{p}, [M_m]^t\mathbf{p}) \geq \gamma\|[M_m]^t\mathbf{p}\|^2 \geq 0,$$

which is a contradiction.

In consequence, there exists a unique  $\{\tilde{h}_m\}$  verifying equation (4.53a) and the initial condition  $\{\tilde{h}_m(0)\} = \|\tilde{\theta}_0\|_2\mathbf{e}_1$ . Now, we can define  $\{\tilde{g}_m\}$  through the relation (4.53b). Since  $\{\tilde{g}_m(0)\} = \mathbf{e}_1$  thanks to property (4.48), we have proved that problem  $(\widehat{VP}_m)_a$  admits a unique solution verifying (4.50). □

### Step III: A priori estimates

Our aim is to obtain the limit of these sequences as  $m \rightarrow \infty$ . For that purpose, it is necessary to deduce some *a priori* estimates independent of  $m$ . From now on,  $c_l$ ,  $l \geq 1$  will denote a positive constant.

**A priori estimates I** Considering Problem  $(\widehat{VP}_m)$  and taking  $\mathbf{v} = \dot{\tilde{\mathbf{u}}}_m(t)$ ,  $\phi = \tilde{\theta}_m(t)$ , we obtain for any  $t \in (0, t_f]$ :

$$\begin{cases} a(\tilde{\mathbf{u}}_m(t), \dot{\tilde{\mathbf{u}}}_m(t)) - m(\tilde{\theta}_m(t), \dot{\tilde{\mathbf{u}}}_m(t)) = \langle L_{\mathbf{u}}(t), \dot{\tilde{\mathbf{u}}}_m(t) \rangle_{\mathbf{u}}, \\ (\dot{\tilde{\theta}}_m(t), \tilde{\theta}_m(t))_2 + \kappa(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + m(\tilde{\theta}_m(t), \dot{\tilde{\mathbf{u}}}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_m^2(t)}{\theta_r} d\Gamma = \langle L_{\theta}(t), \tilde{\theta}_m(t) \rangle_{\theta}. \end{cases}$$

Adding these equations, we have:

$$\begin{aligned} & a(\tilde{\mathbf{u}}_m(t), \dot{\tilde{\mathbf{u}}}_m(t)) + (\dot{\tilde{\theta}}_m(t), \tilde{\theta}_m(t))_2 + \kappa(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_m^2(t)}{\theta_r} d\Gamma \\ & = \langle L_{\mathbf{u}}(t), \dot{\tilde{\mathbf{u}}}_m(t) \rangle_{\mathbf{u}} + \langle L_{\theta}(t), \tilde{\theta}_m(t) \rangle_{\theta}. \end{aligned}$$

Taking into account definitions (4.31) and (4.33), the previous equality can be rewritten as follows:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [a(\tilde{\mathbf{u}}_m(t), \tilde{\mathbf{u}}_m(t))] + \frac{1}{2} \frac{d}{dt} \|\tilde{\theta}_m(t)\|_2^2 + \kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + \kappa_2(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_m^2(t)}{\theta_r} d\Gamma \\ & = \frac{d}{dt} [\langle L_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t) \rangle_{\mathbf{u}}] - \langle \dot{L}_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t) \rangle_{\mathbf{u}} + \langle L_{\theta}(t), \tilde{\theta}_m(t) \rangle_{\theta}. \end{aligned}$$

Integrating over  $(0, t)$ , we arrive at

$$\begin{aligned} & \int_0^t \frac{1}{2} \frac{d}{ds} [a(\tilde{\mathbf{u}}_m(s), \tilde{\mathbf{u}}_m(s))] ds + \int_0^t \frac{1}{2} \frac{d}{ds} \|\tilde{\theta}_m(s)\|_2^2 ds + \int_0^t \kappa_1(\tilde{\theta}_m(s), \tilde{\theta}_m(s)) ds \\ & + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_m^2(s)}{\theta_r} d\Gamma ds = - \int_0^t \kappa_2(\tilde{\theta}_m(s), \tilde{\theta}_m(s)) ds + \int_0^t \frac{d}{ds} [\langle L_{\mathbf{u}}(s), \tilde{\mathbf{u}}_m(s) \rangle_{\mathbf{u}}] ds \\ & - \int_0^t \langle \dot{L}_{\mathbf{u}}(s), \tilde{\mathbf{u}}_m(s) \rangle_{\mathbf{u}} ds + \int_0^t \langle L_{\theta}(s), \tilde{\theta}_m(s) \rangle_{\theta} ds; \end{aligned}$$

and hence we can deduce,

$$\begin{aligned}
& \frac{1}{2}a(\tilde{\mathbf{u}}_m(t), \tilde{\mathbf{u}}_m(t)) + \frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}_m(s), \tilde{\theta}_m(s))ds + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c}{\theta_r} \tilde{\theta}_m^2(s) d\Gamma ds \\
&= \frac{1}{2}a(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_0) + \frac{1}{2}\|\tilde{\theta}_0\|_2^2 - \int_0^t \kappa_2(\tilde{\theta}_m(s), \tilde{\theta}_m(s)) ds + \langle L_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t) \rangle_{\mathbf{u}} - \langle L_{\mathbf{u}}(0), \tilde{\mathbf{u}}_0 \rangle_{\mathbf{u}} \\
&- \int_0^t \langle \dot{L}_{\mathbf{u}}(s), \tilde{\mathbf{u}}_m(s) \rangle_{\mathbf{u}} ds - \int_0^t \langle L_{\theta}(s), \tilde{\theta}_m(s) \rangle_{\theta} ds. \tag{4.56}
\end{aligned}$$

Since  $a$ ,  $L_{\mathbf{u}}$ ,  $L_{\theta}$ ,  $\dot{L}_{\mathbf{u}}$  are continuous and the terms on the left-hand side of equality (4.56) are all non negative, taking into account hypotheses (H2), (H4), (H10), Hölder's inequality and definition of  $\kappa_2$  from expression (4.33), we obtain:

$$\begin{aligned}
& \frac{1}{2}a(\tilde{\mathbf{u}}_m(t), \tilde{\mathbf{u}}_m(t)) + \frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}_m(s), \tilde{\theta}_m(s))ds + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \\
&\leq \frac{1}{2}a_{max}\|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}^2 + \frac{1}{2}\|\tilde{\theta}_0\|_2^2 + \frac{\|k\|_{L^\infty(\Omega)}\|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\nabla\tilde{\theta}_m(s)\|_{\mathbf{L}^2(\Omega)}\|\tilde{\theta}_m(s)\|_{\mathbf{L}^2(\Omega)}ds \\
&+ \|L_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}\|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)} + \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}\|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)} \\
&+ \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}\|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}ds + \int_0^t \|L_{\theta}(s)\|_{H_{0,\Gamma_{\theta},D}^{1'}(\Omega)}\|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta},D}^1(\Omega)}ds, \tag{4.57}
\end{aligned}$$

$a_{max}$  being the constant of continuity of  $a(\cdot, \cdot)$ . Using inequality

$$2\sqrt{\alpha}\sqrt{\beta}ab \leq \alpha a^2 + \beta b^2, \quad \text{where } \alpha, \beta > 0, \tag{4.58}$$

with  $\alpha = \frac{1}{2}$  and  $\beta = \frac{1}{2}$  for the fifth term on the right-hand side of inequality (4.57), we have

$$\begin{aligned}
& \frac{1}{2}a(\tilde{\mathbf{u}}_m(t), \tilde{\mathbf{u}}_m(t)) + \frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}_m(s), \tilde{\theta}_m(s))ds + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \\
&\leq \frac{1}{2} \left[ (a_{max} + 1)\|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}^2 + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}^2 \right] \\
&+ \frac{\|k\|_{L^\infty(\Omega)}\|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\nabla\tilde{\theta}_m(s)\|_{\mathbf{L}^2(\Omega)}\|\tilde{\theta}_m(s)\|_{\mathbf{L}^2(\Omega)}ds + \|L_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}\|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)} \\
&+ \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}\|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}ds + \int_0^t \|L_{\theta}(s)\|_{H_{0,\Gamma_{\theta},D}^{1'}(\Omega)}\|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta},D}^1(\Omega)}ds. \tag{4.59}
\end{aligned}$$

Taking into account properties (4.36) and (4.37), we rewrite expression (4.59) in the following form:

$$\begin{aligned}
& \frac{a_{min}}{2} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \frac{1}{2} \|\tilde{\theta}_m(t)\|_2^2 + \frac{k_{min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds \\
& + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \leq \frac{1}{2} \left[ (a_{max} + 1) \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 \right] \\
& + \frac{\|k\|_{L^\infty(\Omega)} \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)} \|\tilde{\theta}_m(s)\|_{L^2(\Omega)} ds \\
& + \|L_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)} + \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)} \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)} ds \\
& + \int_0^t \|L_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)} \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)} ds. \tag{4.60}
\end{aligned}$$

Now, we apply again inequality (4.58) to the following terms on the right-hand side of inequality (4.60):

- the second term with  $\alpha = \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}}$  and  $\beta = \frac{\|k\|_{L^\infty(\Omega)} \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)} \|\theta_r\|_{L^\infty(\Omega)}}{\theta_{r,min}^4 k_{min}}$ ,
- the third term with  $\alpha = \frac{1}{a_{min}}$  and  $\beta = \frac{a_{min}}{4}$ ,
- the fourth term with  $\alpha = \frac{1}{4}$  and  $\beta = 1$  and
- the fifth term with  $\alpha = \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}}$  and  $\beta = \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}}$ .

So, we get

$$\begin{aligned}
& \frac{a_{min}}{2} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \frac{1}{2} \|\tilde{\theta}_m(t)\|_2^2 + \frac{k_{min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds \\
& + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \leq \frac{1}{2} \left[ (a_{max} + 1) \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 \right] \\
& + \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds + \frac{\|k\|_{L^\infty(\Omega)} \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)} \|\theta_r\|_{L^\infty(\Omega)}}{\theta_{r,min}^4 k_{min}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Omega)}^2 ds \\
& + \frac{1}{a_{min}} \|L_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 + \frac{a_{min}}{4} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \frac{1}{4} \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 ds \\
& + \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds \\
& + \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}} \int_0^t \|L_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 ds. \tag{4.61}
\end{aligned}$$

From equality

$$L_{\mathbf{u}}(t) = L_{\mathbf{u}}(0) + \int_0^t \dot{L}_{\mathbf{u}}(s) ds,$$

and applying inequality (4.58), we deduce:

$$\begin{aligned} \|L_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 &= \left\| L_{\mathbf{u}}(0) + \int_0^t \dot{L}_{\mathbf{u}}(s) ds \right\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 \\ &\leq 2\|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 + 2\left\| \int_0^t \dot{L}_{\mathbf{u}}(s) ds \right\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 \\ &\leq 2\|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 + 2t_f \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 ds. \end{aligned} \quad (4.62)$$

Then, thanks to the previous expression, inequality (4.61) can be rewritten as follows:

$$\begin{aligned} &\frac{a_{min}}{4} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1}^2 + \frac{1}{2} \|\tilde{\theta}_m(t)\|_2^2 + \frac{k_{min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta},D}^1}^2 ds \\ &+ \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \leq \frac{1}{2} \left[ (a_{max} + 1) \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1}^2 + \|\tilde{\theta}_0\|_2^2 \right] \\ &+ \left( \frac{a_{min} + 4}{2a_{min}} \right) \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 + \frac{\|k\|_{L^\infty(\Omega)}^2 \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)}}{\theta_{r,min}^4 k_{min}} \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Omega)}^2 ds \\ &+ \left( \frac{a_{min} + 8t_f}{4a_{min}} \right) \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 ds + \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1}^2 ds \\ &+ \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}} \int_0^t \|L_\theta(s)\|_{H_{0,\Gamma_{\theta},D}^{1'}}^2 ds. \end{aligned}$$

Summing up and applying Remark 4.4.4 to the second term on the left-hand side of the previous inequality, we have

$$\begin{aligned} &c_1 \left( \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1}^2 + \|\tilde{\theta}_m(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta},D}^1}^2 ds + \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \right) \\ &\leq c_2 \left[ \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1}^2 + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 \right] \\ &+ c_3 \left[ \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1}^2 ds + \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Omega)}^2 ds \right] \\ &+ c_4 \left[ \int_0^t \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}}^2 ds + \int_0^t \|L_\theta(s)\|_{H_{0,\Gamma_{\theta},D}^{1'}}^2 ds \right], \end{aligned} \quad (4.63)$$

with

$$\begin{aligned} c_1 &= \min \left\{ \frac{a_{min}}{4}, \frac{\rho_0 CF}{2\|\theta_r\|_{L^\infty(\Omega)}}, \frac{k_{min}}{2\|\theta_r\|_{L^\infty(\Omega)}}, \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \right\}, \quad c_2 = \max \left\{ \frac{a_{max} + 1}{2}, \frac{a_{min} + 4}{2a_{min}} \right\}, \\ c_3 &= \max \left\{ \frac{\|k\|_{L^\infty(\Omega)}^2 \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)}}{\theta_{r,min}^4 k_{min}}, 1 \right\}, \quad c_4 = \max \left\{ \frac{a_{min} + 8t_f}{4a_{min}}, \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}} \right\}. \end{aligned}$$

We introduce the notation

$$\begin{aligned} |||L|||^2 &= \int_0^{t_f} \left( \|\dot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}^2 + \|L_{\theta}(s)\|_{H_{0,\Gamma_{\theta},D}^{1'}(\Omega)}^2 \right) ds, \\ \Phi_m(t) &= \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}^2 + \|\tilde{\theta}_m(t)\|_{L^2(\Omega)}^2, \end{aligned}$$

and we rewrite expression (4.63) as follows:

$$\begin{aligned} \Phi_m(t) &+ \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta},D}^1(\Omega)}^2 ds + \int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \\ &\leq \frac{c_2}{c_1} \left[ \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}^2 + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{1'}(\Omega)}^2 \right] + \frac{c_3}{c_1} \int_0^t \Phi_m(s) ds + \frac{c_4}{c_1} |||L|||^2. \end{aligned} \quad (4.64)$$

Since  $\int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta},D}^1(\Omega)}^2 ds \geq 0$ ,  $\int_0^t \|\tilde{\theta}_m(s)\|_{L^2(\Gamma_{\theta,R})}^2 ds \geq 0$ ,  $\tilde{\mathbf{u}}_0$ ,  $\tilde{\theta}_0$  and  $|||L|||^2$  are known, we deduce:

$$\Phi_m(t) \leq c_5 + \frac{c_3}{c_1} \int_0^t \Phi_m(s) ds.$$

In consequence, thanks to Gronwall's lemma we obtain

$$\Phi_m(t) \leq c_5 e^{\frac{c_3}{c_1} t}, \quad \text{with } t \in [0, t_f].$$

Therefore, there exists  $M_{1,\mathbf{u},\theta}$  independent of  $m$ , such that

$$\|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)}^2 \leq M_{1,\mathbf{u},\theta}, \quad \|\tilde{\theta}_m(t)\|_{L^2(\Omega)}^2 \leq M_{1,\mathbf{u},\theta}, \quad 0 \leq t \leq t_f. \quad (4.65)$$

We can now establish the following result:

**Corollary 4.4.7.** a) The sequence  $\{\tilde{\mathbf{u}}_m\}$  is bounded in  $L^\infty(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega))$ .

b) The sequence  $\{\tilde{\theta}_m\}$  is bounded in  $L^\infty(0, t_f; L^2(\Omega))$  and  $L^2(0, t_f; H_{0,\Gamma_{\theta},D}^1(\Omega))$ .

*Proof.* The bounds of  $\{\tilde{\mathbf{u}}_m\}$  in  $L^\infty(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega))$  and  $\{\tilde{\theta}_m\}$  in  $L^\infty(0, t_f; L^2(\Omega))$  are obtained directly from (4.65). Taking into account these bounds in inequality (4.64), we can also deduce that  $\{\tilde{\theta}_m\}$  is bounded in  $L^2(0, t_f; H_{0,\Gamma_{\theta},D}^1(\Omega))$ .  $\square$

**A priori estimates II** Thanks to Lemma 4.4.6 we can derive in time the first equation of Problem  $(\widetilde{VP}_m)$  and we obtain:

$$\begin{cases} a(\dot{\tilde{\mathbf{u}}}_m(t), \mathbf{v}) - m(\dot{\tilde{\theta}}_m(t), \mathbf{v}) = \langle \dot{L}_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}}, & \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{\mathbf{u}m}, \\ (\dot{\tilde{\theta}}_m(t), \phi)_2 + \kappa(\tilde{\theta}_m(t), \phi) + m(\phi, \dot{\tilde{\mathbf{u}}}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_m(t)}{\theta_r} \phi d\Gamma = \langle L_{\theta}(t), \phi \rangle_{\theta}, & \forall \phi \in H_{0,\Gamma_{\theta},D}^{\theta m}. \end{cases}$$

Taking  $\mathbf{v} = \dot{\tilde{\mathbf{u}}}_m(t)$ ,  $\phi = \dot{\tilde{\theta}}_m(t)$  and adding the previous equations, we get for any  $t \in (0, t_f]$ :

$$\begin{aligned} & a(\dot{\tilde{\mathbf{u}}}_m(t), \dot{\tilde{\mathbf{u}}}_m(t)) + (\dot{\tilde{\theta}}_m(t), \dot{\tilde{\theta}}_m(t))_2 + \kappa(\tilde{\theta}_m(t), \dot{\tilde{\theta}}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c}{\theta_r} \tilde{\theta}_m(t) \dot{\tilde{\theta}}_m(t) d\Gamma \\ & = \langle \dot{L}_{\mathbf{u}}(t), \dot{\tilde{\mathbf{u}}}_m(t) \rangle_{\mathbf{u}} + \langle L_{\theta}(t), \dot{\tilde{\theta}}_m(t) \rangle_{\theta} . \end{aligned}$$

Applying definition (4.33), the above equality can be rewritten as follows:

$$\begin{aligned} & a(\dot{\tilde{\mathbf{u}}}_m(t), \dot{\tilde{\mathbf{u}}}_m(t)) + \|\dot{\tilde{\theta}}_m(t)\|_2^2 + \frac{1}{2} \frac{d}{dt} [\kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t))] + \frac{d}{dt} \left[ \int_{\Gamma_{\theta,R}} \frac{\alpha_c}{2\theta_r} \tilde{\theta}_m^2(t) d\Gamma \right] \\ & = -\kappa_2(\tilde{\theta}_m(t), \dot{\tilde{\theta}}_m(t)) + \frac{d}{dt} [\langle \dot{L}_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t) \rangle_{\mathbf{u}}] - \langle \ddot{L}_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t) \rangle_{\mathbf{u}} \\ & + \frac{d}{dt} [\langle L_{\theta}(t), \tilde{\theta}_m(t) \rangle_{\theta}] - \langle \dot{L}_{\theta}(t), \tilde{\theta}_m(t) \rangle_{\theta} . \end{aligned}$$

Integrating over  $(0, t)$ , we have

$$\begin{aligned} & \int_0^t a(\dot{\tilde{\mathbf{u}}}_m(s), \dot{\tilde{\mathbf{u}}}_m(s)) ds + \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_2^2 ds + \frac{1}{2} \kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c}{2\theta_r} \tilde{\theta}_m^2(t) d\Gamma \\ & = \frac{1}{2} \kappa_1(\tilde{\theta}_0, \tilde{\theta}_0) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c}{2\theta_r} \tilde{\theta}_0^2 d\Gamma - \int_0^t \kappa_2(\tilde{\theta}_m(s), \dot{\tilde{\theta}}_m(s)) ds + \langle \dot{L}_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t) \rangle_{\mathbf{u}} \\ & - \langle \dot{L}_{\mathbf{u}}(0), \tilde{\mathbf{u}}_0 \rangle_{\mathbf{u}} - \int_0^t \langle \ddot{L}_{\mathbf{u}}(s), \tilde{\mathbf{u}}_m(s) \rangle_{\mathbf{u}} ds + \langle L_{\theta}(t), \tilde{\theta}_m(t) \rangle_{\theta} - \langle L_{\theta}(0), \tilde{\theta}_0 \rangle_{\theta} \\ & - \int_0^t \langle \dot{L}_{\theta}(s), \tilde{\theta}_m(s) \rangle_{\theta} ds . \end{aligned}$$

Thanks to hypotheses (H1)-(H12), all terms on the left-hand side of the previous expression are all non negative and  $a$ ,  $L_{\mathbf{u}}$ ,  $\dot{L}_{\mathbf{u}}$ ,  $\ddot{L}_{\mathbf{u}}$ ,  $L_{\theta}$ ,  $\dot{L}_{\theta}$  are continuous. So, taking into account Remark 4.4.2 and applying Hölder's inequality, we get

$$\begin{aligned} & \int_0^t a(\dot{\tilde{\mathbf{u}}}_m(s), \dot{\tilde{\mathbf{u}}}_m(s)) ds + \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_2^2 ds + \frac{1}{2} \kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \\ & \leq \frac{\|k\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\ & + \frac{\|k\|_{L^\infty(\Omega)} \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\nabla\tilde{\theta}_m(s)\|_{\mathbf{L}^2(\Omega)} \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)} ds + \|\dot{L}_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)} \\ & + \|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)} + \int_0^t \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)} \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)} ds \\ & + \|L_{\theta}(t)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)} \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)} + \|L_{\theta}(0)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)} \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)} \\ & + \int_0^t \|\dot{L}_{\theta}(s)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)} \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)} ds . \end{aligned}$$

Taking into account Remark 4.4.4, this inequality can be rewritten as follows

$$\begin{aligned}
& \int_0^t a(\dot{\tilde{\mathbf{u}}}_m(s), \dot{\tilde{\mathbf{u}}}_m(s)) ds + \frac{\rho_0 c_F}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2} \kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) \\
& + \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \frac{\|k\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
& + \frac{\|k\|_{L^\infty(\Omega)} \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)} ds \\
& + \|\dot{\mathbf{L}}_{\mathbf{u}}(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\dot{\mathbf{L}}_{\mathbf{u}}(0)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \\
& + \int_0^t \|\ddot{\mathbf{L}}_{\mathbf{u}}(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} ds + \|L_\theta(t)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_m(t)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \\
& + \|L_\theta(0)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} + \int_0^t \|\dot{L}_\theta(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} ds.
\end{aligned}$$

Using inequality (4.58), with  $\alpha = 1$  and  $\beta = \frac{1}{4}$ , to the fourth, fifth, sixth, eighth and ninth terms on the right-hand side of the above equation, we arrive at

$$\begin{aligned}
& \int_0^t a(\dot{\tilde{\mathbf{u}}}_m(s), \dot{\tilde{\mathbf{u}}}_m(s)) ds + \frac{\rho_0 c_F}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2} \kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) \\
& + \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \frac{\|k\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
& + \frac{\|k\|_{L^\infty(\Omega)} \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)} ds + \|\dot{\mathbf{L}}_{\mathbf{u}}(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 \\
& + \frac{1}{4} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 + \|\dot{\mathbf{L}}_{\mathbf{u}}(0)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 + \frac{1}{4} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 + \int_0^t \|\ddot{\mathbf{L}}_{\mathbf{u}}(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 ds \\
& + \frac{1}{4} \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 ds + \|L_\theta(t)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_m(t)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} + \|L_\theta(0)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 \\
& + \frac{1}{4} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 + \int_0^t \|\dot{L}_\theta(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 ds + \frac{1}{4} \int_0^t \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 ds. \tag{4.66}
\end{aligned}$$

Considering properties (4.36) and (4.37), and applying again inequality (4.58) to the following terms on the right-hand side of expression (4.66):

- the third term with  $\alpha = \frac{\|k\|_{L^\infty(\Omega)} \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)} \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 \rho_0 c_F}$  and  $\beta = \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}}$ ,
- the tenth term with  $\alpha = \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}}$  and  $\beta = \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}}$ ,

we can rewrite inequality (4.66) as:

$$\begin{aligned}
& a_{min} \int_0^t \|\dot{\tilde{\mathbf{u}}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \frac{\rho_0 c_F}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{k_{min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 \\
& + \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \left( \frac{2\|k\|_{L^\infty(\Omega)} + \theta_{r,min}}{4\theta_{r,min}} \right) \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
& + \frac{\|k\|_{L^\infty(\Omega)}^2 \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)} \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 \rho_0 c_F} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds + \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds \\
& + \|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 + \frac{1}{4} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \|L_\theta(0)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 + \|\dot{L}_{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 \\
& + \frac{1}{4} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \int_0^t \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 ds + \frac{1}{4} \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds \\
& + \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}} \|L_\theta(t)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 + \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \int_0^t \|\dot{L}_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 ds \\
& + \frac{1}{4} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds. \tag{4.67}
\end{aligned}$$

From equalities

$$\dot{L}_{\mathbf{u}}(t) = \dot{L}_{\mathbf{u}}(0) + \int_0^t \ddot{L}_{\mathbf{u}}(s) ds, \quad L_\theta(t) = L_\theta(0) + \int_0^t \dot{L}_\theta(s) ds,$$

and applying on these equalities the same reasoning used in inequality (4.62), expression (4.67) can be rewritten as follows:

$$\begin{aligned}
& a_{min} \int_0^t \|\dot{\tilde{\mathbf{u}}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 \\
& + \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \left( \frac{2\|k\|_{L^\infty(\Omega)} + \theta_{r,min}}{4\theta_{r,min}} \right) \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
& + \frac{\|k\|_{L^\infty(\Omega)}^2 \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)} \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 \rho_0 c_F} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds + \|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 + \frac{1}{4} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 \\
& + \|L_\theta(0)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 + 2\|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 + 2t_f \int_0^t \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 ds + \frac{1}{4} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 \\
& + \int_0^t \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 ds + \frac{1}{4} \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \frac{2\|\theta_r\|_{L^\infty(\Omega)}}{k_{min}} \|L_\theta(0)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 \\
& + \frac{2t_f \|\theta_r\|_{L^\infty(\Omega)}}{k_{min}} \int_0^t \|\dot{L}_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 ds + \int_0^t \|\dot{L}_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 ds + \frac{1}{4} \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds.
\end{aligned}$$

And rearranging the terms, we get

$$\begin{aligned}
& a_{min} \int_0^t \|\dot{\tilde{\mathbf{u}}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 \\
& + \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \left( \frac{2\|k\|_{L^\infty(\Omega)} + \theta_{r,min}}{4\theta_{r,min}} \right) \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
& + \left( \frac{2\|k\|_{L^\infty(\Omega)}^2 \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)} + \theta_{r,min}^4 \rho_0 c_F}{4\theta_{r,min}^4 \rho_0 c_F} \right) \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds + 3\|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1\nu}(\Omega)}^2 \\
& + \frac{1}{4} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \left( \frac{2\|\theta_r\|_{L^\infty(\Omega)} + k_{min}}{k_{min}} \right) \|L_\theta(0)\|_{H_{0,\Gamma_{\theta,D}}^{1\nu}(\Omega)}^2 + (2t_f + 1) \int_0^t \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1\nu}(\Omega)}^2 ds \\
& + \frac{1}{4} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \frac{1}{4} \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds \\
& + \left( \frac{2t_f \|\theta_r\|_{L^\infty(\Omega)} + k_{min}}{k_{min}} \right) \int_0^t \|\dot{L}_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1\nu}(\Omega)}^2 ds.
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
& c_6 \left[ \int_0^t \|\dot{\tilde{\mathbf{u}}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \right] \\
& \leq c_7 \left[ \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 + \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1\nu}(\Omega)}^2 + \|L_\theta(0)\|_{H_{0,\Gamma_{\theta,D}}^{1\nu}(\Omega)}^2 \right] \\
& + c_8 \left[ \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds + \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds \right. \\
& \quad \left. + \int_0^t \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1\nu}(\Omega)}^2 ds + \int_0^t \|\dot{L}_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1\nu}(\Omega)}^2 ds \right],
\end{aligned}$$

with

$$\begin{aligned}
c_6 &= \min \left\{ a_{min}, \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}}, \frac{k_{min}}{4\|\theta_r\|_{L^\infty(\Omega)}}, \frac{\alpha_{c,min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \right\} \\
c_7 &= \max \left\{ \frac{2\|k\|_{L^\infty(\Omega)} + \theta_{r,min}}{4\theta_{r,min}}, \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,min}}, 3, \frac{2\|\theta_r\|_{L^\infty(\Omega)} + k_{min}}{k_{min}} \right\}
\end{aligned}$$

and

$$c_8 = \max \left\{ \frac{2\|k\|_{L^\infty(\Omega)}^2 \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)} + \theta_{r,min}^4 \rho_0 c_F}{4\theta_{r,min}^4 \rho_0 c_F}, 2t_f + 1, \frac{2t_f \|\theta_r\|_{L^\infty(\Omega)} + k_{min}}{k_{min}} \right\}.$$

If we introduce the notation

$$\|\dot{L}\|^2 = \int_0^{t_f} \left( \|\ddot{L}_{\mathbf{u}}(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1\nu}(\Omega)}^2 + \|\dot{L}_\theta(s)\|_{H_{0,\Gamma_{\theta,D}}^{1\nu}(\Omega)}^2 \right) ds,$$

we can rewrite the previous inequality as follows:

$$\begin{aligned}
& c_6 \left[ \int_0^t \|\dot{\tilde{\mathbf{u}}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds + \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds + \|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \right] \\
& \leq c_9 \left[ \|\tilde{\theta}_0\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 + \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 + \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \|\dot{L}_{\mathbf{u}}(0)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^{1'}(\Omega)}^2 + \|L_{\theta}(0)\|_{H_{0,\Gamma_{\theta,D}}^{1'}(\Omega)}^2 \right. \\
& \quad \left. + \|\dot{L}\|^2 \right] + c_8 \left[ \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 + \int_0^t \|\tilde{\theta}_m(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds + \int_0^t \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds \right], \quad (4.68)
\end{aligned}$$

where

$$c_9 = \max \{c_7, c_8\}.$$

Thanks to Gronwall's lemma and Corollary 4.4.7, we can deduce from expression (4.68), the existence of a constant  $M_{2,\mathbf{u},\theta}$  independent of  $m$ , such that:

$$\begin{aligned}
\int_0^t \|\dot{\tilde{\mathbf{u}}}_m(s)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)}^2 ds & \leq M_{2,\mathbf{u},\theta}, & \int_0^t \|\dot{\tilde{\theta}}_m(s)\|_{L^2(\Omega)}^2 ds & \leq M_{2,\mathbf{u},\theta}, \\
\|\tilde{\theta}_m(t)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 & \leq M_{2,\mathbf{u},\theta}, & a.e. \ t \in (0, t_f]. &
\end{aligned}$$

Finally, we conclude:

**Corollary 4.4.8.** a) The sequence  $\{\dot{\tilde{\mathbf{u}}}_m\}$  is bounded in  $L^2(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega))$ .

b) The sequence  $\{\dot{\tilde{\theta}}_m\}$  is bounded in  $L^2(0, t_f; L^2(\Omega))$ .

c) The sequence  $\{\tilde{\theta}_m\}$  is bounded in  $L^\infty(0, t_f; H_{0,\Gamma_{\theta,D}}^1(\Omega))$ .

#### Step IV: Passage to the limit

Taking into account some well-known results of compactness, we deduce the following result from Corollaries 4.4.7 and 4.4.8:

**Corollary 4.4.9.** Under assumptions (H1)-(H13) there exist

$$\begin{aligned}
\tilde{\mathbf{u}} & \in W^{1,2}(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)) \cap L^\infty(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)), \\
\tilde{\theta} & \in W^{1,2}(0, t_f; L^2(\Omega)) \cap L^\infty(0, t_f; H_{0,\Gamma_{\theta,D}}^1(\Omega)),
\end{aligned}$$

and subsequences (again indexed with  $m$ ) such that, as  $m \rightarrow \infty$ :

$$\begin{aligned}
\tilde{\mathbf{u}}_m & \rightharpoonup \tilde{\mathbf{u}} \text{ in } L^\infty(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)) \text{ weak-star,} \\
\dot{\tilde{\mathbf{u}}}_m & \rightharpoonup \dot{\tilde{\mathbf{u}}} \text{ in } L^2(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)) \text{ weak,} \\
\tilde{\theta}_m & \rightharpoonup \tilde{\theta} \text{ in } L^\infty(0, t_f; H_{0,\Gamma_{\theta,D}}^1(\Omega)) \text{ weak-star,} \\
\dot{\tilde{\theta}}_m & \rightharpoonup \dot{\tilde{\theta}} \text{ in } L^2(0, t_f; L^2(\Omega)) \text{ weak.}
\end{aligned}$$

**Step V: Verifying that  $(\tilde{\mathbf{u}}, \tilde{\theta})$  is a solution of Problem  $(\widetilde{VP})$** 

As consequence of Corollary 4.4.9,  $(\tilde{\mathbf{u}}, \tilde{\theta})$  satisfies properties (4.47).

In order to complete the proof of the existence of a solution, it is necessary to prove that  $(\tilde{\mathbf{u}}, \tilde{\theta})$  verifies Problem  $(\widetilde{VP})$ . For this purpose we use the methodology from Viaño [98].

Let  $j$  be an arbitrary non negative fixed integer, and  $l > j$ . Then,  $(\tilde{\mathbf{u}}_l, \tilde{\theta}_l)$  is a solution of Problem  $(\widetilde{VP}_l)$ , with  $m = l$ . We are going to prove that we can pass to the limit in order to obtain that  $(\tilde{\mathbf{u}}, \tilde{\theta})$  is a solution of Problem  $(\widetilde{VP})$ .

**Verifying the weak formulation (4.44a) for the mechanical submodel.**

In equation (4.49a), considering the test function  $\mathbf{v}_j(t) \in \mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{\mathbf{u}j} \subset \mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^{\mathbf{u}l}$ :

$$\mathbf{v}_j(t) = \xi(t)w_j^{\mathbf{u}}, \quad \xi \in C^1(0, t_f), \quad \xi(t_f) = 0, \quad (4.69)$$

and integrating over  $(0, t_f)$ , we obtain

$$\int_0^{t_f} \left[ a(\tilde{\mathbf{u}}_l(t), \mathbf{v}_j(t)) - m(\tilde{\theta}_l(t), \mathbf{v}_j(t)) - \langle L_{\mathbf{u}}(t), \mathbf{v}_j(t) \rangle_{\mathbf{u}} \right] dt = 0, \quad \text{for all } l > j. \quad (4.70)$$

Thanks to Corollary 4.4.9, we can pass to the limit as  $l \rightarrow \infty$ , and we have

$$\int_0^{t_f} \left[ a(\tilde{\mathbf{u}}(t), w_j^{\mathbf{u}}) - m(\tilde{\theta}(t), w_j^{\mathbf{u}}) - \langle L_{\mathbf{u}}(t), w_j^{\mathbf{u}} \rangle_{\mathbf{u}} \right] \xi(t) dt = 0, \quad (4.71)$$

for all  $\xi \in C^1(0, t_f)$ ,  $\xi(t_f) = 0$ . In particular, we have proved that for all  $\xi \in D(0, t_f)$ ,

$$a(\tilde{\mathbf{u}}(t), w_j^{\mathbf{u}}) - m(\tilde{\theta}(t), w_j^{\mathbf{u}}) = \langle L_{\mathbf{u}}(t), w_j^{\mathbf{u}} \rangle_{\mathbf{u}}, \quad (4.72)$$

in the sense of distributions on  $(0, t_f)$ .

Taking all non negative integers  $j$ , and using the density of the finite linear combinations of  $w_j^{\mathbf{u}}$  in  $\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)$ , we deduce

$$a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) = \langle L_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}}, \quad (4.73)$$

for all  $\mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)$  in the sense of distributions on  $t \in (0, t_f)$ .

**Verifying the weak formulation (4.44b) for the thermal submodel.**

In equation (4.49b) taking the test function  $\phi_j(t) \in H_{0,\Gamma_{\theta},D}^{\theta j} \subset H_{0,\Gamma_{\theta},D}^{\theta l}$ ,  $l > j$ :

$$\phi_j(t) = \zeta(t)w_j^{\theta}, \quad \zeta \in C^1(0, t_f), \quad \zeta(t_f) = 0, \quad (4.74)$$

and integrating over  $(0, t_f)$ , we obtain

$$\int_0^{t_f} \left[ (\dot{\tilde{\theta}}_l(t), \phi_j(t))_2 + \kappa(\tilde{\theta}_l(t), \phi_j(t)) + m(\phi_j(t), \dot{\tilde{\mathbf{u}}}_l(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_l(t)}{\theta_r} \phi_j(t) d\Gamma - \langle L_{\theta}(t), \phi_j(t) \rangle_{\theta} \right] dt = 0, \quad \text{for all } l > j. \quad (4.75)$$

Taking into account equality

$$(\dot{\tilde{\theta}}_l(t), \phi_j(t))_2 = \frac{d}{dt}(\tilde{\theta}_l(t), \phi_j(t))_2 - (\tilde{\theta}_l(t), \dot{\phi}_j(t))_2, \quad (4.76)$$

and since  $\phi_j(t_f) = 0$ , expression (4.75) can be rewritten as:

$$\begin{aligned} & \int_0^{t_f} \left[ -(\tilde{\theta}_l(t), \dot{\phi}_j(t))_2 + \kappa(\tilde{\theta}_l(t), \phi_j(t)) + m(\phi_j(t), \dot{\mathbf{u}}_l(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_l(t)}{\theta_r} \phi_j(t) d\Gamma \right. \\ & \left. - \langle L_\theta(t), \phi_j(t) \rangle_\theta \right] dt = (\tilde{\theta}_l(0), \phi_j(0))_2, \quad \text{for all } l > j. \end{aligned} \quad (4.77)$$

Applying the initial condition for  $\tilde{\theta}_l$  (4.49c), we have

$$\begin{aligned} & \int_0^{t_f} \left[ -(\tilde{\theta}_l(t), \dot{\phi}_j(t))_2 + \kappa(\tilde{\theta}_l(t), \phi_j(t)) + m(\phi_j(t), \dot{\mathbf{u}}_l(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}_l(t)}{\theta_r} \phi_j(t) d\Gamma \right. \\ & \left. - \langle L_\theta(t), \phi_j(t) \rangle_\theta \right] dt = (\tilde{\theta}_0, \phi_j(0))_2, \quad \text{for } l > j, \end{aligned} \quad (4.78)$$

for all  $\phi_j$  given by expression (4.74). So, thanks to Corollary 4.4.9, we can pass to the limit as  $l \rightarrow \infty$ , and we get

$$\begin{aligned} & \int_0^{t_f} -(\tilde{\theta}(t), w_j^\theta)_2 \dot{\zeta}(t) dt + \int_0^{t_f} \left[ \kappa(\tilde{\theta}(t), w_j^\theta) + m(w_j^\theta, \dot{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} w_j^\theta d\Gamma \right. \\ & \left. - \langle L_\theta(t), w_j^\theta \rangle_\theta \right] \zeta(t) dt = (\tilde{\theta}_0, w_j^\theta)_2 \zeta(0), \end{aligned} \quad (4.79)$$

for all  $\zeta \in C^1(0, t_f)$ ,  $\zeta(t_f) = 0$ . In particular, equation (4.76) is true for all  $\zeta \in D(0, t_f)$ , so

$$\frac{d}{dt}(\tilde{\theta}(t), w_j^\theta)_2 + \kappa(\tilde{\theta}(t), w_j^\theta) + m(w_j^\theta, \dot{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} w_j^\theta d\Gamma = \langle L_\theta(t), w_j^\theta \rangle_\theta, \quad (4.80)$$

in the sense of distributions on  $(0, t_f)$ .

Taking all non negative integers  $j$ , and using the density of the finite linear combinations of  $w_j^\theta$  in  $H_{0,\Gamma_{\theta,D}}^1(\Omega)$ , we deduce

$$\frac{d}{dt}(\tilde{\theta}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \dot{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \phi d\Gamma = \langle L_\theta(t), \phi \rangle_\theta, \quad (4.81)$$

for all  $\phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega)$  in the sense of distributions on  $t \in (0, t_f)$ .

As  $\dot{\tilde{\theta}} \in L^2(0, t_f; L^2(\Omega))$ , equality (4.81) is equivalent to

$$(\dot{\tilde{\theta}}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \dot{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \phi d\Gamma = \langle L_\theta(t), \phi \rangle_\theta, \quad (4.82)$$

for all  $\phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega)$  in the sense of distributions on  $(0, t_f)$ .

In consequence, from equalities (4.73) and (4.82) we can conclude that  $(\tilde{\mathbf{u}}, \tilde{\theta})$  verifies weak equalities (4.49a) and (4.49b) of Problem  $(\widetilde{VP})$ . In order to complete the problem, we must prove that  $(\tilde{\mathbf{u}}, \tilde{\theta})$  verifies the initial conditions (4.49c) of Problem  $(\widetilde{VP})$ .

**Verifying the initial condition for temperature.**

Considering the test function of (4.74), in the weak equality (4.82) and integrating over  $(0, t_f)$ , we obtain:

$$\begin{aligned} & \int_0^{t_f} \left[ (\dot{\tilde{\theta}}(t), \zeta(t)w_j^\theta)_2 + \kappa(\tilde{\theta}(t), \zeta(t)w_j^\theta) + m(\zeta(t)w_j^\theta, \dot{\tilde{\mathbf{u}}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \zeta(t)w_j^\theta d\Gamma \right] dt \\ &= \int_0^{t_f} \langle L_\theta(t), \zeta(t)w_j^\theta \rangle_\theta dt, \quad \forall \zeta \in C^1(0, t_f); \zeta(t_f) = 0. \end{aligned}$$

Taking into account equality (4.76) we have,

$$\begin{aligned} & \int_0^{t_f} \left[ -(\tilde{\theta}(t), \dot{\zeta}(t)w_j^\theta)_2 + \kappa(\tilde{\theta}(t), \zeta(t)w_j^\theta) + m(\zeta(t)w_j^\theta, \dot{\tilde{\mathbf{u}}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \zeta(t)w_j^\theta d\Gamma \right. \\ & \quad \left. - \langle L_\theta(t), \zeta(t)w_j^\theta \rangle_\theta \right] dt = (\tilde{\theta}(0), \zeta(0)w_j^\theta)_2, \end{aligned}$$

for all  $\zeta \in C^1(0, t_f)$ ,  $\zeta(t_f) = 0$ . Now, if we compare the previous expression with equality (4.79), we obtain:

$$(\tilde{\theta}(0), w_j^\theta)_2 \zeta(0) = (\tilde{\theta}_0, w_j^\theta)_2 \zeta(0), \quad \forall \zeta \in C^1(0, t_f), \quad \zeta(t_f) = 0,$$

for all non negative integer  $j$ , hence, we conclude  $\tilde{\theta}(0) = \tilde{\theta}_0$ .

**Verifying the initial condition for displacements.**

As  $L_{\mathbf{u}} \in C^1(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega))$  it is possible to derive expression (4.73) in time, since  $\dot{\tilde{\mathbf{u}}} \in L^2(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega))$  and  $\dot{\tilde{\theta}} \in L^2(0, t_f; L^2(\Omega))$ , we have

$$a(\dot{\tilde{\mathbf{u}}}(t), \mathbf{v}) - m(\dot{\tilde{\theta}}(t), \mathbf{v}) = \langle \dot{L}_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega).$$

In particular, taking again the test function of (4.69) and integrating in time, we obtain

$$\begin{aligned} & \int_0^{t_f} \left[ a(\dot{\tilde{\mathbf{u}}}(t), \xi(t)w_j^{\mathbf{u}}) - m(\dot{\tilde{\theta}}(t), \xi(t)w_j^{\mathbf{u}}) - \langle \dot{L}_{\mathbf{u}}(t), \xi(t)w_j^{\mathbf{u}} \rangle_{\mathbf{u}} \right] dt = 0, \\ & \forall \xi \in C^1(0, t_f); \xi(t_f) = 0. \end{aligned}$$

Since  $\xi(t_f) = 0$ , we deduce

$$\begin{aligned} & \int_0^{t_f} \left[ -a(\tilde{\mathbf{u}}(t), \dot{\xi}(t)w_j^{\mathbf{u}}) + m(\tilde{\theta}(t), \dot{\xi}(t)w_j^{\mathbf{u}}) + \langle L_{\mathbf{u}}(t), \dot{\xi}(t)w_j^{\mathbf{u}} \rangle_{\mathbf{u}} \right] dt = a(\tilde{\mathbf{u}}(0), \xi(0)w_j^{\mathbf{u}}) \\ & - m(\tilde{\theta}(0), \xi(0)w_j^{\mathbf{u}}) - \langle L_{\mathbf{u}}(0), \xi(0)w_j^{\mathbf{u}} \rangle_{\mathbf{u}}, \quad \forall \xi \in C^1(0, t_f); \xi(t_f) = 0. \end{aligned} \quad (4.83)$$

Taking into account expression (4.73), the term on the left-hand side of expression (4.83) vanishes, and since  $\tilde{\theta}(0) = \tilde{\theta}_0$ , we get

$$a(\tilde{\mathbf{u}}(0), \xi(0)w_j^{\mathbf{u}}) = [m(\tilde{\theta}_0, w_j^{\mathbf{u}}) + \langle L_{\mathbf{u}}(0), w_j^{\mathbf{u}} \rangle_{\mathbf{u}}] \xi(0), \quad \forall \xi \in C^1(0, t_f), \quad \xi(t_f) = 0.$$

Finally, compatibility condition (4.48) let us write this equation in the following equivalent form:

$$a(\tilde{\mathbf{u}}(0), w_j^{\mathbf{u}}) \xi(0) = a(\tilde{\mathbf{u}}_0, w_j^{\mathbf{u}}) \xi(0), \quad \forall \xi \in C^1(0, t_f), \quad \xi(t_f) = 0, \quad \forall j \in \mathbb{N}.$$

Therefore, we can conclude that  $\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0$  thanks to coercitivity of bilinear form  $a(\cdot, \cdot)$ .

#### 4.4.4 Uniqueness of solution of Problem (VP)

In this section we prove the uniqueness of solution of Problem (VP).

As we have considered the reference temperature, the thermal conductivity of the material and the Lamé's parameters depend on the material point, and we have also considered a convection heat transfer boundary condition, so the methodology used by Viaño [98] can not be applied for his case because following the techniques given in his proof, there appear new terms whose signs are not known.

Therefore, our proof is based on applying Gronwall's lemma following the papers of Gawinecki [56, 58, 59, 60] and Gawinecki *et al.* [62].

**Theorem 4.4.10.** *Under assumptions (H1)-(H13), there exists a unique solution  $(\mathbf{u}, \theta)$  of Problem (VP) such that*

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, t_f; \mathbf{H}^1(\Omega)), \quad \dot{\mathbf{u}} \in L^2(0, t_f; \mathbf{H}^1(\Omega)), \quad \text{and} \\ \theta &\in L^\infty(0, t_f; H^1(\Omega)), \quad \dot{\theta} \in L^2(0, t_f; L^2(\Omega)). \end{aligned}$$

*Proof.* In order to establish the uniqueness of solution of Problem (VP), we take into account that this problem is equivalent to Problem  $(\widetilde{VP})$ , therefore it is enough to prove the uniqueness of solution of Problem  $(\widetilde{VP})$ . For this purpose, we consider that let  $(\tilde{\mathbf{u}}_1, \tilde{\theta}_1)$ ,  $(\tilde{\mathbf{u}}_2, \tilde{\theta}_2)$  be two solutions of Problem  $(\widetilde{VP})$  and let us denote by

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2 \in L^\infty(0, t_f; \mathbf{H}_{0, \Gamma_{\mathbf{u}, D}}^1(\Omega)) \quad \text{and} \quad \tilde{\theta} = \tilde{\theta}_1 - \tilde{\theta}_2 \in L^\infty(0, t_f; H_{0, \Gamma_{\theta, D}}^1(\Omega)).$$

So, they verify:

$$\begin{cases} a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) = \mathbf{0}, & \forall \mathbf{v} \in \mathbf{H}_{0, \Gamma_{\mathbf{u}, D}}^1(\Omega), \\ (\dot{\tilde{\theta}}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \dot{\tilde{\mathbf{u}}}(t)) + \int_{\Gamma_{\theta, R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \phi \, d\Gamma = 0, & \forall \phi \in H_{0, \Gamma_{\theta, D}}^1(\Omega), \\ \tilde{\mathbf{u}}(0) = \mathbf{0}, \quad \tilde{\theta}(0) = 0. \end{cases}$$

Integrating over  $(0, t)$  these equations, taking  $\mathbf{v} = \dot{\tilde{\mathbf{u}}}(t)$ ,  $\phi = \tilde{\theta}(t)$  and adding the resulting equations, we arrive at *a.e.*  $t \in (0, t_f)$ :

$$\int_0^t a(\tilde{\mathbf{u}}(s), \dot{\tilde{\mathbf{u}}}(s)) \, ds + \int_0^t (\dot{\tilde{\theta}}(s), \tilde{\theta}(s))_2 \, ds + \int_0^t \kappa(\tilde{\theta}(s), \tilde{\theta}(s)) \, ds + \int_0^t \int_{\Gamma_{\theta, R}} \frac{\alpha_c}{\theta_r} \tilde{\theta}^2(s) \, d\Gamma \, ds = 0.$$

Taking into account the definition of  $\kappa$  given in expression (4.33), this equation can be rewritten as:

$$\begin{aligned} &\int_0^t \frac{1}{2} \frac{d}{ds} [a(\tilde{\mathbf{u}}(s), \tilde{\mathbf{u}}(s))] \, ds + \int_0^t \frac{1}{2} \frac{d}{ds} \|\tilde{\theta}(s)\|_2^2 \, ds + \int_0^t \kappa_1(\tilde{\theta}(s), \tilde{\theta}(s)) \, ds + \int_0^t \int_{\Gamma_{\theta, R}} \frac{\alpha_c}{\theta_r} \tilde{\theta}^2(s) \, d\Gamma \, ds \\ &= - \int_0^t \kappa_2(\tilde{\theta}(s), \tilde{\theta}(s)) \, ds. \end{aligned}$$

Since  $\tilde{\mathbf{u}}(0) = \mathbf{0}$  and  $\tilde{\theta}(0) = 0$ ,

$$\begin{aligned} & \frac{1}{2}a(\tilde{\mathbf{u}}(t), \tilde{\mathbf{u}}(t)) + \frac{1}{2}\|\tilde{\theta}(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}(s), \tilde{\theta}(s))ds + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c}{\theta_r} \tilde{\theta}^2(s) d\Gamma ds \\ &= \int_0^t \int_{\Omega} \frac{k}{\theta_r^2} \tilde{\theta}(s) \nabla \tilde{\theta}(s) \cdot \nabla \theta_r dp ds. \end{aligned} \quad (4.85)$$

Thanks to hypotheses (H2), (H4), (H10) and Hölder's inequality, we have:

$$\begin{aligned} & \frac{1}{2}a(\tilde{\mathbf{u}}(t), \tilde{\mathbf{u}}(t)) + \frac{1}{2}\|\tilde{\theta}(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}(s), \tilde{\theta}(s))ds + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Gamma_{\theta,R})}^2 \\ & \leq \frac{\|k\|_{L^\infty(\Omega)} \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\tilde{\theta}(s)\|_{L^2(\Omega)} \|\nabla \tilde{\theta}(s)\|_{\mathbf{L}^2(\Omega)} ds. \end{aligned} \quad (4.86)$$

Consequently,

$$\begin{aligned} & \frac{1}{2}a(\tilde{\mathbf{u}}(t), \tilde{\mathbf{u}}(t)) + \frac{1}{2}\|\tilde{\theta}(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}(s), \tilde{\theta}(s))ds + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Gamma_{\theta,R})}^2 \\ & \leq \frac{c\|k\|_{L^\infty(\Omega)} \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,min}^2} \int_0^t \|\tilde{\theta}(s)\|_{L^2(\Omega)} \|\tilde{\theta}(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)} ds, \end{aligned} \quad (4.87)$$

with  $c$  the constant of equivalent norms. Taking into account properties (4.36), (4.37), and using inequality (4.58) with

$$\alpha = \frac{c^2 \|k\|_{L^\infty(\Omega)}^2 \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 k_{min}} \text{ and } \beta = \frac{k_{min}}{2\|\theta_r\|_{L^\infty(\Omega)}},$$

we deduce

$$\begin{aligned} & \frac{a_{min}}{2} \|\tilde{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{u,D}}^1(\Omega)}^2 + \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Omega)}^2 + \frac{k_{min}}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds \\ & + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \frac{c^2 \|k\|_{L^\infty(\Omega)}^2 \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 k_{min}} \int_0^t \|\tilde{\theta}(s)\|_{L^2(\Omega)}^2 ds \\ & + \frac{k_{min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds. \end{aligned} \quad (4.88)$$

This above inequality can be rewritten as follows:

$$\begin{aligned} & \frac{a_{min}}{2} \|\tilde{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{u,D}}^1(\Omega)}^2 + \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Omega)}^2 + \frac{k_{min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\tilde{\theta}(s)\|_{H_{0,\Gamma_{\theta,D}}^1(\Omega)}^2 ds \\ & + \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leq \frac{c^2 \|k\|_{L^\infty(\Omega)}^2 \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 k_{min}} \int_0^t \|\tilde{\theta}(s)\|_{L^2(\Omega)}^2 ds, \end{aligned} \quad (4.89)$$

and we get

$$\frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}(t)\|_{L^2(\Omega)}^2 \leq \frac{c^2 \|k\|_{L^\infty(\Omega)}^2 \|\nabla \theta_r\|_{\mathbf{L}^\infty(\Omega)}^2 \|\theta_r\|_{L^\infty(\Omega)}}{2\theta_{r,min}^4 k_{min}} \int_0^t \|\tilde{\theta}(s)\|_{L^2(\Omega)}^2 ds. \quad (4.90)$$

Thanks to Gronwall's lemma, we can conclude  $\|\tilde{\theta}(t)\|_{L^2(\Omega)}^2 = 0$  *a.e.*  $t \in (0, t_f)$ , hence we deduce  $\tilde{\theta}_1 = \tilde{\theta}_2$ . Finally, using again inequality (4.89) and taking into account that  $\tilde{\theta}_1 = \tilde{\theta}_2$ , we directly deduce that  $\|\tilde{\mathbf{u}}(t)\|_{\mathbf{H}_{0,\Gamma_{\mathbf{u}},D}^1(\Omega)} = 0$  *a.e.*  $t \in (0, t_f)$ . In consequence, we conclude that  $\tilde{\mathbf{u}}_1 = \tilde{\mathbf{u}}_2$ . Therefore, hence we can deduce the result.

□

## Chapter 5

# Regularity of a thermoelastic problem

### 5.1 Introduction

In the previous chapter we have proved the existence and uniqueness of a quasistatic coupled thermoelastic problem with mixed displacement-traction conditions for the mechanical submodel and mixed boundary conditions including a Robin boundary condition for the thermal one. In this chapter, we study the regularity properties of the displacements and temperature with respect to time and space. Assuming additional regularity on the data we prove the  $H_{loc}^2$  regularity in space and the  $W^{r,\infty}$  regularity in time,  $r \in \{0\} \cup \mathbb{N}$ . We present two results for the time regularity. In the first one, the regularity is obtained increasing the smooth properties of the solution at the initial instant. In the second one, the regularity result is restricted to the corresponding homogeneous Dirichlet problem; in this case the regularity properties at the initial instant are obtained by means of an auxiliary linear elasticity problem. These are the main contributions of this chapter.

In the bibliography there are many works that deal with the regularity properties of the solution of coupled thermoelastic problems with respect to space. However there are not many studies which have as subject the regularity properties with respect to time. So, firstly we mention some studies about the regularity properties in space and then some in time.

For example, Kačur and Ženíšek [75] study the regularity in space of approximate solutions of coupled dynamical thermoelasticity and Marzocchi *et al.* [86] formulate a result of regularity in space assuming smoother initial data for a transmission problem in thermoelasticity with symmetry. In these works, they use results given in Athanasiadis and G. Stratis [8], Lions and Magenes [83], Mizohata [87] and Nečas [90]. Nevertheless, they consider the dynamic thermoelastic problems where the coefficients of the mechanical behavior law and the reference temperature are independent of the spatial variable. In the case of quasistatic problems, Copetti and Elliott [32] give regularity properties with respect to space of the solution of a one dimensional linear thermoelastic problem with unilateral contact of the Signorini type. Afterwards, Muñoz and Racke [88] study the interior smoothing effects in a multidimensional quasistatic contact problem in thermoelasticity with Dirichlet boundary conditions; in particular, they prove that the displacements and the temperature are infinitely smooth in time and space, no matter how smooth the initial data are.

On the other hand, with respect to regularity properties in time we emphasize the works of Gawinecki [56, 58, 59, 60, 61] and Gawinecki *et al.* [62], who present results of regularity with respect to space and time for dynamic coupled thermoelastic homogeneous Dirichlet problems. Considering their techniques, we achieve similar regularity properties in time for the quasistatic coupled thermoelastic problem with mixed displacement-traction boundary conditions for the mechanical submodel and mixed boundary conditions including a Robin boundary condition for the thermal one. Furthermore, assuming less smoothness over the solution at the initial instant and with homogeneous Dirichlet boundary conditions, we also obtain the same regularity for the quasistatic coupled thermoelastic problem. Nonetheless, these considerations cause some difficulties such as the obtaining of the initial conditions for an auxiliary problem that we will see hereinafter. Using results of Nečas [90] and Agmon *et al.* [2] we overcome these difficulties. So, in this part, our main contributions are the results of the regularity properties in time.

This chapter is outlined as follows: firstly, in Section 5.2, we will prove the  $H_{Loc}^2$  regularity of displacements and temperature with respect to space. In Section 5.3, we will obtain the regularity properties of the displacements and temperature with respect to time for the complete problem and after for a simplified case where we consider homogeneous Dirichlet conditions on the boundary.

## 5.2 Regularity of the weak solution with respect to space

In this chapter we consider  $\Omega \subset \mathbb{R}^3$  a bounded, open and connected set with smooth enough boundary  $\Gamma$ . We assume that  $\Gamma_{\mathbf{u},D}$ ,  $\Gamma_{\mathbf{u},N}$ ,  $\Gamma_{\theta,D}$ ,  $\Gamma_{\theta,N}$  and  $\Gamma_{\theta,R}$  are open subsets of  $\Gamma$ , such that

- $\Gamma = \bar{\Gamma}_{\mathbf{u},D} \cup \bar{\Gamma}_{\mathbf{u},N} = \bar{\Gamma}_{\theta,D} \cup \bar{\Gamma}_{\theta,N} \cup \bar{\Gamma}_{\theta,R}$ ,
- $\Gamma_{\mathbf{u},D} \cap \Gamma_{\mathbf{u},N} = \emptyset$ ,  $\Gamma_{\theta,D} \cap \Gamma_{\theta,N} = \emptyset$ ,  $\Gamma_{\theta,D} \cap \Gamma_{\theta,R} = \emptyset$ ,  $\Gamma_{\theta,R} \cap \Gamma_{\theta,N} = \emptyset$ .

Furthermore, they verify  $\text{meas}(\Gamma_{\mathbf{u},D}) > 0$  and  $\text{meas}(\Gamma_{\theta,D} \cup \Gamma_{\theta,R}) > 0$ . Let  $[0, t_f]$  be the time interval of interest.

In order to study the regularity properties of the solution of the quasistatic coupled problem introduced in the previous chapter, we recall the weak formulation of Problem (P) (see Subsection 4.2.1, Chapter 4), given in Subsection 4.4.2 of Chapter 4, as follows:

### **Problem** (VP)

Find  $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$  such that  $\mathbf{u}(t) = \mathbf{u}_D(t)$  on  $\Gamma_{\mathbf{u},D}$  and  $\theta(t) = \theta_D(t)$  on  $\Gamma_{\theta,D}$  *a.e.*  $t \in (0, t_f)$ , and verifying:

$$\left\{ \begin{array}{l} a(\mathbf{u}(t), \mathbf{v}) - m(\theta(t) - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \quad (5.1a) \\ (\dot{\theta}(t), \phi)_2 + \kappa(\theta(t), \phi) + m(\phi, \dot{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta(t)}{\theta_r} \phi \, d\Gamma = \int_{\Omega} \frac{f(t)}{\theta_r} \phi \, dp \\ + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi \, d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi \, d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \quad (5.1b) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \text{ in } \Omega. \quad (5.1c) \end{array} \right.$$

Here, we use the following notation:

- $\mathbf{u}$  is the displacement,
- $\mathbf{u}_D$  is the displacement on the Dirichlet mechanical boundary,
- $\theta$  is the absolute temperature,
- $\theta_D$  is the temperature on the Dirichlet thermal boundary,
- $\theta_r$  is the reference temperature,
- $\mathbf{g}$  is the density of surface forces of the mechanical problem,
- $\mathbf{b}$  are the body forces per unit volume at the reference configuration,
- $\alpha_c$  is the coefficient of convective heat transfer,
- $f$  is the body heat per unit volume at the reference configuration,
- $\theta^e$  is the external convection temperature,
- $h$  is the heat flux on the Neumann thermal boundary,
- $a(\cdot, \cdot)$  is the bilinear form defined on  $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$  by:

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) dp, \quad (5.2)$$

where  $\Lambda^{-1}$  is the elasticity tensor and  $\boldsymbol{\varepsilon}(\mathbf{u})$  is the deformation tensor,

- $m(\cdot, \cdot)$  is the bilinear form defined on  $L^2(\Omega) \times \mathbf{H}^1(\Omega)$  by:

$$m(\phi, \mathbf{v}) = \int_{\Omega} \phi \alpha (3\lambda + 2\mu) \mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v}) dp = \int_{\Omega} \phi \alpha (3\lambda + 2\mu) \mathbf{I} : \nabla \mathbf{v} dp, \quad (5.3)$$

$\alpha$  being the coefficient of thermal expansion and  $\lambda$  and  $\mu$  the Lamé's parameters,

- $(\cdot, \cdot)_2$  is the scalar product in  $L^2(\Omega)$  given by

$$(\phi, \psi)_2 = \int_{\Omega} \frac{\rho_0 c_F}{\theta_r} \phi \psi dp, \quad (5.4)$$

where  $\rho_0$  is the reference density and  $c_F$  is the specific heat at constant deformation,

- $\kappa(\cdot, \cdot)$  is the bilinear form on  $H^1(\Omega) \times H^1(\Omega)$  such that:

$$\kappa(\phi, \psi) = \int_{\Omega} k \nabla \phi \cdot \nabla \left( \frac{\psi}{\theta_r} \right) dp, \quad (5.5)$$

$k$  being the thermal conductivity coefficient of the material, and we recall that

$$\kappa(\phi, \psi) = \kappa_1(\phi, \psi) + \kappa_2(\phi, \psi) = \int_{\Omega} \frac{k}{\theta_r} \nabla \phi \cdot \nabla \psi dp - \int_{\Omega} k \nabla \phi \cdot \frac{\nabla \theta_r}{\theta_r^2} \psi dp, \quad (5.6)$$

- the admissible displacements space is

$$\mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : v_i|_{\Gamma_{\mathbf{u},D}} = 0, 1 \leq i \leq 3\},$$

- the admissible temperature space is

$$H_{0,\Gamma_{\theta,D}}^1(\Omega) = \{\phi \in H^1(\Omega) : \phi|_{\Gamma_{\theta,D}} = 0\},$$

- and finally,  $\mathbf{u}_0$  and  $\theta_0$  are the initial conditions.

In Chapter 4, in Theorems 4.4.5 and 4.4.10, we have proved that under assumptions (H1)-(H13) (see Subsection 4.4.2), there exists a unique solution  $(\mathbf{u}, \theta)$  of Problem (VP) such that

$$\mathbf{u} \in L^\infty(0, t_f; \mathbf{H}^1(\Omega)), \quad \dot{\mathbf{u}} \in L^2(0, t_f; \mathbf{H}^1(\Omega)), \quad (5.7)$$

$$\theta \in L^\infty(0, t_f; H^1(\Omega)), \quad \dot{\theta} \in L^2(0, t_f; L^2(\Omega)). \quad (5.8)$$

In this section we are going to prove additional regularity properties with respect to the space of the weak solution of Problem (VP). The proof is based on the methodology used by Kačur and Ženíšek [75], which consists in rewriting our coupled problem as two equations defined by means of elliptic operators. So, we can apply the results given by Nečas [90], Lions and Magenes [83] and Mizohata [87] to these operators in order to obtain the  $H_{Loc}^2$  regularity of displacements and temperature with respect to space.

For  $r \in \{0\} \cup \mathbb{N}$  and  $\iota = (\iota_1, \iota_2, \iota_3)$  a 3-tuple of nonnegative integers, let us denote by  $\mathcal{C}^r(\bar{\Omega})$  the vectorial space consisting of all functions  $\phi$  which, together with all their partial derivatives of orders  $|\iota| \leq r$ , are continuous on  $\bar{\Omega}$ .

**Definition 5.2.1.** *If  $0 < \delta \leq 1$ , we define  $\mathcal{C}^{r,\delta}(\bar{\Omega})$  to be the subspace of  $\mathcal{C}^r(\bar{\Omega})$  consisting of those functions  $\phi$  for which, for  $0 \leq |\iota| \leq r$ , the partial derivative of order  $|\iota|$  satisfies in  $\Omega$  a Hölder condition of exponent  $\delta$ .*

**Remark 5.2.2.** *We notice that  $\mathcal{C}^{0,1}(\bar{\Omega})$  is a subset of Lipschitz continuous functions on  $\Omega$ , which is a Banach space with the usual norm.*

**Remark 5.2.3.** *We notice that if  $\mathbf{v} \in \mathbf{W}^{1,\infty}(\Omega)$  with  $\Omega$  an open and convex set, then  $\mathbf{v}$  is a Hölder function with exponent 1, since*

$$|\mathbf{v}(p_1) - \mathbf{v}(p_2)| \leq \|\nabla \mathbf{v}\|_{[\mathbf{L}^\infty(\Omega)]^3} |p_1 - p_2|, \quad \forall p_1, p_2 \in \Omega,$$

(see Brezis [25] Remark 8, p. 154).

In order to prove the regularity, we increase the smooth properties of the hypotheses (H1), (H2), (H4), (H7) and (H8), considering the following assumptions:

(H1') The elasticity tensor  $\Lambda^{-1} \in [\mathcal{C}^1(\bar{\Omega})]^{3^4} \cap [\mathcal{C}^{0,1}(\bar{\Omega})]^{3^4}$  and there exists  $a_{min} > 0$  such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geq a_{min} |\boldsymbol{\tau}|^2,$$

for all  $\boldsymbol{\tau} \in S_3$ , being  $S_3$  the space of symmetric second-order tensors over  $\mathbb{R}^3$ .

- (H2') The reference temperature  $\theta_r \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ , and there exists  $\theta_{r,min} > 0$  such that  $\theta_r(p) \geq \theta_{r,min}$  in  $\Omega$ .
- (H3) The reference density  $\rho_0 > 0$ , the specific heat at constant deformation  $c_F > 0$  and the coefficient of thermal expansion  $\alpha > 0$ .
- (H4') The thermal conductivity coefficient  $k \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ , and there exists  $k_{min} > 0$  such that  $k(p) \geq k_{min}$  in  $\Omega$ .
- (H5) The body forces  $\mathbf{b} \in W^{2,2}(0, t_f; \mathbf{L}^2(\Omega))$ .
- (H6) The body heat  $f \in W^{1,2}(0, t_f; L^2(\Omega))$ .
- (H7')  $\mathbf{u}_D$  is the restriction to  $\Gamma_{\mathbf{u},D} \times (0, t_f)$  of a function called  $\bar{\mathbf{u}}_D$  such that  $\bar{\mathbf{u}}_D \in W^{2,2}(0, t_f; \mathbf{H}^{\frac{3}{2}}(\Gamma))$ .
- (H8')  $\theta_D$  is the restriction to  $\Gamma_{\theta,D} \times (0, t_f)$  of a function called  $\bar{\theta}_D$  such that  $\bar{\theta}_D \in W^{2,2}(0, t_f; H^{\frac{3}{2}}(\Gamma))$ .
- (H9) The surface forces  $\mathbf{g} \in W^{2,2}(0, t_f; \mathbf{L}^2(\Gamma_{\mathbf{u},N}))$  and  $h \in W^{1,2}(0, t_f; L^2(\Gamma_{\theta,N}))$ .
- (H10) The coefficient of convective heat transfer  $\alpha_c \in L^\infty(\Gamma_{\theta,R})$ , and there exists  $\alpha_{c,min} > 0$  verifying  $\alpha_c(p) \geq \alpha_{c,min}$  a.e. on  $\Gamma_{\theta,R}$ .
- (H11) The external convection temperature  $\theta^e \in W^{1,2}(0, t_f; L^2(\Gamma_{\theta,R}))$ .
- (H12) The initial conditions  $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$  and  $\theta_0 \in H^1(\Omega)$ .
- (H13) The initial conditions  $\mathbf{u}_0$  and  $\theta_0$  verify:

$$a(\mathbf{u}_0, \mathbf{v}) - m(\theta_0 - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(0) \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b}(0) \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega),$$

$$\mathbf{u}_0 = \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D},$$

$$\theta_0 = \theta_D(0) \text{ on } \Gamma_{\theta,D}.$$

**Remark 5.2.4.** Taking into account hypotheses (H1'), (H2') and (H4'), the Lamé's parameters, the reference temperature and the thermal conductivity satisfy a Hölder condition of exponent 1. Thus there exist constants  $a_{\lambda,h} > 0$ ,  $a_{\mu,h} > 0$ ,  $\theta_{r,h} > 0$  and  $k_h > 0$ , such that

$$|\lambda(p_1) - \lambda(p_2)| \leq a_{\lambda,h}|p_1 - p_2|, \quad |\mu(p_1) - \mu(p_2)| \leq a_{\mu,h}|p_1 - p_2|,$$

$$|\theta_r(p_1) - \theta_r(p_2)| \leq \theta_{r,h}|p_1 - p_2| \quad \text{and} \quad |k(p_1) - k(p_2)| \leq k_h|p_1 - p_2|,$$

for all  $p_1, p_2 \in \Omega$ .

**Theorem 5.2.5.** Under assumptions (H1')-(H2'), (H3), (H4'), (H5)-(H6), (H7')-(H8') and (H9)-(H13), the solution of Problem (VP) verifies

$$\mathbf{u} \in L^\infty(0, t_f; \mathbf{H}^1(\Omega) \cap \mathbf{H}_{Loc}^2(\Omega)) \text{ and } \theta \in L^\infty(0, t_f; H^1(\Omega) \cap H_{Loc}^2(\Omega)). \quad (5.9)$$

*Proof.* Firstly, in order to obtain a problem with null Dirichlet boundary conditions, we introduce the change of variable by translation given in the previous chapter (see Subsection 4.4.3). We recall that

$$\tilde{\mathbf{u}} = \mathbf{u} - \underline{\mathbf{u}}, \quad \tilde{\mathbf{u}}_0 = \mathbf{u}_0 - \underline{\mathbf{u}}(0), \quad \tilde{\theta} = \theta - \underline{\theta}, \quad \tilde{\theta}_0 = \theta_0 - \underline{\theta}(0),$$

where  $\underline{\mathbf{u}}$  and  $\underline{\theta}$  are fields whose existence is guaranteed by assumptions (H7')-(H8') (see Duvaut and Lions [43], p. 125), verifying

$$\begin{aligned} \underline{\mathbf{u}} &\in W^{2,2}(0, t_f; \mathbf{H}^2(\Omega)), \quad \underline{\mathbf{u}} = \mathbf{u}_D \text{ on } \Gamma_{\mathbf{u},D} \times (0, t_f], \\ \underline{\theta} &\in W^{2,2}(0, t_f; H^2(\Omega)), \quad \underline{\theta} = \theta_D \text{ on } \Gamma_{\theta,D} \times (0, t_f]. \end{aligned}$$

With respect to these new unknowns, we introduce the following problem:

**Problem** ( $\widetilde{VP}$ )

Find  $(\tilde{\mathbf{u}}(t), \tilde{\theta}(t)) \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega) \times H_{0,\Gamma_{\theta,D}}^1(\Omega)$  a.e.  $t \in (0, t_f)$  verifying:

$$\left\{ \begin{aligned} a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) &= \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} \, dp - a(\underline{\mathbf{u}}(t), \mathbf{v}) \\ &\quad + m(\underline{\theta}(t) - \theta_r, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \end{aligned} \right. \quad (5.10a)$$

$$\left\{ \begin{aligned} (\dot{\tilde{\theta}}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \dot{\tilde{\mathbf{u}}}(t)) &+ \int_{\Gamma_{\theta,R}} \frac{\alpha_c \tilde{\theta}(t)}{\theta_r} \phi \, d\Gamma = \int_{\Omega} \frac{f(t)}{\theta_r} \phi \, dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi \, d\Gamma \\ &+ \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi \, d\Gamma - (\dot{\underline{\theta}}(t), \phi)_2 - \kappa(\underline{\theta}(t), \phi) - m(\phi, \dot{\underline{\mathbf{u}}}(t)) \\ &- \int_{\Gamma_{\theta,R}} \frac{\alpha_c \underline{\theta}(t)}{\theta_r} \phi \, d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \end{aligned} \right. \quad (5.10b)$$

$$\left\{ \begin{aligned} \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0 \quad \tilde{\theta}(0) = \tilde{\theta}_0. \end{aligned} \right. \quad (5.10c)$$

We remind that Problems (VP) and ( $\widetilde{VP}$ ) are equivalent and therefore, thanks to theorems of existence and uniqueness (see Theorems 4.4.5 and 4.4.10), we get

$$\begin{aligned} \tilde{\mathbf{u}} &\in L^\infty(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)), \quad \dot{\tilde{\mathbf{u}}} \in L^2(0, t_f; \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega)), \quad \text{and} \\ \tilde{\theta} &\in L^\infty(0, t_f; H_{0,\Gamma_{\theta,D}}^1(\Omega)), \quad \dot{\tilde{\theta}} \in L^2(0, t_f; L^2(\Omega)). \end{aligned} \quad (5.11)$$

Next, we are going to rewrite equations (5.10a)-(5.10b) in their differential form in order that the members on the left-hand side define two elliptic operators. So, we can apply a result for this kind of operators given in Nečas [90] in order to obtain the  $H_{Loc}^2$  regularity with respect to space.

Thus, taking into account equations (5.10a)-(5.10b) of Problem ( $\widetilde{VP}$ ) and applying the Green's formula, we can deduce that its solution  $(\tilde{\mathbf{u}}(t), \tilde{\theta}(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$  is a weak solution in the sense of distributions of the equations:

$$A_{\mathbf{u}} \tilde{\mathbf{u}}(t) = \mathbf{b}_{\mathbf{u}}(t), \quad (5.12)$$

$$A_{\theta} \tilde{\theta}(t) = f_{\theta}(t), \quad (5.13)$$

being

$$\left\{ \begin{array}{l} A_{\mathbf{u}}\tilde{\mathbf{u}}(t) = -\text{Div}[\Lambda^{-1} : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}(t))], \\ \mathbf{b}_{\mathbf{u}}(t) = -\text{Div}(\alpha\tilde{\theta}(t)(3\lambda + 2\mu)\mathbf{I}) + \mathbf{b}(t) + \text{Div}[\Lambda^{-1} : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}(t))] - \text{Div}(\alpha(\underline{\theta}(t) - \theta_r)(3\lambda + 2\mu)\mathbf{I}), \\ A_{\theta}\tilde{\theta}(t) = \text{Div}\left(\frac{k}{\theta_r}\nabla\tilde{\theta}(t)\right), \\ f_{\theta}(t) = -\frac{\rho_0 c_F}{\theta_r}\dot{\tilde{\theta}}(t) - \alpha(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\dot{\tilde{\mathbf{u}}}(t)) - k\nabla\tilde{\theta}(t) \cdot \frac{\nabla\theta_r}{\theta_r^2} + \frac{f(t)}{\theta_r} - \frac{\rho_0 c_F}{\theta_r}\dot{\theta}(t) \\ \quad + \text{Div}\left(\frac{k}{\theta_r}\nabla\underline{\theta}(t)\right) - k\nabla\underline{\theta}(t) \cdot \frac{\nabla\theta_r}{\theta_r^2} - \alpha(3\lambda + 2\mu)\mathbf{I} : \boldsymbol{\varepsilon}(\dot{\underline{\mathbf{u}}}(t)). \end{array} \right.$$

In order to apply the regularity result given in Nečas [90] (p. 197) to equations (5.12)-(5.13), we verify the hypotheses of that result:

- the operator  $A_{\mathbf{u}}$  is  $\mathbf{H}_0^1(\Omega)$ -elliptic with coefficients of  $\mathcal{C}^{0,1}(\overline{\Omega})$ . This is true taking into account Remark 4.4.1 and hypothesis (H1').
- the operator  $A_{\theta}$  is also  $H_0^1(\Omega)$ -elliptic with coefficients of  $\mathcal{C}^{0,1}(\overline{\Omega})$ . Considering Remarks 4.4.2 and 5.2.4 and assumptions (H2') and (H4'), we get for  $p_1, p_2 \in \Omega$ :

$$\begin{aligned} \left| \frac{k}{\theta_r}(p_1) - \frac{k}{\theta_r}(p_2) \right| &= \left| \frac{k(p_1)\theta_r(p_2) - k(p_2)\theta_r(p_1)}{\theta_r(p_1)\theta_r(p_2)} \right| = \left| \frac{1}{\theta_r(p_1)\theta_r(p_2)} \right| |k(p_1)\theta_r(p_2) - k(p_2)\theta_r(p_1)| \\ &\leq \frac{1}{\theta_{r,\min}^2} |k(p_1)\theta_r(p_2) - k(p_2)\theta_r(p_2) + k(p_2)\theta_r(p_2) - k(p_2)\theta_r(p_1)| \\ &\leq \frac{1}{\theta_{r,\min}^2} \{ \|\theta_r\|_{L^\infty(\Omega)} |k(p_1) - k(p_2)| + \|k\|_{L^\infty(\Omega)} |\theta_r(p_2) - \theta_r(p_1)| \} \\ &\leq \frac{1}{\theta_{r,\min}^2} \{ \|\theta_r\|_{L^\infty(\Omega)} k_h + \|k\|_{L^\infty(\Omega)} \theta_{r,h} \} |p_1 - p_2|. \end{aligned}$$

- $\mathbf{b}_{\mathbf{u}}(t) \in \mathbf{L}^2(\Omega)$  a.e.  $t \in (0, t_f)$  taking into account hypotheses (H1')-(H2'), (H3), (H4'), (H5)-(H6), (H7')-(H8') and (H9)-(H11).
- $f_{\theta}(t) \in L^2(\Omega)$  a.e.  $t \in (0, t_f)$  thanks to assumptions (H1')-(H2'), (H3), (H4'), (H5)-(H6), (H7')-(H8') and (H9)-(H11).

Therefore,  $(\tilde{\mathbf{u}}(t), \tilde{\theta}(t))$  is the weak solution in the sense of distributions of problem (5.12)-(5.13), verifying

$$(\tilde{\mathbf{u}}(t), \tilde{\theta}(t)) \in \mathbf{H}_{Loc}^2(\Omega) \times H_{Loc}^2(\Omega).$$

And hence, we can conclude the regularity properties (5.9) for  $\mathbf{u}$  and  $\theta$ . □

### 5.3 Regularity with respect to time

In this section, we present two results of regularity with respect to time. The main difficulty is in establishing the assumptions of regularity properties at the initial instant. In the first subsection, we study the regularity in time of the previous problem and the regularity result is obtained increasing the smooth properties of the data and the solution at the initial instant. In the following subsection, we consider the regularity with respect to time of the associated homogeneous Dirichlet problem. In this case the regularity properties of the solution are not increased at the initial instant, but we extend the smooth properties in space of the data of the problem.

#### 5.3.1 Regularity of the weak solution with respect to time

In this part, we study the regularity properties of the solution of the quasistatic coupled Problem (VP) (see equations (5.1a)-(5.1c)) with respect to time.

From now on, we consider the notation  $\partial_t^r$  in order to denote the partial derivative with respect to  $t$  of order  $r$ , with  $r \in \{0\} \cup \mathbb{N}$ .

The aim of this section is to prove the  $W^{r,\infty}$  regularity of displacements and temperature with respect to time for  $r \in \{0\} \cup \mathbb{N}$ . To do so, we generalize assumptions (H1), (H5)-(H9) and (H11)-(H13) as follows:

( $\overline{\text{H1}}$ ) The elasticity tensor  $\Lambda^{-1} \in [\mathbf{W}^{1,\infty}(\Omega)]^4$  and there exists  $a_{\min} > 0$  such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geq a_{\min} |\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_3.$$

( $\overline{\text{H5}}$ ) The body forces  $\mathbf{b} \in W^{r+2,2}(0, t_f; \mathbf{L}^2(\Omega))$ .

( $\overline{\text{H6}}$ ) The body heat  $f \in W^{r+1,2}(0, t_f; L^2(\Omega))$ .

( $\overline{\text{H7}}$ )  $\mathbf{u}_D$  is the restriction to  $\Gamma_{\mathbf{u},D} \times (0, t_f)$  of a function called  $\bar{\mathbf{u}}_D$  such that  $\bar{\mathbf{u}}_D \in W^{r+2,2}(0, t_f; \mathbf{H}^{\frac{1}{2}}(\Gamma))$ .

( $\overline{\text{H8}}$ )  $\theta_D$  is the restriction to  $\Gamma_{\theta,D} \times (0, t_f)$  of a function called  $\bar{\theta}_D$  such that  $\bar{\theta}_D \in W^{r+2,2}(0, t_f; H^{\frac{1}{2}}(\Gamma))$ .

( $\overline{\text{H9}}$ ) The surface forces  $\mathbf{g} \in W^{r+2,2}(0, t_f; \mathbf{L}^2(\Gamma_{\mathbf{u},N}))$  and  $h \in W^{r+1,2}(0, t_f; L^2(\Gamma_{\theta,N}))$ .

( $\overline{\text{H11}}$ ) The external convection temperature  $\theta^e \in W^{r+1,2}(0, t_f; L^2(\Gamma_{\theta,R}))$ .

( $\overline{\text{H12}}$ ) The displacements and temperature verify at time  $t = 0$ :

$$\partial_t^l \mathbf{u}(0) \in \mathbf{H}^1(\Omega) \text{ and } \partial_t^l \theta(0) \in H^1(\Omega), \quad 0 \leq l \leq r.$$

(H13) The displacements and temperature verify at time  $t = 0$  for all  $0 \leq l \leq r$ :

$$\begin{aligned} a(\partial_t^l \mathbf{u}(0), \mathbf{v}) - m(\partial_t^l \theta(0) - \partial_t^l \theta_r, \mathbf{v}) &= \int_{\Gamma_{\mathbf{u},N}} \partial_t^l \mathbf{g}(0) \cdot \mathbf{v} \, d\Gamma \\ &\quad + \int_{\Omega} \partial_t^l \mathbf{b}(0) \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \\ \partial_t^l \mathbf{u}(0) &= \partial_t^l \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D}, \\ \partial_t^l \theta(0) &= \partial_t^l \theta_D(0) \text{ on } \Gamma_{\theta,D}. \end{aligned}$$

and for  $0 < l \leq r$ :

$$\begin{aligned} (\partial_t^l \theta(0), \phi)_2 + \kappa(\partial_t^{l-1} \theta(0), \phi) + m(\phi, \partial_t^l \mathbf{u}(0)) &+ \int_{\Gamma_{\theta,R}} \frac{\alpha_c \partial_t^{l-1} \theta(0)}{\theta_r} \phi \, d\Gamma = \int_{\Omega} \frac{\partial_t^{l-1} f(0)}{\theta_r} \phi \, dp \\ + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \partial_t^{l-1} \theta^e(0)}{\theta_r} \phi \, d\Gamma + \int_{\Gamma_{\theta,N}} \frac{\partial_t^{l-1} h(0)}{\theta_r} \phi \, d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned}$$

**Remark 5.3.1.** We notice that in hypothesis (H13) the term  $\partial_t^l \theta_r$  of the first member of the first equality is only not null in the case  $l = 0$ .

**Theorem 5.3.2.** Let  $r \in \{0\} \cup \mathbb{N}$  be a fixed parameter. Under assumptions (H1), (H2)-(H4), (H5)-(H9), (H10) and (H11)-(H13), the solution of Problem (VP) satisfies

$$\begin{aligned} \mathbf{u} &\in W^{r,\infty}(0, t_f; \mathbf{H}^1(\Omega)), \quad \partial_t^{r+1} \mathbf{u} \in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ and} \\ \theta &\in W^{r,\infty}(0, t_f; H^1(\Omega)), \quad \partial_t^{r+1} \theta \in L^2(0, t_f; L^2(\Omega)). \end{aligned}$$

*Proof.* We prove this result using the methodology of mathematical induction. For this purpose, we show the induction from  $r = 0$  to  $r = 1$  and the induction from  $r$  to  $r + 1$  runs in the same way. So, the proof is divided into two steps following the scheme:

- **Step  $r = 0$ .** It is directly deduced from Theorems 4.4.5 and 4.4.10 of existence and uniqueness of solution.
- **Step  $r = 1$ .** In order to obtain the regularity of the first derivative with respect to time we define an auxiliary problem, where the second members are the derivatives in time of the Problem (VP) ones. We will prove that this problem verifies the assumptions of the theorems of existence and uniqueness (see Theorems 4.4.5 and 4.4.10), and, finally, we will prove that the unique solution of the auxiliary problem is the derivative in time of the solution of Problem (VP).

**Auxiliary problem.** Deriving the second member of equations (5.1a) and (5.1b) of Problem (VP) with respect to time, we can define the following problem:

**Problem  $(\widehat{VP})_t$**

Find  $(\widehat{\mathbf{u}}(t), \widehat{\theta}(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$  such that  $\widehat{\mathbf{u}}(t) = \partial_t^1 \mathbf{u}_D(t)$  on  $\Gamma_{\mathbf{u},D}$  and  $\widehat{\theta}(t) = \partial_t^1 \theta_D(t)$

on  $\Gamma_{\theta,D}$  a.e.  $t \in (0, t_f)$ , and verifying:

$$\left\{ \begin{array}{l} a(\widehat{\mathbf{u}}(t), \mathbf{v}) - m(\widehat{\theta}(t) - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \left[ \partial_t^1 \mathbf{g}(t) + (\alpha \theta_r (3\lambda + 2\mu) \mathbf{I}) \mathbf{n} \right] \cdot \mathbf{v} d\Gamma \\ + \int_{\Omega} \left[ \partial_t^1 \mathbf{b}(t) - \alpha \nabla(\theta_r (3\lambda + 2\mu)) \right] \cdot \mathbf{v} dp, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \\ (\partial_t^1 \widehat{\theta}(t), \phi)_2 + \kappa(\widehat{\theta}(t), \phi) + m(\phi, \partial_t^1 \widehat{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \widehat{\theta}(t)}{\theta_r} \phi d\Gamma = \int_{\Omega} \frac{\partial_t^1 f(t)}{\theta_r} \phi dp \\ + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \partial_t^1 \theta^e(t)}{\theta_r} \phi d\Gamma + \int_{\Gamma_{\theta,N}} \frac{\partial_t^1 h(t)}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \\ \widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}_0, \quad \widehat{\theta}(0) = \widehat{\theta}_0 \text{ in } \Omega. \end{array} \right. \quad (5.14a)$$

$$\left\{ \begin{array}{l} (\partial_t^1 \widehat{\theta}(t), \phi)_2 + \kappa(\widehat{\theta}(t), \phi) + m(\phi, \partial_t^1 \widehat{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \widehat{\theta}(t)}{\theta_r} \phi d\Gamma = \int_{\Omega} \frac{\partial_t^1 f(t)}{\theta_r} \phi dp \\ + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \partial_t^1 \theta^e(t)}{\theta_r} \phi d\Gamma + \int_{\Gamma_{\theta,N}} \frac{\partial_t^1 h(t)}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \end{array} \right. \quad (5.14b)$$

$$\left\{ \begin{array}{l} \widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}_0, \quad \widehat{\theta}(0) = \widehat{\theta}_0 \text{ in } \Omega. \end{array} \right. \quad (5.14c)$$

The initial conditions in (5.14c),  $\widehat{\mathbf{u}}_0$  and  $\widehat{\theta}_0$ , are defined as:

$$\widehat{\mathbf{u}}_0 = \partial_t^1 \mathbf{u}(0) \text{ and } \widehat{\theta}_0 = \partial_t^1 \theta(0).$$

These initial conditions are well defined thanks to hypothesis  $(\overline{\text{H12}})$ .

**Existence and uniqueness of solution for Problem  $(\widehat{VP})_t$ .** Next, we prove that the data of Problem  $(\widehat{VP})_t$  verify the assumptions of Theorems 4.4.5 and 4.4.10 of existence and uniqueness. Indeed:

- Taking into account hypotheses  $(\overline{\text{H1}})$ , (H2), (H3) and  $(\overline{\text{H5}})$ , we deduce that the body forces associated to Problem  $(\widehat{VP})_t$  verify:

$$\partial_t^1 \mathbf{b}(t) - \alpha \nabla(\theta_r (3\lambda + 2\mu)) \in W^{2,2}(0, t_f; \mathbf{L}^2(\Omega)).$$

- Thanks to assumption  $(\overline{\text{H6}})$ , we obtain that the body heat for Problem  $(\widehat{VP})_t$  verifies:

$$\partial_t^1 f \in W^{1,2}(0, t_f; L^2(\Omega)).$$

- Considering hypotheses  $(\overline{\text{H1}})$ , (H2), (H3) and  $(\overline{\text{H9}})$ , we have that the density of surface forces associated to Problem  $(\widehat{VP})_t$  verify hypothesis (H9):

$$\partial_t^1 \mathbf{g}(t) + (\alpha \theta_r (3\lambda + 2\mu) \mathbf{I}) \mathbf{n} \in W^{2,2}(0, t_f; \mathbf{L}^2(\Gamma_{\mathbf{u},N})) \text{ and } \partial_t^1 h(t) \in W^{1,2}(0, t_f; L^2(\Gamma_{\theta,N})).$$

- Under hypothesis  $(\overline{\text{H11}})$ , we get that the external convection temperature for Problem  $(\widehat{VP})_t$  verifies:

$$\partial_t^1 \theta^e(t) \in W^{1,2}(0, t_f; L^2(\Gamma_{\theta,R})).$$

- Finally, thanks to hypotheses  $(\overline{\text{H12}})$  and  $(\overline{\text{H13}})$  for  $r = 1$ , the initial conditions  $\widehat{\mathbf{u}}_0$  and  $\widehat{\theta}_0$  verify hypotheses (H12) and (H13) of Theorem 4.4.5:

$$\begin{aligned} a(\widehat{\mathbf{u}}_0, \mathbf{v}) - m(\widehat{\theta}_0 - \theta_r, \mathbf{v}) &= \int_{\Gamma_{\mathbf{u},N}} \left[ \partial_t^1 \mathbf{g}(0) + (\alpha \theta_r (3\lambda + 2\mu) \mathbf{I}) \mathbf{n} \right] \cdot \mathbf{v} d\Gamma \\ &+ \int_{\Omega} (\partial_t^1 \mathbf{b}(0) - \text{Div}(\theta_r \alpha (3\lambda + 2\mu) \mathbf{I})) \cdot \mathbf{v} dp, \text{ for all } \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \\ \widehat{\mathbf{u}}_0 &= \partial_t^1 \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D}, \quad \widehat{\theta}_0 = \partial_t^1 \theta_D(0) \text{ on } \Gamma_{\theta,D}. \end{aligned}$$

Therefore, we can deduce the existence of a unique solution  $(\widehat{\mathbf{u}}, \widehat{\theta})$  of Problem  $(\widehat{VP})_t$  such that

$$\widehat{\mathbf{u}} \in W^{0,\infty}(0, t_f; \mathbf{H}^1(\Omega)), \quad \partial_t^1 \widehat{\mathbf{u}} \in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ and} \quad (5.15)$$

$$\widehat{\theta} \in W^{0,\infty}(0, t_f; H^1(\Omega)), \quad \partial_t^1 \widehat{\theta} \in L^2(0, t_f; L^2(\Omega)). \quad (5.16)$$

**The solution of Problem  $(\widehat{VP})_t$  is the derivative of the solution of Problem  $(VP)$ .** We introduce the helpful functions

$$\mathbf{w}(t) = \mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s) ds \quad \text{and} \quad \Theta(t) = \theta_0 + \int_0^t \widehat{\theta}(s) ds. \quad (5.17)$$

From the regularity properties (5.15) and (5.16), we deduce that

$$\mathbf{w} \in W^{1,\infty}(0, t_f; \mathbf{H}^1(\Omega)), \quad \partial_t^2 \mathbf{w} \in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ with } \mathbf{w}(0) = \mathbf{u}_0,$$

and

$$\Theta \in W^{1,\infty}(0, t_f; H^1(\Omega)), \quad \partial_t^2 \Theta \in L^2(0, t_f; L^2(\Omega)) \text{ with } \Theta(0) = \theta_0.$$

Next, we are going to integrate in time the equations of Problem  $(\widehat{VP})_t$  and we will get that  $\mathbf{w}$  and  $\Theta$  are also solutions of Problem  $(VP)$ . Then, by uniqueness of solution of Problem  $(VP)$  we will obtain that

$$\widehat{\mathbf{u}}(t) = \partial_t^1 \mathbf{u}(t), \quad \widehat{\theta}(t) = \partial_t^1 \theta(t),$$

which will complete the proof. In effect, if we integrate the equations of Problem  $(\widehat{VP})_t$  over  $(0, t)$ , we get

$$\begin{aligned} & \int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v}) ds - \int_0^t m(\widehat{\theta}(s) - \theta_r, \mathbf{v}) ds = \int_0^t \int_{\Gamma_{\mathbf{u},N}} \partial_t^1 \mathbf{g}(s) \cdot \mathbf{v} d\Gamma ds \\ & + \int_0^t \int_{\Gamma_{\mathbf{u},N}} ((\theta_r \alpha (3\lambda + 2\mu) \mathbf{I}) \mathbf{n}) \cdot \mathbf{v} d\Gamma ds + \int_0^t \int_{\Omega} \partial_t^1 \mathbf{b}(s) \cdot \mathbf{v} dp ds \\ & - \int_0^t \int_{\Omega} \alpha \nabla(\theta_r (3\lambda + 2\mu)) \cdot \mathbf{v} dp ds, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \quad (5.18) \\ & \int_0^t (\partial_t^1 \widehat{\theta}(s), \phi)_2 ds + \int_0^t \kappa(\widehat{\theta}(s), \phi) ds + \int_0^t m(\phi, \partial_t^1 \widehat{\mathbf{u}}(s)) ds + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c \widehat{\theta}(s)}{\theta_r} \phi d\Gamma ds \\ & = \int_0^t \int_{\Omega} \frac{\partial_t^1 f(s)}{\theta_r} \phi dp ds + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c \partial_t^1 \theta^e(s)}{\theta_r} \phi d\Gamma ds \\ & + \int_0^t \int_{\Gamma_{\theta,N}} \frac{\partial_t^1 h(s)}{\theta_r} \phi d\Gamma ds, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned}$$

Applying the Green's formula to term  $m(\theta_r, \mathbf{v})$  of the first member of equation (5.18), they can be rewritten as follows:

$$\begin{aligned} & \int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v}) ds - \int_0^t m(\widehat{\theta}(s), \mathbf{v}) ds = \int_{\Omega} (\mathbf{b}(t) - \mathbf{b}(0)) \cdot \mathbf{v} dp \\ & + \int_{\Gamma_{\mathbf{u},N}} (\mathbf{g}(t) - \mathbf{g}(0)) \cdot \mathbf{v} d\Gamma, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \end{aligned} \quad (5.19)$$

$$\begin{aligned} & (\widehat{\theta}(t) - \widehat{\theta}(0), \phi)_2 + \int_0^t \kappa(\widehat{\theta}(s), \phi) ds + m(\phi, \widehat{\mathbf{u}}(t) - \widehat{\mathbf{u}}(0)) + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c \widehat{\theta}(s)}{\theta_r} \phi d\Gamma ds \\ & = \int_{\Omega} \frac{(f(t) - f(0))}{\theta_r} \phi dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c (\theta^e(t) - \theta^e(0))}{\theta_r} \phi d\Gamma \\ & + \int_{\Gamma_{\theta,N}} \frac{(h(t) - h(0))}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned} \quad (5.20)$$

On the other hand, considering hypothesis  $(\overline{\text{H13}})$  for  $l = 0$  in displacements and for  $l = 1$  in temperature, we have:

$$\begin{aligned} & \int_{\Omega} \mathbf{b}(0) \cdot \mathbf{v} dp + \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(0) \cdot \mathbf{v} d\Gamma = a(\mathbf{u}_0, \mathbf{v}) - m(\theta_0 - \theta_r, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \\ & (\widehat{\theta}_0, \phi)_2 = -\kappa(\theta_0, \phi) - m(\phi, \widehat{\mathbf{u}}_0) - \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta_0}{\theta_r} \phi d\Gamma + \int_{\Omega} \frac{f(0)}{\theta_r} \phi dp \\ & + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(0)}{\theta_r} \phi d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h(0)}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned}$$

Thus, if we replace the previous equalities in expressions (5.19)-(5.20), we obtain

$$\begin{aligned} & \int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v}) ds - \int_0^t m(\widehat{\theta}(s), \mathbf{v}) ds = \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp + \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} d\Gamma \\ & - a(\mathbf{u}_0, \mathbf{v}) + m(\theta_0 - \theta_r, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \\ & (\widehat{\theta}(t), \phi)_2 + \kappa(\theta_0, \phi) + \int_0^t \kappa(\widehat{\theta}(s), \phi) ds + m(\phi, \widehat{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta_0}{\theta_r} \phi d\Gamma \\ & + \int_0^t \int_{\Gamma_{\theta,R}} \frac{\alpha_c \widehat{\theta}(s)}{\theta_r} \phi d\Gamma ds = \int_{\Omega} \frac{f(t)}{\theta_r} \phi dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi d\Gamma \\ & + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned}$$

These equations can be rewritten as follows:

$$\begin{aligned} & a(\mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s) ds, \mathbf{v}) - m(\theta_0 + \int_0^t \widehat{\theta}(s) ds - \theta_r, \mathbf{v}) \\ &= \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp + \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} d\Gamma, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \end{aligned} \quad (5.21)$$

$$\begin{aligned} & (\widehat{\theta}(t), \phi)_2 + \kappa(\theta_0 + \int_0^t \widehat{\theta}(s) ds, \phi) + m(\phi, \widehat{\mathbf{u}}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c(\theta_0 + \int_0^t \widehat{\theta}(s) ds)}{\theta_r} \phi d\Gamma \\ &= \int_{\Omega} \frac{f(t)}{\theta_r} \phi dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega). \end{aligned} \quad (5.22)$$

Therefore,  $\mathbf{w}$  and  $\Theta$  are solution of the problem:

$$\begin{aligned} & a(\mathbf{w}(t), \mathbf{v}) - m(\Theta(t) - \theta_r, \mathbf{v}) = \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp + \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v} d\Gamma, \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_{\mathbf{u},D}}^1(\Omega), \\ & (\partial_t^1 \Theta(t), \phi)_2 + \kappa(\Theta(t), \phi) + m(\phi, \partial_t^1 \mathbf{w}(t)) + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \Theta(t)}{\theta_r} \phi d\Gamma \\ &= \int_{\Omega} \frac{f(t)}{\theta_r} \phi dp + \int_{\Gamma_{\theta,R}} \frac{\alpha_c \theta^e(t)}{\theta_r} \phi d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi d\Gamma, \quad \forall \phi \in H_{0,\Gamma_{\theta,D}}^1(\Omega), \\ & \mathbf{w}(0) = \mathbf{u}_0, \quad \Theta(0) = \theta_0, \end{aligned}$$

which coincides with Problem (VP) (see equations (5.1a)-(5.1c)). Since this problem has a unique solution we can conclude that

$$\mathbf{w}(t) = \mathbf{u}(t) \quad \text{and} \quad \Theta(t) = \theta(t).$$

Furthermore, from (5.17) we deduce that

$$\partial_t^1 \mathbf{w}(t) = \widehat{\mathbf{u}}(t) \quad \text{and} \quad \partial_t^1 \Theta(t) = \widehat{\theta}(t),$$

and therefore

$$\widehat{\mathbf{u}}(t) = \partial_t^1 \mathbf{u}(t) \quad \text{and} \quad \widehat{\theta}(t) = \partial_t^1 \theta(t).$$

Finally from the regularity properties (5.15) and (5.16), we obtain

$$\begin{aligned} & \mathbf{u} \in W^{1,\infty}(0, t_f; \mathbf{H}^1(\Omega)), \quad \partial_t^2 \mathbf{u} \in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ and} \\ & \theta \in W^{1,\infty}(0, t_f; \mathbf{H}^1(\Omega)), \quad \partial_t^2 \theta \in L^2(0, t_f; L^2(\Omega)), \end{aligned}$$

and Theorem 5.3.2 for  $r = 1$  is proved. □

To conclude this subsection, we summarize the regularity properties in time and space for the solution of Problem (VP) from Theorems 5.2.5 and 5.3.2.

To do so, we replace  $(\overline{\text{H7}})$  and  $(\overline{\text{H8}})$  by the following hypotheses with  $r \in \{0\} \cup \mathbb{N}$ :

(H7)  $\mathbf{u}_D$  is the restriction to  $\Gamma_{\mathbf{u},D} \times (0, t_f)$  of a function called  $\bar{\mathbf{u}}_D$  such that  $\bar{\mathbf{u}}_D \in W^{r+2,2}(0, t_f; \mathbf{H}^{\frac{3}{2}}(\Gamma))$ .

(H8)  $\theta_D$  is the restriction to  $\Gamma_{\theta,D} \times (0, t_f)$  of a function called  $\bar{\theta}_D$  such that  $\bar{\theta}_D \in W^{r+2,2}(0, t_f; H^{\frac{3}{2}}(\Gamma))$ .

**Theorem 5.3.3.** *Let  $r \in \{0\} \cup \mathbb{N}$  be a fixed parameter. Under assumptions (H1'), (H2'), (H3), (H4'), (H5), (H6), (H7), (H8), (H9), (H10) and (H11)-(H13), we have*

$$\begin{aligned} \mathbf{u} &\in W^{r,\infty}(0, t_f; \mathbf{H}^1(\Omega) \cap \mathbf{H}_{Loc}^2(\Omega)), & \partial_t^{r+1} \mathbf{u} &\in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ and} \\ \theta &\in W^{r,\infty}(0, t_f; H^1(\Omega) \cap H_{Loc}^2(\Omega)), & \partial_t^{r+1} \theta &\in L^2(0, t_f; L^2(\Omega)). \end{aligned}$$

*Proof.* The proof is deduced directly from Theorems 5.2.5 and 5.3.2. □

### 5.3.2 Regularity of the Dirichlet problem with respect to time

In this subsection, we are going to consider a particular case of Problem (P) with homogeneous Dirichlet boundary conditions in displacements and temperature. In the previous subsection we have proved the  $W^{r,\infty}$  regularity in time under hypotheses (H12) and (H13) on the solution at the initial instant. Here, we are going to prove that if we replace those hypotheses by another ones on the initial data, we can obtain the same regularity in time for the Dirichlet case.

From here on, let us denote by  $r \in \mathbb{N}$  a fixed parameter.

In order to prove the regularity properties in time the following results are necessary.

**Definition 5.3.4.** *Let us consider  $\bar{\Lambda}^{-1}$  as the perturbation of  $\Lambda^{-1}$  given by*

$$\bar{\Lambda}^{-1} : \boldsymbol{\tau} = \Lambda^{-1} : \boldsymbol{\tau} + \gamma \boldsymbol{\tau}_s, \quad (5.23)$$

where  $\boldsymbol{\tau}$  belongs to  $S_3$ ,  $\boldsymbol{\tau}_s$  denotes the spherical part of  $\boldsymbol{\tau}$  and  $\gamma$  is a non-negative scalar function.

**Lemma 5.3.5.** *Let us consider  $m \in \mathbb{N}$  a fixed parameter. We assume that the elasticity tensor*

$$\Lambda^{-1} \in \left[ \mathcal{C}^{0,1}(\bar{\Omega}) \right]^{3^4} \text{ and if } m \geq 2 \text{ } \Lambda^{-1} \in \left[ \mathcal{C}^{0,1}(\bar{\Omega}) \right]^{3^4} \cap \left[ \mathcal{C}^{2m-1}(\bar{\Omega}) \right]^{3^4} \quad (5.24)$$

and there exists  $a_{min} > 0$  such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geq a_{min} |\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_3.$$

Then, if the body forces  $\bar{\mathbf{b}} \in \mathbf{H}^{2(m-1)}(\Omega)$  and

$$\gamma \in \mathcal{C}^{0,1}(\bar{\Omega}) \text{ and if } m \geq 2 \text{ } \gamma \in \mathcal{C}^{0,1}(\bar{\Omega}) \cap \mathcal{C}^{2m-1}(\bar{\Omega}), \quad (5.25)$$

there exists a unique weak solution  $\bar{\mathbf{u}} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2m}(\Omega)$  of the following equation:

$$-\text{Div}(\bar{\Lambda}^{-1} : \boldsymbol{\varepsilon}(\bar{\mathbf{u}})) = \bar{\mathbf{b}} \text{ in } \Omega.$$

*Proof.* Notice that, since  $\Lambda^{-1}$  satisfies (5.24) and  $\gamma$  verifies (5.25) then

$$\overline{\Lambda^{-1}} \in \begin{cases} [\mathcal{C}^{0,1}(\overline{\Omega})]^{3^4} & \text{if } m = 1 \\ [\mathcal{C}^{0,1}(\overline{\Omega})]^{3^4} \cap [\mathcal{C}^{2m-1}(\overline{\Omega})]^{3^4} & \text{if } m \geq 2 \end{cases}.$$

Indeed, from definition (5.23) the Lamé's parameters of  $\overline{\Lambda^{-1}}$  are  $\mu$  and  $\lambda + \frac{\gamma}{3}$ , which belong to  $\mathcal{C}^{0,1}(\overline{\Omega})$  if  $m = 1$  and  $\mathcal{C}^{0,1}(\overline{\Omega}) \cap \mathcal{C}^{2m-1}(\overline{\Omega})$  if  $m \geq 2$ .

Thus, we can deduce the result for  $m = 1$  from Nečas [90] (see Theorems 3.7.2, 2.4.10 and Lemma 3.2 of Chapter 5, Section 3.2 about existence and uniqueness of linear elastic systems with variable parameters). Next, for  $m \geq 2$ , the  $\mathbf{H}^{2m}$  regularity is obtained thanks to the estimates for elliptic equations with variables coefficients (Theorem 10.5 of Section 10.2) from Agmon *et al.* [2] (see also for example, Healey [66], Theorem 2.8 of Chapter 2).  $\square$

Let us define the following problem:

**Problem** ( $\bar{P}^r$ )

Find  $\bar{\mathbf{u}}^r(p)$  in  $\Omega$ , verifying:

$$-\text{Div}(\overline{\Lambda^{-1}} : \boldsymbol{\varepsilon}(\bar{\mathbf{u}}^r)) = \bar{\mathbf{b}}^r \quad \text{in } \Omega, \quad (5.26)$$

$$\bar{\mathbf{u}}^r = \mathbf{0} \quad \text{on } \Gamma, \quad (5.27)$$

where  $\overline{\Lambda^{-1}}$  is the perturbed operator defined in expression (5.23), with  $\gamma = \frac{3\theta_r \alpha^2 (3\lambda + 2\mu)^2}{\rho_0 c_F}$  and

$$\bar{\mathbf{b}}^r = \partial_t^r \mathbf{b}(0) - \text{Div} \left( \frac{\alpha(3\lambda + 2\mu)}{\rho_0 c_F} \text{Div}(k \nabla \bar{\theta}^{r-1}) \mathbf{I} \right) - \text{Div} \left( \frac{\alpha(3\lambda + 2\mu) \partial_t^{r-1} f(0)}{\rho_0 c_F} \mathbf{I} \right). \quad (5.28)$$

Here, for  $r \geq 1$

$$\bar{\theta}^r = -\frac{\theta_r \alpha (3\lambda + 2\mu)}{\rho_0 c_F} \text{Div} \bar{\mathbf{u}}^r + \frac{\text{Div}(k \nabla \bar{\theta}^{r-1})}{\rho_0 c_F} + \frac{\partial_t^{r-1} f(0)}{\rho_0 c_F} \quad \text{in } \Omega, \quad (5.29)$$

and  $\bar{\theta}^0 = \theta_0$ .

**Corollary 5.3.6.** *Let  $1 \leq l \leq r$ . Under hypothesis (H3) and the following assumptions:*

(h1)  $\theta_r$ ,  $\lambda$  and  $\mu$  strictly positive functions in  $\mathcal{C}^{2r-1}(\overline{\Omega}) \cap \mathcal{C}^{0,1}(\overline{\Omega})$ ,

(h2)  $k \in H^{2r}(\Omega)$ ,

(h3)  $\theta_0 \in H^{2r+1}(\Omega)$ ,

(h4)  $\partial_t^l \mathbf{b}(0) \in \mathbf{H}^{2r-2l}(\Omega)$ ,

(h5)  $\partial_t^{l-1} f(0) \in H^{2r-2l+1}(\Omega)$ ,

there exists a unique weak solution  $\bar{\mathbf{u}}^l \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r-2l+2}(\Omega)$  of each Problem  $(\bar{P}^l)$ .

*Proof.* This result is proved using the methodology of mathematical induction. For this purpose, we take any fixed parameter  $r \in \mathbb{N}$ .

- If  $r = 1$  then  $l = 1$  and the result is deduced from the previous lemma taking  $m = 1$ . For this purpose, we are going to prove that the body forces verify the hypotheses of Lemma 5.3.5 and that the scalar function  $\gamma$  satisfies a Hölder condition with exponent 1.

Then, thanks to assumptions (H3) and (h1)-(h5) for  $r = 1$ , the body forces associated to Problem  $(\bar{P}^1)$  given by (5.28) belong to  $\mathbf{L}^2(\Omega)$ . And, from assumptions (H3) and (h1), we get:

$$\gamma \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{C}^{0,1}(\bar{\Omega}) \text{ and } \gamma(p) \geq \frac{3\theta_{r,\min}\alpha^2(3\lambda(p) + 2\mu(p))^2}{\rho_0 c_F} \geq 0, \quad \text{for all } p \in \Omega.$$

In effect, to prove that  $\gamma \in \mathcal{C}^{0,1}(\bar{\Omega})$ ; let us take  $p_1$  and  $p_2 \in \Omega$ , we have

$$\begin{aligned} |\gamma(p_1) - \gamma(p_2)| &= \frac{3\alpha^2}{\rho_0 c_F} \left| \theta_r(p_1)(3\lambda(p_1) + 2\mu(p_1))^2 - \theta_r(p_2)(3\lambda(p_1) + 2\mu(p_1))^2 \right. \\ &\quad \left. + \theta_r(p_2)(3\lambda(p_1) + 2\mu(p_1))^2 - \theta_r(p_2)(3\lambda(p_2) + 2\mu(p_2))^2 \right| \\ &\leq \frac{3\alpha^2}{\rho_0 c_F} (3\lambda(p_1) + 2\mu(p_1))^2 |\theta_r(p_1) - \theta_r(p_2)| \\ &\quad + \frac{3\alpha^2}{\rho_0 c_F} \|\theta_r\|_{L^\infty(\Omega)} \left| (3\lambda(p_1) + 2\mu(p_1))^2 - (3\lambda(p_2) + 2\mu(p_2))^2 \right|, \end{aligned}$$

thanks to hypotheses (H3) and (h1). Then, taking into account again assumption (h1), that inequality can be rewritten as follows,

$$\begin{aligned} |\gamma(p_1) - \gamma(p_2)| &\leq \left( \frac{3\alpha^2\theta_{r,h}}{\rho_0 c_F} (9\|\lambda\|_{L^\infty(\Omega)}^2 + 12\|\lambda\|_{L^\infty(\Omega)}\|\mu\|_{L^\infty(\Omega)} + 4\|\mu\|_{L^\infty(\Omega)}^2) \right) |p_1 - p_2| \\ &\quad + \frac{3\alpha^2}{\rho_0 c_F} \|\theta_r\|_{L^\infty(\Omega)} \|6\lambda + 4\mu\|_{L^\infty(\Omega)} |3\lambda(p_1) + 2\mu(p_1) \\ &\quad - 3\lambda(p_2) - 2\mu(p_2)|, \end{aligned}$$

$\theta_{r,h}$  being the constant of Hölder condition for  $\theta_r$ . Finally, we conclude

$$\begin{aligned} |\gamma(p_1) - \gamma(p_2)| &\leq \left( \frac{3\alpha^2\theta_{r,h}}{\rho_0 c_F} (9\|\lambda\|_{L^\infty(\Omega)}^2 + 12\|\lambda\|_{L^\infty(\Omega)}\|\mu\|_{L^\infty(\Omega)} + 4\|\mu\|_{L^\infty(\Omega)}^2) \right) \\ &\quad + \frac{3\alpha^2}{\rho_0 c_F} \|\theta_r\|_{L^\infty(\Omega)} (18\|\lambda\|_{L^\infty(\Omega)}a_{\lambda,h} + 12\|\lambda\|_{L^\infty(\Omega)}a_{\mu,h} + 12\|\mu\|_{L^\infty(\Omega)}a_{\lambda,h} \\ &\quad + 8\|\mu\|_{L^\infty(\Omega)}a_{\mu,h}) |p_1 - p_2|, \end{aligned}$$

$a_{\lambda,h}$  and  $a_{\mu,h}$  being the constants of Hölder condition for  $\lambda$  and  $\mu$ , respectively.

- If  $r \geq 2$ , we prove the result by mathematical induction on the parameter  $l$ . For this purpose, we show the induction from  $l = 1$  to  $l = 2$  and the induction from  $l$  to  $l + 1$  runs in the same way. So, the proof is divided into two steps, following the scheme:

- Step  $l = 1$ . Thanks to assumptions for  $l = 1$ , we can apply Lemma 5.3.5 with  $m = r$  using the same reasoning that in the previous case, and we obtain a unique solution  $\bar{\mathbf{u}}^1 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r}(\Omega)$  of Problem  $(\bar{P}^1)$ .
- Step  $l = 2$ . In this step, we get Problem  $(\bar{P}^2)$  and, thanks to the previous step,  $\bar{\mathbf{u}}^1 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r}(\Omega)$  and  $\bar{\theta}^1$  defined by equality (5.29) belongs to  $H^{2r-1}(\Omega)$ . So,  $\bar{\mathbf{b}}^2 \in \mathbf{H}^{2r-4}(\Omega)$  (see (5.28)). Thus, we can apply again Lemma 5.3.5 with  $m = r - 1$  and we obtain a unique solution  $\bar{\mathbf{u}}^2 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r-2}(\Omega)$  of Problem  $(\bar{P}^2)$ .

We notice that, at each step  $l$ , we can apply Lemma 5.3.5 for  $m = r - l + 1$ .

□

**Remark 5.3.7.** *We notice that the result is also valid taking  $\theta_r$ ,  $\lambda$  and  $\mu$  strictly positive functions in  $W^{2r-1,\infty}(\Omega) \cap C^{0,1}(\bar{\Omega})$ .*

As we have announced, throughout this subsection we consider the following Dirichlet problem:

**Problem**  $(P_D)$

Find  $\mathbf{u}(p, t)$  and  $\theta(p, t)$  in  $\Omega \times (0, t_f]$ , verifying:

$$\begin{cases} -\text{Div } \boldsymbol{\sigma}(\theta, \mathbf{u}) = \mathbf{b} & \text{in } \Omega \times (0, t_f], & (5.30a) \\ \rho_0 c_F \partial_t^1 \theta = -\theta_r \alpha (3\lambda + 2\mu) \text{Div } \partial_t^1 \mathbf{u} + \text{Div}(k \nabla \theta) + f & \text{in } \Omega \times (0, t_f], & (5.30b) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma \times (0, t_f], & (5.30c) \\ \theta = 0 & \text{on } \Gamma \times (0, t_f], & (5.30d) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 & \text{in } \Omega, & (5.30e) \end{cases}$$

with

$$\boldsymbol{\sigma}(\theta, \mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - \alpha(\theta - \theta_r)(3\lambda + 2\mu)\mathbf{I}.$$

Following the reasoning of Subsection 4.4.1 and 4.4.2 of the previous chapter, we consider the following weak variational formulation:

**Problem**  $(VP_D)$

Find  $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ , verifying *a.e.*  $t \in (0, t_f)$ :

$$\begin{cases} a(\mathbf{u}(t), \mathbf{v}) - m(\theta(t) - \theta_r, \mathbf{v}) = \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp, & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), & (5.31a) \\ (\partial_t^1 \theta(t), \phi)_2 + \kappa(\theta(t), \phi) + m(\phi, \partial_t^1 \mathbf{u}(t)) = \int_{\Omega} \frac{f(t)}{\theta_r} \phi dp, & \forall \phi \in H_0^1(\Omega), & (5.31b) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 & \text{in } \Omega. & (5.31c) \end{cases}$$

Under hypotheses (H1)-(H6), (H12)-(H13), Theorems 4.4.5 and 4.4.10 imply the existence and uniqueness of solution  $(\mathbf{u}, \theta)$  for Problem  $(VP_D)$ , such that:

$$\mathbf{u} \in W^{0,\infty}(0, t_f; \mathbf{H}_0^1(\Omega)), \quad \partial_t^1 \mathbf{u} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ and} \quad (5.32)$$

$$\theta \in W^{0,\infty}(0, t_f; H_0^1(\Omega)), \quad \partial_t^1 \theta \in L^2(0, t_f; L^2(\Omega)). \quad (5.33)$$

The aim of this section is to prove the  $W^{r,\infty}$  regularity of displacements and temperature with respect to time for  $r \in \mathbb{N}$ . To do so, we generalize assumptions (H1), (H2), (H4)-(H6), (H12) and (H13) as follows:

( $\widehat{\text{H1}}$ ) The elasticity tensor  $\Lambda^{-1} \in [\mathcal{C}^{2r-1}(\overline{\Omega})]^{3^4} \cap [\mathcal{C}^{0,1}(\overline{\Omega})]^{3^4}$  and there exists  $a_{\min} > 0$  such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geq a_{\min} |\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_3.$$

( $\widehat{\text{H2}}$ ) The reference temperature  $\theta_r \in \mathcal{C}^{2r-1}(\overline{\Omega}) \cap \mathcal{C}^{0,1}(\overline{\Omega})$ , and there exists  $\theta_{r,\min} > 0$  such that  $\theta_r(p) \geq \theta_{r,\min}$  in  $\Omega$ .

( $\widehat{\text{H4}}$ ) The thermal conductivity coefficient  $k \in W^{2r,\infty}(\Omega)$ , and there exists  $k_{\min} > 0$  such that  $k(p) \geq k_{\min}$  in  $\Omega$ .

( $\widehat{\text{H5}}$ ) The body forces  $\mathbf{b} \in W^{r+2,2}(0, t_f; \mathbf{L}^2(\Omega))$ , and  $\partial_t^l \mathbf{b}(0) \in \mathbf{H}^{2r-2l}(\Omega)$ ,  $1 \leq l \leq r$ .

( $\widehat{\text{H6}}$ ) The body heat  $f \in W^{r+1,2}(0, t_f; L^2(\Omega))$ , and  $\partial_t^l f(0) \in H_0^{2r-(2l+1)}(\Omega)$ ,  $0 \leq l \leq r-1$ .

( $\widehat{\text{H12}}$ ) The initial conditions  $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$  and  $\theta_0 \in H_0^{2r+1}(\Omega)$ .

( $\widehat{\text{H13}}$ ) The initial conditions  $\mathbf{u}_0$  and  $\theta_0$  verify:

$$a(\mathbf{u}_0, \mathbf{v}) - m(\theta_0 - \theta_r, \mathbf{v}) = \int_{\Omega} \mathbf{b}(0) \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega).$$

( $\widehat{\text{H14}}$ ) For  $1 \leq l \leq r$ , the solution  $\bar{\mathbf{u}}^l$  of each Problem ( $\bar{P}^l$ ) verifies that

$$\text{Div } \bar{\mathbf{u}}^l \in \mathbf{H}_0^{2r-2l+1}(\Omega).$$

**Remark 5.3.8.** We notice that in hypotheses ( $\widehat{\text{H1}}$ ) and ( $\widehat{\text{H2}}$ ), it would be enough to consider  $\Lambda^{-1} \in [\mathbf{W}^{2r-1,\infty}(\Omega)]^{3^4} \cap [\mathcal{C}^{0,1}(\overline{\Omega})]^{3^4}$  and  $\theta_r \in W^{2r-1,\infty}(\Omega) \cap \mathcal{C}^{0,1}(\overline{\Omega})$ .

**Theorem 5.3.9.** Let  $r \in \mathbb{N}$  be a fixed parameter. Under assumptions ( $\widehat{\text{H1}}$ ), ( $\widehat{\text{H2}}$ ), (H3), ( $\widehat{\text{H4}}$ )-( $\widehat{\text{H6}}$ ) and ( $\widehat{\text{H12}}$ )-( $\widehat{\text{H14}}$ ), the solution of Problem (VP<sub>D</sub>) satisfies

$$\begin{aligned} \mathbf{u} &\in W^{r,\infty}(0, t_f; \mathbf{H}_0^1(\Omega)), \quad \partial_t^{r+1} \mathbf{u} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ and} \\ \theta &\in W^{r,\infty}(0, t_f; H_0^1(\Omega)), \quad \partial_t^{r+1} \theta \in L^2(0, t_f; L^2(\Omega)). \end{aligned}$$

*Proof.* We prove this result using mathematical induction. For this purpose, we prove the result for  $r = 1$  following the methodology of Theorem 5.3.2.

**Auxiliary problem.** Deriving the second member of equations (5.31a) and (5.31b) of Problem  $(VP_D)$  with respect to time, we can define the following problem:

**Problem**  $(\widehat{VP}_D)_t$

Find  $(\widehat{\mathbf{u}}(t), \widehat{\theta}(t)) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ , verifying *a.e.*  $t \in (0, t_f)$ :

$$\left\{ \begin{array}{l} a(\widehat{\mathbf{u}}(t), \mathbf{v}) - m(\widehat{\theta}(t) - \theta_r, \mathbf{v}) = \int_{\Omega} \left[ \partial_t^1 \mathbf{b}(t) - \alpha \nabla(\theta_r(3\lambda + 2\mu)) \right] \cdot \mathbf{v} \, dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (\partial_t^1 \widehat{\theta}(t), \phi)_2 + \kappa(\widehat{\theta}(t), \phi) + m(\phi, \partial_t^1 \widehat{\mathbf{u}}(t)) = \int_{\Omega} \frac{\partial_t^1 f(t)}{\theta_r} \phi \, dp, \quad \forall \phi \in H_0^1(\Omega), \\ \widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}_0, \quad \widehat{\theta}(0) = \widehat{\theta}_0 \text{ in } \Omega. \end{array} \right. \quad (5.34a)$$

$$(5.34b)$$

$$(5.34c)$$

In (5.34c) the initial conditions are defined as:

$$\widehat{\mathbf{u}}_0 = \bar{\mathbf{u}}^1, \quad (5.35)$$

$$\widehat{\theta}_0 = \bar{\theta}^1, \quad (5.36)$$

where  $\bar{\mathbf{u}}^1$  is the weak solution of Problem  $(\bar{P}^1)$  and  $\bar{\theta}^1$  is defined from (5.29).

**Remark 5.3.10.** The initial conditions  $\widehat{\mathbf{u}}_0$  and  $\widehat{\theta}_0$  were defined to coincide formally with the derivatives with respect to time of  $\mathbf{u}$  and  $\theta$  (the weak solution of Problem  $(P_D)$ ) at  $t = 0$ . In effect, if we evaluate energy equation (5.30b) at time  $t = 0$ , we obtain

$$\partial_t^1 \theta(0) = -\frac{\theta_r \alpha (3\lambda + 2\mu) \text{Div } \partial_t^1 \mathbf{u}(0)}{\rho_0 c_F} + \frac{\text{Div}(k \nabla \theta_0)}{\rho_0 c_F} + \frac{f(0)}{\rho_0 c_F} \text{ in } \Omega.$$

In addition, if we formally derive motion equation (5.30a) with respect to time, we consider  $t = 0$  and we replace the previous expression, we obtain equation (5.26) for  $\bar{\mathbf{u}}^1$  playing the role of  $\partial_t^1 \mathbf{u}(0)$  with  $\bar{\mathbf{b}}^1$  given by equality (5.28).

**Lemma 5.3.11.** Under assumptions of Theorem 5.3.9 for  $r = 1$ , the initial conditions  $\widehat{\mathbf{u}}_0$  and  $\widehat{\theta}_0$  given in (5.35)-(5.36), are well defined.

*Proof.* Under hypotheses  $(\widehat{H1})$ ,  $(\widehat{H2})$ ,  $(H3)$ ,  $(\widehat{H4})$ - $(\widehat{H6})$  and  $(\widehat{H12})$  for  $r = 1$ , we obtain the assumptions of Corollary 5.3.6. Thus, we can conclude that  $\widehat{\mathbf{u}}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$  is the unique solution of equation (5.26) for  $r = 1$ . Therefore,  $\widehat{\mathbf{u}}_0$  is well defined. Finally,  $\widehat{\theta}_0 = \bar{\theta}^1$  can be defined from equality (5.29). Notice that  $\bar{\theta}_0 \in H^1(\Omega)$ , then  $\widehat{\theta}_0$  is also well defined.  $\square$

**Existence and uniqueness of solution of Problem  $(\widehat{VP}_D)_t$ .** In the following, we prove the existence and uniqueness of Problem  $(\widehat{VP}_D)_t$ . The main difficulty is to verify that the initial conditions of this problem,  $(\widehat{\mathbf{u}}_0, \widehat{\theta}_0)$  satisfies hypotheses  $(H12)$  and  $(H13)$  of the theorems of existence and uniqueness (Theorems 4.4.5 and 4.4.10).

**Lemma 5.3.12.** *Under assumptions of Theorem 5.3.9 for  $r = 1$ , there exists a unique solution of Problem  $(\widehat{VP}_D)_t$  verifying*

$$\widehat{\mathbf{u}} \in W^{0,\infty}(0, t_f; \mathbf{H}_0^1(\Omega)), \quad \partial_t^1 \widehat{\mathbf{u}} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ and} \quad (5.37)$$

$$\widehat{\theta} \in W^{0,\infty}(0, t_f; H_0^1(\Omega)), \quad \partial_t^1 \widehat{\theta} \in L^2(0, t_f; L^2(\Omega)). \quad (5.38)$$

*Proof.* Taking into account hypotheses  $(\widehat{\mathbf{H1}})$ ,  $(\widehat{\mathbf{H2}})$  and  $(\widehat{\mathbf{H5}})$ , we easily deduce that the body forces associated to Problem  $(\widehat{VP}_D)_t$  verify:

$$\partial_t^1 \mathbf{b} - \alpha \nabla(\theta_r(3\lambda + 2\mu)) \in W^{2,2}(0, t_f; \mathbf{L}^2(\Omega)).$$

In the same way, considering assumption  $(\widehat{\mathbf{H6}})$ , we obtain:

$$\partial_t^1 f \in W^{1,2}(0, t_f; L^2(\Omega)).$$

Furthermore, from Lemma 5.3.11,  $\widehat{\mathbf{u}}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$  and in consequence  $\widehat{\theta}_0 \in H^1(\Omega)$ , so hypothesis  $(\mathbf{H12})$  of Chapter 4 is verified.

In order to complete the proof of the existence and uniqueness of solution for Problem  $(\widehat{VP}_D)_t$  it is necessary to prove hypothesis  $(\mathbf{H13})$  for the initial conditions of this problem. From definition of  $\widehat{\mathbf{u}}_0$ , we can prove that  $(\widehat{\mathbf{u}}_0, \widehat{\theta}_0)$  verifies the weak equality:

$$a(\widehat{\mathbf{u}}_0, \mathbf{v}) - m(\widehat{\theta}_0 - \theta_r, \mathbf{v}) = \int_{\Omega} (\partial_t^1 \mathbf{b}(0) - \text{Div}(\theta_r \alpha(3\lambda + 2\mu) \mathbf{I})) \cdot \mathbf{v} dp \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega). \quad (5.39)$$

Indeed, since  $\widehat{\mathbf{u}}_0$  is the weak solution of Problem  $(\bar{P}^1)$  given by equation (5.26), considering the definition of  $\bar{\mathbf{b}}^1$  (see (5.28)), we get:

$$\begin{aligned} - \int_{\Omega} \text{Div}(\bar{\Lambda}^{-1} : \varepsilon(\widehat{\mathbf{u}}_0)) \cdot \mathbf{v} dp &= \int_{\Omega} \left[ \partial_t^1 \mathbf{b}(0) - \text{Div} \left( \frac{\alpha(3\lambda + 2\mu)}{\rho_0 c_F} \text{Div}(k \nabla \theta_0) \mathbf{I} \right) \right] \cdot \mathbf{v} dp \\ &\quad - \int_{\Omega} \left[ \text{Div} \left( \frac{\alpha(3\lambda + 2\mu) f(0)}{\rho_0 c_F} \mathbf{I} \right) \right] \cdot \mathbf{v} dp, \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

From the definition of  $\bar{\Lambda}^{-1}$  given in (5.23) with  $\gamma = \frac{3\theta_r \alpha^2 (3\lambda + 2\mu)^2}{\rho_0 c_F}$ , we have:

$$\begin{aligned} - \int_{\Omega} \left[ \text{Div}(\Lambda^{-1} : \varepsilon(\widehat{\mathbf{u}}_0)) + \text{Div} \left( \frac{\theta_r \alpha^2 (3\lambda + 2\mu)^2}{\rho_0 c_F} \text{Div}(\widehat{\mathbf{u}}_0) \mathbf{I} \right) \right] \cdot \mathbf{v} dp &= \int_{\Omega} \partial_t^1 \mathbf{b}(0) \cdot \mathbf{v} dp \\ - \int_{\Omega} \left[ \text{Div} \left( \frac{\alpha(3\lambda + 2\mu)}{\rho_0 c_F} \text{Div}(k \nabla \theta_0) \mathbf{I} \right) + \text{Div} \left( \frac{\alpha(3\lambda + 2\mu) f(0)}{\rho_0 c_F} \mathbf{I} \right) \right] \cdot \mathbf{v} dp &\text{, for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

By rearranging the terms, we obtain

$$\begin{aligned} - \int_{\Omega} \text{Div} \left( \Lambda^{-1} : \varepsilon(\widehat{\mathbf{u}}_0) - \alpha(3\lambda + 2\mu) \left[ - \frac{\theta_r \alpha(3\lambda + 2\mu)}{\rho_0 c_F} \text{Div}(\widehat{\mathbf{u}}_0) \mathbf{I} + \frac{\text{Div}(k \nabla \theta_0) \mathbf{I}}{\rho_0 c_F} + \frac{f(0)}{\rho_0 c_F} \mathbf{I} \right] \right) \cdot \mathbf{v} dp \\ = \int_{\Omega} \partial_t^1 \mathbf{b}(0) \cdot \mathbf{v} dp, \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

Furthermore, considering expression of  $\bar{\theta}^1$  (see (5.29)) and taking into account that  $\widehat{\theta}_0 = \bar{\theta}^1$ , this equation can be rewritten as

$$\begin{aligned} & - \int_{\Omega} \text{Div} \left( \Lambda^{-1} : \varepsilon(\widehat{\mathbf{u}}_0) - \alpha(3\lambda + 2\mu)(\widehat{\theta}_0 - \theta_r)\mathbf{I} \right) \cdot \mathbf{v} \, dp \\ & = \int_{\Omega} [\partial_t^1 \mathbf{b}(0) - \text{Div}(\alpha\theta_r(3\lambda + 2\mu)\mathbf{I})] \cdot \mathbf{v} \, dp, \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

Hence, we can easily obtain the weak formulation (5.39).

Furthermore, by definition of Problem  $(\bar{P}^1)$ , we have  $\widehat{\mathbf{u}}_0 = \mathbf{0}$  on  $\Gamma$ . Therefore, it only remains to prove that  $\widehat{\theta}_0 = 0$  on  $\Gamma$ . But, thanks to equality (5.29) and taking into account assumptions  $(\widehat{H}2)$ ,  $(H3)$ ,  $(\widehat{H}6)$ ,  $(\widehat{H}12)$  and  $(\widehat{H}14)$ , we deduce that  $\widehat{\theta}_0 = 0$  on  $\Gamma$ . Then hypothesis  $(H13)$  of the theorems of existence and uniqueness is true when they are applied to Problem  $(\widehat{VP}_D)_t$ . Notice that hypothesis  $(\widehat{H}14)$  is necessary to obtain the boundary condition  $\widehat{\theta}_0 = 0$  on  $\Gamma$ .

Summing up, from Theorems 4.4.5 and 4.4.10, we can conclude the existence of a unique weak solution  $(\widehat{\mathbf{u}}, \widehat{\theta})$  of Problem  $(\widehat{VP}_D)_t$  with regularity (5.37)-(5.38). □

**The solution of Problem  $(\widehat{VP}_D)_t$  is the derivative of the solution of Problem  $(VP_D)$ .** We introduce the helpful functions

$$\mathbf{w}(t) = \mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s) \, ds \quad \text{and} \quad \Theta(t) = \theta_0 + \int_0^t \widehat{\theta}(s) \, ds. \quad (5.40)$$

From the regularity properties (5.37) and (5.38), we deduce that

$$\mathbf{w} \in W^{1,\infty}(0, t_f; \mathbf{H}_0^1(\Omega)), \quad \partial_t^2 \mathbf{w} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ with } \mathbf{w}(0) = \mathbf{u}_0,$$

and

$$\Theta \in W^{1,\infty}(0, t_f; H_0^1(\Omega)), \quad \partial_t^2 \Theta \in L^2(0, t_f; L^2(\Omega)) \text{ with } \Theta(0) = \theta_0.$$

Next, we are going to integrate in time the equations of Problem  $(\widehat{VP}_D)_t$  and we will get that  $\mathbf{w}$  and  $\Theta$  are also solutions of Problem  $(VP_D)$ . Then, by uniqueness of solution of Problem  $(VP_D)$  we will obtain that

$$\widehat{\mathbf{u}}(t) = \partial_t^1 \mathbf{u}(t), \quad \widehat{\theta}(t) = \partial_t^1 \theta(t),$$

which complete the proof. In effect, integrating the equations of Problem  $(\widehat{VP}_D)_t$  over  $(0, t)$ , we get

$$\begin{aligned} & \int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v}) \, ds - \int_0^t m(\widehat{\theta}(s) - \theta_r, \mathbf{v}) \, ds = \int_0^t \int_{\Omega} \partial_t^1 \mathbf{b}(t) \cdot \mathbf{v} \, dp \, ds \\ & - \int_0^t \int_{\Omega} \alpha \nabla(\theta_r(3\lambda + 2\mu)) \cdot \mathbf{v} \, dp \, ds, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (5.41) \\ & \int_0^t (\partial_t^1 \widehat{\theta}(s), \phi)_2 \, ds + \int_0^t \kappa(\widehat{\theta}(s), \phi) \, ds + \int_0^t m(\phi, \partial_t^1 \widehat{\mathbf{u}}(s)) \, ds = \int_0^t \int_{\Omega} \frac{\partial_t^1 f(s)}{\theta_r} \phi \, dp \, ds, \quad \forall \phi \in H_0^1(\Omega). \end{aligned}$$

Applying the Green's formula to term  $m(\theta_r, \mathbf{v})$  of the first member of equation (5.41), this can be rewritten as follows:

$$\int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v}) ds - \int_0^t m(\widehat{\theta}(s), \mathbf{v}) ds = \int_{\Omega} (\mathbf{b}(t) - \mathbf{b}(0)) \cdot \mathbf{v} dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (5.42)$$

$$\begin{aligned} (\widehat{\theta}(t) - \widehat{\theta}(0), \phi)_2 + \int_0^t \kappa(\widehat{\theta}(s), \phi) ds + m(\phi, \widehat{\mathbf{u}}(t) - \widehat{\mathbf{u}}(0)) \\ = \int_{\Omega} \frac{(f(t) - f(0))}{\theta_r} \phi dp, \quad \forall \phi \in H_0^1(\Omega). \end{aligned} \quad (5.43)$$

On the other hand, considering  $(\widehat{\mathbf{H13}})$ , we have:

$$\int_{\Omega} \mathbf{b}(0) \cdot \mathbf{v} dp = a(\mathbf{u}_0, \mathbf{v}) - m(\theta_0 - \theta_r, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega).$$

In addition, due to expression of  $\widehat{\theta}_0 = \bar{\theta}^1$  (see equation (5.29)), we can deduce:

$$(\widehat{\theta}_0, \phi)_2 = -m(\phi, \widehat{\mathbf{u}}_0) - \kappa(\theta_0, \phi) + \int_{\Omega} \frac{f(0)}{\theta_r} \phi dp, \quad \forall \phi \in H_0^1(\Omega).$$

Thus, if we replace the previous equalities in expressions (5.42) and (5.43), we obtain

$$\begin{aligned} \int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v}) ds - \int_0^t m(\widehat{\theta}(s), \mathbf{v}) ds &= \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp - a(\mathbf{u}_0, \mathbf{v}) + m(\theta_0 - \theta_r, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (\widehat{\theta}(t), \phi)_2 + \int_0^t \kappa(\widehat{\theta}(s), \phi) ds + m(\phi, \widehat{\mathbf{u}}(t)) &+ \kappa(\theta_0, \phi) - \int_{\Omega} \frac{f(0)}{\theta_r} \phi dp \\ &= \int_{\Omega} \frac{(f(t) - f(0))}{\theta_r} \phi dp, \quad \forall \phi \in H_0^1(\Omega). \end{aligned}$$

These equations can be rewritten as follows:

$$a(\mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s) ds, \mathbf{v}) - m(\theta_0 + \int_0^t \widehat{\theta}(s) ds - \theta_r, \mathbf{v}) = \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (5.44)$$

$$(\widehat{\theta}(t), \phi)_2 + \kappa(\theta_0 + \int_0^t \widehat{\theta}(s) ds, \phi) + m(\phi, \widehat{\mathbf{u}}(t)) = \int_{\Omega} \frac{f(t)}{\theta_r} \phi dp, \quad \forall \phi \in H_0^1(\Omega). \quad (5.45)$$

Therefore,  $\mathbf{w}$  and  $\Theta$  are solution of the problem:

$$\begin{aligned} a(\mathbf{w}(t), \mathbf{v}) - m(\Theta(t) - \theta_r, \mathbf{v}) &= \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (\partial_t^1 \Theta(t), \phi)_2 + \kappa(\Theta(t), \phi) + m(\phi, \partial_t^1 \mathbf{w}(t)) &= \int_{\Omega} \frac{f(t)}{\theta_r} \phi dp, \quad \forall \phi \in H_0^1(\Omega), \\ \mathbf{w}(0) &= \mathbf{u}_0, \quad \Theta(0) = \theta_0, \end{aligned}$$

which coincides with Problem  $(VP_D)$  (see equations (5.31a)-(5.31c)). Since this problem has a unique solution, we can conclude that

$$\widehat{\mathbf{u}}(t) = \partial_t^1 \mathbf{u}(t) \quad \text{and} \quad \widehat{\theta}(t) = \partial_t^1 \theta(t).$$

Finally from the regularity properties (5.37) and (5.38), we obtain

$$\begin{aligned} \mathbf{u} &\in W^{1,\infty}(0, t_f; \mathbf{H}_0^1(\Omega)), \quad \partial_t^2 \mathbf{u} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ and} \\ \theta &\in W^{1,\infty}(0, t_f; \mathbf{H}_0^1(\Omega)), \quad \partial_t^2 \theta \in L^2(0, t_f; L^2(\Omega)), \end{aligned}$$

and Theorem 5.3.9 for  $r = 1$  is proved. □

In order to conclude this part, we summarize the regularity properties in space and time for the solution of Problem (VP<sub>D</sub>) of Theorems 5.2.5 and 5.3.9.

To do so, we need that the thermal conductivity belongs to  $\mathcal{C}^{0,1}(\bar{\Omega})$ , so, we replace  $(\widehat{\text{H4}})$  by the following hypothesis with  $r \in \mathbb{N}$ :

$(\widehat{\text{H4}})$  The thermal conductivity coefficient  $k \in \mathcal{C}^{2r}(\bar{\Omega}) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ , and there exists  $k_{min} > 0$  such that  $k(p) \geq k_{min}$  in  $\Omega$ .

**Remark 5.3.13.** We notice that it would be enough to consider  $k \in W^{2r,\infty}(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$ .

**Theorem 5.3.14.** Let  $r \in \mathbb{N}$  be a fixed parameter. Under assumptions  $(\widehat{\text{H1}})$ ,  $(\widehat{\text{H2}})$ , (H3),  $(\widehat{\text{H4}})$ ,  $(\widehat{\text{H5}})$ ,  $(\widehat{\text{H6}})$ ,  $(\widehat{\text{H12}})$ ,  $(\widehat{\text{H13}})$  and  $(\widehat{\text{H14}})$ , we have

$$\begin{aligned} \mathbf{u} &\in W^{r,\infty}(0, t_f; \mathbf{H}_0^1(\Omega) \cap \mathbf{H}_{Loc}^2(\Omega)), \quad \partial_t^{r+1} \mathbf{u} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ and} \\ \theta &\in W^{r,\infty}(0, t_f; H_0^1(\Omega) \cap H_{Loc}^2(\Omega)), \quad \partial_t^{r+1} \theta \in L^2(0, t_f; L^2(\Omega)). \end{aligned}$$

*Proof.* The proof is deduced directly from Theorems 5.2.5 and 5.3.9. □



# Conclusions

The aim of this dissertation thesis is the modelling and mathematical analysis of certain nonlinear coupled thermomechanical problems in solid mechanics, arising from real processes under a strong raise in temperature. Model equations for thermoviscoelastic materials with long memory have been obtained, and existence, uniqueness and regularity of solution in two thermomechanical submodels have been shown. More specifically, existence, uniqueness and regularity of solution has been proved for a fully coupled thermoelastic submodel and for a mechanical one considering thermoviscoelastic materials and assuming that the temperature is known in the last case.

The main conclusions drawn from this dissertation thesis are briefly presented:

- General constitutive laws in order to define thermoviscoelastic materials with long memory have been introduced in Part I, including a tensorial internal variable which accounts their viscoplastic history. To guarantee the second principle of thermodynamics, and to satisfy the principle of material frame indifference and the isotropy, some restrictions on the response functions have been obtained. Moreover, the corresponding equilibrium equations -mass, momentum and energy conservation- have been deduced in Eulerian as well as in Lagrangian coordinates.

Given that thermoviscoelastic materials with long memory are mainly employed in materials processing, three different linearizations for their conservation laws have been introduced. These linearizations will allow us:

- to model processes that present very strong changes in temperature -first linearization-;
- to perform numerical simulation using, for example, an incremental method -second linearization-; and
- to simplify the viscoplastic nonlinearities coming from tensorial internal variable -third linearization-.

These three linearizations have been illustrated with Maxwell–Norton materials.

- Existence, uniqueness and regularity of a quasistatic thermoviscoelastic problem with mixed boundary conditions have been proved in Part II. This mechanical problem corresponds to the linearized motion equation according to the third linearization introduced in Part I. The behaviour law is of the Maxwell–Norton type, whose deformation rate tensor is a superposition of elastic, viscoplastic and thermal contributions. Its elastic part is given by

means of a Hooke's law, whereas the viscoplastic term is related to the classical Norton–Hoff law and the thermal part is a generalization of the Arrhenius's law. In addition, all mechanical coefficients depend on temperature.

The methodology for proving the existence and uniqueness of solution is based on:

- (i) a discretization in time of the problem by an implicit scheme and
- (ii) a limit procedure using monotone techniques (see Chapter 2).

Subsequently, local regularity properties of the stress solution of this problem have been obtained. Assuming additional hypotheses on the data, spatial  $H_{Loc}^1$  regularity of stresses and  $H_{Loc}^2$  regularity are also achieved (see Chapter 3).

- A mathematical analysis of a quasistatic coupled thermoelastic problem has been carried out in Part III. Particularly, the existence, uniqueness and regularity of the problem solution has been accomplished with mixed displacement–traction boundary conditions for the mechanical submodel, and mixed boundary conditions, including a Robin boundary condition, for the thermal one. Furthermore, the reference temperature, the thermal conductivity and the Lamé's parameters depend on the material point. All these contributions extend the previous results in the literature.

The proof of the existence of the problem has been obtained applying a Galerkin's method and the uniqueness has been proved using Gronwall's lemma (see Chapter 3).

Regularity properties of the solution with respect to space and time have been obtained in Chapter 5. Specifically, spatial  $H_{Loc}^2$  regularity in displacements and temperature is achieved, along with  $W^{r,\infty}$  ( $r \in \{0\} \cup \mathbb{N}$ ) regularity of displacements and temperature with respect to time. The same is concluded for a simplified case of this problem, where homogeneous Dirichlet conditions on the boundary have been imposed.

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# Resumen

En la actualidad, siguen apareciendo nuevos materiales en la industria cuyo comportamiento a altas temperaturas se aleja del típico material elástico lineal. Además, con más frecuencia se usan este tipo de materiales en el campo de la Ingeniería Civil por lo que estudiar su respuesta cuando se someten a altas temperaturas es muy importante, y cualquier contribución en el avance de este tema es esencial. En consecuencia, un profundo conocimiento y una buena comprensión de la modelización y del análisis matemático en termomecánica es necesario y fundamental. Por este motivo, en esta memoria se plantea avanzar en el conocimiento del comportamiento no lineal de ciertos materiales a altas temperaturas, y estudiar algunos problemas bajo este contexto para este tipo de materiales.

De este modo, el objetivo de esta tesis doctoral es el estudio de ciertos problemas termomecánicos acoplados no lineales de la mecánica de sólidos, los cuales surgen de procesos reales bajo fuertes gradientes de temperatura, tales como el procesado de materiales o el incendio de un edificio. Con esta temática en común, este manuscrito, dividido en tres partes, aborda la modelización y el análisis matemático de varios submodelos en termomecánica. En la primera parte se deducen varios modelos termomecánicos acoplados para materiales termoviscoelásticos con memoria larga y con tensiones dependientes de la temperatura. Estos modelos permiten modelar, por ejemplo, los procesos de solidificación de coladas, de extrusión de piezas de metal o de laminación de metales y aleaciones. La segunda y tercera parte de la memoria, con una estructura similar, se dedican al estudio de la existencia, unicidad y regularidad de solución de dos submodelos termomecánicos. En la segunda parte se analiza el submodelo mecánico con una ley de comportamiento no lineal de tipo Maxwell-Norton, con parámetros dependientes de la temperatura y condiciones de contorno mixtas desplazamiento-tracción. En este caso, se supone que el campo de temperaturas es conocido. Bajo este contexto se pueden modelar, por ejemplo, las deformaciones mecánicas sufridas por una estructura de aleación expuesta al fuego. En la tercera parte de la memoria se analiza un problema termoelástico completamente acoplado, con condiciones de contorno mixtas e incluyendo una condición de contorno tipo Robin para el submodelo térmico. Este modelo permite realizar una primera aproximación de las deformaciones termomecánicas de una estructura expuesta al fuego.

En lo que sigue, veremos cada una de las partes de la tesis más detalladamente.

## *Parte I. Modelización de materiales con memoria larga*

La primera parte de esta tesis, que consta de un único capítulo, está dedicada a la modelización

de materiales termoviscoelásticos con memoria larga, obteniéndose varios submodelos del correspondiente problema termomecánico acoplado. A partir de los principios termodinámicos clásicos se deduce un modelo termomecánico acoplado asociado a estos materiales, tanto en coordenadas Eulerianas como Lagrangianas, considerando variables tensoriales internas. Para adaptarse mejor a distintos contextos reales proponemos tres linealizaciones diferentes para el modelo obtenido en la configuración de referencia:

- una primera linealización bajo la hipótesis de pequeñas deformaciones, que permite, por ejemplo, modelar procesos que presentan variaciones importantes de temperatura;
- a partir de la primera linealización se obtiene una segunda linealizando, con respecto a la temperatura, el término de la función respuesta térmica que depende linealmente del gradiente de deformación. Esta segunda linealización está muy bien adaptada para introducir, por ejemplo, un algoritmo numérico siguiendo las técnicas de métodos incrementales para resolver el correspondiente problema no lineal;
- a partir de la segunda linealización se deduce una tercera, linealizando la parte plástica del tensor de tensiones con respecto a la variable interna tensorial para simplificar las no linealidades. Este submodelo permite recuperar modelos clásicos en la literatura como el asociado al comportamiento de materiales de Maxwell-Norton, con coeficientes dependientes de la temperatura, durante procesos con fuertes variaciones térmicas.

En este contexto, se introduce como variable interna un tensor simétrico de segundo orden cuya evolución temporal está dada por una ecuación diferencial ordinaria que depende de la variación del correspondiente potencial de plasticidad. La introducción de esta variable interna permite controlar la disipación de energía derivada de la historia de las tensiones viscoplásticas desde el instante inicial. La metodología empleada es una generalización de la utilizada por Carlson [26] para materiales elásticos, y por Bermúdez [19] para materiales de Coleman-Noll, introduciendo una variable interna capaz de dar cuenta, en cada instante de tiempo y en cada punto, de la historia plástica del material.

Los resultados de esta primera parte están publicados en Naya-Riveiro y Quintela [89].

En la literatura hay distintos trabajos en los que se modela el comportamiento termodinámico de materiales no lineales introduciendo distintos tipos de variables internas, diferentes, por supuesto, de las clásicas en este ámbito: gradiente de deformación y temperatura. Así, por ejemplo, Simo y Miehe [95] y más tarde Serrano *et al.* [93] introducen una variable interna escalar correspondiente a la parte de la entropía que disipa energía para definir nuevas leyes termoplásticas. También, Coleman y Gurtin [30] consideran variables internas vectoriales para modelizar los fluidos, y en particular los gases ideales, y Alber [3] para describir el comportamiento mecánico de materiales con ecuaciones constitutivas de tipo monótono. Sin embargo, hasta donde nosotras conocemos, solo el trabajo de Lattanzio y Tzavaras [77] utiliza una variable interna tensorial para describir las tensiones viscoelásticas de las ecuaciones elastodinámicas. Esta variable interna viene definida a través de la teoría de memoria evanescente u olvidadiza ("fading memory") con núcleo o kernel que comprende un único tiempo de relajación, mientras que la considerada en esta memoria, como ya se ha comentado anteriormente, viene dada por la variación del potencial de plasticidad.

Los materiales de Maxwell-Norton con coeficientes mecánicos fuertemente dependientes de la temperatura son un ejemplo particular de los materiales termoviscoelásticos con memoria larga modelados en esta parte de la memoria. De hecho, se demuestra que la tercera linealización antes anunciada permite recuperar este modelo, que es el modelo termoviscoelástico más usual. Estos materiales se usan, por ejemplo, en procesos de colada (ver, por ejemplo, Drezet *et al.* [36, 37]), de laminación de metales y aleaciones o de extrusión de piezas, donde existen fuertes gradientes de temperatura. Para su simulación numérica es muy importante incluir los términos de disipación mecánica en la ecuación de la energía y la dependencia de la temperatura en todos los coeficientes termomecánicos.

Esta primera parte de la tesis está organizada como sigue:

- se introduce la notación y una revisión de algunos conceptos y de los principios de conservación de la termomecánica del continuo;
- se definen las funciones respuesta asociadas a los materiales termoviscoelásticos con memoria larga y se deducen las restricciones necesarias para satisfacer el segundo principio de la termodinámica, el principio de la indiferencia material y la isotropía;
- se obtienen las ecuaciones de equilibrio asociadas a estos materiales en coordenadas Eulerianas, y se reescriben en la configuración de referencia, es decir, en coordenadas Lagrangianas;
- asumiendo pequeños desplazamientos y/o pequeñas variaciones de temperatura y/o pequeñas perturbaciones de la variable interna tensorial se muestran tres linealizaciones de las leyes de equilibrio;
- se presentan las funciones respuesta asociadas a los materiales de Maxwell-Norton con coeficientes mecánicos dependientes de la temperatura, como caso particular de materiales termoviscoelásticos con memoria larga;
- y, finalmente, se reescriben los modelos linealizados para los materiales de Maxwell-Norton identificando el modelo clásico asociado.

## *Parte II. Análisis matemático de un problema viscoelástico con coeficientes dependientes de la temperatura*

En esta segunda parte de la memoria se realiza el análisis matemático del submodelo mecánico correspondiente a los materiales de Maxwell-Norton obtenido en la Parte I, suponiendo que el campo de temperaturas es conocido. Se trata, por tanto, de analizar la evolución cuasiestática de un problema termoviscoelástico con memoria larga y con coeficientes mecánicos dependientes de la temperatura. En esta ley, la variación del tensor de deformación es una superposición de las contribuciones elásticas, viscoelásticas y térmicas. La parte elástica viene dada por una ley de Hooke, la parte viscoelástica por la ley no lineal de Norton-Hoff y la parte térmica es una generalización de la ley de Arrhenius.

Esta segunda parte está dividida en dos capítulos, en el primero se demuestra la existencia y unicidad de solución de este problema y en el segundo se obtienen propiedades de regularidad en espacio de su solución.

Los resultados de esta segunda parte están publicados en Barral *et al.* [12, 13].

Para demostrar la existencia y unicidad de solución se siguen las técnicas usadas en Djaoua y Suquet [35] y Barral y Quintela [15], que se basan en discretizar el problema mediante un esquema implícito y probar la convergencia de la solución discretizada a la solución del problema inicial aplicando técnicas de monotonía. Si bien en estos trabajos se dan resultados de existencia y unicidad para problemas viscoelásticos de tipo Maxwell-Norton, en el primero con condiciones de contorno mixtas, y en el segundo incorporando una condición de contacto tipo Signorini, en ambos casos los coeficientes son independientes de la temperatura. Por este motivo, la principal contribución de este capítulo es, no solo considerar una completa dependencia de la temperatura de la ley de comportamiento, sino también incluir la contribución de la parte térmica al tensor de tensiones, lo que hace que el problema tenga una cierta dificultad añadida.

El Capítulo 2 sigue el siguiente esquema:

- se realiza una revisión bibliográfica de los resultados más importantes relativos a la existencia y unicidad de este tipo de modelos;
- se introduce el modelo matemático a considerar;
- se definen los espacios funcionales apropiados siguiendo los trabajos de Geymonat y Suquet [63];
- se presentan las hipótesis que deben verificar los datos del problema;
- se transforma el problema inicial en uno homogéneo mediante un cambio de variable por traslación;
- se discretiza el problema en tiempo mediante un método de Euler implícito obteniendo una inecuación variacional en tensiones;
- se demuestra que el problema débil discretizado tiene una única solución aplicando resultados de análisis convexo de Ekeland y Teman [47]; a partir de esta tensión solución discreta, se obtiene el campo discreto de desplazamientos asociado;
- se obtienen estimaciones a priori que permiten pasar al límite;
- se demuestra que el límite hallado es una solución del problema inicial mediante técnicas de monotonía;
- y, finalmente, se prueba la unicidad del problema continuo.

Como se ha comentado anteriormente, en el segundo capítulo de esta segunda parte de la tesis se estudian las propiedades locales de regularidad en espacio del tensor solución. Para alcanzar este objetivo, se sigue la metodología empleada en Bensoussan y Frehse [16, 17], que se basa en obtener estimaciones a priori de las derivadas espaciales del tensor solución mediante un cálculo formal. En los trabajos citados anteriormente se obtienen propiedades locales de regularidad de la solución del problema viscoelástico de tipo Maxwell-Norton, en el primero considerando el caso estático, y en el segundo incluyendo la dependencia del problema con respecto al tiempo. Pero en

ninguno de ellos se considera la contribución térmica del tensor de tensiones ni la dependencia de los parámetros mecánicos con respecto a la temperatura.

Este Capítulo 3 está organizado como sigue:

- para facilitar la lectura y simplificar el cálculo desarrollado en esta parte se introduce una notación preliminar referente al cálculo diferencial y al producto tensorial similar a la dada por Segel [92];
- se reescribe el problema estudiado en el capítulo previo utilizando la nueva notación introducida;
- se obtienen dos resultados de regularidad  $H^1_{Loc}$  en espacio del tensor solución asumiendo hipótesis adicionales sobre los datos; en el segundo resultado se suavizan determinadas hipótesis sobre los datos a costa de suponer que el desviatorio del tensor de tensiones está acotado;
- usando una desigualdad de Gagliardo-Nirenberg, que se obtiene al aplicar un resultado de Friedman [53], se demuestra la regularidad espacial  $H^2_{Loc}$  para las tensiones.

### *Parte III. Análisis matemático de un problema termoelástico*

En esta tercera parte de la tesis se lleva a cabo un análisis matemático del acoplamiento entre las ecuaciones de conservación de la energía y del movimiento para materiales termoelásticos. Este modelo se deduce de las ecuaciones obtenidas en la primera parte de esta memoria, siendo su parte mecánica una simplificación del modelo estudiado en la segunda parte. Se trata, por tanto, de analizar un problema termoelástico acoplado cuasiestático con condiciones de contorno mixtas desplazamiento-tracción sobre el submodelo mecánico, y con condiciones de contorno mixtas, incluyendo una condición de contorno tipo Robin, para el submodelo térmico. Además, la temperatura de referencia, la conductividad térmica y los parámetros de Lamé dependen del punto material. Esta tercera parte también está dividida en dos capítulos, en el primero se demuestra la existencia y unicidad de solución de este problema y en el segundo se obtienen las propiedades de regularidad de la solución, es decir, de desplazamientos y temperatura, tanto en espacio como en tiempo.

Para probar la existencia de solución se utiliza un método de Galerkin, siguiendo la metodología usada en Bermúdez y Viaño [18], Figueiredo y Trabucho [48, 49], Gawinecki [55, 57] y Viaño [98]. Para demostrar la unicidad de solución se siguen las técnicas empleadas en Gawinecki [56, 57, 61] y Gawinecki *et al.* [62]. En todos estos trabajos se dan resultados de existencia y unicidad para problemas termoelásticos; por ejemplo, en Viaño [98] se consideran materiales elásticos isotrópicos y anisótropos en contacto con otro cuerpo elástico tomando la temperatura de referencia constante; en Figueiredo y Trabucho [48, 49] se analizan problemas con tres tipos diferentes de materiales con contacto, siendo la ecuación del movimiento dinámica. La principal contribución de este capítulo es obtener un resultado de existencia y unicidad de solución considerando que:

- el problema es cuasiestático;

- hay una condición de contorno de transferencia de calor por convección, que viene dada por una condición tipo Robin;
- la temperatura de referencia, la conductividad térmica y los parámetros de Lamé dependen de la variable espacial.

Todas estas aportaciones amplían los trabajos de la literatura, y causan diversas dificultades para calcular las estimaciones a priori y demostrar la unicidad de solución del problema considerado.

El Capítulo 4 tiene el siguiente esquema:

- se hace una revisión bibliográfica de los resultados más importantes relativos a la existencia y unicidad de solución de este tipo de modelos;
- se introduce el modelo matemático a considerar;
- se definen los espacios funcionales apropiados;
- se propone una formulación débil para el problema;
- se presentan las hipótesis que deben verificar los datos del problema;
- se realiza un cambio de variable por traslación para obtener un problema cuyas condiciones de contorno Dirichlet son homogéneas;
- se definen aproximaciones del problema mediante un método de Galerkin;
- se prueba la existencia y unicidad de solución de los problemas aproximados;
- se obtienen algunas estimaciones a priori para la sucesión de Galerkin que permiten pasar al límite;
- se prueba la convergencia de la sucesión de Galerkin;
- se demuestra que el límite de la sucesión de Galerkin es una solución del problema inicial;
- y, finalmente, se obtiene la unicidad de solución aplicando el lema de Gronwall.

El análisis matemático del problema termoelástico se concluye en el segundo capítulo de esta tercera parte de la memoria con el estudio de las propiedades de regularidad de la solución. Para obtener las propiedades de regularidad con respecto al espacio se sigue la metodología empleada en Kačur y Ženíšek [75], que consiste en reescribir el problema acoplado como dos ecuaciones definidas mediante operadores elípticos y aplicar resultados dados en Athanasiadis y G. Stratis [8], Lions y Magenes [83], Mizohata [87] y Nečas [90]. En estos trabajos no se considera el caso en que los coeficientes mecánicos, la conductividad térmica ni la temperatura de referencia dependan de la variable espacial, como es el caso estudiado en esta memoria.

Para demostrar los resultados de regularidad con respecto al tiempo se aplican las técnicas usadas en Gawinecki [56, 58, 59, 60, 61] y Gawinecki *et al.* [62], ampliando los resultados allí obtenidos a condiciones de contorno mixtas e incluyendo una condición de contorno tipo Robin.

Se presentan dos resultados de regularidad con respecto al tiempo: en el primero se demuestra la regularidad  $W^{r,\infty}$  ( $r \in \{0\} \cup \mathbb{N}$ ) del problema analizado en el capítulo anterior; para obtener este resultado se incrementan las propiedades de regularidad de los datos y de las condiciones iniciales; en el segundo resultado se demuestra que en el caso Dirichlet homogéneo se puede obtener la misma regularidad en tiempo sin pedir tanta regularidad a las condiciones iniciales. Para ello, en este segundo resultado, se definen las condiciones iniciales de los problemas asociados a las sucesivas derivadas como solución de un problema de elasticidad lineal auxiliar y se demuestran las propiedades necesarias de regularidad de estas condiciones iniciales así definidas utilizando resultados de Nečas [90] y Agmon *et al.* [2].

Este último capítulo de la tesis, está organizado como sigue:

- asumiendo hipótesis de regularidad espacial sobre algunos datos se obtiene regularidad  $H_{Loc}^2$  en espacio del campo de desplazamientos y de temperaturas del problema estudiado en el capítulo anterior;
- se presenta un resultado de regularidad en tiempo  $W^{r,\infty}$  ( $r \in \{0\} \cup \mathbb{N}$ ) para el mismo problema tal y como se indicó anteriormente;
- se demuestra un resultado de regularidad en tiempo  $W^{r,\infty}$  ( $r \in \{0\} \cup \mathbb{N}$ ) para el problema Dirichlet homogéneo correspondiente, asumiendo hipótesis más suaves sobre las condiciones iniciales;
- y, finalmente, para cada uno de los problemas se agrupan las propiedades de regularidad en espacio y tiempo en un resultado conjunto.

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# Bibliography

- [1] L. Adam and J.-P. Ponthot. Thermomechanical modeling of metals at finite strains: first and mixed order finite elements. *International Journal of Solids and Structures*, 42(21-22):5615 – 5655, 2005. PACAM VIII SPECIAL ISSUE.
- [2] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.
- [3] H.-D. Alber. *Materials with memory. Initial-boundary value problems for constitutive equations with internal variables*, volume 1682 of *Lecture notes in Mathematics*. Springer, Berlin, 1998.
- [4] H.-D. Alber and K. Chelmiński. Quasistatic problems in viscoplasticity theory. Technical Report 2190, Darmstadt University of Technology, 2002.
- [5] H.-D. Alber and K. Chelmiński. Quasistatic problems in viscoplasticity theory. I. Models with linear hardening. In *Operator theoretical methods and applications to mathematical physics*, volume 147 of *Oper. Theory Adv. Appl.*, pages 105–129. Birkhäuser, Basel, 2004.
- [6] H.-D. Alber and K. Chelmiński. Quasistatic problems in viscoplasticity theory. II. Models with nonlinear hardening. *Math. Models Methods Appl. Sci.*, 17(2):189–213, 2007.
- [7] K.A. Ames and L.E. Payne. Uniqueness and continuous dependence of solutions to a multi-dimensional thermoelastic contact problem. *Journal of Elasticity*, 34:139–148, 1994.
- [8] C. Athanasiadis and I.G. Stratis. On some elliptic transmission problems. *Ann. Polon. Math.*, 63(2):137–154, 1996.
- [9] H. Attouch and A. Damlamian. Problèmes d’évolution dans les Hilberts et applications. *J. Math. Pures Appl. (9)*, 54:53–74, 1975.
- [10] M. Badii. Existence of periodic solutions for the quasi-static thermoelastic thermistor problem. *NoDEA Nonlinear Differential Equations Appl.*, 16(1):1–15, 2009.
- [11] P. Barral, A. Bermúdez, M.C. Muñiz, M.V. Otero, P. Quintela, and P. Salgado. Numerical simulation of some problems related to aluminium casting. *Journal of Materials Processing Tech.*, 142(2):383–399, 2003.

- [12] P. Barral, M.C. Naya-Riveiro, and P. Quintela. Mathematical analysis of a viscoelastic problem with temperature-dependent coefficients. I. Existence and uniqueness. *Math. Methods Appl. Sci.*, 30(13):1545–1568, 2007.
- [13] P. Barral, M.C. Naya-Riveiro, and P. Quintela. Mathematical analysis of a viscoelastic problem with temperature-dependent coefficients. II. Regularity. *Math. Methods Appl. Sci.*, 30(13):1569–1592, 2007.
- [14] P. Barral and P. Quintela. A numerical algorithm for prediction of thermomechanical deformation during the casting of aluminium alloy ingots. *Finite Elem. Anal. Des.*, 34(2):125–143, 2000.
- [15] P. Barral and P. Quintela. Existence of a solution for a Signorini contact problem for Maxwell-Norton materials. *IMA J. Appl. Math.*, 67(6):525–549, 2002.
- [16] A. Bensoussan and J. Frehse. Asymptotic behaviour of Norton-Hoff's law in plasticity theory and  $H^1$  regularity. In *Boundary value problems for partial differential equations and applications*, volume 29 of *RMA Res. Notes Appl. Math.*, pages 3–25. Masson, Paris, 1993.
- [17] A. Bensoussan and J. Frehse. Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and  $H^1$  regularity. *Comment. Math. Univ. Carolin.*, 37(2):285–304, 1996.
- [18] A. Bermúdez and J.M. Viaño. Étude de deux schémas numériques pour les équations de la thermoélasticité. *RAIRO Anal. Numér.*, 17(2):121–136, 1983.
- [19] A. Bermúdez de Castro. *Continuum thermomechanics*, volume 43 of *Progress in Mathematical Physics*. Birkhäuser Verlag, Basel, 2005.
- [20] M.A. Biot. Thermoelasticity and irreversible thermodynamics. *J. Appl. Phys.*, 27:240, 1956.
- [21] D. Blanchard and G.A. Francfort. Asymptotic thermoelastic behavior of flat plates. *Quart. Appl. Math.*, 45(4):645–667, 1987.
- [22] D. Blanchard and P. Le Tallec. Numerical analysis of the equations of small strains quasistatic elastoviscoplasticity. *Numer. Math.*, 50(2):147–169, 1986.
- [23] B.A. Boley and J.H. Weiner. *Theory of thermal stresses*. John Wiley & Sons Inc., New York London, 1960.
- [24] P. Bolley and J. Camus. Équations aux dérivées partielles. Régularité pour des systèmes à coefficients constants. *C. R. Acad. Sc. Paris Sér. A*, 281(24):A1091–A1094, 1975.
- [25] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983. Théorie et applications.
- [26] D.E. Carlson. *Linear Thermoelasticity*, volume VIa/2 of *Handbuch der Physik (Encyclopedia of Physics, Mechanics of Solids II)*. Springer, Berlin, 1972.
- [27] A. Chrzęszczyk. On the regularity, uniqueness and continuous dependence for generalized solutions of some coupled problems in nonlinear theory of thermoelastic shells. *Arch. Mech. (Arch. Mech. Stos.)*, 38(1-2):97–102, 1986.

- [28] P.G. Ciarlet. *Élasticité tridimensionnelle*, volume 1 of *Recherches en Mathématiques Appliquées*. Masson, Paris, 1986.
- [29] B.D. Coleman. Thermodynamics of materials with memory. *Arch. Rational Mech. Anal.*, 17:1–46, 1964.
- [30] B.D. Coleman and M.E. Gurtin. Thermodynamics with internal state variables. *J. Chem. Phys.*, 47(2):597–613, 1967.
- [31] M.I.M. Copetti. Numerical analysis of a unilateral problem in planar thermoelasticity. *SIAM J. Numer. Anal.*, 45(6):2637–2652, 2007.
- [32] M.I.M. Copetti and C.M. Elliott. A one-dimensional quasi-static contact problem in linear thermoelasticity. *European J. Appl. Math.*, 4(2):151–174, 1993.
- [33] C.M. Dafermos. On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 29:241–271, 1968.
- [34] W.A. Day. *Heat Conduction within Linear Thermoelasticity*. Springer-Verlag, New York-Berlin, 1985.
- [35] M. Djaoua and P. Suquet. Évolution quasi-statique des milieux visco-plastiques de Maxwell-Norton. *Math. Methods Appl. Sci.*, 6(2):192–205, 1984.
- [36] J. M. Drezet, M. Rappaz, and Y. Krähenbühl. Thermomechanical Effects during Direct Chill and Electromagnetic Casting of Aluminum Alloys. Part II : Numerical Simulation. *Light Metals*, (1995):941, 1995.
- [37] J.M. Drezet and M. Plata. Thermomechanical effects during direct chill and electromagnetic casting of aluminum alloys. Part I: Experimental investigation. *Light Metals*, pages 931–940, 1995.
- [38] J.M.C. Duhamel. Second mémoire sur les phénomènes thermomécaniques. *J. de l'École Polyt.*, 15:1–57, 1837.
- [39] G. Duvaut. Free boundary problem connected with thermoelasticity and unilateral contact. In *Free boundary problems, Vol. II (Pavia, 1979)*, pages 217–236. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1980.
- [40] G. Duvaut and J.-L. Lions. Nouvelles inéquations variationnelles rencontrées en thermique et en thermoélasticité. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A1198–A1201, 1969.
- [41] G. Duvaut and J.-L. Lions. Sur de nouveaux problèmes d'inéquations variationnelles posés par la Mécanique. Le cas d'évolution. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A570–A572, 1969.
- [42] G. Duvaut and J.-L. Lions. Sur de nouveaux problèmes d'inéquations variationnelles posés par la Mécanique. Le cas stationnaire. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A510–A513, 1969.

- [43] G. Duvaut and J.-L. Lions. Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.*, 46:241–279, 1972.
- [44] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972. Travaux et Recherches Mathématiques, No. 21.
- [45] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.
- [46] C. Eck. Existence of solutions to a thermo-viscoelastic contact problem with Coulomb friction. *Math. Models Methods Appl. Sci.*, 12(10):1491–1511, 2002.
- [47] I. Ekeland and R. Temam. *Analyse convexe et problèmes variationnels*. Dunod, 1974. Collection Études Mathématiques.
- [48] I. Figueiredo and L. Trabucho. A class of contact and friction dynamic problems in thermoelasticity and in thermoviscoelasticity. *Internat. J. Engrg. Sci.*, 33(1):45–66, 1995.
- [49] I. Figueiredo and L. Trabucho. Some existence results for contact and friction problems in thermoelasticity and in thermoviscoelasticity. In *Asymptotic methods for elastic structures (Lisbon, 1993)*, pages 223–235. Gruyter, Berlin, 1995.
- [50] J. Frehse and J. Málek. Boundary regularity results for models of elasto-perfect plasticity. *Math. Models Methods Appl. Sci.*, 9(9):1307–1321, 1999.
- [51] A. Friaã. Le matériau de Norton-Hoff généralisé et ses applications en analyse limite. *C. R. Acad. Sci. Paris Sér. A-B*, 286(20):A953–A956, 1978.
- [52] A. Friaã. *La loi de Norton-Hoff généralisée en plasticité et viscoplasticité*. PhD thesis, Université Pierre et Marie Curie, Paris, 1979.
- [53] A. Friedman. *Partial differential equations*. Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.
- [54] S. Fučík and A. Kufner. *Nonlinear differential equations*, volume 2 of *Studies in Applied Mechanics*. Elsevier Scientific Publishing Co., Amsterdam, 1980.
- [55] J. Gawinecki. On the first boundary-initial value problem for thermal stresses equations of generalized thermomechanics. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.*, 29(7-8):405–411, 1981.
- [56] J. Gawinecki. Uniqueness and regularity of the solution of the first boundary-initial value problem for thermal stresses equations of classical and generalized thermomechanics. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.*, 29(11-12):231–238, 1981.
- [57] J. Gawinecki. Existence and uniqueness of the solution of the third boundary-initial value problem for thermal stresses equations of classical and generalized thermomechanics. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.*, 30(3-4):155–163, 1982.

- [58] J. Gawinecki. Existence, uniqueness and regularity of the first boundary-initial value problem for thermal stresses equations of classical and generalized thermomechanics. *J. Tech. Phys.*, 24(4):467–479, 1983.
- [59] J. Gawinecki. Existence, uniqueness and regularity of the solution of the first boundary-initial value problem for the equations of linear thermomicroelasticity. *Bull. Polish Acad. Sci. Tech. Sci.*, 34(7-8):447–460, 1986.
- [60] J. Gawinecki. Existence, uniqueness and regularity of the first boundary-initial value problem for hyperbolic equations system of the thermal stresses theory for temperature-rate-dependent solids. *Bull. Polish Acad. Sci. Tech. Sci.*, 35(7-8):411–419, 1987.
- [61] J. Gawinecki. The faedo-galerkin method in thermal stresses theory. *Comment. Math. Prace Mat.*, 27(1):83–107, 1987.
- [62] J. Gawinecki, T. Kowalski, and K. Litewska. Existence and uniqueness of the solution of the mixed boundary-initial value problem in linear thermoelasticity. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.*, 30(11-12):173–178, 1982.
- [63] G. Geymonat and P. Suquet. Functional spaces for Norton-Hoff materials. *Math. Methods Appl. Sci.*, 8(2):206–222, 1986.
- [64] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences)*. Springer-Verlag, Berlin, second edition, 1983.
- [65] M.E. Gurtin. *An introduction to continuum mechanics*, volume 158 of *Mathematics in Science and Engineering*. Academic Press Inc., New York, 1981.
- [66] T.J. Healey. Global continuation methods in three dimensional elasticity. In *Lecture notes of the NSA-UPRH Regional Conference on Global Continuation Methods in Three Dimensional Elasticity, Humacao (Puerto Rico)*. <http://mate.uprh.edu/pnm/regconf/Lectures.pdf>, 2003.
- [67] D. Helm and P. Haupt. Shape memory behaviour: modelling within continuum thermomechanics. *International Journal of Solids and Structures*, 40(4):827 – 849, 2003.
- [68] R.B. Hetnarski. The fundamental solution of the coupled thermoelastic problem for small times. *Arch. Mech. Stos.*, 16:23–31, 1964.
- [69] I. Hlaváček and J.R. Whiteman. Existence and uniqueness of solutions in strain space of elastoplastic problems with isotropic hardening. *Math. Models Methods Appl. Sci.*, 7(1):31–48, 1997.
- [70] D. Ieşan. Principes variationnels dans la théorie de la thermoélasticité couplée. *Analele științifice ale Universitatii Al.I. Cuza din Iasi. Matematica*, 12:439–456, 1966.
- [71] D. Ieşan. On the linear coupled thermoelasticity with two temperatures. *Z. Angew. Math. Phys.*, 21:583–591, 1970.
- [72] D. Ieşan. On some theorems in thermoelastodynamics. *Rev. Roumaine Sci. Tech. Sér. Méc. Appl.*, 34(2):101–111, 1989.

- [73] V. Ionescu-Cazimir. Problem of linear coupled thermoelasticity. III. Uniqueness theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.*, 12(12):565–573, 1964.
- [74] V. Ionescu-Cazimir. Problem of linear coupled thermoelasticity. IV. Uniqueness theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.*, 12(12):575, 1964.
- [75] J. Kačur and A. Ženíšek. Analysis of approximate solutions of coupled dynamical thermoelasticity and related problems. *Apl. Mat.*, 31(3):190–223, 1986.
- [76] R.J. Knops and L.E. Payne. On uniqueness and continuous dependence in dynamical problems of linear thermoelasticity. *International Journal of Solids and Structures*, 6(8):1173 – 1184, 1970.
- [77] C. Lattanzio and A.E. Tzavaras. Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics. *Arch. Ration. Mech. Anal.*, 180(3):449–492, 2006.
- [78] P. Le Tallec. *Numerical analysis of viscoelastic problems*. Recherches en Mathématiques Appliquées. Masson, Paris, 1990.
- [79] J. Lemaitre and J.L. Chaboche. *Mécanique des matériaux solides*. Dunod, Paris, 1988.
- [80] M. Lessen. The motion of a thermoelastic solid. *Quart. Appl. Math.*, 15:105, 1957.
- [81] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.
- [82] J.L. Lions. *Cours d'analyse numérique*. Cours de L'École Polytechnique. Hermann, Paris, 1973.
- [83] J.L. Lions and E. Magenes. *Problèmes aux limites non-homogènes et applications*. Dunod, Paris, 1968.
- [84] S. Mariaux, M. Rappaz, Y. Krahenbuhl, and M. Plata. Modelling of Thermomechanical Effects During the Start-Up Phase of the Electromagnetic Casting Process. *Advances in Production and Fabrication of Light Metals and Metal Matrix Composites*, pages 175–187, 1992.
- [85] J.A.C. Martins and J.T. Oden. Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws. *Nonlinear Anal.*, 11(3):407–428, 1987.
- [86] A. Marzocchi, J.E. Muñoz Rivera, and M.G. Naso. Transmission problem in thermoelasticity with symmetry. *IMA J. Appl. Math.*, 68(1):23–46, 2003.
- [87] S. Mizohata. *The theory of partial differential equations*. Cambridge University Press, New York, 1973.
- [88] J.E. Muñoz Rivera and R. Racke. Multidimensional contact problems in thermoelasticity. *SIAM J. Appl. Math.*, 58(4):1307–1337, 1998.

- 
- [89] M.C. Naya-Riveiro and P. Quintela. Modelling of materials with long memory. *Internat. J. Solids Structures*, 45(24):6133–6156, 2008.
- [90] J. Nečas. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris, 1967.
- [91] P. Rabier, J.A.C. Martins, J.T. Oden, and L. Campos. Existence and local uniqueness of solutions to contact problems in elasticity with nonlinear friction laws. *Internat. J. Engrg. Sci.*, 24(11):1755–1768, 1986.
- [92] L.A. Segel. *Mathematics applied to continuum mechanics*. Macmillan Publishing Co., Inc., New York, 1977.
- [93] M. Serrano, J.A. Huidobro, and A. Robles. Análisis de la resolución de un modelo termoe-elastoplástico. In *Proceedings of XIV CEDYA/IV Congress of Applied Mathematics, Vic*, 1995.
- [94] P. Shi and M. Shillor. Existence of a solution to the  $N$ -dimensional problem of thermoelastic contact. *Comm. Partial Differential Equations*, 17(9-10):1597–1618, 1992.
- [95] J.C. Simo and C. Miehe. Associative coupled thermoplasticity at finite strains: formulation, numerical analysis and implementation. *Computer Methods in Applied Mechanics and Engineering*, 98(1):41 – 104, 1992.
- [96] J. Sprenger. On a contact problem in thermoelasticity with second sound. *Quart. Appl. Math.*, 67(4):601–615, 2009.
- [97] A. Tzavaras. Materials with internal variables and relaxation to conservation laws. *Arch. Rational Mech. Anal.*, 146:129–155, 1999.
- [98] J.M. Víaño Rey. *Existencia y aproximación de soluciones en termoelasticidad y elastoplasticidad*. PhD thesis, Department of Applied Mathematics, Universidade de Santiago de Compostela, 1981.
- [99] J.H. Weiner. A uniqueness theorem for the coupled thermoelastic problem. *Quart. Appl. Math.*, 15:102–105, 1957.
- [100] X. Wu and X. Xu. Existence for the thermoelastic thermistor problem. *J. Math. Anal. Appl.*, 319(1):124–138, 2006.
- [101] A. Ženíšek. *Nonlinear elliptic and evolution problems and their finite element approximations*. Computational Mathematics and Applications. Academic Press Inc., London, 1990.

