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Research Article

On Some Fixed Point Results for H^+-Type Contraction Mappings in Metric Spaces

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We prove some fixed point theorems for H^+-type multivalued contractive mappings in the setting of Banach spaces and metric spaces. The results provided allow recovering different well-known results.

1. Introduction

The interest in the study of fixed point theory in the frame of multivalued mappings by using the Hausdorff metric has its origin in the contributions of Markin [1] and Nadler [2], which have lead to interesting achievements on this topic with reference to both theoretical results and applications (see the monograph by Singh et al. on fixed point theory [3] and the references therein).

Starting from the work of Nadler [2], many authors have contributed to the huge development of fixed point theory for set-valued contractions. We mention some interesting contributions, such as those made by Reich [4, 5], Lami Dozo [6], Singh [7], Lim [8], Kaneko [9], Mizoguchi and Takahashi [10], Dhompongsa et al. [11], Feng and Liu [12], Klim and Wardowski [13], Suzuki [14], and Pathak and Shahzad [15, 16] (see also the monographs by Goebel and Kirk [17], Petrusel [18], and so forth). We also refer to some recent works on this topic, for instance, those by Hasanzade Asl et al. [19], Samet et al. [20], and Kumam and Sintunavarat [21].

In this paper, we provide an extension of some fixed point results to the context of multivalued H^+-type φ-contraction mappings, by establishing a common frame which allows obtaining as corollaries some well-known fixed point results.

2. Preliminaries

For $(X, \|\cdot\|)$ a normed linear space and K a nonempty subset of X, we denote by 2^K, $\mathscr{CB}(K)$, and $\mathscr{K}(K)$ the collection of all nonempty subsets of K, the collection of all nonempty closed and bounded subsets of K, and the collection of all compact subsets of K, respectively.

For $A, B \in \mathscr{CB}(X)$, we define the following mappings:

$$H(A,B) = \max\{\rho(A,B), \rho(B,A)\},$$
$$H^+(A,B) = \frac{1}{2}\{\rho(A,B) + \rho(B,A)\}, \quad (1)$$

where $\rho(A,B) = \sup_{x\in A}\operatorname{dist}(x,B)$ and $\operatorname{dist}(x,B) = \inf_{y\in B}\|x-y\|$. It is well known that the mapping H is a metric on $\mathscr{CB}(X)$ called the *Hausdorff metric* induced by the norm in X. From Proposition 2.1 [16], we also know that H^+ is a metric on $\mathscr{CB}(X)$. Moreover, H and H^+ are equivalent metrics [22], since $(1/2)H(A,B) \le H^+(A,B) \le H(A,B)$. By the results in the monograph by Kuratowski [22], we deduce that $(\mathscr{CB}(X), H^+)$ is complete provided that (X,d) is complete (d denotes the metric induced by the norm $\|\cdot\|$) and $\mathscr{K}(X)$ is a closed subspace of $(\mathscr{CB}(X), H^+)$ (see Theorem 2.6 [16]). The mapping $H^+ : \mathscr{CB}(X) \times \mathscr{CB}(X) \to \mathbb{R}$ is also continuous and satisfies the following properties (see Proposition 2.2 [16]):

(i) $H^+(\lambda A, \lambda B) = |\lambda| H^+(A, B)$, for any $\lambda \in \mathbb{C}$ and $A, B \in \mathscr{CB}(X)$;

(ii) $H^+(A + a, B + a) = H^+(A, B)$, for any $A, B \in \mathscr{CB}(X)$ and $a \in X$.

Although we have introduced the definitions of H and H^+ for normed linear spaces, we can also do it for metric spaces (X, d) just by using the metric d instead of the norm; this will be enough for our purpose.

The notions of multivalued contraction and H^+-contraction mapping are essential to this work and we include it here for completeness.

Definition 1. One says that a set-valued mapping $T : X \to \mathscr{CB}(X)$ is a multivalued k-contraction mapping if there exists a fixed real number k, $0 < k < 1$, such that

$$H(Tx, Ty) \le k \|x - y\|, \quad \forall x, y \in X. \tag{2}$$

Definition 2 (see [7]). Let (X, d) be a metric space. A multivalued map $T : X \to \mathscr{CB}(X)$ is called H^+-contraction if the following conditions hold:

(C1) there exists k in $(0, 1)$ such that

$$H^+(Tx, Ty) \le k d(x, y), \quad \text{for every } x, y \in X, \tag{3}$$

(C2) for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that

$$d(y, z) \le H^+(Ty, Tx) + \varepsilon. \tag{4}$$

Another important concept is the notion of fixed point for a multivalued map.

Definition 3. An element $x \in X$ is said to be a fixed point of a multivalued map $T : K \subseteq X \to 2^X$ if $x \in T(x)$. One denotes by $F(T)$ the set of fixed points of T.

Concerning the existence of fixed points for multivalued contractions, a highly relevant result was provided by Nadler (see [2]).

Theorem 4 (Theorem 5 [2]). *Let (X, d) be a complete metric space and $T : X \to \mathscr{CB}(X)$ a multivalued contraction mapping. Then T has a fixed point.*

Pathak and Shahzad [16] have given a generalization of Theorem 4 under a condition weaker than multivalued contractivity by using the metric H^+, as recalled below.

Theorem 5 (Theorem 3.2 [16]). *Let (X, d) be a complete metric space. Every H^+-type multivalued contraction mapping $T : X \to \mathscr{CB}(X)$ with Lipschitz constant $k < 1$ has a fixed point.*

In the next section, we obtain some fixed point results for a more general class of multivalued H^+-type contractions. We provide a common structure which allows obtaining as particular cases some well-known fixed point results.

3. Fixed Point Results for Multivalued H^+-Type Weak φ-Contraction Mappings

In this section, we present the main results, in which the existence of fixed points is deduced for multivalued H^+-type weak φ-Lipschitz and φ-contraction mappings, as introduced below.

Definition 6. Let (X, d) be a metric space. A multivalued map $T : X \to \mathscr{CB}(X)$ is called an H^+-type multivalued weak φ-Lipschitz mapping if condition (C2) in Definition 2 holds and there exist $M : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$H^+(Tx, Ty) \le \varphi(\mathscr{M}(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))), \quad \text{for every } x, y \in X. \tag{5}$$

If, moreover, there exists $k \in (0, 1)$ such that $\varphi(t) \le kt$, for every $t \ge 0$, we say that T is an H^+-type multivalued weak φ-contraction.

Remark 7. In Definition 6, if we take

$$\varphi(t) = kt, \quad \text{with } k \in (0, 1), \qquad \mathscr{M}(r_1, r_2, r_3, r_4, r_5) = r_1, \tag{6}$$

we get the notion of H^+-type multivalued contraction mapping.

On the other hand, if we take

$$\varphi(t) = kt, \quad \text{with } k \in (0, 1), \qquad \mathscr{M}(r_1, r_2, r_3, r_4, r_5) = \max\left\{r_1, r_2, r_3, \frac{r_4 + r_5}{2}\right\}, \tag{7}$$

then we obtain the notion of H^+-type multivalued weak contractive mapping (see [23, Definition 3.3]), since inequality (5) is reduced to

$$H^+(Tx, Ty) \le k \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}, \quad \text{for every } x, y \in X. \tag{8}$$

Finally, if we take

$$\varphi(t) = kt, \quad \text{with } k \in (0, 1), \qquad \mathscr{M}(r_1, r_2, r_3, r_4, r_5) = \max\{r_1, r_2, r_3, r_4, r_5\}, \tag{9}$$

then we obtain the notion of H^+-type multivalued quasi-contraction mapping (compare inequality with [23, equation (2.3)])

$$H^+(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad \text{for every } x, y \in X. \tag{10}$$

We start this study by providing a fixed point result for H^+-type multivalued weak φ-Lipschitz mappings. This result

(Theorem 8) will find an extension below in Theorem 18 and further in Theorem 21, both stated for partial H^+-type multivalued weak φ-Lipschitz mappings. It is possible to state first these results and, then, obtain Theorem 8 as a corollary. However, by considering Theorem 8 first, we can focus our attention on the properties of function $\mathscr{M}$, considering a general expression which immediately connects with some well-known fixed point results. Therefore, we present the main results in several steps:

(s.i) First, in Theorem 8, we consider a general expression for function $\mathscr{M}$, while for function φ it is required that $\varphi(t) \leq kt$, $t \geq 0$.

(s.ii) Then, in Theorem 18, we extend this result to partial H^+-type multivalued weak φ-Lipschitz mappings.

(s.iii) In Theorem 21, the same type of variability is allowed for function $\mathscr{M}$ and we explore some other possible general expressions for function φ, also for partial H^+-type multivalued weak φ-Lipschitz mappings.

(s.iv) We include some considerations concerning H^+-type weak φ-Lipschitz mappings with respect to a binary relation (including the case of partial orderings).

(s.v) We complete this study by proving Theorem 32, which considers different restrictions for the selection of functions $\mathscr{M}$ and φ.

Theorem 8. *Let (X, d) be a complete metric space. Let $T : X \to \mathscr{CB}(X)$ be an H^+-type multivalued weak φ-Lipschitz mapping such that the functions φ, $\mathscr{M}$ satisfy the following:*

(i) *There exists $k > 0$ such that $\varphi(t) \leq kt$, for every $t \geq 0$.*

(ii) *For every $i = 2, 3, 4, 5$, the function $\mathscr{M}(r_1, r_2, r_3, r_4, r_5)$ is monotonically increasing in the variable r_i, provided that the other variables r_j, $j \neq i$, remain fixed.*

(iii) *For each $r > 0$ fixed, $\mathscr{M}(r_1, r_2, r, r_4, r_5)$ is continuous at $(r_1, r_2, r_4, r_5) = (0, 0, r, 0)$.*

(iv) *There exists a function $\mathscr{N} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(iv-a) *$\mathscr{M}(r, r, s, r + s, 0) \leq \mathscr{N}(r, s)$, for every $r, s \in \mathbb{R}^+$,*

(iv-b) *$\mathscr{N}(\cdot, s)$ is monotonically increasing for each $s \in \mathbb{R}^+$ fixed,*

(iv-c) *$\mathscr{N}(r, \cdot)$ is monotonically increasing for each $r \in \mathbb{R}^+$ fixed,*

(iv-d) *there exists $\tau > 0$ such that $\mathscr{N}(r, r) \leq \tau r$, for every $r > 0$,*

(iv-e) *there exists $\mathscr{R} \geq 0$ such that $\mathscr{N}(0, s) \leq \mathscr{R}s$, for every $s > 0$.*

(v) *The constants k in (i) and τ, $\mathscr{R}$ in (iv) are such that $k < 1/\max\{\tau, 2\mathscr{R}\}$.*

Then T has a fixed point.

Proof. We denote by $\psi : X \times X \to (\mathbb{R}^+)^5$ the mapping defined as

$$\psi(x, y) := (d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)). \tag{11}$$

By (5) and (i), we get

$$H^+(Tx, Ty) \leq k\mathscr{M}(\psi(x, y)), \quad \text{for every } x, y \in X. \tag{12}$$

We construct a sequence in X in lines similar to [16, Theorem 3.2] or [23, Theorem 3.4] as follows. Let $\varepsilon > 0$ be given and take $x_0 \in X$ to be arbitrary. We fix an element x_1 in Tx_0. Now, from property (C2), it is possible to choose $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq H^+(Tx_0, Tx_1) + \varepsilon$. At this step, we could choose ε depending on x_0 and x_1. In general, for $n \in \mathbb{N}$, if x_n is chosen, then we can select $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \leq H^+(Tx_{n-1}, Tx_n) + \varepsilon. \tag{13}$$

At this step, we could choose ε depending on x_{n-1} and x_n. Note that if $H^+(Tx_{n-1}, Tx_n) = 0$ for some n, then $Tx_{n-1} = Tx_n$ and the proof is complete. By (v), we can take $A, B > 0$ such that $A - Bk > 0$ and $L := A - Bk + k < 1/\max\{\tau, 2\mathscr{R}\}$. In fact, if we set on each step $\varepsilon = (A - Bk)\mathscr{M}(\psi(x_{n-1}, x_n)) > 0$, then, by (13), we obtain, for every $n \in \mathbb{N}$, that

$$d(x_n, x_{n+1}) \leq H^+(Tx_{n-1}, Tx_n) + (A - Bk)\mathscr{M}(\psi(x_{n-1}, x_n)), \tag{14}$$

which implies, for every $n \in \mathbb{N}$, that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq k\mathscr{M}(\psi(x_{n-1}, x_n)) + (A - Bk) \cdot \mathscr{M}(\psi(x_{n-1}, x_n)) = (A - Bk + k) \\ &\cdot \mathscr{M}(\psi(x_{n-1}, x_n)) = L\mathscr{M}(d(x_{n-1}, x_n), \\ &d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), \\ &d(x_n, Tx_{n-1})) = L\mathscr{M}(d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), \\ &d(x_n, Tx_n), d(x_{n-1}, Tx_n), 0) \leq L\mathscr{M}(d(x_{n-1}, x_n), \\ &d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) \\ &\leq L\mathscr{M}(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0) \leq L\mathscr{N}(d(x_{n-1}, x_n), \\ &d(x_n, x_{n+1})), \end{aligned} \tag{15}$$

where we have used (ii) and (iv-a). Hence,

$$d(x_n, x_{n+1}) \leq L\mathscr{N}(d(x_{n-1}, x_n), d(x_n, x_{n+1})), \quad \forall n \in \mathbb{N}. \tag{16}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x_n = x_{n+1} \in Tx_n$, so that x_n is a fixed point of T and the proof is concluded. Suppose, therefore, that $d(x_n, x_{n+1}) > 0$, for every $n \in \mathbb{N}$.

Now, assuming that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, for some $n \in \mathbb{N}$, by (16) and (iv-b), we get

$$d(x_n, x_{n+1}) \leq L\mathscr{N}(d(x_n, x_{n+1}), d(x_n, x_{n+1})), \quad \forall n \in \mathbb{N} \tag{17}$$

and, by (iv-d),

$$d(x_n, x_{n+1}) \leq L\tau d(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \tag{18}$$

which is a contradiction since $d(x_n, x_{n+1}) > 0$ and $L\tau \in (0,1)$ (due to inequality $L < 1/\max\{\tau, 2\mathscr{R}\} \leq 1/\tau$). Hence, we have proved that $d(x_{n-1}, x_n) \geq d(x_n, x_{n+1})$, for every $n \in \mathbb{N}$, so that, by (16) and (iv-c), we have

$$d(x_n, x_{n+1}) \leq L\mathscr{N}(d(x_{n-1}, x_n), d(x_n, x_{n+1})) \leq L\mathscr{N}(d(x_{n-1}, x_n), d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \tag{19}$$

By (iv-d), we obtain

$$d(x_n, x_{n+1}) \leq L\tau d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{20}$$

Repeating the same procedure n-times, we get $d(x_n, x_{n+1}) \leq (L\tau)^n d(x_0, x_1)$, for all $n \in \mathbb{N}$, where $L\tau < 1$ by hypothesis (v). Hence, $\{x_n\}$ is a Cauchy sequence since, for $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} (L\tau)^i d(x_0, x_1) \\ &= \frac{(L\tau)^n - (L\tau)^m}{1 - L\tau} d(x_0, x_1) \\ &= \frac{(L\tau)^n}{1 - L\tau}\left(1 - (L\tau)^{m-n}\right) d(x_0, x_1). \end{aligned} \tag{21}$$

Therefore, by the complete character of X, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Suppose that $d(u, Tu) > 0$. Then

$$\begin{aligned} &\frac{1}{2}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} = H^+(Tx_n, Tu) \\ &\quad \leq k\mathscr{M}(\psi(x_n, u)) = k\mathscr{M}(d(x_n, u), d(x_n, Tx_n), \\ &\quad d(u, Tu), d(x_n, Tu), d(u, Tx_n)) \leq k\mathscr{M}(d(x_n, u), \\ &\quad d(x_n, x_{n+1}), d(u, Tu), d(x_n, Tu), d(u, x_{n+1})). \end{aligned} \tag{22}$$

We note that $d(x_n, Tu) \leq d(x_n, u) + d(u, Tu)$ and $d(u, Tu) \leq d(u, x_n) + d(x_n, Tu)$, so that $d(u, Tu) - d(u, x_n) \leq d(x_n, Tu) \leq d(x_n, u) + d(u, Tu)$, which implies that $d(x_n, Tu) \to d(u, Tu)$ as $n \to \infty$.

Hence, from $d(x_n, u) \to 0$, $d(x_n, x_{n+1}) \to 0$, and $d(x_n, Tu) \to d(u, Tu)$, as $n \to \infty$, and using hypothesis (iii), we get from (22)

$$\begin{aligned} &\frac{1}{2}\liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} \\ &\quad \leq k\mathscr{M}(0, 0, d(u, Tu), d(u, Tu), 0). \end{aligned} \tag{23}$$

Moreover, by (iv-a) and (iv-e), we obtain

$$\begin{aligned} &\frac{1}{2}\liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} \\ &\quad \leq k\mathscr{N}(0, d(u, Tu)) \leq k\mathscr{R}d(u, Tu). \end{aligned} \tag{24}$$

Using that

$$\liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} \leq 2k\mathscr{R}d(u, Tu), \tag{25}$$

we deduce that

$$\begin{aligned} &\liminf_{n\to\infty}\rho(Tx_n, Tu) \\ &\quad \leq \liminf_{n\to\infty}\rho(Tx_n, Tu) + \liminf_{n\to\infty}\rho(Tu, Tx_n) \\ &\quad \leq \liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} \\ &\quad \leq 2k\mathscr{R}d(u, Tu). \end{aligned} \tag{26}$$

Since

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &\leq \rho(Tx_n, Tu) + d(x_{n+1}, u) \end{aligned} \tag{27}$$

and $\lim_{n\to\infty} d(x_{n+1}, u) = 0$, it follows that

$$\begin{aligned} d(u, Tu) &\leq \liminf_{n\to\infty}[\rho(Tx_n, Tu) + d(x_{n+1}, u)] \\ &= \liminf_{n\to\infty}\rho(Tx_n, Tu) + \lim_{n\to\infty} d(x_{n+1}, u) \\ &\leq 2k\mathscr{R}d(u, Tu). \end{aligned} \tag{28}$$

Hence, if $\mathscr{R} = 0$, then $d(u, Tu) = 0$. On the other hand, if $\mathscr{R} > 0$, the assumption $d(u, Tu) > 0$ leads to a contradiction by virtue of hypothesis (v) ($2k\mathscr{R} < 2\mathscr{R}/\max\{\tau, 2\mathscr{R}\} \leq 1$), which implies that $d(u, Tu) = 0$. Finally, since Tu is closed, it is proved that $u \in Tu$; that is, u is a fixed point of T. □

Remark 9. Note that if $\max\{\tau, 2\mathscr{R}\} > 1$, condition (v) in Theorem 8 implies that $k < 1$, so that in this case the H^+-type multivalued weak φ-Lipschitz mapping T in Theorem 8 is in fact an H^+-type multivalued weak φ-contraction. In this sense, both functions φ and $\mathscr{M}$ may contribute to the contractivity of the multivalued mapping through condition (v) which establishes a relation among the constants involved.

Corollary 10. *If one takes $\varphi(t) := kt$, with $k \in (0,1)$, and $\mathscr{M}(r_1, r_2, r_3, r_4, r_5) := r_1$, then $\mathscr{M}$ is monotonically increasing in all the variables and continuous. Besides, for $\mathscr{N}(r, s) := r$, one has*

$$\begin{aligned} &\mathscr{M}(r, r, s, r+s, 0) = r = \mathscr{N}(r, s), \quad \text{for every } r, s \in \mathbb{R}^+, \\ &\mathscr{N}(r, s) \text{ is monotonically increasing in } r \\ &\qquad \text{for } s \in \mathbb{R}^+ \text{ fixed}, \\ &N(r, s) \text{ is monotonically increasing in } s \\ &\qquad \text{for } r \in R^+ \text{ fixed}, \\ &N(r, r) = r, \quad \text{thus (iv} - \text{d) holds for } \tau = 1 \\ &N(0, s) = 0, \\ &\quad \text{for every } s > 0, \text{ so that (iv} - \text{e) holds for } R = 0, \\ &\max\{\tau, 2\mathscr{R}\} = 1, \\ &\qquad \text{so that (v) is fulfilled for } 0 < k < 1. \end{aligned} \tag{29}$$

In this setting of H^+-type multivalued contraction mappings, we have, as a corollary of Theorem 8, Theorem 3.2 [16] (see Theorem 5).

In the context of H^+-type multivalued weak contractive mappings, we obtain the following corollary which corrects Theorem 3.4 [23] [see the proof of Theorem 3.4 [23], where it was assumed that $d(u, Tu) = (1/2)(d(u, Tu) + d(u, Tu)) \leq (1/2)(\rho(Tx_n, Tu) + \rho(Tu, Tx_n)) + d(u, x_{n+1})$].

Corollary 11. *Let (X, d) be a complete metric space and $T : X \to \mathscr{CB}(X)$ an H^+-type multivalued weak contractive mapping with $0 < k < 1/2$. Then T has a fixed point.*

Proof. If we take $\varphi(t) := kt$, with $k \in (0, 1/2)$, and

$$\mathscr{M}(r_1, r_2, r_3, r_4, r_5) := \max\left\{r_1, r_2, r_3, \frac{r_4 + r_5}{2}\right\}, \tag{30}$$

$\mathscr{M}$ is monotonically increasing in all the variables and continuous. Choosing $\mathscr{N}(r, s) := \max\{r, s\}$, we have

$$\begin{aligned}\mathscr{M}(r, r, s, r + s, 0) &= \max\left\{r, r, s, \frac{r + s}{2}\right\} = \max\{r, s\} \\ &= \mathscr{N}(r, s), \quad \text{for every } r, s \in \mathbb{R}^+,\end{aligned} \tag{31}$$

$\mathscr{N}(r, s)$ is monotonically increasing in each variable, $\mathscr{N}(r, r) = \max\{r, r\} = r$, and thus (iv-d) holds for $\tau = 1$, $\mathscr{N}(0, s) = \max\{0, s\} = s$, for every $s > 0$, so that (iv-e) holds for $\mathscr{R} = 1$ and $\max\{\tau, 2\mathscr{R}\} = 2$, so that (v) is fulfilled for the choice $0 < k < 1/2$. Hence Theorem 8 applies. □

In the context of H^+-type multivalued quasi-contraction mappings, we obtain the following corollary which coincides with Theorem 3.6 [23].

Corollary 12. *Let (X, d) be a complete metric space and $T : X \to \mathscr{CB}(X)$ an H^+-type multivalued quasi-contraction mapping with $0 < k < 1/2$. Then T has a fixed point.*

Proof. Taking $\varphi(t) := kt$, with $k \in (0, 1/2)$ and

$$\mathscr{M}(r_1, r_2, r_3, r_4, r_5) := \max\{r_1, r_2, r_3, r_4, r_5\}, \tag{32}$$

we have that $\mathscr{M}$ is monotonically increasing in all the variables and also continuous. Taking $\mathscr{N}(r, s) := r + s$, we get

$$\begin{aligned}\mathscr{M}(r, r, s, r + s, 0) &= \max\{r, r, s, r + s, 0\} = r + s \\ &= \mathscr{N}(r, s), \quad \text{for every } r, s \in \mathbb{R}^+,\end{aligned} \tag{33}$$

$\mathscr{N}(r, s)$ is monotonically increasing in each variable for the other fixed, $\mathscr{N}(r, r) = r + r = 2r$, and thus (iv-d) holds for $\tau = 2$, $\mathscr{N}(0, s) = s$, for every $s > 0$, so that (iv-e) holds for $\mathscr{R} = 1$ and $\max\{\tau, 2\mathscr{R}\} = 2$, so that (v) is fulfilled for $0 < k < 1/2$. □

On the other hand, if a result similar to Theorem 8 was established for H-multivalued contractions, we would have the following theorem.

Theorem 13. *Let (X, d) be a complete metric space. Let $T : X \to \mathscr{CB}(X)$ be an H-type multivalued weak φ-Lipschitz mapping, that is, such that there exist $\mathscr{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$\begin{aligned}H(Tx, Ty) \leq \varphi(\mathscr{M}(d(x, y), d(x, Tx), d(y, Ty), \\ d(x, Ty), d(y, Tx))), \quad \text{for every } x, y \in X.\end{aligned} \tag{34}$$

Suppose that the functions φ, $\mathscr{M}$ satisfy conditions (i)–(iv) in Theorem 8 and

(v*) *the constants k in (i) and τ, $\mathscr{R}$ in (iv) are such that $k < 1/\max\{\tau, \mathscr{R}\}$.*

Then T has a fixed point.

Proof. Identical to the proof of Theorem 8, except the last part, we have

$$\begin{aligned}\liminf_{n\to\infty} \rho(Tx_n, Tu) &\leq \liminf_{n\to\infty} H(Tx_n, Tu) \\ &\leq k\mathscr{N}(0, d(u, Tu)) \\ &\leq k\mathscr{R} d(u, Tu).\end{aligned} \tag{35}$$

Since

$$\begin{aligned}d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &\leq \rho(Tx_n, Tu) + d(x_{n+1}, u)\end{aligned} \tag{36}$$

and $\lim_{n\to\infty} d(x_{n+1}, u) = 0$, then

$$\begin{aligned}d(u, Tu) &\leq \liminf_{n\to\infty} [\rho(Tx_n, Tu) + d(x_{n+1}, u)] \\ &= \liminf_{n\to\infty} \rho(Tx_n, Tu) \leq k\mathscr{R} d(u, Tu).\end{aligned} \tag{37}$$

If $\mathscr{R} = 0$, then $d(u, Tu) = 0$. If $\mathscr{R} > 0$, $d(u, Tu) > 0$ leads to a contradiction in the previous inequality due to (v*) ($k\mathscr{R} < 1$), which implies that $d(u, Tu) = 0$ and the proof is complete. □

Remark 14. As a consequence of Theorem 13, using the Hausdorff metric H, the restriction required on k is

(r.i) for multivalued contractions $0 < k < 1$,

(r.ii) for multivalued weak contractive mappings $0 < k < 1$,

(r.iii) for multivalued quasi-contraction mappings $0 < k < 1/2$.

Remark 15. In Theorems 8 and 13, conditions (iv-b), (iv-c), and (iv-d) can be removed, adding the property that

(iv-d*) there exists $\theta > 0$ such that $\mathscr{N}(r, s) \leq \theta(r + s)$, for every $r, s > 0$,

while condition (v) (resp., (v*)) has to be replaced by the fact that

(v**) the constants k in (i) and $\mathscr{R}$, θ in (iv) are such that $k < 1/\max\{2\theta, 2\mathscr{R}\}$ (for H^+) or $k < 1/\max\{2\theta, \mathscr{R}\}$ (for H).

This comes from the following ideas: from (v**), following the proof of Theorem 8, we can take $A, B > 0$ such that $A - Bk > 0$ and $L := A - Bk + k < 1/\max\{2\theta, 2\mathscr{R}\}$ (for the Hausdorff metric H, we choose them in such a way that $L := A - Bk + k < 1/\max\{2\theta, \mathscr{R}\}$). Supposing that $d(x_n, x_{n+1}) > 0$, for every $n \in \mathbb{N}$, if we do not have monotonicity of $\mathscr{N}$, from (16) and (iv-d*), we get

$$d(x_n, x_{n+1}) \le L\theta(d(x_{n-1}, x_n) + d(x_n, x_{n+1})), \quad \forall n \in \mathbb{N} \tag{38}$$

so that, using $L\theta < 1$,

$$d(x_n, x_{n+1}) \le \frac{L\theta}{1 - L\theta} d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{39}$$

Repeating the same procedure n-times, we get $d(x_n, x_{n+1}) \le K^n d(x_0, x_1)$, for all $n \in \mathbb{N}$, where $K = L\theta/(1 - L\theta) < 1$ by hypothesis (v**). The rest of the proof is valid.

Now, following the lines in [21] for q-set-valued quasi-contractions, we give the following definition.

Definition 16. Let (X, d) be a metric space and let $\alpha : X \times X \to [0, \infty)$ be given. A multivalued map $T : X \to \mathscr{CB}(X)$ is called a partial H^+-type multivalued weak φ-Lipschitz mapping if condition (C2) in Definition 2 holds and there exist $\mathscr{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} H^+(Tx, Ty) \le \varphi(\mathscr{M}(d(x, y), d(x, Tx), d(y, Ty), \\ d(x, Ty), d(y, Tx))), \\ \text{for every } (x, y) \in X \times X \text{ with } \alpha(x, y) \ge 1. \end{aligned} \tag{40}$$

If, moreover, there exists $k \in (0, 1)$ such that $\varphi(t) \le kt$, for every $t \ge 0$, we say that T is a partial H^+-type multivalued weak φ-contraction.

Remark 17. Taking $\varphi(t) = kt$, with $k \in (0, 1)$, and

$$\mathscr{M}(r_1, r_2, r_3, r_4, r_5) = \max\{r_1, r_2, r_3, r_4, r_5\}, \tag{41}$$

then we obtain the notion of partial H^+-type q-set-valued quasi-contraction (compare with [21, Definition 3.1]).

In relation with Definitions 2.21, 3.1 and Theorem 3.2 [21], we can establish the following result.

Theorem 18. *Let (X, d) be a complete metric space, let $\alpha : X \times X \to [0, \infty)$ be given, and let $T : X \to \mathscr{CB}(X)$ be a partial H^+-type multivalued weak φ-Lipschitz mapping such that*

(h1) *T is α-admissible; that is, for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \ge 1$, one has $\alpha(y, z) \ge 1$, for every $z \in Ty$,*

(h2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$,*

(h3) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N}$, and there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$, for every n,*

(h4) *the functions φ, $\mathscr{M}$ are such that conditions (i), (ii), (iii), (iv) [(a)–(e)], and (v) in Theorem 8 hold.*

Then T has a fixed point in X.

Proof. We proceed similarly to the proof of Theorem 8 and also Theorem 3.2 [21]. By (40) and (i), we get

$$\begin{aligned} H^+(Tx, Ty) \le k\mathscr{M}(\psi(x, y)), \\ \text{for every } (x, y) \in X \times X \text{ with } \alpha(x, y) \ge 1. \end{aligned} \tag{42}$$

We take $\varepsilon > 0$ given. We start the sequence with the terms x_0 and $x_1 \in Tx_0$ and then, using (C2), there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \le H^+(Tx_0, Tx_1) + \varepsilon$. Note that since T is α-admissible, we have $\alpha(x_1, x_2) \ge 1$. In general, for $n \in \mathbb{N}$, if $x_n \in Tx_{n-1}$ is chosen such that $\alpha(x_{n-1}, x_n) \ge 1$, then we can select $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \le H^+(Tx_{n-1}, Tx_n) + \varepsilon \tag{43}$$

and we also have $\alpha(x_n, x_{n+1}) \ge 1$. Again, ε could have been chosen at each step depending on x_{n-1} and x_n. For this, we remark that if $H^+(Tx_{n-1}, Tx_n) = 0$ for some n, the proof is concluded, so that we can assume that $\mathscr{M}(\psi(x_{n-1}, x_n)) > 0$, for every n and take on each step $\varepsilon = (A - Bk)\mathscr{M}(\psi(x_{n-1}, x_n)) > 0$, where $A, B > 0$ are fixed, by (v), in such a way that $A - Bk > 0$ and $L := A - Bk + k < 1/\max\{\tau, 2\mathscr{R}\}$. Then, by (43), we get, for every $n \in \mathbb{N}$, that

$$\begin{aligned} d(x_n, x_{n+1}) &\le H^+(Tx_{n-1}, Tx_n) \\ &\quad + (A - Bk)\mathscr{M}(\psi(x_{n-1}, x_n)) \\ &\le L\mathscr{M}(\psi(x_{n-1}, x_n)) \end{aligned} \tag{44}$$

which implies, for every $n \in \mathbb{N}$, that

$$\begin{aligned} d(x_n, x_{n+1}) &\le L\mathscr{M}(\psi(x_{n-1}, x_n)) \\ &\le L\mathscr{M}(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0) \le L\mathscr{N}(d(x_{n-1}, x_n), \\ &\quad d(x_n, x_{n+1})), \end{aligned} \tag{45}$$

using (ii) and (iv-a). As in the proof of Theorem 8, we can suppose that $d(x_n, x_{n+1}) > 0$, for every $n \in \mathbb{N}$. Similarly to the proof of Theorem 8, we deduce that $d(x_{n-1}, x_n) \ge d(x_n, x_{n+1})$, for every $n \in \mathbb{N}$ and

$$d(x_n, x_{n+1}) \le L\tau d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}, \tag{46}$$

so that $\{x_n\}$ is a Cauchy sequence and, by the completeness of X, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Finally, if $d(u, Tu) > 0$, inequality (22) holds; since $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N}$, and $x_n \to u$ as $n \to \infty$, then, by hypotheses, $\alpha(x_n, u) \ge 1$, for every n, so that

$$H^+(Tx_n, Tu) \le k\mathscr{M}(\psi(x_n, u)), \quad \text{for every } n. \tag{47}$$

Similarly to the proof of Theorem 8, we get that

$$\begin{aligned} d(u, Tu) &\le d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &\le \rho(Tx_n, Tu) + d(x_{n+1}, u); \end{aligned} \tag{48}$$

hence

$$d(u,Tu) \le 2k\mathscr{R}d(u,Tu), \tag{49}$$

so that if $\mathscr{R} = 0$, then $d(u,Tu) = 0$ and if $\mathscr{R} > 0$, the assumption $d(u,Tu) > 0$ leads to a contradiction again and the proof is complete. □

The previous result allows formulating a fixed point result for H^+-type partial quasi-contraction mappings while, in [21], the Pompeiu-Hausdorff metric is used in b-metric spaces. The procedure in metric spaces could be adapted to b-metric spaces in the lines of [21].

Remark 19. In Theorem 18, if we consider the Pompeiu-Hausdorff metric H, then condition (v) can be relaxed to hypothesis (v*) in the statement of Theorem 13.

In particular, for partial q-set-valued quasi-contractions, $\varphi(t) := kt$, $k \in (0, 1/2)$, and $\mathscr{M}(r_1, r_2, r_3, r_4, r_5) := \max\{r_1, r_2, r_3, r_4, r_5\}$, we obtain, as a corollary, the assertion of Theorem 3.2 [21] for metric spaces.

Remark 20. Inequality (40) is trivially valid if one of the following conditions hold (see Corollaries 3.4–3.6 [21]):

(r.i) $\alpha(x, y)H^+(Tx, Ty) \le \varphi(\mathscr{M}(\psi(x, y)))$, for every $x, y \in X$;

(r.ii) $(H^+(Tx, Ty) + \varepsilon)^{\alpha(x,y)} \le \varphi(\mathscr{M}(\psi(x, y))) + \varepsilon$, for every $x, y \in X$, where $\varepsilon \ge 1$;

(r.iii) $(\alpha(x, y) - 1 + \varepsilon)^{H^+(Tx,Ty)} \le \varepsilon^{\varphi(\mathscr{M}(\psi(x,y)))}$, for every $x, y \in X$, where $\varepsilon > 1$.

Theorem 21. *Let (X, d) be a complete metric space, let $\alpha : X \times X \to [0, \infty)$ be given, and let $T : X \to \mathscr{CB}(X)$ be a partial H^+-type multivalued weak φ-Lipschitz mapping such that conditions (h1), (h2), and (h3) hold. Suppose also that the functions φ, $\mathscr{M}$ satisfy the following:*

(A1) *φ is monotonically increasing and $\varphi(0) = 0$.*

(A2) *If $z_n \to z$ as $n \to \infty$, then $\liminf_{n\to\infty}\varphi(z_n) \le \varphi(z)$.*

(A3) *Conditions (ii), (iii), and (iv) [(a)–(e)] in Theorem 8 hold.*

(A4) *For $\mathscr{R}$ given in (iv-e), one has that $2\varphi(\mathscr{R}z) < z$, for every $z > 0$.*

(A5) *There exists $\mathcal{S} > 0$ such that the operator $\mathscr{L}_{\mathcal{S}} := \varphi + \mathcal{S}I$ (where I is the identity mapping on X) satisfies the following properties:*

(A5.i) *$\mathscr{L}_{\mathcal{S}}(\tau z) < z$, for every $z > 0$;*

(A5.ii) *$\mathscr{L}_{\mathcal{S}}(\tau z) \le \tau\mathscr{L}_{\mathcal{S}}(z)$, for every $z > 0$;*

(A5.iii) *$\sum_{n=1}^{\infty}\tau^n\mathscr{L}_{\mathcal{S}}{}^n(z) < \infty$, for every $z > 0$, where $\mathscr{L}_{\mathcal{S}}{}^n := \mathscr{L}_{\mathcal{S}} \circ \overset{(n)}{\cdots} \circ \mathscr{L}_{\mathcal{S}}$.*

Then T has a fixed point in X.

Proof. We proceed similarly to the proof of Theorem 18. We take x_0 and $x_1 \in Tx_0$ given by (h2), with $\alpha(x_0, x_1) \ge 1$; then a sequence is obtained by using (C2) and α-admissibility. The process is summarized as follows: for $n \in \mathbb{N}$, if $x_n \in Tx_{n-1}$ is chosen such that $\alpha(x_{n-1}, x_n) \ge 1$, then, by (C2), we can select $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \le H^+(Tx_{n-1}, Tx_n) + \varepsilon, \tag{50}$$

and we also have $\alpha(x_n, x_{n+1}) \ge 1$.

We remark that, due to the monotonically increasing character of φ, the mapping $\mathscr{L}_{\mathcal{S}}$ is always strictly increasing. Note that if $H^+(Tx_{n-1}, Tx_n) = 0$ for some n, the proof is concluded, so that we can assume that $\mathscr{M}(\psi(x_{n-1}, x_n)) > 0$, for every n, so that we can take, on each step, $\varepsilon_n = \mathcal{S}\mathscr{M}(\psi(x_{n-1}, x_n)) > 0$. Then, for every $n \in \mathbb{N}$, by (ii) and (iv-a),

$$\begin{aligned} d(x_n, x_{n+1}) &\le \varphi(\mathscr{M}(\psi(x_{n-1}, x_n))) \\ &\quad + \mathcal{S}\mathscr{M}(\psi(x_{n-1}, x_n)) \\ &= \mathscr{L}_{\mathcal{S}}(\mathscr{M}(\psi(x_{n-1}, x_n))) \\ &\le \mathscr{L}_{\mathcal{S}}(\mathscr{N}(d(x_{n-1}, x_n), d(x_n, x_{n+1}))). \end{aligned} \tag{51}$$

If $x_n = x_{n+1}$, for some n, then x_n is a fixed point of T, so we assume that $x_n \ne x_{n+1}$, for every $n \in \mathbb{N}$.

Assuming that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, for some $n \in \mathbb{N}$, by (iv-b) and (iv-d), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\le \mathscr{L}_{\mathcal{S}}(\mathscr{N}(d(x_n, x_{n+1}), d(x_n, x_{n+1}))) \\ &\le \mathscr{L}_{\mathcal{S}}(\tau d(x_n, x_{n+1})), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{52}$$

which is a contradiction due to (A5) since $d(x_n, x_{n+1}) > 0$. This proves that $d(x_{n-1}, x_n) \ge d(x_n, x_{n+1})$, for every $n \in \mathbb{N}$. Therefore, by (iv-c), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\le \mathscr{L}_{\mathcal{S}}(\mathscr{N}(d(x_{n-1}, x_n), d(x_n, x_{n+1}))) \\ &\le \mathscr{L}_{\mathcal{S}}(\mathscr{N}(d(x_{n-1}, x_n), d(x_{n-1}, x_n))) \\ &\le \mathscr{L}_{\mathcal{S}}(\tau d(x_{n-1}, x_n)) \\ &\le \tau\mathscr{L}_{\mathcal{S}}(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{53}$$

This implies, from (A5) and the property that $\mathscr{L}_{\mathcal{S}}(z) > 0$ for every $z > 0$, that

$$\begin{aligned} d(x_n, x_{n+1}) &\le \tau\mathscr{L}_{\mathcal{S}}(d(x_{n-1}, x_n)) \\ &\le \tau\mathscr{L}_{\mathcal{S}}(\tau\mathscr{L}_{\mathcal{S}}(d(x_{n-2}, x_{n-1}))) \\ &\le \tau^2\mathscr{L}_{\mathcal{S}}{}^2(d(x_{n-2}, x_{n-1})) \\ &\le \tau^n\mathscr{L}_{\mathcal{S}}{}^n(d(x_0, x_1)), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{54}$$

To check that $\{x_n\}$ is a Cauchy sequence, we observe that, for $n, m \in \mathbb{N}$ with $n < m$, we get

$$\begin{aligned} d(x_n, x_m) &\le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1}\tau^i\mathscr{L}_{\mathcal{S}}{}^i(d(x_0, x_1)) \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{55}$$

Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. If $d(u, Tu) > 0$, using that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, and $x_n \to u$ as $n \to \infty$, then, by (h3), we have $\alpha(x_n, u) \geq 1$, for every n; hence

$$H^+(Tx_n, Tu) \leq \varphi(\mathscr{M}(\psi(x_n, u))), \quad \text{for every } n \tag{56}$$

so that

$$\begin{aligned}&\liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\}\\ &\quad\leq 2\liminf_{n\to\infty}\varphi(\mathscr{M}(\psi(x_n, u))).\end{aligned} \tag{57}$$

Therefore, similarly to inequality (22), we have, by the monotonicity of φ,

$$\begin{aligned}\varphi(\mathscr{M}(\psi(x_n, u))) &= \varphi(\mathscr{M}(d(x_n, u), d(x_n, Tx_n),\\ &\quad d(u, Tu), d(x_n, Tu), d(u, Tx_n)))\\ &\leq \varphi(\mathscr{M}(d(x_n, u), d(x_n, x_{n+1}), d(u, Tu),\\ &\quad d(x_n, Tu), d(u, x_{n+1}))).\end{aligned} \tag{58}$$

Analogously to the proof of Theorem 8 and using (iii),

$$\begin{aligned}&\lim_{n\to\infty}\mathscr{M}(d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), d(x_n, Tu),\\ &\quad d(u, x_{n+1})) = \mathscr{M}(0, 0, d(u, Tu), d(u, Tu), 0).\end{aligned} \tag{59}$$

By (A2), we have

$$\begin{aligned}&\liminf_{n\to\infty}\varphi(\mathscr{M}(\psi(x_n, u))) \leq \liminf_{n\to\infty}\varphi(\mathscr{M}(d(x_n, u),\\ &\quad d(x_n, x_{n+1}), d(u, Tu), d(x_n, Tu), d(u, x_{n+1})))\\ &\quad\leq \varphi(\mathscr{M}(0, 0, d(u, Tu), d(u, Tu), 0)).\end{aligned} \tag{60}$$

Hence, from this inequality and (57), by (iv-a) and (iv-e), we get

$$\begin{aligned}&\liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\}\\ &\quad\leq 2\liminf_{n\to\infty}\varphi(\mathscr{M}(\psi(x_n, u)))\\ &\quad\leq 2\varphi(\mathscr{M}(0, 0, d(u, Tu), d(u, Tu), 0))\\ &\quad\leq 2\varphi(\mathscr{N}(0, d(u, Tu))) \leq 2\varphi(\mathscr{R}d(u, Tu)).\end{aligned} \tag{61}$$

Similarly to the proof of Theorem 8, we have $d(u, Tu) \leq 2\varphi(\mathscr{R}d(u, Tu))$. Therefore, if $\mathscr{R} = 0$, then $d(u, Tu) \leq \varphi(0) = 0$ and the proof is concluded. If $\mathscr{R} > 0$, we get to a contradiction due to $d(u, Tu) > 0$ and (A4). Hence, $d(u, Tu) = 0$ and the proof is complete, since Tu is closed, so that $u \in Tu$. □

Remark 22. Theorem 8 is a particular case of Theorem 21. Indeed, if we take $\varphi(t) = kt$, where $k > 0$, and assuming that condition (v) holds, then we have the following:

(r.i) φ is monotonically increasing and $\varphi(0) = 0$; hence (A1) holds.

(r.ii) φ is continuous (so that (A2) holds).

(r.iii) Condition (A4) is valid since $2\varphi(\mathscr{R}z) < z$, for every $z > 0$ (this is true since it is equivalent to $k < 1/2\mathscr{R}$).

(r.iv) Concerning (A5), since we assume that $k\tau < 1$, then we can take $\mathcal{S} > 0$ such that $(k + \mathcal{S})\tau < 1$ and the operator $\mathscr{L}_{\mathcal{S}} := \varphi + \mathcal{S}I$ is monotonically increasing and

(r.a) $\mathscr{L}_{\mathcal{S}}(\tau z) = (k + \mathcal{S})\tau z < z$, for every $z > 0$;

(r.b) $\mathscr{L}_{\mathcal{S}}(\tau z) = (k + \mathcal{S})\tau z = \tau(k + \mathcal{S})z = \tau\mathscr{L}_{\mathcal{S}}(z)$, for every $z > 0$;

(r.c) $\sum_{n=1}^{\infty}\tau^n\mathscr{L}_{\mathcal{S}}{}^n(z) = z\sum_{n=1}^{\infty}(k\tau)^n < \infty$, for every $z > 0$.

It is also obvious that if T is an H^+-type multivalued weak φ-Lipschitz mapping and there exists $k > 0$ such that $\varphi(t) \leq kt$, for $t \geq 0$, then T is also an H^+-type multivalued weak $\widetilde{\varphi}$-Lipschitz mapping for $\widetilde{\varphi}(t) = kt$.

Remark 23. In the proof of Theorems 18 and 21, it is easy to observe that a proper combination of condition (C2) with the α-admissible character of the mapping, (h1), would allow relaxing slightly the definition of partial H^+-type multivalued weak φ-Lipschitz mappings. Indeed, in these theorems, it is possible to replace these hypotheses ((C2) and α-admissibility of T) by the following:

(2^*) for every $x \in X$, $y \in Tx$ with $\alpha(x, y) \geq 1$ and $\varepsilon > 0$, there exists $z \in Ty$ such that $d(y, z) \leq H^+(Ty, Tx) + \varepsilon$ and $\alpha(y, z) \geq 1$.

If we consider self-mappings $T : X \to X$, a procedure similar to Theorem 2.3 [19] gives the following result.

Theorem 24. *Let (X, d) be a complete metric space, let $\alpha : X \times X \to [0, \infty)$ be given, and let $T : X \to X$ be such that*

(t.i) *for each $x \in X$ with $\alpha(x, Tx) \geq 1$, one has $\alpha(Tx, T(Tx)) \geq 1$,*

(t.ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

and there exist $\mathscr{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\begin{aligned}&d(Tx, Ty) \leq \varphi(M(d(x, y), d(x, Tx), d(y, Ty),\\ &\quad d(x, Ty), d(y, Tx))),\\ &\quad \text{for every } (x, y) \in X \times X \text{ with } \alpha(x, y) \geq 1.\end{aligned} \tag{62}$$

Suppose also that (h3) holds and that the functions φ, $\mathscr{M}$ satisfy the following conditions:

(I) *φ is monotonically increasing, $\mathscr{M}$ is increasing in the fourth variable, and conditions (iv) [(a)–(d)] in Theorem 8 hold, where $\tau \leq 1$.*

(II) *Consider $\sum_{n=1}^{\infty}\varphi^n(t) < +\infty$, for every $t > 0$.*

(III) *One of the following conditions holds:*

(III.i) *For $r_3 > 0$ fixed, if r_1, r_2, r_5 are small enough and r_4 is close enough to r_3, then $\mathscr{M}(r_1, r_2, r_3, r_4, r_5) \leq r_3$.*

(III.ii) *(A2) is satisfied and conditions (iii) and (iv-e) in Theorem 8 hold, where* $\mathscr{R} \leq 1$.

Then T has a fixed point in X.

Proof. We take $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$ and define the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$, for some n, then we have a fixed point of T; thus we assume that $x_n \neq x_{n+1}$, for every n (i.e., $d(x_n, x_{n+1}) > 0$, for every $n \in \mathbb{N}$). Using the hypotheses, it is easy to prove that $\alpha(x_n, x_{n+1}) \geq 1$, for every n. Then, for each $n \in \mathbb{N}$, by the nondecreasing character of φ and $\mathscr{M}$ (in the fourth variable) and (iv-a),

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \varphi(\mathscr{M}(\psi(x_{n-1}, x_n))) \\ &= \varphi(\mathscr{M}(d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), \\ &\quad d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}))) \\ &\leq \varphi(\mathscr{M}(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)) \\ &\leq \varphi(\mathscr{N}(d(x_{n-1}, x_n), d(x_n, x_{n+1}))). \end{aligned} \tag{63}$$

If $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, for some $n \in \mathbb{N}$, by the monotonicity of φ, (iv-b), (iv-d), $\tau \leq 1$, and (II), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \varphi(\mathscr{N}(d(x_n, x_{n+1}), d(x_n, x_{n+1}))) \\ &\leq \varphi(\tau d(x_n, x_{n+1})) \leq \varphi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}), \end{aligned} \tag{64}$$

which is a contradiction. Then, $d(x_{n-1}, x_n) \geq d(x_n, x_{n+1})$, for every $n \in \mathbb{N}$; hence, by the monotonicity of φ, (iv-c), (iv-d), and $\tau \leq 1$,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \varphi(\mathscr{N}(d(x_{n-1}, x_n), d(x_{n-1}, x_n))) \\ &\leq \varphi(\tau d(x_{n-1}, x_n)) \leq \varphi(d(x_{n-1}, x_n)). \end{aligned} \tag{65}$$

Using (II), the sequence $\{x_n\}$ is a Cauchy sequence. Then there exists u such that $x_n \to u$ as $n \to \infty$. By (h3), $\alpha(x_n, u) \geq 1$, for every n. Suppose that $d(u, Tu) > 0$; then

$$\begin{aligned} d(u, Tu) &\leq d(u, Tx_n) + d(Tx_n, Tu) \leq d(u, x_{n+1}) \\ &\quad + \varphi(\mathscr{M}(\psi(x_n, u))) = d(x_{n+1}, u) \\ &\quad + \varphi(\mathscr{M}(d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), \\ &\quad d(x_n, Tu), d(u, x_{n+1}))). \end{aligned} \tag{66}$$

In case (III.i), for n being large enough, we have

$$d(u, Tu) \leq d(x_{n+1}, u) + \varphi(d(u, Tu)), \tag{67}$$

so that $d(u, Tu) \leq \varphi(d(u, Tu)) < d(u, Tu)$, which is a contradiction; hence $Tu = u$ and the proof is complete. On the other hand, in case (III.ii), by the monotonicity of φ and (II), we have

$$\begin{aligned} d(u, Tu) &\leq \liminf_{n\to\infty} \varphi(\mathscr{M}(d(x_n, u), d(x_n, x_{n+1}), \\ &\quad d(u, Tu), d(x_n, Tu), d(u, x_{n+1}))) \\ &\leq \varphi\Big(\lim_{n\to\infty} \mathscr{M}(d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), \\ &\quad d(x_n, Tu), d(u, x_{n+1}))\Big) \leq \varphi(\mathscr{M}(0, 0, d(u, Tu), \\ &\quad d(u, Tu), 0)) \leq \varphi(\mathscr{N}(0, d(u, Tu))) \leq \varphi(\mathscr{R} d(u, \\ &\quad Tu)) \leq \varphi(d(u, Tu)) < d(u, Tu), \end{aligned} \tag{68}$$

a contradiction again and the proof is finished. □

Remark 25. If $\mathscr{M}(r_1, r_2, r_3, r_4, r_5) := \max\{r_1, r_2, r_3, (r_4 + r_5)/2\}$ in Theorem 24, then we have Theorem 2.3 [19]. Indeed, all the conditions are satisfied (see the proof of Corollary 11) and, for $r_3 > 0$ fixed, if r_1, r_2, r_5 are small enough and r_4 is close enough to r_3, then $\mathscr{M}(r_1, r_2, r_3, r_4, r_5) = r_3$; hence (III.i) is also fullfilled.

Remark 26. Certain conditions in Theorem 24 also extend some hypotheses in [20].

As indicated in [21], a function α can be defined in connection with a binary relation $\mathfrak{R}$ in X (which could be, e.g., a partial ordering in X). Thus, Theorem 4.7 [21] can be extended to H^+-type multivalued weak φ-Lipschitz mappings with respect to $\mathfrak{R}$, which are defined as follows. Here, we only consider the case of metric spaces, but b-metric spaces could be considered accordingly.

Definition 27. Let (X, d) be a metric space and $\mathfrak{R}$ a binary relation on X. A multivalued map $T : X \to \mathscr{CB}(X)$ is called an H^+-type multivalued weak φ-Lipschitz mapping with respect to $\mathfrak{R}$ if condition (C2) in Definition 2 holds and there exist $\mathscr{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} H^+(Tx, Ty) &\leq \varphi(\mathscr{M}(d(x, y), d(x, Tx), d(y, Ty), \\ &\quad d(x, Ty), d(y, Tx))), \\ &\qquad \text{for every } x, y \in X \text{ with } x\mathfrak{R}y. \end{aligned} \tag{69}$$

If, moreover, there exists $k \in (0, 1)$ such that $\varphi(t) \leq kt$, for every $t \geq 0$, one says that T is an H^+-type multivalued weak φ-contraction with respect to $\mathfrak{R}$.

Theorem 28. *Let (X, d) be a complete metric space, $\mathfrak{R}$ a binary relation on X, and $T : X \to \mathscr{CB}(X)$ an H^+-type multivalued weak φ-Lipschitz mapping with respect to $\mathfrak{R}$ such that*

(H1) *T is weakly preserving, that is, for each $x \in X$ and $y \in Tx$ with $x\mathfrak{R}y$, one has $y\mathfrak{R}z$, for every $z \in Ty$,*

(H2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0\mathfrak{R}x_1$,*

(H3) *if $\{x_n\}$ is a sequence in X such that $x_n\mathfrak{R}x_{n+1}$, for all $n \in \mathbb{N}$, and there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$, then $x_n\mathfrak{R}x$, for every n,* (H4) *one of the following conditions holds:*

(H4-i) *the functions φ, $\mathcal{M}$ satisfy conditions (i), (ii), (iii), (iv) [(a)–(e)], and (v) in Theorem 8 or*

(H4-ii) *the functions φ, $\mathcal{M}$ satisfy (A1)–(A5).*

Then T has a fixed point in X.

Remark 29. The condition of weakly preservance of T, (H1), is fulfilled if the following condition holds:

$$\text{if } x\mathfrak{R}y, \quad \text{then } Tx \leq_{\mathfrak{R}} Ty, \tag{70}$$

where, given $A, B \subseteq X$, we define

$$A \leq_{\mathfrak{R}} B \quad \text{if, for each } a \in A,\ b \in B, \text{we have } a\mathfrak{R}b. \tag{71}$$

Remark 30. In the proof of Theorem 28, we can relax slightly the definition of H^+-type multivalued weak φ-Lipschitz mappings with respect to $\mathfrak{R}$, by replacing hypotheses (C2) and weakly preservance of T, (H1), by the following combination of both:

(2^{**}) for every $x \in X$, $y \in Tx$ with $x\mathfrak{R}y$ and $\varepsilon > 0$, there exists $z \in Ty$ such that $d(y,z) \leq H^+(Ty,Tx) + \varepsilon$ and $y\mathfrak{R}z$.

For self-mappings $T : X \to X$, we have the following corollary of Theorem 24.

Corollary 31. *Let (X,d) be a complete metric space, $\mathfrak{R}$ a binary relation on X, and $T : X \to X$ such that*

(c.i) *for each $x, y \in X$ with $x\mathfrak{R}y$, one has $Tx\mathfrak{R}Ty$,*

(c.ii) *there exists $x_0 \in X$ such that $x_0\mathfrak{R}Tx_0$,*

and there exist $\mathcal{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} d(Tx,Ty) \leq \varphi(M(d(x,y), d(x,Tx), d(y,Ty),\\ d(x,Ty), d(y,Tx))), \end{aligned} \tag{72}$$

$$\textit{for every } x, y \in X \textit{ with } xRy.$$

Suppose also that (H3) holds and that the functions φ, $\mathcal{M}$ satisfy conditions (I), (II), and (III) in Theorem 24.

Then T has a fixed point in X.

In the previous result, if the metric space admits a partial ordering $\preceq$, then the relation $\mathfrak{R}$ can be chosen as $\preceq$, so that the conditions in Corollary 31 reduce to those in Corollary 2.4 [19].

Also in the lines of [19], we consider the family Φ of functions defined as follows: $\varphi \in \Phi$ if and only if $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing and such that

$$\sum_{n=1}^{\infty} \varphi^n(z) < \infty, \quad \forall z > 0. \tag{73}$$

It is clear that if $\varphi \in \Phi$, then $\varphi(t) < t$, for every $t > 0$ (see [19] and the references therein).

We have the following result, where condition (C2) is removed.

Theorem 32. *Let (X,d) be a complete metric space, let $\alpha : X \times X \to [0,\infty)$ be given, and let $T : X \to \mathcal{CB}(X)$ be a multifunction such that conditions (h1), (h2), and (h3) hold. Suppose, further, that there exist $\mathcal{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\begin{aligned} H^+(Tx,Ty) \leq \varphi(M(d(x,y), d(x,Tx), d(y,Ty),\\ d(x,Ty), d(y,Tx))), \end{aligned} \tag{74}$$

$$\textit{for every } (x,y) \in X \times X \textit{ with } \alpha(x,y) \geq 1,$$

where the functions φ, $\mathcal{M}$ satisfy that

(a1) *$\varphi \in \Phi$ is strictly increasing and $\varphi(0) = 0$,*

(a2) *there exists $\beta > 0$ such that $\mathcal{M}(\psi(x,y)) \leq \beta d(x,y)$, for every $x \neq y$, where $\beta \leq 1/2$,*

(a3) *if $\lim_{n\to\infty} x_n = x$, then $\liminf_{n\to\infty} \mathcal{M}(\psi(x_n,x)) = 0$.*

Then T has a fixed point in X.

Proof. We proceed similarly to the proof of Theorem 2.1 [19]. We include it here for completeness. We take $x_0 \in X$ and x_1 in Tx_0 with $\alpha(x_0,x_1) \geq 1$ given by hypothesis. The proof is finished if $x_0 = x_1$ or if $x_1 \in Tx_1$. Then, we assume that $x_0 \neq x_1 \notin Tx_1$ and take $\varepsilon_0 > 1$ arbitrarily fixed. Then, by the definition of H^+ and inequality (74),

$$\begin{aligned} 0 &< \frac{1}{2} d(x_1, Tx_1) \leq \frac{1}{2}[\rho(Tx_0,Tx_1) + \rho(Tx_1,Tx_0)]\\ &= H^+(Tx_0,Tx_1) \leq \varphi(\mathcal{M}(\psi(x_0,x_1)))\\ &< \varepsilon_0 \varphi(\mathcal{M}(\psi(x_0,x_1))), \end{aligned} \tag{75}$$

so that we can choose $x_2 \in Tx_1$ such that $0 < (1/2)d(x_1,x_2) < \varepsilon_0\varphi(\mathcal{M}(\psi(x_0,x_1)))$ ($x_1 \neq x_2$ due to $x_1 \notin Tx_1$). By α-admissibility, we have that $\alpha(x_1,x_2) \geq 1$. Moreover, since $\varphi(0) = 0$, then $z_0 := \mathcal{M}(\psi(x_0,x_1)) > 0$ and $0 < (1/2)d(x_1,x_2) < \varepsilon_0\varphi(z_0)$. Since φ is strictly increasing, we have $\varphi((1/2)d(x_1,x_2)) < \varphi(\varepsilon_0\varphi(z_0))$. Next, we take $\varepsilon_1 = \varphi(\varepsilon_0\varphi(z_0))/\varphi((1/2)d(x_1,x_2)) > 1$. Now, if $x_2 \in Tx_2$, the proof is finished, so we assume that $x_2 \notin Tx_2$ and, similarly, we get

$$\begin{aligned} 0 &< \frac{1}{2} d(x_2, Tx_2) \leq H^+(Tx_1,Tx_2)\\ &\leq \varphi(\mathcal{M}(\psi(x_1,x_2))) < \varepsilon_1 \varphi(\mathcal{M}(\psi(x_1,x_2))); \end{aligned} \tag{76}$$

then there exists $x_3 \in Tx_2$ ($x_3 \neq x_2$) such that

$$\begin{aligned} 0 &< \frac{1}{2} d(x_2,x_3) < \varepsilon_1 \varphi(\mathcal{M}(\psi(x_1,x_2)))\\ &\leq \varepsilon_1 \varphi(\beta d(x_1,x_2)) \leq \varepsilon_1 \varphi\left(\frac{1}{2} d(x_1,x_2)\right)\\ &= \varphi(\varepsilon_0 \varphi(z_0)). \end{aligned} \tag{77}$$

Besides, $\alpha(x_2,x_3) \geq 1$.

Using that φ is strictly increasing, we have $\varphi((1/2)d(x_2, x_3)) < \varphi^2(\varepsilon_0\varphi(z_0))$, and we can take $\varepsilon_2 = \varphi^2(\varepsilon_0\varphi(z_0))/\varphi((1/2)d(x_2, x_3)) > 1$. Again, if $x_3 \in Tx_3$, the proof is finished, so we assume that $x_3 \notin Tx_3$ and, similarly, we get

$$\begin{aligned} 0 &< \frac{1}{2}d(x_3, Tx_3) \leq H^+(Tx_2, Tx_3) \\ &\leq \varphi(\mathcal{M}(\psi(x_2, x_3))) < \varepsilon_2\varphi(\mathcal{M}(\psi(x_2, x_3))), \end{aligned} \tag{78}$$

and then there exists $x_4 \in Tx_3$ $(x_4 \neq x_3)$ such that

$$\begin{aligned} 0 &< \frac{1}{2}d(x_3, x_4) < \varepsilon_2\varphi(\mathcal{M}(\psi(x_2, x_3))) \\ &\leq \varepsilon_2\varphi(\beta d(x_2, x_3)) \leq \varepsilon_2\varphi\left(\frac{1}{2}d(x_2, x_3)\right) \\ &= \varphi^2(\varepsilon_0\varphi(z_0)). \end{aligned} \tag{79}$$

Besides, $\alpha(x_3, x_4) \geq 1$.

In general, for $n \in \mathbb{N}$, if $x_n \in Tx_{n-1}$ $(x_n \neq x_{n-1})$ is chosen such that $\alpha(x_{n-1}, x_n) \geq 1$ and $\varphi((1/2)d(x_{n-1}, x_n)) < \varphi^{n-1}(\varepsilon_0\varphi(z_0))$, we can take $\varepsilon_{n-1} = \varphi^{n-1}(\varepsilon_0\varphi(z_0))/\varphi((1/2)d(x_{n-1}, x_n)) > 1$. If $x_n \in Tx_n$, the proof is finished, so we assume that $x_n \notin Tx_n$ and, similarly, we get

$$\begin{aligned} 0 &< \frac{1}{2}d(x_n, Tx_n) \leq H^+(Tx_{n-1}, Tx_n) \\ &\leq \varphi(\mathcal{M}(\psi(x_{n-1}, x_n))) \\ &< \varepsilon_{n-1}\varphi(\mathcal{M}(\psi(x_{n-1}, x_n))), \end{aligned} \tag{80}$$

and then there exists $x_{n+1} \in Tx_n$ $(x_{n+1} \neq x_n)$ such that

$$\begin{aligned} 0 &< \frac{1}{2}d(x_n, x_{n+1}) < \varepsilon_{n-1}\varphi(\mathcal{M}(\psi(x_{n-1}, x_n))) \\ &\leq \varepsilon_{n-1}\varphi(\beta d(x_{n-1}, x_n)) \leq \varepsilon_{n-1}\varphi\left(\frac{1}{2}d(x_{n-1}, x_n)\right) \\ &= \varphi^{n-1}(\varepsilon_0\varphi(z_0)) \end{aligned} \tag{81}$$

and $\alpha(x_n, x_{n+1}) \geq 1$.

Hence, we can take a sequence $\{x_n\} \subset X$ with $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$, $\alpha(x_n, x_{n+1}) \geq 1$, and $0 < (1/2)d(x_n, x_{n+1}) < \varphi^{n-1}(\varepsilon_0\varphi(z_0))$ for every $n \in \mathbb{N}$.

This allows proving that $\{x_n\}$ is a Cauchy sequence since, for $n, m \in \mathbb{N}$ with $n < m$, we get

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq 2\sum_{i=n}^{m-1} \varphi^{i-1}(\varepsilon_0\varphi(z_0)), \tag{82}$$

which tends to 0 as $n \to \infty$ since $\varphi \in \Phi$.

Now, by completeness of X, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. By hypothesis, $\alpha(x_n, u) \geq 1$, for every n. Suppose that $d(u, Tu) > 0$. Then

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &\leq \rho(Tx_n, Tu) + d(u, x_{n+1}) \\ &\leq 2H^+(Tx_n, Tu) + d(u, x_{n+1}) \\ &\leq 2\varphi(\mathcal{M}(\psi(x_n, u))) + d(u, x_{n+1}), \end{aligned} \tag{83}$$

for every n. Since $\varphi(z) < z$, for all z, then, for every $n \in \mathbb{N}$,

$$0 \leq \varphi(\mathcal{M}(\psi(x_n, u))) \leq \mathcal{M}(\psi(x_n, u)) \tag{84}$$

so that the hypothesis $\liminf_{n\to\infty} \mathcal{M}(\psi(x_n, u)) = 0$ implies that $\liminf_{n\to\infty} \varphi(\mathcal{M}(\psi(x_n, u))) = 0$. This, joint to $d(u, Tu) - d(u, x_{n+1}) \leq 2\varphi(\mathcal{M}(\psi(x_n, u)))$ and the convergence of $\{x_n\}$, implies that $u \in Tu$. □

Remark 33. Note that condition (74) coincides with (40). In Theorem 32, if we consider the Hausdorff distance H in (74) and $\beta \leq 1$, the conclusion holds.

Remark 34. If we take $\mathcal{M}(r_1, r_2, r_3, r_4, r_5) := r_1$, then $\mathcal{M}(\psi(x, y)) = d(x, y)$ and if $\lim_{n\to\infty} x_n = x$, then $\liminf_{n\to\infty} \mathcal{M}(\psi(x_n, x)) = \liminf_{n\to\infty} d(x_n, x) = 0$. Hence, for the Hausdorff distance H, the existence of fixed point follows, such as in Theorem 2.1 [19], since $\beta = 1$. On the other hand, for the metric H^+, if we consider $\mathcal{M}(r_1, r_2, r_3, r_4, r_5) := (1/2)r_1$, then the condition is valid for $\beta = 1/2$ and Theorem 32 applies.

Theorem 35. *Let (X, d) be a complete metric space, $\mathfrak{R}$ a binary relation on X, and $T : X \to \mathcal{CB}(X)$ such that conditions (H1), (H2), and (H3) hold. Suppose that there exist $\mathcal{M} : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\begin{aligned} &H^+(Tx, Ty) \leq \varphi(M(d(x, y), d(x, Tx), d(y, Ty), \\ &\quad d(x, Ty), d(y, Tx))), \\ &\qquad \text{for every } x, y \in X \text{ with } xRy, \end{aligned} \tag{85}$$

where the functions φ, $\mathcal{M}$ satisfy conditions (a1), (a2), and (a3) in Theorem 32. Then T has a fixed point in X.

Remark 36. Conditions (69) and (85) are the same.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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